

IMPERIAL

SPECIAL SPINORS IN GENERALISED
SPIN GEOMETRY

DIEGO ARTACHO DE OBESO

Supervised by

DR MARIE-AMÉLIE LAWN
DR STEVEN SIVEK

Department of Mathematics
Imperial College London
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*Deo, cui omnis gloria, et Domino Iesu Christo,
per cuius sanguinem salvi facti sumus.*

Abstract

Special spinors play a key role in understanding geometric and topological properties of manifolds. This thesis investigates how symmetry helps construct and classify special spinors on homogeneous spaces, both in classical and generalised spin^r settings. A central idea is that symmetry provides algebraic control over spinorial equations, making explicit constructions and classification results possible.

In the first part of the thesis, we establish new results in the classical spin setting. First, we provide a combinatorial description of invariant spinors on flag manifolds in terms of the root system of the corresponding Lie algebras. We then give the first complete classification of invariant generalised Killing spinors on three-dimensional Lie groups equipped with arbitrary left-invariant metrics. Remarkably, their existence and the symmetry properties of the associated endomorphism depend only on the underlying Lie algebra, and not on the left-invariant metric itself. Finally, we construct the first known examples of invariant generalised Killing spinors whose associated endomorphisms have more than four distinct eigenvalues.

The second part of the thesis concerns a generalisation of spin structures known as spin^r structures. This framework allows spinorial techniques to be applied to all oriented manifolds, even if they are not spin. We systematically develop the theory of invariant spin^r structures on homogeneous spaces and provide a representation-theoretic classification. We then compute the minimal r needed for each homogeneous realisation of spheres to admit invariant spin^r structures, and establish an equivalence between certain holonomy lifts and the existence of special invariant spin structures on spheres. Finally, we construct explicit special spin^r spinors on complex and quaternionic projective spaces that encode geometric information about these manifolds.

Bringing together tools from algebra, topology, geometry, and representation theory, this thesis advances the study of classical and spin^r spinors, and opens new directions for exploring geometric structures through spinorial techniques.

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Declarations

Declaration of Originality

I hereby declare that the work presented in this thesis is my own, except where explicitly acknowledged. To the best of my knowledge, the work is original, and collaborative contributions have been appropriately acknowledged. Most of the results of this thesis have been published in [AS25, Art25, AL25, AH25]. I have produced all these works during my PhD.

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Note

For completeness, I note that during my PhD I also produced the works [ALO25, ALO26], which concern topics unrelated to the main focus of this thesis and are not included here.

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1

Introduction

A fundamental question in differential geometry is whether a given smooth manifold admits a *special G -structure*, i.e., a noteworthy reduction of the structure group to a subgroup G of the general linear group. Classical examples include Riemannian, almost-complex, and almost-quaternionic structures. Positive scalar curvature metrics, complex, Kähler, and quaternionic structures are examples of G -structures that satisfy some additional conditions.

Spin geometry offers a particularly elegant and powerful framework for studying these structures. It gives a way of accessing global geometric information about *spin manifolds* via *spinors*, which are sections of a certain naturally defined complex vector bundle, the *spinor bundle*. A central theme in spin geometry is the deep correspondence between geometric structures and the existence of special spinors. More precisely, there are numerous situations in which the following two conditions are equivalent or closely related:

- the existence of a special geometric structure on a spin manifold, and
- the existence of a spinor satisfying a specific differential equation.

This interplay between geometry and analysis lies at the heart of this thesis.

For instance, recall that a Riemannian n -manifold M carrying a non-zero *parallel* spinor, i.e., a spinor ψ satisfying

$$\nabla_X \psi = 0 \tag{1.1}$$

for all vector fields $X \in \Gamma(TM)$, where ∇ is the covariant derivative induced by the Levi-Civita connection, is necessarily Ricci-flat. In fact, Wang characterised the Riemannian holonomy of complete, simply connected, and irreducible spin manifolds in terms of the dimension of the space of parallel spinors [Wan89].

More generally, the existence of a non-zero *Killing* spinor, i.e., a spinor ψ satisfying

$$\nabla_X \psi = \lambda X \cdot \psi \quad (1.2)$$

for all $X \in \Gamma(TM)$, where $\lambda \in \mathbb{C}$ and $\cdot$ is Clifford multiplication, is Einstein with scalar curvature equal to $4n(n-1)\lambda^2$ [Fri80, BFGK91]. These have been a topic of fruitful research in the last decades [GH08, Lic86, FK88, Bau89, Kat90, Lic89, Lic88, CSD24]. Notably, Bär classified complete simply connected Riemannian manifolds that carry Killing spinors with $\lambda \in \mathbb{R}$ [Bär93]. The range of spaces that admit these spinors is rather restricted, and so different relaxations of the Killing condition have been considered, such as spinors that satisfy (1.2) where λ is allowed to be a smooth complex-valued function [Rad91]. In this thesis, however, we use the term *generalised Killing spinors* to refer to a specific class introduced below.

Special spinors are of interest in submanifold theory [KS96, Fri98, BGM05, BLR17, CSD24]. One can restrict a spinor ψ on an ambient manifold to a spinor ψ' on a hypersurface. If ψ satisfies an equation $\mathcal{E}$, then ψ' satisfies a modified equation $\mathcal{E}'$, which incorporates information on the embedding of the hypersurface. It is interesting to ask whether this construction can be reversed: given a spinor ψ' satisfying the equation $\mathcal{E}'$, is it possible to embed the manifold with codimension one so that the previous construction recovers ψ' ?

An example of this situation is the following. The restriction of a parallel spinor to a hypersurface satisfies the so-called *generalised Killing equation*

$$\nabla_X \psi = A(X) \cdot \psi$$

for all $X \in \Gamma(TM)$, where $A \in \Gamma(\text{End}(TM))$ is a symmetric endomorphism of the tangent bundle. In this particular case, it is half the second fundamental form of the embedding of the hypersurface [BGM05]. In the analytic category, the problem of reconstructing the embedding from a generalised Killing spinor was solved in [AMM13]. These spinors are also intimately related to G -structures: they are in one-to-one correspondence with hypo $SU(2)$ -, half-flat $SU(3)$ -, and co-calibrated G_2 -structures in dimensions five, six, and seven, respectively [CS07, ACFH15]. Un-

fortunately, they are not yet well understood, and a classification is out of reach even in simple low-dimensional manifolds such as the round sphere S^3 , although progress has been made [MS14a, MS14b].

So far, the spinorial equations we have considered involve the covariant derivative on the spinor bundle induced by the Levi-Civita connection of the Riemannian metric. However, for certain geometric structures, there exist other metric connections with non-vanishing torsion which are more *adapted* to the geometry. For example, there is a naturally defined metric connection with skew-symmetric torsion which interacts nicely with the data of a 3-Sasakian structure [AD20]. In a similar spirit, homogeneous spaces come equipped with the so-called *canonical* or *Ambrose-Singer connection*, which parallelises all invariant tensors. This connection coincides with the Levi-Civita connection on symmetric spaces.

Spinors are complicated objects, and we cannot always expect to obtain explicit expressions for them. However, if M is a homogeneous G -space for a Lie group G and the spin structure is G -invariant, then the group G acts naturally on the space of spinors, and we can consider the spinors that are stabilised under the action of G . Geometrically, this is the space of spinors that are parallelised by the canonical connection. Sometimes, this space is zero. However, in many cases, it is large enough to contain spinors that are special *with respect to the Levi-Civita connection*. This is to be expected, since the canonical connection is related to the Levi-Civita connection of an invariant metric. We introduce the relevant definitions and set up the necessary background in Chapter 2.

In Chapter 3, we study invariant spinors on an important family of homogeneous spaces, namely flag manifolds. These are obtained as quotients of compact simply connected simple Lie groups by their maximal tori. For each complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, its associated maximal flag manifold is denoted by $\mathrm{MF}(\mathfrak{g}_{\mathbb{C}})$, and it admits an invariant spin structure. We prove the following result, which gives a description of the space of invariant spinors in terms of combinatorial properties of the corresponding root system:

Theorem (see Theorem 3.5). *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra with positive roots $a_1, \dots, a_r$. Then, $\mathrm{MF}(\mathfrak{g}_{\mathbb{C}})$ admits non-trivial invariant spinors if and only if $\{a_1, \dots, a_r\}$ is strongly linearly dependent. Moreover, the dimension of the space of invariant spinors is equal to the number of strong linear combinations of the set of positive roots that sum to zero. In particular, it is always even.* $\square$

We then run through the classification of complex simple Lie algebras, and obtain

lower bounds for the dimension of the space of invariant spinors in each case. The result is summarised in Table 1.1 (see Theorem 3.7).

$\mathfrak{g}_{\mathbb{C}}$	$\dim(\Sigma_{\text{inv}})$
$A_n, \quad n \geq 1$	$\geq 2^{\frac{n}{2}}$ if n is even 0 otherwise
$B_n, \quad n \geq 2$	0
$C_n, \quad n \geq 3$	$\geq 2^{\lfloor \frac{n+1}{4} \rfloor}$ if $n \equiv 0, 3 \pmod{4}$ 0 otherwise
$D_n, \quad n \geq 4$	$\geq 2^{\lfloor \frac{n+1}{4} \rfloor}$ if $n \equiv 0, 1 \pmod{4}$ 0 otherwise
E_6	13697920
E_7	0
E_8	≥ 369600
F_4	34432
G_2	4

Table 1.1: *Dimension of the space of invariant spinors Σ_{inv} of the maximal flag manifold $\text{MF}(\mathfrak{g}_{\mathbb{C}})$ of each complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ (see Theorem 3.7).*

In Chapter 4, we turn to the study of generalised Killing spinors in dimension three. We consider arbitrary left-invariant metrics on three-dimensional Lie groups and obtain a complete classification of invariant generalised Killing spinors in this setting. One particularly striking result is the following rigidity phenomenon:

Theorem (see Lemma 4.5). *Let G be a connected three-dimensional Lie group with Lie algebra $\mathfrak{g}$, and let g be a left-invariant metric on G carrying a non-zero invariant generalised Killing spinor. Then every invariant spinor is generalised Killing with the same endomorphism.* $\square$

We then show that the question of existence of invariant generalised Killing spinors in this context is independent of the left-invariant metric. In fact, to each left-invariant metric g on a three-dimensional Lie group G we associate an endomorphism $A(g)$ of the Lie algebra $\mathfrak{g}$ such that *every* invariant spinor satisfies the generalised Killing equation with endomorphism $A(g)$. Surprisingly, the question whether $A(g)$ is symmetric is independent of g .

Moreover, we obtain the first known examples of invariant generalised Killing spinors whose associated endomorphism has more than four distinct eigenvalues, giving an answer to questions in [AFS15, AHL23]. We find these in high-dimensional Heisenberg groups equipped with a deformation of the standard left-invariant metric – see Theorem 4.4.

So far, we have assumed that our manifold M admits a spin structure, so that we can rightfully talk about spinors. However, not every manifold can be endowed with a spin structure. The question then naturally arises whether one can apply the tools of spin geometry to non-spin manifolds. The answer is affirmative, and there are several possible approaches. The unifying idea is to consider suitable *enlargements* of the spin groups, i.e., Lie groups L_n along with homomorphisms

$$\mathrm{Spin}(n) \xrightarrow{\iota_n} L_n \xrightarrow{p_n} \mathrm{SO}(n)$$

such that $p_n \circ \iota_n$ is the usual two-sheeted covering $\mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$. Hence, an oriented Riemannian n -manifold admitting a lift of the structure group to L_n is a weaker condition than being spin. Following ideas of Friedrich and Trautman [FT00], the so-called *spinorial Lipschitz structures* have garnered much attention in recent years [LS18, LS19, LS22]. These appear by following the inverse approach: starting with a suitable generalisation of the concept of *spinor bundle*, one investigates the enlargement L_n of $\mathrm{Spin}(n)$ to which this bundle corresponds. These L_n are called *Lipschitz groups*.

Another choice of L_n corresponds to *complexifying* or *quaternionifying* the spin group, i.e., taking

$$\mathrm{Spin}^{\mathbb{C}}(n) := \frac{\mathrm{Spin}(n) \times \mathrm{U}(1)}{\langle(-1, -1)\rangle}, \quad \mathrm{Spin}^{\mathbb{H}}(n) := \frac{\mathrm{Spin}(n) \times \mathrm{Sp}(1)}{\langle(-1, -1)\rangle}$$

with the obvious homomorphisms. These give rise to $\mathrm{spin}^{\mathbb{C}}$ and $\mathrm{spin}^{\mathbb{H}}$ structures respectively, which have been at the centre of much research in recent decades [LeB21, Nag95, LS19, OT95, Che17, Law23, DK25].

While every spin manifold is $\mathrm{spin}^{\mathbb{C}}$ and every $\mathrm{spin}^{\mathbb{C}}$ manifold is in turn $\mathrm{spin}^{\mathbb{H}}$, the converses are not true. For example, even-dimensional complex projective spaces $\mathbb{CP}^{2n}$ are not spin, but they are $\mathrm{spin}^{\mathbb{C}}$. Similarly, the Wu manifold $W = \mathrm{SU}(3)/\mathrm{SO}(3)$ is not $\mathrm{spin}^{\mathbb{C}}$, yet it is $\mathrm{spin}^{\mathbb{H}}$ [Che17]. Moreover, there exist examples of manifolds that are not even $\mathrm{spin}^{\mathbb{H}}$, such as $W \times W$ [AM21].

It is a well-known fact that every almost-complex Riemannian manifold is $\mathrm{spin}^{\mathbb{C}}$, and every almost-quaternionic Riemannian manifold is $\mathrm{spin}^{\mathbb{H}}$. Furthermore, it is known that every oriented Riemannian manifold of dimension ≤ 3 is spin. In 2005, Teichner and Vogt [TV94] completed the work started by Hirzebruch and Hopf in 1958 [HH58], showing that every oriented Riemannian manifold of dimension ≤ 4 admits a $\mathrm{spin}^{\mathbb{C}}$ structure. These structures are crucial for Seiberg-Witten theory, which has become

an essential tool in the study of smooth four-manifolds [Mor96]. In 2021, Albanese and Milivojević [AM21, AM23] showed that every oriented Riemannian manifold of dimension ≤ 5 is $\text{spin}^{\mathbb{H}}$, and for closed manifolds, this is true for dimensions ≤ 7 . In [OT95], in analogy to Seiberg-Witten theory, the authors construct quaternionic monopoles, which are associated to $\text{spin}^{\mathbb{H}}$ structures on four-manifolds.

Finally, an alternative choice of L_n was introduced by Espinosa and Herrera [EH16], who had the idea of *spinorially twisting* the spin group. In our setting, this corresponds to taking, for $r \in \mathbb{N}$, the groups

$$\text{Spin}^r(n) := \frac{\text{Spin}(n) \times \text{Spin}(r)}{\langle(-1, -1)\rangle}$$

with the obvious homomorphisms. We say that an oriented Riemannian n -manifold is spin^r if it admits a lift of the structure group to $\text{Spin}^r(n)$. The case $r = 1$ is the classical spin case, and the cases $r = 2$ and $r = 3$ correspond to $\text{spin}^{\mathbb{C}}$ and $\text{spin}^{\mathbb{H}}$ respectively. They call these structures *spinorially twisted spin structures* [EH16, HS19, HNT18]. Albanese and Milivojević [AM21] refer to them as *generalised spin^k structures*. They show that an oriented Riemannian manifold M is spin^r if and only if it can be embedded into a spin manifold with codimension r . They also proved that there is no $r \in \mathbb{N}$ such that every oriented Riemannian manifold is spin^r . This result can be interpreted as ensuring that this chain of structures does not *stabilise*. As pointed out by Lawson in [Law23], this is remarkable, as every manifold embeds into an orientable manifold with codimension 1.

Spin^r structures come equipped with naturally associated spinor bundles, whose special sections encode rich geometric information. For example, all special Riemannian holonomies in Berger's list can be characterised in terms of parallel pure spin^r spinors - see [HS19, §4].

In Chapter 5, we carry out a general study of invariant spin^r structures on homogeneous spaces, and give a classification in terms of representation-theoretical data:

Theorem (see Theorem 5.13). *Let G/H be an n -dimensional oriented Riemannian homogeneous space with H connected and isotropy representation $\sigma: H \rightarrow \text{SO}(n)$. Then, there is a bijective correspondence between*

- *G -invariant spin^r structures on G/H modulo G -equivariant equivalence of spin^r structures, and*

- Lie group homomorphisms $\varphi: H \rightarrow \mathrm{SO}(r)$ such that $\sigma \times \varphi: H \rightarrow \mathrm{SO}(n) \times \mathrm{SO}(r)$ lifts to $\mathrm{Spin}^r(n)$ modulo conjugation by an element of $\mathrm{SO}(r)$. $\square$

This extends results described in [CGT93, §5]. We then find, for each homogeneous realisation of the spheres [MS43], the minimal value of r for which there exists an invariant spin^r structure, completing the picture started in [DSKL22], which only considered the classical spin case - see Table 1.2.

Sphere	Acting group G	Minimal r for G -invariant spin^r structure
S^n	$\mathrm{SO}(n+1)$	$r = n,$ if $n \neq 4$ $r = 3,$ if $n = 4$
S^{2n+1}	$\mathrm{U}(n+1)$	$r = 2$
S^{2n+1}	$\mathrm{SU}(n+1)$	$r = 1$
S^{4n+3}	$\mathrm{Sp}(n+1)$	$r = 1$
S^{4n+3}	$\mathrm{Sp}(n+1) \cdot \mathrm{U}(1)$	$r = 1,$ if n odd $r = 2,$ if n even
S^{4n+3}	$\mathrm{Sp}(n+1) \cdot \mathrm{Sp}(1)$	$r = 1,$ if n odd $r = 3,$ if n even
S^6	G_2	$r = 1$
S^7	$\mathrm{Spin}(7)$	$r = 1$
S^{15}	$\mathrm{Spin}(9)$	$r = 1$

Table 1.2: *Minimal value of r such that the given homogeneous realisation of the sphere admits an invariant spin^r structure.*

Jointly with Lawn, we also proved a result that links certain invariant spin^r structures on spheres to lifts of the holonomy representation to the spin^r group:

Theorem (see Theorem 5.21). *Let G be the holonomy group of a simply connected irreducible non-symmetric Riemannian manifold of dimension $n+1 \geq 4$. Then, G acts transitively on the sphere S^n , by Berger’s classification [Ber55] – see also [Sim62]. Then, the following are equivalent:*

1. *There exists a homomorphic lift of the holonomy group G to $\mathrm{Spin}^r(n+1)$.*
2. *S^n has a G -invariant spin^r structure with strongly G -trivial auxiliary bundle.*

$\square$

Finally, in Chapter 6, we focus on a family of examples. The idea is to extend the computational and representation-theoretical techniques developed in [AHL23] to the broader spin^r setting. Specifically, for each homogeneous realisation of the complex and quaternionic projective spaces equipped with a generic left-invariant

M	G	r	m	$\dim(\Sigma_{*,r}^m)_{\text{inv}}$	Special spinors	Geometry
$\mathbb{CP}^n$	$\text{SU}(n+1)$	2	1	2	pure, parallel	Kähler-Einstein
$\mathbb{CP}^{2n+1}$	$\text{Sp}(n+1)$	2, if n even	1	2	pure, parallel	Kähler-Einstein
		1, if n odd	1	2	generalised Killing	Einstein, nearly-Kähler ($n = 1 \uparrow$)
$\mathbb{HP}^n$	$\text{Sp}(n+1)$	3	n	1	pure, parallel	quaternionic Kähler

Table 1.3: *For each compact, simple, and simply connected Lie group G acting transitively on M : the minimum values of r, m such that M admits a G -invariant spin^r structure that carries a non-zero invariant m -twisted spin^r spinor, the dimension of the space of such invariant spinors, and the geometric significance of these. For $\uparrow$, see [BFGK91, p. 146].*

metric, we determine the minimal values of $r, m \in \mathbb{N}$ for which non-trivial invariant m -twisted spin^r spinors exist (see Definition 5.24), and we construct explicit bases for these spaces. We then show that well-known geometric properties of these spaces are neatly captured by these special invariant spin^r spinors. A summary of the main results is presented in Table 1.3.

The spin^r framework is universal in the sense that *every* orientable manifold admits a spin^r structure for some r . This flexibility opens the door to applying spinorial techniques in broader contexts, and this work demonstrates that much of the algebraic machinery developed for spin geometry extends to this more general setting.

In summary, the main contributions of this thesis are the following:

1. A combinatorial characterisation of invariant spinors on maximal flag manifolds, in terms of the root system of the corresponding Lie algebra – Theorem 3.5.
2. A complete classification of invariant generalised Killing spinors on three-dimensional Lie groups with arbitrary left-invariant metrics – Chapter 4.
3. The first known examples of invariant generalised Killing spinors whose associated endomorphisms have more than four distinct eigenvalues – Theorem 4.4.
4. A systematic study and representation-theoretic classification of invariant spin^r structures on homogeneous spaces – Chapter 5.
5. A computation of the minimal values of r for which each homogeneous realisation of the sphere admits an invariant spin^r structure – Table 1.2.
6. Equivalence between certain lifts of holonomy representations and existence of special invariant spin^r structures on spheres – Theorem 5.21.
7. Explicit constructions of invariant spin^r spinors that encode geometric information on complex and quaternionic projective spaces – Chapter 6.

2

Spin geometry on homogeneous spaces

For a comprehensive introduction to spin geometry and homogeneous spaces, the author profited from [LM89] and [Arv03], respectively.

Let (M, g) be an oriented Riemannian manifold of dimension n , and let FM denote its principal $\mathrm{SO}(n)$ -bundle of oriented orthonormal frames. A *spin structure* on M is a lift of the structure group of FM along the double covering

$$\lambda_n: \mathrm{Spin}(n) \rightarrow \mathrm{SO}(n).$$

In other words, a spin structure is a pair (P, Φ) , where P is a principal $\mathrm{Spin}(n)$ -bundle over M , and $\Phi: P \rightarrow FM$ is a $\mathrm{Spin}(n)$ -equivariant bundle map covering the identity, with $\mathrm{Spin}(n)$ acting on FM via λ_n . One can easily prove that such Φ is a two-sheeted covering map.

Given a spin structure, the complex vector bundle ΣM associated to it via the complex $2^{\lfloor n/2 \rfloor}$ -dimensional spin representation

$$\Delta_n: \mathrm{Spin}(n) \rightarrow \mathrm{End}_{\mathbb{C}}(\Sigma_n)$$

is called the *spinor bundle*: its sections are known as *spinors*. The spinor bundle is in fact a module over the Clifford bundle of TM : tangent vectors act fibrewise on

spinors via *Clifford multiplication*, characterised by the identity

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$$

for all vector fields $X, Y \in \Gamma(TM)$ and spinors $\psi \in \Gamma(\Sigma M)$.

As λ_n and Φ are covering maps, the pullback along Φ of a connection on FM naturally induces a connection on P , and hence a covariant derivative on the associated vector bundle ΣM . In particular, the Levi-Civita connection of g induces the so-called *spin connection* on ΣM .

We are interested in the study of *special spinors*, i.e., sections of the spinor bundle that satisfy some additional property. In this thesis, we will only be concerned with those spinors $\psi \in \Gamma(\Sigma M)$ which satisfy the *generalised Killing equation*

$$\nabla_X \psi = A(X) \cdot \psi \tag{2.1}$$

for all vector fields $X \in \Gamma(TM)$, where ∇ is the spin connection, $A \in \Gamma(\text{End}(TM))$ is symmetric with respect to g , and $\cdot$ denotes Clifford multiplication. If A is a constant multiple of the identity, ψ is said to be *Killing*, and if $A = 0$ we call ψ *parallel*.

The problem of finding generalised Killing spinors becomes particularly tractable when M is a homogeneous space of a Lie group G and the relevant geometric structures are compatible with the action of G . In this setting, the manifold's symmetries often allow for concrete algebraic descriptions of objects that are otherwise defined abstractly. The goal of this chapter is to develop the necessary tools to study spinors that are invariant under the action of G . We begin by explaining how invariant geometric structures – such as Riemannian metrics, spin structures, spinor bundles, and connections – can be encoded in terms of algebraic data. In particular, we show how questions about invariant spinors can often be translated into purely algebraic problems.

2.1 Homogeneous spaces

Let G be a connected Lie group acting smoothly and transitively on an oriented smooth manifold M of dimension n . Choose a point $x \in M$, and let $H \subseteq G$ be its stabiliser. Then, the natural map $G \rightarrow M$, $g \mapsto g \cdot x$, descends to a G -equivariant

diffeomorphism

$$G/H \cong M, \quad gH \mapsto g \cdot x =: L_g(x). \quad (2.2)$$

We call M an *oriented homogeneous G -space*.

In all the examples considered in this thesis, the isotropy group H will be compact and connected, so we shall assume this throughout. This allows us to use the standard averaging trick to construct an inner product on the Lie algebra $\mathfrak{g}$ of G that is invariant under the adjoint action of H on $\mathfrak{g}$. The orthogonal complement $\mathfrak{m}$ of the Lie algebra $\mathfrak{h}$ of H is invariant under the adjoint action of H , and we obtain a *reductive decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \quad (2.3)$$

The vector space $\mathfrak{m}$ can be identified with the tangent space of M at the point x :

$$\mathfrak{m} \cong T_x M, \quad X \mapsto \left. \frac{d}{dt} \right|_{t=0} \exp(tX) \cdot x. \quad (2.4)$$

As H stabilises x , each $h \in H$ gives a linear isomorphism

$$\sigma(h) := dL_h(x): T_x M \rightarrow T_x M$$

which under the identification (2.4) corresponds to the adjoint action of h on $\mathfrak{m}$. This is because, for each $X \in \mathfrak{m}$ and $h \in H$,

$$\begin{aligned} \sigma(h) \left(\left. \frac{d}{dt} \right|_{t=0} \exp(tX) \cdot x \right) &= \left. \frac{d}{dt} \right|_{t=0} h \exp(tX) \cdot x \\ &= \left. \frac{d}{dt} \right|_{t=0} h \exp(tX) h^{-1} \cdot x \\ &= \left. \frac{d}{dt} \right|_{t=0} \exp(t \operatorname{Ad}(h)(X)) \cdot x. \end{aligned}$$

This representation $\sigma: H \rightarrow \operatorname{GL}(\mathfrak{m})$ is called the *isotropy representation*.

In the remainder of this chapter, we describe how the relevant global geometric objects can be encoded in purely algebraic terms. This reformulation is what makes it possible to translate the geometric problem of finding special spinors into a more tractable question in representation theory.

- In Section 2.2, we recall the well-known correspondence between G -invariant Riemannian metrics on G/H and H -invariant inner products on $\mathfrak{m}$, and give a complete description of the space of such inner products under mild hypothe-

ses. Although this result is frequently used, we are not aware of a complete proof in the literature.

- In Section 2.3, we introduce the notion of equivariance of principal and vector bundles over homogeneous spaces.
- In Section 2.4, we define invariant spin structures and spinors.
- In Section 2.5, we introduce G -invariant connections and show how they are encoded algebraically via Nomizu maps.
- In Section 2.6, we show how the global problem of finding invariant generalised Killing spinors can be reduced to an algebraic problem.
- Finally, in Section 2.7, we give an explicit description of the spin representation which will allow us to carry out spinorial computations in an explicit manner.

2.2 Invariant metrics

Any Riemannian metric on $M = G/H$ induces an inner product B on $T_x M \cong \mathfrak{m}$. If the metric is G -invariant, then B is H -invariant, i.e.,

$$B(\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y) = B(X, Y),$$

for each $h \in H$ and $X, Y \in \mathfrak{m}$. As H is connected, this is equivalent to B being invariant under the differential action of the Lie algebra $\mathfrak{h}$.

Conversely, any H -invariant inner product B on $\mathfrak{m}$ naturally induces a Riemannian metric on M by translating B using the action of G .

In this section, we will not need H to be compact. We only require the existence of a reductive decomposition (2.3).

Suppose that the adjoint representation of H on $\mathfrak{m}$ decomposes as a direct sum of irreducible components

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_k, \tag{2.5}$$

and assume that there exists a H -invariant inner product on $\mathfrak{m}$. These two conditions always hold if H is compact. We now aim to classify all H -invariant inner products on $\mathfrak{m}$. More specifically, we would like to show that invariant inner products on each irreducible component are unique up to positive scaling (Proposition 2.6), and

that any invariant inner product on $\mathfrak{m}$ makes the decomposition (2.5) orthogonal (Proposition 2.9).

Note that this is false in general: for example, if $k = 2$,

$$\mathfrak{m}_1 = \mathfrak{m}_2 =: \mathfrak{m}',$$

and B' is an invariant inner product on $\mathfrak{m}'$, we can construct an invariant inner product B for which $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are not orthonormal. Indeed, for $\varepsilon > 0$ small enough,

$$B((v_1, w_1), (v_2, w_2)) := B'(v_1, v_2) + B'(w_1, w_2) + \varepsilon (B'(v_1, w_2) + B'(w_1, v_2))$$

defines an inner product on $\mathfrak{m}$ which is obviously invariant, but the two irreducible components of $\mathfrak{m}$ are not B -orthogonal. As we shall see in Proposition 2.9, the existence of isotypic (i.e. mutually isomorphic) components of $\mathfrak{m}$ is essentially the only obstruction to our desired result.

Proposition 2.6. *Let $\mathfrak{h}$ be a finite-dimensional real Lie algebra and*

$$\rho: \mathfrak{h} \rightarrow \text{End}_{\mathbb{R}}(V)$$

a finite-dimensional irreducible real representation of $\mathfrak{h}$. Suppose that there exists a ρ -invariant inner product on V . Then, it is unique up to positive scaling.

Proof. Let B_1, B_2 be two ρ -invariant inner products on V , i.e., two inner products on V satisfying

$$B_i(\rho(X)v, w) + B_i(v, \rho(X)w) = 0, \quad i = 1, 2,$$

for all $X \in \mathfrak{h}$ and $v, w \in V$. Then, for each $i \in \{1, 2\}$, B_i defines an isomorphism of representations φ_i between ρ and its dual representation $\rho^*: \mathfrak{h} \rightarrow \text{End}_{\mathbb{R}}(V^*)$:

$$\begin{aligned} \varphi_i: V &\rightarrow V^* \\ v &\mapsto B_i(v, \cdot). \end{aligned}$$

Now consider the endomorphism of representations given by $\varphi_1^{-1} \circ \varphi_2 \in \text{End}(\rho)$. By Schur's lemma, the endomorphism ring of an irreducible representation (over *any* ground field) is a division ring. In particular, the endomorphism ring of ρ is a finite-dimensional associative division algebra over $\mathbb{R}$. By Frobenius' theorem, these

are, up to isomorphism, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. We will consider these three cases separately.

1. If $\text{End}(\rho) \cong \mathbb{R}$, then there exists $\lambda \in \mathbb{R}$ such that $\varphi_1^{-1} \circ \varphi_2 = \lambda \text{Id}_V$, which implies that $\varphi_2 = \lambda \varphi_1$, which in turn means that $B_2 = \lambda B_1$. As both B_1, B_2 are inner products, we must have that $\lambda > 0$. This completes the proof for the case $\text{End}(\rho) \cong \mathbb{R}$.
2. If $\text{End}(\rho) \cong \mathbb{C}$, then there exists $J \in \text{End}(\rho)$, with $J^2 = -\text{Id}_V$. And any $\varphi \in \text{End}(\rho)$ is of the form

$$\varphi = a \text{Id}_V + bJ,$$

for some $a, b \in \mathbb{R}$. In particular, applying this to $\varphi = \varphi_1^{-1} \circ \varphi_2$, we obtain

$$B_2(\cdot, \cdot) = aB_1(\cdot, \cdot) + bB_1(J\cdot, \cdot), \quad (2.7)$$

Now we claim that, for every $v, w \in V$, $B_1(Jv, Jw) = B_1(v, w)$. Indeed, define $\tilde{B}: V \times V \rightarrow \mathbb{R}$ by

$$\tilde{B}(v, w) = B_1(Jv, Jw).$$

As $J^2 = -\text{Id}_V$, $\tilde{B}$ is non-degenerate. And, as J and B_1 are ρ -invariant and B_1 is symmetric, $\tilde{B}$ is ρ -invariant and symmetric. Now define

$$\tilde{\varphi}: V \rightarrow V^*, \quad v \mapsto \tilde{B}(v, \cdot).$$

Consider the endomorphism of representations $\varphi_1^{-1} \circ \tilde{\varphi}$. Then, there exist $c, d \in \mathbb{R}$ such that $\varphi_1^{-1} \circ \tilde{\varphi} = c \text{Id}_V + dJ$. Hence, for every $v, w \in V$,

$$\tilde{B}(v, w) = cB_1(v, w) + dB_1(Jv, w).$$

As $\tilde{B}, B_1$ are symmetric, we have that, for every $v, w \in V$,

$$dB_1(Jv, w) = dB_1(v, Jw).$$

Suppose $d \neq 0$. Then, for every $v, w \in V$, $B_1(Jv, w) = B_1(v, Jw)$. In particular, if $v \neq 0$, we would have that

$$B_1(Jv, Jv) = B_1(v, J^2v) = -B_1(v, v) < 0,$$

which contradicts positive-definiteness of B_1 . Hence, $\tilde{B} = cB_1$, for some $c \in \mathbb{R}$.

By positive-definiteness and non-degeneracy, $c > 0$. Moreover, if $v \neq 0$,

$$B_1(v, v) = B_1(J^2v, J^2v) = c^2 B_1(v, v).$$

Hence, $c = 1$. This completes the proof of the fact that, for every $v, w \in V$, $B_1(Jv, Jw) = B_1(v, w)$. Equivalently, $B_1(J\cdot, \cdot)$ is skew-symmetric.

Going back to (2.7), as B_1 and B_2 are symmetric and $B_1(J\cdot, \cdot)$ is skew-symmetric, $b = 0$. And positive-definiteness implies that $a > 0$, finishing the proof for the case $\text{End}(\rho) \cong \mathbb{C}$.

3. Finally, suppose $\text{End}(\rho) \cong \mathbb{H}$. Then, there exist $J_1, J_2, J_3 \in \text{End}(\rho)$ linearly independent, with

$$J_i^2 = -\text{Id}_V, \quad J_1 J_2 J_3 = -\text{Id}_V,$$

such that any element $\varphi \in \text{End}(\rho)$ is of the form

$$a \text{Id}_V + b_1 J_1 + b_2 J_2 + b_3 J_3,$$

for some $a, b_1, b_2, b_3 \in \mathbb{R}$. Hence, in particular, $\varphi_1^{-1} \circ \varphi_2$ is of this form, and thus

$$B_2(\cdot, \cdot) = a B_1(\cdot, \cdot) + b_1 B_1(J_1 \cdot, \cdot) + b_2 B_1(J_2 \cdot, \cdot) + b_3 B_1(J_3 \cdot, \cdot), \quad (2.8)$$

for some $a, b_1, b_2, b_3 \in \mathbb{R}$. As in the previous case, one shows that, for every i , $B_1(J_i \cdot, \cdot)$ is skew-symmetric. For completeness, let us show it for J_1 . Define

$$\tilde{B}: V \times V \rightarrow \mathbb{R}, \quad \tilde{B}(v, w) = B_1(J_1 v, J_1 w).$$

As $J_1^2 = -\text{Id}_V$, $\tilde{B}$ is non-degenerate. And, as J_1 and B_1 are ρ -invariant and B_1 is symmetric, $\tilde{B}$ is ρ -invariant and symmetric. Now define

$$\tilde{\varphi}: V \rightarrow V^*, \quad \tilde{\varphi}(v) = \tilde{B}(v, \cdot).$$

Consider the endomorphism of representations $\varphi_1^{-1} \circ \tilde{\varphi}$. Then, there exist $c, d_1, d_2, d_3 \in \mathbb{R}$ such that

$$\varphi_1^{-1} \circ \tilde{\varphi} = c \text{Id}_V + d_1 J_1 + d_2 J_2 + d_3 J_3.$$

Hence, for every $v, w \in V$,

$$\tilde{B}(v, w) = cB_1(v, w) + d_1B_1(J_1v, w) + d_2B_1(J_2v, w) + d_3B_1(J_3v, w).$$

As $\tilde{B}$ and B_1 are symmetric, $d_1B_1(J_1\cdot, \cdot) + d_2B_1(J_2\cdot, \cdot) + d_3B_1(J_3\cdot, \cdot)$ is symmetric. Define

$$Q := d_1J_1 + d_2J_2 + d_3J_3,$$

so that $B_1(Q\cdot, \cdot)$ is symmetric. Note that $Q^2 = -(d_1^2 + d_2^2 + d_3^2)\text{Id}_V$. Now, for $0 \neq v \in V$,

$$B_1(Qv, Qv) = B_1(v, Q^2v) = -(d_1^2 + d_2^2 + d_3^2)B_1(v, v) \leq 0.$$

By positive-definiteness of B_1 , we must have $d_1^2 + d_2^2 + d_3^2 = 0$, and hence $\tilde{B}(v, w) = cB_1(v, w)$. Again by positive definiteness of B_1 , $c > 0$. And, for $0 \neq v \in V$,

$$B_1(v, v) = B_1(J_1^2v, J_1^2v) = \tilde{B}(J_1v, J_1v) = cB_1(J_1v, J_1v) = c\tilde{B}(v, v) = c^2B_1(v, v),$$

so $c = 1$. This shows that $B_1(J_1\cdot, J_1\cdot) = B_1(\cdot, \cdot)$ or, equivalently, that $B_1(J_1\cdot, \cdot)$ is skew-symmetric. One can repeat this reasoning with J_2 and J_3 to show that $B_1(J_2\cdot, \cdot)$ and $B_1(J_3\cdot, \cdot)$ are skew-symmetric.

Going back to (2.8), we have that

$$b_1B_1(J_1\cdot, \cdot) + b_2B_1(J_2\cdot, \cdot) + b_3B_1(J_3\cdot, \cdot)$$

is skew-symmetric. And, as B_1, B_2 are symmetric, this bilinear form is also symmetric. Hence, it is identically zero. Therefore, by positive-definiteness of B_1 , $b_1J_1 + b_2J_2 + b_3J_3 = 0$, showing that $b_1 = b_2 = b_3 = 0$, and hence that $B_2 = aB_1$. And, again by positive-definiteness, $a > 0$, which concludes the proof for the case $\text{End}(\rho) \cong \mathbb{H}$.

□

Proposition 2.9. *Let $\mathfrak{h}$ be a finite-dimensional Lie algebra and $(\rho_1, V_1), (\rho_2, V_2)$ two non-isomorphic self-dual irreducible real representations of $\mathfrak{h}$. Then, $V_1 \perp V_2$ with respect to any $(\rho_1 \oplus \rho_2)$ -invariant inner product on $V_1 \oplus V_2$.*

Proof. Let B be a $(\rho_1 \oplus \rho_2)$ -invariant inner product on $V_1 \oplus V_2$. If V_1 is not B -

orthogonal to V_2 , then we get a non-zero morphism of representations

$$V_1 \rightarrow V_2^*, \quad v \mapsto B(v, \cdot)|_{V_2}.$$

By Schur's lemma, this is an isomorphism, which is a contradiction since V_2 is self-dual and V_1 is not isomorphic to V_2 . $\square$

Finally, we arrive at the main result of this section:

Theorem 2.10. *Let G/H be a homogeneous space with H connected and reductive decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Suppose that the adjoint representation of G restricted to H on $\mathfrak{m}$ decomposes as a direct sum of pairwise non-isomorphic irreducible components

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_k.$$

Suppose that $\mathfrak{m}$ admits a H -invariant inner product. Then, H -invariant inner products on each irreducible component exist and are unique up to positive scaling, and any H -invariant inner product on $\mathfrak{m}$ is a positive linear combination of H -invariant inner products on the irreducible components.

Proof. A H -invariant inner product on $\mathfrak{m}$ restricts to a H -invariant inner product on each irreducible component $\mathfrak{m}_i$. Hence, $\mathfrak{m}_i$ is self-dual. Now, apply Propositions 2.6 and 2.9. $\square$

2.3 Equivariant bundles

Using the diffeomorphism (2.2), the Lie group G can be seen as the total space of a principal H -bundle over M . A useful guiding principle is that any principal K -bundle over M which is suitably *compatible* with the G -action arises as an associated bundle to the principal H -bundle $G \rightarrow G/H$, via a Lie group homomorphism $H \rightarrow K$. We now make this precise.

Definition 2.11. *Let X be a smooth manifold equipped with a left action of a Lie group G (not necessarily transitive). A G -equivariant structure on a principal K -bundle $\pi: P \rightarrow X$ is a left G -action on P by bundle homomorphisms such that*

π is G -equivariant. A G -equivariant bundle is principal bundle with a choice of G -equivariant structure.

Example 2.12. Let G/H be a homogeneous space, and let K be a Lie group.

1. If $\varphi: H \rightarrow K$ is a Lie group homomorphism, the principal K -bundle

$$G \times_{\varphi} K \rightarrow G/H, \quad [g, k] \mapsto gH \quad (2.13)$$

has a natural G -equivariant structure given by

$$g' \cdot [g, k] := [g'g, k].$$

In part 1 of Lemma 2.15, we will see that every G -equivariant bundle over a homogeneous G -space is G -equivariantly isomorphic to one of this form.

2. G -equivariant structures on the trivial bundle $G/H \times K \rightarrow G/H$ are of the form

$$g' \cdot (gH, k) := (g'gH, \alpha(g, g')k),$$

where $\alpha: G \times G \rightarrow K$ is a smooth map satisfying $\alpha(\cdot, e_G) = e_K$ and

$$\alpha(g, g'g'') = \alpha(g''g, g')\alpha(g, g''),$$

for all $g, g', g'' \in G$. If α is independent of the first argument, we call the corresponding G -equivariant structure special. In this case, the G -equivariant structure is given by

$$g' \cdot (gH, k) := (g'gH, \beta(g')k),$$

where β is a Lie group homomorphism.

Definition 2.14. Let P be a G -equivariant principal bundle over the homogeneous space G/H . We say that P is strongly G -trivial if it is G -equivariantly isomorphic to the trivial bundle equipped with a special G -equivariant structure – see part 2 of Example 2.12.

We now prove that every G -equivariant bundle over G/H is of the form (2.13), and characterise triviality and strong G -triviality in terms of the homomorphism φ :

Lemma 2.15. Let G/H be a homogeneous space, K a Lie group, and $P \rightarrow G/H$ a G -equivariant principal K -bundle. Then, there exists a Lie group homomorphism $\varphi: H \rightarrow K$ such that:

1. P is G -equivariantly bundle-isomorphic to

$$G \times_{\varphi} K \rightarrow G/H, \quad [g, k] \mapsto gH.$$

Moreover, φ is unique up to conjugation by elements of K .

2. P is trivial if and only if φ extends to a smooth map $\tilde{\varphi}: G \rightarrow K$ satisfying

$$\tilde{\varphi}(gh) = \tilde{\varphi}(g)\tilde{\varphi}(h), \tag{2.16}$$

for all $g \in G$ and $h \in H$.

3. P is strongly G -trivial if and only if φ extends to a Lie group homomorphism $\tilde{\varphi}: G \rightarrow K$.

Proof. A proof of part 1 can be found in [ČS09, Lem. 1.4.5]. As we use ideas from this proof later, we include it here. Note that the Lie group $G \times K$ acts on P by

$$(g, k) \cdot p := g \cdot p \cdot k^{-1},$$

for every $g \in G$, $k \in K$ and $p \in P$. As the G -action on P is by K -bundle homomorphisms, this is a well-defined action of $G \times K$ on P . Moreover, this action is transitive, as the base space is a homogeneous G -space and K acts transitively on each fibre. Fix $p_0 \in P$ with projection $o := e_G H \in G/H$, and let $H_{p_0} \subseteq G \times K$ be the stabiliser of p_0 . The first projection $H_{p_0} \rightarrow G$ lands in H , because the right K -action does not change the fibre. And the resulting homomorphism $H_{p_0} \rightarrow H$ is in fact an isomorphism, since the K -action is simply transitive on the fibres. Denote by $\hat{\varphi}$ the inverse isomorphism, and define $\varphi := \text{pr}_2 \circ \hat{\varphi}: H \rightarrow K$. Now, it is straightforward to check that the map

$$\begin{aligned} G \times_{\varphi} K &\rightarrow P \\ [g, k] &\mapsto gp_0k \end{aligned}$$

is a G -equivariant K -bundle isomorphism.

For the last assertion, suppose $\varphi_1, \varphi_2: H \rightarrow K$ are Lie group homomorphisms, and suppose that there exists a G -equivariant K -bundle isomorphism

$$f: G \times_{\varphi_1} K \rightarrow G \times_{\varphi_2} K.$$

Then, for each $g \in G$ and $k \in K$,

$$f([g, k]) = f(g \cdot [e_G, e_K] \cdot k) = g \cdot f([e_G, e_K]) \cdot k,$$

and so f is determined by $f([e_G, e_K])$. As f is a bundle homomorphism, $f([e_G, e_K])$ lies over o , and hence there exists $t \in K$ such that $f([e_G, e_K]) = [e_G, t]$. Now, for every $g \in G$, $k \in K$ and $h \in H$, we have that

$$\begin{aligned} [g, tk] &= g \cdot [e_G, t] \cdot k = g \cdot f([e_G, e_K]) \cdot k = f(g \cdot [e_G, e_K] \cdot k) \\ &= f([g, k]) = f([gh, \varphi_1(h)^{-1}k]) = [gh, t\varphi_1(h)^{-1}k] = [g, \varphi_2(h)t\varphi_1(h)^{-1}k]. \end{aligned}$$

Therefore, for each $h \in H$, $\varphi_2(h) = t\varphi_1(h)t^{-1}$.

For part 2, suppose first that P is a trivial bundle, so that there is a bundle isomorphism $F: G \times_{\varphi} K \rightarrow G/H \times K$. Such a map has the form $[g, k] \mapsto (gH, \tilde{\varphi}(g)k)$, for some smooth map $\tilde{\varphi}: G \rightarrow K$, which we can take so that $\tilde{\varphi}(e_G) = e_K$. Note that, for $g \in G$, $h \in H$, and $k \in K$,

$$\begin{aligned} (gH, \tilde{\varphi}(g)k) &= F([g, k]) = F([gh, \varphi(h)^{-1}k]) \\ &= (ghH, \tilde{\varphi}(gh)\varphi(h)^{-1}k) = (gH, \tilde{\varphi}(gh)\varphi(h)^{-1}k). \end{aligned}$$

Hence, $\tilde{\varphi}(gh) = \tilde{\varphi}(g)\varphi(h)$. In particular, as $\tilde{\varphi}(e_G) = e_K$, $\tilde{\varphi}$ is an extension of φ .

Conversely, suppose that φ admits an extension to G satisfying (2.16). Then, we can trivialise the bundle $G \times_{\varphi} K$ via the map F defined above.

Let us now prove part 3. If such a homomorphism $\tilde{\varphi}$ exists, we can equip the trivial bundle $G/H \times K$ with the special G -action

$$g' \cdot (gH, k) := (g'gH, \tilde{\varphi}(g')k),$$

and the proof of 1 shows that this bundle is G -equivariantly isomorphic to P .

Conversely, suppose that P is strongly G -trivial. Then, P is G -equivariantly isomorphic to the trivial bundle $G/H \times K$ equipped with a special G -equivariant structure, i.e., one given by

$$g' \cdot (gH, k) := (g'gH, \alpha(g')k),$$

for a Lie group homomorphism $\alpha: G \rightarrow K$. Running the proof of 1 for the trivial bundle with this G -action, we see that this bundle is also G -equivariantly isomorphic to $G \times_{\alpha|_H} K$. Hence, by the final assertion of part 1, there exists $t \in K$ such that

$\alpha|_H = t\varphi t^{-1}$. Taking $\tilde{\varphi} = t^{-1}\alpha t$ finishes the proof. $\square$

Example 2.17. *If G/H is an n -dimensional oriented homogeneous space equipped with a G -invariant metric, its positively oriented orthonormal frame bundle carries a natural G -equivariant structure. The Lie group homomorphism $\varphi: H \rightarrow \mathrm{SO}(n)$ given by part 1 of Lemma 2.15 is precisely the isotropy representation.*

We can extend part 1 of Lemma 2.15 to obtain a very general result on G -equivariant lifts of the structure group:

Lemma 2.18. *Let G/H be a homogeneous space, and let K, L be Lie groups. Let $\lambda: K \rightarrow L$ and $\eta: H \rightarrow L$ be Lie group homomorphisms, with λ surjective. Let P be a G -equivariant lift of the structure group of $G \times_\eta L$ to K along λ . Then, there exists a Lie group homomorphism $\tilde{\eta}: H \rightarrow K$ such that P is G -equivariantly bundle-isomorphic to $G \times_{\tilde{\eta}} K$ covering the identity on $G \times_\eta L$, and $\eta = \lambda \circ \tilde{\eta}$. Moreover, $\tilde{\eta}$ is unique up to conjugation by elements of $\ker(\lambda)$.*

Proof. Let $f: P \rightarrow G \times_\eta L$ be a G -equivariant lift of the structure group along λ . As λ is surjective, there exists $p_0 \in P$ such that $f(p_0) = [e_G, e_L]$. Applying the same argument of the proof of part 1 of Lemma 2.15 with this choice of p_0 , one obtains a Lie group homomorphism $\tilde{\eta}: H \rightarrow K$ such that P is G -equivariantly bundle-isomorphic to $G \times_{\tilde{\eta}} K$, and we can take this isomorphism to send p_0 to $[e_G, e_K]$. Hence, we have a G -equivariant lift of the structure group

$$\begin{aligned} G \times_{\tilde{\eta}} K &\xrightarrow{\cong} P \rightarrow G \times_\eta L \\ [g, k] &\mapsto gp_0k \mapsto [g, \lambda(k)]. \end{aligned}$$

One readily checks that this map is well-defined if and only if $\lambda \circ \tilde{\eta} = \eta$.

Now, suppose that $\tilde{\eta}_1, \tilde{\eta}_2$ are such that

$$\lambda \circ \tilde{\eta}_1 = \lambda \circ \tilde{\eta}_2 = \eta.$$

Suppose further that there exists a G -equivariant K -bundle isomorphism f such that the diagram

$$\begin{array}{ccc} G \times_{\tilde{\eta}_1} K & \xrightarrow{f} & G \times_{\tilde{\eta}_2} K \\ & \searrow & \swarrow \\ & G \times_\eta L & \end{array}$$

commutes. Then, there exists $t \in \ker(\lambda)$ such that

$$f([e_G, e_K]) = [e_G, t].$$

Hence, for every $h \in H$,

$$[e_G, t] = f([e_G, e_K]) = f([h, \tilde{\eta}_1(h)^{-1}]) = [h, t\tilde{\eta}_1(h)^{-1}] = [e_G, \tilde{\eta}_2(h)t\tilde{\eta}_1(h)^{-1}].$$

This shows that

$$\tilde{\eta}_2 = t\tilde{\eta}_1 t^{-1}.$$

Conversely, if $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are conjugate by an element $t \in \ker(\lambda)$, we can define f by

$$f([g, k]) = [g, tk].$$

□

Remark 2.19. *We recover part 1 of Lemma 2.15 by taking L to be the trivial one-element Lie group.*

Let us now consider vector bundles. If a vector bundle over M with fibre V admits a left G -action by vector bundle homomorphisms for which the projection is G -equivariant, applying the previous discussion to its frame bundle, we deduce that the vector bundle is isomorphic to

$$G \times_{\alpha} V \rightarrow M, \quad [g, v] \mapsto g \cdot x,$$

for some representation $\alpha: H \rightarrow \mathrm{GL}(V)$. Sections of this bundle are in bijective correspondence with smooth H -equivariant maps $v: G \rightarrow V$, which means that

$$v(gh) = \alpha(h)^{-1}v(g),$$

for all $g \in G$ and $h \in H$. Explicitly, to such a map corresponds the section

$$M \rightarrow G \times_{\alpha} V, \quad g \cdot x \mapsto [g, v(g)].$$

There is a natural G -action on the space of sections, given by

$$(g \cdot v)(g') := v(g^{-1}g'),$$

for $g, g' \in G$. Sections that are invariant under this action are constant H -equivariant

maps, which can be viewed as H -invariant elements of the vector space V .

For example, the fundamental vector field $\widehat{X} \in \Gamma(TM)$ corresponding to $X \in \mathfrak{g}$ is given by the H -equivariant map

$$\widehat{X}: G \rightarrow \mathfrak{m}, \quad g \mapsto (\mathrm{Ad}(g)^{-1}X)_{\mathfrak{m}}, \quad (2.20)$$

where $(\cdot)_{\mathfrak{m}}$ denotes projection to $\mathfrak{m}$.

2.4 Invariant spin structures and spinors

We are now in a position to define and describe invariant spin structures:

Definition 2.21. *Let $M = G/H$ be an oriented homogeneous space equipped with a G -invariant Riemannian metric, and let FM be its bundle of oriented orthonormal frames equipped with its natural G -equivariant structure. A G -invariant spin structure is a spin structure (P, Φ) on M such that both P and Φ are G -equivariant.*

Applying Lemma 2.18 with $K = \mathrm{Spin}(n)$, $L = \mathrm{SO}(n)$, and $\lambda = \lambda_n$, and recalling Example 2.17, we see that G -invariant spin structures are in bijective correspondence with lifts of the isotropy representation to the spin group – see [DSKL22, Prop. 1.3].

After fixing a positively oriented orthonormal basis of $\mathfrak{m}$, we can identify

$$\mathrm{SO}(\mathfrak{m}) \cong \mathrm{SO}(n).$$

Now suppose that there exists a G -invariant spin structure (P, Φ) . Then, the isotropy representation admits a lift $\tilde{\sigma}$ to $\mathrm{Spin}(n)$ along the double covering λ_n

$$\begin{array}{ccc} & & \mathrm{Spin}(n) \\ & \nearrow \tilde{\sigma} & \downarrow \lambda_n \\ H & \xrightarrow{\sigma} & \mathrm{SO}(n) \end{array}$$

so that the principal $\mathrm{Spin}(n)$ -bundle P is given by

$$G \times_{\tilde{\sigma}} \mathrm{Spin}(n) \rightarrow M, \quad [g, \mu] \mapsto g \cdot x.$$

P , together with its natural bundle homomorphism down to the frame bundle

$$\Phi: G \times_{\tilde{\sigma}} \text{Spin}(n) \rightarrow G \times_{\sigma} \text{SO}(n), \quad [g, \mu] \mapsto [g, \lambda_n(\mu)],$$

defines a spin structure on M . The associated spinor bundle is then

$$\Sigma M = G \times_{\Delta_n \circ \tilde{\sigma}} \Sigma_n \rightarrow M,$$

where $\Delta_n: \text{Spin}(n) \rightarrow \text{End}_{\mathbb{C}}(\Sigma_n)$ is the spin representation.

Clifford multiplication between vector fields and spinors is defined fibrewise using the standard action of $\mathbb{R}^n$ on Σ_n . If $v: G \rightarrow \mathbb{R}^n$ is a H -equivariant map representing a vector field and $\psi: G \rightarrow \Sigma_n$ is a spinor, their Clifford product is the spinor

$$(v \cdot \psi)(g) := v(g) \cdot \psi(g),$$

where the dot on the right denotes the usual Clifford multiplication $\mathbb{R}^n \otimes \Sigma_n \rightarrow \Sigma_n$. One easily checks that this product is H -equivariant and hence defines a G -equivariant $\mathcal{C}^\infty(M)$ -linear map

$$\Gamma(TM) \otimes \Gamma(\Sigma M) \rightarrow \Gamma(\Sigma M).$$

2.5 Invariant covariant derivatives and Nomizu maps

Let $E := G \times_{\alpha} V \rightarrow M = G/H$ be the G -equivariant vector bundle corresponding to a real or complex representation $\alpha: H \rightarrow \text{GL}(V)$. We say that a covariant derivative ∇ on E is G -invariant if

$$g \cdot (\nabla_X \psi) = \nabla_{g \cdot X} (g \cdot \psi),$$

for all $g \in G$, $X \in \Gamma(TM)$, and $\psi \in \Gamma(E)$.

The *Lie derivative* of a section $\psi \in \Gamma(E)$ along the fundamental vector field $\hat{X}$ corresponding to $X \in \mathfrak{g}$ is the section $\mathcal{L}_{\hat{X}}\psi$ of E defined by

$$\mathcal{L}_{\hat{X}}\psi(g) := \left. \frac{d}{dt} \right|_{t=0} \psi(\exp(tX)g).$$

Now, observe that the map

$$\nabla_{\hat{X}} - \mathcal{L}_{\hat{X}} : \Gamma(E) \rightarrow \Gamma(E)$$

is $\mathcal{C}^\infty(M)$ -linear. Indeed, if $g \in G$, $\psi \in \Gamma(E)$ and $f \in \mathcal{C}^\infty(M)$,

$$\begin{aligned} (\nabla_{\hat{X}} - \mathcal{L}_{\hat{X}})(f\psi)(g) &= \nabla_{\hat{X}}(f\psi)(g) - \mathcal{L}_{\hat{X}}(f\psi)(g) \\ &= \frac{d}{dt} \Big|_{t=0} f(\exp(tX)g)\psi(g) + f(g)\nabla_{\hat{X}}\psi(g) \\ &\quad - \frac{d}{dt} \Big|_{t=0} (f(\exp(tX)g)\psi(\exp(tX)g)) \\ &= f(g)(\nabla_{\hat{X}} - \mathcal{L}_{\hat{X}})\psi(g). \end{aligned}$$

Hence, it can be viewed as a section $\Lambda'(X)$ of $\text{End}(E)$, i.e., as a H -equivariant map $G \rightarrow \text{End}(V)$. Now, if ∇ is G -invariant, using that $g^{-1} \cdot \hat{X}$ is the fundamental vector field corresponding to $\text{Ad}(g^{-1})X$, we have:

$$\begin{aligned} \Lambda'(X)(g)\psi(g) &= \nabla_{\hat{X}}\psi(g) - \mathcal{L}_{\hat{X}}\psi(g) = g^{-1} \cdot (\nabla_{\hat{X}}\psi - \mathcal{L}_{\hat{X}}\psi)(e_G) \\ &= \left(\nabla_{g^{-1} \cdot \hat{X}}(g^{-1} \cdot \psi) - \mathcal{L}_{g^{-1} \cdot \hat{X}}(g^{-1} \cdot \psi) \right)(e_G) \\ &= \Lambda'(\text{Ad}(g^{-1})X)(e_G)(g^{-1} \cdot \psi)(e_G) \\ &= \Lambda'(\text{Ad}(g^{-1})X)(e_G)\psi(g). \end{aligned}$$

Therefore,

$$\Lambda'(X)(g) = \Lambda'(\text{Ad}(g^{-1})X)(e_G),$$

which implies that Λ' is fully determined by $\Lambda'(X)(e_G)$. The map

$$\Lambda : \mathfrak{g} \rightarrow \text{End}(V), \quad X \mapsto \Lambda'(X)(e_G)$$

is clearly linear, and a calculation shows that it satisfies the properties

$$\forall X \in \mathfrak{h}: \quad \Lambda(X) = \alpha_*(X), \tag{2.22}$$

$$\forall h \in H: \quad \Lambda \circ \text{Ad}(h) = \alpha(h)\Lambda\alpha(h)^{-1}. \tag{2.23}$$

This is called the *Nomizu map* of ∇ . Conversely, a linear map satisfying these properties determines a G -invariant connection on E via

$$\nabla_{\hat{X}}\psi(g) := \mathcal{L}_{\hat{X}}\psi(g) + \Lambda(\text{Ad}(g^{-1})X)\psi(g).$$

In particular, if $\psi \in V$ is an invariant section of E , then

$$\nabla_{\hat{X}}\psi(g) = \Lambda(\text{Ad}(g^{-1})X)\psi. \quad (2.24)$$

Given a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

the Nomizu map of an invariant connection is fully determined by its restriction $\Lambda_{\mathfrak{m}}$ to $\mathfrak{m}$, which by (2.23) satisfies

$$\Lambda_{\mathfrak{m}} \circ \text{Ad}(h) = \alpha(h)\Lambda_{\mathfrak{m}}\alpha(h)^{-1},$$

for all $h \in H$.

The *canonical connection* ∇^{can} is the one corresponding to $\Lambda_{\mathfrak{m}} \equiv 0$. The Nomizu map of the Levi-Civita connection of an invariant Riemannian metric can be computed explicitly:

Proposition 2.25. *([Nom54, Thm. 13.1]) Let G/H be an oriented homogeneous space, with a G -invariant metric g corresponding to an invariant inner product B on a reductive complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$.*

Then, the Nomizu map $\Lambda: \mathfrak{g} \rightarrow \mathfrak{so}(\mathfrak{m})$ of the Levi-Civita connection of g is given by

$$\Lambda(X)(Y) = \frac{1}{2} [X, Y]_{\mathfrak{m}} + U(X, Y), \quad X \in \mathfrak{g}, Y \in \mathfrak{m},$$

where U is uniquely determined by

$$B(U(X, Y), W) = \frac{1}{2} (B([W, X]_{\mathfrak{m}}, Y) + B(X, [W, Y]_{\mathfrak{m}}))$$

for all $X, Y, W \in \mathfrak{m}$, and $(\cdot)_{\mathfrak{m}}$ denotes projection to $\mathfrak{m}$. □

Recall that the Levi-Civita connection determines the spin connection on the spinor bundle. If the Riemannian metric is invariant, the spin connection is also invariant, and the corresponding Nomizu maps Λ and $\tilde{\Lambda}$ are related by

$$\tilde{\Lambda} = (\Delta_n)_* \circ \rho \circ \Lambda,$$

where $\rho: \mathfrak{so}(\mathfrak{m}) \rightarrow \mathfrak{spin}(\mathfrak{m})$ is the inverse of the derivative of the double covering $\text{Spin}(\mathfrak{m}) \rightarrow \text{SO}(\mathfrak{m})$.

As we shall see later, the condition (2.23) is quite restrictive, as it forces the kernel of $\Lambda|_{\mathfrak{m}}$ to be quite large. The following is a useful criterion that will be used in Chapter 6:

Lemma 2.26. *Let G/H be a homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and let $\phi: H \rightarrow K$ be a Lie group homomorphism. Let $H_0 \subseteq H$ be the kernel of $\text{Ad}_K \circ \phi: H \rightarrow \text{GL}(\mathfrak{k})$. If $X \in \text{span}_{\mathbb{R}}[\mathfrak{h}_0, \mathfrak{m}]$, then $\Lambda(X) = 0$ for the Nomizu map $\Lambda: \mathfrak{g} \rightarrow \mathfrak{k}$ associated to any invariant connection on $G \times_{\phi} K$.*

Proof. By linearity of Λ , it suffices to consider $X = [v, Y]$ for some $v \in \mathfrak{h}_0$ and $Y \in \mathfrak{m}$. Let $\gamma: \mathbb{R} \rightarrow H_0$ be a curve with $\gamma(0) = e_G$ and $\gamma'(0) = v$. By (2.23), the Nomizu map Λ of any invariant connection on $G \times_{\phi} K$ satisfies $\Lambda \circ \text{Ad}_H(\gamma(t)) = \Lambda$, and hence

$$0 = \frac{d}{dt}\bigg|_{t=0} \Lambda(Y) = \frac{d}{dt}\bigg|_{t=0} \Lambda(\text{Ad}_H(\gamma(t))Y) = \Lambda([v, Y]) = \Lambda(X).$$

□

2.6 Invariant generalised Killing spinors

We now show that if an invariant spinor satisfies the generalised Killing equation (2.1) at a single point, then the equation holds everywhere on the manifold.

Proposition 2.27. *Let G/H be an n -dimensional oriented homogeneous space with an invariant Riemannian metric g , and let $\mathfrak{m}$ be a reductive complement of $\mathfrak{h}$ in $\mathfrak{g}$.*

Let $\psi \in \Sigma_n$ be a non-zero invariant spinor which satisfies

$$\tilde{\Lambda}(X)\psi = A(X) \cdot \psi,$$

for all $X \in \mathfrak{m}$, where $\tilde{\Lambda}: \mathfrak{g} \rightarrow \text{End}(\Sigma_n)$ is the Nomizu map of the spin connection of g , and $A \in \text{End}(\mathfrak{m})$.

Then, the spinor ψ satisfies the global equation

$$\nabla_Y \psi = \hat{A}(Y) \cdot \psi,$$

for all vector fields $Y \in \Gamma(TM)$, where $\hat{A}$ is the invariant section of the endomorphism bundle $\text{End}(TM)$ determined by A .

Proof. We first show that $\widehat{A}$ is a well-defined section of $\text{End}(TM)$. We need to check that

$$A = \sigma(h)^{-1} \circ A \circ \sigma(h),$$

for all $h \in H$, where σ is the isotropy representation. Indeed, for each $X \in \mathfrak{m}$,

$$\begin{aligned} A(\sigma(h)(X)) \cdot \psi &= A(\text{Ad}(h)(X)) \cdot \psi = \widetilde{\Lambda}(\text{Ad}(h)(X))\psi = \widetilde{\sigma}(h)\widetilde{\Lambda}(X)\widetilde{\sigma}(h)^{-1}\psi \\ &= \widetilde{\sigma}(h) (A(X) \cdot (\widetilde{\sigma}(h)^{-1}\psi)) \\ &= \widetilde{\sigma}(h)\widetilde{\sigma}(h)^{-1} ((\sigma(h)(A(X))) \cdot \psi) \\ &= \sigma(h)(A(X)) \cdot \psi, \end{aligned}$$

where the penultimate equality holds by equivariance of Clifford multiplication. As $\psi \neq 0$, we conclude that A is invariant. Therefore, the constant map $g \mapsto A$ is a well-defined section $\widehat{A}$ of $\text{End}(TM)$. Now, for each $X \in \mathfrak{g}$,

$$\begin{aligned} \nabla_{\widehat{X}}\psi(g) &= \widetilde{\Lambda}(\text{Ad}(g)^{-1}X)\psi = \widetilde{\Lambda}((\text{Ad}(g)^{-1}X)_{\mathfrak{h}})\psi + \widetilde{\Lambda}((\text{Ad}(g)^{-1}X)_{\mathfrak{m}})\psi \\ &= \widetilde{\Lambda}((\text{Ad}(g)^{-1}X)_{\mathfrak{m}})\psi = A((\text{Ad}(g)^{-1}X)_{\mathfrak{m}}) \cdot \psi = \widehat{A}(g)(\widehat{X}(g)) \cdot \psi(g), \end{aligned}$$

where the first equality is true by (2.24), the second one is obvious, the third one follows from (2.22) together with invariance of ψ , the fourth one is true by hypothesis and the fifth one only uses the definition of $\widehat{A}$ and the definition (2.20) of $\widehat{X}$. The proposition follows because $\nabla_Y\psi(g)$ only depends on $Y(g)$, and $Y(g) = \widehat{X}(g)$, for some $X \in \mathfrak{g}$. $\square$

2.7 Differential forms approach to the spin representation

Finally, to conclude this introductory chapter, we exhibit a concrete realisation of the spin representation that will allow us to carry out spinorial computations in a very explicit fashion.

It is well-known – see e.g. [LM89] – that the complexification $\mathbb{Cl}(n)$ of the real Clifford algebra $\text{Cl}(n)$ is isomorphic to a complex matrix algebra

$$\mathbb{Cl}(n) \cong \begin{cases} M_{2^k}(\mathbb{C}) & \text{if } n = 2k, \\ M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C}) & \text{if } n = 2k + 1. \end{cases} \quad (2.28)$$

For $n = 2k$ or $2k + 1$, define

$$\Sigma_n := \mathbb{C}^{2^k},$$

and let $s_k: M_{2^k}(\mathbb{C}) \rightarrow \text{End}_{\mathbb{C}}(\Sigma_n)$ be the standard representation of $M_{2^k}(\mathbb{C})$. The *spin representation* $\Delta_n: \text{Cl}(n) \rightarrow \text{End}_{\mathbb{C}}(\Sigma_n)$ is defined by

$$\Delta_n := \begin{cases} s_k & \text{if } n = 2k, \\ s_k \circ \text{pr}_1 & \text{if } n = 2k + 1, \end{cases}$$

where pr_1 is the projection onto the first factor. We will also denote by Δ_n its restriction to $\text{Spin}(n)$ when there is no risk of confusion. This restriction is equivalent to $s_k \circ \text{pr}_2$ for odd n .

It is useful to have an explicit description of the spin representation that does not use the isomorphism (2.28). We will describe one here, which we refer to as the *differential forms approach* to the spin representation. More details can be found e.g. in [GW09, AHL23].

Suppose $n = 2k + 1$, and let $(e_0, \dots, e_{2k})$ be the standard basis of $\mathbb{R}^n$. Its complexification decomposes as

$$\mathbb{C}_0 \oplus L \oplus L',$$

where $\mathbb{C}_0 = \text{span}_{\mathbb{C}}\{u_0 := ie_0\}$ and

$$L := \text{span}_{\mathbb{C}} \left\{ x_j := \frac{1}{\sqrt{2}} (e_{2j-1} - ie_{2j}) \right\}_{j=1}^k, \quad L' := \text{span}_{\mathbb{C}} \left\{ y_j := \frac{1}{\sqrt{2}} (e_{2j-1} + ie_{2j}) \right\}_{j=1}^k.$$

Note that $\dim_{\mathbb{C}}(\Lambda^{\bullet} L') = 2^k$, and $\text{Cl}(n)$ acts on $\Lambda^{\bullet} L'$ by extending

$$u_0 \cdot \eta := -\eta_{\text{even}} + \eta_{\text{odd}}, \quad x_j \cdot \eta := i\sqrt{2}x_j \lrcorner \eta, \quad y_j \cdot \eta := i\sqrt{2}y_j \wedge \eta,$$

where η_{even} and η_{odd} are, respectively, the even and odd parts of $\eta \in \Lambda^{\bullet} L'$. Hence, for $1 \leq j \leq k$,

$$e_0 \cdot \eta = i\eta_{\text{even}} - i\eta_{\text{odd}}, \quad e_{2j-1} \cdot \eta = i(x_j \lrcorner \eta + y_j \wedge \eta), \quad e_{2j} \cdot \eta = y_j \wedge \eta - x_j \lrcorner \eta. \quad (2.29)$$

Example 2.30. *To illustrate the action (2.29), here are a few examples:*

$$\begin{aligned} e_0 \cdot y_1 \wedge y_2 &= iy_1 \wedge y_2, \\ e_1 \cdot y_1 \wedge y_2 &= i(x_1 \lrcorner (y_1 \wedge y_2) + y_1 \wedge y_1 \wedge y_2) = iy_2, \\ e_4 \cdot y_1 \wedge y_2 &= y_2 \wedge y_1 \wedge y_2 - x_2 \lrcorner (y_1 \wedge y_2) = y_1, \end{aligned}$$

$$e_2 \cdot 1 = y_1.$$

This representation is isomorphic to Δ_n for $n = 2k + 1$. To obtain it for $n = 2k$, repeat all the above ignoring everything with a zero subscript. These representations have an invariant Hermitian product, which we shall denote by $\langle \cdot, \cdot \rangle$, and an associated norm $|\cdot|$, for which the basis

$$\{y_{j_1, \dots, j_l} := y_{j_1} \wedge \cdots \wedge y_{j_l} : 0 \leq l \leq n, 1 \leq j_1 < \cdots < j_l \leq n\}$$

is orthonormal.

Remark 2.31. *Note that there is an embedding*

$$SU(2)^k \subseteq \text{Spin}(2k + 1),$$

where the j -th factor corresponds to the Lie subalgebra of $\mathfrak{spin}(2k + 1)$ generated by $\{e_0 e_{2j-1}, e_0 e_{2j}, e_{2j-1} e_{2j}\}$, which is isomorphic to $\mathfrak{su}(2)$. In this section, we have essentially decomposed the spin representation as

$$\Sigma_{2k+1} \cong \bigotimes_{j=1}^k (\mathbb{C}1 \oplus \mathbb{C}y_j),$$

and each tensor factor is the standard two-dimensional $SU(2)$ -module for the j -th copy of $SU(2)$ inside $\text{Spin}(2k + 1)$.

3

Invariant spinors on flag manifolds

The objective of this chapter is to give a combinatorial characterisation of the existence of invariant spinors on maximal flag manifolds of complex simple Lie algebras. We also give bounds for the dimension of the space of invariant spinors in each case. The results of this chapter have been published in [AS25].

Let us start by recalling the construction of flag manifolds and some of their properties. For additional details, see e.g. [Bes08, Ch. 8]. Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra with compact real form $\mathfrak{g}$, and let G be the compact simply connected Lie group with Lie algebra $\mathfrak{g}$. Fix a maximal torus $T \subseteq G$.

Definition 3.1. *The maximal flag manifold associated to $\mathfrak{g}_{\mathbb{C}}$ is*

$$\mathrm{MF}(\mathfrak{g}_{\mathbb{C}}) := G/T.$$

Recall that all maximal tori of G are conjugate, and hence this definition does not depend on the choice of maximal torus. These manifolds have nice spinorial properties:

Lemma 3.2. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra. Then, $\mathrm{MF}(\mathfrak{g}_{\mathbb{C}})$ admits a unique spin structure. Moreover, this spin structure is G -invariant, where G is the simply connected Lie group corresponding to the compact real form of $\mathfrak{g}_{\mathbb{C}}$.*

Proof. The fact that $\mathrm{MF}(\mathfrak{g}_{\mathbb{C}})$ admits a spin structure follows from the fact that it is diffeomorphic to a generic adjoint orbit in $\mathfrak{g}$, and such an orbit has trivial normal

bundle – see e.g. [Bes08, §8.23]. The fact that this structure is unique and G -invariant follows from [HS90, p. 327] – see also [DSKL22] and [AH70, Prop. 2.1]. $\square$

Note that Definition 3.1 deals only with *maximal* flag manifolds. In fact, we can make a more general definition:

Definition 3.3. *A flag manifold associated to $\mathfrak{g}_{\mathbb{C}}$ is an adjoint orbit of G in $\mathfrak{g}$.*

These arise as quotients of G by the centraliser of a torus, which explains the terminology *maximal* in Definition 3.1. Note that the definition of flag manifold in [Bes08] only includes those we call *maximal*.

We now study the existence of non-trivial invariant spinors on maximal flag manifolds. First, we need a definition:

Definition 3.4. *Let V be a vector space. A strong linear combination of a finite subset $S = \{v_1, \dots, v_r\} \subseteq V$ is one of the form*

$$\sum_{i=1}^r \varepsilon_i v_i,$$

where $\varepsilon_1, \dots, \varepsilon_r \in \{\pm 1\}$. S is said to be strongly linearly dependent if there exists a strong linear combination of S that sums to zero.

We now prove a combinatorial characterisation of existence of invariant spinors on maximal flag manifolds in terms of strong linear dependence of positive roots:

Theorem 3.5. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra with positive roots $a_1, \dots, a_r$. Then, the dimension of the space of invariant spinors of $\text{MF}(\mathfrak{g}_{\mathbb{C}})$ is equal to the number of strong linear combinations of the set of positive roots that sum to zero. In particular, it is always even.*

Proof. Let $\mathfrak{g}$ be the compact real form of $\mathfrak{g}_{\mathbb{C}}$, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Then, as a representation of $\mathfrak{h}$, the Lie algebra $\mathfrak{g}$ decomposes as an orthogonal direct sum with respect to the negative of the Killing form of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}_1 \oplus \dots \oplus \mathfrak{a}_r,$$

where $\dim(\mathfrak{a}_i) = 2$ and, for each $X \in \mathfrak{h}$,

$$\text{ad}_{\mathfrak{g}}(X) \begin{pmatrix} e_1^{(i)} \end{pmatrix} = a_i(X) e_2^{(i)}, \quad \text{and} \quad \text{ad}_{\mathfrak{g}}(X) \begin{pmatrix} e_2^{(i)} \end{pmatrix} = -a_i(X) e_1^{(i)},$$

where $\text{ad}_{\mathfrak{g}}$ is the adjoint action of $\mathfrak{g}$, and $(e_1^{(i)}, e_2^{(i)})$ is an orthonormal basis of $\mathfrak{a}_i$. Note that r does not stand for the rank. Hence, under the standard identification of the Lie algebra $\mathfrak{so}$ with the space of two-forms, we can write

$$\text{ad}_{\mathfrak{g}}(X) = \sum_{i=1}^r a_i(X) e_1^{(i)} \wedge e_2^{(i)}. \quad (3.6)$$

Observe that, by Theorem 2.10 and the fact that all the roots are different, another choice of invariant inner product on $\mathfrak{g}$ would yield the same decomposition, modulo rescaling of each pair $\{e_1^{(i)}, e_2^{(i)}\}$. Note also that fixing a partial order in the set of roots, i.e., determining which of $\pm a_i$ is positive, determines an orientation on each $\mathfrak{a}_i$. Thus, we have a positively-oriented orthonormal basis of the orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$:

$$(e_1^{(1)}, e_2^{(1)}, \dots, e_1^{(r)}, e_2^{(r)}),$$

which gives an identification $\mathfrak{m} \cong \mathbb{R}^{2r}$. We now define, as in Section 2.7, two subspaces of the complexification of $\mathfrak{m}$:

$$\begin{aligned} L &:= \text{span}_{\mathbb{C}} \left\{ x_j := \frac{1}{\sqrt{2}} (e_1^{(j)} - ie_2^{(j)}) \mid 1 \leq j \leq r \right\}, \\ L' &:= \text{span}_{\mathbb{C}} \left\{ y_j := \frac{1}{\sqrt{2}} (e_1^{(j)} + ie_2^{(j)}) \mid 1 \leq j \leq r \right\}, \end{aligned}$$

and recall that the spin representation of the complex Clifford algebra $\text{Cl}(2r)$ is isomorphic to the action on the exterior algebra $\Lambda^{\bullet} L'$ defined by extending

$$x_j \cdot \eta := i\sqrt{2}x_j \lrcorner \eta, \quad y_j \cdot \eta := i\sqrt{2}y_j \wedge \eta,$$

for $1 \leq j \leq r$ and $\eta \in \Lambda^{\bullet} L'$. Hence, if G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and T is its maximal torus with Lie algebra $\mathfrak{h}$, the spinor bundle of $\text{MF}(\mathfrak{g}_{\mathbb{C}})$ is given by

$$G \times_{\tilde{\sigma}} \Lambda^{\bullet} L' \rightarrow \text{MF}(\mathfrak{g}_{\mathbb{C}}), \quad [g, \psi] \mapsto gT,$$

where $\tilde{\sigma}: T \rightarrow \text{Spin}(2r)$ is the lift of the isotropy representation $\sigma: T \rightarrow \text{SO}(2r)$ to $\text{Spin}(2r)$, which exists by Lemma 3.2. In particular, spinors correspond to T -equivariant maps $G \rightarrow \Lambda^{\bullet} L'$, and *invariant* spinors correspond to *constant* such maps. Invariant spinors, then, can be seen as those elements of $\Lambda^{\bullet} L'$ which are annihilated by the differential action of $\mathfrak{h}$.

The differential of the isotropy representation is just the adjoint action $\text{ad}_{\mathfrak{g}}$ restricted to $\mathfrak{h}$ on its reductive complement $\mathfrak{m}$. Hence, the action of an element $X \in \mathfrak{h}$ on $\Lambda^{\bullet} L'$

is given by the action of $\widetilde{\text{ad}}_{\mathfrak{g}}(X) := \rho(\text{ad}_{\mathfrak{g}}(X)) \in \mathbb{C}l(2r)$, where ρ is the inverse of the Lie algebra isomorphism induced by the double covering $\lambda_{2r}: \text{Spin}(2r) \rightarrow \text{SO}(2r)$:

$$\rho: \mathfrak{so}(2r) \rightarrow \mathfrak{spin}(2r) \subseteq \mathbb{C}l(2r), \quad e_i \wedge e_j \mapsto \frac{1}{2}e_i \cdot e_j.$$

By (3.6), the action of $X \in \mathfrak{h}$ on $\eta \in \Lambda^\bullet L'$ is then given by

$$\widetilde{\text{ad}}_{\mathfrak{g}}(X) \cdot \eta = \frac{1}{2} \sum_{j=1}^r a_j(X) e_1^{(j)} \cdot e_2^{(j)} \cdot \eta.$$

Note that for $1 \leq j_1 < \dots < j_k \leq r$ and $1 \leq j \leq r$,

$$e_1^{(j)} \cdot e_2^{(j)} \cdot (y_{j_1} \wedge \dots \wedge y_{j_k}) = \begin{cases} -i(y_{j_1} \wedge \dots \wedge y_{j_k}) & \text{if } j \in \{j_1, \dots, j_k\}, \\ i(y_{j_1} \wedge \dots \wedge y_{j_k}) & \text{if } j \notin \{j_1, \dots, j_k\}, \end{cases}$$

generalising Example 2.30. In particular,

$$2\widetilde{\text{ad}}_{\mathfrak{g}}(X) \cdot (y_{j_1} \wedge \dots \wedge y_{j_k}) = -i \left(\sum_{j \in \{j_1, \dots, j_k\}} a_j - \sum_{j \notin \{j_1, \dots, j_k\}} a_j \right) (X) \cdot (y_{j_1} \wedge \dots \wedge y_{j_k}),$$

and the theorem follows. $\square$

We now apply Theorem 3.5 in turn to each complex simple Lie algebra. The following theorem is summarised in Table 1.1.

Theorem 3.7. *For each of the complex simple Lie algebras, the following holds:*

1. $\text{MF}(A_n)$, $n \geq 1$, admits non-trivial invariant spinors if and only if n is even. In this case, the dimension of the space of invariant spinors is at least $2^{\frac{n}{2}}$.
2. $\text{MF}(B_n)$, $n \geq 2$, does not admit non-trivial invariant spinors for any n .
3. $\text{MF}(C_n)$, $n \geq 3$, admits non-trivial invariant spinors if and only if $n \equiv 0, 3 \pmod{4}$. In this case, the dimension of the space of invariant spinors is at least $2^{\lfloor \frac{n+1}{4} \rfloor}$.
4. $\text{MF}(D_n)$, $n \geq 4$, admits non-trivial invariant spinors if and only if $n \geq 4$ and $n \equiv 0, 1 \pmod{4}$. In this case, the dimension of the space of invariant spinors is at least $2^{\lfloor \frac{n+1}{4} \rfloor}$.
5. $\text{MF}(E_6)$ admits a 13 697 920-dimensional space of invariant spinors.

6. $\text{MF}(E_7)$ does not admit non-trivial invariant spinors.
7. $\text{MF}(E_8)$ admits non-trivial invariant spinors. The space of invariant spinors has dimension at least 369 600.
8. $\text{MF}(F_4)$ admits a 34 432-dimensional space of invariant spinors.
9. $\text{MF}(G_2)$ admits a four-dimensional space of invariant spinors.

Proof. By Theorem 3.5, we need to discuss strong linear dependence of the set of positive roots of each complex simple Lie algebra. Their positive roots are given in Table 3.1.

$\mathfrak{g}_{\mathbb{C}}$	Positive roots of $\mathfrak{g}_{\mathbb{C}}$	
A_n	$\lambda_i - \lambda_j$,	$1 \leq i < j \leq n$
	$\lambda_i + \sum_{j=1}^n \lambda_j$,	$1 \leq i \leq n$
B_n	$\lambda_i - \lambda_j$,	$1 \leq i < j \leq n$
	$\lambda_i + \lambda_j$,	$1 \leq i < j \leq n$
	λ_i ,	$1 \leq i \leq n$
C_n	$\lambda_i - \lambda_j$,	$1 \leq i < j \leq n$
	$\lambda_i + \lambda_j$,	$1 \leq i < j \leq n$
	$2\lambda_i$,	$1 \leq i \leq n$
D_n	$\lambda_i - \lambda_j$,	$1 \leq i < j \leq n$
	$\lambda_i + \lambda_j$,	$1 \leq i < j \leq n$
E_6	$\lambda_i - \lambda_j$,	$1 \leq i < j \leq 6$
	$\lambda_i + \lambda_j + \lambda_k$,	$1 \leq i < j < k \leq 6$
	$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6$	
E_7	$\lambda_i - \lambda_j$,	$1 \leq i < j \leq 7$
	$\lambda_i + \lambda_j + \lambda_k$,	$1 \leq i < j < k \leq 7$
	$\left(\sum_{j=1}^7 \lambda_j\right) - \lambda_i$,	$1 \leq i \leq 7$
E_8	$\lambda_i - \lambda_j$,	$1 \leq i < j \leq 8$
	$\lambda_i + \lambda_j + \lambda_k$,	$1 \leq i < j < k \leq 8$
	$\left(\sum_{j=1}^8 \lambda_j\right) + \lambda_i$,	$1 \leq i \leq 8$
	$\left(\sum_{k=1}^8 \lambda_k\right) - \lambda_i - \lambda_j$,	$1 \leq i < j \leq 8$
F_4	$\lambda_i - \lambda_j$,	$1 \leq i < j \leq 4$
	$\lambda_i + \lambda_j$,	$1 \leq i < j \leq 4$
	λ_i ,	$1 \leq i \leq 4$
	$\frac{1}{2}(\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4)$	
G_2	$\lambda_1, \lambda_2, -\lambda_1 - \lambda_2, \lambda_1 - \lambda_2, \lambda_1 + 2\lambda_2, 2\lambda_1 + \lambda_2$	

Table 3.1: Complex simple Lie algebras and their positive roots (see e.g. [Sam90, §2.14]). The roots of X_m are elements of $\mathbb{R}^m$, and we take $(\lambda_1, \dots, \lambda_m)$ to be its standard basis.

A_n First, we show that for every $k \geq 1$ the set of positive roots of A_{2k} is strongly linearly dependent. For each $1 \leq l < k$, note that

$$S_l := \sum_{j=2l+1}^{2k} (-1)^j (\lambda_{2l} - \lambda_j) - \sum_{j=2l+2}^{2k} (-1)^j (\lambda_{2l+1} - \lambda_j) = 0.$$

Letting now $\mu = \sum_{j=1}^{2k} \lambda_j$, observe that

$$T := (\lambda_1 + \mu) - \sum_{j=2}^{2k} (-1)^j ((\lambda_1 - \lambda_j) + (\lambda_j + \mu)) = 0.$$

Finally, then, $\pm S_1 \pm \dots \pm S_{k-1} \pm T = 0$ is a strong linear combination of the positive roots which equals zero, for every choice of sign at each $\pm$. In particular, the dimension of the space of invariant spinors of $\text{MF}(A_{2k})$ is at least 2^k .

We now turn to the case when n is odd. One can express the positive roots of A_n in terms of the basis $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \dots, \lambda_{n-1} - \lambda_n, \lambda_n + \sum_{j=1}^n \lambda_j)$ given by a fundamental system. There are exactly n positive roots of A_n whose first coordinate in this basis is 1, namely $\lambda_1 - \lambda_2, \dots, \lambda_1 - \lambda_n$, and $\lambda_1 + \sum_{j=1}^n \lambda_j$. The rest have vanishing first coordinate. Therefore, if n is odd, the set of positive roots of A_n cannot be strongly linearly dependent.

B_n Every strong linear combination of the positive roots of B_n is a sum of terms of the form $\pm \lambda_i$, and there is an odd number of those with $i = 1$.

C_n Given any strong linear combination of the positive roots of C_n , grouping together the terms $\pm(\lambda_i - \lambda_j)$ and $\pm(\lambda_i + \lambda_j)$, there are $n(n+1)/2$ terms of the form $\pm 2\lambda_i$. In particular, if $n \equiv 1, 2 \pmod{4}$, there is an odd number of these, and hence the strong linear combination cannot be zero.

If $n = 4k + \varepsilon \geq 3$ with $\varepsilon \in \{0, 3\}$, observe that for each $0 \leq l < k$:

$$\begin{aligned} S'_l := & (\lambda_{4l+1} - \lambda_{4l+2}) + (\lambda_{4l+1} + \lambda_{4l+2}) + \sum_{j=4l+3}^{4k+\varepsilon} ((\lambda_{4l+1} + \lambda_j) - (\lambda_{4l+1} - \lambda_j)) \\ & + (\lambda_{4l+2} - \lambda_{4l+3}) + (\lambda_{4l+2} + \lambda_{4l+3}) - \sum_{j=4l+4}^{4k+\varepsilon} ((\lambda_{4l+2} + \lambda_j) - (\lambda_{4l+2} - \lambda_j)) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=4l+4}^{4k+\varepsilon} ((\lambda_{4l+3} + \lambda_j) - (\lambda_{4l+3} - \lambda_j)) - \sum_{j=4l+5}^{4k+\varepsilon} ((\lambda_{4l+4} + \lambda_j) - (\lambda_{4l+4} - \lambda_j)) \\
& - 2\lambda_{4l+1} - 2\lambda_{4l+2} - 2\lambda_{4l+3} - 2\lambda_{4l+4} = 0.
\end{aligned}$$

Note that $\pm S'_0 \pm \dots \pm S'_l$ is a strong linear combination of those positive roots which contain $\lambda_1, \lambda_2, \dots, \lambda_{4l+4}$. Hence, if $\varepsilon = 0$, we are done, as every positive root appears exactly once in $\pm S'_0 \pm \dots \pm S'_{k-1} = 0$. If $\varepsilon = 3$, note that the remaining roots are strongly linearly dependent, as

$$\begin{aligned}
T' := & (\lambda_{4k+1} - \lambda_{4k+2}) - (\lambda_{4k+1} - \lambda_{4k+3}) + (\lambda_{4k+2} - \lambda_{4k+3}) + (\lambda_{4k+1} + \lambda_{4k+2}) \\
& + (\lambda_{4k+1} + \lambda_{4k+3}) + (\lambda_{4k+2} + \lambda_{4k+3}) - 2\lambda_{4k+1} - 2\lambda_{4k+2} - 2\lambda_{4k+3} = 0,
\end{aligned}$$

and thus $\pm S'_0 \pm \dots \pm S'_{k-1} \pm T' = 0$. This proof shows that, when it is not zero, the dimension of the space of invariant spinors is at least $2^{\lfloor \frac{n+1}{4} \rfloor}$.

D_n In a similar way as for C_n , any strong linear combination of the positive roots of D_n is the sum of $n(n-1)/2$ terms of the form $\pm 2\lambda_i$, which shows that for $n \equiv 2, 3 \pmod{4}$ no such linear combination can be zero.

If $n = 4k + \varepsilon \geq 4$ with $\varepsilon \in \{0, 1\}$, note that for all $0 \leq l < k$:

$$\begin{aligned}
S''_l := & \sum_{j=4l+2}^{4k+\varepsilon} (-1)^j ((\lambda_{4l+1} - \lambda_j) - (\lambda_{4l+1} + \lambda_j)) \\
& + (\lambda_{4l+2} - \lambda_{4l+3}) + (\lambda_{4l+2} + \lambda_{4l+3}) - \sum_{j=4l+4}^{4k+\varepsilon} (-1)^j ((\lambda_{4l+2} - \lambda_j) - (\lambda_{4l+2} + \lambda_j)) \\
& - (\lambda_{4l+3} - \lambda_{4l+4}) - (\lambda_{4l+3} + \lambda_{4l+4}) + \sum_{j=4l+5}^{4k+\varepsilon} (-1)^j ((\lambda_{4l+3} - \lambda_j) - (\lambda_{4l+3} + \lambda_j)) \\
& - \sum_{j=4l+5}^{4k+\varepsilon} (-1)^j ((\lambda_{4l+4} - \lambda_j) - (\lambda_{4l+4} + \lambda_j)) = 0,
\end{aligned}$$

As $\varepsilon \in \{0, 1\}$, all the positive roots appear exactly once in $\pm S''_0 \pm \dots \pm S''_{k-1} = 0$. It is clear, then, that the number of strong linear combinations of the roots that equal zero is at least 2^k .

E₆ We exhibit an explicit strong linear combination of the positive roots of E_6 that does the job:

$$\begin{aligned}
& (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) - (\lambda_1 - \lambda_2) + (\lambda_1 - \lambda_3) + (\lambda_1 - \lambda_4) + (\lambda_1 - \lambda_5) \\
& - (\lambda_1 - \lambda_6) + (\lambda_2 - \lambda_3) + (\lambda_2 - \lambda_4) - (\lambda_2 - \lambda_5) + (\lambda_2 - \lambda_6) - (\lambda_3 - \lambda_4) + (\lambda_3 - \lambda_5) \\
& - (\lambda_3 - \lambda_6) - (\lambda_4 - \lambda_5) - (\lambda_4 - \lambda_6) + (\lambda_5 - \lambda_6) - (\lambda_1 + \lambda_2 + \lambda_3) - (\lambda_1 + \lambda_2 + \lambda_4) \\
& - (\lambda_1 + \lambda_2 + \lambda_5) - (\lambda_1 + \lambda_2 + \lambda_6) - (\lambda_1 + \lambda_3 + \lambda_4) + (\lambda_1 + \lambda_3 + \lambda_5) + (\lambda_1 + \lambda_3 + \lambda_6) \\
& + (\lambda_1 + \lambda_4 + \lambda_5) + (\lambda_1 + \lambda_4 + \lambda_6) - (\lambda_1 + \lambda_5 + \lambda_6) + (\lambda_2 + \lambda_3 + \lambda_4) - (\lambda_2 + \lambda_3 + \lambda_5) \\
& + (\lambda_2 + \lambda_3 + \lambda_6) + (\lambda_2 + \lambda_4 + \lambda_5) - (\lambda_2 + \lambda_4 + \lambda_6) - (\lambda_2 + \lambda_5 + \lambda_6) + (\lambda_3 + \lambda_4 + \lambda_5) \\
& + (\lambda_3 + \lambda_4 + \lambda_6) - (\lambda_3 + \lambda_5 + \lambda_6) - (\lambda_4 + \lambda_5 + \lambda_6) = 0.
\end{aligned}$$

The value of the dimension of the space of invariant spinors is obtained with the aid of a computer – see Appendix B.

E₇ In any strong linear combination of the positive roots of E_7 , expressed as a sum of terms of the form $\pm\lambda_i$, there are 27 terms of the form $\pm\lambda_1$, so the strong linear combination cannot be zero.

E₈ Define $\nu := \sum_{i=1}^8 \lambda_i$, and note that:

$$\begin{aligned}
\sum_{1 \leq i < j < k \leq 8} (\lambda_i + \lambda_j + \lambda_k) &= \sum_{1 \leq i < j \leq 8} (-\lambda_i - \lambda_j + \nu) = 21\nu, \\
\sum_{j=2}^8 (-1)^j (\lambda_1 - \lambda_j) &= - \sum_{j=1}^8 (-1)^j (\lambda_j + \nu).
\end{aligned}$$

Therefore, the set

$$\begin{aligned}
& \{\lambda_1 - \lambda_j : 2 \leq j \leq 8\} \cup \{\lambda_i + \nu : 1 \leq i \leq 8\} \\
& \cup \{\lambda_i + \lambda_j + \lambda_k : 1 \leq i < j < k \leq 8\} \cup \{-\lambda_i - \lambda_j + \nu : 1 \leq i < j \leq 8\}
\end{aligned}$$

is strongly linearly dependent. It is enough, then, to show that the set $\{\lambda_i - \lambda_j : 2 \leq i < j \leq 8\}$ is as well. But this follows from the proof of the case A_{2k} for $k = 3$, by shifting the indices and taking $\mu = \lambda_8$. Using a computer – see Appendix B – we obtain the claimed lower bound for the dimension of the space of invariant spinors: $2 \times 70 \times 2640 = 369600$.

F₄ We exhibit an explicit strong linear combination of the positive roots of F_4 that does the job:

$$\begin{aligned}
& \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - (\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_3) - (\lambda_1 - \lambda_4) - (\lambda_2 - \lambda_3) - (\lambda_2 - \lambda_4) \\
& - (\lambda_3 - \lambda_4) - (\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_3) - (\lambda_1 + \lambda_4) + (\lambda_2 + \lambda_3) + (\lambda_2 + \lambda_4) \\
& - (\lambda_3 + \lambda_4) + \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) - \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4) + \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4) \\
& + \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4) + \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4) + \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4) \\
& + \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4) + \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) = 0.
\end{aligned}$$

The claimed dimension is easily obtained with the aid of a computer – see Appendix B.

G₂ There are exactly four strong linear combinations of the positive roots of G_2 that equal zero:

$$\begin{aligned}
& \pm (\lambda_1 + \lambda_2 + (-\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_2) - (\lambda_1 + 2\lambda_2) + (2\lambda_1 + \lambda_2)) = 0, \\
& \pm (\lambda_1 + \lambda_2 + (-\lambda_1 - \lambda_2) + (\lambda_1 - \lambda_2) + (\lambda_1 + 2\lambda_2) - (2\lambda_1 + \lambda_2)) = 0.
\end{aligned}$$

This finishes the proof of Theorem 3.7. □

Remark 3.8. *One can computationally obtain the exact value of the dimension of the space of invariant spinors. For example, the values for A_4 and A_6 are 24 and 2640 respectively.*

Until now, we have only considered *maximal* flag manifolds, and one might ask whether similar results can be obtained for *general* flag manifolds. Given a complex simple Lie algebra, its flag manifolds can behave very differently, as we show now. The following examples show that, when it comes to the spinorial properties we have been considering so far, general flag manifolds can have a wild behaviour.

Example 3.9. *(Non-spin flag manifolds) Complex projective space $\mathbb{CP}^n$ is a flag manifold of A_n , and this space is spin if and only if n is odd.*

Example 3.10. *(Two flag manifolds of the same Lie algebra with all possible spinorial behaviours) Complex projective space $\mathbb{CP}^{2n+1}$ is also a flag manifold of C_{n+1} . We know that this space has non-trivial invariant spinors if and only if n is odd – see [AH25]. However, by Theorem 3.7, $\text{MF}(C_{n+1})$ has invariant spinors if and only if $n \equiv 2, 3 \pmod{4}$. In particular:*

- $\mathbb{CP}^3$ is a (non-maximal) flag manifold of B_2 which admits non-trivial invariant spinors, however the maximal flag manifold of B_2 does not.
- $\mathbb{CP}^5$ is a (non-maximal) flag manifold of C_3 which does not admit non-trivial invariant spinors, however, the maximal flag manifold of C_3 does.
- $\mathbb{CP}^7$ is a (non-maximal) flag manifold of C_4 which admits non-trivial invariant spinors, and the maximal flag manifold of C_4 does as well.
- $\mathbb{CP}^9$ is a (non-maximal) flag manifold of C_5 which does not admit non-trivial invariant spinors, and neither does the maximal flag manifold of C_5 .

4

Generalised Killing spinors on Lie groups

The contents of this chapter and Appendix A have been published in [Art25]. In Section 4.1, we construct the first known examples of invariant generalised Killing spinors whose associated endomorphisms have more than four distinct eigenvalues. We then give a full classification of invariant generalised Killing spinors on three-dimensional Lie groups in Section 4.2.

4.1 The Heisenberg groups

In this section, we exhibit the first known examples of invariant generalised Killing spinors with an arbitrary number of distinct eigenvalues on a homogeneous manifold. For each $n \in \mathbb{N}$, we will find a left-invariant metric on the n -th Heisenberg group H_{2n+1} which carries an invariant generalised Killing spinor with $(n + 1)$ distinct eigenvalues. The calculations carried out in this section are crucial for Section 4.2, as they exemplify the methods required there.

We start by recalling the definition of the Heisenberg groups:

Definition 4.1. For $n \in \mathbb{N}$, the n -th (real) Heisenberg group H_{2n+1} is defined as

$$H_{2n+1} := \left\{ \begin{pmatrix} 1 & v^T & x \\ 0 & \text{Id}_n & w \\ 0 & 0 & 1 \end{pmatrix} \in M_{n+2}(\mathbb{R}) : v, w \in \mathbb{R}^n, x \in \mathbb{R} \right\},$$

where we think of elements $v \in \mathbb{R}^n$ as column vectors and v^T denotes the transpose of v .

With the obvious topology and smooth structure, H_{2n+1} is a Lie group which is diffeomorphic to $\mathbb{R}^{2n+1}$. The Lie algebra of H_{2n+1} is

$$\mathfrak{h}_{2n+1} := \text{span}_{\mathbb{R}} \left\{ \hat{Z}, \hat{E}_p, \hat{F}_p \right\}_{p=1, \dots, n},$$

where

$$\hat{Z} := \begin{pmatrix} 0 & 0^T & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{E}_p := \begin{pmatrix} 0 & e_p^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{F}_p := \begin{pmatrix} 0 & 0^T & 0 \\ 0 & 0 & e_p \\ 0 & 0 & 0 \end{pmatrix}$$

and e_p denotes the p -th vector of the standard basis of $\mathbb{R}^n$. These generators of $\mathfrak{h}_{2n+1}$ satisfy, for $1 \leq p, q \leq n$,

$$[\hat{E}_p, \hat{F}_q] = \delta_{p,q} \hat{Z}, \quad [\hat{E}_p, \hat{Z}] = [\hat{F}_p, \hat{Z}] = [\hat{E}_p, \hat{E}_q] = [\hat{F}_p, \hat{F}_q] = 0,$$

where δ is the Kronecker delta. Observe that this Lie algebra has one-dimensional centre. We will fix the orientation determined by the ordered basis

$$(\hat{Z}, \hat{E}_1, \hat{F}_1, \dots, \hat{E}_n, \hat{F}_n).$$

Now, let g be the left-invariant metric on H_{2n+1} determined by the inner product B on $\mathfrak{h}_{2n+1}$ for which the basis

$$(Z := \hat{Z}, E_1 := a_1 \hat{E}_1, F_1 := \hat{F}_1, \dots, E_n := a_n \hat{E}_n, F_n := \hat{F}_n) \quad (4.2)$$

is B -orthonormal, for some $(a_1, \dots, a_n) \in \mathbb{R}_{>0}^n$. We can obtain an expression for the Nomizu map of the Levi-Civita connection of g .

Lemma 4.3. The Nomizu map $\Lambda: \mathfrak{h}_{2n+1} \rightarrow \mathfrak{so}(\mathfrak{h}_{2n+1})$ of the Levi-Civita connection

of g is given by

$$\Lambda(Z) = -\frac{1}{2} \sum_{p=1}^n a_p E_p \wedge F_p, \quad \Lambda(F_p) = -\frac{1}{2} a_p E_p \wedge Z, \quad \Lambda(E_p) = \frac{1}{2} a_p F_p \wedge Z.$$

Proof. Recall from Proposition 2.25 that the Nomizu map Λ of the Levi-Civita connection of the invariant metric g on H_{2n+1} induced by the inner product B on $\mathfrak{h}_{2n+1}$ satisfies, for every $X, Y \in \mathfrak{h}_{2n+1}$,

$$\Lambda(X)(Y) = \frac{1}{2} [X, Y] + U(X, Y),$$

where the bilinear map U is determined by imposing, for all $W \in \mathfrak{h}_{2n+1}$,

$$B(U(X, Y), W) = \frac{1}{2} (B([W, X], Y) + B(X, [W, Y])).$$

Fix $1 \leq p \leq n$. Let us first compute $\Lambda(E_p)$. Note that, for $1 \leq q \leq n$,

$$\begin{aligned} \Lambda(E_p)(E_q) &= \frac{1}{2} [E_p, E_q] + U(E_p, E_q) = 0, \\ \Lambda(E_p)(F_q) &= \frac{1}{2} [E_p, F_q] + U(E_p, F_q) = \frac{1}{2} [E_p, F_q] = \\ &= \frac{1}{2} a_p [\hat{E}_p, \hat{F}_q] = \delta_{p,q} \frac{1}{2} a_p Z, \\ \Lambda(E_p)(Z) &= \frac{1}{2} [E_p, Z] + U(E_p, Z) = U(E_p, Z) = \\ &= \frac{1}{2} B([F_p, E_p], Z) F_p = -\frac{1}{2} a_p F_p. \end{aligned}$$

This proves that $\Lambda(E_p)$ is as claimed. A similar computation yields that $\Lambda(F_p)$ is as claimed. Finally, let us compute $\Lambda(Z)$. For each $1 \leq p \leq n$, we obtain

$$\begin{aligned} \Lambda(Z)(E_p) &= \frac{1}{2} [Z, E_p] + U(Z, E_p) = U(Z, E_p) = \\ &= \frac{1}{2} B(Z, [F_p, E_p]) F_p = -\frac{1}{2} a_p F_p, \\ \Lambda(Z)(F_p) &= \frac{1}{2} [Z, F_p] + U(Z, F_p) = U(Z, F_p) = \\ &= \frac{1}{2} B(Z, [E_p, F_p]) E_p = \frac{1}{2} a_p E_p, \\ \Lambda(Z)(Z) &= \frac{1}{2} [Z, Z] + U(Z, Z) = 0. \end{aligned}$$

This concludes the proof. $\square$

Recalling the notation of Section 2.7, we can now state the following result:

Theorem 4.4. *The invariant spinor on (H_{2n+1}, g) determined by $1 \in \Sigma$ is a generalised Killing spinor whose associated endomorphism has the following set of eigenvalues:*

$$\left\{ \lambda_p = \frac{1}{4}a_p, \mu = -\sum_{q=1}^n \lambda_q \right\}_{p=1, \dots, n}.$$

In particular, choosing $a_p := p$, there are $n+1$ distinct constant eigenvalues.

Proof. For $X \in \mathfrak{h}_{2n+1}$, let $\tilde{\Lambda}(X) \in \mathfrak{spin}(2n+1)$ be the image of $\Lambda(X) \in \mathfrak{so}(2n+1)$ under the isomorphism

$$e_i \wedge e_j \mapsto \frac{1}{2}e_i \cdot e_j.$$

We will check that, for all $1 \leq p \leq n$,

$$\tilde{\Lambda}(E_p) \cdot 1 = \lambda_p E_p \cdot 1, \quad \tilde{\Lambda}(F_p) \cdot 1 = \lambda_p F_p \cdot 1, \quad \tilde{\Lambda}(Z) \cdot 1 = \mu Z \cdot 1.$$

From Lemma 4.3, we get that the spin lift $\tilde{\Lambda}$ of the Nomizu map Λ is given by

$$\tilde{\Lambda}(E_p) = \frac{1}{4}a_p F_p \cdot Z, \quad \tilde{\Lambda}(F_p) = -\frac{1}{4}a_p E_p \cdot Z, \quad \tilde{\Lambda}(Z) = -\frac{1}{4} \sum_{p=1}^n a_p E_p \cdot F_p.$$

Hence, applying these operators to the spinor $1 \in \Sigma$, we obtain

$$\begin{aligned} \tilde{\Lambda}(E_p) \cdot 1 &= \frac{1}{4}a_p F_p \cdot Z \cdot 1 = \frac{i}{4}a_p F_p \cdot 1 = \frac{i}{4}a_p y_p, \\ \tilde{\Lambda}(F_p) \cdot 1 &= -\frac{1}{4}a_p E_p \cdot Z \cdot 1 = -\frac{i}{4}a_p E_p \cdot 1 = \frac{1}{4}a_p y_p, \\ \tilde{\Lambda}(Z) \cdot 1 &= -\frac{1}{4} \sum_{p=1}^n a_p E_p \cdot F_p \cdot 1 = -\frac{i}{4} \sum_{p=1}^n a_p \cdot 1. \end{aligned}$$

And because

$$E_p \cdot 1 = i y_p, \quad F_p \cdot 1 = y_p, \quad Z \cdot 1 = i 1,$$

we conclude. $\square$

4.2 Three-dimensional Lie groups

In this section, we study the existence of left-invariant metrics on three-dimensional Lie groups which admit invariant generalised Killing spinors. We use Bianchi's classification of three-dimensional real Lie algebras [Bia01, Mub63] –see Appendix A.

For a general spin manifold, the set of generalised Killing spinors does not have a good algebraic structure. It is clear that this set is closed under multiplication by scalars in $\mathbb{C}$, but in general it is not closed under addition. Not even the set of *invariant* generalised Killing spinors on a homogeneous space can be endowed with a good structure. However, in the case of a connected three-dimensional Lie group G , this set is a complex vector space, as we show in the following lemma. Obviously, G has a complex two-dimensional space of invariant spinors.

Lemma 4.5. *Let G be a connected three-dimensional Lie group with Lie algebra $\mathfrak{g}$, and let g be a left-invariant metric on G carrying a non-zero invariant generalised Killing spinor with associated endomorphism A . Then, every invariant spinor is generalised Killing with endomorphism A .*

Proof. Let (e_1, e_2, e_3) be a g -orthonormal basis of $\mathfrak{g}$. Let $\tilde{\Lambda}$ be the lift to $\mathfrak{spin}(3)$ of the Nomizu map of the Levi-Civita connection of g . There exist linear forms

$$\lambda, \mu, \nu: \mathfrak{g} \rightarrow \mathbb{R}$$

such that, for every $X \in \mathfrak{g}$,

$$\tilde{\Lambda}(X) = \lambda(X)e_1e_2 + \mu(X)e_1e_3 + \nu(X)e_2e_3,$$

and hence

$$\begin{aligned} \tilde{\Lambda}(X) \cdot 1 &= \lambda(X)e_1e_2 \cdot 1 + \mu(X)e_1e_3 \cdot 1 + \nu(X)e_2e_3 \cdot 1 \\ &= i\lambda(X)e_1 \cdot (y_1) + \mu(X)e_1 \cdot (y_1) + \nu(X)e_2 \cdot (y_1) \\ &= \lambda(X)y_1 - i\mu(X)y_1 + i\nu(X)1 \\ &= \lambda(X)e_3 \cdot 1 - \mu(X)e_2 \cdot 1 + \nu(X)e_1 \cdot 1 \\ &= (\lambda(X)e_3 - \mu(X)e_2 + \nu(X)e_1) \cdot 1. \end{aligned}$$

Similarly,

$$\tilde{\Lambda}(X) \cdot y_1 = \lambda(X)e_1e_2 \cdot y_1 + \mu(X)e_1e_3 \cdot y_1 + \nu(X)e_2e_3 \cdot y_1$$

$$\begin{aligned}
&= i\lambda(X)e_1 \cdot (1) - \mu(X)e_1 \cdot (1) - \nu(X)e_2 \cdot (1) \\
&= -\lambda(X)1 - i\mu(X)1 - i\nu(X)y_1 \\
&= \lambda(X)e_3 \cdot y_1 - \mu(X)e_2 \cdot y_1 + \nu(X)e_1 \cdot y_1 \\
&= (\lambda(X)e_3 - \mu(X)e_2 + \nu(X)e_1) \cdot y_1.
\end{aligned}$$

Define $B \in \text{End}(\mathfrak{g})$ by

$$X \mapsto \lambda(X)e_3 - \mu(X)e_2 + \nu(X)e_1.$$

It is clear that, for each $X \in \mathfrak{g}$, and every invariant spinor $\phi \in \Sigma$,

$$\tilde{\Lambda}(X) \cdot \phi = B(X) \cdot \phi. \quad (4.6)$$

But, by hypothesis, there exists a non-zero invariant generalised Killing spinor ψ with associated symmetric endomorphism $A \in \text{End}(\mathfrak{g})$. Hence, for every $X \in \mathfrak{g}$,

$$A(X) \cdot \psi = \tilde{\Lambda}(X) \cdot \psi = B(X) \cdot \psi.$$

As $\psi \neq 0$, this implies that $A = B$, and hence B is symmetric. So, by (4.6), every invariant spinor is generalised Killing with endomorphism A . $\square$

We can now state the main result of this section, which gives a complete picture of the set of invariant generalised Killing spinors on connected three-dimensional Lie groups.

Theorem 4.7. *Let G be a connected three-dimensional Lie group with Lie algebra $\mathfrak{g}$. For any choice of orientation and left-invariant metric on G , the following holds:*

1. *If $\mathfrak{g} = \text{Bian}(I)$, then every left-invariant metric on G carries a two-dimensional space of invariant generalised Killing spinors with one eigenvalue (in fact, the spinors are parallel).*
2. *If $\mathfrak{g} = \text{Bian}(II)$, then every left-invariant metric on G carries a two-dimensional space of invariant generalised Killing spinors with two distinct eigenvalues.*
3. *If $\mathfrak{g} = \text{Bian}(III)$, $\text{Bian}(IV)$, $\text{Bian}(V)$, $\text{Bian}(VI_x)$ for some $0 < |x| < 1$ or $\text{Bian}(VII_y)$ for some $y > 0$, then no left-invariant metric on G carries non-trivial invariant generalised Killing spinors.*

4. If $\mathfrak{g} = \text{Bian}(VI_{-1})$, then every left-invariant metric on G carries a two-dimensional space of invariant generalised Killing spinors with three distinct eigenvalues.
5. If $\mathfrak{g} = \text{Bian}(VII_0)$, $\text{Bian}(VIII)$ or $\text{Bian}(IX)$, then every left-invariant metric on G carries a two-dimensional space of invariant generalised Killing spinors. Moreover, the set of left-invariant metrics for which the number of distinct eigenvalues of these spinors is strictly less than 3 has measure zero in the set of all left-invariant metrics on G .

This is summarised in Table 4.1.

$\mathfrak{g}$	$\dim(\Sigma_{\text{inv}}^{GK})$	l
$\text{Bian}(I)$	2	1
$\text{Bian}(II)$	2	2
$\text{Bian}(III), \text{Bian}(IV), \text{Bian}(V)$	0	—
$\text{Bian}(VI_x)$	2, if $x = -1$ 0, if $x \neq -1$	3 —
$\text{Bian}(VII_y)$	2, if $y = 0$ 0, if $y \neq 0$	3 —
$\text{Bian}(VIII), \text{Bian}(IX)$	2	3

Table 4.1: Maximum (and generic) number of distinct eigenvalues, denoted by l , of invariant generalised Killing spinors on a connected three-dimensional Lie group with Lie algebra $\mathfrak{g}$ equipped with any orientation and any left-invariant metric. Σ_{inv}^{GK} denotes the complex dimension of the vector space of invariant generalised Killing spinors.

Proof. The strategy in every case will be the following. We start by fixing a basis (f_1, f_2, f_3) of $\mathfrak{g}$ for which the structure constants take the form given in Appendix A. We fix the orientation defined by this ordered basis (a different choice of orientation does not affect the subsequent results). Then, we consider an invertible matrix of the form

$$P = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \varepsilon & \zeta \\ 0 & 0 & \iota \end{pmatrix} \in \text{GL}^+(3, \mathbb{R}),$$

and define B_P to be the unique inner product on $\mathfrak{g}$ such that the basis (e_1, e_2, e_3) defined by

$$e_1 = \alpha f_1, \quad e_2 = \beta f_1 + \varepsilon f_2, \quad \text{and} \quad e_3 = \gamma f_1 + \zeta f_2 + \iota f_3$$

is (positively oriented and) B_P -orthonormal. Note that this covers all possible inner products on $\mathfrak{g}$. Then, we compute the endomorphism A of $\mathfrak{g}$ such that, for every $X \in \mathfrak{g}$,

$$\tilde{\Lambda}(X) \cdot 1 = A(X) \cdot 1.$$

This endomorphism exists by the proof of Lemma 4.5, and it is obtained after a computation analogous to the one done in detail in Section 4.1. We give a general expression later, in equation (4.9). The explicit results in terms of P are collected in Appendix A.

Finally, by Lemma 4.5, there are only two cases: if A is symmetric, then all invariant spinors are generalised Killing with endomorphism A ; if not, the only invariant generalised Killing spinor is 0. In Table 4.2, we show the value of $A - A^T$ in each case –see also Remark 4.8.

$\mathfrak{g}$	$A - A^T$
Bian(I), Bian(II), Bian(VIII), Bian(IX)	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Bian(III)	$\frac{1}{2} \begin{pmatrix} 0 & -\zeta & \varepsilon \\ \zeta & 0 & 0 \\ -\varepsilon & 0 & 0 \end{pmatrix}$
Bian(IV), Bian(V)	$\begin{pmatrix} 0 & -\iota & 0 \\ \iota & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Bian(VI _{x})	$\frac{x+1}{2} \begin{pmatrix} 0 & -\iota & 0 \\ \iota & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Bian(VII _{y})	$y \begin{pmatrix} 0 & -\iota & 0 \\ \iota & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Table 4.2: For each Lie algebra $\mathfrak{g}$, value of $A - A^T$ in the basis (e_1, e_2, e_3) .

□

Remark 4.8. *It is very surprising that the symmetry of A is completely independent of the choice of left-invariant metric – see Table 4.2.*

In fact, we can obtain a general expression for the endomorphism A as follows. Fix a connected three-dimensional Lie group G , and fix an orientation and an inner product B on its Lie algebra $\mathfrak{g}$. Let (e_1, e_2, e_3) be a positively oriented B -orthonormal

basis of $\mathfrak{g}$. Then, for each $1 \leq i, j \leq 3$,

$$[e_i, e_j] = \sum_{k=1}^3 c_{ij}^k e_k,$$

for some $c_{ij}^k \in \mathbb{R}$. A calculation similar to the one detailed in Section 4.1 yields the matrix of A in the basis (e_1, e_2, e_3) :

$$A = \begin{pmatrix} \frac{1}{4}(c_{12}^3 - c_{13}^2 - c_{23}^1) & -\frac{1}{2}c_{23}^2 & -\frac{1}{2}c_{23}^3 \\ \frac{1}{2}c_{13}^1 & \frac{1}{4}(c_{12}^3 + c_{13}^2 + c_{23}^1) & \frac{1}{2}c_{13}^3 \\ -\frac{1}{2}c_{12}^1 & -\frac{1}{2}c_{12}^2 & \frac{1}{4}(-c_{12}^3 - c_{13}^2 + c_{23}^1) \end{pmatrix}. \quad (4.9)$$

This matrix is symmetric if and only if

$$c_{13}^1 + c_{23}^2 = 0, \quad c_{12}^1 - c_{23}^3 = 0, \quad \text{and} \quad c_{12}^2 + c_{13}^3 = 0.$$

In this case, every invariant spinor $\psi \in \Sigma$ is a generalised Killing spinor with endomorphism A . In particular, every $\psi \in \Sigma$ is an eigenspinor of the Dirac operator with eigenvalue

$$\text{tr}(A) = \frac{1}{4}(c_{12}^3 - c_{13}^2 + c_{23}^1).$$

The endomorphism A commutes with the Ricci endomorphism, as shown in the following:

Theorem 4.10. *Let G be a connected three-dimensional Lie group equipped with an orientation and a left-invariant metric g carrying a non-trivial invariant generalised Killing spinor with endomorphism A . Then, A commutes with the Ricci endomorphism of g .*

Proof. Let ψ be such a spinor, and let Ric be the Ricci endomorphism of g . As, by hypothesis, A is symmetric, using the well-known expression

$$\sum_{j=1}^3 e_j \text{R}(X, e_j) \psi = -\frac{1}{2} \text{Ric}(X) \cdot \psi,$$

and the explicit expression of A in (4.9), one obtains the matrix of Ric in the basis (e_1, e_2, e_3) :

$$\begin{pmatrix} \frac{1}{2}(-(c_{12}^3)^2 - (c_{13}^2)^2 - 4(c_{13}^1)^2 + (c_{23}^1)^2) & -(c_{12}^3 + c_{13}^2 - c_{23}^1)c_{23}^2 - 2c_{13}^3c_{23}^3 & (c_{12}^3 + c_{13}^2 + c_{23}^1)c_{23}^3 - 2c_{13}^3c_{23}^2 \\ -(c_{12}^3 + c_{13}^2 - c_{23}^1)c_{23}^2 - 2c_{13}^3c_{23}^3 & \frac{1}{2}((c_{13}^2)^2 - (c_{12}^3 - c_{23}^1)^2 - 4(c_{23}^3)^2) & c_{13}^3(-c_{12}^3 + c_{13}^2 + c_{23}^1) + 2c_{23}^2c_{23}^3 \\ (c_{12}^3 + c_{13}^2 + c_{23}^1)c_{23}^3 - 2c_{13}^3c_{23}^2 & c_{13}^3(-c_{12}^3 + c_{13}^2 + c_{23}^1) + 2c_{23}^2c_{23}^3 & \frac{1}{2}((c_{12}^3)^2 - (c_{13}^2 + c_{23}^1)^2 - 4(c_{23}^3)^2) \end{pmatrix} \quad (4.11)$$

– see also the classical paper [Mil76]. Using the Jacobi identity, one can easily check that Ric and A commute. $\square$

Remark 4.12. *As a consequence of Theorem 4.10, there exists a global orthonormal frame of left-invariant vector fields on G which are eigenvectors of both A and Ric . This illustrates one of the main features of spin geometry, namely that it reduces curvature calculations from a second-order problem to a first-order one.*

For example, consider the case of the Heisenberg group H_3 , which has Lie algebra $\mathfrak{h}_3 \cong \text{Bian}(\text{II})$, with an arbitrary orientation and left-invariant metric. Take $\{Z, E, F\}$ an orthonormal basis of $\mathfrak{h}_3$ as in (4.2), satisfying $[E, F] = aZ$, $[E, Z] = [F, Z] = 0$, for some $a > 0$. Identify now the Lie algebra with the space of left-invariant vector fields in the usual way. We saw – see the proof of Theorem 4.4 – that in this case the endomorphism A is symmetric and has two distinct eigenvalues, which give rise to two left-invariant subbundles $\Xi = \text{span}\{Z\}$ and $\Xi^\perp = \text{span}\{E, F\}$ of the tangent bundle of H_3 . The distribution $\Xi^\perp$ defines a contact metric structure on H_3 , and Ξ is the corresponding Reeb distribution, because $[E, F] \notin \Xi$. The matrix of the Ricci endomorphism in this basis is obtained by substituting the structure constants into (4.11), yielding

$$\frac{a^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which shows that any left-invariant metric on H_3 is η -Einstein.

5

A generalisation of spin: spin^r geometry

Most of the content of this chapter appears in the co-authored works [AL25, AH25]. The portions of this chapter based on joint work reflect my own contributions, and the roles of my co-authors are duly acknowledged.

Section 5.1 formally introduces the spin^r groups and structures. Section 5.2 extends the homogeneous-space setup of Chapter 2 to spin^r , culminating in a new classification result (Theorem 5.13). In Section 5.3 we study invariant spin^r structures on homogeneous spheres in detail, and Section 5.4 introduces the tools to work with invariant spin^r spinors in Chapter 6.

5.1 Preliminaries

5.1.1 Spin^r groups

For $n, r \in \mathbb{N}$, we define the group

$$\text{Spin}^r(n) := \frac{\text{Spin}(n) \times \text{Spin}(r)}{\langle (-1, -1) \rangle}.$$

Note that, as

$$\text{Spin}(1) \cong \{\pm 1\}, \quad \text{Spin}(2) \cong \text{U}(1), \quad \text{and} \quad \text{Spin}(3) \cong \text{Sp}(1),$$

we recover

$$\text{Spin}^1(n) = \text{Spin}(n), \quad \text{Spin}^2(n) = \text{Spin}^{\mathbb{C}}(n), \quad \text{and} \quad \text{Spin}^3(n) = \text{Spin}^{\mathbb{H}}(n).$$

There are natural Lie group homomorphisms

$$\begin{aligned} \lambda_n^r: \text{Spin}^r(n) &\rightarrow \text{SO}(n) & \xi_n^r: \text{Spin}^r(n) &\rightarrow \text{SO}(r) \\ [\mu, \nu] &\mapsto \lambda_n(\mu) & [\mu, \nu] &\mapsto \lambda_r(\nu) \end{aligned}$$

For $1 \leq r \leq s$, the obvious inclusion $\mathbb{R}^r \hookrightarrow \mathbb{R}^s$ induces an inclusion of $\text{Spin}(r)$ into $\text{Spin}(s)$. This in turn yields an inclusion $\iota_n^{rs}: \text{Spin}^r(n) \hookrightarrow \text{Spin}^s(n)$ which makes the following diagram commute:

$$\begin{array}{ccccc} & & \text{Spin}^r(n) & & \\ & \swarrow \iota_n^{1r} & \downarrow \iota_n^{rs} & \searrow \lambda_n^r & \\ \text{Spin}(n) & \xrightarrow{\lambda_n} & & \xrightarrow{\lambda_n^s} & \text{SO}(n) \\ & \searrow \iota_n^{1s} & \downarrow & \swarrow \lambda_n^s & \\ & & \text{Spin}^s(n) & & \end{array} \quad .$$

We need to first make some topological considerations, which we shall use later. Define the map

$$\begin{aligned} \varphi^{r,n}: \text{Spin}^r(n) &\rightarrow \text{SO}(n) \times \text{SO}(r). \\ [\mu, \nu] &\mapsto (\lambda_n(\mu), \lambda_r(\nu)) \end{aligned}$$

The following result shows that this map is a two-sheeted covering, and describes it at the level of fundamental groups. We will use this result later to apply the lifting criterion.

Proposition 5.1. *For $n, r \in \mathbb{N}$, let $\varphi_{\sharp}^{r,n}$ be the map induced by $\varphi^{r,n}$ at the level of fundamental groups. Then,*

1. $\varphi^{r,n}$ is a two-sheeted covering map;
2. $\varphi_{\sharp}^{2,2}(\pi_1(\text{Spin}^2(n))) = \langle (1, \pm 1) \rangle \subseteq \mathbb{Z} \times \mathbb{Z} \cong \pi_1(\text{SO}(2) \times \text{SO}(2));$

3. if $n \geq 3$, then $\varphi_{\sharp}^{2,n}(\pi_1(\text{Spin}^2(n))) = \langle(1, 1)\rangle \subseteq \mathbb{Z}_2 \times \mathbb{Z} \cong \pi_1(\text{SO}(n) \times \text{SO}(2))$;

4. if $r, n \geq 3$, then $\varphi_{\sharp}^{r,n}(\pi_1(\text{Spin}^r(n))) = \langle(1, 1)\rangle \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \pi_1(\text{SO}(n) \times \text{SO}(r))$.

Proof. Consider the composition

$$\text{Spin}(n) \times \text{Spin}(r) \xrightarrow{p} \text{Spin}^r(n) \xrightarrow{\varphi^{r,n}} \text{SO}(n) \times \text{SO}(r),$$

where p is the obvious projection.

Point 1 follows easily from the fact that λ_n, λ_r and p are two-sheeted covering maps.

For 2, suppose $r = n = 2$. Recall that $\text{Spin}(2) \cong \text{SO}(2) \cong S^1$, and that the map $\varphi^{2,2} \circ p$ is given by $(z, w) \mapsto (z^2, w^2)$. Consider the loops

$$\alpha_{\pm}: [0, 1] \rightarrow \text{Spin}^2(2), \quad \alpha_{\pm}(t) := p(e^{i\pi t}, e^{\pm i\pi t}).$$

It is clear that the image of $\alpha_{\pm}$ in the fundamental group of $\text{SO}(2) \times \text{SO}(2)$, which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, is $(1, \pm 1)$. Moreover, the lift of every loop β based at $[1, 1]$ in $\text{Spin}^2(2)$ along p starting at $(1, 1)$ is of the form (β_1, β_2) , where either both β_i are loops based at 1 or both are paths from 1 to -1 in the corresponding factors of $\text{Spin}(2) \times \text{Spin}(2)$. In the first (resp. second) case, the image of β in the fundamental group of $\text{SO}(2) \times \text{SO}(2)$, which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, is of the form $(2r, 2s)$ (resp. $(2r + 1, 2s + 1)$), for some $r, s \in \mathbb{Z}$. In both cases, the image of β is an element of $\langle(1, \pm 1)\rangle$. This proves point 2.

Let us prove 3. Suppose $n \geq 3$. Let β be a loop in $\text{Spin}^2(n)$ based at $[1, 1]$. Its lift along p starting at $(1, 1)$ is a path with endpoint either $(1, 1)$ or $(-1, -1)$. Hence, this lift is of the form (β_1, β_2) , where either both β_i are loops based at 1 or both are paths from 1 to -1 in the corresponding factors of $\text{Spin}(n) \times \text{Spin}(2)$. In the first case, the image of β in the fundamental group of $\text{SO}(n) \times \text{SO}(2)$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$, is of the form $(0, 2m)$, for some $m \in \mathbb{Z}$. In the other case, the image of β is of the form $(1, 2l + 1)$, for some $l \in \mathbb{Z}$. This proves point 3.

The proof of 4 is analogous.

□

5.1.2 Spin^r structures on manifolds

As in the classical spin , $\text{spin}^{\mathbb{C}}$ and $\text{spin}^{\mathbb{H}}$ cases, one can define a family of structures on oriented Riemannian manifolds:

Definition 5.2. *Let M be an oriented Riemannian n -manifold with principal $\text{SO}(n)$ -bundle of positively oriented orthonormal frames FM . A spin^r structure on M is a lift of the structure group of FM along the Lie group homomorphism*

$$\lambda_n^r: \text{Spin}^r(n) \rightarrow \text{SO}(n).$$

In other words, it is a pair (P, Φ) consisting of

- *a principal $\text{Spin}^r(n)$ -bundle P over M , and*
- *a $\text{Spin}^r(n)$ -equivariant bundle homomorphism $\Phi: P \rightarrow FM$, where $\text{Spin}^r(n)$ acts on FM via λ_n^r .*

The associated principal $\text{SO}(r)$ -bundle to P along ξ_n^r is called the auxiliary bundle of the spin^r structure, and is denoted by $\widehat{P}$. The natural map

$$\theta: P \rightarrow FM \widetilde{\times} \widehat{P},$$

where $\widetilde{\times}$ denotes fibre product, is a two-sheeted covering.

If (P_1, Φ_1) and (P_2, Φ_2) are spin^r structures on M , an equivalence of spin^r structures from (P_1, Φ_1) to (P_2, Φ_2) is a $\text{Spin}^r(n)$ -equivariant diffeomorphism

$$f: P_1 \rightarrow P_2 \quad \text{such that} \quad \Phi_1 = \Phi_2 \circ f.$$

Remark 5.3. *For $r < s$, a spin^r structure naturally induces a spin^s structure via the associated bundle construction along the inclusion ι^{rs} of $\text{Spin}^r(n)$ into $\text{Spin}^s(n)$.*

Given an oriented Riemannian manifold M , it is therefore natural to ask which is the minimum value of $r \in \mathbb{N}$ such that M has a spin^r structure.

Definition 5.4. *Let M be an oriented Riemannian manifold. The spin type of M is defined as*

$$\Sigma(M) := \min \{r \in \mathbb{N}: M \text{ admits a } \text{spin}^r \text{ structure}\}.$$

The spin type measures the failure of a manifold to be spin. For example, every spin manifold has spin type 1, and $\Sigma(\mathbb{CP}^{2n}) = 2$ for all $n \geq 1$. Every oriented Riemannian manifold of dimension ≤ 4 has spin type ≤ 2 [TV94]. In [AM21, AM23], the authors show that every oriented Riemannian manifold of dimension ≤ 5 has spin type ≤ 3 (in the closed case, this holds for dimension ≤ 7). They also prove that in every dimension ≥ 8 there is a closed manifold with spin type ≥ 4 . For example, the product of the Wu manifold with itself is a 10-dimensional manifold with spin type ≥ 5 . This is because it does not admit a $\text{spin}^{\mathbb{H}}$ structure, and hence by results of [AM21] it cannot admit a spin^4 structure either.

Being spin^r turns out to be a topological property, as shown in the next result.

Proposition 5.5. ([AM21, Prop. 3.2]) *For an orientable Riemannian manifold M , the following are equivalent:*

1. M is spin^r ,
2. there is an orientable rank- r real vector bundle $E \rightarrow M$ such that $TM \oplus E$ is spin ,
3. M immerses in a spin manifold with codimension r ,
4. M embeds in a spin manifold with codimension r . □

This lets us prove that Definition 5.4 is a good definition, in the sense that the spin type is defined for all oriented Riemannian manifolds:

Corollary 5.6. *An oriented Riemannian n -manifold M has spin type*

$$1 \leq \Sigma(M) \leq n.$$

Proof. By Whitney's embedding theorem, such a manifold can be smoothly embedded into $\mathbb{R}^{2n}$, which is spin. Now apply Proposition 5.5. Equivalently, one can take $E = TM$ in point 2 of Proposition 5.5, and note that $w_1(TM \oplus TM) = 0$ and $w_2(TM \oplus TM) = 0$, as TM is oriented. □

Remark 5.7. *Note that, if $n > 1$ and M is a closed oriented Riemannian n -manifold, one can improve the bound in Corollary 5.6 to $1 \leq \Sigma(M) \leq n - \alpha(n)$, where $\alpha(n)$ is the number of ones in the binary expression of n , by Cohen's theorem [Coh85].*

There is a way of constructing an explicit spin^n structure on any connected and oriented Riemannian n -manifold M . Fix $x \in M$, and let G be the holonomy group of M at x . Let $h: G \rightarrow \text{SO}(n)$ be the holonomy representation at x . The positively oriented orthonormal frame bundle of M admits a reduction of the structure group from $\text{SO}(n)$ to G via h . Hence, if h lifts to $\text{Spin}^r(n)$, we obtain a spin^r structure on M via the associated bundle construction. This is always the case for $r = n$:

Proposition 5.8. (see [EH16, Prop. 3.3]) *Let M be a connected oriented Riemannian n -manifold, let $x \in M$, and let $h: G \rightarrow \text{SO}(n)$ be the holonomy representation at x . Then, h lifts to $\text{Spin}^n(n)$, i.e., there exists a Lie group homomorphism $\tilde{h}: G \rightarrow \text{Spin}^n(n)$ such that the diagram*

$$\begin{array}{ccc} & & \text{Spin}^n(n) \\ & \nearrow \tilde{h} & \downarrow \lambda_n^n \\ G & \xrightarrow{h} & \text{SO}(n) \end{array}$$

commutes.

Proof. Recall that $\lambda_n^n = \text{pr}_1 \circ \varphi^{n,n}$. The map $h \times h: G \rightarrow \text{SO}(n) \times \text{SO}(n)$ always lifts to $\text{Spin}^n(n)$ along $\varphi^{n,n}$, by Proposition 5.1. $\square$

This construction gives us a naturally preferred spin^n structure on any connected oriented Riemannian n -manifold. Note that its auxiliary vector bundle is the tangent bundle TM itself (compare with the proof of Corollary 5.6). It is natural to ask whether this spin^n structure on M^n is *proper*, in the sense that it is not induced by a spin^r structure on M , for some $r < n$. To answer this question, we first generalise a result by Moroianu [Mor97, Lem. 2.1], who focused on the case $r = 2, s = 1$:

Lemma 5.9. *Let M be an oriented Riemannian n -manifold admitting a spin^r structure for some $r \geq 1$, and let $1 \leq s \leq r$. Then, the spin^r structure is induced by a spin^s structure along ι_n^{sr} if and only if the auxiliary $\text{SO}(r)$ -bundle of the spin^r structure admits a reduction of the structure group to $\text{SO}(s)$.*

Proof. The proof generalises the one in [Mor97]. One implication follows easily from

the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Spin}^s(n) & \xrightarrow{\iota_n^{sr}} & \mathrm{Spin}^r(n) \\ \downarrow \xi_n^s & & \downarrow \xi_n^r \\ \mathrm{SO}(s) & \xrightarrow{j^{sr}} & \mathrm{SO}(r) \end{array},$$

where j^{sr} is the inclusion of $\mathrm{SO}(s)$ into $\mathrm{SO}(r)$ as the top left-hand block. For the other implication, let E be a reduced $\mathrm{SO}(s)$ -subbundle of the auxiliary bundle $\widehat{P}$ of the spin^r -structure (P, Φ) . Then, the preimage of $FM \widetilde{\times} E$ under the map

$$\theta: P \rightarrow FM \widetilde{\times} \widehat{P}$$

is a spin^s structure from which the original spin^r structure is induced. $\square$

A principal $\mathrm{SO}(r)$ -bundle admits a reduction of the structure group to $\mathrm{SO}(s)$ if and only if the associated rank- r vector bundle admits $r - s$ global orthonormal sections. In the construction of Proposition 5.8, the auxiliary rank- n vector bundle is the tangent bundle TM . Take, for instance, an even-dimensional sphere S^{2n} . By the hairy ball theorem, it does not admit a nowhere-vanishing vector field, so in particular the spin^{2n} structure constructed in Proposition 5.8 is not induced by any spin^r structure, for any $r < 2n$. This argument generalises to a wide class of manifolds, namely those closed oriented Riemannian manifolds with non-zero Euler class. Examples of these include S^{2n} , $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$. For more homogeneous examples, see [Wan49]. Note also that every odd-dimensional manifold admits a nowhere-vanishing vector field, and hence the spin^n structure given by the holonomy construction of Proposition 5.8 is induced by a spin^{n-1} structure.

Corollary 5.10. *Let M be an oriented Riemannian n -manifold. If $r > n$, every spin^r structure on M is induced by a spin^n structure on M along the inclusion*

$$\iota_n^{n,r}: \mathrm{Spin}^n(n) \hookrightarrow \mathrm{Spin}^r(n).$$

Proof. Recall that, if $r > n$, every rank- r vector bundle over an n -manifold has a nowhere-vanishing section [Hir76]. Apply this inductively to the auxiliary vector bundle of a spin^r structure on M . $\square$

Using Lemma 5.9, one can refine Corollary 5.6:

Corollary 5.11. *Let $n > 1$. An oriented Riemannian n -manifold M has spin type $1 \leq \Sigma(M) \leq n - 1$.*

Proof. If M is compact, this follows from Cohen's theorem, as in Remark 5.7. If M is non-compact, recall that every rank- n vector bundle over M has a nowhere-vanishing section [Hir76]. Hence, by Lemma 5.9, every spin^n structure on M comes from a spin^{n-1} structure on M . $\square$

5.2 Spin^r geometry on homogeneous spaces

We now turn to the case where M is an oriented Riemannian homogeneous space, extending the setup of Chapter 2. The main result of this section is Theorem 5.13, which gives a classification of G -invariant spin^r structures on homogeneous G -spaces up to G -equivariant equivalence. Its proof relies on Lemmata 2.15 and 2.18, which we proved back in Chapter 2.

5.2.1 Invariant spin^r structures

Let G be a Lie group acting by orientation-preserving isometries on an oriented Riemannian n -manifold M with positively oriented orthonormal frame bundle FM .

Definition 5.12. *A G -invariant spin^r structure on M is a spin^r structure (P, Φ) where P and Φ are G -equivariant.*

Two G -invariant spin^r structures $(P_1, \Phi_1), (P_2, \Phi_2)$ on M are said to be G -equivariantly equivalent if there exists a G -equivariant $\text{Spin}^r(n)$ -bundle isomorphism $f: P_1 \rightarrow P_2$ making the following diagram commute:

$$\begin{array}{ccc} P_1 & \xrightarrow{f} & P_2 \\ & \searrow \Phi_1 & \swarrow \Phi_2 \\ & FM & \end{array} .$$

For $r = 1$, we recover Definition 2.21. Finally, we can state our main result:

Theorem 5.13. *Let G/H be an n -dimensional oriented Riemannian homogeneous space with H connected and isotropy representation $\sigma: H \rightarrow \text{SO}(n)$. Then, there is a bijective correspondence between*

- G -invariant spin^r structures on G/H modulo G -equivariant equivalence of spin^r structures, and
- Lie group homomorphisms $\varphi: H \rightarrow \text{SO}(r)$ such that $\sigma \times \varphi: H \rightarrow \text{SO}(n) \times \text{SO}(r)$ lifts to $\text{Spin}^r(n)$ along λ_n^r modulo conjugation by an element of $\text{SO}(r)$.

Proof. A G -invariant spin^r structure is, by definition, a G -equivariant reduction of the structure group of the positively oriented orthonormal frame bundle $G \times_\sigma \text{SO}(n)$. Hence, by Lemma 2.18, G -invariant spin^r structures on G/H modulo G -equivariant equivalence of spin^r structures are in bijection with Lie group homomorphisms $\tilde{\sigma}: H \rightarrow \text{Spin}^r(n)$ such that $\lambda_n^r \circ \tilde{\sigma} = \sigma$ modulo conjugation by elements of $\ker(\lambda_n^r)$. But, as H is connected and λ_n^r is a covering map, these are in bijection with Lie group homomorphisms $\varphi: H \rightarrow \text{SO}(r)$ such that $\sigma \times \varphi$ lifts to $\text{Spin}^r(n)$ modulo conjugation by elements of $\text{SO}(r)$. $\square$

Remark 5.14. For the case $r = 1$, as $\text{SO}(1)$ is trivial, we get two cases: either σ lifts to $\text{Spin}(n)$ or it does not. So we have either a unique such structure or none. This recovers the well-known result that, if a G -invariant spin structure exists, it is unique (see e.g. [DSKL22, Cor. 1.4]).

In the light of Theorem 5.13, it is clear that, if $r < s$, a G -invariant spin^r structure naturally induces a G -invariant spin^s structure. So it makes sense to introduce a G -invariant analogue of the spin type of Definition 5.4:

Definition 5.15. Let G be a Lie group acting smoothly by orientation-preserving isometries on an oriented Riemannian manifold M . The G -invariant spin type of M , denoted by $\Sigma(M, G)$, is the least $r \in \mathbb{N}$ such that M admits a G -invariant spin^r structure.

For example, we know that the unique spin structure of S^n is not G -invariant for $G = \text{SO}(n+1), \text{U}((n+1)/2)$ [DSKL22], i.e.,

$$\Sigma(S^n, \text{SO}(n+1)), \Sigma(S^n, \text{U}((n+1)/2)) > 1.$$

But there might exist a G -invariant $\text{spin}^{\mathbb{C}}$ structure. Let us explore this in the next section.

5.3 Spheres

The aim of this section is to find the G -invariant spin type $\Sigma(S^n, G)$ of S^n for each $n \in \mathbb{N}$ and each compact and connected Lie group G acting smoothly, transitively and effectively by orientation-preserving isometries on S^n . These groups were classified in [MS43]. In [DSKL22], the authors determine when $\Sigma(S^n, G) = 1$. We will compute the rest (see Table 5.1).

Theorem 5.16. *The $\text{SO}(n+1)$ -invariant spin type of S^n is n , for $n \neq 4$, and 3, for $n=4$ (compare with [AM21, Prop. 3.9]). In other words,*

$$\Sigma(S^n, \text{SO}(n+1)) = \begin{cases} n, & n \neq 4 \\ 3, & n = 4. \end{cases}$$

Moreover, the $\text{SO}(n+1)$ -invariant spin^n structures on S^n up to $\text{SO}(n+1)$ -equivariant equivalence are classified as follows:

1. $n = 1$: there is a unique $\text{SO}(2)$ -invariant spin structure on S^1 .
2. $n = 2$: the $\text{SO}(3)$ -invariant $\text{spin}^{\mathbb{C}}$ structures on S^2 are the ones corresponding to $\varphi_s: S^1 \cong \text{SO}(2) \rightarrow \text{SO}(2) \cong S^1$ given by $z \mapsto z^s$, with s odd.
3. $n \geq 3$ odd: there exists a unique $\text{SO}(n+1)$ -invariant spin^n structure on S^n , and it is the one corresponding to the identity map $\text{SO}(n) \rightarrow \text{SO}(n)$.
4. $n = 4$: There are two $\text{SO}(5)$ -invariant $\text{spin}^{\mathbb{H}}$ structures on S^4 , corresponding to the Lie group homomorphisms

$$\text{SO}(4) \rightarrow \text{SO}(4)/\mathbb{Z}_2 \cong \text{SO}(3) \times \text{SO}(3) \xrightarrow{\text{pr}_i} \text{SO}(3)$$

for $i = 1, 2$.

5. $n \geq 6$ even: there exist two different $\text{SO}(n+1)$ -invariant spin^n structures on S^n , namely the ones corresponding to the identity map $\text{SO}(n) \rightarrow \text{SO}(n)$ and to conjugation by a fixed element of $\text{O}(n) \setminus \text{SO}(n)$.

Proof. We can build an $\text{SO}(n+1)$ -invariant spin^n structure on S^n as follows. Con-

sider the diagram

$$\begin{array}{ccc} & & \text{Spin}^n(n) \\ & & \downarrow \varphi^{n,n} \\ \text{SO}(n) & \xrightarrow{\sigma=id} \text{SO}(n) & \xrightarrow{\Delta} \text{SO}(n) \times \text{SO}(n) \end{array} ,$$

where Δ is the diagonal map. By Proposition 5.1, $\Delta \circ \sigma$ lifts to $\text{Spin}^n(n)$, yielding an $\text{SO}(n+1)$ -invariant spin^n structure on S^n .

Suppose $n \neq 1, 2, 4$. Let $2 \leq r < n$, and suppose S^n has an $\text{SO}(n+1)$ -invariant spin^r structure. Then, there exists a lift $\tilde{\sigma}: \text{SO}(n) \rightarrow \text{Spin}^r(n)$, making the following diagram commute:

$$\begin{array}{ccc} & \text{Spin}^r(n) & \\ \tilde{\sigma} \nearrow & & \searrow \varphi^{r,n} \\ \text{SO}(n) & & \text{SO}(n) \times \text{SO}(r) \\ \searrow \sigma=id & & \nwarrow \text{pr}_1 \\ & \text{SO}(n) & \end{array}$$

Now consider the Lie group homomorphism $\text{pr}_2 \circ \varphi^{r,n} \circ \tilde{\sigma}: \text{SO}(n) \rightarrow \text{SO}(r)$. This induces a Lie algebra homomorphism $\mathfrak{so}(n) \rightarrow \mathfrak{so}(r)$. The kernel of this map is an ideal of $\mathfrak{so}(n)$, which is a simple Lie algebra (as $n \geq 3$ and $n \neq 4$), and hence it is either the whole of $\mathfrak{so}(n)$ or 0. As $r < n$, it cannot be 0 for dimensional reasons. Hence, it has to be the whole of $\mathfrak{so}(n)$. As $\text{SO}(n)$ is connected, the Lie exponential map generates the whole group, and hence the Lie group homomorphism has to be trivial. Now, let α_k be a generator of the fundamental group of $\text{SO}(k)$. Then, $\alpha_n \in \pi_1(\text{SO}(n))$ goes, on the one hand, to $(\alpha_n, 0) \in \pi_1(\text{SO}(n) \times \text{SO}(r))$ and, on the other hand, to $(\alpha_n, s\alpha_r)$, for some odd s , by Proposition 5.1, which is a contradiction.

Now let us consider the case $n = 4$. In this case, the Lie algebra $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, so it is not simple, and the previous argument does not work. First, let us show that S^4 does not admit an $\text{SO}(5)$ -invariant spin^2 structure. Indeed, if such a structure existed, as before, we would have a Lie group homomorphism $\text{SO}(4) \rightarrow \text{SO}(2)$, and hence a Lie algebra homomorphism $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \rightarrow \mathfrak{so}(2)$. The kernel of this map is an ideal of $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$, and so it is either 0, $\mathfrak{so}(3)$ or the whole of $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. It cannot be 0 for dimensional reasons. If the kernel is the whole Lie algebra, then

the Lie group homomorphism has to be trivial, as before, and hence concluding that the isotropy cannot lift to $\text{Spin}^2(4)$, by Proposition 5.1. So, the only possibility is that the kernel is $\mathfrak{so}(3)$, which again is impossible for dimensional reasons. Hence, the $\text{SO}(5)$ -invariant spin type of S^4 is either 3 or 4. We shall show now that it is 3.

We know that $\text{SO}(4)/\mathbb{Z}_2 \cong \text{SO}(3) \times \text{SO}(3)$. Hence, we have a chain of Lie group homomorphisms:

$$\text{SO}(4) \xrightarrow{\pi} \text{SO}(4)/\mathbb{Z}_2 \xrightarrow{\cong} \text{SO}(3) \times \text{SO}(3) \xrightarrow{\text{pr}_i} \text{SO}(3).$$

At the level of fundamental groups, a generator of $\pi_1(\text{SO}(4))$ goes to a generator of $\pi_1(\text{SO}(3))$, for a suitable choice of $i \in \{1, 2\}$. Hence, the isotropy representation lifts to $\text{Spin}^3(4)$.

The rest of the theorem follows from standard facts of representation theory together with Proposition 5.1 and Theorem 5.13. $\square$

This is an instance of a more general result:

Theorem 5.17. *Let $n \geq 3$ and let G/H be an n -dimensional oriented Riemannian homogeneous space. Then, there exists a G -invariant spin^n structure on G/H . In particular, its G -invariant spin type satisfies $1 \leq \Sigma(G/H, G) \leq n$.*

Proof. Let $\sigma: H \rightarrow \text{SO}(n)$ be its isotropy representation. The induced homomorphism at the level of fundamental groups is $\sigma_{\#}: \pi_1(H) \rightarrow \pi_1(\text{SO}(n)) \cong \mathbb{Z}_2$. Consider $\sigma \times \sigma: H \rightarrow \text{SO}(n) \times \text{SO}(n)$. Then, $(\sigma \times \sigma)_{\#}(\pi_1(H)) \subseteq \langle (1, 1) \rangle \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \pi_1(\text{SO}(n) \times \text{SO}(n))$, and $(\varphi^{n,n})_{\#}(\pi_1(\text{Spin}^n(n))) = \langle (1, 1) \rangle$, by Proposition 5.1. Hence, the isotropy representation lifts to $\text{Spin}^n(n)$, yielding a G -invariant spin^n structure on G/H . $\square$

Note that Theorem 5.16 shows that the bound in Theorem 5.17 is sharp (compare with Corollary 5.11). Moreover, recall that, if the isotropy H is connected and G/H has a G -invariant spin structure, then such a structure is unique [DSKL22]. This has the advantage of singling out a *preferred* spin structure to work with. We observed that, in general, we cannot expect to have a unique G -invariant spin^r structure. Theorem 5.17 gives a G -invariant spin^n structure which is built in a very natural way using only the isotropy representation. So, in general, an n -dimensional oriented Riemannian homogeneous G -space comes equipped with a *natural* G -invariant spin^n structure.

Let us now continue with the study of homogeneous spheres. We know that the unique spin structure of S^{2n+1} is not $U(n+1)$ -invariant [DSKL22]. So, the $U(n+1)$ -invariant spin type of S^{2n+1} is not 1. However, we have the following:

Theorem 5.18. *The $U(n+1)$ -invariant spin type of S^{2n+1} is 2:*

$$\Sigma(S^{2n+1}, U(n+1)) = 2.$$

Moreover, the $U(n+1)$ -invariant $\text{spin}^{\mathbb{C}}$ structures on S^{2n+1} are the ones corresponding to

$$\begin{aligned} \varphi_s: U(n) &\rightarrow U(1) \\ A &\mapsto \det(A)^s \end{aligned}$$

with s odd.

Proof. The isotropy representation $\sigma: U(n) \rightarrow \text{SO}(2n+1)$ is given by the natural inclusions $U(n) \subseteq \text{SO}(2n) \subseteq \text{SO}(2n+1)$. Consider the map $\sigma \times \det: U(n) \rightarrow \text{SO}(2n+1) \times \text{SO}(2)$, given by $\sigma(A) = (A, \det(A))$. The image of a generator of the fundamental group of $U(n)$ is an element of the form $(\alpha_{2n+1}, \alpha_2)$, where α_s is a generator of the fundamental group of $\text{SO}(s)$. Hence, by Proposition 5.1, this map lifts to $\text{Spin}^2(2n+1)$, yielding a $U(n+1)$ -invariant spin^2 structure on S^{2n+1} .

For the final assertion of the theorem, note that the only Lie group homomorphisms $U(n) \rightarrow \text{SO}(2) \cong U(1)$ are φ_s , for $s \in \mathbb{Z}$. Moreover, by Proposition 5.1, $\sigma \times \varphi_s$ lifts to $\text{Spin}^{\mathbb{C}}(2n+1)$ if and only if s is odd. And, as $U(1)$ is abelian, the $\text{spin}^{\mathbb{C}}$ structures defined by these are pairwise non- $U(n+1)$ -equivariantly equivalent. $\square$

We know that the $\text{Sp}(n+1) \cdot U(1)$ -invariant spin type of S^{4n+3} is 1 for n odd, and that it is not 1 for n even [DSKL22].

Theorem 5.19. *The $\text{Sp}(2n+1) \cdot U(1)$ -invariant spin type of S^{8n+3} is 2. Hence,*

$$\Sigma(S^{4n+3}, \text{Sp}(n+1) \cdot U(1)) = \begin{cases} 1, & n \text{ odd} \\ 2 & n \text{ even} . \end{cases}$$

Moreover, the $\text{Sp}(2n+1) \cdot U(1)$ -invariant $\text{spin}^{\mathbb{C}}$ structures on S^{8n+3} are the ones corresponding to

$$\varphi_s: \text{Sp}(2n+1) \cdot U(1) \rightarrow U(1)$$

$$[A, z] \mapsto z^s$$

with $s \equiv 2 \pmod{4}$.

Proof. Take the description of the isotropy representation $\sigma: \text{Sp}(2n) \cdot \text{U}(1) \rightarrow \text{SO}(8n+3)$ given in [DSKL22]. Then, the Lie group homomorphism $\text{Sp}(2n) \cdot \text{U}(1) \rightarrow \text{U}(1)$ given by $[A, z] \mapsto z^2$ is well-defined, and, by looking at generators of the corresponding fundamental groups and applying Proposition 5.1, one concludes.

For the final assertion of the theorem, note that the only Lie group homomorphisms $\text{Sp}(2n+1) \cdot \text{U}(1) \rightarrow \text{SO}(2) \cong \text{U}(1)$ are φ_s , for $s \in \mathbb{Z}$ even. And $\sigma \times \varphi_s$ lifts to $\text{Spin}^{\mathbb{C}}(8n+3)$ if and only if $s/2$ is odd, by Proposition 5.1. $\square$

We also know that the $\text{Sp}(n+1) \cdot \text{Sp}(1)$ -invariant spin type of S^{4n+3} is 1 for n odd, and that it is not 1 for n even [DSKL22]. Now, we obtain:

Theorem 5.20. *The $\text{Sp}(2n+1) \cdot \text{Sp}(1)$ -invariant spin type of S^{8n+3} is 3. Hence,*

$$\Sigma(S^{4n+3}, \text{Sp}(n+1) \cdot \text{Sp}(1)) = \begin{cases} 1, & n \text{ odd} \\ 3 & n \text{ even} . \end{cases}$$

Moreover, there exists a unique $\text{Sp}(2n+1) \cdot \text{Sp}(1)$ -invariant $\text{spin}^{\mathbb{H}}$ structure on S^{8n+3} up to equivariant equivalence, namely the one corresponding to

$$\begin{aligned} \varphi: \text{Sp}(2n+1) \cdot \text{Sp}(1) &\rightarrow \text{SO}(3) \\ [A, q] &\mapsto \lambda_3(q) \end{aligned}$$

where $\lambda_3: \text{Spin}(3) \cong \text{Sp}(1) \rightarrow \text{SO}(3)$ is the standard double covering.

Proof. Take the description of the isotropy representation $\sigma: \text{Sp}(2n) \cdot \text{Sp}(1) \rightarrow \text{SO}(8n+3)$ in [DSKL22]. Then, the Lie group homomorphism $\text{Sp}(2n) \cdot \text{Sp}(1) \cong \text{Sp}(2n) \cdot \text{Spin}(3) \rightarrow \text{SO}(3)$ given by $[A, z] \mapsto \lambda_3(z)$ is well-defined, and by looking at generators of the corresponding fundamental groups one concludes that the $\text{Sp}(2n+1) \cdot \text{Sp}(1)$ -invariant spin type of S^{8n+3} is either 2 or 3. But it cannot be 2, because then we would have a non-trivial Lie group homomorphism $\text{SO}(3) \rightarrow \text{SO}(2)$, which is impossible, as in the proof of Proposition 5.1.

The last assertion of the theorem follows from the fact that there is only one non-trivial three-dimensional representation of $\text{Sp}(1)$ up to conjugation by elements of $\text{SO}(3)$, together with Proposition 5.1 and Theorem 5.13. $\square$

We summarise our results in Table 5.1 (see also Table 1.2), which shows the G -invariant spin type of the spheres for all connected Lie groups G acting transitively and effectively on them, and thus completing the study of invariant generalised spin^r structures on homogeneous spheres.

Space M	Group G	$\Sigma(M, G)$
S^n	$\text{SO}(n+1)$	n , for $n \neq 4$ 3 , for $n = 4$
S^{2n+1}	$\text{U}(n+1)$	2
S^{2n+1}	$\text{SU}(n+1)$	1
S^{4n+3}	$\text{Sp}(n+1)$	1
S^{4n+3}	$\text{Sp}(n+1) \cdot \text{U}(1)$	1 , for n odd 2 , for n even
S^{4n+3}	$\text{Sp}(n+1) \cdot \text{Sp}(1)$	1 , for n odd 3 , for n even
S^6	G_2	1
S^7	$\text{Spin}(7)$	1
S^{15}	$\text{Spin}(9)$	1

Table 5.1: G -invariant spin type of homogeneous spheres.

Table 5.1 has an interesting application, in analogy with [DSKL22, Prop. 1.6]:

Theorem 5.21. *Let G be the holonomy group of a simply connected irreducible non-symmetric Riemannian manifold of dimension $n+1 \geq 4$. Then, G acts transitively on the sphere S^n , by Berger’s classification [Ber55] – see also [Sim62]. Then, the following are equivalent:*

1. *There exists a homomorphic lift of the holonomy group G to $\text{Spin}^r(n+1)$.*
2. *S^n has a G -invariant spin^r structure with strongly G -trivial auxiliary bundle.*

Proof. ¹ Let $h: G \rightarrow \text{SO}(n+1)$ be the inclusion of the holonomy group, and let $\sigma: H \rightarrow \text{SO}(n)$ be the isotropy representation of the corresponding sphere $S^n = G/H$. We have a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\iota} & G \\ \downarrow \sigma & & \downarrow h \\ \text{SO}(n) & \longrightarrow & \text{SO}(n+1). \end{array}$$

First, suppose that there exists a lift $\tilde{h}: G \rightarrow \text{Spin}^r(n+1)$ of h . Define $\psi: G \rightarrow \text{SO}(r)$ by $\psi = \text{pr}_2 \circ \varphi^{r,n} \circ \tilde{h}$, and consider $\sigma \times (\psi \circ \iota): H \rightarrow \text{SO}(n) \times \text{SO}(r)$. Then, we get

¹The original idea of the theorem is due to M.-A. Lawn. I came up with the notion of strong G -triviality, and we wrote the proof in collaboration.

the long exact sequences of homotopy groups

$$\begin{array}{ccccccc} \pi_2(G/H) = 0 & \longrightarrow & \pi_1(H) & \xrightarrow{\iota_\#} & \pi_1(G) & \longrightarrow & \pi_1(G/H) = 0 \\ & & \downarrow \sigma_\# \times (\psi \circ \iota)_\# & & \downarrow h_\# \times \psi_\# & & \\ \pi_2(S^n) = 0 & \longrightarrow & \pi_1(\text{SO}(n) \times \text{SO}(r)) & \longrightarrow & \pi_1(\text{SO}(n+1) \times \text{SO}(r)) & \longrightarrow & \pi_1(S^n) = 0. \end{array}$$

This implies we have a commuting square where the horizontal maps are isomorphisms:

$$\begin{array}{ccc} \pi_1(H) & \xrightarrow{\cong} & \pi_1(G) \\ \downarrow \sigma_\# \times (\psi \circ \iota)_\# & & \downarrow h_\# \times \psi_\# \\ \pi_1(\text{SO}(n) \times \text{SO}(r)) & \xrightarrow{\cong} & \pi_1(\text{SO}(n+1) \times \text{SO}(r)). \end{array} \quad (5.22)$$

To conclude, we need to show that the image of $\pi_1(\text{Spin}^r(n))$ in $\pi_1(\text{SO}(n) \times \text{SO}(r))$ maps to the image of $\pi_1(\text{Spin}^r(n+1))$ in $\pi_1(\text{SO}(n+1) \times \text{SO}(r))$ by the isomorphism of the bottom row. This follows by applying direct image to the commuting square:

$$\begin{array}{ccc} \text{Spin}^r(n) & \longrightarrow & \text{Spin}^r(n+1) \\ \downarrow & & \downarrow \\ \text{SO}(n) \times \text{SO}(r) & \longrightarrow & \text{SO}(n+1) \times \text{SO}(r). \end{array}$$

This guarantees the existence of a lift of $\sigma \times (\psi \circ \iota)$ to $\text{Spin}^r(n)$. By part 3 of Lemma 2.15, the auxiliary bundle of the corresponding spin^r structure is strongly G -trivial.

Conversely, suppose that the sphere G/H has a G -invariant spin^r structure. This means that there exists a homomorphism $\varphi: H \rightarrow \text{SO}(r)$ such that $\sigma \times \varphi$ lifts to $\text{Spin}^r(n)$. If, moreover, the auxiliary bundle is strongly G -trivial, there exists a Lie group homomorphism ψ extending φ to G . Now, the same reasoning as above using diagram (5.22) concludes the proof. $\square$

Remark 5.23. *If we only require the auxiliary bundle to be trivial in point 2 of Theorem 5.21 and not necessarily strongly G -trivial, the holonomy representation still lifts for topological reasons, but the lift need not be a homomorphism.*

This result places the study of invariant spin^r structures on spheres in a much wider context. For example, as a consequence of Theorem 5.21 and Table 5.1, the holonomy representation of a simply connected irreducible non-symmetric $(8k+4)$ -dimensional quaternionic Kähler manifold (or more generally any $(8k+4)$ -dimensional manifold with holonomy containing $\text{Sp}(2k+1) \cdot \text{Sp}(1)$) does not lift to $\text{Spin}^\mathbb{C}(8k+4)$.

5.4 Special invariant spin^r spinors

A classical spin structure allows one to build a spinor bundle. Similarly, a spin^r structure naturally induces a family of complex vector bundles as follows:

Definition 5.24. *Let M be an n -dimensional oriented Riemannian manifold with a spin^r structure (P, Φ) . For $m \in \mathbb{N}$ odd, its m -twisted spin^r spinor bundle is defined as*

$$\Sigma_{n,r}^m M := P \times_{\Delta_{n,r}^m} \Sigma_{n,r}^m,$$

with the natural projection to M induced by that of P , where

$$\Delta_{n,r}^m := \Delta_n \otimes_{\mathbb{C}} \Delta_r^{\otimes m}, \quad \Sigma_{n,r}^m := \Sigma_n \otimes_{\mathbb{C}} \Sigma_r^{\otimes m},$$

where $\Delta_{n,r}^m$ is viewed as a representation of $\text{Spin}^r(n)$.

The requirement that m be odd in Definition 5.24 comes from the fact that $\Delta_{n,r}^m$, which is a representation of $\text{Spin}(n) \times \text{Spin}(r)$, descends to a representation of $\text{Spin}^r(n)$ if and only if m is odd.

If, moreover, $M = G/H$ is a homogeneous space and (P, Φ) is G -invariant, then there exists a homomorphism $\varphi: H \rightarrow \text{SO}(r)$ such that $\sigma \times \varphi$ lifts to a homomorphism $\phi: H \rightarrow \text{Spin}^r(n)$ (see Theorem 5.13), and this bundle takes the form

$$\Sigma_{n,r}^m M = G \times_{\Delta_{n,r}^m \circ \phi} \Sigma_{n,r}^m.$$

Sections of $\Sigma_{n,r}^m M$ are called *m -twisted spin^r spinors* – if there is no risk of confusion, we will just refer to them as *spin^r spinors* or simply *spinors*. As in Chapter 2, they are identified with H -equivariant maps $\psi: G \rightarrow \Sigma_{n,r}^m$, and G acts on the space of spinors by

$$(g \cdot \psi)(g') := \psi(g^{-1}g').$$

G -invariant spinors correspond, then, to *constant* H -equivariant maps $G \rightarrow \Sigma_{n,r}^m$, which in turn correspond to elements of $\Sigma_{n,r}^m$ which are stabilised by H . If H is connected, these are just elements of $\Sigma_{n,r}^m$ which are annihilated by the differential action of the Lie algebra of H . We denote the space of invariant m -twisted spin^r spinors by $(\Sigma_{n,r}^m)_{\text{inv}}$.

In a similar spirit to Definition 5.4, we make the following definition:

Definition 5.25. For an oriented Riemannian homogeneous space $M = G/H$, the G -invariant spinor type of M is defined by

$$\sigma(M, G) := \min \left\{ r \in \mathbb{N} \mid \begin{array}{l} M \text{ admits a } G\text{-invariant } \text{spin}^r \text{ structure} \\ \text{with } (\Sigma_{n,r}^m)_{\text{inv}} \neq 0 \text{ for some odd } m \end{array} \right\}.$$

Remark 5.26. The G -invariant spinor type $\sigma(M^n, G)$ is well-defined, and it satisfies

$$1 \leq \sigma(M, G) \leq n.$$

This is because the G -invariant spin^n structure on M obtained by taking $\varphi: H \rightarrow \text{SO}(n)$ to be equal to the isotropy representation always carries a non-zero invariant 1-twisted spin^n spinor – see [EH16, Prop. 3.3].

In the classical spin setting, it is well-known that spinors satisfying certain additional properties (e.g. parallel, Killing, or generalised Killing) carry geometric information about the manifold, as explained in Chapter 1. Another class of important special spinors is that of *pure spinors*, which are defined by the algebraic condition that their annihilator inside the complexified tangent bundle (with respect to Clifford multiplication) is a maximal isotropic subbundle. Such spinors correspond, uniquely up to scaling, with orthogonal almost complex structures on the manifold – see [LM89, Ch. IV, §9] for details.

As in the classical spin case, special spin^r spinors also encode geometric properties. In analogy to pure spinors, we define:

Definition 5.27. ([HS19, Def. 3.5]) Let $\psi \in \Sigma_{n,r}^m$, $X, Y \in \mathbb{R}^n$ and $1 \leq k < l \leq r$, and let $(\widehat{e}_1, \dots, \widehat{e}_r)$ be the standard basis of $\mathbb{R}^r$. The real two-form η_{kl}^ψ and the skew-symmetric endomorphism $\widehat{\eta}_{kl}^\psi$ associated to ψ are defined by

$$\eta_{kl}^\psi(X, Y) := \Re \langle (X \wedge Y) \cdot (\widehat{e}_k \cdot \widehat{e}_l) \cdot \psi, \psi \rangle, \quad \widehat{\eta}_{kl}^\psi(X) := \left(\eta_{kl}^\psi(X, \cdot) \right)^\sharp, \quad (5.28)$$

where $\Re$ denotes real part, $X \wedge Y = X \cdot Y + \langle X, Y \rangle \in \mathfrak{spin}(n)$, and $\widehat{e}_k \cdot \widehat{e}_l \in \mathfrak{spin}(r)$.

We say that ψ is pure if

$$\left(\widehat{\eta}_{kl}^\psi \right)^2 = -\text{Id}_{\mathbb{R}^n} \quad \text{and} \quad \left(\eta_{kl}^\psi + 2\widehat{e}_k \cdot \widehat{e}_l \right) \cdot \psi = 0 \quad (\text{only for } r \geq 3),$$

for all $1 \leq k < l \leq r$, where η_{kl}^ψ is seen as an element of $\mathfrak{spin}(n)$. An m -twisted spin^r spinor on a manifold is pure if it is pure at every point.

It is clear that an invariant spin^r spinor on a homogeneous space is pure if and only if it is pure at one point.

We are also interested in various differential equations that a spin^r spinor might satisfy. Recall that the Levi-Civita connection on a spin manifold naturally induces a connection on the spinor bundle. Similarly, the Levi-Civita connection of a spin^r manifold together with a connection θ on the auxiliary bundle defines a connection ∇^θ on each twisted spin^r spinor bundle. There are obvious analogues of the usual special spinorial field equations (including the classical Killing spinor equation mentioned above) to the spin^r setting:

Definition 5.29. *Let ψ be a twisted spin^r spinor on M and θ a connection on the auxiliary bundle of the spin^r structure.*

1. ψ is θ -parallel if $\nabla^\theta \psi = 0$;
2. ψ is θ -Killing if for all vector fields X one has that $\nabla_X^\theta \psi = \lambda X \cdot \psi$, for some constant $\lambda \in \mathbb{R}$;
3. ψ is θ -generalised Killing if there exists a symmetric endomorphism field $A \in \text{End}(TM)$ such that, for all vector fields X , one has $\nabla_X^\theta \psi = A(X) \cdot \psi$.

We collect here a number of results that relate the existence of special spin^r spinors to geometric properties of the manifold:

Theorem 5.30 ([EH16, HS19]). *Let M be an n -dimensional spin^r manifold, and let θ be a connection on its auxiliary bundle.*

1. *If M carries a θ -parallel spinor ψ , then the Ricci tensor decomposes as*

$$\text{Ric} = \frac{1}{|\psi|^2} \sum_{k < l} \hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\psi,$$

where $\hat{\Theta}_{kl}$ is the skew-symmetric endomorphism associated to the two-form on TM given by

$$\Theta_{kl}(X, Y) := \langle \Omega(X, Y) (\hat{e}_k), \hat{e}_l \rangle,$$

where Ω is the curvature two-form of the connection θ on the auxiliary bundle.

2. *If θ is flat and M carries a θ -Killing spinor, then M is Einstein.*

3. If M carries a θ -parallel m -twisted pure spinor for some $m \in \mathbb{N}$, $r \geq 3$, $r \neq 4$, $n \neq 8$, $n + 4r - 16 \neq 0$ and $n + 8r - 16 \neq 0$, then M is Einstein.
4. If $r = 2$ and M carries a θ -parallel pure spinor, then M is Kähler.
5. If $r = 3$ and M admits a θ -parallel pure spinor, then M is quaternionic Kähler. $\square$

As explained in Chapter 2, if $M = G/H$ is a Riemannian homogeneous space, an invariant connection on a homogeneous bundle over M is determined by its Nomizu map. In the case of an invariant connection on the m -twisted spin^r spinor bundle of M , this correspondence is given by the following result, whose proof is immediate.

Proposition 5.31. *Let $(G/H, g)$ be an n -dimensional Riemannian homogeneous space equipped with a G -invariant spin^r structure P . Let $\Lambda: \mathfrak{g} \rightarrow \mathfrak{so}(n)$ be the Nomizu map of the Levi-Civita connection of g , and let $\Lambda': \mathfrak{g} \rightarrow \mathfrak{so}(r)$ be the Nomizu map of an invariant connection θ on the associated bundle $\widehat{P}$. Let $\widetilde{\Lambda}$ be the spin lift of Λ to $\mathfrak{spin}(n)$ and let $\widetilde{\Lambda}'$ be the spin lift of Λ' to $\mathfrak{spin}(r)$. Then, $\widetilde{\Lambda} \otimes (\widetilde{\Lambda}')^{\otimes m}$ is the Nomizu map of the invariant connection ∇^θ on the m -twisted spin^r spinor bundle. Moreover, if $\psi \in (\Sigma_{n,r}^m)_{\text{inv}}$ and $\widehat{X}$ is the fundamental vector field on G/H defined by $X \in \mathfrak{m}$, then*

$$(\nabla_{\widehat{X}}^\theta \psi)_{eH} = \left(\widetilde{\Lambda} \otimes (\widetilde{\Lambda}')^{\otimes m} \right) (X) \cdot \psi.$$

In particular, an invariant m -twisted spin^r spinor ψ is θ -parallel if and only if it satisfies the equation

$$\left(\widetilde{\Lambda} \otimes (\widetilde{\Lambda}')^{\otimes m} \right) (X) \cdot \psi = 0,$$

for all $X \in \mathfrak{m}$. $\square$

Finally, we examine the differential equations satisfied by invariant spin^r spinors on symmetric spaces. The following proposition is analogous to the familiar fact in the spin setting that invariant spinors on symmetric spaces are ∇^g -parallel, since the Levi-Civita and the Ambrose-Singer connections coincide:

Proposition 5.32. *Let $(M = G/H, g)$ be a Riemannian symmetric space admitting a G -invariant spin^r structure P . Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition such that $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$. Let E be the natural vector bundle associated to the auxiliary bundle $\widehat{P}$, and let ∇^E be the unique G -invariant connection on E whose Nomizu map vanishes identically on $\mathfrak{m}$. Let $\nabla := \nabla^g \otimes (\nabla^E)^{\otimes m}$ be the corresponding twisted connection on $\Sigma_{n,r}^m M$.*

If $\psi \in \Sigma_{n,r}^m$ is a G -invariant spin^r spinor, then $\nabla \psi = 0$.

Proof. With respect to the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the Nomizu map of the Levi-Civita connection vanishes identically on the reductive complement $\mathfrak{m}$, i.e., $\Lambda^g|_{\mathfrak{m}} \equiv 0$. The Nomizu map of ∇ then vanishes identically on $\mathfrak{m}$, and the result follows. $\square$

This result will be useful in Chapter 6, where the limited number of low-dimensional representations of the isotropy groups will force the auxiliary bundles to be isomorphic to familiar tensor bundles.

6

Invariant spin^r spinors on projective spaces

Oniščik [Oni66, p. 163] classified the compact, simple, and simply connected Lie groups which act transitively on complex and quaternionic projective spaces – see also [Zil82, p. 356]. We exhibit them in Table 6.1.

Space	G	H
$\mathbb{CP}^n$	$\text{SU}(n+1)$	$\text{S}(\text{U}(1) \times \text{U}(n))$
$\mathbb{CP}^{2n+1}$	$\text{Sp}(n+1)$	$\text{U}(1) \times \text{Sp}(n)$
$\mathbb{HP}^n$	$\text{Sp}(n+1)$	$\text{Sp}(1) \times \text{Sp}(n)$

Table 6.1: *Compact, simple and simply connected Lie groups G acting transitively with isotropy H on complex and quaternionic projective spaces – see e.g. [Zil82, p. 356].*

The aim of this chapter is to find, for each homogeneous realisation of complex and quaternionic projective space, the minimum values of $r, m \in \mathbb{N}$ for which there exists an invariant spin^r structure carrying non-trivial invariant m -twisted spin^r spinors. We find explicit expressions for these, and deduce geometric properties of the corresponding spaces from the properties of the spinors.

In this thesis, we shall only consider complex and quaternionic projective spaces. The Cayley plane $\mathbb{OP}^2 \cong F_4/\text{Spin}(9)$ is treated in [AH25], and was dealt with mainly by Hofmann.

Throughout the computations carried out in this chapter, we will repeatedly use some matrix notation and identities, taken from [AHL23, p. 9]. We will denote by

$E_{i,j}^{(n)}$ (resp. $F_{i,j}^{(n)}$) the elementary $n \times n$ skew-symmetric (resp. symmetric) matrix given by

$$E_{i,j}^{(n)} = \begin{matrix} & i & j \\ i & & \\ j & \begin{pmatrix} & \vdots & \\ & -1 & \dots \\ \dots & 1 & \\ & \vdots & \end{pmatrix} \end{matrix}, \quad F_{i,j}^{(n)} = \begin{matrix} & i & j \\ i & & \\ j & \begin{pmatrix} & \vdots & \\ & 1 & \dots \\ \dots & 1 & \\ & \vdots & \end{pmatrix} \end{matrix}.$$

By convention, the matrix $F_{i,i}^{(n)}$ has all the entries equal to zero except for the (i, i) entry, which is 1. We will denote by B_0 the bilinear form on the space of matrices of appropriate size given by

$$B_0(X, Y) := -\Re(\text{tr}(XY)),$$

where $\Re(\cdot)$ denotes real part and $\text{tr}(A)$ is the trace of the matrix A . Finally, if $\{e_i\}_i$ is an orthonormal basis for some vector space V with respect to an inner product B , we shall denote by $e_{i,j} := e_i \wedge e_j$ the standard basis elements for $\mathfrak{so}(V, B) \cong \Lambda^2 V$, sending $e_i \mapsto e_j$ and $e_j \mapsto -e_i$.

6.1 Hermitian complex projective space

We begin by viewing complex projective space $\mathbb{CP}^n$ as a homogeneous space of the special unitary group:

$$\mathbb{CP}^n \cong \text{SU}(n+1)/H,$$

where $H \subseteq \text{SU}(n+1)$ is the subgroup preserving a fixed complex line in $\mathbb{C}^{n+1}$. More concretely, H consists of block diagonal matrices of the form

$$\left(\begin{array}{c|c} z & 0 \\ \hline 0 & B \end{array} \right) \quad \text{with } z \in \text{U}(1), B \in \text{U}(n), \text{ and } z \det B = 1.$$

This subgroup is sometimes denoted $\text{S}(\text{U}(1) \times \text{U}(n))$, and it fixes the line $\mathbb{C}e_1$.

To study $\text{SU}(n+1)$ -invariant spin^r structures and spinors, we first need to understand the corresponding Lie algebras. Let $\mathfrak{su}(n+1)$ be the Lie algebra of $\text{SU}(n+1)$, and

denote by $\mathfrak{h}$ the Lie algebra of H . Then $\mathfrak{h}$ splits as

$$\mathfrak{h} = \mathbb{R}\xi \oplus \mathfrak{su}(n),$$

where $\mathfrak{su}(n)$ is embedded in the lower-right $n \times n$ block, and ξ is the diagonal matrix

$$\xi = i \left(-nF_{1,1}^{(n+1)} + \sum_{l=2}^{n+1} F_{l,l}^{(n+1)} \right).$$

The isotropy subalgebra $\mathfrak{h} \subseteq \mathfrak{su}(n+1)$ has a reductive complement

$$\mathfrak{m} := (\mathfrak{h})^{\perp_{B_0}} = \text{span}_{\mathbb{R}} \left\{ iF_{1,p}^{(n+1)}, E_{1,p}^{(n+1)} \right\}_{p=2,\dots,n+1},$$

and the adjoint representation of H on $\mathfrak{m}$ is irreducible.

It follows from Theorem 2.10 that the space of $\text{SU}(n+1)$ -invariant metrics on $\mathbb{CP}^n$ is one-dimensional: each such metric corresponds to a scalar multiple of the restriction of B_0 to $\mathfrak{m}$. We write this family as

$$g_a = aB_0|_{\mathfrak{m} \times \mathfrak{m}}, \quad a > 0.$$

A g_a -orthonormal basis for $\mathfrak{m}$ is then

$$\left\{ e_{2p-1} := \frac{i}{\sqrt{2a}} F_{1,p+1}^{(n+1)}, e_{2p} := \frac{1}{\sqrt{2a}} E_{1,p+1}^{(n+1)} \right\}_{p=1,\dots,n}, \quad (6.1)$$

and we choose the orientation determined by the ordered basis $(e_1, e_2, \dots, e_{2n})$.

Invariant spin^r structures

We now determine the $\text{SU}(n+1)$ -invariant spin type of complex projective space $\mathbb{CP}^n$. It follows from [HS90, p. 327] that $\mathbb{CP}^n$ admits an $\text{SU}(n+1)$ -invariant spin structure if and only if it admits a spin structure at all, which happens precisely when n is odd. One can describe all invariant $\text{spin}^{\mathbb{C}}$ structures as follows:

Theorem 6.2. *The $\text{SU}(n+1)$ -invariant $\text{spin}^{\mathbb{C}}$ structures on $\mathbb{CP}^n$ are given by*

$$\text{SU}(n+1) \times_{\phi_s} \text{Spin}^{\mathbb{C}}(2n), \quad s \in \mathbb{Z} \text{ such that } n \not\equiv s \pmod{2},$$

where ϕ_s is the unique lift of $\sigma \times \varphi_s$ to $\text{Spin}^{\mathbb{C}}(2n)$. Here, $\sigma: H \rightarrow \text{SO}(2n)$ is the isotropy representation, and $\varphi_s: H \rightarrow \text{SO}(2) \cong \text{U}(1)$ is the homomorphism

$$\left(\begin{array}{c|c} z & 0 \\ \hline 0 & B \end{array} \right) \mapsto \det(B)^s.$$

In particular, the $\text{SU}(n+1)$ -invariant spin type of $\mathbb{CP}^n$ is

$$\Sigma(\mathbb{CP}^n, \text{SU}(n+1)) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. First, observe that $H \cong \text{U}(n)$ via $\text{diag}(z, B) \mapsto B$, and every homomorphism $\text{U}(n) \rightarrow \text{U}(1)$ is of the form $B \mapsto \det(B)^s$ for some $s \in \mathbb{Z}$. The fundamental group $\pi_1(H) \cong \mathbb{Z}$ is generated by the loop

$$\alpha(t) = \text{diag}(e^{-2\pi it}, 1, \dots, 1, e^{2\pi it}).$$

We compute the image of α under the map $\sigma \times \varphi_s$. The representation σ is just the adjoint action of H on $\mathfrak{m}$, and using the orthonormal basis (6.1) one can easily see that the image of $\alpha(t)$ under σ is

$$\sigma(\alpha(t)) = \text{diag}(e^{2\pi it}, \dots, e^{2\pi it}, e^{4\pi it}) \in \text{U}(n) \subseteq \text{SO}(2n),$$

where $e^{2\pi it}$ appears $n-1$ times.

Thus, the induced map on fundamental groups satisfies

$$(\sigma \times \varphi_s)_\#([\alpha]) = (n-1, s) \in \pi_1(\text{SO}(2n)) \times \pi_1(\text{SO}(2)) \cong \mathbb{Z}_2 \times \mathbb{Z}.$$

By Proposition 5.1, this lifts to $\text{Spin}^{\mathbb{C}}(2n)$ if and only if $n \not\equiv s \pmod{2}$. This gives the stated condition on s .

Finally, since $\text{U}(1)$ is abelian, the homomorphisms φ_s are pairwise non-conjugate, and we conclude using Theorem 5.13. $\square$

Invariant spin^r spinors

We now investigate the existence of $\text{SU}(n+1)$ -invariant spin^r spinors on $\mathbb{CP}^n$. In the classical case $r=1$, there are no non-trivial invariant spinors, as the following result shows:

Theorem 6.3. *For n odd, there are no non-trivial $\mathrm{SU}(n+1)$ -invariant spinors on $\mathbb{CP}^n$.*

Proof. We need the explicit expression of the action of $\xi \in \mathfrak{h}$ on $\mathfrak{m}$. Letting $\sigma: H \rightarrow \mathrm{SO}(2n)$ be the isotropy representation and $\tilde{\sigma}$ its lift to $\mathrm{Spin}(2n)$, and e.g. using the commutation relations in [AHL23, p. 9], one can readily see that, for each $p = 1, \dots, n$,

$$\mathrm{ad}(\xi)|_{\mathfrak{m}}(e_{2p}) = [\xi, e_{2p}]_{\mathfrak{m}} = (n+1)e_{2p-1}, \quad \mathrm{ad}(\xi)|_{\mathfrak{m}}(e_{2p-1}) = [\xi, e_{2p-1}]_{\mathfrak{m}} = -(n+1)e_{2p}.$$

Hence,

$$\sigma_*(\xi) = \mathrm{ad}(\xi)|_{\mathfrak{m}} = (n+1) \sum_{p=1}^n e_{2p} \wedge e_{2p-1} \in \mathfrak{so}(2n),$$

and the spin lift is given by

$$\tilde{\sigma}_*(\xi) = \frac{n+1}{2} \sum_{p=1}^n e_{2p} \cdot e_{2p-1} \in \mathfrak{spin}(2n) \subseteq \mathrm{Cl}(2n).$$

A direct computation using (2.29) shows that, for each $0 \leq k \leq n$ and $1 \leq j_1 < \dots < j_k \leq n$,

$$\tilde{\sigma}_*(\xi) \cdot (y_{j_1} \wedge \dots \wedge y_{j_k}) = \frac{i(n+1)}{2} (2k-n) y_{j_1} \wedge \dots \wedge y_{j_k}.$$

Since $(2k-n) \neq 0$ for all k when n is odd, no non-zero invariant spinors exist in this case. $\square$

This result motivates the investigation of $\mathrm{spin}^{\mathbb{C}}$ spinors.

Theorem 6.4. *Let $n, s \in \mathbb{N}$ with $n \not\equiv s \pmod{2}$. Then the space of $\mathrm{SU}(n+1)$ -invariant 1-twisted $\mathrm{spin}^{\mathbb{C}}$ spinors on $\mathbb{CP}^n$ associated to the $\mathrm{spin}^{\mathbb{C}}$ structure*

$$\mathrm{SU}(n+1) \times_{\phi_s} \mathrm{Spin}^{\mathbb{C}}(2n)$$

is given by

$$(\Sigma_{2n,2}^1)_{\mathrm{inv}} = \begin{cases} \mathrm{span}_{\mathbb{C}} \left\{ 1 \otimes \widehat{1}, (y_1 \wedge \dots \wedge y_n) \otimes \widehat{y}_1 \right\} & s = n+1, \\ \mathrm{span}_{\mathbb{C}} \left\{ (y_1 \wedge \dots \wedge y_n) \otimes \widehat{1}, 1 \otimes \widehat{y}_1 \right\} & s = -(n+1), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the $\text{SU}(n+1)$ -invariant spinor type of $\mathbb{CP}^n$ is

$$\sigma(\mathbb{CP}^n, \text{SU}(n+1)) = 2.$$

Proof. Recall that $\mathfrak{h} = \mathbb{R}\xi \oplus \mathfrak{su}(n)$ as Lie algebras. Now, observe that

$$(\phi_s)_* = (\tilde{\sigma})_* \otimes \text{Id} + \text{Id} \otimes (\varphi_s)_*,$$

and the definition of φ_s forces $(\varphi_s)_*(\mathfrak{su}(n))$ to annihilate the whole of Σ_2 . Hence, a spinor $\psi \in \Sigma_{2n,2}^1 = \Sigma_{2n} \otimes \Sigma_2$ satisfies

$$(\phi_s)_*(X) \cdot \psi = 0$$

for all $X \in \mathfrak{su}(n)$ if and only if

$$\psi \in V \otimes \Sigma_2,$$

where V is the subspace of Σ_{2n} annihilated by $\mathfrak{su}(n)$. By [AHL23, Thm. 3.7],

$$V = \text{span}_{\mathbb{C}}\{1, y_1 \wedge \cdots \wedge y_n\}.$$

Moreover,

$$(\phi_s)_*(\xi) = \left(\frac{n+1}{2} \sum_{p=1}^n e_{2p} \cdot e_{2p-1}, \frac{sn}{2} \widehat{e}_1 \cdot \widehat{e}_2 \right) \in \mathfrak{spin}(2n) \oplus \mathfrak{spin}(2) \cong \mathfrak{spin}^{\mathbb{C}}(2n).$$

Finally, an easy calculation shows that, for $0 \leq k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$,

$$\begin{aligned} (\phi_s)_*(\xi) \cdot \left((y_{j_1} \wedge \cdots \wedge y_{j_k}) \otimes \widehat{1} \right) &= \frac{i}{2} ((n+1)(2k-n) + sn) \left((y_{j_1} \wedge \cdots \wedge y_{j_k}) \otimes \widehat{1} \right), \\ (\phi_s)_*(\xi) \cdot \left((y_{j_1} \wedge \cdots \wedge y_{j_k}) \otimes \widehat{y}_1 \right) &= \frac{i}{2} ((n+1)(2k-n) - sn) \left((y_{j_1} \wedge \cdots \wedge y_{j_k}) \otimes \widehat{y}_1 \right). \end{aligned}$$

The only possible values of k for which these eigenvalues vanish for some integer value of s are $k = 0$ and $k = n$, which incidentally are the only ones that are of interest by the previous reasoning. Plugging these into the expressions yields the conditions $s = \pm(n+1)$, giving the stated result. $\square$

Special spin^r spinors

The aim now is to show that the $\text{SU}(n+1)$ -invariant $\text{spin}^{\mathbb{C}}$ spinors on $\mathbb{CP}^n$ found in Theorem 6.4 are pure and parallel with respect to a suitable connection on the

auxiliary bundle.

Proposition 6.5. *The four non-zero $\mathrm{SU}(n+1)$ -invariant $\mathrm{spin}^{\mathbb{C}}$ spinors on $\mathbb{CP}^n$ found in Theorem 6.4 are pure. Moreover, they are parallel with respect to the invariant connection on the corresponding auxiliary bundle determined by the Nomizu map $\Lambda|_{\mathfrak{m}} = 0$.*

Proof. We will only show the calculations for the spinor $\psi = (y_1 \wedge \cdots \wedge y_n) \otimes \widehat{1}$, as the other three are analogous. Since $r = 2 < 3$, we only need to show that $(\widehat{\eta}_{12}^{\psi})^2 = -\mathrm{Id}$. Indeed, using the definitions (5.28), we obtain

$$\begin{aligned} \eta_{12}^{\psi}(e_{2p}, e_{2q}) &= \Re \langle e_{2p} \cdot e_{2q} \cdot \widehat{e}_1 \cdot \widehat{e}_2 \cdot \psi, \psi \rangle + \delta_{p,q} \Re \langle \widehat{e}_1 \cdot \widehat{e}_2 \cdot \psi, \psi \rangle \\ &= \Re \langle ie_{2p} \cdot e_{2q} \cdot \psi, \psi \rangle + \delta_{p,q} \Re \langle i\psi, \psi \rangle = 0, \\ \eta_{12}^{\psi}(e_{2p-1}, e_{2q-1}) &= 0, \\ \eta_{12}^{\psi}(e_{2p}, e_{2q-1}) &= \Re \langle e_{2p} \cdot e_{2q-1} \cdot \widehat{e}_1 \cdot \widehat{e}_2 \cdot \psi, \psi \rangle = \Re \langle ie_{2p} \cdot e_{2q-1} \cdot \psi, \psi \rangle = -\delta_{p,q}. \end{aligned}$$

Hence, for each $1 \leq p \leq n$,

$$\widehat{\eta}_{12}^{\psi}(e_{2p}) = -e_{2p-1} \quad \text{and} \quad \widehat{\eta}_{12}^{\psi}(e_{2p-1}) = e_{2p}.$$

The last assertion of the proposition follows by noting that, as $\mathbb{CP}^n \cong \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$ is a symmetric space, the Levi-Civita connection coincides with the Ambrose-Singer connection, whose Nomizu map satisfies $\Lambda^g|_{\mathfrak{m}} \equiv 0$. $\square$

In light of the Ricci decomposition in [EH16, Thm. 3.1], the existence of parallel pure $\mathrm{spin}^{\mathbb{C}}$ spinors encodes a very well-known fact – see e.g. [Zil82]:

Theorem 6.6. *The $\mathrm{SU}(n+1)$ -invariant metrics g_a on $\mathbb{CP}^n$ are Kähler-Einstein.*

Proof. Take $s = -(n+1)$. Consider the $\mathrm{spin}^{\mathbb{C}}$ structure defined by φ_s , and endow its auxiliary bundle with the connection described in Proposition 6.5. We have seen that this $\mathrm{spin}^{\mathbb{C}}$ structure carries a non-zero parallel pure $\mathrm{spin}^{\mathbb{C}}$ spinor $\psi = (y_1 \wedge \cdots \wedge y_n) \otimes \widehat{1}$. This implies that the metric is Kähler [HS19, Cor. 4.10] with respect to the invariant complex structure defined by $\widehat{\eta}_{12}^{\psi}$. Now, by Theorem 5.30(1), the Ricci tensor decomposes as

$$\mathrm{Ric} = \frac{1}{|\psi|^2} \widehat{\Theta}_{12} \circ \widehat{\eta}_{12}^{\psi},$$

where $\widehat{\Theta}_{12}$ is the endomorphism associated to the two-form on $\mathfrak{m}$

$$\Theta_{12}(X, Y) := \langle \Omega(X, Y) (\widehat{e}_1), \widehat{e}_2 \rangle, \quad X, Y \in \mathfrak{m},$$

where Ω is the curvature two-form of the connection on the auxiliary bundle. Recall [ANT23] that, if Λ is the Nomizu map of the connection on the auxiliary bundle, then

$$\Omega(X, Y) = [\Lambda(X), \Lambda(Y)]_{\mathfrak{so}(2)} - \Lambda([X, Y]) = -\Lambda([X, Y]),$$

for all $X, Y \in \mathfrak{m}$. As $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$, and by the property (2.22) of Nomizu maps,

$$\Omega(X, Y) = -(\varphi_s)_*([X, Y]).$$

It is now easy to see that, for all $1 \leq p, q \leq n$,

$$\Omega(e_{2p-1}, e_{2q}) = \delta_{p,q} \frac{s}{a} \widehat{e}_1 \wedge \widehat{e}_2, \quad \Omega(e_{2p-1}, e_{2q-1}) = \Omega(e_{2p}, e_{2q}) = 0.$$

Hence, as $s = -(n+1)$,

$$\widehat{\Theta}_{12} = -\frac{n+1}{a} \sum_{p=1}^n e_{2p-1} \wedge e_{2p},$$

and finally, using the expression of $\widehat{\eta}_{12}^\psi$ obtained in the proof of Proposition 6.5, we obtain

$$\text{Ric} = \frac{1}{|\psi|^2} \widehat{\Theta}_{12} \circ \widehat{\eta}_{12}^\psi = \frac{n+1}{a} \text{Id}. \quad (6.7)$$

□

Remark 6.8. Recall that the Fubini-Study metric g_{FS} on $\mathbb{CP}^n$ is $\text{SU}(n+1)$ -invariant, and that its Ricci constant is $2(n+1)$. From equation (6.7), we can deduce that $g_{FS} = g_{1/2}$.

6.2 Symplectic complex projective space

We now consider an alternative homogeneous realisation of complex projective space, this time using the symplectic group. For $n \geq 1$, the odd-dimensional complex projective space $\mathbb{CP}^{2n+1}$ can be expressed as the quotient

$$\mathbb{CP}^{2n+1} \cong \text{Sp}(n+1) / \text{U}(1) \times \text{Sp}(n),$$

where the subgroup $H := \mathrm{U}(1) \times \mathrm{Sp}(n)$ is embedded in $\mathrm{Sp}(n+1)$ via block inclusion: the $\mathrm{U}(1)$ factor occupies the upper left 1×1 block, while $\mathrm{Sp}(n)$ sits in the lower right $n \times n$ block. The Lie algebra $\mathfrak{h}$ of H splits as

$$\mathfrak{h} = \mathbb{R}\xi_1 \oplus \mathfrak{sp}(n),$$

where $\xi_1 := iF_{1,1}^{(n+1)}$ spans the $\mathfrak{u}(1)$ factor of $\mathfrak{h}$. A basis for $\mathfrak{sp}(n)$ is given by

$$\left\{ iF_{p,p}^{(n+1)}, jF_{p,p}^{(n+1)}, kF_{p,p}^{(n+1)}, iF_{r,s}^{(n+1)}, jF_{r,s}^{(n+1)}, kF_{r,s}^{(n+1)}, E_{r,s}^{(n+1)} \right\}_{\substack{2 \leq r < s \leq n+1 \\ p=2, \dots, n+1}}.$$

The subalgebra $\mathfrak{h} \subseteq \mathfrak{sp}(n+1)$ admits a reductive complement $\mathfrak{m} = \mathfrak{h}^{\perp_{B_0}}$, which decomposes further into two irreducible $\mathfrak{h}$ -submodules:

$$\mathfrak{m} = \mathcal{V} \oplus \mathcal{H},$$

where $\mathcal{V}$ is the two-dimensional space

$$\mathcal{V} := \mathrm{span}_{\mathbb{R}} \left\{ \xi_2 := -kF_{1,1}^{(n+1)}, \xi_3 := jF_{1,1}^{(n+1)} \right\},$$

and $\mathcal{H}$ has dimension $4n$ and is given by

$$\mathcal{H} := \mathrm{span}_{\mathbb{R}} \left\{ e_{4p} := jF_{1,p+1}^{(n+1)}, e_{4p+1} := kF_{1,p+1}^{(n+1)}, e_{4p+2} := iF_{1,p+1}^{(n+1)}, e_{4p+3} := E_{1,p+1}^{(n+1)} \right\}_{p=1, \dots, n}.$$

This decomposition into $\mathcal{V}$ and $\mathcal{H}$ corresponds geometrically to the vertical and horizontal distributions of the twistor fibration

$$S^2 \hookrightarrow \mathbb{CP}^{2n+1} \rightarrow \mathbb{HP}^n.$$

It follows from Theorem 2.10 that the space of $\mathrm{Sp}(n+1)$ -invariant metrics on $\mathbb{CP}^{2n+1}$ is parametrised by two positive real numbers a and t :

$$g_{a,t} := aB_0|_{\mathcal{H} \times \mathcal{H}} + 2at B_0|_{\mathcal{V} \times \mathcal{V}}, \quad a, t > 0.$$

A $g_{a,t}$ -orthonormal basis of $\mathfrak{m}$ is then given by

$$\left\{ \xi_2^{a,t} := \frac{1}{\sqrt{2ta}} \xi_2, \quad \xi_3^{a,t} := \frac{1}{\sqrt{2ta}} \xi_3, \quad e_{4p+\varepsilon}^{a,t} := \frac{1}{\sqrt{2a}} e_{4p+\varepsilon} \right\}_{\substack{\varepsilon=0,1,2,3 \\ p=1, \dots, n}}.$$

We fix the orientation defined by the ordered basis

$$(\xi_2^{a,t}, \xi_3^{a,t}, e_4^{a,t}, \dots, e_{4n+3}^{a,t}).$$

Invariant spin^r structures

We now turn to the classification of $\text{Sp}(n+1)$ -invariant spin^r structures on complex projective space $\mathbb{CP}^{2n+1}$. By [HS90, p. 327], this space admits a unique spin structure, which is $\text{Sp}(n+1)$ -invariant. More generally, we may describe all $\text{Sp}(n+1)$ -invariant $\text{spin}^{\mathbb{C}}$ structures using the algebraic criterion provided by Theorem 5.13.

Theorem 6.9. *The $\text{Sp}(n+1)$ -invariant $\text{spin}^{\mathbb{C}}$ structures on $\mathbb{CP}^{2n+1}$ are in one-to-one correspondence with the even integers:*

$$\text{Sp}(n+1) \times_{\phi_s} \text{Spin}^{\mathbb{C}}(4n+2), \quad s \in 2\mathbb{Z},$$

where ϕ_s denotes the unique lift of $\sigma \times \varphi_s$ to $\text{Spin}^{\mathbb{C}}(4n+2)$. Here, $\sigma: H \rightarrow \text{SO}(4n+2)$ is the isotropy representation, and $\varphi_s: H \rightarrow \text{SO}(2) \cong \text{U}(1)$ is defined by

$$(z, A) \mapsto z^s.$$

Proof. This follows from Theorem 5.13, together with two key observations. First, $\text{Sp}(n)$ is a simple Lie group of dimension ≥ 3 , and hence any group homomorphism $H \rightarrow \text{SO}(2)$ has to be trivial on the $\text{Sp}(n)$ factor. Second, any homomorphism $\text{U}(1) \rightarrow \text{U}(1)$ is of the form $z \mapsto z^s$ for some $s \in \mathbb{Z}$.

By Proposition 5.1, the combined map $\sigma \times \varphi_s: H \rightarrow \text{SO}(4n+2) \times \text{U}(1)$ lifts to $\text{Spin}^{\mathbb{C}}(4n+2)$ if and only if s is even. Since $\text{U}(1)$ is abelian, the representations φ_s are pairwise non-equivalent, and hence the resulting $\text{spin}^{\mathbb{C}}$ structures are pairwise non- $\text{Sp}(n+1)$ -equivariantly equivalent. $\square$

Invariant spin^r spinors

We begin by classifying the $\text{Sp}(n+1)$ -invariant spinors associated with the unique spin structure on $\mathbb{CP}^{2n+1}$:

Theorem 6.10. *The space Σ_{inv} of $\text{Sp}(n+1)$ -invariant spinors on $\mathbb{CP}^{2n+1}$ is given*

by

$$\Sigma_{\text{inv}} = \begin{cases} \text{span}_{\mathbb{C}} \{ \psi_+ := \omega^{(n+1)/2}, \psi_- := y_1 \wedge \omega^{(n-1)/2} \} & n \text{ odd}, \\ 0 & n \text{ even}, \end{cases}$$

where $\omega := \sum_{p=1}^n y_{2p} \wedge y_{2p+1}$.

Proof. By [AHL23, Thm. 4.11], the space of spinors annihilated by $\mathfrak{sp}(n)$ is

$$V := \text{span}_{\mathbb{C}} \{ \omega^k, y_1 \wedge \omega^k \}_{k=0, \dots, n},$$

and hence the space Σ_{inv} of invariant spinors is a subspace of V . It remains to determine which of these are annihilated by the element $\xi_1 \in \mathfrak{h}$. As in the proof of Theorem 6.3, let $\tilde{\sigma}$ denote the lift of the isotropy representation $\sigma: H \rightarrow \text{SO}(4n+2)$ to $\text{Spin}(4n+2)$. Then one has:

$$\tilde{\sigma}_*(\xi_1) = \xi_2^{a,t} \cdot \xi_3^{a,t} + \frac{1}{2} \sum_{p=1}^n (e_{4p}^{a,t} \cdot e_{4p+1}^{a,t} + e_{4p+2}^{a,t} \cdot e_{4p+3}^{a,t}).$$

A direct computation shows that

$$\tilde{\sigma}_*(\xi_1) \cdot \omega^k = i(n+1-2k) \omega^k, \quad \tilde{\sigma}_*(\xi_1) \cdot (y_1 \wedge \omega^k) = i(n-1-2k) (y_1 \wedge \omega^k).$$

The invariant spinors are therefore the ones for which these eigenvalues vanish, leading to the stated result. $\square$

We now consider $\text{spin}^{\mathbb{C}}$ spinors. Using the same method as in the Hermitian case (compare with Theorem 6.4), we obtain:

Theorem 6.11. *Let $n \in \mathbb{N}$ and $s = 2s' \in 2\mathbb{Z}$. Then the space $(\Sigma_{4n+2,2}^1)_{\text{inv}}$ of $\text{Sp}(n+1)$ -invariant 1-twisted $\text{spin}^{\mathbb{C}}$ spinors on $\mathbb{CP}^{2n+1}$ associated with the $\text{spin}^{\mathbb{C}}$ structure*

$$\text{Sp}(n+1) \times_{\phi_s} \text{Spin}^{\mathbb{C}}(4n+2)$$

is given by

$$(\Sigma_{4n+2,2}^1)_{\text{inv}} = \begin{cases} \text{span}_{\mathbb{C}} \{ \omega^{(n+1+s')/2}, y_1 \wedge \omega^{(n-1+s')/2} \} \otimes \hat{1} \oplus \\ \oplus \text{span}_{\mathbb{C}} \{ \omega^{(n+1-s')/2}, y_1 \wedge \omega^{(n-1-s')/2} \} \otimes \hat{y}_1 & n \not\equiv s' \pmod{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Here, negative powers of ω are defined to be zero. In particular, the $\text{Sp}(n+1)$ -invariant spinor type of $\mathbb{CP}^{2n+1}$ is

$$\sigma(\mathbb{CP}^{2n+1}, \text{Sp}(n+1)) = \begin{cases} 1 & n \text{ odd}, \\ 2 & n \text{ even}. \end{cases}$$

Remark 6.12. The $\text{spin}^{\mathbb{C}}$ structure corresponding to $s = 0$ coincides with that induced by the underlying spin structure. Indeed, setting $s = 0$ in Theorem 6.11 recovers the spinors from Theorem 6.10, tensored with Σ_2 .

Special spin^r spinors

In order to differentiate these spinors, one can see, using the formulas for the Nomizu map from [BFGK91, p. 141], that the spin lift $\tilde{\Lambda}^{a,t}$ of the Nomizu map of the Levi-Civita connection of $g_{a,t}$ is given by

$$\begin{aligned} \tilde{\Lambda}^{a,t}(\xi_2^{a,t}) &= \frac{1-t}{2\sqrt{2at}} \sum_{p=1}^n (e_{4p}^{a,t} \cdot e_{4p+2}^{a,t} - e_{4p+1}^{a,t} \cdot e_{4p+3}^{a,t}), \\ \tilde{\Lambda}^{a,t}(\xi_3^{a,t}) &= \frac{1-t}{2\sqrt{2at}} \sum_{p=1}^n (e_{4p}^{a,t} \cdot e_{4p+3}^{a,t} + e_{4p+1}^{a,t} \cdot e_{4p+2}^{a,t}), \\ \tilde{\Lambda}^{a,t}(e_{4p}^{a,t}) &= \frac{1}{2} \sqrt{\frac{t}{2a}} (-\xi_2^{a,t} \cdot e_{4p+2}^{a,t} - \xi_3^{a,t} \cdot e_{4p+3}^{a,t}), \\ \tilde{\Lambda}^{a,t}(e_{4p+1}^{a,t}) &= \frac{1}{2} \sqrt{\frac{t}{2a}} (\xi_2^{a,t} \cdot e_{4p+3}^{a,t} - \xi_3^{a,t} \cdot e_{4p+2}^{a,t}), \\ \tilde{\Lambda}^{a,t}(e_{4p+2}^{a,t}) &= \frac{1}{2} \sqrt{\frac{t}{2a}} (\xi_2^{a,t} \cdot e_{4p}^{a,t} + \xi_3^{a,t} \cdot e_{4p+1}^{a,t}), \\ \tilde{\Lambda}^{a,t}(e_{4p+3}^{a,t}) &= \frac{1}{2} \sqrt{\frac{t}{2a}} (-\xi_2^{a,t} \cdot e_{4p+1}^{a,t} + \xi_3^{a,t} \cdot e_{4p}^{a,t}). \end{aligned} \tag{6.13}$$

Baum et al. proved in [BFGK91, p. 146] that $\mathbb{CP}^3$ admits non-trivial $\text{Sp}(2)$ -invariant generalised Killing spinors, given by $\psi_+ \pm i\psi_-$. For the Fubini-Study metric, the two eigenvalues of these generalised Killing spinors coincide, yielding real Killing spinors which are related to the nearly-Kähler geometry of $\mathbb{CP}^3$. We now show that this does not occur in higher dimensions:

Theorem 6.14. *The space $\mathbb{CP}^{2n+1}$ admits non-trivial $\text{Sp}(n+1)$ -invariant generalised Killing spinors if and only if $n = 1$.*

Proof. First, we recall from Theorem 6.10 that there are no invariant spinors when n is even, so it remains only to consider the case where n is odd. Let n be odd, and suppose that $n \geq 3$ so that $\omega^{(n+3)/2} \neq 0$. Using the above formulas (6.13) for the Nomizu map, we get, for $\alpha, \beta \in \mathbb{C}$,

$$\tilde{\Lambda}^{a,t}(e_{4p}^{a,t}) \cdot (\alpha\psi_+ + \beta\psi_-) = -\sqrt{\frac{t}{2a}} \left\{ \alpha \frac{n+1}{2} y_1 \wedge y_{2p} + \beta y_{2p+1} \right\} \wedge \omega^{(n-1)/2}.$$

Writing a general element $X \in \mathfrak{m}$ as a (real) linear combination

$$X = \mu_2 \xi_2^{a,t} + \mu_3 \xi_3^{a,t} + \sum_{p=1}^n (\mu_{4p} e_{4p}^{a,t} + \mu_{4p+1} e_{4p+1}^{a,t} + \mu_{4p+2} e_{4p+2}^{a,t} + \mu_{4p+3} e_{4p+3}^{a,t})$$

of the basis vectors, we find that the Clifford product with an arbitrary invariant spinor is given by

$$\begin{aligned} X \cdot (\alpha\psi_+ + \beta\psi_-) = & \alpha \left\{ (i\mu_2 + \mu_3)y_1 + \sum_{p=1}^n [(i\mu_{4p} + \mu_{4p+1})y_{2p} + (i\mu_{4p+2} + \mu_{4p+3})y_{2p+1}] \right\} \wedge \omega^{(n+1)/2} + \\ & + \alpha \frac{n+1}{2} \left\{ \sum_{p=1}^n [(i\mu_{4p} - \mu_{4p+1})y_{2p+1} + (-i\mu_{4p+2} + \mu_{4p+3})y_{2p}] \right\} \wedge \omega^{(n-1)/2} + \\ & + \beta \{i\mu_2 - \mu_3\} \omega^{(n-1)/2} + \\ & + \beta y_1 \wedge \left\{ \sum_{p=1}^n [(-i\mu_{4p} - \mu_{4p+1})y_{2p} + (-i\mu_{4p+2} - \mu_{4p+3})y_{2p+1}] \right\} \wedge \omega^{(n-1)/2} + \\ & + \beta \frac{n-1}{2} \left\{ \sum_{p=1}^n [(-i\mu_{4p} + \mu_{4p+1}) + (i\mu_{4p+2} - \mu_{4p+3})] \right\} y_1 \wedge \omega^{(n-3)/2}. \end{aligned}$$

Hence, by equating coefficients in

$$\tilde{\Lambda}^{a,t}(e_{4p}^{a,t}) \cdot (\alpha\psi_+ + \beta\psi_-) = X \cdot (\alpha\psi_+ + \beta\psi_-),$$

one easily concludes (using crucially that $n \geq 3$) that the only possibility is $\alpha = \beta = 0$, and the result then follows from the preceding discussion about the case $n = 1$. $\square$

We now turn to the study of the $\mathrm{Sp}(n+1)$ -invariant $\mathrm{spin}^{\mathbb{C}}$ spinors on $\mathbb{CP}^{2n+1}$ found in Theorem 6.11. The aim is to show that, when $s' = s/2 = \pm n \pm 1$ and $t = 1$, there is a pure spinor which is parallel with respect to a suitable connection on the

auxiliary bundle. This encodes the fact that, for $t = 1$, the metric $g_{a,t}$ is Kähler.

Lemma 6.15. *Let $k \in \mathbb{N}$ and $\psi \in \left\{ \omega^k \otimes \widehat{1}, (y_1 \wedge \omega^k) \otimes \widehat{1}, \omega^k \otimes \widehat{y}_1, (y_1 \wedge \omega^k) \otimes \widehat{y}_1 \right\}$. Then, a scalar multiple of ψ is pure if and only if $k = 0$ or $k = n$.*

Proof. We will only prove it for $\psi = \omega^k \otimes \widehat{1}$, as the other cases are analogous. Since $r = 2 < 3$, we only need to show that $\left(\widehat{\eta}_{12}^\psi \right)^2 = -\text{Id}$. Indeed, for all $1 \leq p, q \leq n$ and $\varepsilon \in \{0, 1, 2, 3\}$, one calculates:

$$\begin{aligned} \eta_{12}^\psi(\xi_2^{a,t}, \xi_3^{a,t}) &= \Re \langle \xi_2^{a,t} \cdot \xi_3^{a,t} \cdot \widehat{e}_1 \cdot \widehat{e}_2 \cdot \psi, \psi \rangle = \Re \langle i \xi_2^{a,t} \cdot \xi_3^{a,t} \cdot \psi, \psi \rangle \\ &= -\Re \langle \omega^k \otimes \widehat{1}, \omega^k \otimes \widehat{1} \rangle = -(k!)^2 \binom{n}{k}, \\ \eta_{12}^\psi(e_{4p}^{a,t}, e_{4q+1}^{a,t}) &= \Re \langle e_{4p}^{a,t} \cdot e_{4q+1}^{a,t} \cdot \widehat{e}_1 \cdot \widehat{e}_2 \cdot \psi, \psi \rangle = \Re \langle i \cdot e_{4p}^{a,t} \cdot e_{4q+1}^{a,t} \cdot \omega^k, \omega^k \rangle \\ &= -\delta_{p,q} \langle \omega^k - 2ky_{2p} \wedge y_{2p+1} \wedge \omega^{k-1}, \omega^k \rangle = -\delta_{p,q} (k!)^2 \left[\binom{n}{k} - 2 \binom{n-1}{k-1} \right], \\ \eta_{12}^\psi(e_{4p+2}^{a,t}, e_{4q+3}^{a,t}) &= -\delta_{p,q} (k!)^2 \left[\binom{n}{k} - 2 \binom{n-1}{k-1} \right], \\ \eta_{12}^\psi(\xi_2^{a,t}, e_{4p+\varepsilon}^{a,t}) &= \eta_{12}^\psi(\xi_3^{a,t}, e_{4p+\varepsilon}^{a,t}) = \eta_{12}^\psi(e_{4p}^{a,t}, e_{4q+2}^{a,t}) = \eta_{12}^\psi(e_{4p}^{a,t}, e_{4q+3}^{a,t}) = \eta_{12}^\psi(e_{4p+1}^{a,t}, e_{4q+3}^{a,t}) = 0, \end{aligned}$$

where $\binom{n-1}{k-1}$ is understood to be 0 if $k = 0$. Altogether, we have

$$\widehat{\eta}_{12}^\psi = -(k!)^2 \left[\binom{n}{k} \xi_2^{a,t} \wedge \xi_3^{a,t} + \left[\binom{n}{k} - 2 \binom{n-1}{k-1} \right] \sum_{p=1}^n (e_{4p}^{a,t} \wedge e_{4p+1}^{a,t} + e_{4p+2}^{a,t} \wedge e_{4p+3}^{a,t}) \right],$$

which is easily seen to square to a multiple of $-\text{Id}$ if and only if $k = 0$ or $k = n$. $\square$

The preceding lemma, together with Theorem 6.11 describing the invariant $\text{spin}^{\mathbb{C}}$ spinors, implies that the $\text{Sp}(n+1)$ -invariant $\text{spin}^{\mathbb{C}}$ structure corresponding to $s = 2s' \in 2\mathbb{Z}$ admits invariant pure $\text{spin}^{\mathbb{C}}$ spinors if and only if $s' \in \{n+1, -n-1, n-1, -n+1\}$, which are given in each case by

$$\begin{array}{ll} \{(y_1 \wedge \omega^n) \otimes \widehat{1}, 1 \otimes \widehat{y}_1\} & s' = n+1, \\ \{1 \otimes \widehat{1}, (y_1 \wedge \omega^n) \otimes \widehat{y}_1\} & s' = -n-1, \\ \{\omega^n \otimes \widehat{1}, y_1 \otimes \widehat{y}_1\} & s' = n-1, \\ \{y_1 \otimes \widehat{1}, \omega^n \otimes \widehat{y}_1\} & s' = -n+1. \end{array}$$

In order to differentiate these spinors, one needs to fix a connection on the auxiliary bundle. Applying the criterion in Lemma 2.26, one sees that the only $\text{Sp}(n+1)$ -

invariant connection on the auxiliary bundle is the one with Nomizu map $\Lambda|_{\mathfrak{m}} = 0$. This connection, together with the Levi-Civita connection of the metric $g_{a,t}$ (with Nomizu map $\Lambda^{a,t}$), induces a connection $\nabla^{a,t}$ on the corresponding $\text{spin}^{\mathbb{C}}$ spinor bundle. The following lemma is a straightforward calculation using the expression of the spin lift of the Nomizu map (6.13):

Lemma 6.16. *The invariant pure $\text{spin}^{\mathbb{C}}$ spinor $1 \otimes \widehat{1}$ is $\nabla^{a,t}$ -parallel if and only if $t = 1$. $\square$*

These $\text{spin}^{\mathbb{C}}$ spinors encode some well-known geometric information of $\mathbb{CP}^{2n+1}$ – see e.g. [Zil82]:

Theorem 6.17. *The metric $g_{a,1}$ on $\mathbb{CP}^{2n+1}$ is Kähler-Einstein, for all $a > 0$.*

Proof. Let $s = -2(n+1)$, and consider the $\text{spin}^{\mathbb{C}}$ structure on $\mathbb{CP}^{2n+1}$ determined by φ_s . By Lemmata 6.15 and 6.16, this structure carries a $\nabla^{a,1}$ parallel pure $\text{spin}^{\mathbb{C}}$ spinor $\psi = 1 \otimes \widehat{1}$. Hence, by [HS19, Cor. 4.10], the metric $g_{a,1}$ is Kähler, for all $a > 0$.

Let us now see that these metrics are also Einstein. By Theorem 5.30(1), the Ricci tensor decomposes as

$$\text{Ric} = \frac{1}{|\psi|^2} \widehat{\Theta}_{12} \circ \widehat{\eta}_{12}^{\psi}. \quad (6.18)$$

By calculations similar to those in the proof of Theorem 6.6, one finds that, for $1 \leq p, q \leq n$, $0 \leq \varepsilon \leq 3$ and $2 \leq l \leq 3$,

$$\begin{aligned} \Omega(\xi_2^{a,1}, \xi_3^{a,1}) &= -\Lambda([\xi_2^{a,1}, \xi_3^{a,1}]) = -\frac{1}{a} \Lambda(\xi_1) = -\frac{1}{a} (\varphi_{-2(n+1)})_*(\xi_1) = \frac{2(n+1)}{a} \widehat{e}_1 \wedge \widehat{e}_2, \\ \Omega(e_{4p}^{a,1}, e_{4q+1}^{a,1}) &= \Omega(e_{4p+2}^{a,1}, e_{4q+3}^{a,1}) = \delta_{p,q} \frac{2(n+1)}{a} \widehat{e}_1 \wedge \widehat{e}_2, \\ \Omega(e_{4p}^{a,1}, e_{4q+2}^{a,1}) &= \Omega(e_{4p}^{a,1}, e_{4q+3}^{a,1}) = \Omega(e_{4p+1}^{a,1}, e_{4q+2}^{a,1}) = \Omega(e_{4p+1}^{a,1}, e_{4q+3}^{a,1}) = \Omega(\xi_l^{a,1}, e_{4p+\varepsilon}^{a,1}) = 0. \end{aligned}$$

Hence, using the definition of $\widehat{\Theta}_{12}$ in terms of Ω and taking $k = 0$ in the proof of Lemma 6.15,

$$\begin{aligned} \widehat{\Theta}_{12} &= \frac{2(n+1)}{a} \left(\xi_2^{a,1} \wedge \xi_3^{a,1} + \sum_{p=1}^n (e_{4p}^{a,1} \wedge e_{4p+1}^{a,1} + e_{4p+2}^{a,1} \wedge e_{4p+3}^{a,1}) \right), \\ \widehat{\eta}_{12}^{\psi} &= - \left(\xi_2^{a,1} \wedge \xi_3^{a,1} + \sum_{p=1}^n (e_{4p}^{a,1} \wedge e_{4p+1}^{a,1} + e_{4p+2}^{a,1} \wedge e_{4p+3}^{a,1}) \right). \end{aligned}$$

Finally, substituting everything into equation (6.18), we get

$$\text{Ric} = \frac{2(n+1)}{a} \text{Id},$$

which completes the proof. $\square$

6.3 Quaternionic projective space

Quaternionic projective space admits a natural realisation as a homogeneous space

$$\mathbb{H}\mathbb{P}^n \cong \text{Sp}(n+1) / \text{Sp}(1) \times \text{Sp}(n),$$

where the subgroup $H := \text{Sp}(1) \times \text{Sp}(n)$ is embedded in $\text{Sp}(n+1)$ as follows: the $\text{Sp}(1)$ factor occupies the upper left 1×1 block, while the $\text{Sp}(n)$ factor sits in the lower right $n \times n$ block. The Lie algebra $\mathfrak{h}$ of H decomposes as a direct sum

$$\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n).$$

Explicitly, we may take

$$\begin{aligned} \mathfrak{sp}(1) &= \text{span}_{\mathbb{R}} \left\{ \xi_1 := iF_{1,1}^{(n+1)}, \xi_2 := -kF_{1,1}^{(n+1)}, \xi_3 := jF_{1,1}^{(n+1)} \right\}, \\ \mathfrak{sp}(n) &= \text{span}_{\mathbb{R}} \left\{ iF_{p,p}^{(n+1)}, jF_{p,p}^{(n+1)}, kF_{p,p}^{(n+1)}, iF_{r,s}^{(n+1)}, jF_{r,s}^{(n+1)}, kF_{r,s}^{(n+1)}, E_{r,s}^{(n+1)} \right\}_{\substack{2 \leq r < s \leq n+1 \\ p=2, \dots, n+1}}. \end{aligned}$$

The Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{sp}(n+1)$ admits a reductive complement $\mathfrak{m} = \mathfrak{h}^{\perp_{B_0}}$, given by

$$\mathfrak{m} := \text{span}_{\mathbb{R}} \left\{ e_{4p} := jF_{1,p+1}^{(n+1)}, e_{4p+1} := kF_{1,p+1}^{(n+1)}, e_{4p+2} := iF_{1,p+1}^{(n+1)}, e_{4p+3} := E_{1,p+1}^{(n+1)} \right\}_{p=1, \dots, n}.$$

The adjoint action of $\mathfrak{h}$ on $\mathfrak{m}$ is irreducible, and hence, by Theorem 2.10, the space of $\text{Sp}(n+1)$ -invariant Riemannian metrics on $\mathbb{H}\mathbb{P}^n$ is one-dimensional. These metrics form the family

$$g_a := aB_0|_{\mathfrak{m} \times \mathfrak{m}}, \quad a > 0.$$

The basis of $\mathfrak{m}$ introduced above becomes g_a -orthonormal upon rescaling by $1/\sqrt{2a}$. To simplify notation, and without essential loss of generality, we henceforth fix the normalisation $g := g_{1/2}$. With this choice, the basis $\{e_4, e_5, \dots, e_{4n+3}\}$ is orthonormal, and we take the orientation defined by this ordering.

Invariant spin^r structures

Since $(\mathbb{H}\mathbb{P}^1, g_a)$ is homothetic to the round sphere S^4 , which is studied in detail in [AHL23], we assume throughout that $n > 1$. Our goal is to describe the $\text{Sp}(n+1)$ -invariant spin^r structures on quaternionic projective space. According to Theorem 5.13, this reduces to finding all Lie group homomorphisms

$$\varphi: H \rightarrow \text{SO}(r)$$

such that the homomorphism $\sigma \times \varphi$ lifts to $\text{Spin}^r(4n)$, where σ is the isotropy representation. Since H is simply connected, any such homomorphism automatically admits a lift. In particular, there is a unique spin structure, which is also $\text{Sp}(n+1)$ -invariant - see [HS90, p. 327].

For $r = 2$, the situation is trivial: the simplicity of both $\text{Sp}(1)$ and $\text{Sp}(n)$ and dimensional reasons imply that the only homomorphism $\text{Sp}(1) \times \text{Sp}(n) \rightarrow \text{SO}(2)$ is the trivial one. Thus, the corresponding $\text{Sp}(n+1)$ -invariant $\text{spin}^{\mathbb{C}}$ structure on $\mathbb{H}\mathbb{P}^n$ is simply the one induced by its unique spin structure.

The first genuinely interesting case is $r = 3$, which corresponds to $\text{spin}^{\mathbb{H}}$ structures. To classify these, we must understand the homomorphisms from $\text{Sp}(1)$ to $\text{SO}(3)$:

Proposition 6.19. *Up to conjugation by elements of $\text{SO}(3)$, there are exactly two Lie group homomorphisms $\text{Sp}(1) \rightarrow \text{SO}(3)$: the trivial homomorphism and the standard double covering λ_3 .*

Proof. Let $\varphi: \text{Sp}(1) \rightarrow \text{SO}(3)$ be a non-trivial homomorphism. Recall that $\text{Sp}(1) \cong \text{Spin}(3)$, and that $\mathfrak{so}(3)$ is a simple Lie algebra. Therefore, the kernel of φ is discrete.

Since $\text{Sp}(1)$ is compact, the image of φ is a closed three-dimensional Lie subgroup of $\text{SO}(3)$, and hence also open. By connectedness of $\text{SO}(3)$, we conclude that φ is surjective.

In particular, the representation of $\text{Sp}(1)$ on $\mathbb{R}^3$ induced by composing φ with the standard representation of $\text{SO}(3)$ must be irreducible, since otherwise the image of φ would be contained inside a subgroup isomorphic to $\{1\} \times \text{SO}(2)$. But, up to isomorphism, $\text{Sp}(1)$ admits exactly one irreducible real three-dimensional representation, namely the standard spin representation [IRS91, Prop. 11]. Therefore, any two non-trivial homomorphisms $\varphi_1, \varphi_2: \text{Sp}(1) \rightarrow \text{SO}(3)$ are conjugate by an element of $\text{GL}(3, \mathbb{R})$.

To see that they are also conjugate by an element of $\text{SO}(3)$, suppose $T \in \text{GL}(3, \mathbb{R})$ satisfies $T^{-1}\varphi_1(A)T = \varphi_2(A)$ for all $A \in \text{Sp}(1)$. Then:

$$TT^t = \varphi_1(A)^{-1}T\varphi_2(A)T^t = \varphi_1(A)^{-1}TT^t\varphi_1(A).$$

This shows that TT^t commutes with the image of φ_1 , which is $\text{SO}(3)$, and hence must be a scalar multiple of the identity. Thus, T is proportional to an orthogonal matrix, and rescaling gives $\hat{T} \in \text{SO}(3)$ with the desired conjugation property. $\square$

This classification allows us to describe all $\text{Sp}(n+1)$ -invariant $\text{spin}^{\mathbb{H}}$ structures on $\mathbb{H}\mathbb{P}^n$, using Theorem 5.13:

Theorem 6.20. *Let $n > 1$. Then the $\text{Sp}(n+1)$ -invariant $\text{spin}^{\mathbb{H}}$ structures on $\mathbb{H}\mathbb{P}^n$ are precisely*

$$\text{Sp}(n+1) \times_{\phi_i} \text{Spin}^{\mathbb{H}}(4n), \quad i = 0, 1,$$

where $\sigma: H \rightarrow \text{SO}(4n)$ is the isotropy representation, φ_0 is the trivial homomorphism, $\varphi_1(x, y) := \lambda_3(x)$, and ϕ_i denotes the unique lift of $\sigma \times \varphi_i$ to $\text{Spin}^{\mathbb{H}}(4n)$. $\square$

We note that the $\text{spin}^{\mathbb{H}}$ structure corresponding to φ_0 is induced by the standard spin structure. For the remainder of our discussion, we shall focus on the $\text{spin}^{\mathbb{H}}$ structure determined by the non-trivial homomorphism φ_1 .

Remark 6.21. *The auxiliary vector bundle associated to the $\text{spin}^{\mathbb{H}}$ structure given by φ_1 is $\text{Sp}(n+1)$ -equivariantly isomorphic to the rank-3 subbundle of $\text{End}(T\mathbb{H}\mathbb{P}^n)$ defined by the quaternionic Kähler structure of $\mathbb{H}\mathbb{P}^n$.*

Invariant spin^r spinors

To begin, it is easy to see that this homogeneous realisation carries no invariant spinors: as the homogeneous realisation of $\mathbb{H}\mathbb{P}^n$ that we are considering is that of a symmetric space, invariant spinors are parallel, and we know that $\mathbb{H}\mathbb{P}^n$ cannot have any non-trivial parallel spinors, since it is not Ricci-flat. However, we shall see in the next proposition that there are always non-trivial invariant $\text{spin}^{\mathbb{H}}$ spinors, for sufficient twistings of the spinor bundle, when n is odd.

Remark 6.22. *In Definition of m -twisted spin^r spinor bundle 5.24, we required the number of twistings m to be odd, in order to ensure that the resulting bundle is globally defined. In the particular case of $\mathbb{H}\mathbb{P}^n$, these bundles are well-defined for all*

values of m , because the principal $\mathrm{Spin}^{\mathbb{H}}(4n)$ -bundle of the $\mathrm{spin}^{\mathbb{H}}$ structure lifts to a principal $\mathrm{Spin}(4n) \times \mathrm{Spin}(3)$ -bundle, as H is simply connected.

Proposition 6.23. *The $\mathrm{Sp}(n+1)$ -invariant spinor type of $\mathbb{H}\mathbb{P}^n$ is*

$$\sigma(\mathbb{H}\mathbb{P}^n, \mathrm{Sp}(n+1)) = 3.$$

Furthermore, the number of twistings of the spinor bundle which realises this is n , and the space of non-trivial spinors in this case has dimension one.

Proof. The preceding discussion shows that the unique spin structure carries no non-trivial invariant spinors. As noted above, the only invariant $\mathrm{spin}^{\mathbb{C}}$ structure is the one coming from the spin structure, and it is clear that there are also no invariant $\mathrm{spin}^{\mathbb{C}}$ spinors in this case (since H acts trivially on Σ_2 and hence each $\Sigma_{4n,2}^m$ is equivalent as an H -module to a direct sum of copies of Σ_{4n}). This shows that

$$\sigma(\mathbb{H}\mathbb{P}^n, \mathrm{Sp}(n+1)) \geq 3.$$

We will prove that the subspace of $\Sigma_{4n,3}^n$ annihilated by $\mathfrak{h}$ is non-zero.

Now, by definition of the homomorphism $\varphi_1: H \rightarrow \mathrm{SO}(3)$ defining the $\mathrm{spin}^{\mathbb{H}}$ structure we are considering, the $\mathfrak{sp}(n)$ factor of $\mathfrak{h}$ acts trivially on $\Sigma_3^{\otimes m}$. Hence, the subspace of $\Sigma_{4n,3}^m$ annihilated by $\mathfrak{h}$ is contained in $\mathcal{S} \otimes \Sigma_3^{\otimes m}$, where $\mathcal{S}$ is the subspace of Σ_{4n} annihilated by $\mathfrak{sp}(n)$, which is the same as the subspace annihilated by its complexification $\mathfrak{sp}(2n, \mathbb{C})$.

From [AHL23, Thm. 4.11], we deduce that

$$\mathcal{S} = \mathrm{span}_{\mathbb{C}}\{\omega^k\}_{k=0}^n,$$

where ω is defined by

$$\omega := \sum_{j=1}^n y_{2j} \wedge y_{2j+1}.$$

A straightforward calculation shows that the adjoint action of the $\mathfrak{sp}(1)$ factor is given by

$$\begin{aligned} \widetilde{\mathrm{ad}}(\xi_1)|_{\mathfrak{m}} &= \frac{1}{2} \sum_{p=1}^n (e_{4p} \cdot e_{4p+1} + e_{4p+2} \cdot e_{4p+3}), \\ \widetilde{\mathrm{ad}}(\xi_2)|_{\mathfrak{m}} &= \frac{1}{2} \sum_{p=1}^n (e_{4p} \cdot e_{4p+2} - e_{4p+1} \cdot e_{4p+3}), \end{aligned}$$

$$\widetilde{\mathrm{ad}}(\xi_3)|_{\mathfrak{m}} = \frac{1}{2} \sum_{p=1}^n (e_{4p} \cdot e_{4p+3} + e_{4p+1} \cdot e_{4p+2}).$$

Hence, $\mathfrak{sp}(1)$ factor of $\mathfrak{h}$ acts on $\mathcal{S}$ as follows:

$$\widetilde{\mathrm{ad}}(\xi_1)|_{\mathfrak{m}} \cdot \omega^k = i(n - 2k) \omega^k, \quad (6.24)$$

$$\begin{aligned} \widetilde{\mathrm{ad}}(\xi_2)|_{\mathfrak{m}} \cdot \omega^k &= k(n - k + 1) \omega^{k-1} - \omega^{k+1}, \\ \widetilde{\mathrm{ad}}(\xi_3)|_{\mathfrak{m}} \cdot \omega^k &= i(\omega^{k+1} + k(n - k + 1) \omega^{k-1}). \end{aligned} \quad (6.25)$$

The standard basis element for the Cartan subalgebra of the complexification $\mathfrak{sp}(1)^{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C})$ of $\mathfrak{sp}(1)$ is $-i\xi_1$, and (6.24) shows that the action of this element on $\mathcal{S}$ has highest eigenvalue n . Denote by V_t the unique irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ of dimension $t + 1$. Then, $\mathcal{S}$ contains a copy of V_n , and by dimensional reasons we have $\mathcal{S} \cong V_n$ as $\mathfrak{sl}(2, \mathbb{C})$ -modules. Hence, the subspace of $\Sigma_{4n} \otimes \Sigma_3^{\otimes m}$ annihilated by $\mathfrak{h}$ corresponds to the maximal trivial $\mathfrak{sl}(2, \mathbb{C})$ -subrepresentation of $V_n \otimes V_1^{\otimes m}$. Thus, it suffices to show that the smallest odd tensor power of Σ_3 which contains a copy of V_n is $\Sigma_3^{\otimes n}$. Recalling the well-known decomposition

$$V_s \otimes V_t \simeq V_{s+t} \oplus V_{s+t-2} \oplus \cdots \oplus V_{s-t} \quad (s \geq t)$$

of $\mathfrak{sl}(2, \mathbb{C})$ -representations (see e.g. [FH91, §11.1, §11.2]), the result follows by repeatedly using (6.3) to decompose tensor powers of $\Sigma_3 \cong V_1$. Finally, as the trivial representation appears with multiplicity one, we conclude that the space of invariant $\mathrm{spin}^{\mathbb{H}}$ spinors is one-dimensional. $\square$

Remark 6.26. *Alternatively, in the proof of Proposition 6.23, one could use the decomposition of the spin representation*

$$\Sigma_{4n} \cong \bigoplus_{k=0}^n \Lambda_0^k E \otimes V_{n-k}$$

as a representation of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ in Salamon's $E - H$ formalism – see [BS83].

In the next proposition, we give an explicit non-trivial $\mathrm{Sp}(n+1)$ -invariant n -twisted $\mathrm{spin}^{\mathbb{H}}$ spinor on $\mathbb{H}\mathbb{P}^n$. By Proposition 6.23, this is unique up to scaling.

Theorem 6.27. *The space of $\mathrm{Sp}(n+1)$ -invariant n -twisted $\mathrm{spin}^{\mathbb{H}}$ spinors is spanned over $\mathbb{C}$ by*

$$\psi := \sum_{j=0}^n i^{n-j} (n-j)! \omega^j \otimes \eta_j,$$

where $\eta_j \in \Sigma_3^{\otimes n}$ is defined as the symmetrisation

$$\eta_j := \text{Sym} \left(\widehat{1}^{\otimes j} \otimes \widehat{y}_1^{\otimes (n-j)} \right).$$

In other words, η_j is the sum of all tensor products with j copies of $\widehat{1}$ and $n - j$ copies of $\widehat{y}_1$, taken in all possible orders.

Proof. We only need to check that ψ is annihilated by ξ_1 , ξ_2 , and ξ_3 . Equations (6.24) – (6.25) give the action of these elements on powers of ω . We now calculate how they act on η_j . Recall that, by definition of φ_1 ,

$$\begin{aligned} \xi_1 \cdot \widehat{1} &= i\widehat{1}, & \xi_1 \cdot \widehat{y}_1 &= -i\widehat{y}_1, \\ \xi_2 \cdot \widehat{1} &= i\widehat{y}_1, & \xi_2 \cdot \widehat{y}_1 &= i\widehat{1}, \\ \xi_3 \cdot \widehat{1} &= \widehat{y}_1, & \xi_3 \cdot \widehat{y}_1 &= -\widehat{1}. \end{aligned}$$

Using this, we can easily deduce the action of these elements on η_j :

$$\begin{aligned} \xi_1 \cdot \eta_j &= -i(n - 2j)\eta_j, \\ \xi_2 \cdot \eta_j &= i((n - j + 1)\eta_{j-1} + (j + 1)\eta_{j+1}), \\ \xi_3 \cdot \eta_j &= (n - j + 1)\eta_{j-1} - (j + 1)\eta_{j+1}. \end{aligned}$$

The result now follows from a straightforward calculation. To simplify the notation, let

$$c_j := i^{n-j}(n - j)!,$$

so that

$$\psi = \sum_{j=0}^n c_j \omega^j \otimes \eta_j.$$

Then, we can directly see that ξ_1 , ξ_2 , and ξ_3 annihilate ψ :

$$\begin{aligned} \xi_1 \cdot \psi &= \sum_{j=0}^n c_j (\xi_1 \cdot \omega^j) \otimes \eta_j + \sum_{j=0}^n c_j \omega^j \otimes (\xi_1 \cdot \eta_j) \\ &= \sum_{j=0}^n c_j (i(n - 2j)\omega^j) \otimes \eta_j + \sum_{j=0}^n c_j \omega^j \otimes (-i(n - 2j)\eta_j) = 0, \\ \xi_2 \cdot \psi &= \sum_{j=0}^n c_j (\xi_2 \cdot \omega^j) \otimes \eta_j + \sum_{j=0}^n c_j \omega^j \otimes (\xi_2 \cdot \eta_j) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^n c_j (j(n-j+1)\omega^{j-1} - \omega^{j+1}) \otimes \eta_j + \\
&\quad + \sum_{j=0}^n c_j \omega^j \otimes (i((n-j+1)\eta_{j-1} + (j+1)\eta_{j+1})) \\
&= \sum_{j=1}^n (j(n-j+1)c_j + ij c_{j-1}) \omega^{j-1} \otimes \eta_j + \\
&\quad + \sum_{j=0}^{n-1} (-c_j + i(n-j)c_{j+1}) \omega^{j+1} \otimes \eta_j = 0, \\
\xi_3 \cdot \psi &= \sum_{j=0}^n c_j (\xi_3 \cdot \omega^j) \otimes \eta_j + \sum_{j=0}^n c_j \omega^j \otimes (\xi_3 \cdot \eta_j) \\
&= \sum_{j=0}^n c_j (i(\omega^{j+1} + j(n-j+1)\omega^{j-1})) \otimes \eta_j + \\
&\quad + \sum_{j=0}^n c_j \omega^j \otimes ((n-j+1)\eta_{j-1} - (j+1)\eta_{j+1}) \\
&= \sum_{j=1}^n (ij(n-j+1)c_j - j c_{j-1}) \omega^{j-1} \otimes \eta_j + \\
&\quad + \sum_{j=0}^{n-1} (ic_j + (n-j)c_{j+1}) \omega^{j+1} \otimes \eta_j = 0.
\end{aligned}$$

□

Remark 6.28. *The spinor ψ in the statement of Theorem 6.27 corresponds to the one in [HS19, §3.4.1], which the authors show to be pure.*

Finally, we give the differential equation satisfied by ψ . Recall that

$$\mathbb{HP}^n = \text{Sp}(n+1) / \text{Sp}(1) \times \text{Sp}(n)$$

is a symmetric space, and that the auxiliary bundle of the $\text{spin}^{\mathbb{H}}$ structure under consideration is isomorphic to the rank-3 bundle spanned by the (locally-defined) endomorphisms Φ_1, Φ_2, Φ_3 . This bundle inherits a connection ∇^E from the Levi-Civita connection, and its Nomizu map vanishes identically when restricted to $\mathfrak{m}$. The following is an immediate consequence of Proposition 5.32:

Corollary 6.29. *The invariant n -twisted $\text{spin}^{\mathbb{H}}$ spinor ψ in Proposition 6.27 is parallel with respect to the invariant connection*

$$\nabla^{g,E} := \nabla^g \otimes (\nabla^E)^{\otimes n}.$$

□

This $\text{spin}^{\mathbb{H}}$ spinor encodes, via [HS19, Cor. 4.12], a well-known geometric fact – see e.g. [Bes08]:

Theorem 6.30. *The metric g_a on $\mathbb{H}\mathbb{P}^n$ is quaternionic Kähler.*

□

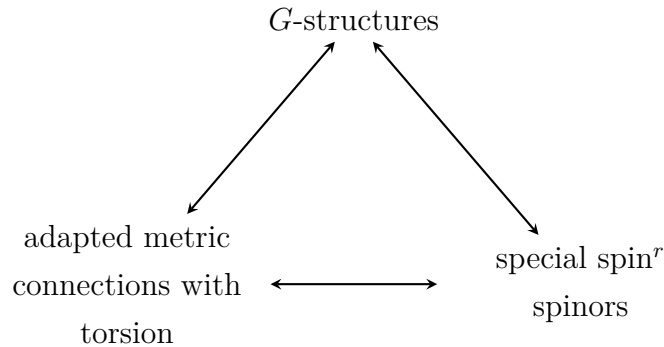
Further lines of research

The role of spin^r spinors in characterising special geometric structures on manifolds is still far from fully understood. To date, the only results in this direction are those in [HS19], which provide a characterisation of special holonomy via special spin^r spinors. These results, however, are restricted to *integrable geometries*, and the corresponding spinors merely satisfy the parallelism condition (1.1).

We now give a list of potential lines of future research that would shed some light on this problem:

1. *Study parallel, Killing and generalised Killing spinors with respect to metric connections with torsion.* In this thesis, we have mainly focused on spinors that satisfy a differential equation involving the Levi-Civita connection – e.g. (2.1) – and the canonical connection of homogeneous spaces.

If the manifold is equipped with additional structure (e.g. a contact metric structure), other metric connections with non-vanishing torsion are somehow *adapted* to the structure, and often turn out to be more suitable – see e.g. [AD20]. An interesting question is the study of parallel, Killing and generalised Killing equations with respect to these connections with torsion. The underlying philosophy is a triangular correspondence between



A natural first step would be to search for such spinors among the invariant ones on homogeneous spaces, using the methods developed in this thesis.

2. *Study generalised Killing spin^r spinors, for $r \geq 3$.* So far, the only result

involving generalised Killing spin^r spinors for general $r \geq 3$ is the decomposition of the Ricci tensor given in [EH16]. Nothing else is known about their geometric significance. A systematic study of these objects would provide new insights into geometric structures with torsion.

3. *Generalise Bär's cone construction to the spin^r setting:* The characterisation of manifolds admitting Killing spinors was achieved by Bär in [Bär93]. His main idea was to use submanifold theory and to consider the Riemannian cone over a manifold with Killing spinors to show that the spin representation of the holonomy of the cone has a fixed point for a suitable scalar renormalisation of the metric on the base following an idea of Bryant [Bry87]. This implies that the cone carries a parallel spinor, and one just has to use Wang's classification result [Wan89] and express the obtained geometric data in terms of the initial manifold. Moroianu later generalised this construction to the setting of $\text{spin}^{\mathbb{C}}$ structures [Mor97], shedding new light on the geometry of Sasaki manifolds.
4. *Study quaternionic Kähler manifolds through their $\text{spin}^{\mathbb{H}}$ spinors.* It was shown in [HS19] that quaternionic Kähler geometry can be characterised by the existence of parallel pure $\text{spin}^{\mathbb{H}}$ spinors. Although twisted Dirac operators have been the main tool in this area, spinors themselves are rarely discussed explicitly in the literature. Investigating the properties of invariant spinors on Wolf spaces – which are parallel, as these spaces are symmetric – could be a tractable first step. Such an analysis may reveal structural features of the class of positive quaternionic Kähler manifolds. While this approach is speculative, it could ultimately contribute to a deeper understanding of this class of manifolds and inform future work on conjectures such as that of LeBrun–Salamon [LS94], though substantial groundwork remains to be done.
5. The maximal flag manifold $\text{MF}(A_2) = \text{SU}(3)/T^2$ is known to have an $\text{SU}(3)$ -invariant nearly-Kähler metric. An invariant spinor in the setting of Theorem 3.7 presumably corresponds to a Killing spinor with respect to the Levi-Civita connection of the nearly-Kähler metric. It would be an interesting problem to characterise other families of invariant spinors of flag manifolds relative to noteworthy Levi-Civita connections.

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Endomorphisms of generalised Killing spinors on three-dimensional Lie groups

Let G be a connected three-dimensional Lie group with Lie algebra $\mathfrak{g}$.

$$\mathfrak{g} = \text{span}_{\mathbb{R}}\{f_1, f_2, f_3\}.$$

A left-invariant metric on G corresponds to an inner product B on $\mathfrak{g}$. By applying the Gram-Schmidt process to the oriented basis (f_1, f_2, f_3) , we obtain an oriented B -orthonormal basis (e_1, e_2, e_3) . More specifically, there exists a matrix

$$P = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \varepsilon & \zeta \\ 0 & 0 & \iota \end{pmatrix} \in \text{GL}^+(3, \mathbb{R})$$

such that

$$e_1 = \alpha f_1, \quad e_2 = \beta f_1 + \varepsilon f_2, \quad \text{and} \quad e_3 = \gamma f_1 + \zeta f_2 + \iota f_3.$$

We saw in the proof of Lemma 4.5 that there exists $A \in \text{End}(\mathfrak{g})$ such that, for all invariant spinors $\phi \in \Sigma$ and all $X \in \mathfrak{g}$,

$$\tilde{\Lambda}(X) \cdot \phi = A(X) \cdot \phi.$$

For each three-dimensional Lie algebra in Bianchi's list [Bia01], we give an explicit expression of the matrix of A in the basis (e_1, e_2, e_3) . These matrices are obtained by following the same procedure we illustrated in Section 4.1. We used Wolfram Mathematica to perform symbolic computations and solve linear systems of equations that depend on parameters. To ensure the reliability of the solutions, we manually verified them to address any potential limitations associated with the use of `Solve`. We use the structure constants given in [Mub63].

A.1 Bian(I)

Bian(I) is the three-dimensional abelian Lie algebra, whose associated matrix A is clearly the zero matrix.

A.2 Bian(II)

The Lie bracket is given by

$$\begin{array}{c|ccc} & f_1 & f_2 & f_3 \\ \hline f_1 & 0 & 0 & 0 \\ f_2 & 0 & 0 & f_1 \\ f_3 & 0 & -f_1 & 0 \end{array}$$

meaning that $[f_1, f_2] = [f_1, f_3] = 0$, $[f_2, f_3] = f_1$, and so on. This is the three-dimensional Heisenberg algebra $\mathfrak{h}_3$. In this case,

$$A = \frac{\det(P)}{4\alpha^2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

As $\det(P) \neq 0$, this matrix always has exactly two distinct eigenvalues.

A.3 Bian(III)

The Lie bracket is given by

$$\begin{array}{c|ccc} & f_1 & f_2 & f_3 \\ \hline f_1 & 0 & f_1 & 0 \\ f_2 & -f_1 & 0 & 0 \\ f_3 & 0 & 0 & 0 \end{array}$$

This is a direct sum of the only non-abelian two-dimensional Lie algebra and the (trivial) one-dimensional one. Then,

$$A = \frac{1}{4\alpha} \begin{pmatrix} \gamma\varepsilon - \beta\zeta & 0 & 0 \\ 2\alpha\zeta & \beta\zeta - \gamma\varepsilon & 0 \\ -2\alpha\varepsilon & 0 & \beta\zeta - \gamma\varepsilon \end{pmatrix},$$

which is not symmetric, as P is non-singular.

A.4 Bian(IV)

The Lie bracket is given by

$$\begin{array}{c|ccc} & f_1 & f_2 & f_3 \\ \hline f_1 & 0 & 0 & f_1 \\ f_2 & 0 & 0 & f_1 + f_2 \\ f_3 & -f_1 & -f_1 - f_2 & 0 \end{array}$$

In this case,

$$A = \frac{1}{4\alpha} \begin{pmatrix} -\iota\varepsilon & -2\alpha\iota & 0 \\ 2\alpha\iota & \iota\varepsilon & 0 \\ 0 & 0 & \iota\varepsilon \end{pmatrix}.$$

The condition $\det(P) \neq 0$ implies that A is not symmetric.

A.5 Bian(V)

We denote by Bian(V) the Lie algebra with Lie bracket given by

	f_1	f_2	f_3
f_1	0	0	f_1
f_2	0	0	f_2
f_3	$-f_1$	$-f_2$	0

The matrix A in this case is given by

$$A = \frac{1}{2} \begin{pmatrix} 0 & -\iota & 0 \\ \iota & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is not symmetric, as $\iota \neq 0$.

A.6 $\text{Bian}(\mathbf{VI}_x)$, $-1 \leq x < 1$, $x \neq 0$

For each $-1 \leq x < 1$, $x \neq 0$, we denote by $\text{Bian}(\mathbf{VI}_x)$ the Lie algebra with Lie bracket given by

	f_1	f_2	f_3
f_1	0	0	f_1
f_2	0	0	xf_2
f_3	$-f_1$	$-xf_2$	0

Note that, for $x = -1$, this is the Poincaré algebra $\mathfrak{p}(1, 1)$, *i.e.*, the Lie algebra of the Poincaré group $P(1, 1)$ –see [Hal03, Section 1.2.5]. For general x ,

$$A = \begin{pmatrix} \frac{(x-1)\beta\iota}{4\alpha} & -\frac{\iota x}{2} & 0 \\ \frac{\iota}{2} & -\frac{(x-1)\beta\iota}{4\alpha} & 0 \\ 0 & 0 & -\frac{(x-1)\beta\iota}{4\alpha} \end{pmatrix}.$$

This matrix is symmetric if and only if $x = -1$. In this case, its eigenvalues are

$$\frac{\beta\iota}{2\alpha}, \quad \pm \frac{\sqrt{\alpha^2\iota^2 + \beta^2\iota^2}}{2\alpha},$$

which are always different.

A.7 Bian(VII_y), $y \geq 0$

For $y \geq 0$, the Lie bracket is given by

	f_1	f_2	f_3
f_1	0	0	$yf_1 - f_2$
f_2	0	0	$f_1 + yf_2$
f_3	$-yf_1 + f_2$	$-f_1 - yf_2$	0

For $y = 0$, this is the Euclidean algebra $\mathfrak{e}(2)$, *i.e.*, the Lie algebra of the Lie group $E(2)$ –see [Hal03, Section 1.2.5]. For general $y \geq 0$,

$$A = \frac{1}{4 \det(P)} \begin{pmatrix} \iota^2(\alpha^2 - \beta^2 - \varepsilon^2) & 2\alpha\iota^2(\beta - \varepsilon y) & 0 \\ 2\alpha\iota^2(\beta + \varepsilon y) & \iota^2(-\alpha^2 + \beta^2 + \varepsilon^2) & 0 \\ 0 & 0 & \iota^2(\alpha^2 + \beta^2 + \varepsilon^2) \end{pmatrix},$$

which is symmetric if and only if $y = 0$. In this case, A has eigenvalues

$$\lambda, \quad \pm \sqrt{\lambda^2 - \frac{1}{4}\iota^2}, \quad \text{where} \quad \lambda = \frac{1}{4 \det(P)} \iota^2 (\alpha^2 + \beta^2 + \varepsilon^2).$$

These eigenvalues are generically different.

A.8 Bian(VIII)

The Lie bracket is given by

	f_1	f_2	f_3
f_1	0	f_1	$2f_2$
f_2	$-f_1$	0	f_3
f_3	$-2f_2$	$-f_3$	0

This is the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. In this case,

$$A = \frac{1}{2 \det(P)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$$\begin{aligned}
a_{11} &= \iota(-\alpha^2\iota + \beta^2\iota + \gamma\varepsilon^2 - \beta\varepsilon\zeta), \\
a_{12} &= \alpha\iota(-2\beta\iota + \varepsilon\zeta), \\
a_{13} &= -\alpha\iota\varepsilon^2, \\
a_{21} &= \alpha\iota(-2\beta\iota + \varepsilon\zeta), \\
a_{22} &= \iota(\alpha^2\iota - \beta^2\iota - \gamma\varepsilon^2 + \beta\varepsilon\zeta), \\
a_{23} &= 0, \\
a_{31} &= -\alpha\iota\varepsilon^2, \\
a_{32} &= 0, \\
a_{33} &= -\iota((\alpha^2 + \beta^2)\iota + \gamma\varepsilon^2 - \beta\varepsilon\zeta).
\end{aligned}$$

This matrix is symmetric, and has generically three distinct eigenvalues.

A.9 Bian(IX)

The Lie bracket is given by

$$\begin{array}{c|ccc}
& f_1 & f_2 & f_3 \\
\hline
f_1 & 0 & f_3 & -f_2 \\
f_2 & -f_3 & 0 & f_1 \\
f_3 & f_2 & -f_1 & 0
\end{array}$$

This is the Lie algebra $\mathfrak{sp}(1) \cong \mathfrak{su}(2) \cong \mathfrak{so}(3)$. In this case,

$$A = \frac{1}{4 \det(P)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$$\begin{aligned}
a_{11} &= \alpha^2 (\iota^2 + \varepsilon^2 + \zeta^2) - \iota^2 (\beta^2 + \varepsilon^2) - (\gamma\varepsilon - \beta\zeta)^2, \\
a_{12} &= 2\alpha (\beta (\iota^2 + \zeta^2) - \gamma\varepsilon\zeta), \\
a_{13} &= 2\alpha\varepsilon(\gamma\varepsilon - \beta\zeta), \\
a_{21} &= 2\alpha (\beta (\iota^2 + \zeta^2) - \gamma\varepsilon\zeta), \\
a_{22} &= -\alpha^2 (\iota^2 - \varepsilon^2 + \zeta^2) + \iota^2 (\beta^2 + \varepsilon^2) + (\gamma\varepsilon - \beta\zeta)^2,
\end{aligned}$$

$$a_{23} = 2\alpha^2\varepsilon\zeta,$$

$$a_{31} = 2\alpha\varepsilon(\gamma\varepsilon - \beta\zeta),$$

$$a_{32} = 2\alpha^2\varepsilon\zeta,$$

$$a_{33} = \alpha^2(\iota^2 - \varepsilon^2 + \zeta^2) + \iota^2(\beta^2 + \varepsilon^2) + (\gamma\varepsilon - \beta\zeta)^2.$$

This matrix is symmetric, and has generically three distinct eigenvalues.

B

Code used to compute the dimension of the space of invariant spinors on flag manifolds

The Wolfram Mathematica code below performs the combinatorial calculation from Chapter 3, enumerating all strong linear combinations of positive roots of F_4 that sum to zero. The same method is used for the other Lie algebras, and it relies on a direct check of all possible strong linear combinations of positive roots.

```
(* Number of strong linear combinations to evaluate *)
m = 2^24;

(* Write down positive roots , from Samelson *)

indexCombinations = Subsets[Range[4] , {2}];

roots1 = IdentityMatrix[4];
roots2 = Table[
  Subtract[IdentityMatrix[4][[indexCombinations[[k, 1]]]] ,
    IdentityMatrix[4][[indexCombinations[[k, 2]]]]] ,
  {k, 1, Length[indexCombinations]}];
roots3 = Table[
```

```

Subtract[IdentityMatrix[4][[indexCombinations[[k, 1]]]],
        -IdentityMatrix[4][[indexCombinations[[k, 2]]]]],
{k, 1, Length[indexCombinations]}}];

(*Define the list of positive roots*)

roots = Join[roots1, roots2, roots3, {
  {1/2, 1/2, 1/2, 1/2},
  {1/2, -1/2, 1/2, 1/2},
  {1/2, 1/2, -1/2, 1/2},
  {1/2, 1/2, 1/2, -1/2},
  {1/2, -1/2, -1/2, 1/2},
  {1/2, -1/2, 1/2, -1/2},
  {1/2, 1/2, -1/2, -1/2},
  {1/2, -1/2, -1/2, -1/2}}];

(* Initialise counter for valid strong linear combinations *)
counter = 0;

(* Generate all possible strong linear combinations
and count how many sum to zero *)
Do[
  coeff = 2*IntegerDigits[i - 1, 2, 24] - 1;
  If[Total[coeff*roots] == ConstantArray[0, 4],
    counter += 1;],
{i, 1, m}];

counter

(* Output: 34432 *)

```