# Convergent and overconvergent $A$-expansions of Drinfeld modular forms

Galen Voysey

*Department of Mathematics*

*Imperial College London*

*London School of Geometry and Number Theory*

October 2021

*A thesis submitted to Imperial College London in partial fulfilment of the requirements of the degree of Doctor of Philosophy.*

## Declaration of Originality

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

## Copyright Declaration

The copyright of this thesis rests with the author. Unless otherwise indicated, its contents are licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International Licence (CC BY NC-SA).

Under this licence, you may copy and redistribute the material in any medium or format. You may also create and distribute modified versions of the work. This is on the condition that; you credit the author, do not use it for commercial purposes and share any derivative works under the same licence.

When reusing or sharing this work, ensure you make the licence terms clear to others by naming the licence and linking to the licence text. Where a work has been adapted, you should indicate that the work has been changed and describe those changes.

Please seek permission from the copyright holder for uses of this work that are not included in this licence or permitted under UK Copyright Law.

**Abstract**

In this thesis we present a novel geometric construction of convergent and overconvergent $\mathfrak{p}$-adic Drinfeld modular forms of level 1 and $\mathbb{Z}_p$-weights. We reinterpret Hattori's construction of convergent $\mathfrak{p}$-adic Drinfeld modular forms using our framework. Following the work of López and Petrov, we define convergent and overconvergent Drinfeld modular forms with $A$-expansions, and prove analogues of some standard theorems in this setting.

**Acknowledgements**

I would like to thank first and foremost my advisor Ambrus Pál for his guidance and support throughout my PhD degree. Their belief in me bolstered my motivation. They were always generous and inspiring with their mathematical ideas.

I have had the pleasure of good mentorship over the years. Besides Ambrus, I would like to thank the other senior mathematicians who gave me some of their time and advice, including Eyal Goren, Payman Kassaei, Kevin Buzzard, and Giovanni Rosso.

My PhD was done at the London School of Geometry and Number Theory (LSGNT). I would like to thank my host institution, Imperial College London. I benefited from the kind assistance of Nicky Townsend, the LSGNT Center Manager. The best part of the LSGNT was the other students, and I am indebted to many of them for valuable conversations, study groups, and friendships. Thank you Andrew, Pol, Ashwin, Lorenzo, Joe, Giulia, Kristina, Ben, Luigi, Jenny, Giada, Gregorio, Gabriel, Angela, Udhav, Daniel, my friends in LMO, and everyone else who made my time as a PhD student brighter. Thank you Jane for hosting me and Josh for driving me all the way to Tuscon.

Outside of math, I would like to thank Sam and Clara for their friendship during my time in London. My support network of friends and family (wherever you are) is deeply appreciated. Thank you Mara for sticking around this whole time and being so supportive. Finally, I would like to thank my cat, Artemis, and apologize to anyone who I may have missed.

During the first three years of my PhD studies I was supported by a scholarship from the National Sciences and Engineering Research Council of Canada (NSERC). I thank this institution for its generosity.

This work was carried out while the author was supported by the Engineering and Physical Sciences Research Council [EP/L015234/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London.

# Contents

**1 Drinfeld modules and Drinfeld modular forms** **12**

1.1 Notation . . . . . 12

1.2 Definition of Drinfeld modules . . . . . 13

**2 A-expansions and $\mathfrak{p}$-adic Drinfeld modular forms** **23**

2.1 $A$-expansions . . . . . 23

2.2 $\mathfrak{p}$-adic modular forms in the sense of Serre . . . . . 29

2.3 $\mathfrak{p}$-adic modular forms in the sense of Katz . . . . . 32

**3 Convergent and overconvergent $\mathfrak{p}$-adic Drinfeld modular forms** **35**

3.1 Weight spaces . . . . . 35

3.2 A construction of $\mathfrak{p}$-adic and overconvergent Drinfeld modular forms . . . . . 44

3.3 Principal $D$-bundles . . . . . 58

3.3.1 Arbitrary weights and étale covers . . . . . 68

**4 Properties and applications of the construction** **71**

# Introduction

The study of modular forms over number fields is a significant area of research in modern number theory. These objects satisfy congruence relations containing deep arithmetic data. A fruitful method of studying these objects and congruences is to use overconvergent $p$-adic modular forms [Cal13]. These were first introduced by Katz [Kat73]. The interested reader is invited to take a look at the references and discussion therein for a description of the history behind the idea. The area is vast and we cannot give everyone due credit, but would like to mention that besides Katz it saw extensive contributions from Hida, Gouvea, and Coleman, among others. Roughly speaking, convergent $p$-adic modular forms are $p$-adic limits of classical modular forms. Overconvergent modular forms also contain the space of classical forms and make up an infinite dimensional sub-Banach space of the convergent $p$-adic modular forms. Although it is not finite dimensional, the space comes with a compact operator, usually denoted $U_p$, and we can study its spectral theory. One can then relate results about overconvergent forms to classical ones using the "control theorem" of Coleman and Kassaei [Col96; Kas06]. There are deep connections between the spectral theory of $U_p$ and $p$-adic families of overconvergent eigenforms [CM98; Buz07].

This "overconvergent" technique has been applicable and useful in a variety of situations, such as in the Hilbert, Seigel, quaternionic, and mixed characteristic cases [AIS14; AIP15; AIP18b; AIP18a]. It has been widely expected that a similar theory should apply in the function field setting, i.e., to Drinfeld modular forms. However, it was only recently that $\mathfrak{p}$-adic Drinfeld modular forms were constructed [Hat21b]. Their construction was in the same spirit as Katz's, but some complications arising in the function field setting had to be overcome. We will give a more extensive discussion of this work in section 2.3.

To explain more about our work we introduce some basic notation of the Drinfeld setting. We will give a more detailed introduction to the area later in section 1. Let $p$ be a prime number, $q$ a power of $p$, and $A = \mathbb{F}_q[T]$, the ring of polynomials with entries in the finite field of $q$ elements. Let $K$ be the fraction field of $A$, which is a discretely valued field when equipped with the negative degree valuation. Let $K_\infty$ be the completion with respect to this valuation. The algebraic closure of $K_\infty$ may not be complete, but the completion of that is both complete and algebraically closed. We denote this field by $C = \mathbb{C}_\infty$. A Drinfeld module over $C$ is, roughly, an $A$-module structure on $\mathbb{A}^1_C$, $\phi : A \to \mathrm{End}(\mathbb{G}_a)$. The base can be globalized from $C$ to more general $\mathbb{F}_q$-schemes. If $\phi$ is a Drinfeld module of rank $r$ over $C$ (definition 1.3), there is an associated $A$-lattice of rank $r$ in $C$. This association is a bijection modulo equivalences (isomorphism and homothety) on both sides. In $C$, there exist $A$-lattices of arbitrary rank, unlike $\mathbb{Z}$-lattices in $\mathbb{C}$ which have rank at most 2. Isomorphism classes of rank 2 Drinfeld modules are also in bijection with the $\mathrm{GL}_2(A)$-orbits of $\Omega$, where $\Omega = \mathbb{P}^1(C) \setminus \mathbb{P}^1(K_\infty)$ is the Drinfeld half-plane. A Drinfeld modular form of level 1 is a holomorphic (in the sense of rigid geometry) function $f : \Omega \to C$ that satisfies a certain transformation property under the action of $\mathrm{GL}_2(A)$ and an auxiliary holomorphy condition "at infinity." A Drinfeld modular form has a power series expansion $f = \sum a_n t^n$ around infinity in $\Omega$. This is unique by analytic continuation. For every prime ideal $\mathfrak{p} \lhd A$, there is a Hecke operator $T_\mathfrak{p}$ acting on Drinfeld modular forms. This is enough to cover the case of an arbitrary ideal in $A$, as, unlike over number fields, we have $T_{\mathfrak{p}\mathfrak{p}'} = T_\mathfrak{p} T_{\mathfrak{p}'}$ for all prime ideals $\mathfrak{p}, \mathfrak{p}' \lhd A$. For the convenience of the reader more familiar with modular forms over number fields, we include the following partial dictionary.

| Number field | Function field |
|---|---|
| $\mathbb{Z}$ | $A = \mathbb{F}_q[T]$ |
| $\mathbb{Q}$ | $K = \mathbb{F}_q(T)$ |
| $\mathbb{R}$ | $K_\infty = \mathbb{F}_q((T^{-1}))$ |
| $\mathbb{C}$ | $C = \widehat{\overline{K_\infty}}$ |
| $\mathbb{H} = \{\mathrm{Im}(z) > 0\}$ | $\Omega = C \setminus K_\infty$ |
| $q = e^{2\pi i z}$ | $t = e_{\bar{\pi}A}(\bar{\pi}z)^{-1}$ |
| elliptic curve | Drinfeld module of rank 2 |
| modular form | Drinfeld modular form |

Brought to life in [Dri74], much of the early development in the theory of Drinfeld modular forms is due to Goss [Gos80a; Gos80b]. Another major contributor to the subject is Gekeler, who in [Gek86] gave a detailed geometric analysis of modular curves parametrizing certain collections of Drinfeld modules.

In [Gek88], Gekeler provides a beautiful exposition and many examples of $t$-expansions and Hecke operators acting thereon. In the last section of ibid., they examine Drinfeld modular forms modulo $\mathfrak{p}$, showing some results analogous to those obtained by Swinnerton-Dyer over number fields [Swi73]. For example, they show that the ring of mod $\mathfrak{p}$ modular forms is the global sections of a nonsingular affine curve, and that if two modular forms are congruent mod $\mathfrak{p}$ then their weights are congruent mod $(q^{\deg(\mathfrak{p})} - 1)$.

Over number fields, $p$-adic modular forms can be presented in two manners. The geometric construction of Katz has as an antecedent an analytic construction due to Serre [Ser73]. Serre's version is built on the theory of modular forms modulo $p$ and properties of $q$-expansions. The order of these developments has been mirrored in the function field case.

Besides the work of Gekeler mentioned above, mod $\mathfrak{p}$ and $\mathfrak{p}$-adic Drinfeld modular forms in the sense of Serre have been studied by Vincent [Vin12], Goss [Gos14], and Papanikolas-Zeng [PZ17], among others.

A geometric construction of $\mathfrak{p}$-adic Drinfeld modular forms, i.e., $\mathfrak{p}$-adic Drinfeld modular forms in the sense of Katz, was recently given by Hattori [Hat21b; Hat20]. To get information on $\mathfrak{p}$-adic families of Drinfeld modular forms (and hence on congruences), we would like to have a theory of $\mathfrak{p}$-adic analytic families as in [Col97] or [Buz07]. In particular, we would like to have a Drinfeld eigencurve, i.e., a rigid geometric object $\mathcal{E}$ over a suitable space of weights $\mathcal{W}$ whose points parametrize systems of Hecke eigenvalues of overconvergent forms. Unfortunately, the function field setting provides ample obstructions. For starters, the analogue of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ is the completed group ring $A_\mathfrak{p}[\![A_\mathfrak{p}^\times]\!]$ (where $A_\mathfrak{p}$ is the completion of $A$ at $\mathfrak{p}$) which is not finitely topologically generated, so the eigenvariety machine of Buzzard breaks down; specifically, the theory of characteristic power series is no longer valid in the non-Noetherian setting [Buz07, §2]. Moreover, the naive analogue of the universal character is no longer locally analytic, which presents significant problems when trying to copy the approach of constructing analytic families of invertible sheaves of different weights as in [AIP18b], or using overconvergent cohomology following Ash and Stevens, for instance. This different behaviour of weights in the function field setting is significant. In contrast to the number field setting, the only locally analytic weights are $\mathbb{Z}_p$-powers [Jeo09]. This makes even defining a "correct" weight space a non-trivial task. We investigate this in section 3.1.

Although there are new barriers in the function field setting, there has been some progress on Drinfeld eigenvarieties and overconvergent forms. Nicole and Rosso describe in [NR18] a

method of adapting the eigenvariety machine to the non-Noetherian setting. In a separate paper [NR20], the authors construct a perfectoid Drinfeld modular curve of infinite level and use this to give an alternative definition of overconvergent Drinfeld modular forms following [CHJ17].

We should mention that while progress on construction and applications of a Drinfeld eigencurve has been restrained so far, there has been rewarding investigation about the spectral theory of $U_{\mathfrak{p}}$ on classical Drinfeld modular forms. Bandini and Valentino have a series of papers examining the action of $U_T$ on cusp forms of level $\Gamma_1(T)$ and $\Gamma_0(T)$ [BV18; BV19a; BV19b]. Their slope calculations prompted them to formulate a conjecture of a similar nature to that of Gouvêa-Mazur [GM92, Conjecture 1] on dimension variation for cuspforms of level $\Gamma_1(T)$. This was proven by Hattori [Hat19]. Hattori has also recently constructed some families of Hecke eigenforms that, while not analytic, have slopes that vary in a $\mathfrak{p}$-adically continuous manner [Hat21a].

We would like to point out another quality specific to the function field setting. Namely, there is a disconnect between the $t$-expansions of Drinfeld modular forms, indexed by $\mathbb{N}$, and the Hecke operators, indexed by monic polynomials in $A$ (which we denote $A_+$). Unlike in the number field setting, where for any normalized eigenform $f = \sum a_n q^n$ the coefficient $a_n$ is precisely the $T_n$-eigenvalue, the coefficients of the $t$-expansion of a Drinfeld modular form have a more opaque arithmetic significance. One method to address this is the theory of $A$-expansions [Lóp10; Lóp11; Pet13]. Taking inspiration from Eisenstein series, it was noticed that some Drinfeld modular forms can be given as elements of $C[\![t]\!]$ indexed by $A_+$. For these forms "with $A$-expansion," the Hecke action is easily computable. This has been used to give novel examples of non-trivial congruences between eigenforms [Pet13, §3.3]. In [Gos14],

Goss used a construction of Petrov to give an example of a $\mathfrak{p}$-adic family of eigenforms with $A$-expansion.

The main goal of this thesis is to place the $\mathfrak{p}$-adic eigenforms of Goss within an analytic overconvergent context. We present a novel geometric construction of convergent and overconvergent $\mathfrak{p}$-adic Drinfeld modular forms of level 1 and $\mathbb{Z}_p$-weights. The restriction to $\mathbb{Z}_p$-weights rather than $\mathbb{S} = \mathbb{Z}/(q^d - 1)\mathbb{Z} \times \mathbb{Z}_p$ as used by Goss is purely for simplicity. We begin in section 1 by reviewing background on Drinfeld modules and modular forms. In section 2 we give more details about $A$-expansions and $\mathfrak{p}$-adic modular forms. We collect relevent results of Petrov, Goss, and Hattori. The body of this thesis is in section 3. In section 3.1 we examine some possible choices of weight space and justify ours. In section 3.2 we give our definition of convergent and overconvergent $\mathfrak{p}$-adic modular forms. Inspired by Pilloni's "fibration en boules" [Pil13], we define $\mathfrak{p}$-adic modular forms of weight $a \in \mathbb{Z}_p$ as bounded holomorphic functions on the open unit disc around 1 in the total space of $\omega$ over the ordinary locus that are homogeneous of weight $a$. Using $D$-torsors, in section 3.3 we reinterpret Hattori's definition in our framework and show the two definitions are compatible. In section 4 we show some basic properties of our construction, such as a $q$-expansion principle. We prove a rigid Vitali-Montel-style result which we believe will be useful to conclude that Goss's eigenforms are overconvergent. We also extend Petrov's results on cuspidal modulo double cuspidal forms to our overconvergent setting.

# 1 Drinfeld modules and Drinfeld modular forms

We start with a basic description of Drinfeld modules and Drinfeld modular forms, focusing on the case over fields. Rank two Drinfeld modules are similar to elliptic curves, and Drinfeld modular forms are defined analogously to the standard construction over $\mathbb{C}$. Similarly to the case of elliptic curves there are distinct but equivalent analytic and algebraic formulations. For further reading we recommend [Gek88] and [Gos80a].

## 1.1 Notation

Let $\mathbb{F}_q$ be the finite field with $q = p^r$ elements for $r \in \mathbb{Z}_{>0}$, $A = \mathbb{F}_q[T]$, and $K = \mathrm{Frac}(A) = \mathbb{F}_q(T)$. We have a valuation $\nu = \nu_\infty : K \to \mathbb{Z} \cup \{\infty\}$ associated to the infinite place/point at infinity in $\mathbb{P}^1_{\mathbb{F}_q}$ (as opposed to a finite prime ideal of $A$),

$$f = g/h \mapsto -\deg(f) = \deg(h) - \deg(g),$$

and associated (normalized) absolute value $|f| = q^{-\nu(f)}$. The valuation ring is

$$\{g/h : \deg(g) \leq \deg(h)\},$$

with maximal ideal $\{\deg(g) < \deg(h)\}$, uniformizer $\pi = T^{-1}$, and units $\{\deg(g) = \deg(h)\}$. Completing with respect to the norm gives us

$$K_\infty := \widehat{K} = \{T^k \sum_{n=0}^{\infty} a_n (T^{-1})^n : a_n \in \mathbb{F}_q, k \in \mathbb{Z}\} = \mathbb{F}_q((T^{-1})) = \mathbb{F}_q((\pi)).$$

This complete nonarchimedian field has valuation ring which we identify with $\mathbb{F}_q[\![\pi]\!]$ and denote by $\mathcal{O}_\infty$.

Let $C := \mathbb{C}_\infty := \widehat{\overline{K}}_\infty$ be the completion of a fixed algebraic closure of $K_\infty$. This is itself complete and algebraically closed, and our valuation/norm extends uniquely to this field. Finally, $\Omega := C \backslash K_\infty$ is the Drinfeld upper half plane. It is an important fact that $\Omega$ has the structure of a smooth geometrically connected rigid analytic subspace of $\mathbb{P}^1$, c.f., [Gos80a, Thm. 1.35].

## 1.2 Definition of Drinfeld modules

Let $\mathbb{G}_a \cong \mathbb{A}^1$ be the additive group scheme. For any commutative ring $k$, we can identify the endomorphisms of $\mathbb{G}_a$, considered as a $k$-group scheme, with polynomials $f \in k[x]$ such that $f(a+b) = f(a) + f(b)$ in $k[a,b]$. We call such polynomails **additive**. If $k$ has characteristic 0 then the only such polynomials are constants. When $\mathrm{char}(k) = p > 0$, let $\tau_p := x^p$. This is clearly an additive polynomial. Write $k\{\tau_p\}$ for the subring of $k[x]$ generated by $\left\{x^{p^i}\right\}_{i \geq 0}$. The ring structure of the endomorphisms has composition of polynomials serve as multiplication. We take multiplication to be composition for the ring $k\{\tau_p\}$. This is non-commutative unless $k = \mathbb{F}_p$, as for any $\alpha \in k \subset k\{\tau_p\}$ we have $\tau_p \cdot \alpha = \tau_p \circ \alpha = (\alpha x)^p = \alpha^p x^p = \alpha^p \cdot \tau_p$. By [Gos98, Prop. 1.1.5] (note the different definition of additive in the reference), we can identify the set of additive polynomials, and hence $\mathrm{End}_k(\mathbb{G}_a)$, with $k\{\tau_p\}$.

Suppose $k$ is an $\mathbb{F}_q$-algebra, where $q = p^r$, set $\tau := \tau_p^r$, and let $k\{\tau\}$ as before with our new $\tau$. It's easy to see that $k\{\tau\}$ is the $\mathbb{F}_q$-algebra of $\mathbb{F}_q$-linear polynomials (i.e. additive *and* elements commute with $\mathbb{F}_q$). This corresponds to looking at endomorphisms of $\mathbb{G}_{a,k}$ an an

$\mathbb{F}_q$-module scheme rather than just a group scheme.

**Definition 1.1.** An $A$**-lattice** in $C$ is a finitely generated, discrete (i.e. its intersection with any ball of finite radius in $C$ is finite) $A$-submodule $L$. Equivalently, $L = A\omega_1 + \cdots + A\omega_r$ where the $\omega_i$'s are $K_\infty$-linearly independent.

We associate to a lattice $L$ the function $e_L : C \to C$ defined by

$$e_L(z) := z \prod\nolimits'_{\lambda \in L} \left(1 - \frac{z}{\lambda}\right), \tag{1}$$

where the prime means we omit the $\lambda = 0$ term. We call $e_L$ the **exponential function associated to the lattice** $L$.

**Proposition 1.2.** *For any lattice $L$, $e_L$ is entire, surjective, $\mathbb{F}_q$-linear, and $L$-periodic. It's zeros, all simple, are precisely the points of $L$.*

*Proof.* Using that $L$ is discrete and [Gos98, Thm. 2.14] shows $e_L$ is entire and non-constant. The theorem we just cited also shows the claim about the zeros of $e_L$. For any $w \in C$, $e_L(z) - w$ is entire and non-constant, so must vanish somewhere by [Gos98, Thm. 2.13]. This shows $e_L$ is surjective. For $\mathbb{F}_q$-linearity see [Gos98, Prop. 4.2.5]. □

Since $e_L$ is entire and $\mathbb{F}_q$-linear, it is given by an everywhere convergent power series

$$e_L(z) = \sum_{k \geq 0} \alpha_k(L) z^{q^k}, \tag{2}$$

where $\alpha_k(L) \in C$ is a coefficient depending on the lattice $L$.

Now, fix an element $a = \sum_{n=0}^{d} a_n T^n \in A$ and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & C & \xrightarrow{e_L} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \vdots{\scriptstyle \phi_a} & & \\ 0 & \longrightarrow & A & \longrightarrow & C & \xrightarrow{e_L} & C & \longrightarrow & 0 \end{array}$$

where the first two vertical maps are multiplication by $a$, the rows are exact, and the map $\phi_a$ is uniquely determined by commutativity. In particular, for all $z \in C$, we have $\phi_a(e_L(z)) = e_L(az)$.

By [Gos98, Thm. 4.3.1] we have an equality of entire functions

$$e_L(az) = a e_L(z) \prod_{0 \neq \alpha \in a^{-1}L/L} \left(1 - \frac{e_L(z)}{e_L(\alpha)}\right),$$

and we see that

$$\phi_a(z) = az \prod_{0 \neq \alpha \in a^{-1}L/L} \left(1 - \frac{z}{e_L(\alpha)}\right)$$

is a $\mathbb{F}_q$-linear polynomial (hence a polynomial in $\tau = z^q$), of degree $d_{L,a} := \mathrm{rank}_A(L) \cdot \deg(a)$ in $\tau$, or equivalently $q^{d_{L,a}}$ in $z$ [Gos98, p. 67].

As shown at the beginning of this section, $\mathrm{End}_{\mathrm{ModSch}}(\mathbb{G}_{a,\mathbb{C}_\infty})$ is the non-commutative ring of polynomials of the form $\sum a_i z^{p^i}$ where multiplication is substitution (in our notation this is $\mathbb{C}_\infty\{\tau_p\}$). Hence, $\phi_a$ is in the sub-ring $\mathbb{C}_\infty\{\tau\}$ (again recall $\tau = z^{p^r} = z^q$), and for any $a \in A$ of degree $d$, we have

$$\phi_a = \sum_{i=0}^{rd} l_i(a, L)\tau^i, \text{ where } l_0 = a \text{ and } l_{rd} \neq 0, \tag{3}$$

and we emphasize the dependence of the coefficients on $L$ and $a$. The map $a \mapsto \phi_a$ is a ring

homomorphism from $A$ to $C\{\tau\}$ [Gos98, Prop. 4.3.2].

**Definition 1.3.** We call any ring homomorphism $\phi : A \to C\{\tau\}$ satisfying eq. (3) for all $a \in A$ a **Drinfeld module of rank** $r$ **(over** $C$**)**.

Such a morphism gives the additive group scheme $\mathbb{G}_{a,C}$ a new $A$-module structure. It is useful to note that since $\phi$ is $\mathbb{F}_q$-linear, it is determined by $\phi_T$.

More generally, we can define a Drinfeld module of rank $r$ over any $\mathbb{F}_q$-scheme $S$. The above definition is equivalent to the following when $S = \mathrm{Spec}(C)$.

**Definition 1.4.** A **(standard) Drinfeld module of rank** $r$ **over an** $\mathbb{F}_q$**-scheme** $S$ is a pair $(\varphi, E)$ where $E$ is a commutative group scheme over $S$, Zariski-locally isomorphic to $\mathbb{G}_{a,S}$, and

$$\varphi : A \to \mathrm{End}_S(E), \quad a \mapsto \varphi_a,$$

is an $\mathbb{F}_q$-algebra homomorphism such that over every affine open $\mathrm{Spec}(B) \subset E$ in some cover where $E$ is locally isomorphic to $\mathbb{G}_{a,S}$ (so $\mathrm{End}(E)$ is locally isomorphic to $B\{\tau\}$) we have $\varphi_a$ is a polynomial in $\tau$ with $\tau$-degree $r\deg(a)$, the top coefficient is a unit, and the constant coefficient is the image of $a$ under the structure map. The map $\varphi$ makes $E$ into an $A$-module scheme.

A **morphism of Drinfeld modules of rank** $r$, $(\varphi, E) \to (\varphi', E')$, is a homomorphism of group schemes over $S$, $E \to E'$, that is equivariant with respect to the actions of $\varphi, \varphi'$. Equivalently, it's a morphism of $A$-module schemes $E \to E'$ where we have given $E, E'$ the induced $A$-module scheme structures from $\varphi, \varphi'$.

Note that although we only defined a standard Drinfeld module here we don't lose anything as any Drinfeld module is isomorphic to a standard Drinfeld module of the same rank.

The above association $L \mapsto \phi^L$ takes $A$-lattices of rank $r$ to Drinfeld modules of rank $r$ over $C$. This is actually a bijection, i.e. given a Drinfeld module $\phi : A \to C\{\tau\}$ we can reconstruct an $A$-lattice $L$ such that $\phi = \phi^L$. Furthermore, two Drinfeld modules are isomorphic if and only if their associated lattices are homothetic [Gek88, (2.6), (2.7)]. We remind the reader of the similarity here with elliptic curves.

**Example 1.5.** Consider the rank 1 Drinfeld module over $C$ given by the equation

$$\rho_T = T\tau^0 + \tau = TX + X^q.$$

This is called the **Carlitz module**. Under the bijection with lattices described above, it corresponds to a lattice $L = \bar{\pi}A$ for some $\bar{\pi} \in C$ unique up to multiplication by an element of $\mathbb{F}_q^\times = \mathrm{Aut}(A)$ (a $(q-1)^{th}$ root of unity). We fix a choice of $\bar{\pi}$ and call this the **Carlitz period**. It can be thought of as analogous to $2\pi i$ in the classical setting.

The exponential function associated to the Carlitz lattice is particularly useful in defining a uniformizer at infinity in the Drinfeld upper half plane, as we will see later. As seen in [Gek88, (4.3)], its everywhere-convergent power series can be computed and we find

$$e_{\bar{\pi}A}(z) = \sum_{i \geq 0} D_i^{-1} z^{p^i}. \tag{4}$$

Here we follow Gekeler's notation by setting

$$[i] := T^{q^i} - T \qquad \text{for } i \geq 0,$$

$$D_i := [i][i-1]^q \cdots [1]^{q^{i-1}} = \prod_{j=1}^{i} [j]^{q^{i-j}} \qquad \text{for } i \geq 1,$$

and $D_0 = 1$. Here the fact that $\rho_T(e_L(z)) = e_L(Tz)$ gives the functional equation

$$\sum_{i \geq 0} D_i^{-1}(Tz)^{p^i} = e_L(Tz) = Te_L(z) + e_L(z)^q.$$

Consider the case of a rank 2 Drinfeld module $\phi = \phi^L$ given by

$$\phi_T(X) = TX + gX^q + \Delta X^{q^2} \tag{5}$$

where $g, \Delta \in C$ and $\Delta \neq 0$. If we define a function $j : \{\text{rank 2 lattices}\} \to C$ by $j(L) = g^{q+1}/\Delta$, we can see it descends to isomorphism classes of rank 2 Drinfeld modules. Indeed, a homothety $L \mapsto \lambda L$ with $\lambda \in C^\times$ changes $g \mapsto \lambda^{1-q} g$ and $\Delta \mapsto \lambda^{1-q^2}\Delta$, hence leaving $j$ invariant. So we can scale $L = A\omega_1 + A\omega_2$ to $Az + A$ where $z = \omega_1/\omega_2 \in \Omega$ as $\omega_1, \omega_2$ must be $K_\infty$-linearly independent. Now we can parametrize homothety classes of lattices by a quotient of the Drinfeld upper half plane $\Omega$. The group $\mathrm{GL}_2(K_\infty)$ acts on $\Omega$ in the usual way, and two elements of $\Omega$ define isomorphic Drinfeld modules if and only if they are equivalent under $\Gamma = \mathrm{GL}_2(A)$. Hence we have

$$j : \Gamma\backslash\Omega = \{\text{rank 2 Drinfeld modules over } C\}_{/\simeq} \xrightarrow{\sim} C, \tag{6}$$

similarly to the classical case. This is both a bijection of sets and an analytic isomorphism $\Gamma\backslash\Omega \cong \mathbb{A}^1_C$ of rigid spaces.

The Drinfeld upper half plane, $\Omega$, has the structure of a rigid-analytic space [Gv80, Chapter $X$]. In particular, $\Omega$ is a connected open admissible subspace of the rigid analytic

space $\mathbb{P}^1_C$ [Gek88, (5.4)]. The weight $k$ Eisenstein series is defined as

$$E_k(z) := \sum_{0 \neq \lambda \in L} \lambda^{-k},$$

where $L$ is the lattice associated to $z \in \Omega$ (as above), is holomorphic in the rigid-analytic sense. It also satisfies $E_k(\gamma z) = (cz+d)^k E_k(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)$. We want $E_k$ to be a weight $k$ modular form of level 1 (i.e. level $\Gamma$), but we need to make sense of "holomorphicity at infinity" first. We follow the explanation in [Gek99, p. 494]. For $z \in C$ we define

$$|z|_i := \inf_{w \in K_\infty} |z - w| = \min_{w \in K_\infty} |z - w|$$

as a replacement for the imaginary part of $z \in \mathbb{H}$ from the classical setting. The greater the value of $|z|_i$, the closer we say $z$ is to infinity, again borrowing from the classical case. In this sense we give a local parameter around infinity (i.e. for $|z|_i$ large). For any $c \in q^{\mathbb{Q}}$ (i.e. $c$ in the value group of $C$) we can define $\Omega_c := \{z \in \Omega : |z|_i \geq c\}$, and this is an open admissible subspace of $\Omega$. Furthermore, $\Omega_c$ is stable under the action of

$$\Gamma_\infty := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{F}_q^\times, b \in A \right\},$$

and satisfies $\Omega_c \cap \gamma(\Omega_c) \neq \emptyset \implies \gamma \in \Gamma_\infty$ if $c > 1$. So for $c > 1$ the canonical map $\Gamma_\infty \backslash \Omega_c \to \Gamma \backslash \Omega$ is injective, and an open immersion of rigid spaces. We want to determine

the quotient $\Gamma_\infty \backslash \Omega_c$. We will first divide out by the group

$$\Gamma_\infty^u := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in A \right\}.$$

Note that the associated quotient map is the restriction of $e_A(z)$ to $\Omega_c$. We factor modulo the action of $\Gamma_\infty / \Gamma_\infty^u = \left\{ \mathrm{diag}(a,d) : a,d \in \mathbb{F}_q^\times \right\}$. We define

$$t(z) := 1/e_{\bar{\pi}A}(\bar{\pi}z) = 1/\bar{\pi}e_A(z)$$

to be the local parameter on $\Gamma_\infty^u \backslash \Omega_c = A \backslash \Omega_c$. Note that the group $\Gamma_\infty / \Gamma_\infty^u$ acts like the group of $(q-1)^{th}$ roots of unity on $t(\Omega_c) \subset C$ as $t(az/d) = d/a \cdot t(z)$. So $t^{q-1} : \Gamma_\infty \backslash \Omega_c \hookrightarrow C$ is an injection, and moreover an open immersion of rigid analytic spaces. By [Gek88, (5.5), (5.6)], we know that $log_q |t(z)|$ depends only on $|z|_i$ when $|z|_i > 1$ and the map $z \mapsto t^{q-1}(z)$ identifies $\Gamma_\infty \backslash \Omega_c$ with a punctured disc $B_r \backslash \{0\}$ for some $r = r(c)$ tending to 0 as $c \to \infty$. So indeed $t$ is a uniformizing parameter at infinity.

**Definition 1.6.** A **Drinfeld modular form** of weight $k \in \mathbb{Z}_{\geq 0}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ is a function $f : \Omega \to C$ such that

1. $f$ is holomorphic,

2. for $\gamma \in \Gamma$, $f(\gamma z) = \frac{(cz+d)^k}{det(\gamma)^m} f(z)$,

3. $f$ is holomorphic at infinity, i.e., if $f = \sum a_n t^n$ is the $t$-expansion of $f$, then $a_n = 0$ for all $n < 0$ (note this always exists because $f(z+b) = f(z)$ for $b \in A$ by 2).

We let $M_{k,m}$ be the $C$-vector space of modular forms of weight $k$ and type $m$. We will mostly work with type 0 modular forms and may drop the subscript corresponding to type.

**Remark 1.7.** (i) $\mathrm{GL}_2(A)$ admits the non-trivial characters $\gamma \mapsto det(\gamma)^m$, explaining the more general transformation rule in (2).

(ii) Applying (2) with $\gamma = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ shows us that $k \equiv 2m \pmod{q-1}$.

(iii) The $t$-expansion $f(z) = \sum a_n t^n(z)$ will only converge for $|t(z)|$ small (i.e. $|z|_i$ large), but $\Omega$ is connected in the rigid analytic sense so the coefficients $a_n$ fully determine $f$.

(iv) Weight 0 modular forms are just constants [Gek88, (5.8)(ii)], [Gv80].

(v) Defining modular forms of different levels works analogously to the classical case.

(vi) The group $\mathrm{GL}_2(A)$ is generated by matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It suffices to check the modularity transformation condition on these.

**Example 1.8.** As hinted at earlier, $E_k(z) = E_k(L_z) = \sum_{a,b \in A}(az+b)^{-k}$ is a modular form of weight $k$ and type 0. The first two conditions are easily shown and holomorphicity at infinity can be proven by computing the $t$-expansion as in [Gek88, (6.3)].

**Example 1.9.** For a rank 2 Drinfeld module and fixed $a \in A$, we can consider the coefficient functions in eq. (3) as functions of $z \in \Omega$. Furthermore, we can express them recursively as

sums of products of Eisenstein series and lower coefficient functions [Gek88, (2.10)]. This is one way of showing that $l_n(a, z) \in M_{q^n-1}$. In particular for $a = T$ (eq. (5)) we have $g \in M_{q-1}$ and $\Delta \in M_{q^2-1}$.

# 2 A-expansions and $\mathfrak{p}$-adic Drinfeld modular forms

As mentioned in the previous section, a Drinfeld modular form has a $t$-expansion, where $t$ is a uniformizer at infinity in $\Omega$. As in the case of Fourier series of classical modular forms over number fields, the coefficients of this expansion are indexed by the non-negative integers. However, in the function field setting, the algebra of Hecke operators is generated by a set indexed by prime ideals of $A = \mathbb{F}_q[T]$ (equivalently by the set of monic polynomials in $A$, which we will denote by $A_+$), rather than those of $\mathbb{Z}$. This creates a disconnect when we look at the action of Hecke operators on $t$-expansions. It was known to Goss and Gekeler (and presumably others) that any weight $k$ Eisenstein series has a convergent series expansion indexed by $A_+$, namely,

$$\bar{\pi}^{-k} E_k(z) = c_0 - \sum_{a \in A_+} G_k(t_a), \tag{7}$$

where $c_0$ is some constant, $\bar{\pi}$ is the Carlitz period and $G_k(t_a)$ is a power series in $t$ that will be defined below. We say that the Eisenstein series has an $A$-expansion. In the work of López and Petrov it was shown that there are many more examples of Drinfeld modular forms with $A$-expansion. In particular, Petrov constructed a family of cuspidal eigenforms with $A$-expansion, and showed that forms with $A$-expansion transform nicely with respect to Hecke operators. We will examine this next in section 2.1.

## 2.1 $A$-expansions

We keep the same notation as the previous section.

**Definition 2.1.** We write $A_+$ for the set of monic polynomials in $A$. We define an **$\mathbb{F}_q$-lattice** $L$ in $C$ as a finitely generated discrete $\mathbb{F}_q$-submodule of $C$, similarly to definition 1.1.

The following exposition of Goss polynomials is taken from [Gek88, §3]. We recall Newton's formulae. Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ be a monic polynomial of degree $n$. Let $\alpha_1, \ldots, \alpha_n$ be the roots of $f$, with multiplicity, and set $S_k := \sum_{i=1}^{n} \alpha_i^k$. Then one can show

$$\begin{aligned} S_k + a_1 S_{k-1} + \cdots + a_{k-1} S_1 + k a_k &= 0 \qquad && \text{for } k \leq n, \\ S_k + a_1 S_{k-1} + \cdots + a_n S_{k-n} &= 0 && \text{for } k > n. \end{aligned}$$

Let $L$ be an $\mathbb{F}_q$-lattice in $\mathbb{C}_\infty$. For now we assume $L$ is finite of dimension $d$, but the following definitions hold for any lattice. Recall that to $L$ we associated a lattice function $e = e_L$ in eq. (1) and eq. (2). Here $e(z)$ is an $\mathbb{F}_q$-linear polynomial of degree $q^d$, so we can expand it as a series in $\tau^i = z^{q^i}$,

$$e(z) = e_L(z) = \sum_{i=0}^{d} \alpha_i z^{q^i}.$$

The coefficients $\alpha_i$ depend on the lattice. We also set

$$t(z) = t_L(z) := \sum_{\lambda \in L} (z - \lambda)^{-1} = e_L^{-1}(z).$$

The polynomial $e(X - z) = e(X) - e(z) \in \mathbb{C}_\infty(z)[X]$ has roots $\{z + \lambda : \lambda \in L\}$, and the polynomial $\tilde{f}(X) := e(X^{-1} - z)X^{q^d}$ has roots $\{(z + \lambda)^{-1} : \lambda \in L\}$. Multiplying this by $-e^{-1}(z)$ gives us a monic polynomial $-e^{-1}(z)\tilde{f}(X) \in \mathbb{C}_\infty(z)[X]$ with the same set of zeros. So we get

$$S_k = S_{k,L} = \sum_{\lambda \in L} (z + \lambda)^{-k},$$

for $k \geq 1$, and set $S_0 = 0$. We have for $k \geq 2$

$$S_k = t(z) \sum_{k-q^i \geq 0} \alpha_i S_{k-q^i}. \tag{8}$$

When $L$ is infinite, we can still define $S_k$ in the same way, and it converges outside of $L$, i.e., it is a meromorphic function with poles at most at $L$. One can prove the following proposition using eq. (8), a couple computations, and taking limits.

**Proposition 2.2** ([Gek88, (3.4)])**.** *Let $L$ be a $\mathbb{F}_q$-lattice in $\mathbb{C}_\infty$. Then for $k \geq 1$, $\exists G_k = G_{k,L}(X)$ a polynomial such that*

1. *$S_k = G_k(t_L)$,*
2. *$G_k(X) = X(G_{k-1} + \alpha_1 G_{k-q} + \cdots + \alpha_i G_{k-q^i} + \cdots)$ for $k \geq 2$ where the sum is taken over $i$ s.t. $k - q^i \geq 0$,*
3. *$G_k$ is monic of degree $k$,*
4. *$G_k(0) = 0$, i.e. $X | G_k(X)$,*
5. *$k \leq q \implies G_k(X) = X^k$,*
6. *$G_{pk} = (G_k)^p$ where $p = \mathrm{char}(\mathbb{F}_q)$,*
7. *$X^2 G_k'(X) = k G_{k+1}(X)$.*

**Definition 2.3.** The polynomial $G_{k,L}(X)$ from proposition 2.2 is called the $k^{th}$ **Goss polynomial of** $L$. We set $G_0 := 0$. Note we always have $G_1(x) = x$ and part 2 of proposition 2.2 gives us a recursive definition of $G_k$ as long as we know the coefficients $\alpha_i$ of the exponential function of our lattice.

Recall from example 1.5 the Carlitz module is the rank 1 Drinfeld module defined by $\rho_T = TX + X^q$, and corresponding to the $A$-lattice $L = \bar{\pi}A$, where $\bar{\pi}$ is the fixed Carlitz period. Also, $\forall a \in A$, $\rho_a(X) = \sum_{i=0}^{\deg(a)} l_i(a)X^{q^i}$ is a polynomial in the variable $X^q$.

**Lemma 2.4.** *The Carlitz module has coefficients in $A$, i.e. $\rho_a(X) \in A\{\tau\}$ for all $a \in A$.*

*Proof.* This follows from knowing $\rho_T(X) = TX + X^q$, recursion, and $\mathbb{F}_q$-linearity. See [Gos98, Cor. 3.3.2]. Note that following this argument one can easily show $l_{\deg(a)}(a)$ is the leading coefficient of $a$. □

**Definition 2.5.** The $a^{th}$ **inverse cyclotomic polynomial**, $f_a(X) \in A[X]$, is defined by

$$f_a(X) := \rho_a(X^{-1})X^{|a|}. \tag{9}$$

Recall that for $a \in A$, $|a| = q^{\deg(a)}$.

Notice that $f_a(X) \in A[X]$ by the above lemma. In fact, $f_a$ is a polynomial in $X^{q-1}$ with leading coefficient $a$ and constant coefficient equal to the leading coefficient of $a$ (i.e., $l_{\deg(a)}(a)$).

**Definition 2.6.** We define

$$t_a(z) := t(az) = \frac{t^{|a|}}{f_a(t)} = \frac{t^{|a|}}{\rho_a(t^{-1})t^{|a|}}. \tag{10}$$

**Proposition 2.7.** *For all $a \in A$, $t_a$ is a power series in $t$ with coefficients in $A$. Furthermore, $t_a \in t^{|a|}A[\![t]\!]$.*

*Proof.* By lemma 2.4 and the above discussion, we know that for all $a = a(T) \in A$, $f_a(t)$ is a polynomial in $t$ with constant coefficient equal to the leading coefficient of $a(T)$. This means

the constant coefficient of $f_a(t)$ is in $\mathbb{F}_q^\times = A^\times$. By standard arguments, this implies that $f_a(t)$ has an inverse power series. The other statement is then clear from the definition. $\square$

**Definition 2.8.** A Drinfeld modular form $f \in M_{k,m}(\mathrm{GL}_2(A))$ has an **$A$-expansion** if $\exists n \geq 0$ and coefficients $c_0(f), c_a(f) \in \mathbb{C}_\infty$ for $a \in A_+$, s.t.

$$f = \sum_{n \geq 0} a_n t^n = c_0(f) + \sum_{a \in A^+} c_a(f) G_n(t_a) \tag{11}$$

is a formal equality in $\mathbb{C}_\infty[\![t]\!]$, and $G_n(X) = G_{n,\bar{\pi}A}(X)$ is the Goss polynomial of the Carlitz lattice as in definition 2.3. We call the integer $n$ the **$A$-exponent of** $f$ and $c_a(f)$ the $a^{th}$ **coefficient** of $f$.

**Proposition 2.9.** *Let $G_k = G_{k,\bar{\pi}A}$ be the $k^{th}$ Goss polynomial of the Carlitz lattice. Then $G_k(X) \in K[X]$, i.e. $G_k$ has coefficients in $K$, for any $k \geq 0$.*

*Proof.* We induct on $k$. If $k \leq q$, this is trivial by proposition 2.2 part 5. Suppose we know the proposition for $n \leq k-1$. By proposition 2.2 part 2 and our induction hypothesis, we can write $G_k$ as a linear combination of polynomials in $K[X]$, with coefficients lying in the set $\{\alpha_i\}_{i \geq 1}$ of coefficients of the Carlitz exponential function. As we mentioned in example 1.5 (see eq. (4), from [Gek88, (4.3)]), we know that $\alpha_i = D_i^{-1} \in K$. This implies that $G_k(X) \in K[X]$. $\square$

**Corollary 2.10.** *Given a modular form $f$ with $A$-expansion, if the coefficients $c_a(f)$ lie in $A$ for all $a \in A$, then the $A$-expansion lies in the ring $A[\![t]\!] \otimes_A K$.*

*Proof.* By definition, $f$ has $A$-exponent $n$ for some $n \in \mathbb{Z}_{\geq 0}$. By proposition 2.9, $G_n$ is a polynomial with coefficients in $K$, so we can find a common denominator, say $\beta \in A$,

such that $\beta G_n(X) \in A[X]$. By proposition 2.2 part 4 and proposition 2.7, this means $G_n(t_a) \in \beta^{-1}A[\![t]\!]$ for all $a \in A$. As $\beta$ doesn't depend on $a \in A$, we can conclude. $\square$

From now on, we will refer to the $k^{th}$ Goss polynomial, $G_k$, to mean the $k^{th}$ Goss polynomial of the Carlitz lattice $\bar{\pi}A$.

Eisenstein series have $A$-expansions, as shown in eq. (7). López showed that the cusp forms $h$ and $\Delta$ also have $A$-expansions.

**Example 2.11** ([Lóp10])**.** Let $\Delta \in S_{q^2-1}$ be the cusp form from example 1.9. It has $A$-expansion

$$\Delta = \sum_{a \in A_+} a^{q(q-1)} G_{q-1}(t_a) = \sum_{a \in A_+} a^{q(q-1)} t_a^{q-1}.$$

Petrov gave an example of a family of cusp forms with $A$-expansions.

**Theorem 2.12** ([Pet13, Thm. 1.3])**.** *Let $k, n \in \mathbb{Z}_{>0}$ s.t. $k - 2n > 0$, $k - 2n \equiv 0 \ mod\ q-1$, and $n \leq p^{\nu_p(k-n)}$ where $\nu_p$ is the p-adic valuation. Then*

$$f_{k,n} := \sum_{a \in A^+} a^{k-n} G_n(t_a) \tag{12}$$

*is in $S_{k,m}(\mathrm{GL}_2(A))$, where $m \equiv n \ mod\ q-1$.*

**Remark 2.13.** Each cusp form $f_{k,n}$ as above is a simultaneous eigenform for the operators $\{T_{\mathfrak{p}}\}$ as $\mathfrak{p}$ varies over all prime ideals of $A$. It's eigensystem is $\{\lambda_{\mathfrak{p}} = \mathfrak{p}^n\}$ where we write $\mathfrak{p}$ also for the unique monic irreducible polynomial generating $\mathfrak{p}$. This can be seen by applying the calculation of Petrov on the action of $T_{\mathfrak{p}}$ on $A$-expansions done in the proof of [Pet13, Thm. 2.3].

## 2.2 $\mathfrak{p}$-adic modular forms in the sense of Serre

**Definition 2.14.** For $\mathfrak{p} \lhd A$ a nonzero prime ideal with monic generator $\wp(T)$ of degree $d$, we write $A_\mathfrak{p}$, $K_\mathfrak{p}$ the completions of $A, K$ at $\mathfrak{p}$. Let $A_{\mathfrak{p},+} \subset A_+$ be the set of monic elements coprime to $\mathfrak{p}$.

For what follows we fix a prime $\mathfrak{p}$. Notice that $A_{\mathfrak{p},+} \subset A_\mathfrak{p}^\times$.

**Definition 2.15.** We define $\mathfrak{p}$-adic weight space as

$$\mathbb{S} = \mathbb{S}_\mathfrak{p} := \varprojlim_n \mathbb{Z}/(q^d-1)p^n\mathbb{Z} \cong \mathbb{Z}/(q^d-1)\mathbb{Z} \times \mathbb{Z}_p. \tag{13}$$

We have a decomposition of the unit group, $A_\mathfrak{p}^\times \cong \mathbb{Z}/(q^d-1)\mathbb{Z} \times (1+\mathfrak{p}A_\mathfrak{p})$. Thus, if $a \in A_{\mathfrak{p},+} \subset A_\mathfrak{p}^\times$, then we may write $a = a_0 a_1$ where $a_0$ is a $(q^d-1)^{st}$-root of unity and $a_1 \equiv 1$ mod $\mathfrak{p}$, and it makes sense to take the $\mathbb{S}$-powers of the element $a$. That is, for $s \in \mathbb{S}$, with $s = (s_0, s_1)$ in the decomposition of eq. (13), we define

$$a^s := a_0^{s_0} a_1^{s_1}. \tag{14}$$

**Proposition 2.16.** *For all $a \in A_\mathfrak{p}^\times$, the function $a^x : \mathbb{S} \to A_\mathfrak{p}^\times$, taking $x \mapsto a^x$ with $a^x$ as defined above, is well-defined and continuous.*

*Proof.* As $a_0$ is a $(q^d-1)^{th}$ root of unity, the choice of $s_0 \in \mathbb{Z}/(q^d-1)\mathbb{Z}$ doesn't matter. As $a_1 \in 1+\mathfrak{p}A_\mathfrak{p}$, we can write, for all $n \in \mathbb{Z}$, $a_1 = 1 + a_1'$ with $a_1' \in \mathfrak{p}A_\mathfrak{p}$, and

$$a_1^n = (1+a_1')^n = \sum_{k=0}^{n} \binom{n}{k} (a_1')^k.$$

It's easy to see exponentiation is $p$-adically continuous on $\mathbb{Z}$, and we can extend it by defining, for any $x \in \mathbb{Z}_p$,

$$a_1^x := \sum_{k=0}^{\infty} \binom{x}{k} (a_1 - 1)^k.$$

Note that for $x \in \mathbb{Z}_p, k \in \mathbb{Z}_{\geq 0}$, the binomial coefficient

$$\binom{x}{k} := \frac{x(x-1)\cdots(n-k+1)}{k!}$$

is well-defined, and continuous as a function of $x$. Using the classical argument of Mahler expansions, we have $p$-adically interpolated exponentiation on $1 + \mathfrak{p}A_{\mathfrak{p}}$. We see the function $a_1^x : \mathbb{Z}_p \to A_{\mathfrak{p}}^{\times}$ is well-defined and continuous. As the first factor in eq. (13) is discrete the function $a^x$ is continuous. □

We record a corollary that will be used in the next section here.

**Corollary 2.17.** *For any $s \in \mathbb{S}$, the exponentiation-by-$s$ map $A_{\mathfrak{p}}^{\times} \to A_{\mathfrak{p}}^{\times} \subseteq \mathbb{C}_{\mathfrak{p}}^{\times}$, $a \mapsto a^s$, is a locally analytic character.*

*Proof.* It suffices to prove this for $s \in \mathbb{Z}_p$. By the proof of proposition 2.16, we know the power series $p(z) = \sum_{k \geq 0} \binom{s}{k} (z-1)^k$ converges and equals $a^s$ when $z = a$. Furthermore, as the binomial coefficients are all bounded by 1, this converges on all of $1 + \mathfrak{p}A_{\mathfrak{p}}$. This shows the function is locally analytic and (since $A_{\mathfrak{p}}$ is complete) has image in $A_{\mathfrak{p}}^{\times}$. As exponentiation by an integer is multiplicative, and $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, continuity implies that $p(z)$ is also multiplicative. Thus, the exponentiation map defines a character. □

**Definition 2.18.** For any power series $f = \sum a_n t^n \in K \otimes A_{\mathfrak{p}}[\![t]\!]$, we set

$$\nu_{\mathfrak{p}}(f) = \inf_{n \geq 0} \{\nu_{\mathfrak{p}}(a_n)\}. \tag{15}$$

We say that $f$ as above is a **$\mathfrak{p}$-adic modular form in the sense of Serre** if $\exists(f_i)$ a sequence of Drinfeld modular forms with $f_i \in M_{k_i,m}(\mathrm{GL}_2(A))$ converging to $f$ in the $\nu_{\mathfrak{p}}$-topology, i.e. $\nu_{\mathfrak{p}}(f - f_i) \to \infty$ as $i \to \infty$. We say $f$ has **weight** $s \in \mathbb{S}$ if $k_i \to s$ in $\mathbb{S}$, and we say $f$ has **type** $m$.

One can show letting the type $m$ vary with the index $i$ doesn't change anything in the definition above, as $m_i$ will eventually be constant (if $f \neq 0$) [PZ17, Lemma 5.7]. We will soon see that due to a result of Hattori the weight of a Serre $\mathfrak{p}$-adic modular form is well-defined (corollary 2.22).

In [PZ17, Prop. 5.12], the authors prove that the Hecke operators $U_{\mathfrak{p}}, V_{\mathfrak{p}}$, and $T_{\mathfrak{l}}$ for $\mathfrak{l} \neq \mathfrak{p}$ preserve the space of $\mathfrak{p}$-adic modular forms (in the sense of Serre) of a fixed weight and type, as well as the subspace of $A_{\mathfrak{p}}$-integral forms.

Using the cusp forms $f_{k,n}$ of theorem 2.12, Goss constructed a family of $\mathfrak{p}$-adic modular forms in the sense of Serre.

**Definition 2.19.** Let $s \in \mathbb{S}$. We define

$$\hat{f}_{s,n} := \sum_{a \in A_{\mathfrak{p},+}} a^s G_n(t_a) \in K \otimes A_{\mathfrak{p}}[\![t]\!]. \tag{16}$$

**Theorem 2.20** ([Gos14, Thm. 2], [PZ17, Thm. 6.3])**.** *Let $n \geq 1$. Let $\mathbb{S}(n)$ be the set of weights $s = (x, y) \in \mathbb{S}$ with $x \equiv n \pmod{q-1}$ and $y \equiv 0 \pmod{q^{\lceil \log_q(n) \rceil}}$. Then for $s \in \mathbb{S}(n)$,*

*we have that $\hat{f}_{s,n}$ is a Serre $\mathfrak{p}$-adic modular form of weight $s+n$ and type $m$ where $m \equiv n$ (mod $q-1$).*

## 2.3 $\mathfrak{p}$-adic modular forms in the sense of Katz

In the setting of classical modular forms, there are two somewhat equivalent notions of $p$-adic modular forms. A beautiful analytic definition using power series was given in [Ser73]. We also have a geometric construction of $p$-adic modular forms of arbitrary weight $\chi \in \mathrm{End}(\mathbb{Z}_p^\times)$ due to Katz. In [Kat73, Thm. 4.5.1], it is shown that this construction agrees with the analytic definition of Serre. In their paper [Hat21b], Hattori defines $\mathfrak{p}$-adic Drinfeld modular forms of arbitrary weight $s \in \mathbb{S}$, in the same geometric spirit as Katz. They also show this agrees with the analytic definition. We will now summarize some key points of this work that we need.

In [Hat21b], the author uses the duality theory of Taguchi to define an analogue of the Hodge-Tate map. They then use this to make the same sort of construction of geometric forms as Katz. This relies on the geometry of the Drinfeld modular curves examined in the separate paper [Hat20]. There, the author conducts a careful study of the formal completion along the cusps of integral models of geometrically connected Drinfeld modular curves. The method is similar to [KM85, (8.11)], but the function field setting provides some added difficulty. In this setting, for any flat $A_{\mathfrak{n}}$-algebra $R_0$ which is an excellent Noetherian domain, we have a map $x_\infty^\Delta : \mathrm{Spec}(R_0[\![x]\!]) \to X_1^\Delta(\mathfrak{n})_{R_0}$. The $R_0$-algebra homomorphism $x \mapsto 0$ gives a point $P_\infty^\Delta \in X_1^\Delta(\mathfrak{n})_{R_0}$ which we call the $\infty$-cusp. A modular form is a section of (an integer tensor power of) an invertible sheaf $\omega$ on $X_1^\Delta(\mathfrak{n})_{R_0}$, and its expansion at the $\infty$-cusp is its image in the completed local ring $\hat{\mathscr{O}}_{X_1^\Delta(\mathfrak{n})_{R_0}, P_\infty^\Delta}$, which is identified with $R_0[\![x]\!]$ in [Hat20, Thm.

5.3(1)]. This agrees with the classical $t$-expansion at infinity in $\Omega$ given in definition 1.6.

**Theorem 2.21** ([Hat21b, Thm. 4.16])**.** *Let $L/K_{\mathfrak{p}}$ be finite, and $\mathcal{O}_L$ the DVR with respect to the unique valuation extending $\nu_{\wp}$. For $i = 1, 2$, let $f_i \in M_{k_i}(\Gamma_1^{\Delta}(\mathfrak{n}))_{\mathcal{O}_L}$. Suppose their $x$-expansions at the $\infty$-cusp satisfy*

$$(f_1)_\infty(x) \equiv (f_2)_\infty(x) \ mod \ \wp^n$$

*and $(f_2)_\infty(x) \not\equiv 0 \ mod \ \wp$. Then we have*

$$k_1 \equiv k_2 \ mod \ (q^d - 1)p^{l_p(n)}, \tag{17}$$

*where $l_p(n) := \min\left\{N \in \mathbb{Z} : p^N \geq n\right\} = \lceil \log_p(n) \rceil$.*

**Corollary 2.22** ([Hat21b, Cor. 4.17])**.** *Let $L/K_{\mathfrak{p}}$ be a finite extension. Let $F_\infty(x)$ be a non-zero element of $\mathcal{O}_L[\![x]\!][1/\wp]$. Suppose $\exists (f_n)_{n\geq 0}$ a sequence with $f_n \in M_{k_n}(\Gamma_1^{\Delta}(\mathfrak{n}))_L$ for some $(k_n)_{n\geq 0} \subset \mathbb{Z}$ and*

$$\lim_{n\to\infty} (f_n)_\infty(x) = F_\infty(x)$$

*with respect to the $\wp$-adic topology on $\mathcal{O}_L[\![x]\!]$. Then $k_n \to \chi \in \mathbb{S}$ as $n \to \infty$. Moreover, the element $\chi$ depends only on $F_\infty(x)$.*

An element $F_\infty(x)$ as in the above corollary is a $\wp$-adic modular form in the sense of Serre, by definition 2.18. This shows that such a form has a well-defined weight which is an element of $\mathbb{S}$. Compare this with [Ser73, Thm. 2, p. 202], which shows $p$-adic modular forms have a well-defined weight using properties of the reduction of the algebra of modular forms.

**Definition 2.23.** A **$\mathfrak{p}$-adic Drinfeld modular form in the sense of Katz** is a $\mathfrak{p}$-adic modular form in the sense of [Hat21b, Def. 4.21] for some weight $s \in \mathbb{S}$. We denote the set of such forms of weight $s$ by $M_s^{\mathfrak{p}-\mathrm{ad}}$.

If $f$ is a $\mathfrak{p}$-adic Drinfeld modular form in the sense of Katz of weight $s \in \mathbb{S}$, then $f$ is a section of an invertible sheaf $\omega^s$ over the formal ordinary locus. For any $n \in \mathbb{Z}_{>0}$, $\exists k_n \in \mathbb{Z}$ sufficiently close to $s$ in $\mathbb{S}$ giving an isomorphism between $\omega^s$ and $\omega^{\otimes k_n}$ over the $\mathfrak{p}^n$-reduction of the modular curve. This gives a series of modular forms $(f_n)_{n\geq 1}$, with $f_n$ of weight $k_n$, and hence a series of expansions at the $\infty$-cusp.

**Definition 2.24.** We call the limit of these in $\mathcal{O}_L[\![x]\!][1/_{\wp}]$ the $t$-expansion of our $\mathfrak{p}$-adic modular form $f$.

**Proposition 2.25** ([Hat21b, Prop. 4.22])**.** *The map taking a $\mathfrak{p}$-adic modular form $f$ of weight $s \in \mathbb{S}$ to its $t$-expansion is injective, and the image is precisely the set of power series that can be written as the $\wp$-adic limit of forms $(f_{k_n})$ of integer weights $k_n \in \mathbb{Z}$ with $k_n \to s$ in $\mathbb{S}$.*

By corollary 2.22 and proposition 2.25 we see that a Serre $\mathfrak{p}$-adic modular form has a well-defined weight. In particular, we know that the family $\hat{f}_{s,n}$, for admissible weights $s$, is a Katz $\mathfrak{p}$-adic Drinfeld modular form.

# 3 Convergent and overconvergent $\mathfrak{p}$-adic Drinfeld modular forms

Consider the specific $\mathfrak{p}$-adic family of modular forms with $A$-expansion given by Goss (definition 2.19 and theorem 2.20). We wish to show this family is overconvergent, giving the first non-trivial example of such a phenomenon in the function field setting. Although a definition of overconvergent Drinfeld modular forms was recently proposed in [NR18], we choose to create our own machinery. This is the main goal of this section. Our definitions will initially only be presented in the special case of level 1 modular forms. However, the constructions can be formulated in complete generality using principal $D$-bundles as shown in section 3.3. Still, we find the ideas in the level 1 setting to be engaging and accessible due to the simpler geometry of the situation. In section 3.1 we discuss weight spaces in preparation for section 3.2 where we detail our main construction.

## 3.1 Weight spaces

In preparation for working with $\mathfrak{p}$-adic and overconvergent Drinfeld modular forms, we must find a weight space to work over. This should be a reasonable space in the sense that it should contain all the classical weights, and have some additional structure so that we can define continuous families of modular forms with varying weights. In the number field setting, there are multiple similar/equivalent definitions. In [CM98, §1.4] it is defined as a rigid analytic space $\mathcal{W}$ given by applying Berthelot's generic fibre functor to the formal scheme $\mathrm{Spf}(\Lambda)$, where $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \cong \mathbb{Z}_p[\Delta][\![T]\!]$ (here $\Delta$ is the finite torsion part of $\mathbb{Z}_p^\times$) is the completed group algebra of the topological group $\mathbb{Z}_p^\times$ with $\mathbb{Z}_p$-coefficients. Geometrically, this is a finite

disjoint union (parametrized by $\Delta$) of open 1-dimensional unit balls. The $\mathbb{C}_p$-points of $\mathcal{W}$ are identified with $\mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$. The integer weights are embedded as the continuous characters $z \mapsto z^k$ for $k \in \mathbb{Z}$.

In contrast to the number field setting, the group $A_\mathfrak{p}^\times$ is not finitely topologically generated, so the completed group ring $A_\mathfrak{p}[\![A_\mathfrak{p}^\times]\!]$ doesn't have the required finiteness properties for us to apply a rigid generic fibre functor. Thus, we must use a different definition of weight space in the function field setting. Recall in definition 2.15 we defined a $\mathfrak{p}$-adic weight space $\mathbb{S}$. At the moment, we believe this is the best definition. In this subsection we examine some alternative possibilities and justify our choice of weight space.

We give our first possible alternative definition of weight space now.

**Definition 3.1.** Let $\mathcal{W}$ be the set of locally analytic characters $\chi : A_\mathfrak{p}^\times \to \mathbb{C}_\mathfrak{p}^\times$. Here, locally analytic means that for every $z \in A_\mathfrak{p}^\times$, there is a non-empty open disc $D = D(z, r_z) \subseteq \mathbb{C}_\mathfrak{p}$ containing $z$ such that $\chi$ is given by a convergent power series on $D$.

We showed in corollary 2.17 that elements of $\mathbb{S}$ give locally analytic characters, so $\mathbb{S} \subseteq \mathcal{W}$. We can show furthermore that this is all of $\mathcal{W}$. We acknowledge that this was proven by Jeong in [Jeo09] using analytic methods. Jeong mentions at the bottom on page 1915 of ibid that this result also follows from the general theory of formal groups.

We now give an alternative proof of this result. We will need a small lemma. First, we recall that if $a \in \mathbb{Z}_\mathfrak{p}$, we can consider the formal power series

$$(1+x)^a := \sum_{n \geq 0} \binom{a}{n} x^n \in \mathbb{Z}_p[\![x]\!].$$

Here $\binom{a}{n}$ denotes the continuous extension of the usual binomial coefficient/choose function

to $\mathbb{Z}_p$. We have, for all $a, b \in \mathbb{Z}_p$, formal equality of power series

$$(1+x)^a(1+x)^b = (1+x)^{a+b},$$
$$((1+x)^a)^b = (1+x)^{ab}.$$

Indeed, for $a, b \in \mathbb{Z}$ this is straightforward to show; for any $a \in \mathbb{Z}_p$ there is a sequence $(a_i) \subset \mathbb{Z}$ converging to $a$ in the $p$-adic topology, and this implies $(1+x)^{a_i}$ converges to $(1+x)^a$ in the $x$-adic topology, which gives the result. With this in mind, it is easy to check that the map

$$a \mapsto (1+x)^a - 1 \tag{18}$$

is a ring homomorphism from $\mathbb{Z}_p$ to the endomorphism ring of the formal group $\hat{\mathbb{G}}_m$.

**Lemma 3.2.** *The map $\mathbb{Z}_p \to \mathrm{End}_k(\hat{\mathbb{G}}_m)$ in eq. (18) is an isomorphism.*

*Proof.* For the sake of this proof we will identify $\hat{\mathbb{G}}_m$ with the multiplicative formal group scheme, and similarly identify the corresponding endomorphism rings. Let $\varphi \in \mathrm{End}_k(\hat{\mathbb{G}}_m)$. Then $\varphi$ induces endomorphisms of the $p^n$-torsion $\hat{\mathbb{G}}_m[p^n] = \mu_{p^n}$ for all $n \geq 1$ compatibly. The Cartier dual of $\mu_{p^n}$ is the constant group scheme $\underline{\mathbb{Z}/p^n\mathbb{Z}}$, and it's endomorphism ring is precisely $\mathbb{Z}/p^n\mathbb{Z}$. Using that the $p^{th}$-power torsion subgroups are schematically dense in $\hat{\mathbb{G}}_m$, we get an injection

$$\mathrm{End}(\hat{\mathbb{G}}_m) \hookrightarrow \varprojlim \mathrm{End}(\mu_{p^n}) \cong \varprojlim \mathrm{End}(\underline{\mathbb{Z}/p^n\mathbb{Z}}) \cong \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p.$$

One can check that this map is the identity on $\mathbb{Z}_p$, so the injection of $\mathbb{Z}_p$ into $\mathrm{End}(\hat{\mathbb{G}}_m)$ must be an isomorphism. □

**Remark 3.3.** For an alternate proof of lemma 3.2, one can use the classification of endomorphism rings of certain formal groups as in [Haz78, Corollary 20.2.14].

**Theorem 3.4.** *We have* $\mathcal{W} = \mathbb{S}$.

*Proof.* We already have $\mathbb{S} \subseteq \mathcal{W}$ by corollary 2.17 as mentioned above. It remains to show $\mathcal{W} \subseteq \mathbb{S}$. Let $\chi : A_{\mathfrak{p}}^{\times} \to \mathbb{C}_{\mathfrak{p}}^{\times}$ be a locally analytic character. It suffices to show that the restriction of $\chi$ to $1 + \mathfrak{p}A_{\mathfrak{p}}$ is given by the map $z \mapsto z^a$ for some $a \in \mathbb{Z}_p$. By definition, there is some disk $D = D(0, \epsilon)$ around 0 such that

$$\chi(1 + x) = p(x) = \sum_{n=1}^{\infty} c_n (z - 1)^n \in \mathbb{C}_{\mathfrak{p}}[\![z - 1]\!]$$

for all $x \in D$. Define $q(z) = p(z) - 1$. Then

$$\begin{aligned} q(x + y + xy) &= p(x + y + xy) - 1 = \chi(1 + x + y + xy) - 1 = \chi((1 + x)(1 + y)) - 1 \\ &= \chi(1 + x)\chi(1 + y) - 1 = p(x)p(y) - 1 = (q(x) + 1)(q(y) + 1) - 1 \\ &= q(x) + q(y) + q(x)q(y). \end{aligned}$$

We see that $q$ gives an element of $\mathrm{End}(\hat{\mathbb{G}}_m)$, so $q(x) = (1 + x)^a - 1$ for some $a \in \mathbb{Z}_p$, so $p(x) = (1 + x)^a$. Without loss of generality we may shrink $D$ so that $1 + \mathfrak{p}A_{\mathfrak{p}}/(1 + D)$ is a group of order $p^r$, so for all $1 + x \in 1 + \mathfrak{p}A_{\mathfrak{p}}$ we have $(1 + x)^{p^r} \in 1 + D$. Thus,

$$\chi((1 + x))^{p^r} = \chi|_{1+D}((1 + x)^{p^r}) = ((1 + x)^{p^r})^a = ((1 + x)^a)^{p^r},$$

and since Frobenius is injective we must have $\chi(1 + x) = (1 + x)^a$ on all of $1 + \mathfrak{p}A_{\mathfrak{p}}$.

□

Next, we examine the $\mathbb{C}_\mathfrak{p}$-Banach algebra of continuous functions $\mathbb{Z}_p \to \mathbb{C}_\mathfrak{p}$, denoted $\mathcal{C} := C(\mathbb{Z}_p, \mathbb{C}_\mathfrak{p})$, equipped with the supremum topology.

**Definition 3.5.** For $f \in \mathbb{C}_\mathfrak{p}$ with $|f-1| < 1$, define the function

$$\exp_f : \mathbb{Z}_p \to \mathbb{C}_\mathfrak{p}, \qquad x \mapsto f^x.$$

Let $\mathcal{E} \subseteq \mathcal{C}$ be the closure of the $\mathbb{C}_\mathfrak{p}$-algebra generated by $\left\{\exp_f : |f-1| < 1, f \in A\right\}$, i.e., the smallest Banach sub-algebra containing all our exponential functions coming from elements of $A$.

**Remark 3.6.** We note that it would have been equivalent to take functions coming from elements $f \in A_\mathfrak{p}$ (so the norm condition implies $f \in 1 + \mathfrak{p}A_\mathfrak{p}$) in the above definition of $\mathcal{E}$. Indeed, $A$ is dense in its completion $A_\mathfrak{p}$, and convergence $f_i \to f$ in the $\mathfrak{p}$-adic topology implies convergence $\exp_{f_i} \to \exp_f$ in the suprepum topology.

Rewording what we just said, and using that $\exp_f \exp_g = \exp_{fg}$, if $\mathcal{E}_0$ is the closure of the $\mathbb{C}_\mathfrak{p}$-span of $\left\{\exp_f : f \in 1 + \mathfrak{p}A_\mathfrak{p}\right\}$, then $\mathcal{E} = \mathcal{E}_0$.

Note that if $\phi : (\mathbb{Z}_p, +) \to (\mathbb{C}_\mathfrak{p}^\times, \cdot)$ is a continuous homomorphism, then as $\mathbb{Z}_p$ is compact $\phi$ is uniformly continuous and hence determined by its values on the dense subset $\mathbb{Z}$. As $\phi$ is a homomorphism, $\phi(z) = f^z$ where $f = \phi(1)$, so $\phi = \exp_f$. With this in mind, we can interpret the following theorem as showing Fourier expansions exist for $\mathbb{C}_\mathfrak{p}$-valued continuous functions on $\mathbb{Z}_p$.

**Theorem 3.7.** *We have $\mathcal{E} = \mathcal{C}$.*

We will need a couple of lemmas.

**Lemma 3.8.** *If $f \in \mathcal{E}$, then $\forall a \in \mathbb{Z}_p$, the function $x \mapsto f(x+a)$ is also in $\mathcal{E}$.*

*Proof.* It is enough to show the same for $f \in \mathcal{E}_0$. If $f \in \mathcal{E}_0$, we can write

$$f(x) = \sum_{g \in I} c_g \exp_g(x),$$

where $I$ is a finite index set and $c_g \in \mathbb{C}_\mathfrak{p}$. For any $a \in \mathbb{Z}_p$,

$$f(x+a) = \sum_{g \in I} c_g \exp_g(x+a) = \sum_{g \in I} c_g \exp_g(a) \exp_g(x),$$

and this is clearly still in $\mathcal{E}_0$. □

**Lemma 3.9.** *If $f \in \mathcal{C}$ is idempotent and $\exists g \in \mathcal{E}$ such that $\|f-g\|_{\sup} < 1$, then $f \in \mathcal{E}$.*

*Proof.* As $f$ is idempotent,

$$\left\|f - g^{p^n}\right\| = \left\|f^{p^n} - g^{p^n}\right\| = \left\|(f-g)^{p^n}\right\| = \|f-g\|^{p^n}.$$

As $\|f-g\| < 1$, we see $g^{p^n} \to f$ as $n \to \infty$. As $\mathcal{E}$ is a closed subalgebra containing $g$, we can conclude $f \in \mathcal{E}$. □

**Lemma 3.10.** *Let $\chi_n : \mathbb{Z}_p \to \mathbb{C}_\mathfrak{p}$ be the characteristic function of the subgroup $p^n\mathbb{Z}_p$. Then $\chi_n \in \mathcal{E}$.*

*Proof.* We prove this by induction on $n$. The base case $n = 0$ is clear as $\chi_0 = \exp_1$ is the constant function 1. Now suppose we know the claim for all natural numbers less than some $n$. Recall $A_\mathfrak{p}$ is of the form $\mathbb{F}_{q^d}[\![\pi]\!]$, where $\pi$ is a uniformizer, so contains the subring $\mathbb{F}_p[\![\pi]\!]$.

For $m \in \mathbb{Z}_{\geq 1}$, define

$$U_m := \{f \in \mathbb{F}_p[\![\pi]\!] : f \equiv 1 \bmod \pi^m \mathbb{F}_p[\![\pi]\!]\}.$$

We know $U_1$ is a subgroup of $\mathbb{F}_p[\![\pi]\!]^\times$, and the $\pi$-adic topology makes it into a pro-$p$ group, isomorphic to infinitely many copies of $\mathbb{Z}_p$ [Iwa86, §2.2]. For all $m \geq 1$, $U_m$ is a subgroup of $U_1$ of finite index $p^{m-1}$. For some $m \geq 2$ we can find a group homomorphism $\phi : \mathbb{Z}_p \to U_1/U_m$ with kernel $p^n\mathbb{Z}_p$.

Pick the minimal such $m$. Then $\phi^{-1}(U_{m-1}/U_m)$ is not contained in the kernel of $\phi$, so must be $p^{n-1}\mathbb{Z}_p$. We see $\phi(p^{n-1}) = 1 + \alpha\pi^{m-1} + O(\pi^m)$ for some $\alpha \in \mathbb{F}_p^\times$. Moreover, $\phi(kp^{n-1}) \equiv 1 + k\alpha\pi^{m-1} \bmod U_m$ for $k \in \{0, 1, \ldots, p-1\}$. Let $f \in U_1$ be a lift of $\phi(1) \in U_1/U_m$. Note that the function $(\exp_f - 1)$ is congruent to $\phi$ modulo $\pi^m$. Define a function

$$u := \pi^{1-m}(\exp_f - 1)\chi_{n-1} \in \mathcal{E}.$$

Let us examine the possible values of $u$. If $x \notin p^{n-1}\mathbb{Z}_p$ then $u(x) = 0$. If $x \in p^n\mathbb{Z}_p$ then by definition of $f$, $\exp_f(x) \in U_m$, so $u(x) \equiv 0 \bmod \pi$. If $x \in p^{n-1}\mathbb{Z}_p \backslash p^n\mathbb{Z}_p$, then $x = p^{n-1}k + x'$ for some $k \in \{1, \ldots, p-1\}$ and $x' \in p^n\mathbb{Z}_p$. In this case

$$u(x) = \pi^{1-m}(f^{kp^{n-1}} f^{x'} - 1) = \pi^{1-m}(k\alpha\pi^{m-1} + O(\pi^m)) = k\alpha + O(\pi),$$

so $u(x) \equiv k\alpha \bmod \pi$. Note that $u$ always lands in $\mathbb{F}_p[\![\pi]\!]$.

Define another function

$$v := -\chi_{n-1} \cdot (1 - \alpha^{-1}u)(2 - \alpha^{-1}u) \cdots ((p-1) - \alpha^{-1}u) = -\chi_{n-1} \prod_{k=1}^{p-1} (k - \alpha^{-1}u) \in \mathcal{E}.$$

To finish the proof, we will show $\|v - \chi_n\| < 1$. As $\chi_n \in \mathcal{C}$ is idempotent and $v \in \mathcal{E}$, we may conclude by applying lemma 3.9. We examine the possible values of $v$ in different cases. If $x \notin p^{n-1}\mathbb{Z}_p$ then $v(x) = 0$. If $x \in p^{n-1}\mathbb{Z}_p \backslash p^n\mathbb{Z}_p$, then $u(x) = k\alpha + O(\pi^m)$ for some $k \in \{1, \ldots, p-1\}$, so $v(x) \equiv 0 \bmod \pi$ (also $\pi^m$). If $x \in p^n\mathbb{Z}_p$, then

$$v(x) = -\prod_{k=1}^{p-1} (k - O(\pi)) = -(p-1)! + O(\pi) = 1 + O(\pi),$$

where we applied Wilson's Theorem to get the final equality. □

We can now prove the theorem.

*Proof of theorem 3.7.* It is enough to show that the characteristic functions of cosets of open subgroups of $\mathbb{Z}_p$ are all in $\mathcal{E}$, as linear combinations of these are dense in $\mathcal{C}$. By lemma 3.10, we know this for characteristic functions of a fundamental system of open neighbourhoods of 0, and by lemma 3.8, we know this for all cosets. □

To finish this section, we examine another possible definition of weight space, and show it does not have our desired properties.

**Definition 3.11.** Let $\mathcal{B} := \mathrm{Hom}_{\mathrm{TopRing}}(A_{\mathfrak{p}}, \mathbb{C}_{\mathfrak{p}})$ be the set of continuous ring homomorphisms from $A_{\mathfrak{p}}$ to $\mathbb{C}_{\mathfrak{p}}$.

It can be easily checked that any element $b \in \mathcal{B}$ gives rise (via restriction) to a continuous character $\chi_b : A_{\mathfrak{p}}^{\times} \to \mathbb{C}_{\mathfrak{p}}^{\times}$. Furthermore, the character $\chi_b$ uniquely determines $b$ (as $A_{\mathfrak{p}}^{\times}$

certainly generates $A_{\mathfrak{p}}$ as a ring). Thus, we have an injection $\mathcal{B} \hookrightarrow \mathcal{W}$, and consider $\mathcal{B}$ as a subset of $\mathcal{W}$. It is interesting to note that $\mathcal{B}$ is bijective to the unit disc in $\mathbb{C}_{\mathfrak{p}}$ under the map $b \mapsto b(T^{-1})$.

**Theorem 3.12.** *We have* $\mathcal{W} \cap \mathcal{B} = \left\{x \mapsto x^{p^k} : k \in \mathbb{N}\right\}$.

*Proof.* Recall for all $a \in \mathbb{Z}_p$ we have the formal power series

$$(1+x)^a = \sum_{k \geq 0} \binom{a}{k} x^k \in \mathbb{F}_p[\![x]\!].$$

Note that it's easy to see the coefficients lie in $\mathbb{F}_p$ as such a series defines a character $\mathbb{A}_{\mathfrak{p}}^{\times} \to \mathbb{C}_{\mathfrak{p}}^{\times}$ - compare the coefficients of the series $(1+x^p)^a = ((1+x)^a)^p$. We showed above that any element of $\mathcal{W}$ is given by such a power series for some $a \in \mathbb{Z}_p$. Suppose this element came from an element of $\mathcal{B}$. Then the power series would be additive as elements of $\mathcal{B}$ are ring homomorphisms. We can write $a = p^n u$ for some $u \in \mathbb{Z}_p^{\times}$ and $n \in \mathbb{Z}_{\geq 0}$, so

$$(1+x)^a = (1+x)^{p^n u} = (1+x^{p^n})^u.$$

It's easy to check this implies that $(1+x)^u$ is also additive. As we discussed at the start of section 1.2, this means that $(1+x)^u = \sum_{k \geq 0} c_k x^{p^k} \in \mathbb{F}_p[\![x]\!]$. Differentiating both sides gives

$$u(1+x)^{u-1} = c_0.$$

By comparing coefficients we immediately see that $u = c_0$ and $(1+x)^{u-1}$ is the constant power series 1, which is only true if $u = 1$. Thus, we have shown $a = p^n$, which means that everything in $\mathcal{W}$ coming from $\mathcal{B}$ is just a $p^{th}$-power exponentiation, as claimed in the

theorem. □

The above theorem shows that $\mathcal{B}$ does not even contain all the classical weights, so is a poor choice for our weight space.

## 3.2 A construction of $\mathfrak{p}$-adic and overconvergent Drinfeld modular forms

We would like to have spaces of modular forms, overconvergent modular forms, and $\mathfrak{p}$-adic modular forms, with Hecke-equivariant inclusions,

$$M \subset M^\dagger \subset M^{\mathfrak{p}-ad}.$$

We will use $\mathbb{S}$ as our $\mathfrak{p}$-adic weight space, as justified in the previous subsection. The overconvergent and $\mathfrak{p}$-adic spaces should contain the weight $s$ subspaces for all weights $s \in \mathbb{S}$. We will give a geometric construction of these spaces, taking inspiration from [Pil13]. More specifically, using tubes and $D$-structures we give a naive construction of Pilloni's "fibrations en boules." We get a Banach space of bounded functions on a rigid variety that contains the weight $a$ $\mathfrak{p}$-adic modular forms as a closed subspace for any $a \in \mathbb{Z}_p$. An extension of this does the same for overconvergent forms. We will focus on $\mathbb{Z}_p$-weights in this thesis, but explain in section 3.3.1 how to extend this to all of $\mathbb{S}$.

Throughout this section, and the rest of the thesis, we may write the same symbol for a scheme/sheaf and its analytification/reduction/rigidification, but will try to be explicit when we feel its required.

Let $\mathcal{O}_K := A_\mathfrak{p}$, $K = K_\mathfrak{p}$ its fraction field, and $\mathbf{k} = \mathbb{F}_\mathfrak{p} := \mathbb{F}_{q^d} = \mathcal{O}_K/\mathfrak{m}$ its residue field.

Because we are primarily interested in Drinfeld modular forms for the full modular group $\mathrm{GL}_2(A)$, we will let $X$ be the compactified modular curve of level 1 for rank 2 Drinfeld modules, defined as a scheme over $\mathcal{O}_K$. We note that the moduli problem for level 1 is not representable, but we may still put ourselves in this situation using descent. We can identify $X$, as a scheme, with $\mathbb{P}^1_{\mathcal{O}_K}$. In particular, there is an invertible sheaf $\omega$ on $X$ such that, for $k \in \mathbb{Z}_{\geq 0}$, the global sections of $\omega^{\otimes -k}$ are precisely the weight $k$ Drinfeld modular forms over $\mathcal{O}_K$. We write $X_{\mathbf{k}}$ for the special fibre, and $X_K$ for the generic fibre. Let $\overline{U}_{\mathfrak{p}} \subset X_{\mathbf{k}}$ be the ordinary locus of the reduction. For our construction we crucially use the geometry of $\mathbb{P}^1$. In this setting, since there is a $\mathbf{k}$-rational point in the complement (a supersingular point, by lemma 3.28), $\overline{U}_{\mathfrak{p}}$ has trivial Picard group. Because we will use this again we phrase it as a lemma.

**Lemma 3.13.** *Let $K$ be a field and $U \subset \mathbb{P}^1_K$ a non-empty open subscheme such that there is a $K$-rational point in the compliment. Then* $\mathrm{Pic}(U)$ *is trivial.*

*Proof.* See [Vak17, (14.2.8.1)], for instance. □

Note that $K$ is a non-archimedian local field, and recall we have an analytification functor, $X \mapsto X^{\mathrm{an}}$, from locally finite type $K$-schemes to rigid $K$-analytic spaces [Bos14, §5.4], [Con99, §5]. Recall we also have a rigid generic fibre functor, $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{rig}}$, from the category of locally Noetherian adic formal schemes over $\mathrm{Spf}(\mathcal{O}_K)$ whose reduction is locally finite type over $\mathrm{Spec}(\mathbf{k})$ to rigid spaces over $K$ [Jon95, §7].

**Definition 3.14.** For a locally free sheaf $\mathscr{F}$ of finite rank on a scheme $S$, we consider the relative spectrum of the symmetric algebra of its dual, denoted by $\mathbb{V}_*(\mathscr{F}) := \underline{\mathrm{Spec}}_S(\mathrm{Sym}^\bullet(\mathscr{F}^\vee))$, which we call the **total space of** $\mathscr{F}$. It is equipped with a canonical map $\pi : \mathbb{V}_*(\mathscr{F}) \to S$

and is unique up to unique isomorphism. We use the same notation when working with rigid spaces.

For example, if we let $\mathscr{F}$ be the structure sheaf, then $\mathbb{V}_*(\mathscr{O}_S) \cong \mathbb{G}_{a,S} = \mathbb{A}^1_S = \mathbb{A}^1 \times S$ and the structure map $\pi$ is the projection to $S$. This construction is functorial.

**Definition 3.15.** Let $\mathcal{X}$ be a rigid analytic variety over $K$, $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$, and $s \in H^0(\mathcal{X}, \mathcal{L})^\times$ be a nowhere vanishing section. Let $D := \{z \in \mathbb{A}^1_K : |1-z|_K < 1\} \subset \mathbb{A}^{1,\mathrm{an}}_K$ (where $z$ is the coordinate on $\mathbb{A}^1$). It's easy to see $D$ is an admissible open subset of the affine line. Next, let $\mathcal{D} := D \times_K \mathcal{X}$, considered as an open subspace of $\mathbb{V}_*(\mathscr{O}_\mathcal{X}) \cong \mathbb{A}^{1,\mathrm{an}}_K \times \mathcal{X}$. There is a unique isomorphism of $\mathscr{O}_\mathcal{X}$-modules, $\mathcal{L} \xrightarrow{\sim} \mathscr{O}_\mathcal{X}$ taking $s \mapsto 1$. This induces an isomorphism $i = i_s \colon \mathbb{V}_*(\mathcal{L}) \xrightarrow{\sim} \mathbb{V}_*(\mathscr{O}_\mathcal{X})$. We define the **tube of** $s$ as $T(s) := i_s^{-1}(\mathcal{D})$, an admissible open subspace of the total space of $\mathcal{L}$.

**Remark 3.16.** Consider the rigid space $D = \{z : |1-z| < 1\}$ as above. Note that $D$ is a group object in the category of rigid spaces, respresenting the functor taking an affinoid $K$-space $\mathrm{Sp}R$ to the set $\{r \in R : |1-r| < 1\}$. For all $a \in \mathbb{Z}_p$, we may consider the power series $(1+x)^a = \sum_{k=0}^\infty \binom{a}{k} x^k \in \mathbb{Z}_p[\![x]\!]$. If $w \in D$, then $|w-1| < 1$, and $w = z+1$ for some $|z| < 1$, so $w^a := (1+z)^a$ converges for any $a \in \mathbb{Z}_p$ and $|1-w^a| < 1$. This defines a natural transformation of group functors from $D$ to itself, and thus, a homomorphism of rigid groups $(-)^a : D \to D$, via $w \mapsto w^a$, for all $a \in \mathbb{Z}_p$.

**Definition 3.17.** Let $\mathcal{X}$ be a rigid $K$-space and $\mathcal{L}$ an invertible sheaf on $\mathcal{X}$. For $k \in \mathbb{Z}_{\geq 0}$, let $M_k(\mathcal{L}) = M_k(\mathcal{X}, \mathcal{L})$ be the $K$-vector space of holomorphic functions (i.e., global sections of the structure sheaf) on $\mathbb{V}_*(\mathcal{L})$ whose restriction to each fibre (over $\mathcal{X}$) is given by a homogeneous polynomial of degree $k$. If $s$ is a nowhere zero section of $\mathcal{L}$, we define

$M(\mathcal{L}, T(s)) = M(\mathcal{L}, \mathcal{X}, T(s))$ to be the $K$-Banach space of bounded holomorphic functions on the tube $T(s)$. We will drop the $\mathcal{X}$ from the notation when the space is clear.

**Remark 3.18.** In definition 3.17 the sheaf $\mathcal{L}$ may be defined on a larger space but we are implicitly taking the restriction to $\mathcal{X}$ when we include it in the notation. We will use this to define $\mathfrak{p}$-adic Drinfeld modular forms as bounded sections on a tube over the ordinary locus.

A global section $f$ of a rigid space $\mathcal{X}$ over $K$ has a well-defined norm at every point $x \in \mathcal{X}$. If $x \in \mathcal{X}$ then we may pick an admissible open affinoid $\mathrm{Sp}A$ containing $\mathfrak{m}_x$ a maximal ideal corresponding to $x$. Then the residue field $K(x) := A/\mathfrak{m}_x$ is a finite extension of $K$, and thus the norm on $K$ uniquely extends to a norm on $A/\mathfrak{m}_x$. The restriction of $f$ to $\mathrm{Sp}A$ gives an element of $A$, and we define $|f(x)|$ to be the norm of the image of this in the quotient $A/\mathfrak{m}_x$. This construction is independent of choice of affinoid containing $x$. Thus, it makes sense to ask when a section is bounded on a subset of a rigid space.

**Lemma 3.19.** *If $f \in M_k(\mathcal{L})$, then by definition $f$ is a homogeneous polynomial of degree $k$ on every fibre. Assuming $\mathcal{X}$ is reduced, this is equivalent to $f$ being a homogeneous degree $k$ polynomial locally for the G-topology.*

*Proof.* Working locally we have $f : \mathbb{A}^1 \times \mathrm{Sp}A \to \mathbb{A}^1$, and if $\pi : \mathbb{A}^1 \times \mathrm{Sp}A \to \mathrm{Sp}A$ is the projection map then by assumption $f$ is a homogeneous polynomial of degree $k$ on every fibre of $\pi$. Let $\lambda : \mathbb{A}^1 \times \mathrm{Sp}A \to \mathbb{A}^1$ be the other projection. Let $i : \mathrm{Sp}A \to \mathbb{A}^1 \times \mathrm{Sp}A$ be the unique section of $\pi$ such that $\lambda \circ i$ is the constant function 1. Let $a := f \circ i : \mathrm{Sp}A \to \mathbb{A}^1$. Then $f$ is equal to the function $(a \circ \pi)\lambda^k$ at every point, and assuming $\mathrm{Sp}A$ is reduced we can conclude these two functions are equal. Conversely, if $\exists a : A \to \mathbb{A}^1$ such that $f = (a \circ \pi)\lambda^k$, then it's easy to see that $f$ is fibrewise a homogeneous polynomial of degree $k$. $\square$

**Lemma 3.20.** *There is a natural isomorphism of $K$-vector spaces $M_k(\mathcal{L}) \xrightarrow{\sim} H^0(\mathcal{X}, \mathcal{L}^{\otimes -k})$.*

*Proof.* We may take an open affinoid cover of $\mathcal{X}$ where $\mathcal{L}^\vee$ is locally on some $U = \mathrm{Sp}A$ in the cover generated by an element $m$, i.e., $\mathcal{L}^\vee|_U \cong \mathscr{O}_U \cdot m$ as $\mathscr{O}_U$-modules. If $V$ is another element of the cover and $\mathcal{L}^\vee$ is locally generated by $m'$ over $V$, then the generators are related by an isomorphism taking $m \mapsto \lambda m'$ for some $\lambda \in \mathscr{O}^\times_{U\cap V}$. Let $f \in M_k(\mathcal{L})$. Then by lemma 3.19, $f$ is locally, on the pullback of our cover, a homogeneous polynomial of weight $k$, say $F_U(X)$ over $\pi^{-1}(U)$. We define a map $\varphi : M_k(\mathcal{L}) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes -k})$ taking $f$ to the section locally given by $F_U(m)$, where multiplication is given by tensoring, e.g., $aX^2 \mapsto a(m \otimes m)$. As $F$ is homogeneous of weight $k$, we see that the images agree on overlaps and hence give a well-defined element of $H^0(\mathcal{X}, \mathcal{L}^{\otimes -k})$. Indeed, if $\mathcal{L}^\vee$ is locally generated by $m$ then $\mathcal{L}^{\otimes -k}$ is generated by $m$ tensored with itself $k$ times, and the isomorphism on overlaps takes $m \otimes \cdots \otimes m \mapsto \lambda m' \otimes \cdots \otimes \lambda m' = \lambda^k(m' \otimes \cdots \otimes m')$. An inverse to $\varphi$ is given by taking $a(m \otimes \cdots \otimes m) \mapsto aX^k$. □

In particular, if we take $\mathcal{L}$ to be the analytification of $\omega$ over the modular curve, then $M_k(\omega^{an})$ is isomorphic to the space of weight $k$ Drinfeld modular forms with coefficients in $K$.

**Definition 3.21.** Let $\mathcal{X}, \mathcal{L}, s$ be as in definition 3.17. Let $r \in M(\mathcal{L}, T(s))$. We say $r$ has **weight** $a \in \mathbb{Z}_p$ if

$$r(wx) = w^a r(x) \tag{19}$$

for all $w \in D, x \in \mathbb{V}_*(\mathcal{L})$. Let $M_a(\mathcal{L}, T(s)) \subset M(\mathcal{L}, T(s))$ denote the closed subspace of functions of weight $a$.

**Remark 3.22.** In definition 3.21 we defined bounded weight $a$ sections of the tube $T(s) \subset$

$\mathbb{V}_*\mathcal{L}$ to be those that are $a$-equivariant for the action of $D$. We now explain how $D$ acts. As $\mathbb{V}_*\mathcal{L}$ is a geometric vector bundle over $\mathcal{X}$, it comes with a natural action of $\mathbb{G}_m^{\mathrm{an}}$. This is locally multiplication by an invertible element on the $\mathbb{A}^1$-part of the vector bundle. We get the $D$-action by viewing $D$ as a subgroup of $\mathbb{G}_m^{\mathrm{an}}$. The action on the right hand side of eq. (19) is again the restriction of the action of $\mathbb{G}_m^{\mathrm{an}}$, but here we're acting on $\mathbb{A}^1_K$.

**Proposition 3.23.** *Let $\mathcal{X}$ be a reduced connected affinoid. Then $M_k(\mathcal{L}) = M_k(\mathcal{L}, T(s))$ for all $k \in \mathbb{Z}_{\geq 0}$.*

*Proof.* Let $f \in M_k(\mathcal{L})$. As $\mathcal{X}$ is affinoid it is quasi-compact so we may pick a finite affinoid trivializing cover for $\mathcal{L}$ over $\mathcal{X}$. Let $U = \mathrm{Sp}A$ be an element of this cover. Then over $U$ the total space is $\mathbb{A}^1 \times U$ and $f = F_U(X)$ is a homogeneous degree $k$ polynomial. The action of $D$ multiplies the coordinate $X$ so it's clear that $f(wx) = w^k f(x)$. Using the notation from lemma 3.19, we can write $f = (a \circ \pi)\lambda^k$. As $s$ is invertible, $s$ and $s^{-1}$ both have nonzero norm everywhere on $\mathcal{X}$, and achieve their maximum somewhere by the maximum modulus principle. Thus, to show boundedness, using the isomorphism $i_s : \mathbb{V}_*\mathcal{L} \xrightarrow{\sim} \mathbb{A}^1 \times \mathcal{X}$, we may assume the tube $T(s)$ is $D \times U$ above $U$. As $U$ is affinoid and $a$ is a function on $U$, again by the maximum modulus principle $a$ must achieve it's max on $U$, which is finite. Thus, for any $(x, y) \in D \times U$, we have

$$|f(x,y)| = \left|\lambda^k(x)\right| \cdot |a(y)| = |a(y)| \leq \|a\|_U ,$$

where $\|\cdot\|_U$ is the sup norm on $U$, and we used that $\lambda^k$ maps $D$ to $D$, so has norm 1 for any $x \in D$. As our cover was finite this shows $f$ is bounded on $T(s)$. Thus, the restriction of $f$ to $T(s)$ lies in $M_k(\mathcal{L}, T(s))$.

Conversely, suppose $f \in M_k(\mathcal{L}, T(s))$. As above we may reduce to the case that $f$ is a holomorphic function on $D \times U$. It is enough to show $f$ extends to a function on $\mathbb{A}^1 \times U$ that is given by a homogeneous degree $k$ polynomial. However, by a similar argument to lemma 3.19, we find that $f = (a \circ \pi|_D)(\lambda|_D)^k$ for some holomorphic function $a$ on $U$, where we've restricted the projection maps. Of course, taking the function $g = (a \circ \pi)\lambda^k$, defined on all of $\mathbb{A}^1 \times U$, we see $f$ and $g$ agree on the open $D \times U$, so by analytic continuation $g$ is the unique extension of $f$ to $\mathbb{A}^1 \times U$. Applying lemma 3.19 again shows $g$ is a homogeneous polynomial of degree $k$ on every fibre. □

We would like to use these ideas to define $\mathfrak{p}$-adic modular forms. Here is some motivation. We will take a nowhere vanishing section $s$ over the ordinary locus $\mathcal{X}^{ord}$ inside our analytified modular curve $\mathcal{X}$, and consider $M(\omega, T(s))$. By proposition 3.23, for any $k \in \mathbb{Z}_{\geq 0}$, the weight $k$ part of this is isomorphic to $M_k(\omega)$, and by lemma 3.20, this is isomorphic to $H^0(\mathcal{X}^{ord}, \omega^{\otimes -k})$, which is the natural definition of weight $k$ $\mathfrak{p}$-adic modular forms. In order to ensure our definition doesn't depend on extraneous factors, we first need some lemmas.

**Lemma 3.24.** *Let $\mathfrak{Y}$ be an affine formal scheme, given by completing an affine scheme $Y = \mathrm{Spec}(A)$, with $A$ Noetherian, over $\mathcal{O}_K$ along its special fibre. Let $\mathfrak{L}$ be a coherent sheaf on $\mathfrak{Y}$. Then every section $s \in H^0(\mathfrak{Y}_0, \mathfrak{L}_0)$ has a lift to $\mathfrak{Y}$.*

*Proof.* Note that in this situation the quasicoherent sheaf corresponding to the special fibre is $\mathscr{I} = \tilde{\mathfrak{a}}$ for some ideal $\mathfrak{a} \lhd A$, and

$$\mathscr{O}_{\mathfrak{Y}} = \varprojlim_n \mathscr{O}_Y / \mathscr{I}^{n+1}.$$

Also, $\mathfrak{Y}_n$ is simply the scheme $\mathrm{Spec}(A/\mathfrak{a})$ equipped with the structure sheaf $\mathscr{O}_Y/\mathscr{I}^{n+1}$.

Finally, $\mathfrak{L}_n = \mathfrak{L} \otimes_{\mathscr{O}_\mathfrak{Y}} \mathscr{O}_\mathfrak{Y}/\mathscr{I}^{n+1}$, and $\mathfrak{L} = \varprojlim \mathfrak{L}_n$.

To prove the lemma, it is enough to show, $\forall n \geq 0$, that we can lift a section $s_n \in H^0(\mathfrak{Y}_n, \mathfrak{L}_n)$ to $s_{n+1} \in H^0(\mathfrak{Y}_{n+1}, \mathfrak{L}_{n+1})$. Then, using [Gro61, 4.1.7], the projective system $(s_0 = s, s_1, s_2, \dots)$ corresponds uniquely to an element of $H^0(\mathfrak{Y}, \mathfrak{L})$ which is a lift of $s$. The transition maps in the limit giving the structure sheaf are precisely the canonical surjections $\mathscr{O}_Y/\mathscr{I}^{n+2} \twoheadrightarrow \mathscr{O}_Y/\mathscr{I}^{n+1}$. By taking the tensor project with $\mathfrak{L}$, we get surjective maps $\mathfrak{L}_{n+1} \twoheadrightarrow \mathfrak{L}_n$. The global sections functor is exact on affine schemes, so we get a surjection $\varphi_{n+1} : H^0(\mathfrak{Y}_{n+1}, \mathfrak{L}_{n+1}) \twoheadrightarrow H^0(\mathfrak{Y}_{n+1}, \mathfrak{L}_n)$, but the target is just equal to $H^0(\mathfrak{Y}_n, \mathfrak{L}_n)$, as $\mathfrak{L}_n$ is the same when considered as a $\mathscr{O}_Y/\mathscr{I}^{n+2}$-module or a $\mathscr{O}_Y/\mathscr{I}^{n+1}$-module. Using this identification, given an $s_n \in H^0(\mathfrak{Y}_n, \mathfrak{L}_n)$, we can define $s_{n+1} = \varphi_{n+1}^{-1}(s_n)$, which gives us the lift we want. □

**Lemma 3.25.** *Let $\mathfrak{Y}, \mathfrak{L}$ be as in lemma 3.24, with the additional condition that $\mathfrak{L}$ is invertible. Let $\bar{s} \in H^0(\mathfrak{Y}_0, \mathfrak{L}_0)^\times$ be a nowhere zero section. Let $s \in H^0(\mathfrak{Y}, \mathfrak{L})$ be a lift. Then $s$ is also nowhere zero. Furthermore, if $\mathcal{Y}$ is the rigid generic fibre of $\mathfrak{Y}$ with sheaf $\mathcal{L}$ coming from $\mathfrak{L}$, and we also use $s$ to denote the corresponding section of $\mathcal{L}$, then the tube $T(s)$ is independent of the choice of lift.*

*Proof.* The coherent sheaf $\mathfrak{L}$ corresponds to a finite type $A$-module $M$. The section $\bar{s}$ can be thought of as an element of $M/\mathfrak{m}M$ that is non-zero at all the stalks $(M/\mathfrak{m}M)_\mathfrak{q} \cong M_\mathfrak{q}/\mathfrak{m}_\mathfrak{q}M_\mathfrak{q}$. The lift is an element of $M$ reducing to $\bar{s}$ mod $\mathfrak{m}M$. We know $s$ is non-zero in all $M_\mathfrak{q}/\mathfrak{m}_\mathfrak{q}M_\mathfrak{q}$, so also nonzero in all $M_\mathfrak{q}$, so $s$ is nowhere zero.

Let $s'$ be another lift of $\bar{s}$. Then $s' = s + r$ for some $r \in \mathfrak{m}M$. As $s$ is nowhere zero in $M$, we have $s + r = s(1 + s^{-1}r)$. The rigid generic fibre is $\mathcal{Y} = \mathrm{Sp}(A \otimes_{\mathcal{O}_K} K)$ and the sections $s, s'$ are mapped to their images under the canonical map $M \to M \otimes_{\mathcal{O}_K} K$ taking $m \mapsto m \otimes 1$.

Let $r' = 1 + s^{-1}r$. As $r$ is topologically nilpotent, $|r| < 1$, so $|1 - r'| \leq |s^{-1}|\,|r| < 1$ at all points. We have $s' = sr'$ with $r' \in D$.

To finish we must show $T(s) = T(s')$. We may work locally and assume $\mathbb{V}_*\mathcal{L}$ is the trivial vector bundle over $\mathcal{Y}$. The map $i_s : \mathbb{V}_*\mathcal{L} \to \mathbb{A}^1_K \times \mathcal{Y}$ is given by multiplication by $s^{-1}$ on $\mathbb{A}^1$ and identity on $\mathcal{Y}$. Thus, the fuctorial points of $T(s)$ are pairs $(s^{-1}x, y)$ with $x \in D$, $y \in \mathcal{Y}$. By above, the functorial points of $T(s')$ are $((r')^{-1}s^{-1}x, y)$ with $x, y$ as before. Replacing $x$ with $x' = (r')^{-1}x \in D$ (as $r' \in D$) shows that the points are the same, so we may conclude $T(s) = T(s')$. □

We are now ready to define $\mathfrak{p}$-adic modular forms.

**Definition 3.26.** Let $X$ be the level 1 modular curve over $\mathcal{O}_K$ with line bundle $\omega$ as described as the start of this subsection. Let $U \subset X$ be an affine open subscheme whose intersection with the special fibre is $\bar{U}_\mathfrak{p}$ (defined above as the non-supersingular points of the reduction). Let $\mathfrak{X}^{ord}$ be the completion of $U$ along the closed subscheme $\bar{U}_\mathfrak{p}$, and $\mathcal{X}^{ord} := (\mathfrak{X}^{ord})^{rig} \subset X_K^{an}$ the associated rigid space. We call $\mathcal{X}^{ord}$ the **(analytic) ordinary locus**. By lemma 3.13 and lemma 3.28, the reduction of $\omega$ is trivial on $\bar{U}_\mathfrak{p}$, so we can pick some nonvanishing section $\bar{s} \in \mathrm{H}^0(\bar{U}_\mathfrak{p}, \omega)$. By lemma 3.25, taking any lift, we get a tube $T(\bar{s})$, which is an open subdomain of $\mathcal{X}^{ord}$ and depends only on $\bar{s}$ (hence the notation). We define

$$M^\mathfrak{p} := M(\omega, \mathcal{X}^{ord}, T(\bar{s})) \tag{20}$$

as our **space of $\mathfrak{p}$-adic modular forms**, and $M_a^\mathfrak{p}$ as the subspace of $\mathfrak{p}$-adic modular forms of weight $a \in \mathbb{Z}_p$ as per definition 3.21.

Note that the formal scheme $\mathfrak{X}^{ord}$, so also its generic fibre $\mathcal{X}^{ord}$, is independent of the

particular subscheme $U$. A priori it is not obvious that such an affine subscheme $U$ exists. We now show it does.

**Lemma 3.27.** *Let $R$ be a DVR with perfect residue field $\mathbf{k}$. Let $X$ be a smooth projective curve over $R$ and $X_{\mathbf{k}}$ its special fibre. Let $U \subset X_{\mathbf{k}}$ be a non-empty affine open subscheme. Then there is a non-empty affine open $V \subset X$ whose intersection with the special fibre is $U$ (i.e., $V_{\mathbf{k}} = U$).*

*Proof.* By assumption, the complement of $U$ is non-empty ($X_{\mathbf{k}}$ is projective). We may assume it is a single point. Indeed, if we knew, for all $x \in X$, that there was an open affine $V_x \subset X$ whose intersection with the special fibre was the complement of $x$, then we could take the intersection of these to get the affine open $V$ that we desire as $X$ is separated. Let $K$ be the fraction field of $R$ and $\mathbf{k}'$ the field of definition of $x$. Then $\mathbf{k}'$ is a finite separable (as $\mathbf{k}$ is perfect) extension of $\mathbf{k}$. By [Ser79, III, §5], we get a corresponding finite unramified extension $K'$ of $K$ with valuation ring $R'$. By Hensel's lemma ([Poo17, 3.5.63]), we can lift $x \in X(\mathbf{k}')$ to some $\tilde{x} \in X(R')$. We can view $\tilde{x}$ as a closed point of the generic fibre $X_K$. Let $S$ be the union of the $\mathrm{Gal}(K^{\mathrm{nr}}/K)$-conjugates of $\tilde{x}$ in $X_K$. Call $\bar{S}$ the closure of $S$ in $X$. It is a finite étale scheme over $\mathrm{Spec}R$ whose intersection with $X_{\mathbf{k}}$ is $x$. By [KM85, Lemma 1.2.3], $\bar{S}$ is an effective Cartier divisor in $X/\mathrm{Spec}R$. Thus, the associated line bundle is ample, and gives a closed embedding of $X$ into $\mathbb{P}^N_R$ identifying the complement of $\bar{S}$ with the complement of a hyperplane section, which is affine. □

**Lemma 3.28.** *There is a $\mathbf{k}$-rational supersingular point on the special fibre of the coarse modular curve.*

*Proof.* By definition of the coarse moduli space it suffices to find a supersingular Drinfeld

module of rank 2 defined over $\mathbf{k}$. It's $j$-invariant will give the point. We do this by considering the minimal polynomial of the Frobenius endomorphism of a Drinfeld module. In the following paragraph note that the notation of [Gos98, §4.12] differs from ours.

Suppose $\mathfrak{p} = (f) \lhd A$. By [Gos98, Thm. 4.12.15], there is a bijection between isogeny classes of rank 2 Drinfeld $A$-modules over $\mathbf{k}$ and conjugacy classes of Weil numbers of rank 2. This bijection takes a Drinfeld module to the roots of $m_\phi$, the minimal polynomial of Frobenius. Consider the polynomial $X^2 - f$. It has roots $\pm\sqrt{f}$, which clearly form a conjugacy class of rank 2 Weil numbers. Thus, there exists an isogeny class of rank 2 Drinfeld $A$-modules over $\mathbf{k}$ with minimal polynomial of Frobenius equal to $X^2 - f$. Pick one $\phi$ from this class. We claim that $\phi$ is supersingular. This follows directly from the equivalence between 2 and 3 in [Gos98, Prop. 4.12.17] as $F^2 = f \in A$. □

**Corollary 3.29.** *A subscheme $U$ as in definition 3.26 exists.*

*Proof.* We have an $\mathbf{k}$-rational supersingular point in the special fibre by lemma 3.28, so the complement of that point is affine. Thus, the ordinary locus of the reduction is the complement in this affine of the vanishing locus of the Hasse invariant, which is affine. We may conclude that $U$ as in definition 3.26 exists. □

**Remark 3.30.** One could choose, for example, $\bar{s}$ to be the Hasse invariant. The following theorem will show our space of $\mathfrak{p}$-adic modular forms is actually independent of the choice of section $\bar{s}$.

**Theorem 3.31.** *For all $a \in \mathbb{Z}_p$, the space $M_a^{\mathfrak{p}} = M_a(\omega, \mathcal{X}^{\mathrm{ord}}, T(\bar{s}))$ isomorphic as a Banach space, independent of choice of $\bar{s}$, to the space of weight $a$ $\mathfrak{p}$-adic modular forms defined by Hattori.*

We will prove theorem 3.31 in section 3.3, using the theory of principal $D$-bundles.

The geometric construction of $\mathfrak{p}$-adic Drinfeld modular forms we gave above is easy to extend to overconvergent forms using the notion of strict containment of connected rational subdomains of $\mathbb{P}^1$.

Let $C = \mathbb{C}_\mathfrak{p}$ be the completion of the algebraic closure of $K = \mathrm{Frac}(A_\mathfrak{p})$. Let $D \subset \mathbb{P}^1_C$ be a non-empty connected rational subdomain. We recall from [Fv04, Prop. 2.2.6, Ex. 3.3.5] the structure of $D$. As a set, $D$ is the complement of finitely many pairwise disjoint open discs, which we shall denote by $\delta D$. If $D$ contains $\infty$, then we may write $\partial D = \{D(z_i, r_i)\}_{i=1,\ldots,m}$, where $D(z_i, r_i) = \{z : |z - z_i| < r_i\}$ with $z_i \in C$ and $r_i \in \sqrt{|C^\times|}$. A holomorphic function $f$ on $D$ has a generalized Laurent series expansion

$$f(z) = a_0 + \sum_{i=1}^{m} \sum_{k=1}^{\infty} a_{i,k}(z - z_i)^{-k}.$$

The supremum norm of $f$ on $D$ is given by

$$\|f\|_D = \sup_{k \geq 1, 1 \leq i \leq m} (|a_0|, |a_{i,k}|\, r_i^{-k}).$$

**Definition 3.32.** Let $D_1, D_2$ be connected rational subdomains of $\mathbb{P}^1_C$. We say $D_1$ is **strictly contained** in $D_2$, and write $D_1 \Subset D_2$, if $D_1 \subseteq D_2$ and $\forall B \in \partial D_1,\ B \cap D_2 \neq \emptyset$.

It is well-known to the experts that the ordinary locus $\mathcal{X}^{ord}$ is the complement of open discs of radius 1 centred around lifts of the supersingular points. This can be deduced from [Ber96, 0.2.7, 0.3.5]. Thus, $\mathcal{X}^{ord}$ is a connected rational subdomain.

**Lemma 3.33.** *There is a connected rational subdomain $\mathcal{V}$ strictly containing $\mathcal{X}^{ord}$ and a nowhere zero section $r \in H^0(\mathcal{V}, \omega)$ such that $T(r|_{\mathcal{X}^{ord}}) = T(\bar{s})$.*

*Proof.* Lift $\bar{s}$ to a nowhere zero section $s$ of $\omega$ over $\mathcal{X}^{ord}$. As $\omega$ is trivial over $\mathcal{X}^{ord}$, we identify $s$ with an invertible holomorphic function on $\mathcal{X}^{ord}$. Since $\mathscr{O}(\mathcal{X}^{ord})$ is the completion of the ring of rational functions with poles outside $\mathcal{X}^{ord}$, we may pick a rational function $r$ with poles outside $\mathcal{X}^{ord}$ such that $\|r-s\| < \|s^{-1}\|$, where here we're taking the supremum norm on $\mathcal{X}^{ord}$. This implies

$$\|1-(r/s)\| = \left\|s^{-1}(s-r)\right\| \leq \left\|s^{-1}\right\| \|s-r\| < \left\|s^{-1}\right\|^2 \leq 1,$$

since we know $s^{-1}$ comes from a formal scheme. Thus, $\left|\frac{r}{s}(x)\right| = 1$ for all $x \in \mathcal{X}^{ord}$, so $r$ is invertible on $\mathcal{X}^{ord}$. Furthermore, $\left|1-\frac{r}{s}(x)\right| < 1$ for all $x \in \mathcal{X}^{ord}$, so $T(r|_{\mathcal{X}^{ord}}) = T(s) = T(\bar{s})$. Finally, we explain how to construct $\mathcal{V}$. Let $D = D(a,R)$ be an open disc in the complement of $\mathcal{X}^{ord}$. Then since $r$ is rational, it has finitely many zeros in $D$, say $x_1, \ldots, x_n$. Set $R' = \max_i |a - x_i|$, and note $R' < R$. We can replace $D$ by the open disc $D' = D(a, R'')$ for any $R' < R'' < R$, and $D'$ will still contain $x_1, \ldots, x_n$. Continue this procedure and let $\mathcal{V}$ be the complement of the new discs. Clearly $\mathcal{V}$ is a connected rational subdomain and $r$ doesn't vanish on $\mathcal{V}$.

□

**Definition 3.34.** Let $\mathcal{V}, r$ be as in the previous proposition. For all $a \in \mathbb{Z}_p$, we define the **space of overconvergent $\mathfrak{p}$-adic Drinfeld modular forms of weight** $a$ as the colimit

$$M_a^\dagger := \varinjlim_{\mathcal{W}} M_a(\omega, T(r|_W)), \tag{21}$$

taken over all connected rational subdomains $\mathcal{X}^{ord} \Subset \mathcal{W} \subseteq \mathcal{V}$, with the obvious restriction maps.

The following lemma shows that the above definition is independent of the choice of $\mathcal{V}$.

**Lemma 3.35.** *Let $D_1, D_2 \subseteq \mathbb{P}^1$ be connected rational subdomains such that $D_1$ is strictly contained in $D_2$. Let $\mathcal{L}$ be an invertible sheaf on $\mathbb{P}^1$ and $r_1, r_2 \in H^0(D_2, \mathcal{L})$ nowhere zero sections such that $T(r_1|_{D_1}) = T(r_2|_{D_1})$. Then $\exists D_3 \ni D_1$ a connected rational subdomain such that $D_3 \subseteq D_2$ and $T(r_1|_{D_3}) = T(r_2|_{D_3})$.*

*Proof.* Consider the invertible holomorphic function $s = \frac{r_2}{r_1}$ on $D_2$. Since $T(r_1|_{D_1}) = T(r_2|_{D_1})$, we know $|1 - s(x)| < 1$ for all $x \in D_1$, so since $D_1$ is affinoid $|1 - s|_{D_1} < 1$ by maximum modulus principle. So $|1 - s|_{D_1} = 1 - \epsilon$ for some $\epsilon > 0$. We define

$$D_3 := \{x \in D_2 : |1 - s(x)| \leq 1 - \varepsilon/2\}.$$

This is a rational subdomain of $D_2$, so a connected rational subdomain. By definition $\|1 - s\|_{D_3} \leq 1 - \varepsilon/2 < 1$, so $T(r_1|_{D_3}) = T(r_2|_{D_3})$. It remains to show $D_1 \Subset D_3$. Clearly $D_1 \subseteq D_3$. Let $B \in \partial D_1$ be one of the open discs that form the complement of $D_1$. Assume $B \cap D_3 = \emptyset$. We define

$$D := (\{B' \in \partial D_1 : B' \neq B\} \cup \{C \in \partial D_2 : C \subseteq B\})^c.$$

Clearly $D$ is a connected rational subdomain and $D_1 \subseteq D \subseteq D_2$. We will show $D$ is disconnected, giving a contradiction. We have $D = D_1 \cup D_1'$ where

$$D_1' := \left\{x \in D : |1 - s(x)| \geq 1 - \frac{2\varepsilon}{3}\right\}.$$

Indeed, if $x \in D$ and $|1 - s(x)| < 1 - \frac{2\varepsilon}{3}$, then $x \in D_3$, but by assumption $D_3 \cap B = \emptyset$

so $x \in D_1$. This cover is admissible as both elements are rational subdomains. Finally, if $x \in D_1 \cap D_1'$ then $|1 - s(x)| \leq 1 - \varepsilon$ and $|1 - s(x)| \geq 1 - \frac{2\varepsilon}{3} > 1 - \varepsilon$, so the intersection is empty. $\square$

## 3.3 Principal $D$-bundles

We review the theory of principal $D$-bundles in the rigid analytic setting and prove some results we will need later to show our construction of $\mathfrak{p}$-adic modular forms agrees with that of Hattori. Roughly, we will prove theorem 3.31 by showing our construction of $\mathfrak{p}$-adic modular forms generalizes that of Hattori once interpreted using $D$-bundles.

**Definition 3.36.** Let $G$ be a rigid analytic group over a nonarchimedean field $F$, i.e., a group object in the category of rigid $F$-analytic spaces. Let $X$ be a rigid $F$-space. A **trivial principal $G$-bundle over** $X$ is a tuple $(P, \pi, \alpha, \phi)$ where $\pi : P \to X$ is a morphism of rigid spaces, $\alpha : P \times G \to P$ is a right $G$-action on $P$, and $\phi : P \xrightarrow{\sim} X \times G$ is an isomorphism of rigid spaces such that the following diagrams commute.

$$\begin{array}{ccc} P & \xrightarrow{\phi} & X \times G \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p_X} \\ X & \xrightarrow{\mathrm{id}_X} & X \end{array} \qquad\qquad \begin{array}{ccc} P \times G & \xrightarrow{\phi \times \mathrm{id}_G} & X \times G \times G \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X \times m} \\ P & \xrightarrow{\phi} & X \times G \end{array} \tag{22}$$

Here $p_X$ is the projection to $X$ and $m : G \times G \to G$ is the multiplication map of $G$.

A **principal $G$-bundle over** $X$ is a triple $(P, \pi, \alpha)$ where $\pi : P \to X$ is a morphism of

rigid spaces, $\alpha : P \times G \to P$ is a right $G$-action on $P$, the following diagram commutes

$$\begin{array}{ccc} P \times G & \xrightarrow{\alpha} & P \\ \downarrow{\scriptstyle \pi \times \mathrm{id}} & & \downarrow{\scriptstyle \pi} \\ X \times G & \xrightarrow{p_X} & X \end{array} \tag{23}$$

and $X$ has an admissible cover $\{X_i\}_{i \in I}$ such that $\pi^{-1}(X_i) \subseteq P$ is a trivial principal $G$-bundle over $X_i$ for all $i \in I$, i.e., $\forall i \in I, \exists \phi_i : \pi^{-1}(X_i) \xrightarrow{\sim} X_i \times G$ an isomorphism such that $(\pi^{-1}(X_i), \pi|_{\pi^{-1}(X_i)}, \alpha|_{\pi^{-1}(X_i)\times G}, \phi_i)$ is a trivial principal $G$-bundle as defined above. We write $\mathrm{Prin}_G(X)$ for the set of isomorphism classes of principal $G$-bundles.

We may write only $P$ instead of $(P, \pi, \alpha)$ when the situation is clear. For simplicity we assume all our rigid groups are commutative. We identify Čech and sheaf cohomology. The set $\mathrm{Prin}_G(X)$ has a distinguished element given by the trivial principal $G$-bundle.

**Definition 3.37.** If $(P, \pi, \alpha)$ is a principal $G$-bundle, the isomorphisms $\phi_i : \pi^{-1}(X_i) \to X_i \times G$ as in definition 3.36 are called **local trivializations**. We will denote $X_{ij} := X_i \cap X_j$. Given such a collection of local trivializations, we define the **transition maps** $\phi_{ij} : X_{ij} \to G$ to be the unique maps such that the diagram

$$\begin{array}{ccc} X_{ij} \times G & \xrightarrow{\mathrm{id}_{X_{ij}} \times m_{ij}} & X_{ij} \times G \\ {\scriptstyle \phi_i|_{\pi^{-1}(X_{ij})}}\uparrow & & \uparrow{\scriptstyle \phi_j|_{\pi^{-1}(X_{ij})}} \\ \pi^{-1}(X_{ij}) & \xrightarrow{\mathrm{id}} & \pi^{-1}(X_{ij}) \end{array}$$

commutes, where $m_{ij}$ is defined as the composition $X_{ij} \times G \xrightarrow{\phi_{ij} \times \mathrm{id}_G} G \times G \xrightarrow{m} G$.

Given a principal $G$-bundle $P$, its transition maps $\{\phi_{ij}\}$ form a Čech 1-cocycle with values in $G$ for the cover $\{X_i\}$, and hence give an element of $\check{H}^1_{\{X_i\}}(X, G)$ that we will call $c(P)$.

**Remark 3.38.** One can show the element $c(P) \in \check{H}^1(X, G)$ from definition 3.37 depends only on the isomorphism class of $P$. We have the standard isomorphism between the first Čech cohomology group and the first sheaf cohomology group. Thus, we get a map of sets $\mathrm{Prin}_G(X) \to H^1(X, G)$, $P \mapsto c(P)$. It's easy to see this map takes the trivial bundle to the identity element of $H^1(X, G)$, so we have a map of pointed sets. One can show that this map is a bijection. Explicitly, any element of $H^1(X, G)$ gives us transition maps and hence a principal $G$-bundle by the standard glueing construction. As $G$ is commutative, $H^1(X, G)$ is an abelian group. We use this to give $\mathrm{Prin}_G(X)$ a group structure. We identify the groups $\mathrm{Prin}_G(X)$ and $H^1(G, X)$ in what follows.

**Definition 3.39.** Let $P_1, P_2$ be principal $G$-bundles. The **tensor product** $P_1 \otimes P_2$ is the principal $G$-bundle corresponding to the cohomology class $c(P_1) \cdot c(P_2)$. It is the principal $G$-bundle whose transition maps are the products of the transition maps of $P_1$ and $P_2$.

If we let $G$ act on the product $P_1 \times_X P_2$ by $(p_1, p_2) \cdot g := (p_1 \cdot g, p_2 \cdot g^{-1})$, then $P_1 \otimes P_2$ is the categorical quotient of $P_1 \times_X P_2$ with respect to this action. The tensor product is the group operation on $\mathrm{Prin}_G(X)$ from remark 3.38.

**Definition 3.40.** Let $(P, \pi, \alpha)$ be a principal $G$-bundle over $X$. Using remark 3.38, this has an inverse. We denote the inverse by $(P^\vee, \pi^\vee, \alpha^\vee)$.

**Definition 3.41.** Let $\phi : G \to H$ be a homomorphism of rigid groups over $F$. Let $X$ be a rigid $F$-space and $\phi_* \colon H^1(X, G) \to H^1(X, H)$ the induced group homomorphism. Let $(P, \pi, \alpha)$ be a principal $G$-bundle over $X$. Consider the (right) action of $G$ on $P \times H$ via $(p, h) \cdot g := (\alpha(p, g), h \cdot \phi(g^{-1}))$. We define $\phi_* P$ to be the categorical quotient of $P \times H$ by this action. Let $\pi' = \pi \circ p_P : P \times H \to X$, where $p_P$ is the projection to $P$. Let $\alpha'$ be the right

action of $H$ on $P \times H$, acting by multiplication on $H$ and trivially on $P$. Then $\pi'$ and $\alpha'$ both factor through the quotient map and we get a candidate principal $H$-bundle $(\phi_* P, \pi', \alpha')$, or $\phi_* P$ for short. If $\{\phi_{ij}\}$ are transition maps for $P$, then the quotient is representable by the rigid space given by glueing the transition maps $\{\phi \circ \phi_{ij}\}$, so $\phi_* P$ is truly a principal $H$-bundle.

**Remark 3.42.** When $G = \mathbb{G}_m$, principal $\mathbb{G}_m$-bundles over $X$ are equivalent to invertible sheaves on $X$, which are equivalent to geometric line bundles over $X$ [Lüt16, Rem. 1.7.2].

**Example 3.43.** If $\mathcal{L}$ is an invertible sheaf on a rigid space $X$, then the total space of $\mathcal{L}$, denoted $\mathbb{V}_* \mathcal{L}$, is a geometric vector bundle of rank 1 over $X$. This has a natural $\mathbb{G}_m$-action. If $0_{\mathcal{L}}$ is the image of the zero section in $\mathbb{V}_* \mathcal{L}$, then $\mathbb{V}_* \mathcal{L} \backslash 0_{\mathcal{L}}$ has the structure of a principal $\mathbb{G}_m$-bundle where the action is the restriction of the natural $\mathbb{G}_m$-action.

**Definition 3.44.** Let $D := \{z \in \mathbb{A}^1_F : |1 - z|_F < 1\} \subset \mathbb{A}^{1,\mathrm{an}}_F$, where $z$ is the coordinate on $\mathbb{A}^1$. This is a rigid group over $F$. Let $\iota : D \to \mathbb{G}_m$ be the natural inclusion (a homomorphism of rigid groups). Let $P$ be a principal $D$-bundle. As $\iota_* P$ is a quotient of $P$ we get a map of rigid spaces $P \to \iota_* P$. We write $\mathcal{L}(P)$ for the invertible $\mathscr{O}_X$-module corresponding to $\iota_* P$ from remark 3.42. Then $\iota_* P$ embeds in the geometric line bundle $\mathbb{V}_* \mathcal{L}(P)$ over $X$. The image of $P$ in this line bundle will be called the **tube corresponding to** $P$.

This is a generalization of our definition of a tube in definition 3.15.

**Definition 3.45.** Let $X$ be a rigid space. For any $a \in \mathbb{Z}_p$ we have a group homomorphism $(-)^a : D \to D$. This induces a group homomorphism $a_* : H^1(X, D) \to H^1(X, D)$. When $P \in H^1(X, D)$ is a principal $D$-bundle over $X$, we define $P^{\otimes a} := a_* P$. We define an $\mathscr{O}_X$-module $\mathcal{M}_a P$ on $X$ whose sections over $U \subseteq X$ is the set of holomorphic functions

$f : (\pi^\vee)^{-1}(U) \to C$ such that $\forall z \in (\pi^\vee)^{-1}(U), w \in D(U)$, we have $f(\alpha^\vee(z,w)) = w^a \cdot f(z)$, where the last action is the usual $\mathbb{G}_m$-action on $\mathbb{A}^1$.

**Remark 3.46.** To justify our notation, when $a \in \mathbb{Z}$ in definition 3.45, $P^{\otimes a}$ is really the tensor product of $P$ with itself $a$ times as in definition 3.39.

**Lemma 3.47.** *There is an isomorphism of $\mathscr{O}_X$-modules between $\mathcal{M}_a P$ and $\mathcal{L}(P^{\otimes a})$, $\forall a \in \mathbb{Z}_p$.*

*Proof.* First suppose $P$ is the trivial $D$-bundle. Consider $f \in \mathcal{M}_a P(X)$. By similar arguments to lemma 3.19, we can, locally on $U \subset X$, write $f = (h \circ \pi)\lambda^a$ for some function $h : U \to \mathbb{A}^1$. This is well-defined on overlaps so $h$ glues to a global section over $X$. This defines a morphism of sheaves taking $f$ to this $h$. The inverse map is clearly given by $h \mapsto (h \circ \pi)\lambda^a$. The general case follows from glueing, along the lines of lemma 3.20. □

**Definition 3.48.** Let $K$ be a discretely valued field, $\mathcal{O}_K$ is ring of integers, and $\mathfrak{m} \lhd \mathcal{O}_K$ the maximal ideal. For any $n \in \mathbb{Z}_{\geq 0}$, scheme $X$ over $\mathrm{Spec}(\mathcal{O}_K/\mathfrak{m}^n)$, and non-negative integer $m \leq n$, we write $X_m := X \times_{\mathrm{Spec}(\mathcal{O}_K/\mathfrak{m}^n)} \mathrm{Spec}(\mathcal{O}_K/\mathfrak{m}^m)$. We define

$$D_n := \ker(\varphi : \mathscr{O}_X^\times \to \mathscr{O}_{X_1}^\times), \tag{24}$$

where $\varphi$ is the reduction mod $\mathfrak{m}$ map.

Note that $X_m \hookrightarrow X$ as in definition 3.48 is a thickening, i.e., a closed embedding that is a homeomorphism, so it induces an equivalence of étale sites $(X_m)_{ét} \cong X_{ét}$ [Stacks, 03SI]. We use this to equate étale sheaves on $X$ with those on $X_m$, and consider $D_n$ as a sheaf on $X_{ét}$.

**Lemma 3.49.** *Let $X$ be an affine scheme over* $\mathrm{Spec}(\mathcal{O}_K/\mathfrak{m}^n)$. *Then $H^i(X_{\acute{e}t}, D_n) = 0$ for all $i > 0$.*

*Proof.* Using induction and taking long exact sequences in cohomology we reduce to showing $D_n$ has a filtration

$$0 = D_n^1 \subseteq D_n^2 \subseteq \cdots \subseteq D_n^n = D_n$$

such that $H^i(X_{\acute{e}t}, D_n^m/D_n^{m-1}) = 0$ for all $m$. Set $D_n^m := \ker(\varphi : D_n \to \mathscr{O}_{X_m}^\times)$, where $\varphi$ is the $\mathfrak{m}^{n+1-m}$-reduction map. Then $D_n^m/D_n^{m-1} \cong \mathscr{O}_{X_1}$ for all $m$, but this is a coherent sheaf so by affine cohomology vanishing we get $H^i(X_{\acute{e}t}, \mathscr{O}_{X_1}) = 0$ for all $i > 0$. □

**Lemma 3.50.** *Let $X$ be a scheme over* $\mathrm{Spec}(\mathcal{O}_K/\mathfrak{m}^n)$. *Then $H^i(X_{\acute{e}t}, D_n) \cong H^i(X_{Zar}, D_n)$ for all $i \geq 0$.*

*Proof.* Let $f : X_{\acute{e}t} \to X_{Zar}$ be the natural continuous map of sites. Since every cover of $X$ has an affine refinement, the higher direct image sheaves $R^i f_* D_n$ vanish for $i > 0$ by lemma 3.49. This implies the Leray spectral sequence $H^p(X_{\acute{e}t}, R^q f_* D_n) \implies H^{p+q}(X_{Zar}, D_n)$ degenerates and induces the desired isomorphism. □

**Definition 3.51.** We now assume $K$ is complete with respect to it's discrete valuation. If $X$ is a formal scheme over $\mathrm{Spf}\mathcal{O}_K$ and $n \geq 1$ we write $X_n := X \times_{\mathrm{Spf}\mathcal{O}_K} \mathrm{Spf}(\mathcal{O}_K/\mathfrak{m}^n)$. Let $\mathbf{D}$ be the kernel of the mod $\mathfrak{m}$ reduction map $\mathscr{O}_X^\times \to \mathscr{O}_{X_1}^\times$.

As above, $X_n \hookrightarrow X$ induces an equivalence of étale sites.

**Theorem 3.52.** *Let $X$ be a formal scheme over* $\mathrm{Spf}\mathcal{O}_K$.

1. *If $X$ is the completion of an affine scheme over* $\mathrm{Spec}\mathcal{O}_K$*, we have* $H^i(X_{\acute{e}t}, \mathbf{D}) = 0$ *for all* $i > 0$.

2. *We have* $H^i(X_{\acute{e}t}, \mathbf{D}) \cong H^i(X_{Zar}, \mathbf{D})$ *for all* $i \geq 0$.

*Proof.* Since $\mathcal{O}_K$ is complete, $\mathbf{D} = \varprojlim D_n$, so we get $H^i(X_{\acute{e}t}, \mathbf{D}) = \varprojlim H^i((X_n)_{\acute{e}t}, D_n)$ and $H^i(X_{Zar}, \mathbf{D}) = \varprojlim H^i((X_n)_{Zar}, D_n)$. When $X$ is the completion of an affine scheme then each $X_n$ is affine so each element of the limit is zero by lemma 3.49 which proves (1). Similarly, lemma 3.50 proves (2). □

We now give a cohomological description of the construction of $\mathfrak{p}$-adic Drinfeld modular forms used in [Hat21b].

**Definition 3.53.** Let $X$ be a formal scheme over $\mathrm{Spf}\mathcal{O}_K$. Let $G = \varprojlim G_n$ be a profinite group. We can consider each $G_n$ as a constant étale group sheaf on $X$, and hence $G$ is a projective system of finite étale group schemes over $X$. Similarly, we may consider $\mathscr{O}_X^\times = \varprojlim \mathscr{O}_{X_n}^\times$ as a projective limit of étale sheaves on $X$. Suppose the profinite group $G$ admits a morphism of projective systems of étale sheaves $\phi : G \to \mathscr{O}_X^\times$, i.e., a compatible collection of maps $\phi_n : G_n \to \mathscr{O}_{X_n}^\times$. Taking cohomology, we get a morphism

$$H^1(\phi) : H^1(X_{\acute{e}t}, G) \to H^1(X_{\acute{e}t}, \mathscr{O}_X^\times). \tag{25}$$

If we assume $X$ is connected and fix a geometric point $x \in X$, then $H^1(\phi)$ induces a morphism

$$\phi_* : \mathrm{Hom}(\pi_1^{\acute{e}t}(X, x), G) \to H^1(X_{Zar}, \mathscr{O}_X^\times) \tag{26}$$

by pre and post-composing with the natural isomorphisms $H^1(X_{\acute{e}t}, G) \cong \mathrm{Hom}(\pi_1^{\acute{e}t}(X, x), G)$ and $H^1(X_{\acute{e}t}, \mathscr{O}_X^\times) \cong H^1(X_{Zar}, \mathscr{O}_X^\times)$ respectively.

As $\mathcal{O}_K$ is the valuation ring of a local field, $\mathcal{O}_K^\times$ is a projective system of discrete groups

**Definition 3.54.** Let $\phi : G \to \mathcal{O}_K^\times$ be a morphism of pro-groups. We can think of this as a morphism of projective systems of étale sheaves on $\mathrm{Spf}\mathcal{O}_K$. Pulling back by the structure map $X \to \mathrm{Spf}\mathcal{O}_K$, this gives rise to a morphism $G \to \mathscr{O}_X^\times$ of projective systems of étale sheaves on $X$. Let $\phi_* : \mathrm{Hom}(\pi_1^{\acute{e}t}(X, x), G) \to H^1(X_{Zar}, \mathscr{O}_X^\times)$ be the induced map from definition 3.53. Let $\mathcal{O}_K^{\times,1} := \left\{ z \in \mathcal{O}_K^\times : z \equiv 1 \bmod \mathfrak{m} \right\}$. If the image of $\phi$ is in $\mathcal{O}_K^{\times,1}$, then we may post-compose $\phi$ with the endomorphism of $\mathcal{O}_K^{\times,1}$ taking $z \mapsto z^a$ for any $a \in \mathbb{Z}_p$, and we denote this composition by $\phi^a$.

**Proposition 3.55.** *Let $X$ be a connected formal scheme over $\mathrm{Spf}\mathcal{O}_K$ with a geometric point $x$. Let $\phi : G \to \mathcal{O}_K^{\times,1}$ be a morphism of pro-groups. For any $c \in \mathrm{Hom}(\pi_1^{\acute{e}t}(X, x), G)$, there is a $D$-structure $D(c)$ on $\phi_*(c)^{rig}$, such that for any $a \in \mathbb{Z}_p$, the rigid line bundle $\mathcal{L}(D(c)^{\otimes a})$ is isomorphic to $\phi_*^a(c)^{rig}$.*

*Proof.* Recall we have a sheaf $\mathbf{D}$ on $X$ defined in definition 3.51. This is the limit of the projective system of sheaves $D_n$ defined in definition 3.48. The morphism $\phi$ induces a morphism of projective systems of étale sheaves on $X$ via pullback from $\mathrm{Spec}\mathcal{O}_K$. This induces a homomorphism $H^1(\phi) : H^1(X_{\acute{e}t}, G) \to H^1(X_{\acute{e}t}, \mathbf{D})$. We may pre-compose with the natural isomorphism $H^1(X_{\acute{e}t}, G) \cong \mathrm{Hom}(\pi_1^{\acute{e}t}(X, x), G)$, and post-compose with the isomorphism $H^1(X_{\acute{e}t}, \mathbf{D}) \cong H^1(X_{Zar}, \mathbf{D})$ from theorem 3.52, to get

$$[\phi] : \mathrm{Hom}(\pi_1^{\acute{e}t}(X, x), G) \to H^1(X_{Zar}, \mathbf{D}). \tag{27}$$

Let $i : \mathbf{D} \to \mathscr{O}_X^\times$ be the inclusion map on sheaves of abelian groups. This induces $i_* : H^1(X_{Zar}, \mathbf{D}) \to H^1(X_{Zar}, \mathscr{O}_X^\times)$, and $\phi_* = i_* \circ [\phi]$.

Any section of $\mathbf{D}$ over $X$ corresponds to a map $X \to \mathbb{G}_m$ whose rigid generic fibre is a holomorphic invertible function on $X^{\mathrm{rig}}$ with image in $D \subset \mathbb{A}^1$. Similarly, a Čech 1-cocyle on $X$ with values in $\mathbf{D}$ gives rise to a 1-cocycle on $X^{\mathrm{rig}}$ with values in $D$. Furthermore, the cohomology class of the latter depends only on the cohomology class of the former. In other words, we have a natural *generic fibre* homomorphism $b$ such that

$$\begin{array}{ccc} H^1(X_{Zar}, \mathbf{D}) & \xrightarrow{b} & H^1(X^{\mathrm{rig}}_{Zar}, D) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \iota_*} \\ H^1(X_{Zar}, \mathscr{O}_X^\times) & \longrightarrow & H^1(X^{\mathrm{rig}}_{Zar}, \mathscr{O}_{X^{\mathrm{rig}}}^\times) \end{array} \tag{28}$$

is commutative, where the bottom horizontal arrow is the generic fibre homomorphism for the cohomology of $\mathbb{G}_m$. The map $\iota$ is the inclusion as defined in definition 3.44. We define $D(c) := b([\phi](c)) \in H^1(X^{\mathrm{rig}}_{Zar}, D)$.

For all $a \in \mathbb{Z}_p$, there is an $a$-power homomorphism of pro-sheaves on $X$, $(-)^a : \mathbf{D} \to \mathbf{D}$, given by pullback from the $a$-power map on the constant pro-sheaf $\mathcal{O}_K^{\times,1}$. This induces a homomorphism $a_* : H^1(X_{Zar}, \mathbf{D}) \to H^1(X_{Zar}, \mathbf{D})$ compatible with our other definition of $a_*$ in definition 3.45 and our generic fibre map $b$ in the sense that

$$\begin{array}{ccc} H^1(X_{Zar}, \mathbf{D}) & \xrightarrow{b} & H^1(X^{\mathrm{rig}}_{Zar}, D) \\ \downarrow{\scriptstyle a_*} & & \downarrow{\scriptstyle a_*} \\ H^1(X_{Zar}, \mathbf{D}) & \xrightarrow{b} & H^1(X^{\mathrm{rig}}_{Zar}, D) \end{array} \tag{29}$$

is commutative.

Notice that $\phi_*^a(c)$ is precisely $i_*(a_*([\phi](c)))$ by definition 3.53. Using this observation along with the commutativity of eq. (28) and eq. (29), we see $\phi_*^a(c)^{\mathrm{rig}} = \iota_*(a_*(b([\phi](c)))) = \iota_*(a_*D(c)) = \iota_*(D(c)^{\otimes a})$, which shows the corresponding rigid geometric line bundles are isomorphic. □

**Remark 3.56.** In the setting/notation of [Hat21b], we have $c = \varprojlim r_{n,n}$, $\phi : \mathcal{O}_K^\times \twoheadrightarrow \mathcal{O}_K^{\times,1}$ the projection, and $\phi_*^a(c)$ corresponds to the sheaf $\omega_{\mathrm{un}}^{a,\mathrm{ord}}$.

**Definition 3.57.** We call the object $D(c)$ constructed in proposition 3.55 **Hattori's $D$-structure**.

We now apply proposition 3.55 to the case when $X = \mathfrak{X}^{ord}$ is the formal ordinary locus to prove that our definition of $\mathfrak{p}$-adic modular forms is compatible with Hattori's.

*Proof of theorem 3.31.* Note that for Hattori's $\mathfrak{p}$-adic modular forms to be defined we work over a modular curve with sufficiently large level structure. There is a natural map from this curve to $\mathbb{P}^1$ and we can use this to pull back our construction. By construction and lemma 3.47, we see that $M_a^{\mathfrak{p}}$ (definition 3.26) is the sheaf associated to the $(-a)$-twist of the trivial $D$-bundle over the ordinary locus. Working over the formal ordinary locus, which is the formal completion of an affine scheme, we see from theorem 3.52 that $H^1(\mathfrak{X}_{\acute{e}t}^{ord}, \mathbf{D})$ is trivial, so by the proof of proposition 3.55 Hattori's $D$-structure is trivial. As both constructions give trivial $D$-bundles we see they must give equivariantly isomorphic sheaves.

□

### 3.3.1 Arbitrary weights and étale covers

So far we have been focusing only on $\mathbb{Z}_p$ weights rather than all of $\mathbb{S}$. We now discuss how to deal with this using a cohomological approach, and show the corresponding line bundles also extend to an overconvergent domain using facts about finite étale covers of rigid spaces.

As usual we let $K$ be a complete discretely valued field, $\mathcal{O}_K$ its valuation ring, and $k$ the residue field. For the rest of this subsection we assume $k$ is finite, so $\mathcal{O}_K^\times \cong k^\times \times \mathcal{O}_K^{\times,1}$.

Let $X$ be a connected formal scheme over $\mathrm{Spf}\mathcal{O}_K$ with geometric point $x \in X$. Every homomorphism $\phi : G \to \mathcal{O}_K^\times$ of pro-groups decomposes into a direct product of homomorphisms $\overleftarrow{\phi} \times \overrightarrow{\phi}$, where $\overleftarrow{\phi} : G \to k^\times$ and $\overrightarrow{\phi} : G \to \mathcal{O}_K^\times$. If we consider the map $\phi_* : \mathrm{Hom}(\pi_1^{\acute{e}t}(X,x), G) \to H^1(X_{Zar}, \mathscr{O}_X^\times)$ from definition 3.53, we see for all $c$ in the source, $\phi_*(c) = \overleftarrow{\phi}_*(c) + \overrightarrow{\phi}_*(c)$. As the group operation in $H^1(X_{Zar}, \mathscr{O}_X^\times)$ corresponds to taking the tensor product of principal $\mathbb{G}_m$-bundles (definition 3.39), we see $\phi_*(c)^{\mathrm{rig}} = \overleftarrow{\phi}_*(c)^{\mathrm{rig}} \otimes \overrightarrow{\phi}_*(c)^{\mathrm{rig}}$. We have given a characterization of $\overrightarrow{\phi}_*(c)^{\mathrm{rig}}$ in proposition 3.55. We now examine $\overleftarrow{\phi}_*(c)^{\mathrm{rig}}$.

First, $\overleftarrow{\phi}_*(c)^{\mathrm{rig}}$ depends only on $\overleftarrow{c} \in \mathrm{Hom}(\pi_1^{\acute{e}t}(X,x), k^\times)$, where $\overleftarrow{c}$ is the composition of $c$ with $\phi$ and then the projection $\mathcal{O}_K^\times \to k^\times$. This corresponds to a finite étale cover $Y \to X$ with Galois group $k^\times$, which we also call $\overleftarrow{c}$. The rigid generic fibre $\overleftarrow{c}^{\mathrm{rig}} : X^{rig} \to Y^{rig}$ is also finite étale with Galois group $k^\times$. This corresponds to a cohomology class $[\overleftarrow{c}^{\mathrm{rig}}] \in H^1(X^{rig}_{\acute{e}t}, k^\times)$, where $k^\times$ is thought of as the constant étale sheaf. If we let $j : k^\times \to \mathscr{O}_X^\times$ be the natural inclusion of étale sheaves, we get an induced map $j_* : H^1(X^{\mathrm{rig}}_{\acute{e}t}, k^\times) \to H^1(X^{\mathrm{rig}}_{\acute{e}t}, \mathscr{O}_X^\times)$. Using the natural isomorphism $H^1(X^{\mathrm{rig}}_{\acute{e}t}, \mathscr{O}_X^\times) \cong H^1(X^{\mathrm{rig}}_{Zar}, \mathscr{O}_X^\times)$, we can think of $j_*([\overleftarrow{c}^{\mathrm{rig}}])$ as a geometric line bundle over $X^{\mathrm{rig}}$. By naturality of everything involved, this line bundle is isomorphic to the line bundle corresponding to $\overleftarrow{\phi}_*(c)^{\mathrm{rig}}$.

We conclude that to give an extension of (the line bundle associated to) $\overleftarrow{\phi}_*(c)^{\mathrm{rig}}$ from

$X^{\mathrm{rig}}$ to a larger domain, it is enough to give an extension of the finite étale cover $\overleftarrow{C}^{\mathrm{rig}}$.

Before we move on let us fix some notation for the rest of this subsection. Let $X$ be a smooth projective curve over $\mathcal{O}_K$, and $\mathfrak{X}$ the formal completion of $X$ along its special fibre $X_1 = X \times_{\mathcal{O}_K} \mathrm{Spec}k$. Let $U_1 \subset X_1$ be some non-empty affine open subcurve in the special fibre. By lemma 3.27, $\exists U \subset X$ an open affine subscheme whose intersection with the special fibre is $U_1$. Let $\mathfrak{U}$ be the formal completion of $U$ along $U_1$. Note this depends only on $U_1$ and not the choice of $U$.

**Definition 3.58.** The closed embedding $U_1 \hookrightarrow \mathfrak{U}$ induces an equivalence between the étale topologies. This implies every finite étale cover $p_1 : Y_1 \to U_1$ has a lift to a finite étale cover $p : \mathfrak{Y} \to \mathfrak{U}$, whose special fibre is isomorphic to $p_1$, and this lift is unique up to unique isomorphism. We call $p$ the **canonical lift** of $p_1$. An **overconvergent extension** of $p$ is a triple $(\tilde{U}, \tilde{Y}, \tilde{p})$ where $\tilde{U} \subseteq \mathfrak{X}^{rig}$ is a connected affinoid admissible subdomain such that $\mathfrak{U}^{rig} \Subset \tilde{U}$, and $\tilde{p} : \tilde{Y} \to \tilde{U}$ if a finite étale cover whose restriction to $\mathfrak{U}^{rig}$ is isomorphic to the rigid generic fibre of $p$.

**Theorem 3.59.** *Every finite étale cover $p_1 : Y_1 \to U_1$ has an overconvergent extension of its canonical lift. Furthermore, this is unique in the sense that if $(\tilde{U}, \tilde{Y}, \tilde{p})$ and $(\hat{U}, \hat{Y}, \hat{p})$ are overconvergent extensions, then $\exists V \subseteq \tilde{U} \cap \hat{U}$ a connected affinoid open subdomain such that $\mathfrak{U}^{rig} \Subset V$, and the restrictions of $\tilde{p}$ and $\hat{p}$ to $V$ are isomorphic.*

This is well-known to the experts. Let us sketch a proof. By standard arguments, e.g.,

as in [Ked08, §7], we can reduce to the affine local case. Define the rings

$$\mathcal{E} := \left\{ \sum_{i \in \mathbb{Z}} c_i t^i : c_i \in K, \sup_i |c_i| < \infty, \lim_{i \to -\infty} |c_i| = 0 \right\},$$
$$\mathcal{E}^\dagger := \left\{ \sum_{i \in \mathbb{Z}} c_i t^i : c_i \in K, \sup_i |c_i| < \infty, \exists \alpha \in (0,1) \text{ s.t. } \lim_{i \to -\infty} |c_i| \, \alpha^i = 0 \right\}.$$

One can show that $\mathcal{E}^\dagger$ is Henselian local with residue field $\mathbf{k}((t))$, which is the same residue field as $\mathcal{E}$. We can (roughly) identify these as the rings of functions on $\mathfrak{U}^{rig}$ and $\tilde{U}$ respectively (notation as in definition 3.58). A finite étale cover thus corresponds to an unramified extension. The existence and uniqueness then follow from the equivalence of categories between unramified extensions of $\mathcal{E}$ and $\mathcal{E}^\dagger$.

# 4 Properties and applications of the construction

We first prove a partial analogue of the $q$-expansion principle. Let $\mathscr{L}$ be an invertible sheaf of degree $d$ on $\mathbb{P}^1_K$, and $\mathscr{I}$ the ideal sheaf of the closed point $\infty \in \mathbb{P}^1_K$. Let $I(d)$ be the closed subscheme cut out by $\mathscr{I}^d$, and $i_d : I(d+1) \hookrightarrow \mathbb{P}^1_K$ the inclusion map.

**Lemma 4.1.** *The map $i_d^* : H^0(\mathbb{P}^1_K, \mathscr{L}) \to H^0(I(d+1), i_d^*\mathscr{L})$ induced by pullback is an isomorphism.*

*Proof.* We know $\dim H^0(\mathbb{P}^1_K, \mathscr{L}) = d+1$ by Riemann-Roch. As $I(d+1)$ is the spectrum of an Artinian $K$-algebra of dimension $d+1$ and $i_d^*\mathscr{L}$ is invertible we have $\dim H^0(I(d+1), i_d^*\mathscr{L}) = d+1$ as well. Thus, it suffices to show $i_d^*$ is injective. Consider the SES

$$0 \to \mathscr{L}(-d-1) \to \mathscr{L} \to \mathscr{L}|_{I(d+1)} \to 0.$$

Taking the LES in cohomology gives

$$H^0(\mathbb{P}^1_K, \mathscr{L}(-d-1)) \to H^0(\mathbb{P}^1_K, \mathscr{L}) \xrightarrow{i_d^*} H^0(I(d+1), i_d^*\mathscr{L}) \to \cdots$$

and we have that $\deg \mathscr{L}(-d-1) = -1 < 0$, so $i_d^*$ is injective. □

**Lemma 4.2.** *Let $L/K$ be a field extension and $f \in H^0(\mathbb{P}^1_L, \mathscr{L})$. If the image of $f$ under $i_{d,L}^*$ lies in the image of the base change map $H^0(I(d+1), i_d^*\mathscr{L}) \to H^0(I(d+1)_L, i_{d,L}^*\mathscr{L})$, then $f$ comes from a section of $\mathscr{L}$ over $\mathbb{P}^1_K$.*

*Proof.* This follows immediately from lemma 4.1 applied to the commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^1_L, \mathscr{L}) & \xrightarrow{i^*_{d,L}} & H^0(I(d+1)_L, i^*_{d,L}\mathscr{L}) \\ \uparrow & & \uparrow \\ H^0(\mathbb{P}^1_K, \mathscr{L}) & \xrightarrow{i^*_d} & H^0(I(d+1), i^*_d\mathscr{L}) \end{array}$$

where the vertical maps come from base change. □

**Proposition 4.3.** *Let $f$ be a cuspidal Drinfeld modular form of weight $k$ and level 1 over $L$ whose $t$-expansion has coefficients in $K$. Then $f$ is defined over $K$.*

*Proof.* Let $f$ be such a modular form. Then $f$ is a section of (a tensor power of) an invertible sheaf $\omega$ on $\mathbb{P}^1_L$ and the $t$-expansion of $f$ is given by the pullback to the completion of the local ring at $\infty$. By assumption, this lies in the completed local ring for $\mathbb{P}^1_K$, so we conclude $f$ is defined over $\mathbb{P}^1_K$ by lemma 4.2. □

In order to work integrally we must refine proposition 4.3. Suppose $\mathscr{L}$ is an invertible sheaf on $\mathbb{P}^1_{\mathcal{O}_K}$ with constant fibre degree $d$, where $\mathcal{O}_K$ is the discrete valuation ring of $K$.

**Lemma 4.4.** *The $\mathcal{O}_K$-module $H^0(\mathbb{P}^1_{\mathcal{O}_K}, \mathscr{L})$ is free of rank $d+1$.*

*Proof.* Consider the structure map $\pi : \mathbb{P}^1_{\mathcal{O}_K} \to \mathrm{Spec}\mathcal{O}_K$. By Grauert's theorem [Vak17, §28.1.5], if we knew that $h^0(\mathbb{P}^1_{k(s)}, \mathscr{L}|_{\mathbb{P}^1_{k(s)}}) = d+1$ for all $s \in \mathrm{Spec}\mathcal{O}_K$, this would imply that $R^0\pi_*\mathscr{L}$ is locally free of rank $d+1$ over $\mathrm{Spec}\mathcal{O}_K$, hence free of rank $d+1$ as $\mathcal{O}_K$ is a PID. By assumption $\mathscr{L}|_{\mathbb{P}^1_{k(s)}}$ has degree $d$. By Riemann-Roch, as the genus of $\mathbb{P}^1_{k(s)}$ is zero, we find $h^0(\mathbb{P}^1_{k(s)}, \mathscr{L}|_{\mathbb{P}^1_{k(s)}}) = d+1$. □

Let $\mathscr{I}$ now be the ideal sheaf cutting out the closed point $\infty \in \mathbb{P}^1_{\mathcal{O}_K}$, $I(d)$ the closed subscheme cut out by $\mathscr{I}^d$, and $i_d : I(d+1) \hookrightarrow \mathbb{P}^1_{\mathcal{O}_K}$ the inclusion map.

**Lemma 4.5.** *The $\mathcal{O}_K$-module $H^0(I(d+1), i_d^*\mathscr{L})$ is free of rank $d+1$.*

*Proof.* As $i_d$ is a closed embedding, the structure map $I(d+1) \to \mathrm{Spec}\mathcal{O}_K$ is projective, so we can apply Grauert's theorem again. We apply lemma 4.1 to see that $H^0(I(d+1)_{k(s)}, i_d^*\mathscr{L}|_{\mathbb{P}^1_{k(s)}})$ is free of dimension $d+1$ for any $s \in \mathrm{Spec}\mathcal{O}_K$. □

**Lemma 4.6.** *The pullback map $i_d^* : H^0(\mathbb{P}^1_{\mathcal{O}_K}, \mathscr{L}) \to H^0(I(d+1), i_d^*\mathscr{L})$ is an isomorphism.*

*Proof.* Let $\mathfrak{m} \in \mathrm{Spec}\mathcal{O}_K$ be the unique closed point. We have a commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^1_{\mathcal{O}_K}, \mathscr{L}) \otimes_{\mathcal{O}_K} k(\mathfrak{m}) & \longrightarrow & H^0(\mathbb{P}^1_{k(\mathfrak{m})}, \mathscr{L}|_{\mathbb{P}^1_{k(\mathfrak{m})}}) \\ \downarrow & & \downarrow \\ H^0(I(d+1), i_d^*\mathscr{L}) \otimes_{\mathcal{O}_K} k(\mathfrak{m}) & \longrightarrow & H^0(I(d+1)_{k(\mathfrak{m})}, i_d^*\mathscr{L}|_{I(d+1)_{k(\mathfrak{m})}}) \end{array}$$

where the horizontal arrows are both isomorphisms by Grauert's theorem. The right vertical arrow is an isomorphism by lemma 4.1. Thus the left vertical map is an isomorphism. We know from lemma 4.4 and lemma 4.5 that this is a map of free $\mathcal{O}_K$-modules of rank $d+1$ that has been tensored with $k(\mathfrak{m})$. In fact, this is precisely $i_d^* \otimes \mathrm{id}_{k(\mathfrak{m})}$. But then we may view $i_d^*$ as a $(d+1) \times (d+1)$ matrix with entries in $\mathcal{O}_K$ which is invertible mod $\mathfrak{m}$, so must have been invertible to begin with. □

**Proposition 4.7.** *Let $f$ be a cuspidal Drinfeld modular form of weight $k \in \mathbb{Z}$ and level 1 defined over $K$ (respectively, $K_\mathfrak{p}$) whose $t$-expansion has coefficients in $A$ (respectively, in $A_\mathfrak{p}$). Then $f$ is defined over $A$ (respectively, $A_\mathfrak{p}$).*

*Proof.* For any nonzero prime $\mathfrak{p} \lhd A$ we have an embedding $K \hookrightarrow K_\mathfrak{p}$. If $x \in A \subset K$ then the image of $x$ under this embedding lies in the valuation ring $A_\mathfrak{p}$. Furthermore, if $x \in K$ and the image of $x$ is in $A_\mathfrak{p}$ for all such embeddings, then $x \in A$. We see that a Drinfeld

modular form is defined over $A \iff$ it is defined over $A_{\mathfrak{p}}$ for all primes. Thus, it suffices to show the latter claim. By assumption $f$ is a section of a line bundle of positive degree over $\mathbb{P}^1_{K_{\mathfrak{p}}}$ whose pullback to a formal neighbourhood of $\infty$ has coefficients in $A_{\mathfrak{p}}$. Using lemma 4.6, the same argument as in lemma 4.2 shows that $f$ is defined over $A_{\mathfrak{p}}$. □

**Remark 4.8.** Proposition 4.7 shows, for example, that if $f = \sum_{a \in A_+} c_a(f) G_n(t_a)$ is a cusp form with $A$-expansion and $c_a(f) \in A$ for all $a \in A_+$, then by corollary 2.10 some $A$-multiple of $f$ is defined over $A$.

Now that we have some basic facts about convergent and overconvergent forms we would like to exhibit the $\mathfrak{p}$-adic family of Goss as overconvergent. We were unfortunately unable to complete this goal, but believe our approach to the proof should go through with a bit more work. We present it as a conjecture and then give some of our argument.

**Conjecture 4.9.** Let $\hat{f}_{s,n}$ be as in definition 2.19 with $n \geq 1$ and $s \in \mathbb{S}(n) \cap \mathbb{Z}_p$ as in theorem 2.20. Then $\hat{f}_{s,n}$ is an overconvergent cuspidal eigenform of weight $s + n$.

Our approach begins with a rigid Vitali-Montel-style theorem.

**Definition 4.10.** Let $U$ be an affinoid subspace of $\mathbb{P}^1_K$. Let $f, f_1, f_2, \ldots$ be holomorphic functions on $U$. We say the sequence $(f_n)_{n \geq 1}$ **converges uniformly** to $f$ on $U$ if $\|f_n - f\|_U \to 0$ as $n \to \infty$, where $\|\cdot\|_U$ is the supremum norm on $U$.

**Theorem 4.11.** *Let $D_1, D_2, D_3 \subset \mathbb{P}^1_C$ be connected rational subdomains such that $D_1$ is strictly contained in $D_2$ is strictly contained in $D_3$. Let $(f_n)$ be a sequence of holomorphic functions on $D_3$ such that*

*(a) $(f_n)$ converges uniformly to a holomorphic function $f$ on $D_1$,*

*(b)* $\sup_{n\geq 1} \|f_n\|_{D_3} < \infty$, *i.e., the sequence is bounded on* $D_3$.

*Then the sequence* $(f_n)$ *converges uniformly to a unique holomorphic function on* $D_2$ *extending* $f$.

*Proof.* We may assume $D_1$ contains $\infty$. Write the complements of our connected rational subdomains as $\partial D_j = \{D(z_i, r_{i,j})\}_{i=1,\dots,m}$, for $j = 1, 2, 3$, $z_i \in C$, and $r_{i,j} \in \mathbb{R}_{\geq 0}$. By assumption of strict containment, $r_{i,1} > r_{i,2} > r_{i,3} > 0$ for all $i$. Write

$$f_n(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} a_{i,k}^{(n)} (z - z_i)^{-k}$$

and

$$f(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} a_{i,k} (z - z_i)^{-k}.$$

Since $(f_n)$ is bounded on $D_3$, there is some $R > 0$ such that for all $i, k, n$ we have $\left|a_{i,k}^{(n)}\right| r_{i,3}^{-k} \leq R$. Since $f_n$ converges uniformly to $f$ on $D_1$, we see

$$\|f_n - f\|_{D_1} = \left\| \sum_{i=1}^{m} \sum_{k=0}^{\infty} (a_{i,k}^{(n)} - a_{i,k})(z - z_i)^{-k} \right\|_{D_1} = \sup_{i,k} \left|a_{i,k}^{(n)} - a_{i,k}\right| r_{i,1}^{-k} \to 0$$

as $n \to \infty$.

We claim that $f$ is also bounded by $R$ on $D_3$. Suppose $\|f\|_{D_3} = \sup_{i,k} |a_{i,k}| \, r_{i,3}^{-k} > R$. Then $\exists i_0, k_0$ such that $|a_{i_0,k_0}| \, r_{i_0,3}^{-k_0} > R$, so $|a_{i_0,k_0}| > R r_{i_0,3}^{k_0}$. This shows $\forall n \geq 1$, $\left|a_{i_0,k_0}^{(n)}\right| < |a_{i_0,k_0}|$, as

$\left|a_{i_0,k_0}^{(n)}\right| \leq R r_{i_0,3}^{k_0}$. Thus,

$$\left|a_{i_0,k_0}^{(n)} - a_{i_0,k_0}\right| r_{i_0,1}^{-k_0} = \left|a_{i_0,k_0}\right| r_{i_0,1}^{-k_0} > R\left(\frac{r_{i_0,3}}{r_{i_0,1}}\right)^{k_0}.$$

This implies $\|f_n - f\|_{D_1}$ is bounded below, which is a contradiction. We conclude $\|f\|_{D_3} \leq R$.

Next, we claim $f$ extends to a function on $D_2$. It suffices to check $|a_{i,k}|\, r_{i,2}^{-k} \to 0$. We know

$$|a_{i,k}|\, r_{i,2}^{-k} = |a_{i,k}|\, r_{i,3}^{-k}\left(\frac{r_{i,3}}{r_{i,2}}\right)^{k} \leq R\left(\frac{r_{i,3}}{r_{i,2}}\right)^{k} \to 0$$

as $r_{i,3}/r_{i,2} < 1$.

Finally, we must show $f_n$ converges uniformly to $f$ on $D_2$. Consider

$$\|f_n - f\|_{D_2} = \sup_{i,k}\left|a_{i,k}^{(n)} - a_{i,k}\right| r_{i,2}^{-k}.$$

Let $\epsilon > 0$. We know $\exists k_0 > 0$ such that $(r_{i,3}/r_{i,2})^k R < \epsilon$ for all $k \geq k_0$ and all $i$. This implies

$$\left|a_{i,k}^{(n)} - a_{i,k}\right| r_{i,2}^{-k} \leq \max\left(\left|a_{i,k}^{(n)}\right|, |a_{i,k}|\right) r_{i,2}^{-k} \leq R\left(\frac{r_{i,3}}{r_{i,2}}\right)^{k} < \epsilon$$

for all $i$ and all $k \geq k_0$. We only have finitely many terms left to bound, corresponding to the pairs $(i,k)$ where $k < k_0$. As $f_n$ converges uniformly to $f$ on $D_1$, $\exists n_0 > 0$ such that $\forall n \geq n_0$,

$$\sup_{i,k}\left|a_{i,k}^{(n)} - a_{i,k}\right| r_{i,1}^{-k} < \epsilon\left(\frac{r_{i,2}}{r_{i,1}}\right)^{k_0}.$$

This implies, for $n \geq n_0$,

$$\max_{1\leq i\leq m, k<k_0} \left|a_{i,k}^{(n)} - a_{i,k}\right| r_{i,2}^{-k} \leq \left(\frac{r_{i,1}}{r_{i,2}}\right)^{k_0} \max_{1\leq i\leq m, k<k_0} \left|a_{i,k}^{(n)} - a_{i,k}\right| r_{i,1}^{-k}$$
$$\leq \left(\frac{r_{i,1}}{r_{i,2}}\right)^{k_0} \sup_{i,k} \left|a_{i,k}^{(n)} - a_{i,k}\right| r_{i,1}^{-k} < \left(\frac{r_{i,1}}{r_{i,2}}\right)^{k_0} \left(\frac{r_{i,2}}{r_{i,1}}\right)^{k_0} \epsilon = \epsilon.$$

Putting this together we see $\|f_n - f\|_{D_2} < \epsilon$ for all $n \geq n_0$. □

By [Gos14, Thm. 2], we have a sequence of modular forms $(f_i)$ converging to $\hat{f}_{s,n}$. Specifically, $f_i = f_{k_i,n}$ as in theorem 2.12, with $(k_i)$ a sequence of integers converging to $s+n$ in $\mathbb{S}$. This shows that $\hat{f}_{s,n}$ has the correct weight. These $f_{k_i,n}$ are all eigenforms with eigensystem $\{\lambda_{\mathfrak{p}} = \mathfrak{p}^n\}$.

By theorem 4.11 it suffices to show the sequence $(f_{k_i,n})_{i\geq 0}$ is bounded on some connected rational subdomain strictly containing the ordinary locus. By corollary 2.10, $\exists \alpha \in A$ such that $\alpha f_{k_i,n} \in A[\![t]\!]$ for all $i$, so by submultiplicativity of the Banach norm we may assume our sequence lies in $A[\![t]\!]$. By taking all our forms to the power $q^d - 1$, we may assume they all have type 0 and weight in $\mathbb{Z}_p \subset \mathbb{S}$. Since $A$ is dense in its completion $A_{\mathfrak{p}}$, we can check boundedness on $A_{\mathfrak{p}}$, or, what is the same, on $K_{\mathfrak{p}}$. Taking a finite extension of $K_{\mathfrak{p}}$ is also fine, so we consider a finite extension such that the residue field contains all the (finitely many) supersingular $j$-invariants. This implies the complement of the ordinary locus is the disjoint union of open unit disks.

In this setting, we believe that $\exists V \subset X^{an} = \mathcal{X}$ an open connected rational subdomain such that $\mathcal{X}^{ord} \Subset V$ and if $f$ is any classical (level 1) Drinfeld modular form with coefficients in $A$, then $f|_V$ is bounded above by 1. This would imply our sequence is bounded and finish the proof. Using our construction, specifically proposition 3.23, we can consider such a form

$f$ as an element of $M_k(\omega, V, T(s))$ for some tube depending on $s$. We would like to argue that this must come from a "formal tube," and hence be bounded, but as of this writing have not been able to complete the argument.

Now that we have identified our space of $\mathfrak{p}$-adic Drinfeld modular forms with that of Hattori, and these are identified with $\mathfrak{p}$-adic forms in the sense of Serre (definition 2.18), we can define Hecke operators in the naive way on power series, extending the usual definition. We do the same for the subspace of overconvergent $\mathfrak{p}$-adic Drinfeld modular forms.

Much of Petrov's results on Drinfeld modular forms with $A$-expansion extend to the $\mathfrak{p}$-adic setting. To start with, we show that $A$-expansions of $\mathfrak{p}$-adic modular forms are unique. The main idea is from [Lóp11, §3]. For lack of reference we prove the following lemma.

**Lemma 4.12.** *Let $k$ be a countable field and $K, L$ algebraically closed extensions of the same uncountable cardinality. Then there is an isomorphism of fields $K \xrightarrow{\sim} L$ that restricts to the identity on $k$.*

*Proof.* The idea is to use the back-and-forth method. Let $\omega$ be the unique countably infinite cardinal. For any set $X$ we write $|X|$ for the cardinality of $X$, i.e., the unique minimal ordinal bijective to $X$. Addition and multiplication is always applied to cardinals and defined in the regular way, i.e., the sum is the cardinality of the union and the product is the cardinality of the cross product. With this in mind, recall that if $\kappa, \kappa'$ are infinite cardinals, then $\kappa + \kappa' \leq \max(\kappa, \kappa')$ and $\kappa \cdot \kappa' \leq \max(\kappa, \kappa')$.

Let $\kappa$ be the cardinal of $K$ and $L$, and fix some bijections $f_K : \kappa \to K$ and $f_L : \kappa \to L$. We will use transfinite induction to show there is a collection of subfields of $K$, $\{K_\alpha\}_{\alpha \in \kappa}$, and embeddings (injective morphisms of fields) $\{i_\alpha : K_\alpha \hookrightarrow L\}_{\alpha \in \kappa}$, indexed by the cardinal $\kappa$, such that

1. $K_0 = k$ and $i_0$ is the identity map on $k$;

2. if $\alpha \in \kappa$ and $\alpha = succ(\beta)$ then $f_K(\alpha) \in K_\beta$ and $f_L(\alpha) \in i_\beta(K_\beta)$;

3. $\forall \alpha \in \kappa$, $|K_\alpha| \leq |\alpha| + \omega$;

4. for all $\alpha, \beta \in \kappa$, if $\alpha < \beta$ then $K_\alpha \subseteq K_\beta$ and $i_\beta|_{K_\alpha} = i_\alpha$.

This is enough to prove the claim. Indeed, if $x \in K$, then $x = f_K(\alpha)$ for some $\alpha \in \kappa$, so by (2) $x \in K_\alpha$. This implies $K = \cup_{\alpha\in\kappa} K_\alpha$. By (4), the embeddings $i_\alpha$ are compatible, so the union defines an embedding $i = \cup_{\alpha\in\kappa} i_\alpha : K \hookrightarrow L$, which is the identity on $k$ by (1). Finally, by (2) again we see that $i$ is surjective, and thus an isomorphism of fields.

As mentioned, our construction is by transfinite induction. We start by setting $K_0 = k$ and taking $i_0$ to be the identity on $k$. This definition immediately shows that (1) holds, and (3) is true for $\alpha = 0$ since $|k| = \omega$ by assumption.

Suppose $\alpha \in \kappa$ is a limit ordinal. We set $K_\alpha = \cup_{\beta\in\alpha} K_\beta$ and $i_\alpha = \cup_{\beta\in\alpha} i_\beta$. By our induction hypothesis ((4) in particular), the map $i_\alpha$ is a well-defined embedding of fields fixing $k$. By construction we have (4). As $\alpha$ is a limit ordinal it is not the successor of any ordinal so (2) is trivially verified. To see (3), we note that

$$|K_\alpha| = \sup_{\beta<\alpha} |K_\beta| \leq \sup_{\beta<\alpha} |\beta| + \omega = |\alpha| + \omega.$$

To finish the construction, it remains to consider the case when $\alpha \in \kappa$ is a successor ordinal, i.e. when $\alpha = succ(\beta)$ for some $\beta \in \kappa$. We start by extending $i_\beta$ to $K_\beta(f_K(\beta))$. We do this in two cases. First, we assume $f_K(\beta)$ is transcendental over $K_\beta$. Then, since

$$\left|\overline{i_\beta(K_\beta)}\right| = \left|\overline{K_\beta}\right| = |K_\beta| \leq |\beta| + \omega < \kappa,$$

(recall that if $F$ is an infinite field then $\left|\bar{F}\right| = |F|$) we can find an element $y \in L$ that is transcendental over $i_\beta(K_\beta)$. By standard field theory, we get an isomorphism of fields $i'_\beta : K_\beta(f_K(\beta)) \xrightarrow{\sim} i_\beta(K_\beta)(y)$. Indeed, consider the map $K_\beta[f_K(\beta)] \to L$ defined by $i_\beta$ on $K_\beta$ and taking $f_K(\beta)$ to $y$. As $f_K(\beta)$ is transcendental this extends to a field homomorphism from $K_\beta(f_K(\beta))$, and has image $i_\beta(K_\beta)(y)$.

If $f_K(\beta)$ is not transcendental, then it is algebraic over $K_\beta$. In this case let $p(t) \in K_\beta[t]$ be its minimal polynomial. As $i_\beta$ is injective it is an isomorphism onto its image, and this extends to an isomorphism $K_\beta[t] \to i_\beta(K_\beta)[t]$. Under this isomorphism, the image of $p(t)$ is again an irreducible polynomial, which we will call $q(t)$ to distinguish it from $p(t)$. As $L$ is algebraically closed, $q(t)$ has a root $y \in L$. Then $y$ is algebraic over $i_\beta(K_\beta)$, with minimal polynomial $q(t)$. By the same argument as above we get an isomorphism $i'_\beta : K_\beta(f_K(\beta)) \xrightarrow{\sim} i_\beta(K_\beta)(y)$ such that $i'_\beta|_{K_\beta} = i_\beta$ and $i'_\beta(f_K(\beta)) = y$.

At this point we've shown, regardless of $f_K(\beta)$, that we can choose an element $y \in L$ and an isomorphism $i'_\beta : K_\beta(f_K(\beta)) \xrightarrow{\sim} i_\beta(K_\beta)(y)$ extending $i_\beta$ and taking $f_K(\beta) \mapsto y$. Let $L_\beta = i_\beta(K_\beta)(y)$ and $\lambda_\beta = (i'_\beta)^{-1} : L_\beta \to K$. Since $L_\beta$ is a simple extension of $i_\beta(K_\beta)$ (i.e. it's generated by a single element), we have (using (3))

$$|L_\beta| \leq |i_\beta(K_\beta)| \cdot \omega = |K_\beta| \cdot \omega \leq (|\beta| + \omega) \cdot \omega \leq |\beta| + \omega \leq |\alpha| + \omega.$$

Now, using the exact same argument as above, we can extend $\lambda_\beta$ to a map $\lambda'_\beta : L_\beta(f_L(\beta)) \to K$. We set $K_\alpha = \lambda'_\beta(L_\beta(f_L(\beta)))$, and let $i_\alpha$ be the inverse map $K_\alpha \to L_\beta(f_L(\beta)) \subseteq L$. We must show this satisfies our conditions. Conditions (2) and (4) are easily seen to follow from our definition of $K_\alpha$ and $i_\alpha$. To see (3), note that $K_\alpha$ is bijective to a simple extension of

$L_\beta$, so

$$|K_\beta| \leq |L_\beta| \cdot \omega \leq (|\alpha| + \omega) \cdot \omega \leq |\alpha| + \omega.$$

□

**Theorem 4.13.** *Let $f$ be a $\mathfrak{p}$-adic modular form with $A$-expansion $f_\infty(t) = c_0 + \sum_{a \in A_+} c_a G_n(t_a)$. If $f = 0$ then $c_a = 0$ for all $a \in A_+ \cup \{0\}$.*

*Proof.* Note that the coefficients of a $\mathfrak{p}$-adic modular form with $A$-expansion lie in the algebraically closed field $\mathbb{C}_\mathfrak{p}$. By lemma 4.12, we have an isomorphism $\mathbb{C}_\mathfrak{p} \xrightarrow{\sim} \mathbb{C}_\infty$ which is the identity on $K = \mathbb{F}_q(T)$. We know that Goss polynomials have coefficients in $K$, and thus the proof that $\{G_n(t_a)\}_{a \in A_+}$ is linearly independent over $\mathbb{C}_\infty$, given in the special case of exponent $n = q - 1$ in [Lóp11, Thm. 3.1], but which is easily seen to generalize to any $n$, implies the same over $\mathbb{C}_\mathfrak{p}$. Once we have this, the rest of Lopez's proof works in the same way. □

Using our machinery we may extend results of Petrov about the space of cuspidal modulo double cuspidal forms to the $\mathfrak{p}$-adic setting.

**Theorem 4.14.** *Let $s \in \mathbb{Z}_p \subset \mathbb{S}(1)$ and consider the Goss-Petrov form $\hat{f}_{s,1}$ defined in definition 2.19. Then this is a simultaneous eigenform for our Hecke operators with eigensystem $\{a\}_{a \in A_{\mathfrak{p},+}}$. Furthermore, there is a direct sum decomposition*

$$S_s = S_s^2 \bigoplus \langle \hat{f}_{s,1} \rangle.$$

*Proof.* As a cusp form in $S_s$ is determined by its $t$-expansion and $S_s^2$ is the kernel of a linear functional (namely, taking the coefficient of $t$), it's codimension is at most 1. However, we

know $\hat{f}_{s,1}$ is in $S_s \backslash S_s^2$ by looking at the $t$-expansion, so the codimension is 1. The claim about the eigensystem follows from the fact that $\hat{f}_{s,1}$ is given as the limit of cusp forms $f_{k_n,1}$ as in theorem 2.12, which are all eigenforms with this eigensystem as mentioned in remark 2.13. $\square$

# References


[AIP15] F. Andreatta, A. Iovita, and V. Pilloni. "$p$-adic families of Siegel modular cuspforms". In: *Ann. of Math.* 181.2 (2015), pp. 623–697. DOI: 10.4007/annals.2015.181.2.5.

[AIP18a] F. Andreatta, A. Iovita, and V. Pilloni. "$p$-adic variation of automorphic sheaves". In: *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*. World Sci. Publ., Hackensack, NJ, 2018, pp. 249–276.

[AIP18b] F. Andreatta, A. Iovita, and V. Pilloni. "Le halo spectral". In: *Ann. Sci. Éc. Norm. Supér. (4)* 51.3 (2018), pp. 603–655. DOI: 10.24033/asens.2362.

[AIS14] F. Andreatta, A. Iovita, and G. Stevens. "Overconvergent modular sheaves and modular forms for $\mathrm{GL}_{2/F}$". In: *Israel J. Math.* 201.1 (2014), pp. 299–359. DOI: 10.1007/s11856-014-1045-8.

[Ber96] P. Berthelot. *Cohomologie rigide et cohomologie rigide avec à supports propres*. Permière partie, preprint IRMAR 96-03. available at https://perso.univ-rennes1.fr/pierre.berthelot/. 1996.

[Bos14] S. Bosch. *Lectures on formal and rigid geometry*. Vol. 2105. Lecture Notes in Mathematics. Springer, Cham, 2014, pp. viii+254. DOI: 10.1007/978-3-319-04417-0.

[Buz07] K. Buzzard. "Eigenvarieties". In: *L-functions and Galois representations*. Vol. 320. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2007, pp. 59–120. DOI: 10.1017/CBO9780511721267.004.

[BV18] A. Bandini and M. Valentino. "On the Atkin $U_t$-operator for $\Gamma_1(t)$-invariant Drinfeld cusp forms". In: *Int. J. Number Theory* 14.10 (2018), pp. 2599–2616. DOI: 10.1142/S1793042118501555.

[BV19a] A. Bandini and M. Valentino. "On the Atkin $U_t$-operator for $\Gamma_0(t)$-invariant Drinfeld cusp forms". In: *Proc. Amer. Math. Soc.* 147.10 (2019), pp. 4171–4187. DOI: https://doi.org/10.1090/proc/14491.

[BV19b] A. Bandini and M. Valentino. "On the Structure and Slopes of Drinfeld Cusp Forms". In: *Exp. Math.* (2019). DOI: 10.1080/10586458.2019.1671921.

[Cal13] F. Calegari. *Congruences between modular forms*. Notes from Arizona Winter School 2013, available at https://www.math.arizona.edu/~swc/aws/2013/2013CalegariLectureNotes.pdf. 2013.

[CHJ17] P. Chojecki, D. Hansen, and C. Johansson. "Overconvergent Modular Forms and Perfectoid Shimura Curves". In: *Doc. Math.* 22 (2017), pp. 191–262. DOI: 10.25537/dm.2017v22.191-262.

[CM98] R. Coleman and B. Mazur. "The Eigencurve". In: *Galois Representations in Arithmetic Algebraic Geometry*. Ed. by A. J. Scholl and R. L. Taylor. Vol. 254. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1998, pp. 1–114. DOI: 10.1017/CBO9780511662010.003.

[Col96] R. F. Coleman. "Classical and overconvergent modular forms". In: *Invent. Math.* 124.1-3 (1996), pp. 215–241.

[Col97] R. Coleman. "$p$-adic Banach spaces and families of modular forms". In: *Invent. Math.* 127.3 (1997), pp. 417–479.

[Con99] B. Conrad. "Irreducible components of rigid spaces". In: *Ann. Inst. Fourier (Grenoble)* 49.2 (1999), pp. 473–541. DOI: 10.5802/aif.1681.

[Dri74] V. G. Drinfeld. "Elliptic modules". In: *Mat. Sb. (N.S.)* 94(136) (1974), pp. 594–627, 656.

[Fv04] J. Fresnel and M. van der Put. *Rigid analytic geometry and its applications*. Vol. 218. Progress in Mathematics. Birkhäuser Basel, 2004.

[Gek86] E.-U. Gekeler. *Drinfeld modular curves*. Vol. 1231. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986, pp. xiv+107. DOI: 10.1007/BFb0072692.

[Gek88] E.-U. Gekeler. "On the coefficients of Drinfeld modular forms". In: *Invent. Math.* 93.3 (1988), pp. 667–700.

[Gek99] E.-U. Gekeler. "A survey on Drinfeld modular forms". In: *Turkish J. Math.* 23.4 (1999). [Online, accessed 04-24-2017] http://journals.tubitak.gov.tr/math/issues/mat-99-23-4/mat-23-4-3-98032.pdf, pp. 485–518.

[GM92] F. Q. Gouvêa and B. Mazur. "Families of modular eigenforms". In: *Math. Comp.* 98.198 (1992), pp. 793–805.

[Gos14] D. Goss. "A construction of $\mathfrak{v}$-adic modular forms". In: *J. Number Theory* 136 (2014), pp. 330–338. DOI: 10.1016/j.jnt.2013.10.010.

[Gos80a] D. Goss. "$\pi$-adic Eisenstein series for function fields". In: *Compos. Math.* 41.1 (1980), pp. 3–38. URL: http://www.numdam.org/item/CM_1980__41_1_3_0.

[Gos80b] D. Goss. "The algebraist's upper half-plane". In: *Bull. Amer. Math. Soc. (N.S.)* 2.3 (May 1980), pp. 391–415. URL: https://projecteuclid.org:443/euclid.bams/1183546349.

[Gos98] D. Goss. *Basic structures of function field arithmetic*. Vol. 35. Ergeb. Math. Grenzgeb. Originally published as volume 35 in the series: Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, Berlin, Heidelberg, 1998. DOI: 10.1007/978-3-642-61480-4.

[Gro61] A. Grothendieck. "Éléments de géométrie algébrique : III. Étude cohomologique des faisceaux cohérents, Première partie". In: *Publications Mathématiques de l'IHÉS* 11 (1961), pp. 5–167. URL: http://www.numdam.org/item/PMIHES_1961__11__5_0/.

[Gv80] L. Gerritzen and M. van der Put. *Schottky Groups and Mumford Curves*. Vol. 817. Lecture notes in Mathematics. Springer, Berlin, 1980, pp. viii+317.

[Hat19] S. Hattori. "Dimension Variation of Gouvêa–Mazur Type for Drinfeld Cuspforms of Level $\Gamma_1(t)$". In: *Int. Math. Res. Not.* 2021.3 (2019), pp. 2389–2402. DOI: 10.1093/imrn/rnz104.

[Hat20] S. Hattori. "On the compactification of the Drinfeld modular curve of level $\Gamma_1^{\Delta}(\mathfrak{n})$". In: *J. Number Theory* (2020). DOI: 10.1016/j.jnt.2020.07.015.

[Hat21a] S. Hattori. "$\wp$-adic continuous families of Drinfeld eigenforms of finite slope". In: *Adv. Math.* 380 (2021). DOI: 10.1016/j.aim.2021.107594.

[Hat21b] S. Hattori. "Duality of Drinfeld modules and $\wp$-adic properties of Drinfeld modular forms". In: *J. Lond. Math. Soc.* 103.1 (2021), pp. 35–70. DOI: 10.1112/jlms.12366.

[Haz78] M. Hazewinkel. *Formal Groups and Applications*. Vol. 78. Pure Appl. Math. Academic Press, Inc., New York, 1978.

[Iwa86] K. Iwasawa. *Local class field theory*. Oxford Science Publications. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1986, pp. viii+155.

[Jeo09] S. Jeong. "On a question of Goss". In: *J. Number Theory* 129.8 (2009), pp. 1912–1918. DOI: https://doi.org/10.1016/j.jnt.2008.08.006.

[Jon95] A. J. de Jong. "Crystalline Dieudonné module theory via formal and rigid geometry". In: *Publications Mathématiques de l'IHÉS* 82 (1995). (1996), pp. 5–96. URL: http://www.numdam.org/item?id=PMIHES_1995__82__5_0.

[Kas06] P. Kassaei. "A gluing lemma and overconvergent modular forms". In: *Duke Mathematical Journal* 132.3 (2006), pp. 509–529.

[Kat73] N. M. Katz. "P-ADIC properties of modular schemes and modular forms". In: *Modular Functions of One Variable III*. Ed. by W. Kuijk and J.-P. Serre. Vol. 350. Springer, Berlin, Heidelberg, 1973, pp. 69–190. DOI: 10.1007/978-3-540-37802-0_3.

[Ked08] K. S. Kedlaya. "Full faithfulness for overconvergent F-isocrystals". In: *Geometric Aspects of Dwork Theory*. Ed. by A. Adolphson et al. De Gruyter, 2008, pp. 819–836. DOI: 10.1515/9783110198133.2.819.

[KM85] N. M. Katz and B. Mazur. *Arithmetic Moduli of Elliptic Curves*. Vol. 108. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1985, pp. xiv+514. DOI: 10.1515/9781400881710.

[Lóp10] B. López. "A non-standard Fourier expansion for the Drinfeld discriminant function". In: *Arch. Math. (Basel)* 95.2 (2010), pp. 143–150. DOI: 10.1007/s00013-010-0148-7.

[Lóp11] B. López. "Action of Hecke operators on two distinguished Drinfeld modular forms". In: *Arch. Math. (Basel)* 97.5 (2011), pp. 423–429. DOI: 10.1007/s00013-011-0318-2.

[Lüt16] W. Lütkebohmert. *Rigid Geometry of Curves and their Jacobians*. Vol. 61. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer International Publishing, 2016.

[NR18] M.-H. Nicole and G. Rosso. "Familles de formes modulaires de Drinfeld pour le groupe général linéaire". In: *arXiv e-prints*, arXiv:1805.08793 (2018). arXiv: 1805.08793 [math.NT].

[NR20] M.-H. Nicole and G. Rosso. *Perfectoid Drinfeld Modular Forms*. arXiv:1912.07738v2. 2020. arXiv: 1912.07738v2 [math.NT].

[Pet13] A. Petrov. "$A$-expansions of Drinfeld modular forms". In: *J. Number Theory* 133.7 (2013), pp. 2247–2266. DOI: https://doi.org/10.1016/j.jnt.2012.12.012.

[Pil13] V. Pilloni. "Overconvergent modular forms". In: *Ann. Inst. Four. (Grenoble)* 63.1 (2013), pp. 219–239. DOI: 10.5802/aif.2759.

[Poo17] B. Poonen. *Rational points on varieties*. Vol. 186. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2017, pp. xv+337.

[PZ17] M. A. Papanikolas and G. Zeng. "Theta operators, Goss polynomials, and $\mathfrak{v}$-adic modular forms". In: *J. Théor. Nombres Bordeaux* 29.3 (2017), pp. 729–753. URL: http://www.jstor.org/stable/26274096.

[Ser73] J.-P. Serre. "Formes modulaires et fonctions zêta $p$-adiques". In: *Modular Functions of One Variable III*. Ed. by J.-P. Kuijk Willem and Serre. Vol. 350. Springer, Berlin, Heidelberg, 1973, pp. 191–268. DOI: 10.1007/978-3-540-37802-0_4.

[Ser79] J.-P. Serre. *Local fields*. Vol. 67. Graduate Texts in Mathematics. Translated from the French by Marvin Jay Greenberg. Springer-Verlag, New York-Berlin, 1979, pp. viii+241.

[Stacks] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu. 2018.

[Swi73] H. P. F. Swinnerton-Dyer. "On $\ell$-adic representations and congruences for coefficients of modular forms". In: *Modular functions of one variable III*. Ed. by W. Kuijk and J.-P. Serre. Vol. 350. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, 1973, pp. 1–55. DOI: 10.1007/978-3-540-37802-0_1.

[Vak17] R. Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. Nov. 18, 2017 draft, available at math216.wordpress.com. 2017.

[Vin12] C. Vincent. "Drinfeld modular forms modulo $\mathfrak{p}$ and Weierstrass points on Drinfeld modular curves". PhD thesis. Univ. of Wisc., 2012.