

Journal Pre-proof

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PII: S0947-3580(26)00026-9
DOI: <https://doi.org/10.1016/j.ejcon.2026.101473>
Reference: EJCON 101473

To appear in: *European Journal of Control*

Received date: 12 July 2025
Revised date: 13 December 2025
Accepted date: 7 February 2026

Please cite this article as: Enrico Franco, Passivity-based control of underactuated systems with non-integrable state-dependent matched disturbances, *European Journal of Control* (2025), doi: <https://doi.org/10.1016/j.ejcon.2026.101473>



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Highlights

Passivity-based control of underactuated systems with non-integrable state-dependent matched disturbances

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- New passivity-based controllers for port-Hamiltonian and Lagrangian formulations.
- Constructive solutions for two classes of non-integrable matched disturbances.
- Numerical simulations for two examples.

Passivity-based control of underactuated systems with non-integrable state-dependent matched disturbances

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ARTICLE INFO

Keywords:

disturbances, energy shaping, non-linear systems, port-Hamiltonian systems, Lagrangian systems, underactuated mechanical systems.

ABSTRACT

This work investigates the passivity-based control of a class of underactuated mechanical systems subject to matched disturbances that enter the dynamics through a state-dependent vector-valued function that is not integrable. The main contributions include a new passivity-based controller with a dynamic extension, designed with the port-Hamiltonian formalism, and, most importantly, a suitably defined function of the states, serving the purpose of estimating the disturbance parameters while circumventing the usual integrability assumption. A corresponding controller is designed with the Lagrangian formalism, and key differences are discussed. Numerical simulations on two examples demonstrate the effectiveness of the new controllers.

1. Introduction

The control of underactuated mechanical systems has been approached with various methodologies, including sliding-mode controllers [1], model-predictive controllers [2, 3], optimal controllers [4], and energy-shaping controllers, each having its respective advantages and drawbacks. The most notable energy-shaping methodologies include the *Interconnection and damping assignment Passivity based control* (IDA-PBC) [5], which employs the port-controlled Hamiltonian (PCH) formalism, and the *Controlled Lagrangians* (CL) [6, 7, 8], which employs the Lagrangian formalism. In both approaches, the control action is designed so that the closed-loop dynamics preserves the structure of a mechanical system, the passivity properties, and is characterized by a desired total energy. Several research works have investigated the effect of matched disturbances on underactuated systems, resulting in the so-called *Integral IDA-PBC* (iIDA-PBC) methodology, see [9, 10, 11, 12]. **This approach achieves robustification to constant additive matched disturbances through a dynamic extension while preserving the energy-based interpretation of IDA-PBC, without relying on persistency of excitation.** More recent works have extended the iIDA-PBC methodology to state-dependent matched disturbances, see [13], to time-dependent matched disturbances, see [14], to unmatched disturbances, see [15, 16], and to systems with disturbances and unmeasured momentum, see [17]. In parallel, an equivalent of the iIDA-PBC for the CL methodology, termed *integral CL* (iCL), was proposed for system subject to a class of matched and unmatched disturbances, see [18, 19]. **One important limitation of the previous works is that an integrability assumption is imposed on the vector-valued function of the system states through which the disturbance enters the system dynamics, see [13]. In other words, the ability to reject state-dependent disturbances hinges on solving a partial-differential equation (PDE), which represents a bottleneck in engineering practice.** This same limitation affected the original *Immersion & Invariance* methodology, see [20], and was later overcome with the introduction of a dynamic scaling term, see [21], which circumvents the integrability obstacle by employing an additional low-pass filter. Nevertheless, the *Immersion & Invariance* methodology does not provide the same energy-based interpretation of the control action as iIDA-PBC or iCL. **In summary, to the best of the authors' knowledge, an implementation of the iIDA-PBC and iCL methodologies for state-dependent disturbances that does not impose an integrability assumption is currently lacking. This is limiting in engineering practice, since common types of disturbances, such as viscous friction and Coulomb friction, may not verify the integrability assumption for some systems.**

This paper investigates the passivity-based control of a class of underactuated mechanical systems subject to linearly parameterized matched disturbances that enter the dynamics through a state-dependent vector-valued function that is not integrable. **This allows addressing a broader class of disturbances, and it is achieved here by defining a**

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dynamic extension and, most importantly, a suitable function of the states that verifies structural and stability properties, without introducing additional PDEs. The main contributions of this work include the following points.

- A new iIDA-PBC design is proposed that does not rely on the usual integrability assumption, see [13, 14], and constructive solutions for two classes of state-dependent disturbances are detailed.
- It is shown that the iCL methodology also relies on a similar integrability assumption, and a new iCL design that overcomes this limitation is proposed.

The effectiveness of the new controllers is demonstrated with numerical simulations on two examples, that is, an Acrobot system and a VTOL system. Compared to the *Immersion & Invariance* methodology, see [21], the proposed controllers do not require dynamic scaling and preserve the interpretation of the control action in terms of system energy.

A brief overview of the iIDA-PBC methodology and of the iCL methodology is provided in Section 2 for completeness. The new iIDA-PBC design is then detailed in Section 3. The corresponding iCL design is outlined in Section 4. Simulation results are presented in Section 5. Concluding remarks are given in Section 6.

Notation. Function arguments are specified on first use and subsequently omitted in equations for conciseness. Given a vector-valued function $f(x) \in \mathbb{R}^s$, $x \in \mathbb{R}^n$, the notation $\nabla_x f \in \mathbb{R}^{n \times s}$ indicates a matrix-valued function.

2. Problem formulation

2.1. Overview of IDA-PBC

The dynamics of an underactuated mechanical system with n degrees of freedom (DOFs) and the control input $u \in \mathbb{R}^m$ applied through the input matrix $G(q) \in \mathbb{R}^{n \times m}$, where $\text{rank}(G) = m < n$ for all $q \in \mathbb{R}^n$, and subject to the disturbances $\delta \in \mathbb{R}^n$, is described in port-Hamiltonian form as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & -D \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} (u + v) - \begin{bmatrix} 0 \\ \delta \end{bmatrix} \quad (1)$$

where $v \in \mathbb{R}^m$ is the auxiliary control corresponding to the iIDA-PBC methodology, and $D = D^\top \geq 0$ is the physical damping matrix. The system states are the position $q \in \mathbb{R}^n$ and the momenta $p = M(q)\dot{q} \in \mathbb{R}^n$. The passive output of (1) is $y = G^\top \nabla_p H$ and the total mechanical energy of the system is

$$H(q, p) = \frac{1}{2} p^\top M(q)^{-1} p + \Omega(q), \quad (2)$$

with the inertia matrix $M(q) = M(q)^\top > 0$ and the potential energy $\Omega(q)$. The remaining terms in (1) are the identity matrix I , and the vectors of partial derivatives of H with respect to q and p , that is, $\nabla_q H$ and $\nabla_p H$ respectively. The controller design aims at stabilizing the prescribed equilibrium $(q, p) = (q^*, 0)$, which should be assignable, that is, $G^\perp (\nabla_q \Omega(q^*) + \delta) = 0$. This is achieved, in the absence of disturbances (i.e., $\delta = 0$), by using $v = 0$ and the IDA-PBC control law [5]

$$u_{IDA} = G^\dagger (\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 \nabla_p H_d) + u_d, \quad u_d = -K_v G^\top \nabla_p H_d, \quad (3)$$

where $H_d(q, p) = \frac{1}{2} p^\top M_d(q)^{-1} p + \Omega_d(q)$ is the prescribed Hamiltonian, $G^\dagger = (G^\top G)^{-1} G^\top$ and $K_v = K_v^\top > 0$ is a design parameter, while the matrix $J_2(q, p) = -J_2^\top(q, p)$ is used as free parameter, see [5] for details. The control law (3) is implementable provided that the potential energy $\Omega_d(q)$ and the inertia matrix $M_d(q) = M_d^\top(q) > 0$ verify for all $(q, p) \in \mathbb{R}^{2n}$ the PDEs

$$G^\perp (\nabla_q (p^\top M^{-1} p) - M_d M^{-1} \nabla_q (p^\top M_d^{-1} p) + 2J_2 M_d^{-1} p) = 0, \quad (4)$$

$$G^\perp (\nabla_q \Omega - M_d M^{-1} \nabla_q \Omega_d) = 0, \quad (5)$$

where $G^\perp$ is defined such that $G^\perp G = 0$ and $\text{rank}(G^\perp) = n - m$. The resulting closed-loop dynamics is

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & J_2 - D_d \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}, \quad (6)$$

where $D_d = GK_v G^\top + DM^{-1}M_d$, and $y_d = G^\top \nabla_p H_d$ is a passive output of (6), see [5]. To stabilize the prescribed equilibrium $(q, p) = (q^*, 0)$, the potential energy Ω_d should admit a strict minimizer in q^* hence verifying the conditions $\Omega_d(q^*) = 0$, $\nabla_q \Omega_d(q^*) = 0$ and $\nabla_q^2 \Omega_d(q^*) > 0$. Computing the time-derivative of H_d along the trajectories of the closed-loop system (6) yields then

$$\dot{H}_d = -\nabla_p H_d^\top (GK_v G^\top + DM^{-1}M_d) \nabla_p H_d \leq 0. \quad (7)$$

The equilibrium $(q, p) = (q^*, 0)$ is asymptotically stable if $\Theta_1 = GK_v G^\top + \frac{1}{2}(DM^{-1}M_d + M_d M^{-1}D) > 0$, where $GK_v G^\top$ is rank-deficient, see [22]. If $D = 0$, we have $\Theta_1 = GK_v G^\top \geq 0$, and the equilibrium is asymptotically stable for all $K_v > 0$, provided that $y_d = G^\top \nabla_p H_d$ is a detectable output of (6), that is, $y_d \rightarrow 0 \implies (q, p) \rightarrow (q^*, 0)$, see [5]. Asymptotic stability is global if $H_d(q, p)$ is radially unbounded, see [23].

If the system is subject to constant matched disturbances $\delta = G\delta_M \neq 0$, $\delta_M \in \mathbb{R}^m$, the input matrix G and the matrix M_d are constant, and the matrix M is independent of the unactuated coordinates, the iIDA-PBC design proposed in [9] can be used to compensate the disturbance. Subsequent versions of iIDA-PBC have extended the results to systems with non-constant matrix M_d [24, 25]. In our recent work [15, 13, 16] we have proposed iIDA-PBC designs for underactuated mechanical systems subject to state-dependent matched and unmatched disturbances. In particular, if $\delta = \phi(p)\delta_M$, where $\phi(p) = GG^\top f(p)$, $\delta_M \in \mathbb{R}$ is constant and unknown, while $f(p) \in \mathbb{R}^n$ is a vector-valued function of p , and $D = 0$, the disturbance can be compensated by the iIDA-PBC controller

$$u = u_{IDA} + v, \quad v = (K_v G^\top G + \Gamma_1) G^\top f(p) K_I (\zeta_M - F(p)), \quad (8)$$

where $\Gamma_1 > 0$ is a tuning parameter, the scalar function $F(p)$ is such that $\nabla_p F(p) = GG^\top f(p)$, see [13], and the time-derivative of $\zeta_M \in \mathbb{R}$ is defined as

$$\dot{\zeta}_M = -\left(M^{-1}M_d GG^\top f(p)\right)^\top \left(\nabla_q \Omega_d + \frac{1}{2} \nabla_q (p^\top M_d^{-1} p)\right) - \left((G\Gamma_1 + J_2 G) G^\top f(p)\right)^\top M_d^{-1} p. \quad (9)$$

In practice, the existence of the scalar function $F(p)$ is a strong assumption that is not generally verified. A common example is viscous friction in revolute joints of inverted pendulum systems, which is typically a function of the angular velocity $\dot{q}$ rather than of the momenta $p = M(q)\dot{q}$, where $M(q)$ might not be diagonal, see Section 5.1.

2.2. Overview of CL

The system dynamics (1) can be expressed alternatively using the Lagrangian formalism, see [6, 7], as

$$\frac{d}{dt} \nabla_{\dot{q}} L(q, \dot{q}) - \nabla_q L(q, \dot{q}) + \nabla_{\dot{q}} \mathcal{R} = G(q)u - \delta, \quad (10)$$

where the system states are the position $q \in \mathbb{R}^n$ and the velocity $\dot{q} \in \mathbb{R}^n$, and the Lagrangian function is defined as

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^\top M \dot{q} - \Omega(q). \quad (11)$$

Physical damping is accounted for by the dissipation function $\mathcal{R}(\dot{q})$, where $\dot{q}^\top \nabla_{\dot{q}} \mathcal{R} \geq 0$, see [8]. If dissipation is due to viscous friction defined by the matrix $D = D^\top \geq 0$, we have $\mathcal{R}(\dot{q}) = \frac{1}{2} \dot{q}^\top D \dot{q} \geq 0$. In the absence of disturbances and damping (i.e., $\delta = 0$, $\mathcal{R}(\dot{q}) = 0$), the CL methodology [6] stabilizes the desired equilibrium $(q, \dot{q}) = (q^*, 0)$ by enforcing the prescribed Lagrangian $L_c(q, \dot{q}) = \frac{1}{2} \dot{q}^\top M_c \dot{q} - \Omega_c(q)$ and the closed-loop dynamics

$$\frac{d}{dt} \nabla_{\dot{q}} L_c(q, \dot{q}) - \nabla_q L_c(q, \dot{q}) = 0, \quad (12)$$

with the inertia matrix $M_c(q) = M_c^\top(q) > 0$, and the potential energy $\Omega_c(q) \geq 0$, where $q = q^*$ is a strict minimizer, that is, $\Omega_c(q^*) = 0$, $\nabla_q \Omega_c(q^*) = 0$ and $\nabla_q^2 \Omega_c(q^*) > 0$. The dynamics (12) is achieved by the control law

$$u_{CL} = G^\dagger (\nabla_q (M \dot{q}) \dot{q} - \nabla_q L) - G^\dagger M M_c^{-1} (\nabla_q (M_c \dot{q}) \dot{q} - \nabla_q L_c), \quad (13)$$

see [7], where M_c and Ω_c should verify for all $(q, \dot{q}) \in \mathbb{R}^{2n}$ the PDEs

$$G^\perp \left(-\nabla_q \left(\frac{1}{2} \dot{q}^\top M \dot{q} \right) + M M_c^{-1} \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right) + G^\perp (\nabla_q (M \dot{q}) \dot{q} - M M_c^{-1} \nabla_q (M_c \dot{q})) = 0, \quad (14)$$

$$G^\perp (\nabla_q \Omega - M M_c^{-1} \nabla_q \Omega_c) = 0. \quad (15)$$

These are equivalent to the PDEs (4) and (5) provided that

$$M_c = M M_d^{-1} M, \quad \Omega_c = \Omega_d, \quad J_2 = M_d M^{-1} (\nabla_q (M M_d^{-1} p)^\top - \nabla_q (M M_d^{-1} p)) M^{-1} M_d, \quad (16)$$

see [26]. Similarly to (3), a damping injection term can be included in (13) as

$$u = u_{CL} + u_{DI}, \quad u_{DI} = -K_v G^\top M^{-1} M_c \dot{q}, \quad (17)$$

which yields the closed-loop dynamics

$$\frac{d}{dt} \nabla_q L_c(q, \dot{q}) - \nabla_q L_c(q, \dot{q}) + \nabla_q \mathcal{R}_c = 0, \quad (18)$$

where $\nabla_q \mathcal{R}_c = M_c M^{-1} (D + G K_v G^\top M^{-1} M_c) \dot{q}$ in case of viscous friction. Defining the storage function $W = \Omega_c + \frac{1}{2} \dot{q}^\top M_c \dot{q} > 0$ and computing its time-derivative along the trajectories of the closed-loop system (18) yields then

$$\dot{W} = -\dot{q}^\top M_c M^{-1} (D + G K_v G^\top M^{-1} M_c) \dot{q} \leq 0. \quad (19)$$

The equilibrium $(q, \dot{q}) = (q^*, 0)$ is asymptotically stable provided that $M_c M^{-1} D + D M^{-1} M_c > 0$ and $K_v > 0$, where $G K_v G^\top$ is rank-deficient. If $D = 0$ (19) yields $\dot{W} = -\dot{q}^\top M_c M^{-1} G K_v G^\top M^{-1} M_c \dot{q} \leq 0$, and asymptotic stability is ensured for all $K_v > 0$ provided that $y_c = G^\top M^{-1} M_c \dot{q}$ is a detectable output of (18), that is, $y_c \rightarrow 0 \implies (q, \dot{q}) \rightarrow (q^*, 0)$, see [26, 8].

If the disturbances are only matched, that is, $\delta = \delta_M G G^\top f(q)$ where $\delta_M \in \mathbb{R}$ is an unknown scalar constant, while $f(q) \in \mathbb{R}^n$ is a known and continuously differentiable vector-valued function of q , they can be compensated with the iCL control law

$$u = u_{CL} + u_{DI} + v, \quad v = G^\top f(q) K_I (\zeta_M - F(q, \dot{q})), \quad (20)$$

see [18], where $F(q, \dot{q}) = f(q)^\top G G^\top M^{-1} M_c \dot{q}$, $K_I = k_I I$ is a tuning parameter, and $\zeta_M \in \mathbb{R}$ is

$$\dot{\zeta}_M = -(\nabla_q F)^\top M_c^{-1} \left(\nabla_q \Omega_c + \nabla_q \mathcal{R}_c + \nabla_q (M_c \dot{q}) \dot{q} - \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right) - (\nabla_q F)^\top \dot{q} + (\nabla_q F)^\top \dot{q}. \quad (21)$$

Differently from (8), the controller (20) accounts for disturbances dependent on the generalized position. However, extending this approach to velocity-dependent disturbances requires a similar integrability assumption as the iIDA-PBC (8), that is, a function $F(q, \dot{q})$ is required such that $\nabla_q F = M_c M^{-1} G G^\top f(q, \dot{q})$. This result is detailed in the following Lemma.

Lemma 1. Consider the system (10) with $\delta = G G^\top f(q, \dot{q}) \delta_M$ where $\delta_M \in \mathbb{R}$ is an unknown scalar constant, while $f(q, \dot{q}) \in \mathbb{R}^n$ is a continuously differentiable vector-valued function of $(q, \dot{q})$. Assume that the PDEs (14) and (15) are solved analytically by M_c and Ω_c , and define the closed-loop dynamics

$$\begin{aligned} \frac{d}{dt} \nabla_q L_c(q, \dot{q}) - \nabla_q L_c(q, \dot{q}) + \nabla_q \mathcal{R}_c(q, \dot{q}) + \nabla_q \Psi(q, \dot{q}, \zeta_M) &= 0, \quad L_c(q, \dot{q}) = \frac{1}{2} \dot{q}^\top M_c \dot{q} - V_c, \\ \nabla_q \mathcal{R}_c(q, \dot{q}) &= M_c M^{-1} (\nabla_q \mathcal{R} + G K_v G^\top M^{-1} M_c \dot{q}), \quad \Psi = \frac{k_I}{2} \left(\zeta_M - F(q, \dot{q}) - \frac{\delta_M}{k_I} \right)^2, \end{aligned} \quad (22)$$

where there exists a scalar function $F(q, \dot{q})$ such that $\nabla_q F = M_c M^{-1} G G^\top f(q, \dot{q})$. The following claims are in order.

1. The dynamics (22) is achieved with the control law

$$u = u_{CL} + u_{DI} + v, \quad v = G^\top f(q, \dot{q}) k_I (\zeta_M - F(q, \dot{q})), \quad (23)$$

where $k_I > 0$ is a scalar tuning parameter, and the time-derivative of ζ_M is given by (21).

2. The equilibrium point $(q, \dot{q}) = (q^*, 0)$ is asymptotically stable for all $k_I > 0$, $K_v > 0$ provided that either $\dot{q}^\top \nabla_q \mathcal{R}_c > 0, \forall \dot{q} \neq 0$ or that the passive output $y_c = G^\top M^{-1} M_c \dot{q}$ is detectable.

The proof is given in Appendix A.

Remark 1. Note that $\nabla_q F$ also depends on the matrix M_c , which is a parameter in the CL design. As such, the integrability assumption required by the iCL design (23) is very restrictive. To clarify this point, consider a system with $n = 2, m = 1$, $G^\top = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $f(q, \dot{q}) = \dot{q}$, where the disturbance is representative of viscous friction. It follows that $\nabla_q F = M_c M^{-1} \begin{bmatrix} \dot{q}_1 \\ 0 \end{bmatrix}$. The former PDE is not solvable analytically, unless the product $M_c M^{-1}$ yields a diagonal matrix. This imposes restrictions on the CL design (e.g., the CL controller (17) would only enforce potential energy shaping) which might not be sufficient for some systems (e.g., the stabilization of inverted pendulum systems also requires kinetic energy shaping, see Section 5.1).

3. Integral IDA-PBC design

3.1. System class

This section considers a class of underactuated mechanical systems with dynamics (1) that are subject to state-dependent matched disturbances as defined in the following assumptions.

Assumption 1. The PDEs (4) and (5) are solvable analytically. The model parameters are exactly known and the states (q, p) are measured. The equilibrium q^* is assignable, that is, $G^\perp (\nabla_q \Omega(q^*) + \delta) = 0$, and $q^* = \operatorname{argmin} (\Omega_d)$. In addition, $\Theta_1 \geq 0$ and the passive output $y_d = G^\top M_d^{-1} p$ of (6) is detectable.

This assumption is a standard requirement for the IDA-PBC methodology, see [5], where the condition on Θ_1 follows from [22]. Solving the PDEs (4) and (5) is a challenging task which has motivated a number of works including a method to simplify the PDEs for mechanical systems, see [27], the study of kinetic energy shaping for mechanical systems, see [28], the transformation of PDEs into Pfaffian differential equations, see [29], the algebraic solutions of the PDEs, see [30], the numerical solution of the PDEs with reinforcement learning, see [31]. Consequently, this aspect is not investigated in this work.

Assumption 2. The disturbances $\delta \in \mathbb{R}^n$ are defined as $\delta = \phi(q, p)\delta_M$, where $\phi(q, p) = GG^\top f(q, p) \in \mathbb{R}^n$, $\delta_M \in \mathbb{R}$ is constant and unknown, while $f(q, p) \in \mathbb{R}^n$ is a vector-valued function of (q, p) such that $\phi(q, p) = 0$ if and only if $p = 0$ (i.e., the case $\phi(q, 0) \neq 0$ is allowed).

To facilitate the comparison with previous works, see [13], the unknown parameter δ_M is a scalar constant. Notably, the requirement on $\phi(q, p) \in \mathbb{R}^n$ to be integrable, that is, the existence of a scalar function $F(q, p)$ such that $\nabla_p F(q, p) = \phi(q, p)$, has been discarded. Consequently, a wide range of functions $\phi(q, p)$ is admitted. In particular, $\phi(q, p)$ does not have to be monotonic and does not have to vanish at equilibrium, which is in contrast with [14]. This allows addressing a broader class of disturbances, which were not tractable in prior works, see Section 5. For completeness, the extension to $\delta_M \in \mathbb{R}^s$ is discussed in Section 3.3.

3.2. Controller design

A new iIDA-PBC design that achieves the regulation goal $q = q^*$ is detailed for the class of systems defined by *Assumptions 1* and *2*. The target closed-loop dynamics proposed in this work is

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\zeta}_M \end{bmatrix} = \begin{bmatrix} 0 & S_{12} & S_{13} \\ -S_{12}^\top & -S_{22} & S_{23} \\ -S_{13}^\top & -S_{23}^\top & -S_{33} \end{bmatrix} \begin{bmatrix} \nabla_q W_d \\ \nabla_p W_d \\ \nabla_{\zeta_M} W_d \end{bmatrix}, \quad (24)$$

where the matrices S_{ij} are defined to achieve matching with the open-loop dynamics (1) while satisfying passivity and structural conditions (see *Proposition 1*), that is,

$$\begin{aligned} S_{12} &= M^{-1} M_d, \quad S_{13} = S_{12} \nabla_p \psi, \quad S_{22} = GK_v G^\top + DS_{12} - J_2, \\ S_{23} &= -S_{12}^\top \nabla_q \psi - S_{22} \nabla_p \psi + \Gamma_1 \phi, \quad S_{33} = S_{13}^\top \nabla_q \psi + S_{23}^\top \nabla_p \psi. \end{aligned} \quad (25)$$

The additional state $\zeta_M \in \mathbb{R}$ is employed to estimate the unknown parameter δ_M . Differently from [13, 16], the task of finding a scalar function $F(q, p)$ such that $\nabla_p F(q, p) = \phi(q, p)$, which is a bottleneck in practice, is replaced by the task of designing the new function $\psi(q, p)$. Design requirements are provided in *Proposition 1*, and the new function $\psi(q, p)$ is in general not unique, that is, it can serve as a design parameter. Constructive solutions, which do not require solving additional PDEs, are provided in *Corollary 1* and *Corollary 2* for two classes of functions $\phi(q, p)$.

The storage function $W_d(q, p, \zeta_M)$ is given by

$$W_d = H_d(q, p) + \frac{1}{2} K_I (\zeta_M - \psi - \beta)^\top (\zeta_M - \psi - \beta), \quad (26)$$

where $\beta = \delta_M / (k_I \gamma_1)$ is an unknown constant, while $\gamma_1 > 0$ and $k_I > 0$ are constant scalar parameters, and $K_I = k_I I$, $\Gamma_1 = \gamma_1 I$ are diagonal matrices. The control law that achieves the target closed-loop dynamics (24) is

$$u_{IDA} = u_{IDA} + v, \quad v = \gamma_1 k_I G^\top f(q, p) (\zeta_M - \psi). \quad (27)$$

The time-derivative of ζ_M is defined as

$$\dot{\zeta}_M = - (M^{-1} M_d \nabla_p \psi)^\top (\nabla_q H_d) - (-M_d M^{-1} \nabla_q \psi - (G K_v G^\top + D S_{12} - J_2) \nabla_p \psi + \Gamma_1 \phi)^\top M_d^{-1} p. \quad (28)$$

Proposition 1. Consider the system (1) that satisfies *Assumptions 1* and 2, in closed-loop with the control law (27), where the time-derivative of ζ_M is (28). Design the function $\psi(q, p)$ such that for some $\gamma_1 > 0$

$$\gamma_1 \phi^\top \nabla_p \psi \geq (\nabla_p \psi)^\top \Theta_1 \nabla_p \psi, \quad (29)$$

and $\psi(q, 0) = 0$ if and only if $\phi(q, 0) = 0$. The following claims are in order.

1. The closed-loop dynamics is given by (24) with the parameters (25) and the storage function (26).
2. The equilibrium point $(q, p, \zeta_M) = (q^*, 0, \beta)$ of the closed-loop system (24) is asymptotically stable.

Proof. To prove the first claim, the partial derivatives of W_d are computed from (26) as

$$\begin{aligned} \nabla_q W_d &= \nabla_q H_d - K_I \nabla_q \psi (\zeta_M - \psi - \beta), \\ \nabla_p W_d &= M_d^{-1} p - K_I \nabla_p \psi (\zeta_M - \psi - \beta), \\ \nabla_{\zeta_M} W_d &= K_I (\zeta_M - \psi - \beta). \end{aligned} \quad (30)$$

Equating the corresponding rows of (1) and of (24) yields the matching equations

$$M^{-1} p = S_{12} \nabla_p W_d + S_{13} \nabla_{\zeta_M} W_d, \quad (31a)$$

$$-\nabla_q H - D \nabla_p H + G(u + v) - \phi \delta_M = -S_{12}^\top \nabla_q W_d - S_{22} \nabla_p W_d + S_{23} \nabla_{\zeta_M} W_d, \quad (31b)$$

$$\dot{\zeta}_M = -S_{13}^\top \nabla_q W_d - S_{23}^\top \nabla_p W_d - S_{33} \nabla_{\zeta_M} W_d. \quad (31c)$$

Step 1. Substituting S_{13} from (25) and the partial derivatives of W_d from (30) into (31a) yields

$$M^{-1} p = S_{12} (M_d^{-1} p - K_I \nabla_p \psi (\zeta_M - \psi - \beta)) + S_{12} (\nabla_p \psi) K_I (\zeta_M - \psi - \beta).$$

Refactoring the former expression while substituting S_{12} yields $M^{-1} p = M^{-1} M_d M_d^{-1} p$ which is verified for all $M_d > 0$.

Step 2. Substituting S_{22} and S_{23} from (25), and the partial derivatives of W_d from (30) into (31b) yields

$$\begin{aligned} -\nabla_q H - D \nabla_p H + G(u + v) - \phi \delta_M &= -S_{12}^\top (\nabla_q H_d - K_I \nabla_q \psi (\zeta_M - \psi - \beta)) \\ &\quad - \underbrace{(G K_v G^\top + D S_{12} - J_2)}_{S_{22}} (M_d^{-1} p - K_I \nabla_p \psi (\zeta_M - \psi - \beta)) \\ &\quad + (-S_{12}^\top \nabla_q \psi - S_{22} \nabla_p \psi + \Gamma_1 \phi) K_I (\zeta_M - \psi - \beta). \end{aligned}$$

Refactoring terms while substituting β cancels the unknown constant δ_M and the physical damping D yielding

$$G(u + v) = \nabla_q H - S_{12}^\top \nabla_q H_d + (J_2 - G K_v G^\top) M_d^{-1} p + K_I \gamma_1 \phi (\zeta_M - \psi).$$

Multiplying the former expression by $G^\perp$ yields the sum of PDEs (4) and (5), which are verified by *Assumptions 1*. Conversely, multiplying the former expression by $G^\dagger$ yields the control law (27).

Step 3. Substituting S_{33} from (25) and the partial derivatives of W_d from (30) into (31c) yields

$$\begin{aligned} \dot{\zeta}_M = & -S_{13}^\top (\nabla_q H_d - K_I \nabla_q \psi (\zeta_M - \psi - \beta)) - S_{23}^\top (M_d^{-1} p - K_I \nabla_p \psi (\zeta_M - \psi - \beta)) \\ & - (S_{13}^\top \nabla_q \psi + S_{23}^\top \nabla_p \psi) K_I (\zeta_M - \psi - \beta). \end{aligned}$$

Refactoring terms cancels β yielding (28), which completes the proof of the first claim.

To prove the second claim, proceed to computing the time-derivative of (26) along the trajectories of the closed-loop system (24), which yields

$$\dot{W}_d = -\nabla_p W_d^\top S_{22} \nabla_p W_d - \nabla_{\zeta_M} W_d^\top S_{33} \nabla_{\zeta_M} W_d, \quad (32)$$

where it follows from (25) that

$$\frac{1}{2} (S_{22} + S_{22}^\top) = \Theta_1, \quad S_{33} = -(\nabla_p \psi)^\top \Theta_1 (\nabla_p \psi) + \phi^\top \Gamma_1 (\nabla_p \psi). \quad (33)$$

It follows from *Assumption 1* that $\Theta_1 \geq 0$, while $\Gamma_1 = \gamma_1 I$ and $\gamma_1 > 0$ by design. Therefore, $S_{33} \geq 0$ if and only if $\phi^\top \gamma_1 (\nabla_p \psi) \geq (\nabla_p \psi)^\top \Theta_1 (\nabla_p \psi)$. If $\phi = 0$ at equilibrium, then also $\psi = 0$ by design, and it follows that $S_{33} = 0$. Conversely, $S_{33} \geq 0$ for some $\gamma_1 > 0$ such that (29) is verified. Therefore, $\dot{W}_d \leq 0$, and the equilibrium is stable.

If $\Theta_1 > 0$ and $S_{33} > 0$, invoking LaSalle's theorem (see Theorem 3.4 in [23]), it follows that the trajectories of (24) converge asymptotically to the invariant set $p = 0 \cap (\zeta_M - \psi - \beta) = 0$. Computing $\dot{p} = 0$ from (24) within this set yields $\nabla_q \Omega_d = 0$, where $q^* = \text{argmin}(\Omega_d)$ by design. Thus, the equilibrium point $(q, p, \zeta_M) = (q^*, 0, \beta)$ is asymptotically stable. Stability is global if (29) holds globally and $H_d(q, p)$ is radially unbounded (i.e., since W is quadratic in ζ_M), see [23].

If $\Theta_1 > 0$ and $S_{33} = 0$, the trajectories of (24) converge to the invariant set $p = 0$ (i.e., the disturbance vanishes at equilibrium, hence $\phi = 0$ and $\psi = 0$). Computing $\dot{p} = 0$ from (24) within this set yields again $\nabla_q \Omega_d = 0$.

If $\Theta_1 \geq 0$, that is, $D = 0$, and $S_{33} > 0$, the trajectories of (24) converge asymptotically to the invariant set $G^\top M_d^{-1} p = 0 \cap (\zeta_M - \psi - \beta) = 0$. Comparing (7) and (32) and employing basic signal chasing it follows that $\dot{W} \leq y_d^\top u_d$ for some $\gamma_1 > \epsilon > 0$, that is, $y_d = G^\top M_d^{-1} p$ is also a passive output of (24) and is a detectable output by *Assumption 1*. Computing $\dot{p} = 0$ from (24) within this set yields again $\nabla_q \Omega_d = 0$.

If $\Theta_1 \geq 0$ and $S_{33} = 0$ (i.e., at the equilibrium), the trajectories of (24) converge to the invariant set $G^\top M_d^{-1} p = 0$, where $y_d = G^\top M_d^{-1} p = 0$ is a detectable output by *Assumption 1*. Computing $\dot{p} = 0$ from (24) within this set yields again $\nabla_q \Omega_d = 0$. Note that, in case $S_{33} = 0$, the convergence of ζ_M to δ_M/k_I is not guaranteed $\square$

Constructive solutions for $\psi(q, p)$ are provided for two distinct classes of functions $\phi(q, p)$.

i) ϕ monotonic increasing (i.e., without loss of generality, since monotonic decreasing can be accommodated with a change of sign of δ_M). Some examples are: (a) $\phi = \tanh(Gy)$, which is representative of Coulomb friction, where $y = G^\top M^{-1} p \in \mathbb{R}^m$ and $\tanh(\cdot)$ is applied to each element of the vector Gy ; (b) $\phi = Gy$ which is representative of viscous friction.

ii) $\phi \geq 0$ (i.e., without loss of generality, since $\phi \leq 0$ can be accommodated with a change of sign of δ_M), where the inequality holds element-wise, but not monotonic. Assuming $m = 1$, some examples are $\phi = Gy^2$ and $\phi = G \frac{1}{1+y^2}$, where $y \in \mathbb{R}$. These functions are not representative of friction, but could be due to external forces.

Corollary 1. Consider system (1) that satisfies *Assumptions 1* and 2, in closed-loop with the control law (27), where $\dot{\zeta}_M$ is given in (28) and ϕ is a monotonic function of Gy such that $\phi^\top \tanh(Gy) \geq \kappa (\tanh(Gy))^\top \tanh(Gy)$, with $\kappa > 0$. Setting $\psi(q, p) = (\tanh(Gy))^\top \tanh(Gy)$ verifies (29) for some $\gamma_1 > \epsilon > 0$.

Proof. By design, $0 \leq \psi \leq 1$ with $\psi = 0$ at the equilibrium. Computing (29) yields

$$2(1 - \psi) \gamma_1 \phi^\top \underbrace{M^{-1} G G^\top}_{\Theta_2} \tanh(Gy) \geq 4(1 - \psi)^2 (\tanh(Gy))^\top \underbrace{G G^\top M^{-1} \Theta_1 M^{-1} G G^\top}_{\Theta_2^\top \Theta_1 \Theta_2} \tanh(Gy),$$

where the square matrix $\Theta_2 = M^{-1} G G^\top \in \mathbb{R}^{n \times n}$ is rank deficient. Define $\lambda_1 > 0$ and $\lambda_2 > 0$ the maximum eigenvalues of Θ_1 and Θ_2 respectively. Simplifying terms, while recalling that $0 \leq (1 - \psi) \leq 1$, and substituting the inequality $\phi^\top \tanh(Gy) \geq \kappa (\tanh(Gy))^\top \tanh(Gy)$ yields then

$$\gamma_1 \lambda_2 \phi^\top \tanh(Gy) \geq \gamma_1 \lambda_2 \kappa (\tanh(Gy))^\top \tanh(Gy) \geq 2 \lambda_1 \lambda_2^2 \tanh(Gy)^\top \tanh(Gy),$$

where the first inequality holds by hypothesis and the second holds for all $\gamma_1 > 2 \frac{\lambda_1 \lambda_2}{\kappa}$, concluding the proof $\square$

Corollary 2. Consider system (1) that satisfies *Assumptions 1* and 2, in closed-loop with the control law (27), where ξ_M is given in (28) and ϕ is a non-monotonic function of y such that $\phi^\top G I_m \geq \kappa I_m G^\top G I_m$, with $\kappa > 0$. Setting $\psi(q, p) = I_m^\top \tanh(y)$, where $I_m \in \mathbb{R}^m$ is a column vector with all elements equal to one, verifies (29) for some $\gamma_1 > \epsilon > 0$.

Proof. By design, $-1 \leq \psi \leq 1$ with $\psi = 0$ at the equilibrium. Computing (29) yields

$$\gamma_1(1 - \psi^2)\phi^\top M^{-1} G I_m \geq (1 - \psi^2)^2 I_m^\top G^\top M^{-1} \Theta_1 M^{-1} G I_m.$$

Noting that $0 \leq (1 - \psi^2) \leq 1$, simplifying terms and substituting the inequality $\phi^\top G I_m \geq \kappa I_m G^\top G I_m$ yields then

$$\gamma_1 \phi^\top M^{-1} G I_m \geq \gamma_1 \kappa I_m^\top G^\top M^{-1} G I_m \geq I_m^\top G^\top M^{-1} \Theta_1 M^{-1} G I_m.$$

Define $\lambda_1 > 0$ and $\lambda_2 > 0$ the maximum of $\Theta_1 = I_m^\top G^\top M^{-1} G I_m$ and $\Theta_2 = I_m^\top G^\top M^{-1} \Theta_1 M^{-1} G I_m$ respectively. Then, the former inequality yields $\gamma_1 \kappa \lambda_1 \geq \lambda_2$, which holds for all $\gamma_1 > \frac{\lambda_2}{\kappa \lambda_1}$ concluding the proof $\square$

Remark 2. Defining a bounded function ψ with bounded derivative is convenient for verifying (29) globally in both corollaries. To this end, the function $\tanh(Gy)$ has been used for illustrative purposes. Note that alternative parameterizations of ψ can be used in *Corollary 1* and 2, that is, $\psi(q, p) = (\tanh(y))^\top G^\top G \tanh(y)$ and $\psi(q, p) = I_n^\top G \tanh(y)$ respectively. In general, if ϕ is a monotonic function of Gy and $\phi^\top h(Gy) \geq \kappa (h(Gy))^\top h(Gy)$ for some known monotonic function $h(x)$ such that $0 \leq \nabla_x h(x) \leq \epsilon$, then setting $\psi = (h(Gy))^\top h(Gy)$ verifies (29) for some $\gamma_1 > \epsilon > 0$. Similarly, if ϕ is a non-monotonic function of y such that $\phi^\top G I_m \geq \kappa I_m G^\top G I_m$, with $\kappa > 0$, then setting $\psi = I_m^\top h(y) \geq 0$, where $0 \leq \nabla_x h(x) \leq \epsilon$, verifies (29) for some $\gamma_1 > \epsilon > 0$. If there exists $F(q, p)$ such that $\nabla_p F(q, p) = \phi(q, p)$, then setting $\psi = F(q, p)$ verifies (29) for all $\gamma_1 > \lambda_M$, where λ_M is the maximum eigenvalue of Θ_1 (i.e., $\lambda_M = k_v$ in case $D = 0$), which is in accordance with [13].

Note finally that the formulation employed for the new controller (27) is more general than [13], since in the latter it is assumed $D = 0$ and the disturbance is dependent on the momentum but not on the position. In addition, a different parameterization is employed in [13], where β is defined such that $G(K_v G^\top G + \Gamma_1) \beta K_I = G \delta_M$.

3.3. Extension to s disturbances

For completeness, this section extends the result of *Proposition 1* to s additive disturbances, as defined by the following assumption.

Assumption 3. The disturbance $\delta \in \mathbb{R}^n$ is defined as $\delta = \phi(q, p) \delta_M$, where $\phi(q, p) = G f(q, p) \in \mathbb{R}^{n \times s}$, $\delta_M \in \mathbb{R}^s$ is constant and unknown, while $f(q, p) \in \mathbb{R}^{m \times s}$ is a matrix-valued function of (q, p) such that $\phi = 0$ if and only if $p = 0$. This represents the extension of *Assumption 2* to dimension s : δ_M is a vector of s unknown constants; each of the s columns of $\phi(q, p)$ is a vector-valued function of (q, p) ; the disturbance $\delta = \phi(q, p) \delta_M$ is the sum of s components, each the product of an element of δ_M and a column of $\phi(q, p)$. Notably, the columns of $\phi(q, p)$ are not required to be integrable in p .

Proposition 2. Consider the system (1) that satisfies *Assumptions 1* and 3, in closed-loop with the control law

$$u_{IDA} = u_{IDA} + v, \quad v = \gamma_1 k_I f(q, p) (\xi_M - \psi). \quad (34)$$

where the time-derivative of $\xi_M \in \mathbb{R}^s$ is given by (28). Design each element of the vector-valued function $\psi(q, p) \in \mathbb{R}^s$ such that the inequality (29) for the corresponding column of $\phi(q, p) \in \mathbb{R}^{n \times s}$ is verified by some $\gamma_1 > 0$, where $\psi(q, 0) = 0$ if and only if $\phi(q, 0) = 0$. The following claims are in order.

1. The closed-loop dynamics is given by (24) with the parameters (25) and the storage function (26).
2. The equilibrium point $(q, p, \xi_M) = (q^*, 0, \beta)$ of the closed-loop system (24) is asymptotically stable.

The proof is identical to that of *Proposition 1*, where ξ_M and $\psi(q, p)$ are now vectors of s components.

4. Integral CL design

Due to their different formulation, the CL methodology and the IDA-PBC methodology typically result in slightly different PDEs, that is, (4)-(5) and (14)-(15) respectively, which become identical under certain conditions, see [22]. The iCL methodology bears similarity to the corresponding iIDA-PBC, where the latter provides a more intuitive interpretation in terms of system energy. Conversely, the iCL methodology might result in a simpler implementation (i.e., it may not require input transformations or partial feedback linearization) for some systems, see [19]. Similarly to the iIDA-PBC, the iCL methodology also hinges on a restrictive integrability assumption, see *Lemma 1*.

4.1. System class

This section considers a class of underactuated mechanical systems with dynamics (10), subject to state-dependent matched disturbances, that are qualified by the following assumptions.

Assumption 4. The PDEs (14) and (15) are solvable analytically. The equilibrium q^* is assignable, that is, $q^* = \text{argmin}(\Omega_c)$ and $G^\perp(\nabla_q \Omega(q^*) + \delta) = 0$. In addition, $M_c M^{-1} D + D M^{-1} M_c \geq 0$ and the passive output $y_c = G^\top M^{-1} M_c \dot{q}$ of (18) is detectable. The model parameters are exactly known and the states $(q, \dot{q})$ are measured.

Assumption 5. The disturbances $\delta \in \mathbb{R}^n$ are defined as $\delta = \phi(q, \dot{q}) \delta_M$, where $\phi(q, \dot{q}) = G G^\top f(q, \dot{q})$, $\delta_M \in \mathbb{R}$ is constant and unknown, while $f(q, \dot{q}) \in \mathbb{R}^n$ is a vector-valued function of $(q, \dot{q})$ such that $\phi(q, \dot{q}) = 0$ if and only if $\dot{q} = 0$ (i.e., the case $\phi(q, 0) \neq 0$ is allowed).

The above assumptions are the counterparts of *Assumption 1* and *Assumption 2* respectively for the proposed iCL design. Notably, the existence of a scalar function $F(q, \dot{q})$ such that $\nabla_{\dot{q}} F = M_c M^{-1} G G^\top f(q, \dot{q})$, which is essential in *Lemma 1*, is not required. Instead, a new function $\psi(q, \dot{q})$ is defined such that $(\nabla_{\dot{q}} \psi)^\top M^{-1} \phi \geq 0$ where $(\nabla_{\dot{q}} \psi)^\top M^{-1} \phi = 0$ if and only if $\dot{q} = 0$. Note that the design requirements for $\psi(q, \dot{q})$ are verified by several functions, hence the choice of $\psi(q, \dot{q})$ is not unique.

4.2. Controller design

In this section, a new iCL design inspired by the new iIDA-PBC (27) is outlined.

Proposition 3. Consider the system (10) that verifies *Assumption 4* and *5*, and define the closed-loop dynamics

$$\frac{d}{dt} \nabla_{\dot{q}} L_c(q, \dot{q}) - \nabla_q L_c(q, \dot{q}) + \nabla_{\dot{q}} \mathcal{R}_c(q, \dot{q}) - M_c M^{-1} \phi k_I \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right) = 0, \quad (35)$$

where $(\nabla_{\dot{q}} \psi)^\top M^{-1} \phi \geq 0$. The following claims are in order.

1. The dynamics (35) is achieved with the control law

$$u = u_{CL} + u_{DI} + v, \quad v = G^\top f(q, \dot{q}) k_I (\zeta_M - \psi(q, \dot{q})), \quad (36)$$

where $k_I > 0$ is a scalar tuning parameter, and the time-derivative of ζ_M is given by

$$\dot{\zeta}_M = -\nabla_{\dot{q}} \psi^\top M_c^{-1} \left(\nabla_q \Omega_c + \nabla_{\dot{q}} \mathcal{R}_c + \nabla_q (M_c \dot{q}) \dot{q} - \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right) - (\dot{q}^\top M_c M^{-1} \phi)^\top + \nabla_q \psi^\top \dot{q}. \quad (37)$$

2. The equilibrium point $(q, \dot{q}, \zeta_M) = (q^*, 0, \delta_M/k_I)$ is asymptotically stable for all $k_I > 0$, $K_v > 0$.

The proof is given in Appendix C.

Remark 3. The new iIDA-PBC design (27) and the corresponding new iCL design (36) are structurally very similar, which is in agreement with prior works, see [18]. Conversely, the time-derivative $\dot{\zeta}_M$ in (28), which results from enforcing a port-Hamiltonian structure of the closed-loop dynamics (24), is structurally different from (37). In this respect, the iIDA-PBC design (27) provides a clearer interpretation in terms of system energy compared to the iCL design (36), which is in agreement with [19]. A further difference between the two designs is the condition on ψ imposed by *Proposition 1*, that is, (29), and the corresponding condition in *Proposition 3*, that is, $(\nabla_{\dot{q}} \psi)^\top M^{-1} \phi \geq 0$, which is comparatively less restrictive (i.e., $\nabla_{\dot{q}} \psi$ is not required to be upper bounded). Constructive solutions of $\psi(q, \dot{q})$ that verify the condition $(\nabla_{\dot{q}} \psi)^\top M^{-1} \phi \geq 0$ can be provided for different classes of functions ϕ in a similar way to *Corollary 1* and *2*. In particular, if ϕ is a monotonic function of Gy , where $y = G^\top \dot{q}$, such that $\phi^\top h(Gy) \geq \kappa (h(Gy))^\top h(Gy)$, where $\kappa > 0$ and the monotonic function $h(x)$ is such that $\nabla_x h(x) \geq 0$, then a suitable

design is $\psi(q, \dot{q}) = (h(Gy))^T h(Gy)$. Conversely, if ϕ is a non-monotonic function of y such that $\phi^T G I_m \geq \kappa I_m G^T G I_m$, with $\kappa > 0$, a suitable design is $\psi(q, \dot{q}) = I_m^T h(Gy)$. Note that, differently from the new iIDA-PBC design (27), $\nabla_x h(x)$ does not have to be upper bounded. If $\phi(q, 0) = 0$ (i.e., the disturbance is vanishing at equilibrium), the conditions of Proposition 3 can be trivially verified by $\nabla_{\dot{q}} \psi = 0$, that is, if ψ does not depend on $\dot{q}$. In such case, however, the convergence of ζ_M to δ_M/k_I is not guaranteed. Finally, if a scalar function $F(q, \dot{q})$ exists such that $\nabla_{\dot{q}} F = M_c M^{-1} G G^T f(q, \dot{q})$, then setting $\psi = F(q, \dot{q})$ recovers the result of Lemma 1 (i.e., (37) becomes coincident to (21)) and the corresponding interpretation in terms of system energy.

4.3. Extension to s disturbances

Following the same logic of Assumption 3, this section extends the result of Proposition 3 to s additive disturbances, as defined by the following assumption.

Assumption 6. The disturbance $\delta \in \mathbb{R}^n$ is defined as $\delta = \phi(q, \dot{q})\delta_M$, where $\phi(q, \dot{q}) = G f(q, \dot{q}) \in \mathbb{R}^{n \times s}$, $\delta_M \in \mathbb{R}^s$ is constant and unknown, while $f(q, \dot{q}) \in \mathbb{R}^{m \times s}$ is a matrix-valued function of $(q, \dot{q})$ such that $\phi = 0$ if and only if $\dot{q} = 0$.

Proposition 4. Consider the system (10) that satisfies Assumptions 4 and 6, in closed-loop with the control law

$$u = u_{CL} + u_{DI} + v; \quad v = f(q, \dot{q})k_I(\zeta_M - \psi(q, \dot{q})), \quad (38)$$

where the time-derivative of $\zeta_M \in \mathbb{R}^s$ is given by (37). Design the vector-valued function $\psi(q, \dot{q}) \in \mathbb{R}^s$ such that $(\nabla_{\dot{q}} \psi)^T M^{-1} \phi \geq 0$, where $\psi(q, 0) = 0$ if and only if $\phi(q, 0) = 0$. The following claims are in order.

1. The closed-loop dynamics is given by (35).
2. The equilibrium point $(q, \dot{q}, \zeta_M) = (q^*, 0, \delta_M/k_I)$ is asymptotically stable for all $k_I > 0$, $K_v > 0$.

The proof is identical to that of Proposition 3, where ζ_M is now a vector of s components.

5. Simulation Results

5.1. iIDA-PBC for the Acrobot system

The Acrobot system [32] consists of an articulated pendulum with a single actuator at the elbow joint q_2 and an unactuated shoulder joint q_1 , see Figure 1. The open-loop dynamics is given by (1) where the input matrix is $G^T = [0 \quad 1]$, while the potential energy and the inertia matrix are

$$\Omega = g(c_4 \cos(q_1) + c_5 \cos(q_1 + q_2)), \quad M = \begin{bmatrix} c_1 + c_2 + 2c_3 \cos(q_2) & c_2 + c_3 \cos(q_2) \\ c_2 + c_3 \cos(q_2) & c_2 \end{bmatrix}.$$

The terms c_1, c_2, c_3, c_4, c_5 are constant parameters depending on the size of the links, while g is the gravity constant and $D = 0$. The system states are the position $q = (q_1, q_2)$ and the momenta $p = (p_1, p_2)$. The control goal for the unperturbed system is to stabilize the upright position $(q_1, q_2) = (0, 0)$, which is open-loop unstable. The IDA-PBC controller (3) for the Acrobot system yields

$$u = \frac{1}{2} \nabla_{q_2} (p^T M^{-1} p) + \nabla_{q_2} \Omega - [k_2 \quad k_3] M^{-1} \nabla_{q_2} \Omega_d + \frac{k_v}{\Delta_d} (k_2 p_1 - k_1 p_2), \quad (39)$$

where k_1, k_2, k_3, k_v are controller parameters, see [32], while the remaining terms are defined as

$$\begin{aligned} \nabla_{q_1} \Omega_d &= -k_0 \sin(q_1 - \mu q_2) - b_1 \sin(q_1) - b_2 \sin(q_1 + q_2) - b_3 \sin(q_1 + 2q_2) - b_4 \sin(q_1 - q_2) + k_u(q_1 - \mu q_2), \\ \nabla_{q_2} \Omega_d &= k_0 \mu \sin(q_1 - \mu q_2) - b_2 \sin(q_1 + q_2) - 2b_3 \sin(q_1 + 2q_2) + b_4 \sin(q_1 - q_2) - k_u(q_1 - \mu q_2), \\ J_2 &= 0, \quad M_d = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}, \quad b_1 = \frac{g}{2k_2} (c_3 c_4 \pm 2c_4 \sqrt{c_1 c_2}), \quad b_2 = \frac{g\mu}{2k_2(\mu+1)} (c_3 c_4 \pm 2c_5 \sqrt{c_1 c_2}), \\ b_3 &= \frac{g\mu c_3 c_5}{2k_2(\mu+2)}, \quad b_4 = \frac{g\mu c_3 c_4}{2k_2(\mu-1)}, \quad \Delta_d = k_1 k_3 - k_2^2. \end{aligned}$$

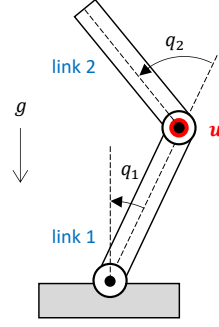


Figure 1: Schematic of Acrobot system with key parameters.

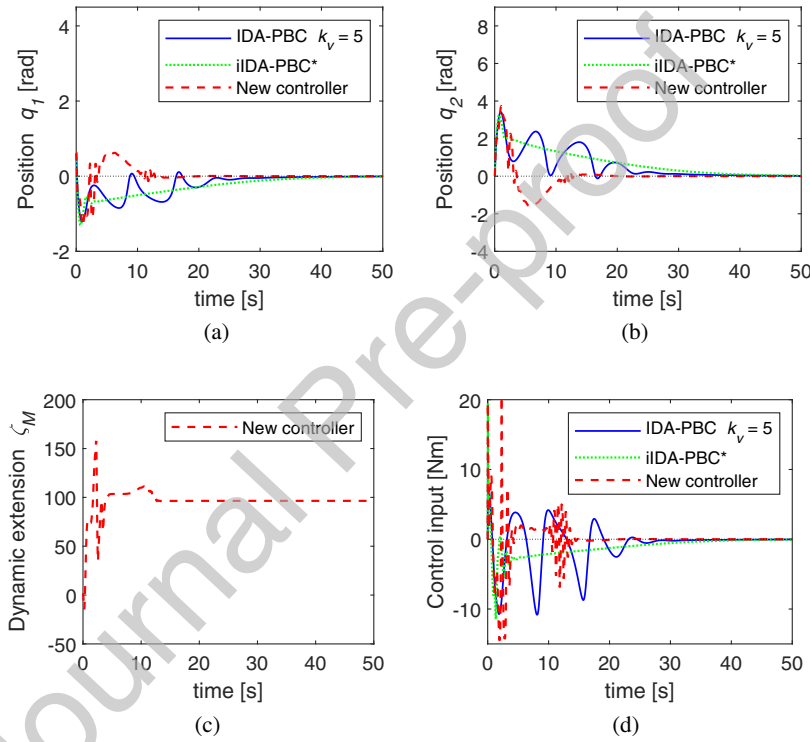


Figure 2: Simulation results for Acrobot system with monotonic matched disturbance representative of Coulomb friction: (a) position q_1 ; (b) position q_2 ; (c) extended state ζ_M ; (d) control input.

Numerical simulations have been performed in MATLAB using an ODE23 solver with the model parameters $c_1 = 0.23333$, $c_2 = 0.53333$, $c_3 = 0.2$, $c_4 = 0.3$, $c_5 = 0.2$, $g = 9.81$, and with the initial conditions $(q_1, q_2, p_1, p_2, \zeta_M) = (\pi/5, 0, 0, 0, 0)$. Two types of state-dependent matched disturbances have been considered.

The first type is representative of position-dependent Coulomb friction on the elbow joint (i.e., due to mechanical tolerances and misalignment of the joint axis with the corresponding bearing), that is, $\delta_M = 5$ and $\phi = G \tanh(y)(1 + \sin(q_2))$, where $y = G^T M^{-1} p = \frac{(c_1 p_2 - c_2 p_1 + c_2 p_2 - c_3 p_1 \cos(q_2) + 2c_3 p_2 \cos(q_2))}{(c_1 c_2 - c_3^2 \cos(q_2)^2)}$ corresponds to $\dot{q}_2$. This disturbance is

monotonic and bounded, that is, $|\phi| \leq 2$, hence we define $\psi = \tanh(y)^2(1 + \sin(q_2))^2$. Note that this disturbance is similar but not identical to that considered in [14] for the same system, that is, $\phi' = G \tanh(p_2)(1 + \sin(q_2))$. The key difference is that there does not exist a function $F(q, p)$ such that $\nabla_p F(q, p) = \phi(q, p)$, therefore the design [14] is not directly applicable in this case. The tuning parameters for the control law (39) have been chosen as in [32] for

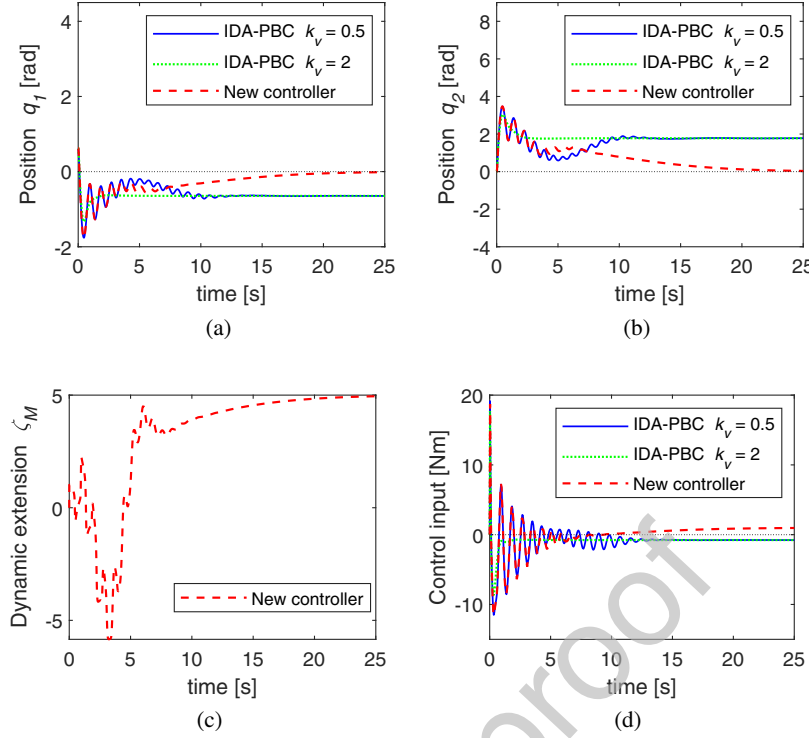


Figure 3: Simulation results for Acrobot system with non-monotonic matched disturbance quadratic in y : (a) position q_1 ; (b) position q_2 ; (c) extended state ζ_M ; (d) control input.

illustrative purposes, that is, $k_0 = -35$, $k_1 = 0.03386$, $k_2 = 0.1$, $k_3 = 0.59073$, $\mu = -0.6019$, $k_u = 1$, $k_v = 0.5$. In addition, we have set $k_I = 0.001$ and $\gamma_1 = 50$ in (27). The system response with various controllers is shown in Figure 2. Employing the new controller (27) the system reaches the prescribed equilibrium $(q_1^*, q_2^*) = (0, 0)$ while ζ_M converges to $\beta = \delta_M / (k_I \gamma_1) = 100$. Instead, employing the IDA-PBC (39) with the same tuning parameters yields instability because the matched disturbance is not compensated. Employing a larger value of k_v in (39), that is, $k_v = 5$, can stabilize the prescribed equilibrium at the cost of a slower response, and yields an oscillatory transient. Employing the controller [14], indicated here as iIDA-PBC* and implemented with the parameters $k_I = 1$, $\gamma_1 = 10$, $k_v = \frac{p_2}{2 \tanh(p_2)}$, results in a similarly slow response but a smoother transient, thanks to the variable damping assignment. In summary, the new controller (27) yields a faster response with comparable control effort.

The second type of disturbance is defined by $\delta_M = 1$ and $\phi = G \frac{1}{1+y^2}$. Also in this case, there does not exist a function $F(q, p)$ such that $\nabla_p F(q, p) = \phi(q, p)$. This disturbance does not vanish at the equilibrium, and it has been chosen for illustrative purposes since it was not tractable in prior works, see [14]. While this disturbance is not representative of friction, it could be due to an external moment. Note that this disturbance is bounded but not monotonic, hence we define $\psi = \tanh(y)$ and set $k_I = 0.01$ and $\gamma_1 = 20$ in (27). The system response with various controllers is shown in Figure 3. Employing the new controller (27) the system reaches the prescribed equilibrium $(q_1^*, q_2^*) = (0, 0)$ while ζ_M converges to $\beta = \delta_M / (k_I \gamma_1) = 5$. Instead, employing the IDA-PBC (39) with the same tuning parameters yields a large steady-state error since the matched disturbance is not compensated. In this case, even employing a larger k_v in (39), that is, $k_v = 2$, is not sufficient to ensure convergence to the prescribed equilibrium. In addition, the controller [14] does not apply to this type of disturbance since ϕ does not vanish at equilibrium.

5.2. iCL for the VTOL system

The vertical-take-off-and-landing aircraft (VTOL) is characterized by two actuators, that is, (u_1, u_2) , and three DOFs, that is, the horizontal and vertical coordinates of the center of mass (x, y) and the roll angle θ , see Figure 4. The mass and the moment of inertia of the VTOL are denoted with m_0 and J respectively. The parameter $0 \leq \epsilon \leq 1$ captures the effect of the “sloped” wings, inducing a coupling between the vertical and the roll dynamics.

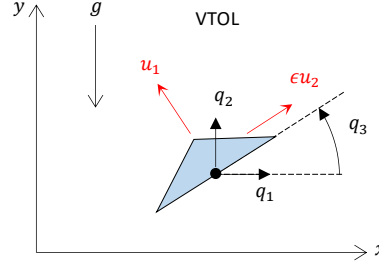


Figure 4: Schematic of the VTOL system with $(q_1, q_2, q_3) = (x, y, \theta)$.

The equations of motion are given by (2) with

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, G = \begin{bmatrix} -\sin(q_3) & \epsilon \cos(q_3) \\ \cos(q_3) & \epsilon \sin(q_3) \\ 0 & 1 \end{bmatrix}, M = \begin{bmatrix} m_0 & 0 & 0 \\ 0 & m_0 & 0 \\ 0 & 0 & J \end{bmatrix}, V = m_0 g q_2, \mathcal{R}(\dot{q}) = 0,$$

where $(q_1, q_2, q_3) = (x, y, \theta)$. The matrix M_c and the potential energy V_c that solve the PDEs (14) and (15) are

$$M_c = \begin{bmatrix} m_{11} & 0 & \sigma_2 \\ 0 & m_{22} & \sigma_1 \\ \sigma_2 & \sigma_1 & \frac{J\epsilon^2\gamma_{33} + \gamma_{11}(\gamma_{11}m_{11} - m_0)}{\epsilon^2\gamma_{33}^2} \end{bmatrix}, \sigma_1 = \frac{(\gamma_{11}m_{22} - m_0)}{\epsilon\gamma_{33}} \sin(q_3), \sigma_2 = \frac{(\gamma_{11}m_{11} - m_0)}{\epsilon\gamma_{33}} \cos(q_3),$$

$$V_c = \frac{k_p}{2} \left(q_1 - q_1^* + \frac{\gamma_{11} \sin(q_3)}{\epsilon\gamma_{33}} \right)^2 + \frac{k_p}{2} \left(q_2^* - q_2 + \frac{\gamma_{22}(\cos(q_3) - 1)}{\epsilon\gamma_{33}} \right)^2 + \frac{gm_0(\cos(q_3) - 1)}{\epsilon\gamma_{33}},$$

where $k_p, m_{11}, m_{22}, \gamma_{11}, \gamma_{22}, \gamma_{33}$ are tuning parameters, see [33].

Simulations have been performed in MATLAB using an ODE23 solver with the same model parameters and controller parameters as [33], that is, $\epsilon = 1, m_0 = 1, J = 1$, and $k_p = 0.9, k_v = 3.8I, \gamma_{11} = \gamma_{22} = 3, \gamma_{33} = -1, m_{11} = m_{22} = 1.5$ and $k_I = 0.1$ for the new iCL controller (36). The initial conditions are $(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, \zeta_M) = (-5, 5, 0.1, -0.1, -0.1, 0.1, 0)$. The prescribed equilibrium is $(q_1^*, q_2^*, q_3^*) = (0, 0, 0)$. Two types of disturbances have been considered for illustrative purposes. **The first type is representative of viscous friction**, that is, $\delta_M = 6$ and $\phi = GG^T \dot{q}$. This disturbance is monotonic and unbounded, hence we define $\psi = \tanh(\dot{q})^T GG^T \tanh(\dot{q})$. **Note that there does not exist a function $F(q, \dot{q})$ such that $\nabla_{\dot{q}} F(q, \dot{q}) = M_c M^{-1} \phi(q, \dot{q})$.** The system response with various controllers is shown in Figure 5. Employing the new controller (36) the system reaches the prescribed equilibrium, while ζ_M converges to $\beta = \delta_M/k_I = 60$. Instead, employing the baseline CL (17) with the same tuning parameters yields large oscillations because the matched disturbance is not compensated. Finally, employing a larger value of k_v in (17), that is, $k_v = 7$, can reduce oscillations at the cost of a slower response. Note, however, that choosing a sufficiently large k_v to dominate the disturbance is only feasible if the bounds of the uncertain parameter are known, which violates Assumption 5.

The second disturbance is representative of quadratic damping, that is, $\delta_M = -3.5$ and $\phi = GG^T \dot{q}^2$, where the square is applied to each element of the velocity vector. **Quadratic damping is a result of aerodynamic forces, specifically air drag.** This disturbance is not monotonic and is unbounded, hence we define $\psi = I_n^T GG^T \tanh(\dot{q})$, where $I_n \in \mathbb{R}^n$ is a column vector with all elements equal to one. Note that there does not exist a function $F(q, \dot{q})$ such that $\nabla_{\dot{q}} F(q, \dot{q}) = M_c M^{-1} \phi(q, \dot{q})$. The system response with various controllers is shown in Figure 6. Employing the baseline CL (17) with the same tuning parameters yields instability, while setting $k_v = 4.5$ yields a slow response. Employing the new controller (36) the system reaches the prescribed equilibrium, while ζ_M converges to $\beta = \delta_M/k_I = -35$. An even faster response is achieved by employing lower values of k_v , that is, $k_v = 3$, at the cost of a larger control effort.

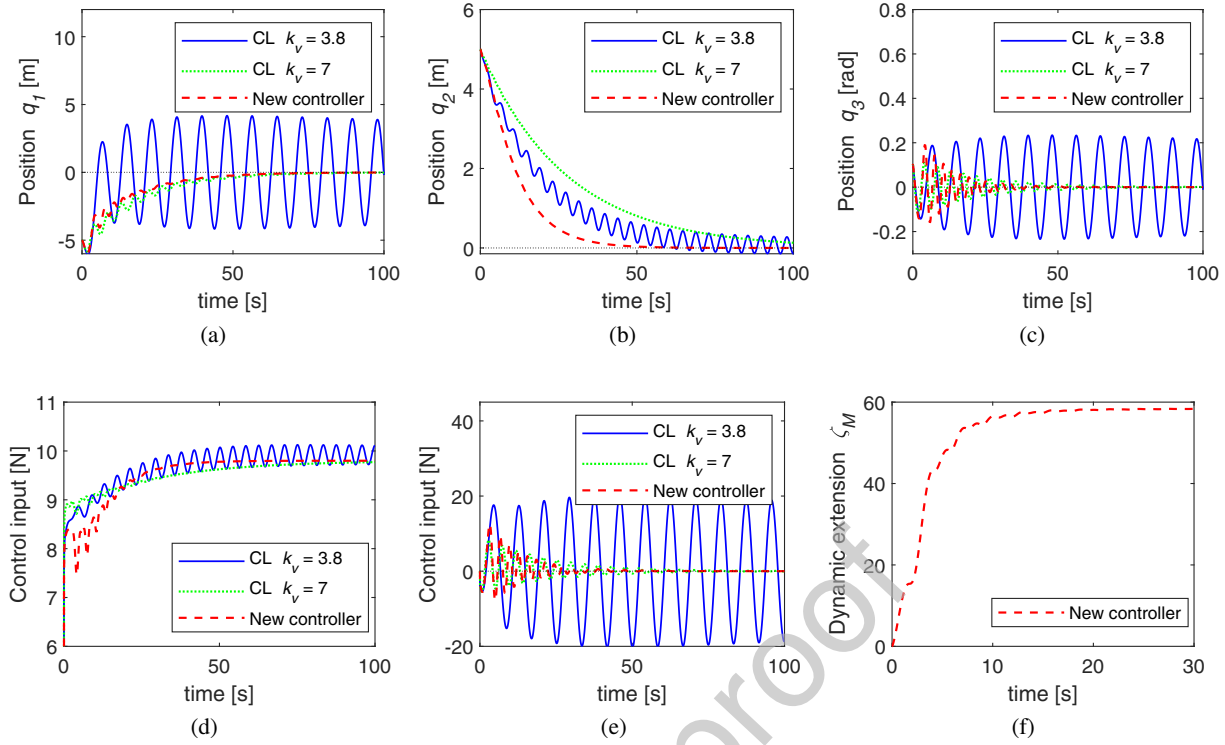


Figure 5: Simulation results for VTOL with monotonic matched disturbance representative of viscous friction: (a) x position; (b) y position; (c) roll angle θ ; (d) control input u_1 ; (e) control input u_2 ; (f) extended state ζ_M .

6. Conclusion

In this work, a new iIDA-PBC design and a corresponding iCL design have been proposed for a class of underactuated mechanical systems with matched disturbances that enter the system dynamics through a state-dependent vector-valued function that is not integrable. The controllers employ a dynamic extension to estimate the disturbance parameters, and, most importantly, a function of the states is designed to avoid integrability assumptions. Constructive solutions have been detailed for two types of disturbances. Simulation results on an Acrobot system and on a VTOL system indicate that the new controllers effectively reject various types of disturbances that were now tractable with prior implementations of iIDA-PBC and iCL. Future work will extend the results to unmatched disturbances.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRedit authorship contribution statement

Enrico Franco: Conceptualization, Formal analysis, Investigation, Software, Visualization, Writing - Original draft preparation.

Acknowledgment

This research was supported by the Engineering and Physical Sciences Research Council (grant agreement No EP/X033546/1).

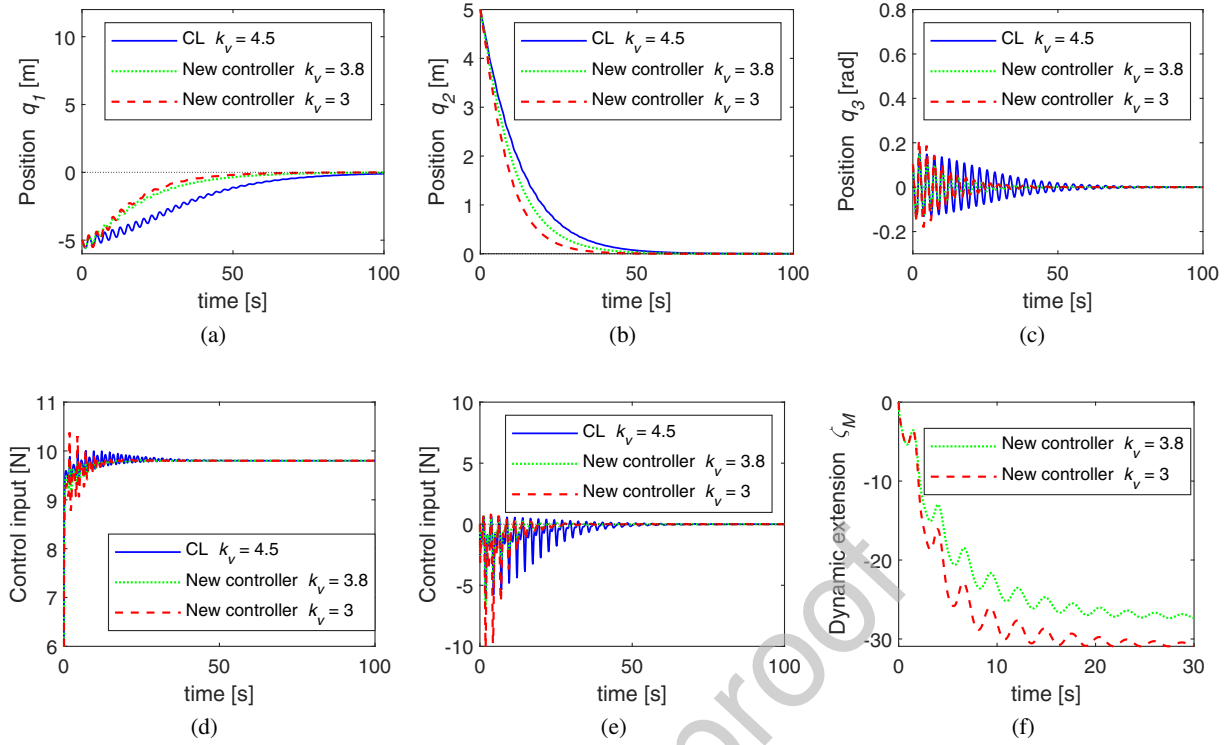


Figure 6: Simulation results for VTOL with non-monotonic matched disturbance representative of quadratic damping: (a) x position; (b) y position; (c) roll angle θ ; (d) control input u_1 ; (e) control input u_2 ; (f) extended state ζ_M .

References

- [1] D. Gutiérrez-Oribio, J. A. Mercado-Urbe, J. A. Moreno, L. Fridman, Robust global stabilization of a class of underactuated mechanical systems of two degrees of freedom, *International Journal of Robust and Nonlinear Control* 31 (2021) 3908–3928.
- [2] G. Bastos, E. Franco, Dynamic tube model predictive control for a class of soft manipulators with fluidic actuation, *International Journal of Robust and Nonlinear Control* (2023).
- [3] G. Bastos, E. Franco, Dynamic Tube-MPC for Underactuated Mechanical Systems With Matched and Unmatched Disturbances, *IEEE Control Systems Letters* (2025) 156 – 161.
- [4] Y. Yang, Y. Pan, C. Z. Xu, D. C. Wunsch, Hamiltonian-Driven Adaptive Dynamic Programming With Efficient Experience Replay, *IEEE Transactions on Neural Networks and Learning Systems* 35 (2024) 3278–3290.
- [5] R. Ortega, M. Spong, F. Gomez-Estern, G. Blankenstein, Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment, *IEEE Transactions on Automatic Control* 47 (2002) 1218–1233.
- [6] A. Bloch, N. Leonard, J. Marsden, Controlled Lagrangians and the stabilization of mechanical systems. I. The first matching theorem, *IEEE Transactions on Automatic Control* 45 (2000) 2253–2270.
- [7] A. Bloch, Dong Eui Chang, N. Leonard, J. Marsden, Controlled Lagrangians and the stabilization of mechanical systems. II. Potential shaping, *IEEE Transactions on Automatic Control* 46 (2001) 1556–1571.
- [8] C. Woolsey, C. K. Reddy, A. M. Bloch, D. E. Chang, N. E. Leonard, J. E. Marsden, Controlled Lagrangian Systems with Gyroscopic Forcing and Dissipation, *European Journal of Control* 10 (2004) 478–496.
- [9] A. Donaire, J. G. Romero, R. Ortega, B. Siciliano, Robust IDA-PBC for underactuated mechanical systems subject to matched disturbances, *International Journal of Robust and Nonlinear Control* 27 (2017) 1000–1016.
- [10] J. Ferguson, A. Donaire, R. Middleton, Integral control of port-Hamiltonian systems: Nonpassive outputs without coordinate transformation, *IEEE Transactions on Automatic Control* 62 (2017) 5947–5953.
- [11] M. Ryalat, D. S. Laila, A Robust IDA-PBC Approach for Handling Uncertainties in Underactuated Mechanical Systems, *IEEE Transactions on Automatic Control* 63 (2018) 3495–3502.
- [12] J. Ferguson, A. Donaire, R. Ortega, R. Middleton, Matched disturbance rejection for a class of nonlinear systems, *IEEE Transactions on Automatic Control* 65 (2019) 1710–1715.
- [13] E. Franco, P. Arpent, A. Donaire, Integral passivity-based control of underactuated mechanical systems with state-dependent matched disturbances, *International Journal of Robust and Nonlinear Control* (2023).
- [14] E. Franco, K. Chen, Integral IDA-PBC for underactuated mechanical systems with unmeasured actuator dynamics and time-varying matched disturbances, *European Journal of Control* 85 (2025) 101256.

- [15] E. Franco, Integral passivity-based control of underactuated mechanical systems with actuator dynamics and constant disturbances, *International Journal of Robust and Nonlinear Control* (2023).
- [16] E. Franco, P. Arpent, A. Donaire, F. Ruggiero, Integral IDA-PBC for Underactuated Mechanical Systems Subject to Matched and Unmatched Disturbances, *IEEE Control Systems Letters* 8 (2024) 1–1.
- [17] E. Franco, M. Ryalat, Position-Feedback Integral IDA-PBC for Constant Matched and Unmatched Disturbances, *International Journal of Robust and Nonlinear Control* (2025).
- [18] E. Franco, F. Forni, Integral Controlled Lagrangians for Underactuated Mechanical Systems Subject to Position-Dependent Matched Disturbances, *IEEE Control Systems Letters* (2024) 466–471.
- [19] E. Franco, Integral Controlled Lagrangians for underactuated mechanical systems subject to matched and unmatched disturbances, *Journal of the Franklin Institute* 362 (2025) 108205.
- [20] A. Astolfi, D. Karagiannis, R. Ortega, *Nonlinear and Adaptive Control with Applications*, Springer, London, 2007.
- [21] D. Karagiannis, M. Sassano, A. Astolfi, Dynamic scaling and observer design with application to adaptive control, *Automatica* 45 (2009) 2883–2889.
- [22] F. Gómez-Estern, A. Van der Schaft, Physical Damping in IDA-PBC Controlled Underactuated Mechanical Systems, *European Journal of Control* 10 (2004) 451–468.
- [23] H. Khalil, *Nonlinear Systems*, 2nd ed., Prentice-Hall, Upper Saddle River, NJ, 1996.
- [24] J. Ferguson, A. Donaire, R. H. Middleton, Integral Control of Port-Hamiltonian Systems: Nonpassive Outputs Without Coordinate Transformation, *IEEE Transactions on Automatic Control* 62 (2017) 5947–5953.
- [25] J. Ferguson, A. Donaire, R. Ortega, R. H. Middleton, Matched disturbance rejection for a class of nonlinear systems, *IEEE Transactions on Automatic Control* 65 (2019) 1710–1715.
- [26] G. Blankenstein, R. Ortega, A. J. Van Der Schaft, The matching conditions of controlled Lagrangians and IDA-passivity based control, *International Journal of Control* 75 (2002) 645–665.
- [27] M. Ryalat, D. S. Laila, A simplified IDA-PBC design for underactuated mechanical systems with applications, *European Journal of Control* 27 (2016) 1–16.
- [28] M. R. J. Harandi, H. D. Taghirad, On the matching equations of kinetic energy shaping in IDA-PBC, *Journal of the Franklin Institute* 358 (2021) 8639–8655.
- [29] M. R. J. Harandi, H. D. Taghirad, Solution of matching equations of IDA-PBC by Pfaffian differential equations, *International Journal of Control* (2021) 1–11.
- [30] K. Nunna, M. Sassano, A. Astolfi, Constructive Interconnection and Damping Assignment for Port-Controlled Hamiltonian Systems, *IEEE Transactions on Automatic Control* 60 (2015) 2350–2361.
- [31] A. Gheibi, A. R. Ghiasi, S. Ghaemi, M. A. Badamchizadeh, Interconnection and damping assignment control based on modified actor–critic algorithm with wavelet function approximation, *ISA Transactions* 101 (2020) 116–129.
- [32] A. D. Mahindrakar, A. Astolfi, R. Ortega, G. Viola, Further constructive results on interconnection and damping assignment control of mechanical systems: the Acrobot example, *International Journal of Robust and Nonlinear Control* 16 (2006) 671–685.
- [33] G. Chen, W. Huo., Adaptive Stabilization of a PVTOL Aircraft with Uncertainties Based on Controlled Lagrangians, in: *Chinese Control Conference, CCC*, volume 2022-July, IEEE Computer Society, 2022, pp. 392–397.

Appendix A

Proof of Lemma 1.

To prove the first claim, start by substituting (11) into (10) while accounting for the dissipation function $\mathcal{R}(\dot{q})$, and compute the acceleration $\ddot{q}$, which yields

$$\ddot{q} = M^{-1} \left(G(u + v) - \delta_M G G^\top f(q, \dot{q}) - \nabla_q \Omega - \nabla_{\dot{q}} \mathcal{R} - \nabla_q (M \dot{q}) \dot{q} + \nabla_q \left(\frac{1}{2} \dot{q}^\top M \dot{q} \right) \right). \quad (\text{A1})$$

Computing the acceleration $\ddot{q}$ from (22) yields instead

$$\ddot{q} = M_c^{-1} \left(-\nabla_q \Omega_c - \nabla_{\dot{q}} \mathcal{R}_c - \nabla_{\dot{q}} \Psi - \nabla_q (M_c \dot{q}) \dot{q} + \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right). \quad (\text{A2})$$

Equating (A1) and (A2) while substituting Ψ and $\mathcal{R}_c$ yields

$$\begin{aligned} M^{-1} \left(G(u + v) - \delta_M G G^\top f(q, \dot{q}) - \nabla_q \Omega - \nabla_{\dot{q}} \mathcal{R} - \nabla_q (M \dot{q}) \dot{q} + \nabla_q \left(\frac{1}{2} \dot{q}^\top M \dot{q} \right) \right) = \\ - M^{-1} \left(\nabla_{\dot{q}} \mathcal{R} + G K_v G^\top M^{-1} M_c \dot{q} \right) + M^{-1} G G^\top f(q, \dot{q}) k_I \left(\zeta_M - F(q, \dot{q}) - \frac{\delta_M}{k_I} \right) \\ - M_c^{-1} \left(\nabla_q \Omega_c + \nabla_q (M_c \dot{q}) \dot{q} - \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right). \end{aligned} \quad (\text{A3})$$

Refactoring (A3) cancels the parameter δ_M and the physical damping $\nabla_{\dot{q}} \mathcal{R}$. Premultiplying (A3) with $G^\perp M$ yields the sum of the PDEs (14) and (15), which are verified by design. Premultiplying (A3) with $G^\top M$ yields (23).

To prove the remaining claim, define the storage function

$$W(q, \dot{q}, \zeta_M) = \Omega_c + \frac{1}{2} \dot{q}^\top M_c \dot{q} + \Psi > 0, \quad (\text{A4})$$

and compute its time-derivative along the trajectories of system (22), which yields

$$\dot{W} = \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right)^\top \dot{q} + \nabla_{\dot{q}} \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right)^\top \ddot{q} + (\nabla_q \Omega_c)^\top \dot{q} + (\nabla_q \Psi)^\top \dot{q} + (\nabla_{\dot{q}} \Psi)^\top \ddot{q} + (\nabla_{\zeta_M} \Psi)^\top \dot{\zeta}_M. \quad (\text{A5})$$

Substituting $\ddot{q}$ from (A2), Ψ from (23), and $\dot{\zeta}_M$ from (21) while noting that $\nabla_q \Psi = -\nabla_{\zeta_M} \Psi \nabla_q F$ and $\nabla_{\dot{q}} \Psi = -\nabla_{\zeta_M} \Psi \nabla_{\dot{q}} F$ yields

$$\begin{aligned} \dot{W} = \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right)^\top \dot{q} + (\nabla_q \Omega_c)^\top \dot{q} - (\nabla_{\zeta_M} \Psi \nabla_q F)^\top \dot{q} + \dot{q}^\top M_c M_c^{-1} \left(-\nabla_q \Omega_c - \nabla_{\dot{q}} \mathcal{R}_c + \nabla_{\zeta_M} \Psi \nabla_{\dot{q}} F \right) \\ + \dot{q}^\top M_c \left(M_c^{-1} \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) - M_c^{-1} \nabla_q (M_c \dot{q}) \dot{q} \right) - (\nabla_{\zeta_M} \Psi \nabla_{\dot{q}} F)^\top M_c^{-1} \left(-\nabla_q \Omega_c - \nabla_{\dot{q}} \mathcal{R}_c + \nabla_{\zeta_M} \Psi \nabla_{\dot{q}} F \right) \\ + (\nabla_{\zeta_M} \Psi \nabla_{\dot{q}} F)^\top M_c^{-1} \left(\nabla_q (M_c \dot{q}) \dot{q} - \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right) + (\nabla_{\zeta_M} \Psi)^\top (\nabla_{\dot{q}} F)^\top M_c^{-1} (-\nabla_q \Omega_c - \nabla_{\dot{q}} \mathcal{R}_c) \\ - (\nabla_{\zeta_M} \Psi)^\top (\nabla_{\dot{q}} F)^\top M_c^{-1} \left(\nabla_q (M_c \dot{q}) \dot{q} - \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right) - (\nabla_{\zeta_M} \Psi)^\top \left((\nabla_{\dot{q}} F)^\top \dot{q} - (\nabla_q F)^\top \dot{q} \right). \end{aligned}$$

Refactoring terms and substituting $\nabla_{\dot{q}} \mathcal{R}_c(q, \dot{q})$, where $\mathcal{R}(\dot{q}) = \frac{1}{2} \dot{q}^\top D \dot{q} \geq 0$, yields finally

$$\dot{W} = -\dot{q}^\top M_c M^{-1} (D + G K_v G^\top M^{-1} M_c) \dot{q} - (\nabla_{\dot{q}} \Psi)^\top M_c^{-1} (\nabla_{\dot{q}} \Psi) \leq 0. \quad (\text{A6})$$

If $M_c M^{-1} D + D M^{-1} M_c > 0$ and $K_v > 0$, invoking LaSalle's theorem (see Theorem 3.4 in [23]) proves that the trajectories of the closed-loop system (22) converge asymptotically to the set $\dot{q} = 0 \cap \nabla_{\dot{q}} \Psi = 0$. Evaluating (A2) within this set yields $\nabla_q \Omega_c = 0$ thus it follows from *Assumption 4* that the equilibrium $(q, \dot{q}) = (q^*, 0)$ is locally asymptotically stable. Stability is global if $H_d(q, p)$ is radially unbounded (i.e., since W is quadratic in ζ_M), see [23].

If $D = 0$, the trajectories of the closed-loop system (22) converge asymptotically to the set $y_c = G^\top M^{-1} M_c \dot{q} = 0 \cap \nabla_{\dot{q}} \Psi = 0$. Comparing (19) and (A6) it follows that $\dot{W} \leq y_c^\top u_{DI}$, that is, y_c is also a passive output of (22). Since y_c is detectable by *Assumption 1*, evaluating (A2) within the set $y_c = 0 \cap \nabla_{\dot{q}} \Psi = 0$ yields $\nabla_q \Omega_c = 0$, hence the equilibrium $(q, \dot{q}) = (q^*, 0)$ is locally asymptotically stable $\square$

Appendix B

Proof of Proposition 3.

To prove the first claim, recall that substituting (11) into (10) while accounting for the dissipation function $\mathcal{R}(\dot{q})$, and computing the acceleration $\ddot{q}$ yields (A1). Computing the acceleration $\ddot{q}$ from (35) yields instead

$$\ddot{q} = M_c^{-1} \left(-\nabla_q \Omega_c - \nabla_{\dot{q}} \mathcal{R}_c + M_c M^{-1} \phi k_I \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right) - \nabla_q (M_c \dot{q}) \dot{q} + \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right). \quad (B1)$$

Equating (A1) and (B1) while substituting $\mathcal{R}_c$ yields

$$\begin{aligned} M^{-1} \left(G(u + v) - \phi \delta_M - \nabla_q \Omega - \nabla_{\dot{q}} \mathcal{R} - \nabla_q (M \dot{q}) \dot{q} + \nabla_q \left(\frac{1}{2} \dot{q}^\top M \dot{q} \right) \right) = \\ - M^{-1} (\nabla_{\dot{q}} \mathcal{R} + G K_v G^\top M^{-1} M_c \dot{q}) + M^{-1} \phi k_I \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right) \\ - M_c^{-1} \left(\nabla_q \Omega_c + \nabla_q (M_c \dot{q}) \dot{q} - \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right). \end{aligned} \quad (B2)$$

Refactoring (B2) cancels the parameter δ_M and the physical damping $\nabla_{\dot{q}} \mathcal{R}$. Premultiplying (B2) with $G^\top M$ yields the sum of the PDEs (14) and (15), which are verified by design. Premultiplying (B2) with $G^\top M$ yields (36).

To prove the remaining claim, define the storage function

$$W(q, \dot{q}, \zeta_M) = \Omega_c + \frac{1}{2} \dot{q}^\top M_c \dot{q} + \frac{1}{2} k_I \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right)^\top \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right) > 0, \quad (B3)$$

and compute its time-derivative along the trajectories of system (35), which yields

$$\begin{aligned} \dot{W} = \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right)^\top \dot{q} + \nabla_{\dot{q}} \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right)^\top \ddot{q} + (\nabla_q \Omega_c)^\top \dot{q} - \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right)^\top (\nabla_q \psi)^\top \dot{q} k_I \\ - \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right)^\top (\nabla_{\dot{q}} \psi)^\top \ddot{q} k_I + k_I \left(\zeta_M - \psi(q, \dot{q}, \zeta_p) - \frac{\delta_M}{k_I} \right)^\top \dot{\zeta}_M \end{aligned} \quad (B4)$$

Substituting $\ddot{q}$ from (B1) yields

$$\begin{aligned} \dot{W} = \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right)^\top \dot{q} + (\nabla_q \Omega_c)^\top \dot{q} + \dot{q}^\top \left(-\nabla_q \Omega_c - \nabla_{\dot{q}} \mathcal{R}_c + \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) - \nabla_q (M_c \dot{q}) \dot{q} \right) \\ + \dot{q}^\top M_c M^{-1} \phi k_I \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right) - \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right)^\top (\nabla_q \psi)^\top \dot{q} k_I \\ - \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right)^\top (\nabla_{\dot{q}} \psi)^\top M_c^{-1} \left(-\nabla_q \Omega_c - \nabla_{\dot{q}} \mathcal{R}_c + \nabla_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) - \nabla_q (M_c \dot{q}) \dot{q} \right) k_I \\ - \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right)^\top (\nabla_{\dot{q}} \psi)^\top M^{-1} \phi k_I^2 \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right) + k_I \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right)^\top \dot{\zeta}_M. \end{aligned}$$

Substituting $\dot{\zeta}_M$ from (37) and $\nabla_{\dot{q}} \mathcal{R}_c(q, \dot{q})$, where $\mathcal{R}(\dot{q}) = \frac{1}{2} \dot{q}^\top D \dot{q} \geq 0$, yields finally

$$\begin{aligned} \dot{W} = -\dot{q}^\top M_c M^{-1} (D + G K_v G^\top M^{-1} M_c) \dot{q} \\ - \left(\zeta_M - \psi(q, \dot{q}, \zeta_p) - \frac{\delta_M}{k_I} \right)^\top (\nabla_{\dot{q}} \psi)^\top M^{-1} \phi k_I^2 \left(\zeta_M - \psi(q, \dot{q}, \zeta_p) - \frac{\delta_M}{k_I} \right) \leq 0. \end{aligned} \quad (B5)$$

If $D > 0$ and $(\nabla_{\dot{q}} \psi)^\top M^{-1} \phi > 0$, invoking LaSalle's theorem (see Theorem 3.4 in [23]) proves that the trajectories of the closed-loop system (35) converge asymptotically to the invariant set $\dot{q} = 0 \cap \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right) = 0$. Evaluating (B1) within this set yields $\nabla_q \Omega_c = 0$ thus it follows from *Assumption 4* that the equilibrium $(q, \dot{q}, \zeta_M) = (q^*, 0, \delta_M/k_I)$

is asymptotically stable. Stability is global if $(\nabla_{\dot{q}}\psi)^\top M^{-1}\phi > 0$ holds globally and $H_d(q, p)$ is radially unbounded (i.e., since $\mathcal{W}$ is quadratic in ζ_M), see [23].

If $D = 0$, the trajectories of the closed-loop system (35) converge asymptotically to the invariant set $y_c = G^\top M^{-1} M_c \dot{q} = 0 \cap \left(\zeta_M - \psi(q, \dot{q}) - \frac{\delta_M}{k_I} \right) = 0$. Comparing (19) and (B5) it follows that $\dot{\mathcal{W}} \leq y_c^\top u_{DI}$, that is, y_c is also a passive output of (35). Since y_c is detectable by *Assumption 4*, evaluating (B1) within the invariant set yields again $\nabla_q \Omega_c = 0$, hence the equilibrium $(q, \dot{q}, \zeta_M) = (q^*, 0, \delta_M/k_I)$ is asymptotically stable.

If $D = 0$ and $(\nabla_{\dot{q}}\psi)^\top M^{-1}\phi \geq 0$ (i.e., the equality applies only at the equilibrium, where the disturbances vanish), then the trajectories of the closed-loop system (35) converge asymptotically to the invariant set $y_c = 0$. Evaluating (B1) within the invariant set yields again $\nabla_q \Omega_c = 0$, hence the equilibrium $(q, \dot{q}) = (q^*, 0)$ is asymptotically stable. In this case however, the convergence of ζ_M to δ_M/k_I is not guaranteed $\square$

Journal Pre-proof