

Splitting of a Gap in the Bulk of the Spectrum of Random Matrices

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Abstract

We consider the probability of having two intervals (gaps) without eigenvalues in the bulk scaling limit of the Gaussian Unitary Ensemble of random matrices. We describe uniform asymptotics for the transition between a single large gap and two large gaps. For the initial stage of the transition, we explicitly determine all the asymptotic terms (up to the decreasing ones) of the logarithm of the probability. We obtain our results by analyzing double-scaling asymptotics of a Toeplitz determinant whose symbol is supported on two arcs of the unit circle.

1 Introduction

Let A be the union of m open disjoint intervals on $\mathbb{R}$, and K_s be the (trace-class) integral operator on $L^2(A, dx)$ given by the kernel

$$K_s(x, y) = \frac{\sin s(x - y)}{\pi(x - y)}. \quad (1)$$

Consider the Fredholm determinant

$$P_s(A) = \det(I - K_s)_A. \quad (2)$$

For a wide class of random matrix ensembles [1], in particular for the Gaussian Unitary Ensemble, $P_s(A)$ is the probability that the set $\frac{s}{\pi}A = \{\frac{s}{\pi}x : x \in A\}$ contains no eigenvalues in the bulk scaling limit where the average distance between the eigenvalues is 1. In this paper, we are interested in the asymptotics of $P_s(A)$ as $s \rightarrow \infty$, and we study the transition between a single interval $A_0 = (\alpha, \beta)$ to the set A composed of 2 disjoint intervals

$$A = A_1 \bigcup A_2, \quad A_1 = (\alpha_1, \beta_1), \quad A_2 = (\alpha_2, \beta_2). \quad (3)$$

Such problems have a rich history, of which we mention some relevant results. For the single interval case $A_0 = (\alpha, \beta)$,

$$\begin{aligned}\log P_s(A_0) &= -\frac{(\beta - \alpha)^2 s^2}{8} - \frac{1}{4} \log s - \frac{1}{4} \log \frac{\beta - \alpha}{2} + c_0 + \mathcal{O}(s^{-1}), \\ c_0 &= \frac{1}{12} \log 2 + 3\zeta'(-1),\end{aligned}\tag{4}$$

as $s \rightarrow \infty$, where ζ is the Riemann zeta-function. The leading term and logarithmic term in (4) were conjectured by des Cloizeaux and Mehta [5] in 1973, while the constant term c_0 remained undetermined until Dyson [9] conjectured an expression for it in 1976, relying on inverse scattering techniques and the work of Widom [16] on Toeplitz determinants (see below). The constant c_0 became known as the Widom-Dyson constant. The first rigorous confirmation of the leading term in (4) was given by Widom [17] in 1994. In a landmark paper of 1997, Deift, Its, Zhou [7] were able to confirm the leading term and the logarithmic term, but the proof of the constant c_0 continued to defy their techniques. Finally, two independent proofs of the constant were later given by Erhardt [10] and the second author [12], and a further third proof given in [8]. The proofs in [12], [8] use Riemann-Hilbert (RH) methods, while [10] uses operator theoretical techniques.

When A is composed of any (fixed) number of intervals, the main term was found and proved by Widom [18] in 1995, where he was also able to identify the next term in the following result:

$$\frac{d}{ds} \log P_s(A) = sC_1 + C_2(s) + o(1),\tag{5}$$

as $s \rightarrow \infty$. The constant C_1 is explicitly computable, while $C_2(s)$ is an oscillatory function given by a Jacobi inversion problem. In [7], which was mentioned above, the authors were also able to find the full asymptotic expansion for the logarithmic derivative of the determinant on any number of intervals and describe the oscillations in terms of θ -functions. Here we present their result when A is composed of 2 intervals as in (3):

$$\frac{d}{ds} \log P_s(A) = -2sG_0 + \frac{d}{ds} \log \theta(sV; \tau) + \mathcal{O}(s^{-1}).\tag{6}$$

More precisely, for any $j = 1, 2, \dots$, the error term here is of the form

$$\frac{G_1(s)}{s} + \frac{G_2(s)}{s^2} + \dots + \frac{G_j(s)}{s^j} + \mathcal{O}(s^{-j-1}).\tag{7}$$

where $G_j(s)$, $j = 1, \dots$, are bounded periodic functions of s . Here

$$\theta(z) = \theta(z; \tau) = \sum_{m \in \mathbb{Z}} e^{2\pi i z m + \pi i \tau m^2}\tag{8}$$

is the Jacobi Theta-function of the third kind, see e.g. [19]. The constants (in s) V, τ, G_0 , are given in terms of elliptic integrals, and $G_1(s), G_2(s), \dots$ are given in terms of θ -functions. Let

$$r(z) = ((z - \alpha_1)(z - \beta_1)(z - \alpha_2)(z - \beta_2))^{1/2}, \quad (9)$$

with branch cuts on A such that $r(z) > 0$ for $z > \beta_2$, and let $q(z)$ be the unique monic polynomial of degree 2 such that

$$\int_{A_j} \frac{q(z)dz}{r_+(z)} = 0, \quad j = 1, 2, \quad (10)$$

where $r_+(z)$ is the limit of $r(z + i\epsilon)$ as $\epsilon \rightarrow 0_+$ (where the "+" side is chosen merely for definiteness). Then q/r has no residue at infinity. Hence as $z \rightarrow \infty$, q/r has the form

$$\frac{q(z)}{r(z)} = 1 + \frac{G_0}{z^2} + \mathcal{O}(z^{-3}), \quad (11)$$

which defines the constant G_0 appearing in (6).

The parameters V, τ appearing in the arguments of the θ -function in (6) are as follows:

$$V = -\frac{1}{\pi} \int_{\beta_1}^{\alpha_2} \frac{q(x)dx}{r(x)}, \quad \tau = i \frac{\int_{\beta_1}^{\alpha_2} \frac{dx}{|r(x)|}}{\int_{\alpha_2}^{\beta_2} \frac{dx}{|r_+(x)|}}. \quad (12)$$

Integrating (6) in s from some large value s_0 to s , and using the properties of $G_1(s)$, Deift, Its, and Zhou concluded that

$$\log P_s(A) = -s^2 G_0 + \log \theta(sV; \tau) + \widehat{G}_1 \log s + c_1 + \mathcal{O}(s^{-1}), \quad \widehat{G}_1 = \lim_{x \rightarrow \infty} \frac{1}{x} \int_{s_0}^x G_1(t) dt. \quad (13)$$

The value of the constant c_1 is unknown as there is no point s_0 for the lower limit of integration where $P_{s_0}(A)$ would be explicitly known.

In this paper we study the transition between the single interval formula (4) and the two-interval formula (13). We obtain an explicit expression (up to the decreasing terms in the expansion of $\log P_s(A)$) for the asymptotics in the regime where the length ℓ of the interval between the gaps decreases sufficiently fast (slightly faster than $1/s$: see below) as $s \rightarrow \infty$. On the other hand we show that the asymptotics of [7], obtained for fixed gaps, can be extended (with proper adjustments) to the regime when ℓ is no longer fixed but decreases sufficiently slowly as $s \rightarrow \infty$. These two regimes overlap. Thus our analysis provides uniform asymptotics for the whole transition. Note, however, that since the constant in (13) is not determined, the expression for the asymptotics in the second regime is not fully explicit.

Obtaining an explicit expression for these asymptotics and establishing the constant in (13) is a separate problem and we plan to address it in future work.

The initial phase of the transition between (4) and (13) resembles the birth of a cut — emergence of an extra interval of support of the limiting eigenvalue density in a unitary ensemble of random matrices: asymptotics for the correlation kernel of the eigenvalues in that case were obtained independently by Bertola & Lee [3], Claeys [4], Mo [14]. From the technical point of view, our analysis is very different as we are dealing with so-called hard edges rather than soft edges in [3, 4, 14] and in the context of a different model, so both the g-function needed in the analysis and the local parametrix are different. Moreover, the works [3, 4, 14] deal with correlation kernels and not determinants.

Consider the Toeplitz determinant whose symbol $f(z)$ is the characteristic function of a subset J of the unit circle C :

$$D_n(J) = \det (f_{j-k})_{j,k=0}^{n-1}, \quad f_k = \int_{e^{i\theta} \in J} e^{-ik\theta} \frac{d\theta}{2\pi}, \quad (14)$$

where integration is in the positive direction around the unit circle. The proofs of the expansion (4) including the constant term c_0 in [8, 12] were based on an analysis of the Toeplitz determinant $D_n(J_2)$ where J_2 is an arc of the unit circle

$$J = J_2 = \{e^{i\theta} | \theta \in (-\pi, \theta_2) \cup (\theta_1, \pi)\}. \quad (15)$$

The asymptotics of $D_n(J_2)$, as $n \rightarrow \infty$, for a fixed arc J_2 were found by Widom [16]. In [8, 12], Widom's result was extended to the case of $J_2 = J_2^{(n)}$ varying with n such that $|\theta_1 - \theta_2| \rightarrow 0$ sufficiently slowly. Namely,

$$\log D_n(J_2^{(n)}) = n^2 \log \cos \frac{\theta_1 - \theta_2}{4} - \frac{1}{4} \log \left(n \sin \frac{\theta_1 - \theta_2}{4} \right) + c_0 + \mathcal{O} \left(\frac{1}{n \sin \frac{\theta_1 - \theta_2}{4}} \right) \quad (16)$$

as $n \rightarrow \infty$, uniformly for $\frac{s_0}{n} \leq \frac{\theta_1 - \theta_2}{2} \leq \pi - \epsilon$, for $\epsilon > 0$ and with s_0 sufficiently large. Asymptotics (4) are obtained from (16) by using the fact that

$$\lim_{n \rightarrow \infty} D_n(J_2^{(n)}) = \det(I - K_s)_{A_0} \quad (17)$$

for fixed s and by taking the limit in (16) as $n \rightarrow \infty$ with $\theta_1 = \frac{2s\beta}{n}$ and $\theta_2 = \frac{2s\alpha}{n}$, where α, β are fixed. The approach of the present paper is based on an analysis of $D_n(J)$ where $J = J^{(n)}$ is the union of 2 arcs $J^{(n)} = J_1^{(n)} \cup J_2^{(n)}$, with $J_1^{(n)} \subset C \setminus J_2^{(n)}$ of sufficiently small length in comparison with $C \setminus J_2^{(n)}$ (see Theorem 1.2 below). We obtain our results on the sine-kernel determinant by taking the limit $n \rightarrow \infty$ of $D_n(J^{(n)})$. However, we believe Theorem 1.2 below to be of independent interest for a future study of Toeplitz determinants with symbols supported on several arcs.

1.1 Results

The kernel (1) is translationally invariant and so we can assume the following form for A :

$$A = (\alpha, -\nu) \bigcup (\nu, \beta), \quad \alpha < 0 < \nu < \beta. \quad (18)$$

In this paper we provide the asymptotics of $\log P_s(A)$ (including the constant term) in the double scaling limit as $s \rightarrow \infty$ while $\nu \rightarrow 0$ in such a way that $s\nu \log \nu^{-1} \rightarrow 0$, and connect these asymptotics with those of [7].

Let $\gamma = \frac{1}{8}(\beta^{-1} - \alpha^{-1})$ and

$$\omega = \frac{s\sqrt{|\alpha\beta|}}{\log(\gamma\nu)^{-1}} > 0. \quad (19)$$

Clearly, ω is uniquely represented in the form

$$\omega = k + x, \quad k = 0, 1, 2, \dots, \quad x \in [-1/2, 1/2). \quad (20)$$

We note that ν has the form

$$\nu = \gamma^{-1} e^{-\frac{s\sqrt{|\alpha\beta|}}{k+x}}. \quad (21)$$

We prove the following:

Theorem 1.1. *As $s \rightarrow \infty$, uniformly for $\nu \in (0, \nu_0)$, where $s\nu_0 \log \nu_0^{-1} \rightarrow 0$,*

$$\begin{aligned} \log P_s(A) &= \log P_s(A_0) + s\sqrt{|\alpha\beta|} \left(\omega - \frac{x^2}{\omega} \right) + c(k) + \delta_k(x) \\ &\quad + \mathcal{O}(\max\{s\nu_0 \log \nu_0^{-1}, 1/\log \nu_0^{-1}, s^{-1}\}), \\ c(k) &= \log \left(\frac{2^{2k^2-k}}{\pi^k} \frac{G(k+1)^4}{G(2k+1)} \right), \\ \delta_k(x) &= \log \left(1 + 2\pi\kappa_{k-1}^2 (\gamma\nu)^{1+2x} \right) + \log \left(1 + (2\pi\kappa_k^2)^{-1} (\gamma\nu)^{1-2x} \right), \end{aligned} \quad (22)$$

where G is the Barnes G -function, and where κ_j is the leading coefficient of the Legendre polynomial of degree j orthonormal on the interval $[-2, 2]$, given by

$$\kappa_j = 4^{-j-1/2} \sqrt{2j+1} \frac{(2j)!}{j!^2}, \quad j = 1, 2, \dots, \quad \kappa_0 = 1/2, \quad \kappa_{-1} = 0, \quad (23)$$

and $\log P_s(A_0)$ is given in (4).

As $s \rightarrow \infty$, uniformly for $\nu \in (\nu_1, \nu_0)$, where $s\nu_0 \log \nu_0^{-1} \rightarrow 0$, $\frac{s}{\log \nu_1^{-1}} \rightarrow \infty$ (i.e., $k \rightarrow \infty$), formula (22) reduces to

$$\begin{aligned} \log P_s(A) = & s^2 \left(-\frac{(\beta - \alpha)^2}{8} + \frac{|\alpha\beta|}{\log(\gamma\nu)^{-1}} \right) - \frac{1}{2} \log s + \frac{1}{4} \log \log(\gamma\nu)^{-1} - x^2 \log(\gamma\nu)^{-1} \\ & + \log(1 + (\gamma\nu)^{1-2|x|}) - \frac{1}{4} \log \left(\frac{\beta - \alpha}{2} \sqrt{|\alpha\beta|} \right) + \frac{1}{6} \log 2 + 6\zeta'(-1) \\ & + \mathcal{O} \left(\max \left\{ s\nu_0 \log \nu_0^{-1}, \frac{1}{\log \nu_0^{-1}}, \frac{\log \nu_1^{-1}}{s} \right\} \right), \end{aligned} \quad (24)$$

where ζ is the Riemann zeta-function.

Remark 1.1. Note that if $k = 0$ and $s \rightarrow \infty$ while $x \in (0, 1/2 - \epsilon)$ for $\epsilon > 0$, then (22) shows that $\frac{P_s(A)}{P_s(A_0)} \rightarrow 1$.

Remark 1.2. As we show in Section 5 (Lemma 5.1), the Deift-Its-Zhou asymptotics (6) for 2 fixed gaps where we set $\alpha_2 = -\beta_1 = \nu$, $\alpha_1 = \alpha$, $\beta_2 = \beta$, can be extended (with a worse error term) to the region where $\nu \rightarrow 0$ in such a way that

$$s\nu^{1/2+\varepsilon} \rightarrow \infty,$$

for any $\varepsilon > 0$. Clearly, this region overlaps with the region of validity

$$s\nu \log(\gamma\nu)^{-1} \rightarrow 0$$

of Theorem 1.1. For example, $\nu = s^{-3/2}$ belongs to both regions. In Remark 5.1 we explicitly show the coincidence of the main (order s^2) asymptotic terms. Full explicit formulas for this matching will be a subject of future work.

Remark 1.3. The function $c(k)$ can alternatively be described in terms of the coefficients (23) of the Legendre polynomials:

$$c(k) = - \sum_{j=0}^{k-1} \log 2\pi \kappa_j^2, \quad k = 1, 2, \dots, \quad c(0) = 0. \quad (25)$$

Formula (25) shows that $\delta_k(x) + c(k)$ is continuous also at the points $|x| = 1/2$.

Remark 1.4. The rescaled sine process (with expected distance between particles π/s) is the determinantal point process with the m 'th correlation function ρ_m , for $m = 1, 2, \dots$, given by

$$\rho_m(x_1, \dots, x_m) = \det(K_s(x_i, x_j))_{i,j=1}^m. \quad (26)$$

Consider the rescaled sine process conditioned to have no eigenvalues in A . Denote this process by $\mathbb{P}_A$ and its m 'th correlation function by ρ_m^A . In Section 4.2, we show that for $x_1, \dots, x_m \in (-1, 1)$,

$$\nu^m \rho_m^A(\nu x_1, \dots, \nu x_m) \rightarrow \det(2K_{\text{Leg}}(2x_i, 2x_j))_{i,j=1}^m \quad (27)$$

as $s \rightarrow \infty$ and $\nu \rightarrow 0$ such that $k \in \mathbb{N}$ and $|x| < 1/2$ remain fixed, where

$$K_{\text{Leg}}(x, y) = \frac{\kappa_{k-1}}{\kappa_k} \frac{L_k(x)L_{k-1}(y) - L_k(y)L_{k-1}(x)}{x - y}, \quad (28)$$

and L_k is the Legendre polynomial of degree k , orthonormal on $[-2, 2]$:

$$\int_{-2}^2 L_j(x)L_i(x)dx = \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases} \quad (29)$$

Recall that for a set $B \subset \mathbb{R}$ and a point process Λ with its m -th correlation function denoted r_m , we have

$$\begin{aligned} \text{Expectation}(\# \text{ ordered } m\text{-tuples in } B) &= \frac{1}{m!} \int_{B^m} r_m(x_1, \dots, x_m) dx_1 \dots dx_m \\ &= \sum_{j=0}^{\infty} \binom{m+j}{m} \text{Prob}(\#(\Lambda \cap B) = m+j). \end{aligned} \quad (30)$$

The process with kernel K_{Leg} is a k -point process. Thus we obtain from (27) and the first equation of (30) that, as $s \rightarrow \infty$ and $\nu \rightarrow 0$ such that $k \in \mathbb{N}$ and $|x| < 1/2$ remain fixed, the expected number of $(k+1)$ -tuples of $\mathbb{P}_A$ on $(-\nu, \nu)$ converges to 0, while the expected number of k -tuples on the same interval converges to 1. It follows from the second equation of (30) that

$$\text{Prob}(\mathbb{P}_A \text{ has } k \text{ particles in } (-\nu, \nu)) \rightarrow 1. \quad (31)$$

Thus the asymptotics of $\log P_s(A)$ as $s \rightarrow \infty$ depends on the value of ν , and we give an overview of the various scaling limits:

- If $\nu = 0$, the asymptotics are given by (4).
- If $\nu \rightarrow 0$ as $s \rightarrow \infty$, such that $s\nu \log \nu^{-1} \rightarrow 0$, the asymptotics are given by Theorem 1.1.

- If $\nu \log \nu^{-1}$ is of order $1/s$ or larger, the asymptotics of Theorem 1.1 breaks down and the transition to the asymptotic formula (13) containing θ -functions takes place. This is discussed in Section 5.
- If $\nu > 0$ is fixed, the asymptotics are given by the θ -function regime (13).

For Toeplitz determinants, we obtain the following result. Let $D_n(J)$ be given by (14) with $J = J^{(n)} = J_1^{(n)} \cup J_2^{(n)}$ where

$$J_1^{(n)} = \left\{ e^{i\theta} : \theta \in \left(-\frac{2s\nu}{n}, \frac{2s\nu}{n} \right) \right\}, \quad J_2^{(n)} = \left\{ e^{i\theta} : \theta \in \left(-\pi, \frac{2s\alpha}{n} \right) \cup \left(\frac{2s\beta}{n}, \pi \right] \right\}, \quad (32)$$

with some $\alpha < 0 < \nu(s) < \beta$. Then, with the notation of Theorem 1.1, we have

Theorem 1.2. *As $s, n \rightarrow \infty$, uniformly for $\nu \in (0, \nu_0)$, where $s\nu_0 \log \nu_0^{-1} \rightarrow 0$ and $s^3/n \rightarrow 0$,*

$$\begin{aligned} \log D_n(J^{(n)}) &= \log D_n(J_2^{(n)}) + s\sqrt{|\alpha\beta|} \left(\omega - \frac{x^2}{\omega} \right) + c(k) + \delta_k(x) \\ &\quad + \mathcal{O}(\max\{s^3/n, s\nu_0 \log \nu_0^{-1}, 1/\log \nu_0^{-1}, s^{-1}\}), \end{aligned} \quad (33)$$

where the expansion of $\log D_n(J_2^{(n)})$ is given in (16) with $\theta_1 = 2s\beta/n$, $\theta_2 = 2s\alpha/n$.

We use Theorem 1.2 to prove Theorem 1.1.

Proof of Theorem 1.1. It is well-known that

$$|D_n(J^{(n)}) - \det(I - K_s)_A| \rightarrow 0 \quad (34)$$

as $n \rightarrow \infty$ for fixed s , a fact which we also prove in the appendix for the reader's convenience. Taking the limit $n \rightarrow \infty$ in (33), we then obtain (22). To obtain (24), we substitute (4) for $P_s(A_0)$, and note that the standard asymptotics of the Barnes G-function $G(z+1)$ as $z \rightarrow \infty$

$$\log G(z+1) = \frac{z^2}{2} \log z - \frac{3}{4}z^2 + \frac{z}{2} \log 2\pi - \frac{1}{12} \log z + \zeta'(-1) + \mathcal{O}(z^{-2}), \quad (35)$$

imply that as $k \rightarrow \infty$,

$$c(k) = -\frac{1}{4} \log k + \frac{1}{12} \log 2 + 3\zeta'(-1) + \mathcal{O}(1/k^2). \quad (36)$$

Furthermore, we note that $2\pi\kappa_k^2 = 1 + \mathcal{O}(k^{-1})$ as $k \rightarrow \infty$. $\square$

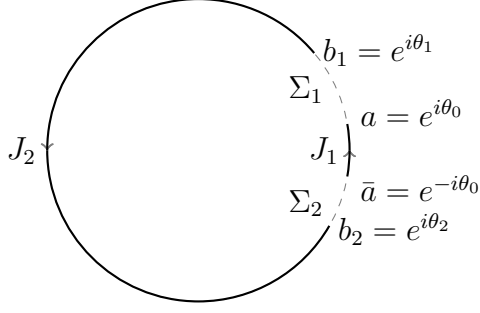


Figure 1: Interval J .

1.2 Outline of the proof of Theorem 1.2

It remains to prove Theorem 1.2.

Let $-\pi < \theta_2 < 0 < \theta_0 < \theta_1 < \pi$ and define $J = J_1 \cup J_2$ where J_1, J_2 are as in Figure 1:

$$J_1 = J_1(\theta_0) = \{e^{i\theta} | \theta \in (-\theta_0, \theta_0)\}, \quad J_2 = \{e^{i\theta} | \theta \in (\theta_1, \pi] \cup (-\pi, \theta_2)\}. \quad (37)$$

We denote the complement of J as $\Sigma = \Sigma_1 \cup \Sigma_2$ where

$$\Sigma_1 = \{e^{i\theta} | \theta \in (\theta_0, \theta_1)\}, \quad \Sigma_2 = \{e^{i\theta} | \theta \in (\theta_2, -\theta_0)\}. \quad (38)$$

It follows from the integral representation for Toeplitz determinants (see (295) in the Appendix) that $D_j(J) > 0$ for all $j \in \mathbb{N}$. Consider the polynomials ϕ_j for $j = 0, 1, 2, \dots$ given by

$$\begin{aligned} \phi_0(z) &= \frac{1}{\sqrt{D_1(J)}}, \\ \phi_j(z) &= \frac{1}{\sqrt{D_j(J)D_{j+1}(J)}} \det \begin{pmatrix} f_0 & f_{-1} & \dots & f_{-j+1} & f_{-j} \\ f_1 & f_0 & \dots & f_{-j+2} & f_{-j+1} \\ & & \ddots & & \\ f_{j-1} & f_{j-2} & \dots & f_0 & f_{-1} \\ 1 & z & \dots & z^{j-1} & z^j \end{pmatrix} = \chi_j z^j + \dots, \quad j > 0, \end{aligned} \quad (39)$$

where the leading coefficient χ_j is given by

$$\chi_j = \sqrt{\frac{D_j(J)}{D_{j+1}(J)}}, \quad j = 0, 1, 2, \dots, \quad (40)$$

and we set $D_0(J) = 1$. The polynomials ϕ_j are orthonormal with weight 1 on J :

$$\int_J \phi_k(z) \overline{\phi_j(z)} \frac{d\theta}{2\pi} = \delta_{jk}, \quad z = e^{i\theta}, \quad j, k = 0, 1, 2, \dots \quad (41)$$

Define a 2×2 matrix $Y(z) = Y_n(z)$ in terms of these orthogonal polynomials as follows:

$$Y(z) = \begin{pmatrix} \chi_n^{-1} \phi_n(z) & \chi_n^{-1} \int_J \frac{\phi_n(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i \zeta^n} \\ -\chi_{n-1} z^{n-1} \overline{\phi_{n-1}(z^{-1})} & -\chi_{n-1} \int_J \frac{\phi_{n-1}(\zeta^{-1})}{\zeta - z} \frac{d\zeta}{2\pi i \zeta} \end{pmatrix}. \quad (42)$$

Then Y is the unique solution of the following Riemann-Hilbert (RH) Problem

- (a) $Y : \mathbb{C} \setminus J \rightarrow \mathbb{C}^{2 \times 2}$ is analytic;
- (b) Y possesses L^2 boundary values Y_+ and Y_- on the $+$ and $-$ side of J , respectively, related by the condition:

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} \\ 0 & 1 \end{pmatrix} \text{ for } z \in J;$$

- (c) $Y(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \rightarrow \infty$.

The fact that orthogonal polynomials satisfy a RH problem was first observed for polynomials orthogonal on the real line by Fokas, Its, Kitaev [11], and extended to polynomials orthogonal on the unit circle by Baik, Deift, Johansson [2]. The RH problem provides an efficient tool, via the Deift-Zhou steepest descent method, for the asymptotic analysis of the polynomials, see e.g. [6].

In Section 2 we express the logarithmic derivative of the Toeplitz determinant $\frac{d}{d\theta_0} \log D_n(J)$ in terms of the polynomials ϕ_n and ϕ_{n-1} . These are, in turn, expressed in terms of Y_n . In Sections 3 and 4 we analyse the RH problem for Y_n as $n \rightarrow \infty$ in a double scaling limit where J depends on n such that $\theta_j = \frac{s}{n} u_j$ for $j = 0, 1, 2$; where $s \rightarrow \infty$ such that $s^3/n \rightarrow 0$; where $u_0 \rightarrow 0$ such that $su_0 \log u_0^{-1} \rightarrow 0$, while u_1 and u_2 remain fixed. As a result, we obtain the asymptotics of Y_n . Substituting these into the differential identity for $\frac{d}{d\theta_0} \log D_n(J)$, and integrating with respect to θ_0 , we obtain the asymptotics of $D_n(J)$, where $u_1 = 2\beta$, $u_2 = 2\alpha$, and $u_0 = u_0(\nu)$ is a function of ν , which proves Theorem 1.2.

2 Differential Identity

We will now obtain the following:

Proposition 2.1. (*Differential identity*) *Let $a = e^{i\theta_0}$. The Toeplitz determinant $D_n(J)$ satisfies*

$$\frac{\partial}{\partial \theta_0} \log D_n(J) = -\frac{1}{2\pi} [F(\bar{a}) + F(a)], \quad (43)$$

where

$$F(z) = n\chi_n^2 |Y_{11}(z)|^2 - 2\chi_n^2 \operatorname{Re} \left(z \overline{Y_{11}(z)} \frac{d}{dz} Y_{11}(z) \right), \quad (44)$$

and J was given in (37).

Proof. From the definition of the orthogonal polynomials it is clear that

$$D_n(J) = \prod_{j=0}^{n-1} \chi_j^{-2}. \quad (45)$$

The orthogonality conditions imply that

$$\frac{1}{2\pi} \int_J \frac{\partial \phi_j(z)}{\partial \theta_0} \overline{\phi_j(z)} d\theta = \frac{1}{2\pi} \int_J \frac{\partial \chi_j}{\partial \theta_0} (z^j + \text{poly of deg } j-1) \overline{\phi_j(z)} d\theta = \frac{1}{\chi_j} \frac{\partial \chi_j}{\partial \theta_0}, \quad (46)$$

and similarly,

$$\frac{1}{2\pi} \int_J \phi_j(z) \frac{\partial \overline{\phi_j(z)}}{\partial \theta_0} d\theta = \frac{1}{\chi_j} \frac{\partial \chi_j}{\partial \theta_0}. \quad (47)$$

By (45)–(47) we obtain:

$$\begin{aligned} \frac{\partial}{\partial \theta_0} \log(D_n(J)) &= -2 \sum_{j=0}^{n-1} \frac{\partial \chi_j}{\partial \theta_0} / \chi_j \\ &= -\frac{1}{2\pi} \int_J \frac{\partial}{\partial \theta_0} \left(\sum_{j=0}^{n-1} |\phi_j(z)|^2 \right) d\theta. \end{aligned} \quad (48)$$

On the other hand, one can express F given in (44) in terms of the orthogonal polynomials:

$$F(z) = n|\phi_n(z)|^2 - 2\operatorname{Re} \left(z \overline{\phi_n(z)} \frac{d}{dz} \phi_n(z) \right). \quad (49)$$

Now the Christoffel-Darboux formula for orthogonal polynomials gives

$$\sum_{k=0}^{n-1} |\phi_k(z)|^2 = -F(z) \quad \text{for } z \in C, \quad (50)$$

(see eg. [12]), and hence (48) can be written as

$$\frac{\partial}{\partial \theta_0} \log D_n(J) = \frac{1}{2\pi} \int_J \frac{\partial}{\partial \theta_0} (F(z)) d\theta. \quad (51)$$

Since, by (50) and orthogonality, $\int_J F(z) \frac{d\theta}{2\pi} = n$, we obtain

$$0 = \frac{\partial}{\partial \theta_0} \left(\int_J F(z) d\theta \right) = F(\bar{a}) + F(a) + \int_J \frac{\partial}{\partial \theta_0} F(z) d\theta, \quad (52)$$

upon which Proposition 2.1 follows immediately. $\square$

3 Analysis of Riemann-Hilbert problem

We start by setting $\theta_0 = 0$ so that J_1 is a point, and then let J_1 develop into an arc. Throughout the rest of the paper we use the notation

$$a = e^{i\theta_0} = e^{iu_0s/n}, \quad b_1 = e^{i\theta_1} = e^{iu_1s/n}, \quad b_2 = e^{i\theta_2} = e^{iu_2s/n}. \quad (53)$$

We let $s, n \rightarrow \infty$ such that $s^3/n \rightarrow 0$, and let $u_0 \rightarrow 0$ as $s \rightarrow \infty$ such that $su_0 \log u_0^{-1} \rightarrow 0$, while $u_2 < 0 < u_1$ remain constant. Denote $\Sigma^o = \Sigma_1^o \cup \Sigma_2^o$ where

$$\Sigma_1^o = \{e^{i\theta} | 0 < \theta < \theta_1\}, \quad \Sigma_2^o = \{e^{i\theta} | \theta_2 < \theta < 0\}. \quad (54)$$

Let g_1 be the function:

$$g_1(z) = \log \left(\frac{1}{b_1^{1/2} + b_2^{1/2}} \left(z + (b_1 b_2)^{1/2} + ((z - b_1)(z - b_2))^{1/2} \right) \right), \quad (55)$$

where the square root has branch cut on J_2 and is positive as $z \rightarrow +\infty$, and the logarithm $\log x$ has a branch cut for $x < 0$ and is positive for $x > 1$. At infinity,

$$g_1(z) = \log z - \log \frac{\sqrt{b_1} + \sqrt{b_2}}{2} + o(1) \quad \text{as } z \rightarrow \infty. \quad (56)$$

The boundary values of the function g_1 satisfy

$$g_{1,+}(z) + g_{1,-}(z) = \log z, \quad \text{for } z \in J_2, \quad (57)$$

and at b_1, b_2 we have

$$g_1(b_1) = \frac{1}{2} \log b_1, \quad g_1(b_2) = \frac{1}{2} \log b_2. \quad (58)$$

Alternatively, for $z = e^{i\theta} \in \Sigma^o$ we can write g_1 in the following form:

$$\exp(g_1(e^{i\theta})) = e^{i\theta/2} \frac{\cos \frac{1}{2} \left(\theta - \frac{\theta_1 + \theta_2}{2} \right) + \sqrt{\left| \sin \frac{\theta - \theta_1}{2} \sin \frac{\theta - \theta_2}{2} \right|}}{\cos \frac{1}{4}(\theta_1 - \theta_2)}. \quad (59)$$

On the $+$ and $-$ side of J_2 , we have

$$\exp((g_1)_\pm(e^{i\theta})) = e^{i\theta/2} \frac{\cos \frac{1}{2} \left(\theta - \frac{\theta_1 + \theta_2}{2} \right) \mp i \sqrt{\left| \sin \frac{\theta - \theta_1}{2} \sin \frac{\theta - \theta_2}{2} \right|}}{\cos \frac{1}{4}(\theta_1 - \theta_2)}, \quad (60)$$

from which it follows that e^{g_1} maps the $+$ side of J_2 to Σ^o and the $-$ side to J_2 , and that e^{g_1} maps $\mathbb{C} \setminus J_2$ to the outside of the unit disc.

Set

$$w = \frac{n}{s}(z - 1), \quad B_j = \frac{n}{s}(1 - b_j) \quad (61)$$

for $j = 1, 2$. Note that B_1 and B_2 remain bounded as $s/n \rightarrow 0$. Then

$$g_1(z) = g_1(1) + \log \left(1 + \frac{s}{n} \frac{\left(1 - \frac{s}{n} B_1\right)^{1/2} + \left(1 - \frac{s}{n} B_2\right)^{1/2}}{e^{g_1(1)}} w H(w) \right), \quad (62)$$

$$H(w) = \frac{1}{w} \left(w + ((w + B_1)(w + B_2))^{1/2} - (B_1 B_2)^{1/2} \right),$$

where $H(w)$ is analytic in w at the point 0. Thus g_1 has the following expansion in w at the point $w = 0$

$$g_1(z) = g_1(1) + \frac{s}{n}(c_1 w + \mathcal{O}(w^2)), \quad (63)$$

where

$$g_1(1) = \log \left(\frac{\cos(\theta_1 + \theta_2)/4 + \sqrt{|\sin \theta_1/2| |\sin \theta_2/2|}}{\cos(\theta_1 - \theta_2)/4} \right) = \frac{s\sqrt{|u_1 u_2|}}{2n} \left(1 + \mathcal{O}\left(\frac{s}{n}\right) \right)$$

$$c_1 = \frac{1}{\sqrt{(1 - b_1)(1 - b_2)}} \left(1 - \frac{\sqrt{b_1} + \sqrt{b_2}}{2e^{g_1(1)}} \right) = \left(\frac{1}{2} + \frac{i}{4}(u_1^{-1} + u_2^{-1})\sqrt{|u_1 u_2|} \right) \left(1 + \mathcal{O}\left(\frac{s}{n}\right) \right), \quad (64)$$

as $\frac{s}{n} \rightarrow 0$.

Define

$$r(z) = ((z - b_1)(z - b_2))^{1/2}, \quad (65)$$

where the square root has a branch cut on J_2 , and is positive as $z \rightarrow +\infty$. Let

$$h(z) = r(z) \int_{\Sigma_2^o} \frac{d\xi}{r(\xi)(\xi - z)}, \quad z \in \mathbb{C} \setminus (J_2 \cup \Sigma_2^o), \quad (66)$$

where integration is taken in counter-clockwise direction. It is easily verified by differentiation that

$$-r(z) \int_{\tilde{C}}^t \frac{d\xi}{r(\xi)(\xi - z)} = \log \left(\frac{2r^2(z) + (2z - b_1 - b_2)(t - z) + 2r(z)r(t)}{t - z} \right) + C(z, \tilde{C}), \quad (67)$$

for any constant $\tilde{C}$ and some function $C(z)$. Thus

$$h(z) = r(z) \int_{b_2}^1 \frac{d\xi}{r(\xi)(\xi - z)}$$

$$= \log \frac{b_2 - b_1}{2}(z - 1) - \log \left(z \left(1 - \frac{b_1 + b_2}{2} \right) + b_1 b_2 - \frac{b_1 + b_2}{2} + r(z)r(1) \right). \quad (68)$$

The function h has a logarithmic singularity at $z = 1$ and a jump on $\Sigma_2^o \cup J_2$, such that

$$h_+ - h_- = \begin{cases} 0 & \text{for } z \in \Sigma_1^o, \\ 2\pi i & \text{for } z \in \Sigma_2^o, \end{cases} \quad (69)$$

$$h_+ + h_- = 0 \quad \text{for } z \in J_2.$$

The jump conditions (69) also imply that

$$\begin{aligned} h(b_1) &= 0 \\ h_+(b_2) &= -h_-(b_2) = \pi i. \end{aligned} \quad (70)$$

As $z \rightarrow \infty$,

$$h(z) \rightarrow \log \frac{b_2 - b_1}{((1 - b_1)^{1/2} + (1 - b_2)^{1/2})^2} \equiv h(\infty). \quad (71)$$

On the interval Σ^o we can alternatively write h in the following form:

$$\exp(h(z)) = \frac{\sin \frac{\theta_1 - \theta_2}{2} \sin \frac{\theta}{2}}{\cos \frac{\theta - \theta_1 - \theta_2}{2} - \cos \frac{\theta_1 - \theta_2}{2} \cos \frac{\theta}{2} + 2\sqrt{|\sin \frac{\theta - \theta_1}{2} \sin \frac{\theta - \theta_2}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}|}}. \quad (72)$$

With the notation of (61) we can write

$$h(z) = \log w \frac{B_1 - B_2}{(B_1 + B_2)w + 2B_1B_2 + 2\sqrt{(w + B_1)(w + B_2)}\sqrt{B_1B_2}}. \quad (73)$$

Then we can expand h at the point $z = 1$:

$$h(z) = \log w + c'_0 + c'_1 w + \mathcal{O}(w^2), \quad (74)$$

where

$$\begin{aligned} c'_0 &= -\log 4 \frac{B_1 B_2}{B_1 - B_2} = \left(-\frac{\pi i}{2} + \log \frac{(u_1^{-1} - u_2^{-1})}{4} \right) + \mathcal{O}\left(\frac{s}{n}\right), \\ c'_1 &= -\frac{B_1 + B_2}{2B_1 B_2} = -\frac{i}{2}(u_1^{-1} + u_2^{-1}) + \mathcal{O}\left(\frac{s}{n}\right), \end{aligned} \quad (75)$$

as $\frac{s}{n} \rightarrow 0$.

We define the g -function by:

$$g(z) = g_1(z) + \frac{\Omega}{2\pi} h(z), \quad (76)$$

where $\Omega > 0$ is a constant yet to be fixed. The jump conditions for g_1 and h imply that

$$\begin{aligned} g_+ + g_- &= \log z & \text{for } z \in J_2, \\ g_+ - g_- &= 0 & \text{for } z \in \Sigma_1^o, \\ g_+ - g_- &= i\Omega & \text{for } z \in \Sigma_2^o. \end{aligned} \quad (77)$$

We define the local variable ζ on a disc U_0 containing the interval J_1 (but not the points b_1, b_2), by

$$\zeta(z) = e^{\frac{2\pi}{\Omega}(g(z) - \frac{1}{2} \log z)} = e^{h(z) + \frac{2\pi}{\Omega}(g_1(z) - \frac{1}{2} \log z)}. \quad (78)$$

The jump conditions for g imply that the function ζ is analytic in U_0 . The precise radius of U_0 will be determined later on by requiring that the mapping ζ be conformal on U_0 .

Since e^{g_1} maps $\mathbb{C} \setminus J_2$ to the exterior of the unit disc, we have

$$e^{g_1(e^{\pm i\theta_0}) \mp i\theta_0/2} > 1. \quad (79)$$

For $u_0 < \epsilon$ with some $\epsilon > 0$, it follows from (72) that

$$e^{h(e^{\pm i\theta_0})} \in \begin{cases} (0, 1) & \text{for “+”,} \\ (-1, 0) & \text{for “-”}. \end{cases} \quad (80)$$

Now consider, as a function of Ω ,

$$\zeta(a) - \zeta(\bar{a}). \quad (81)$$

By (79) and (80) it follows that if we let $\Omega = +\infty$ in (78) then (81) is smaller than 2, and if we instead set $\Omega = +0$ then (81) is equal to $+\infty$. Since (82) is monotone in Ω , there exists, for $u_0 < \epsilon$, a unique value for $\Omega > 0$ such that

$$\zeta(a) - \zeta(\bar{a}) = 4. \quad (82)$$

We define Ω so that ζ satisfies (82). From (59) and (72), it follows that $\zeta(\Sigma^o) \subset \mathbb{R}$. By (63) and (74) we have the following expansion at the point $z = 1$:

$$\begin{aligned} \zeta(z) &= w\zeta_0 \left(1 + \zeta_1 w + \mathcal{O} \left(\left(\frac{sw}{n\Omega} \right)^2 \right) \right), \quad w = \frac{n}{s}(z - 1), \\ \zeta_0 &= e^{c'_0 + \frac{2\pi}{\Omega}g_1(1)}, \\ \zeta_1 &= c'_1 + \frac{2\pi s}{n\Omega}(c_1 - 1/2). \end{aligned} \quad (83)$$

In what follows, it will be apparent that $\Omega \rightarrow 0$ in the limit $s, n \rightarrow \infty$ and $u_0, s/n \rightarrow 0$. Moreover, by (64) and (75),

$$\begin{aligned} \zeta_0 &= -\frac{i}{4}(u_1^{-1} - u_2^{-1})e^{\frac{\pi}{\Omega}\frac{s\sqrt{|u_1 u_2|}}{n}(1 + \mathcal{O}(s/n))}(1 + \mathcal{O}(s/n)) \\ \zeta_1 &= \frac{2\pi s}{n\Omega}(c_1 - 1/2) - \frac{i}{2}(u_1^{-1} + u_2^{-1}) + \mathcal{O}(s/n). \end{aligned} \quad (84)$$

Substituting these expansions into (83), which we in turn substitute into (82), we obtain

$$u_0 \left(1 + \mathcal{O} \left(\frac{1}{\Omega^2} \frac{u_0^2 s^2}{n^2} \right) \right) = \frac{8}{u_1^{-1} - u_2^{-1}} e^{-\frac{2\pi}{n\Omega} \frac{s\sqrt{|u_1 u_2|}}{2} (1 + \mathcal{O}(s/n))} (1 + \mathcal{O}(s/n)), \quad (85)$$

or upon taking the logarithm,

$$\log \left(u_0 \frac{u_1^{-1} - u_2^{-1}}{8} \right)^{-1} = \frac{\pi s \sqrt{|u_1 u_2|}}{\Omega n} \left(1 + \mathcal{O}(s/n) + \mathcal{O}(\Omega) + \mathcal{O} \left(\frac{1}{\Omega} \frac{u_0^2 s}{n} \right) \right). \quad (86)$$

Therefore,

$$\Omega = \frac{\pi s \sqrt{|u_1 u_2|}}{n} / \log \frac{8}{(u_1^{-1} - u_2^{-1}) u_0} (1 + \mathcal{O}(s/n) + \mathcal{O}(u_0^2 \log u_0^{-1})). \quad (87)$$

Using the definition of g_1 , h and ζ in (55), (68), (78), and the expansion of Ω in (87), it is easily seen that there are constants $m_1 < m_2$ independent of s, n, u_0 such that $\zeta'(z)$ has at least one zero in the set

$$\left\{ z : \frac{sm_1}{n \log u_0^{-1}} < |z - 1| < \frac{sm_2}{n \log u_0^{-1}} \right\}. \quad (88)$$

By (84) and expansion (87), m_1 may be chosen such that

$$\left| \zeta_1 w + \mathcal{O} \left(\left(\frac{sw}{n\Omega} \right)^2 \right) \right| < 1 \quad (89)$$

as $\frac{sw}{n\Omega} \rightarrow 0$ and so ζ is conformal on the following disc

$$\left\{ z : |z - 1| < \frac{sm_1}{n \log u_0^{-1}} \right\}, \quad (90)$$

for $u_0, s/n < \epsilon$ for some fixed $\epsilon > 0$. Thus we define U_0 to be the set (90).

We define $\tilde{g}$ as

$$\tilde{g} = \lim_{z \rightarrow \infty} e^{g(z) - \log z} = \frac{2}{b_1^{1/2} + b_2^{1/2}} \left(\frac{b_2 - b_1}{((1 - b_1)^{1/2} + (1 - b_2)^{1/2})^2} \right)^{\frac{\Omega}{2\pi}}, \quad (91)$$

and T as

$$T(z) = \tilde{g}^{n\sigma_3} Y(z) e^{-ng(z)\sigma_3}. \quad (92)$$

It follows by (77) that T satisfies the following RH problem:

- (a) $T : \mathbb{C} \setminus C \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

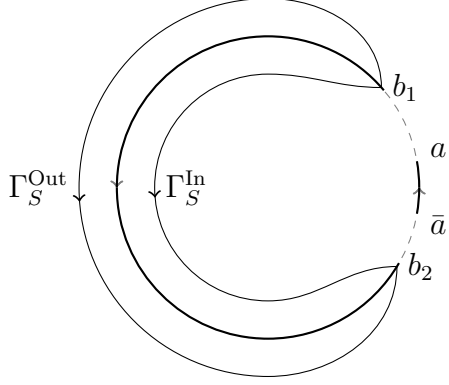


Figure 2: Contour Γ_S .

(b) T has the following jumps on C :

$$\begin{aligned}
T_+ &= T_- \begin{pmatrix} e^{n(g_- - g_+)(z)} & 1 \\ 0 & e^{n(g_+ - g_-)(z)} \end{pmatrix} & \text{for } z \in J_2, \\
T_+ &= T_- e^{-in\Omega\sigma_3} & \text{for } z \in \Sigma_2. \\
T_+ &= T_- \begin{pmatrix} e^{-in\Omega} & z^{-n} e^{n(g_+ + g_-)} \\ 0 & e^{in\Omega} \end{pmatrix} & \text{for } z \in (\bar{a}, 1) = \Sigma_2^o \cap J_1, \\
T_+ &= T_- \begin{pmatrix} 1 & z^{-n} e^{2ng} \\ 0 & 1 \end{pmatrix} & \text{for } z \in (1, a) = \Sigma_1^o \cap J_1. \\
T_+ &= T_- & \text{for } z \in \Sigma_1.
\end{aligned}$$

(c) As $z \rightarrow \infty$,

$$T(z) = I + \mathcal{O}(z^{-1}).$$

The jump of T on J_2 factorizes as

$$\begin{pmatrix} e^{n(g_- - g_+)(z)} & 1 \\ 0 & e^{n(g_+ - g_-)(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{n(g_+ - g_-)(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{n(g_- - g_+)(z)} & 1 \end{pmatrix}. \quad (93)$$

Define

$$\phi(z) = 2g(z) - \log(z), \quad (94)$$

Then the jumps of e^ϕ are induced by g and we obtain by (77) that for $z \in J_2$.

$$\begin{aligned}
\exp(\phi_+(z)) &= \exp[(g_+(z) + g_-(z) - \log(z)) + g_+(z) - g_-(z)] \\
&= \exp(g_+(z) - g_-(z)),
\end{aligned} \quad (95)$$

and similarly

$$\exp(\phi_-(z)) = \exp(g_-(z) - g_+(z)). \quad (96)$$

We proceed to open the lenses around J_2 as in Figure 2.

Proposition 3.1. *For z on the edges of the lense $\Gamma_S^{in} \cup \Gamma_S^{out}$ in Figure 2 such that $|z - b_1|, |z - b_2| > \epsilon s/n$ for some fixed $\epsilon > 0$, there exists a constant $C > 0$ independant of s, n, z such that*

$$e^{-ng(z)} < e^{-sC}, \quad (97)$$

as $s, n \rightarrow \infty$ and for u_0 sufficiently small.

Proof. Since e^{g_1} sends $\mathbb{C} \setminus J_2$ to the outside of the unit disc, it is clear that for $|z| < 1$ we have

$$|e^{2g_1(z) - \log(z)}| > 1. \quad (98)$$

Let $g_1^{(-)}$ denote the function defined as in (55), but with $+(z - b_1)(z - b_2)^{1/2}$ replaced with $-((z - b_1)(z - b_2))^{1/2}$. Then $e^{g_1^{(-)}}$ maps $\mathbb{C} \setminus J_2$ to the inside of the circle and for $z \in \mathbb{C} \setminus J_2$ we have the relation

$$e^{g_1(z) + g_1^{(-)}(z)} = z. \quad (99)$$

It follows that if $|z| > 1$, then

$$|e^{2g_1(z) - \log z}| = |e^{\log z - 2g_1^{(-)}(z)}| > 1. \quad (100)$$

Using (98), (100), the definition of g , and the fact that $\Omega = \mathcal{O}(s/(n \log u_0^{-1}))$, as $s/(n \log u_0^{-1}) \rightarrow 0$, it follows that $e^{-\phi(z)}$ lies in interior of the unit disc for z sufficiently close to the interval J_2 , and in particular that

$$e^{-n\phi(z)} = \mathcal{O}(e^{-cn}), \quad (101)$$

uniformly for z on the lense that is opened around J_2 in Figure 2, except near the endpoints b_1 and b_2 , for some constant $c > 0$.

Consider $h(z)$ and $g_1(z)$ at $z = b_1$. Let $w_1 = \frac{n}{s}(z - b_1)$. From (68) we have, with $B_j = \frac{n}{s}(1 - b_j)$,

$$h(z) = \log \frac{B_1 - B_2}{2}(w_1 - B_1) - \log \left(w_1 \frac{B_1 + B_2}{2} + \frac{B_2 - B_1}{2} B_1 + (w_1(w_1 + B_2 - B_1) B_1 B_2)^{1/2} \right).$$

(102)

It follows from (69) and (102) that $h(z)/w_1^{1/2}$ is analytic at $w_1 = 0$, and we have

$$h(z) = -w_1^{1/2} \frac{2}{(B_2 - B_1)^{1/2}} \left(\frac{B_2}{B_1} \right)^{1/2} (1 + \mathcal{O}(w_1)). \quad (103)$$

Likewise, we let $w_2 = \frac{n}{s}(z - b_2)$. Then at the point $w_2 = 0$ we have the expansion

$$h(z) = \pm \pi i - w_2^{1/2} \frac{2}{(B_1 - B_2)^{1/2}} \left(\frac{B_1}{B_2} \right)^{1/2} (1 + \mathcal{O}(w_2)), \quad (104)$$

where $\pm$ means $+$ on the $+$ -side and $-$ on the $-$ -side of the unit circle C (so the jumps agree with (69)).

We evaluate g_1 using the definition (55):

$$g_1(z) = \log \frac{b_1 + (b_1 b_2)^{1/2} + \frac{s}{n}((w_1(w_1 + B_2 - B_1))^{1/2} + w_1)}{b_1^{1/2} + b_2^{1/2}}. \quad (105)$$

From (105) and (57), we have that $(g_1(z) - \frac{\log z}{2})/w_1^{1/2}$ is analytic at $w_1 = 0$, and that at the point $w_1 = 0$ we have the expansion

$$g_1(z) - \frac{\log z}{2} = \frac{s}{n} w_1^{1/2} \frac{(B_2 - B_1)^{1/2}}{b_1 + (b_1 b_2)^{1/2}} (1 + \mathcal{O}(w_1)). \quad (106)$$

Likewise we expand g_1 at the point $w_2 = 0$:

$$g_1(z) - \frac{\log z}{2} = \frac{s}{n} w_2^{1/2} \frac{(B_1 - B_2)^{1/2}}{b_2 + (b_1 b_2)^{1/2}} (1 + \mathcal{O}(w_2)). \quad (107)$$

Consider now a neighbourhood of b_1 . The error terms in (103), (106) are uniform for $0 < s/n < \delta$ for some sufficiently small δ . By (103), (106) and the fact that $\Omega = \mathcal{O}\left(\frac{s}{n \log u_0^{-1}}\right)$, it follows that there is a constant $C_1 > 0$ independent of s, n, z for u_0 sufficiently small such that

$$\operatorname{Re}(g) > w_1^{1/2} C_1 s/n \quad (108)$$

as $z \rightarrow b_1, s/n \rightarrow 0$. Thus from (108) it follows that there exists $\epsilon, C > 0$ such that for all $|z - b_1| > \epsilon s/n$ we have

$$e^{-ng(z)} < e^{-sC} \quad (109)$$

as $n, s \rightarrow \infty$, and for u_0 sufficiently small. The same may be shown at the point b_2 , concluding the proof. $\square$

Let

$$S(z) = \begin{cases} T(z) & \text{for } z \text{ outside the lenses,} \\ T(z) \begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)} & 1 \end{pmatrix} & \text{for } z \text{ inside the lense and outside the unit disc,} \\ T(z) \begin{pmatrix} 1 & 0 \\ -e^{-n\phi(z)} & 1 \end{pmatrix} & \text{for } z \text{ inside the lense and inside the unit disc.} \end{cases} \quad (110)$$

Then S satisfies the following RH problem:

- (a) $S : \mathbb{C} \setminus \Gamma_S \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where $\Gamma_S = C \cup \Gamma_S^{in} \cup \Gamma_S^{out}$ as shown in Figure 2.
- (b) On $\Gamma_S \setminus \{a, \bar{a}, b_1, b_2\}$, S has the following jumps:

$$\begin{aligned} S_+ &= S_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in J_2, \\ S_+ &= S_- \begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)} & 1 \end{pmatrix} & \text{for } z \in \Gamma_S^{in} \cup \Gamma_S^{out}, \\ S_+ &= S_- e^{-in\Omega\sigma_3} & \text{for } z \in \Sigma_2. \\ S_+ &= S_- \begin{pmatrix} e^{-in\Omega} & z^{-n}e^{n(g_++g_-)} \\ 0 & e^{in\Omega} \end{pmatrix} & \text{for } z \in (\bar{a}, 1) = J_1 \cap \Sigma_2^o, \\ S_+ &= S_- \begin{pmatrix} 1 & z^{-n}e^{2ng} \\ 0 & 1 \end{pmatrix} & \text{for } z \in (1, a) = J_1 \cap \Sigma_1^o. \\ S_+ &= S_- & \text{for } z \in \Sigma_1. \end{aligned}$$

- (c) As $z \rightarrow \infty$,

$$S(z) = I + \mathcal{O}(z^{-1}). \quad (111)$$

3.1 Main parametrix

In the region $\mathbb{C} \setminus (U_0 \cup U_1 \cup U_2)$, we approximate the RH problem for S by a main parametrix M , which satisfies the RH problem:

- (a) $M : \mathbb{C} \setminus \overline{\{J_2 \cup \Sigma_2^o\}} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) On J_2 and Σ_2^o , M has the following jumps:

$$\begin{aligned} M_+(z) &= M_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} && \text{for } z \in J_2, \\ M_+(z) &= M_-(z) e^{-in\Omega\sigma_3} && \text{for } z \in \Sigma_2^o. \end{aligned}$$

(c) As $z \rightarrow \infty$,

$$M(z) = I + \mathcal{O}(z^{-1}).$$

A solution to the RH problem for M is given by

$$M(z) = \left(I + \frac{F}{z-1} \right) D^{-1}(\infty) \begin{pmatrix} \gamma_1(z) & -\gamma_2(z) \\ \gamma_2(z) & \gamma_1(z) \end{pmatrix} D(z), \quad (112)$$

where F is a constant matrix and

$$\begin{aligned} \gamma_1(z) &= \frac{1}{2} \left(\left(\frac{z-b_1}{z-b_2} \right)^{1/4} + \left(\frac{z-b_2}{z-b_1} \right)^{1/4} \right) \\ \gamma_2(z) &= \frac{1}{2i} \left(\left(\frac{z-b_1}{z-b_2} \right)^{1/4} - \left(\frac{z-b_2}{z-b_1} \right)^{1/4} \right) \end{aligned} \quad (113)$$

with branch cuts on J_2 and such that $\gamma_1(z) \rightarrow 1$ and $\gamma_2(z) \rightarrow 0$ as $z \rightarrow \infty$. For $y \in \mathbb{R}$, let $\langle y \rangle$ be defined such that

$$\langle y \rangle \in [-1/2, 1/2), \quad y - \langle y \rangle \in \mathbb{Z}. \quad (114)$$

Then D is given by

$$D(z) = \exp \left(- \left\langle \frac{n\Omega}{2\pi} \right\rangle h(z) \sigma_3 \right), \quad (115)$$

where it follows from the jumps of h (69) that D is analytic for $z \in \mathbb{C} \setminus (\overline{\Sigma_2^o} \cup J_2)$, and

$$\begin{aligned} D_-^{-1} D_+ &= \exp \left(-2\pi i \left\langle \frac{n\Omega}{2\pi} \right\rangle \sigma_3 \right) && \text{for } z \in \Sigma_2^o, \\ D_- D_+ &= I && \text{for } z \in J_2. \end{aligned} \quad (116)$$

The function M defined in (112) will solve the RH problem for M with any constant matrix F , which we will define later in (128). The reason for the prefactor $I + \frac{F}{z-1}$ in (112), which does not affect the jump conditions for M , will become apparent later on.

3.2 Model RH problem Φ

Consider the following RH problem for $\Phi(\zeta; k)$, where $k \in \mathbb{N}$:

- (a) $\Phi(\zeta) : \mathbb{C} \setminus [\eta_1, \eta_2] \rightarrow \mathbb{C}^{2 \times 2}$ is analytic for given $\eta_1 < \eta_2$.
- (b) Φ has L^2 boundary values for $\zeta \in (\eta_1, \eta_2)$ satisfying

$$\Phi_+(\zeta; k) = \Phi_-(\zeta; k) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

- (c) As $\zeta \rightarrow \infty$,

$$\Phi(\zeta; k) = \left(I + \frac{\Phi_1}{\zeta} + \frac{\Phi_2}{\zeta^2} + \mathcal{O}(\zeta^{-3}) \right) \zeta^{k\sigma_3}. \quad (117)$$

It is well-known by the standard theory and is easy to verify that the unique solution to this RH problem is given by

$$\begin{aligned} \Phi(\zeta; 0) &= \begin{pmatrix} 1 & \frac{1}{2\pi i} \log \left(\frac{\zeta - \eta_2}{\zeta - \eta_1} \right) \\ 0 & 1 \end{pmatrix} \\ \Phi(\zeta; k) &= \begin{pmatrix} \frac{1}{\kappa_k} L_k(\zeta) & \frac{1}{2\pi i \kappa_k} \int_{\eta_1}^{\eta_2} \frac{L_k(x)}{x - \zeta} dx \\ -2\pi i \kappa_{k-1} L_{k-1}(\zeta) & -\kappa_{k-1} \int_{\eta_1}^{\eta_2} \frac{L_{k-1}(x)}{x - \zeta} dx \end{pmatrix} \quad \text{for } k \geq 1, \end{aligned} \quad (118)$$

where L_k are the Legendre polynomials of degree k with positive leading coefficients, orthonormal on (η_1, η_2) :

$$\int_{\eta_1}^{\eta_2} L_k(\zeta) L_j(\zeta) d\zeta = \delta_{jk} = \begin{cases} 0 & \text{for } j \neq k, \\ 1 & \text{for } j = k, \end{cases} \quad (119)$$

and we denote the first 3 leading coefficients as follows:

$$L_k(\zeta) = \kappa_k \zeta^k + \mu_k \zeta^{k-1} + \nu_k \zeta^{k-2} + \dots \quad (120)$$

Writing the large ζ expansion of (118) and using orthogonality in the second column, we obtain that

$$\begin{aligned} \Phi_1 &= \begin{pmatrix} \frac{\mu_k}{\kappa_k} & -\frac{1}{2\pi i} \kappa_k^{-2} \\ -2\pi i \kappa_{k-1}^2 & -\frac{\mu_k}{\kappa_k} \end{pmatrix} \quad \text{for } k \geq 1, \\ \Phi_2 &= \begin{pmatrix} \frac{\nu_k}{\kappa_k} & \frac{\mu_{k+1}}{2\pi i \kappa_{k+1} \kappa_k^2} \\ -2\pi i \kappa_{k-1} \mu_{k-1} & \frac{1}{\kappa_k \kappa_{k+1}} (\mu_k \mu_{k+1} - \nu_{k+1} \kappa_k) \end{pmatrix} \quad \text{for } k \geq 2. \end{aligned} \quad (121)$$

When $k = 0$ we have

$$\Phi_1 = \begin{pmatrix} 0 & -\frac{\eta_2 - \eta_1}{2\pi i} \\ 0 & 0 \end{pmatrix}. \quad (122)$$

It is well known that L_k has the explicit representation for $k \geq 0$:

$$\begin{aligned} L_k(\zeta) &= \sqrt{\frac{2k+1}{\eta_2 - \eta_1}} \sum_{j=0}^k \binom{k}{j} \binom{k+j}{j} \left(\frac{\zeta - \eta_2}{\eta_2 - \eta_1} \right)^j \\ &= \sqrt{\frac{2k+1}{\eta_2 - \eta_1}} \sum_{j=0}^k \binom{k}{j} \binom{k+j}{j} \left(\frac{\zeta - \eta_1}{\eta_2 - \eta_1} \right)^j (-1)^{k-j}, \end{aligned} \quad (123)$$

where $\binom{0}{0} = 1$. As a consequence, the coefficients in (120) are given by

$$\begin{aligned} \kappa_k &= (\eta_2 - \eta_1)^{-k-1/2} \sqrt{2k+1} \binom{2k}{k}, \\ \mu_k &= -(\eta_2 - \eta_1)^{-k-1/2} \sqrt{2k+1} \binom{2k}{k} \frac{k}{2} (\eta_1 + \eta_2), \\ \nu_k &= (\eta_2 - \eta_1)^{-k-1/2} \sqrt{2k+1} \binom{2k-2}{k-2} \frac{k}{2} (k(\eta_1 + \eta_2)^2 - (\eta_1^2 + \eta_2^2)), \end{aligned} \quad (124)$$

for $k \geq 0, 1, 2$ respectively. From (118), (123), (124), it follows that for $k \geq 1$,

$$\begin{aligned} \Phi(\zeta) &= \begin{pmatrix} (-1)^k \frac{(\eta_2 - \eta_1)^k}{\binom{2k}{k}} \left(1 - \frac{k(k+1)}{\eta_2 - \eta_1} (\zeta - \eta_1) + \mathcal{O}((\zeta - \eta_1)^2) \right) & * \\ (-1)^k 2\pi i (2k-1) (\eta_2 - \eta_1)^{-k} \binom{2k-2}{k-1} \left(1 - \frac{k(k-1)}{\eta_2 - \eta_1} (\zeta - \eta_1) + \mathcal{O}((\zeta - \eta_1)^2) \right) & * \end{pmatrix}, \quad \text{as } \zeta \rightarrow \eta_1 \\ \Phi(\zeta) &= \begin{pmatrix} \frac{(\eta_2 - \eta_1)^k}{\binom{2k}{k}} \left(1 + \frac{k(k+1)}{\eta_2 - \eta_1} (\zeta - \eta_2) + \mathcal{O}((\zeta - \eta_2)^2) \right) & * \\ -2\pi i (2k-1) (\eta_2 - \eta_1)^{-k} \binom{2k-2}{k-1} \left(1 + \frac{k(k-1)}{\eta_2 - \eta_1} (\zeta - \eta_2) + \mathcal{O}((\zeta - \eta_2)^2) \right) & * \end{pmatrix}, \quad \text{as } \zeta \rightarrow \eta_2 \end{aligned} \quad (125)$$

3.3 Local parametrix at 1

Recall that U_0 defined by (90) is an open disc containing J_1 and that as $s, n \rightarrow \infty, u_0 \rightarrow 0$ the radius of U_0 is of length $\epsilon s / (n \log u_0^{-1})$ for some $\epsilon > 0$. On U_0 we defined a local variable ζ in (78). We define the local parametrix P on U_0 by

$$P(z) = E(z) \Phi(\zeta(z); k) e^{-n(g(z) - \frac{\log(z)}{2})\sigma_3}, \quad k = \frac{\Omega n}{2\pi} - x, \quad x = \left\langle \frac{\Omega n}{2\pi} \right\rangle, \quad (126)$$

where Φ is given by (118), and where E is an analytic function on U_0 given by

$$\begin{aligned} E(z) &= \left(I + \frac{F}{z-1} \right) D(\infty)^{-1} \begin{pmatrix} \gamma_1(z) & -\gamma_2(z) \\ \gamma_2(z) & \gamma_1(z) \end{pmatrix} B(z) \left(I - \frac{X}{\zeta(z)} \right), \\ B(z) &= e^{\frac{2\pi}{\Omega} x (g_1(z) - \frac{1}{2} \log z) \sigma_3}, \end{aligned} \quad (127)$$

with constant matrices F and X defined below. From (59) and (72) we see that $\zeta(J_1) \subset \mathbb{R}$. We let $\eta_1 = \zeta(\bar{a})$ and $\eta_2 = \zeta(a)$. Then $\zeta(J_1) = (\eta_1, \eta_2)$, and so $P(z)$ has a jump on J_1 induced by that of Φ on (η_1, η_2) . F is a constant, nilpotent matrix

$$F = \tilde{h}^{-\sigma_3} \begin{pmatrix} f & \psi f \\ -f/\psi & -f \end{pmatrix} \tilde{h}^{\sigma_3}, \quad (128)$$

where

$$\begin{aligned} \psi &= \begin{cases} -\frac{\gamma_1(1)}{\gamma_2(1)} & \text{for } 0 \leq x < 1/2, \\ \frac{\gamma_2(1)}{\gamma_1(1)} & \text{for } -1/2 \leq x < 0, \end{cases} \\ \tilde{h} &= D_{11}(\infty) = \exp(-xh(\infty)) = \left(\frac{(1-b_1)^{1/2} + (1-b_2)^{1/2}}{b_2 - b_1} \right)^{-x} \\ f &= \begin{cases} -\frac{(1-b_1)^{1/2}(1-b_2)^{1/2}\rho}{1+\rho} & \text{for } 0 \leq x < 1/2, \\ -\frac{(1-b_1)^{1/2}(1-b_2)^{1/2}\rho}{1-\rho} & \text{for } -1/2 \leq x < 0, \end{cases} \\ \rho &= \begin{cases} \frac{1}{2\pi\kappa_k^2} \exp \left[\frac{2\pi}{\Omega} g_1(1) (-1+2x) \right] & \text{for } 0 \leq x < 1/2, \\ -2\pi\kappa_{k-1}^2 \exp \left[\frac{2\pi}{\Omega} g_1(1) (-1-2x) \right] & \text{for } -1/2 \leq x < 0, \end{cases} \end{aligned} \quad (129)$$

where $h(\infty)$ was defined in (71) and X is a constant matrix given in terms of elements of (121)

$$X = \begin{cases} \Phi_{1,12} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{for } 0 \leq x < 1/2, \\ \Phi_{1,21} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \text{for } -1/2 \leq x < 0. \end{cases} \quad (130)$$

The factor $I - X/\zeta(z)$ in $E(z)$ is needed to cancel the would be $u_0^{1-2|x|} \log u_0^{-1}$ non-smallness in the matrix elements of $\Delta_1^{(1)}$ originating from $\Phi_{1,12}B_{11}^2$ for $0 \leq x < 1/2$ and $\Phi_{1,21}B_{11}^{-2}$ for $-1/2 \leq x < 0$ (see Proposition 3.2 below) so that P and M match to the main order on the boundary ∂U_0 for all $x \in [-1/2, 1/2)$. This factor, however, has a pole at $z = 1$, but we need $E(z)$ to be analytic in U_0 . As is easy to verify, the analyticity of $E(z)$ (i.e. the absence of a pole at 1) is achieved by choosing F as defined in (128).

Proposition 3.2. *As $u_0 \rightarrow 0$ and $s, n \rightarrow \infty$, we have the following matching condition uniformly on the boundary ∂U_0 :*

$$P(z)M^{-1}(z) = I + \Delta_1^{(1)}(z) + \Delta_2^{(1)}(z) + \Xi^{(1)} + \mathcal{O}(\tilde{e}u_0^{-2|x|}(u_0 \log u_0^{-1})^3 + \tilde{e}(s^2u_0^4(\log u_0^{-1})^2)),$$

where

$$\tilde{e} = \left(1 + u_0^{1-2|x|} \log u_0^{-1}\right)^2, \quad (131)$$

and $\Delta_1^{(1)}$, $\Delta_2^{(1)}$ and $\Xi^{(1)}$ are given by (138) below. We have, uniformly for z on the boundary ∂U ,

$$\begin{aligned} \Delta_1^{(1)}(z) &= \mathcal{O}(\tilde{e}u_0^{1+2|x|} \log u_0^{-1} + \tilde{e}s u_0^2 \log u_0^{-1}), \\ \Delta_2^{(1)}(z) &= \mathcal{O}\left(\tilde{e}u_0^2 (\log u_0^{-1})^2 \left(\frac{s}{\log u_0^{-1}} + u_0^{1-2|x|} \log u_0^{-1}\right)\right), \\ \Xi^{(1)} &= \mathcal{O}\left(\tilde{e}s u_0^{-2|x|} u_0^3 (\log u_0^{-1})^2\right). \end{aligned} \quad (132)$$

Proof. First, assume that k is bounded. Since

$$\zeta^k(z) e^{-n(g(z) - \frac{\log z}{2})} = e^{-\frac{2\pi x}{\Omega}(g(z) - \frac{\log z}{2})}, \quad x = \frac{n\Omega}{2\pi} - k \quad (133)$$

we have on the boundary ∂U that (recall (76), (112), (117), (127))

$$P(z)M^{-1}(z) = E(z) \left(I + \frac{\Phi_1}{\zeta} + \frac{\Phi_2}{\zeta^2} + \mathcal{O}(\zeta^{-3})\right) \left(I - \frac{X}{\zeta(z)}\right) E^{-1}(z). \quad (134)$$

It follows from (64), (87), (129) that

$$\begin{aligned} \rho &= \mathcal{O}\left(u_0^{1-2|x|}\right), \\ f, F &= \mathcal{O}\left(s u_0^{1-2|x|}/n\right), \end{aligned} \quad (135)$$

as $u_0, s/n \rightarrow 0$ and $s, n \rightarrow \infty$. Denote

$$\tilde{E}(z) = \left(I + \frac{F}{z-1}\right) D(\infty)^{-1} \begin{pmatrix} \gamma_1(z) & -\gamma_2(z) \\ \gamma_2(z) & \gamma_1(z) \end{pmatrix}. \quad (136)$$

From (135), the boundedness of $\tilde{h}, \tilde{h}^{-1}, \psi, \psi^{-1}, \gamma_1(z), \gamma_2(z)$ for $z \in \partial U_0$, and the fact that the radius of U_0 equals $\epsilon \frac{s}{n \log u_0^{-1}}$ for some $\epsilon > 0$, we have

$$\tilde{E}(z) = \mathcal{O}(\sqrt{\tilde{e}}) \quad (137)$$

as $u_0 \rightarrow 0$ and $s, n \rightarrow \infty$, uniformly for $z \in \partial U_0$. From (134) and (136) it follows that $\Delta_1^{(1)}, \Delta_2^{(1)}$ take the form

$$\Delta_1^{(1)}(z) = \begin{cases} \zeta^{-1}(z) \tilde{E}(z) B(z) \begin{pmatrix} \Phi_{1,11} & 0 \\ \Phi_{1,21} & \Phi_{1,22} \end{pmatrix} B^{-1}(z) \tilde{E}^{-1}(z) & 0 \leq x < 1/2, \\ \zeta^{-1}(z) \tilde{E}(z) B(z)(z) \begin{pmatrix} \Phi_{1,11} & \Phi_{1,12} \\ 0 & \Phi_{1,22} \end{pmatrix} B^{-1}(z) \tilde{E}^{-1}(z) & -1/2 \leq x < 0, \end{cases} \quad (138)$$

$$\begin{aligned} \Delta_2^{(1)}(z) &= \zeta^{-2}(z) \tilde{E}(z) \hat{g}_1^{x\sigma_3}(z) (\Phi_2 - X\Phi_1) \hat{g}_1^{-x\sigma_3}(z) \tilde{E}^{-1}(z), \\ \Xi^{(1)}(z) &= -\zeta^{-3}(z) \tilde{E}(z) \hat{g}_1^{x\sigma_3}(z) X\Phi_2 \hat{g}_1^{-x\sigma_3}(z) \tilde{E}^{-1}(z). \end{aligned}$$

From (63)–(64), and recalling (87) and the fact that $w = \mathcal{O}\left(\frac{1}{\log u_0^{-1}}\right)$ as $u_0 \rightarrow 0$, it follows that

$$\widehat{g}_1(z) = \mathcal{O}(u_0^{-1}) \quad (139)$$

as $u_0 \rightarrow 0$, uniformly on the boundary ∂U_0 . Similarly, substituting (84) into (83) and recalling (87), we have

$$\zeta(z) = \mathcal{O}\left(\frac{1}{u_0 \log u_0^{-1}}\right) \quad (140)$$

as $u_0 \rightarrow 0$, uniformly on the boundary ∂U_0 . From (83) and (84) we have

$$\zeta(a) + \zeta(\bar{a}) = \mathcal{O}(u_0 \log u_0^{-1}), \quad (141)$$

as $u_0 \rightarrow 0$. Using (141) it follows from (121) and (124) that

$$\begin{aligned} \Phi_{1,11}, \Phi_{1,22}, \Phi_{2,12}, \Phi_{2,21} &= \mathcal{O}(u_0 \log u_0^{-1}) \\ \Phi_{1,12}, \Phi_{1,21}, \Phi_{2,11}, \Phi_{2,22} &= \mathcal{O}(1) \end{aligned} \quad (142)$$

as $u_0 \rightarrow 0$ (for finite k). Combining (137)–(142) the proposition is proven for bounded k .

Now consider $k \rightarrow \infty$. From Stirling's formula we have

$$2\pi\kappa_k^2 \rightarrow 1 \quad (143)$$

as $k \rightarrow \infty$. Thus (135) holds uniformly for $k \in \mathbb{N}$. We study the particular double scaling limit where $k, \zeta \rightarrow \infty$, and from (85), (141) we have that $\eta_1 + \eta_2 \rightarrow 0$ in such a manner that $k/\zeta, k(\eta_1 + \eta_2) = \mathcal{O}(u_0 s) \rightarrow 0$. Thus using (143) we find that as $k \rightarrow \infty$

$$\Phi_1 = \begin{pmatrix} -\frac{k}{2}(\eta_1 + \eta_2) & i + \mathcal{O}(k^{-1}) \\ -i + \mathcal{O}(k^{-1}) & \frac{k}{2}(\eta_1 + \eta_2) \end{pmatrix}. \quad (144)$$

We also find that as $k \rightarrow \infty$ and $(\eta_1 + \eta_2) \rightarrow 0$ such that $(\eta_1 + \eta_2)k \rightarrow 0$

$$\Phi_2 = \begin{pmatrix} -\frac{k}{8}(\eta_1^2 + \eta_2^2) + \mathcal{O}(1) & \frac{ik}{2}(\eta_1 + \eta_2) + \mathcal{O}(\eta_1 + \eta_2) \\ \frac{ik}{2}(\eta_1 + \eta_2) + \mathcal{O}(\eta_1 + \eta_2) & \frac{k}{8}(\eta_1^2 + \eta_2^2) + \mathcal{O}(1) \end{pmatrix}. \quad (145)$$

Thus we know the large k, ζ behaviour of $\zeta^{-1}\Phi_1$, $\zeta^{-2}\Phi_2$, and upon substituting into (138) this yields (132). It remains to calculate the error terms of order $\zeta^{-3}\Phi_3$ and higher, and in particular establish their behaviour as $k \rightarrow \infty$ with ζ . We rely here on the work by Kuijlaars, McLaughlin, Van Assche and Vanlessen in [13] where the authors found uniform error terms for the Legendre polynomials L_k as $k \rightarrow \infty$. In the remaining part of the proof

of the proposition, we let $\widehat{Y}, \widehat{R}, \widehat{N}$ denote the functions Y, R, N found in [13]. We compare $\widehat{Y}$ to Φ from (118) in the present paper:

$$\Phi(\zeta) = 2^{k\sigma_3} \widehat{Y}(y(\zeta)), \quad y(\zeta) = \frac{1}{2} \left(\zeta - \frac{\eta_1 + \eta_2}{2} \right) \quad (146)$$

where the parameter n in [13] is set to be k here. For y bounded away from $[-1, 1]$ it follows from equations (3.1), (4.2), (5.5), (7.1) in [13] that

$$\begin{aligned} \widehat{Y}(y) &= 2^{-k\sigma_3} \widehat{R}(y) \widehat{N}(y) y^{k\sigma_3} \left(1 + (1 - y^{-2})^{1/2} \right)^{k\sigma_3}, \\ \widehat{N}(y) &= \begin{pmatrix} \frac{1}{2}(a(y) + 1/a(y)) & \frac{1}{2i}(a(y) - 1/a(y)) \\ -\frac{1}{2i}(a(y) - 1/a(y)) & \frac{1}{2}(a(y) + 1/a(y)) \end{pmatrix}, \quad a(y) = \left(\frac{y-1}{y+1} \right)^{1/4}. \end{aligned} \quad (147)$$

By the form of $\widehat{N}$ in (147) above and formula (8.11) in [13] it is clear that

$$\widehat{R}(y(\zeta); k) \widehat{N}(y(\zeta)) = I + \frac{\chi_1}{\zeta} + \frac{\chi_2}{\zeta^2} + \mathcal{O}(\zeta^{-3})$$

as $\zeta \rightarrow \infty$, where χ_1 and χ_2 are bounded for $k \in \mathbb{N}$ and the $\mathcal{O}(\zeta^{-3})$ term is uniform for $k \in \mathbb{N}$. As $\zeta, k \rightarrow \infty$ and $\eta_1 + \eta_2 \rightarrow 0$ such that $k/\zeta \rightarrow 0$, $k(\eta_1 + \eta_2) \rightarrow 0$, we have

$$\left(1 + (1 - y^{-2})^{1/2} \right)^{\pm k} = 1 \mp k\zeta^{-1} \left(\frac{\eta_1 + \eta_2}{2} + \zeta^{-1} \right) + \mathcal{O}(k^2|\zeta|^{-2}(|\eta_1 + \eta_2| + |\zeta|^{-1})^2). \quad (148)$$

It follows from (147)-(148) that as $\zeta, k \rightarrow \infty$ and $\eta_1 + \eta_2 \rightarrow 0$

such that $k/\zeta \rightarrow 0$, $k(\eta_1 + \eta_2) \rightarrow 0$,

$$\begin{aligned} \Phi(\zeta; k) &= \left(I + \chi_1/\zeta + \chi_2/\zeta^2 + \mathcal{O}(\zeta^{-3}) \right) \zeta^{k\sigma_3} \\ &\quad \times \begin{pmatrix} 1 - k\zeta^{-1} \left(\frac{\eta_1 + \eta_2}{2} + \zeta^{-1} \right) + \mathcal{O}(k^2|\zeta|^{-2}(|\eta_1 + \eta_2| + |\zeta|^{-1})^2) & 0 \\ 0 & 1 + k\zeta^{-1} \left(\frac{\eta_1 + \eta_2}{2} + \zeta^{-1} \right) + \mathcal{O}(k^2\zeta^{-2}(|\eta_1 + \eta_2| + |\zeta|^{-1})^2) \end{pmatrix}, \end{aligned} \quad (149)$$

where, in particular, χ_1 and χ_2 are bounded for $k \in \mathbb{N}$ and the $\mathcal{O}(\zeta^{-3})$ term is uniform for $k \in \mathbb{N}$. By comparing (149) with (117) it follows that as $\zeta, k \rightarrow \infty$ and $\eta_1 + \eta_2 \rightarrow 0$ such that $k/\zeta \rightarrow 0$, $k(\eta_1 + \eta_2) \rightarrow 0$,

$$\begin{aligned} \Phi(\zeta; k) \zeta^{-k\sigma_3} - I - \Phi_1/\zeta - \Phi_2/\zeta^2 &= \\ &= \mathcal{O}(\zeta^{-3}) + \begin{pmatrix} \mathcal{O}(k^2(\zeta^{-4} + |\eta_1 + \eta_2|\zeta^{-3})) & 0 \\ 0 & \mathcal{O}(k^2(\zeta^{-4} + |\eta_1 + \eta_2|\zeta^{-3})) \end{pmatrix}. \end{aligned} \quad (150)$$

□

3.4 Model RH problem Ψ

The following RH problem has a solution in terms of Bessel functions.

- (a) $\Psi : \mathbb{C} \setminus \overline{\Gamma_\Psi} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where $\Gamma_\Psi = \mathbb{R}^- \cup \Gamma_\Psi^\pm$, with $\Gamma_\Psi^\pm = \{xe^{\pm \frac{2\pi}{3}i} : x \in \mathbb{R}^+\}$, and with orientation taken in the direction of increasing real part.
- (b) Ψ has continuous boundary values Ψ_+, Ψ_- on Γ_Ψ satisfying the following jump conditions:

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for } \zeta \in \mathbb{R}^-, \quad (151)$$

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{for } \zeta \in \Gamma_\Psi^\pm. \quad (152)$$

- (c) As $\zeta \rightarrow \infty$, Ψ has the following asymptotics:

$$\Psi(\zeta) = \left(\pi\zeta^{\frac{1}{2}}\right)^{-\frac{\sigma_3}{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I + \frac{1}{8\sqrt{\zeta}} \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} + \mathcal{O}(\zeta^{-1})\right) e^{\zeta^{\frac{1}{2}}\sigma_3}. \quad (153)$$

- (d) As $\zeta \rightarrow 0$, the behaviour of Ψ is

$$\Psi(\zeta) = \mathcal{O}(\log |\zeta|). \quad (154)$$

This RH problem has a solution given in [13], in terms of Bessel functions. For definitions and properties of Bessel functions see [15]. We take the principal branches of the Bessel functions. For $|\arg \zeta| < 2\pi/3$, we have

$$\Psi(\zeta) = \begin{pmatrix} I_0(\zeta^{1/2}) & \frac{i}{\pi} K_0(\zeta^{1/2}) \\ \pi i \zeta^{1/2} I'_0(\zeta^{1/2}) & -\zeta^{1/2} K'_0(\zeta^{1/2}) \end{pmatrix}. \quad (155)$$

For $2\pi/3 < \arg \zeta < \pi$ we the solution is given by

$$\Psi(\zeta) = \frac{1}{2} \begin{pmatrix} H_0^{(1)}(e^{\frac{\pi i}{2}} \zeta^{1/2}) & H_0^{(2)}(e^{\frac{\pi i}{2}} \zeta^{1/2}) \\ \pi \zeta^{1/2} \left(H_0^{(1)}\right)'(e^{\frac{\pi i}{2}} \zeta^{1/2}) & \pi \zeta^{1/2} \left(H_0^{(2)}\right)'(e^{\frac{\pi i}{2}} \zeta^{1/2}) \end{pmatrix}. \quad (156)$$

For $-\pi < \arg \zeta < -2\pi/3$ it is defined as

$$\Psi(\zeta) = \frac{1}{2} \begin{pmatrix} H_0^{(2)}(e^{\frac{\pi i}{2}} \zeta^{1/2}) & -H_0^{(1)}(e^{\frac{\pi i}{2}} \zeta^{1/2}) \\ -\pi \zeta^{1/2} \left(H_0^{(2)}\right)'(e^{\frac{\pi i}{2}} \zeta^{1/2}) & \pi \zeta^{1/2} \left(H_0^{(1)}\right)'(e^{\frac{\pi i}{2}} \zeta^{1/2}) \end{pmatrix}. \quad (157)$$

We have the following useful asymptotics as $\zeta \rightarrow 0$ for I_0 and K_0 :

$$I_0(\zeta) = 1 + \frac{\zeta^2}{4} + \frac{\zeta^4}{64} + \mathcal{O}(\zeta^6), \quad (158)$$

$$K_0(\zeta) = -\log \frac{\zeta}{2} \left(1 + \frac{\zeta^2}{4} + \frac{\zeta^4}{64} + \mathcal{O}(\zeta^6)\right). \quad (159)$$

3.5 Local parametrix at b_1 and b_2

Let U_1 and U_2 be discs of radius $\frac{\epsilon s}{n}$ for some fixed but sufficiently small $\epsilon > 0$, centered at b_1 and b_2 respectively. Recalling $w_j = \frac{n}{s}(z - b_j)$ for $j = 1, 2$, we have $|w_j| = \epsilon$ on ∂U_j . For $z \in U_1$, define

$$\zeta_1(z) = \frac{n^2}{4}\phi(z)^2, \quad (160)$$

where ϕ was defined in (94). Recall the notation $B_j = \frac{n}{s}(1 - b_j)$ for $j = 1, 2$. By (103) and (106) we have the following expansion of $\zeta_1(z)$ for w_1 in a neighbourhood of 0:

$$\begin{aligned} \zeta_1(z) &= s^2 \zeta_{1,0} w_1 (1 + \mathcal{O}(w_1)) \\ \zeta_{1,0} &= \frac{B_2 - B_1}{(b_1 + (b_1 b_2)^{1/2})^2} \left(1 - \frac{n\Omega}{\pi s} \frac{b_1 + (b_1 b_2)^{1/2}}{B_2 - B_1} \left(\frac{B_2}{B_1} \right)^{1/2} \right)^2, \end{aligned} \quad (161)$$

and by considering (77) in addition, one verifies that ζ_1 is analytic on U_1 .

Recall from (58), (70) that $\phi_{\pm}(b_2) = \pm\Omega i$ and define

$$\tilde{\phi}(z) = \begin{cases} \phi(z) - \Omega i & \text{for } z \in U_2 \text{ and } z \in \mathcal{D}, \\ \phi(z) + \Omega i & \text{for } z \in U_2 \text{ and } z \notin \mathcal{D}. \end{cases} \quad (162)$$

where $\mathcal{D}$ denotes the unit disc. Then $\tilde{\phi} : U \setminus J_2 \rightarrow \mathbb{C}$ is analytic, with a square root singularity at b_2 . We define the local variable

$$\zeta_2(z) = \frac{n^2}{4}\tilde{\phi}^2(z),$$

which is analytic on U_2 . Then, by (104) and (107), $\zeta_2(z)$ has the following expansion at $w_2 = 0$:

$$\begin{aligned} \zeta_2(z) &= s^2 \zeta_{2,0} w_2 (1 + \mathcal{O}(w_2)) \\ \zeta_{2,0} &= \frac{B_1 - B_2}{(b_2 + (b_1 b_2)^{1/2})^2} \left(1 - \frac{n\Omega}{\pi s} \frac{b_2 + (b_1 b_2)^{1/2}}{B_1 - B_2} \left(\frac{B_1}{B_2} \right)^{1/2} \right)^2, \end{aligned} \quad (163)$$

and by considering (77) in addition, one verifies that ζ_2 is analytic on U_2 .

The local parametrix is given by

$$P_j(z) = E_j(z) \sigma_3^j \Psi(\zeta_j(z)) \sigma_3^j e^{-\frac{n}{2}\phi(z)\sigma_3} \quad (164)$$

$$E_j(z) = M(z) e^{\pm \frac{n}{2}\phi_{\pm}(b_j)\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & (-1)^{j+1}i \\ (-1)^{j+1}i & 1 \end{pmatrix} (\pi \zeta_j(z)^{1/2})^{\frac{1}{2}\sigma_3}, \quad (165)$$

on the $\pm$ side of the contour C , where $\phi_+(b_1) = 0$ and $\phi_+(b_2) = \Omega i$. As a consequence of the expansions of ζ_j above, we have

$$\zeta_{j,-}^{-1/4} \zeta_{j,+}^{1/4} = (-1)^j i, \quad (166)$$

and recalling the definition of M in (112), one may verify that E_j is analytic on U_j . Recalling the jumps of ϕ in (95)–(96) and jumps of g in (77), one verifies that the jumps of P_j match those of S on U_j .

Since, recalling (129), $F = \mathcal{O}(s/n)$ as $s/n \rightarrow 0$ while $D(\infty)$ remains bounded, we have that E_j is uniformly bounded on ∂U_j , and it follows that uniformly for $z \in \partial U_j$ we have the following matching condition of M and P_j

$$(P_j M^{-1})(z) = I + \Delta_1^{(b_j)}(z) + \mathcal{O}(1/s^2), \quad \Delta_1^{(b_j)}(z) = \mathcal{O}(1/s), \quad (167)$$

as $s \rightarrow \infty$. A simple calculation yields

$$\Delta_1^{(b_1)}(z) = \frac{(B_2 - B_1)^{1/2}}{16s\sqrt{\zeta_{1,0}w_1}} \left(I + \frac{F}{b_1 - 1} \right) D(\infty)^{-1} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} D(\infty) \left(I - \frac{F}{b_1 - 1} \right) + \mathcal{O}(1) \quad (168)$$

as $z \rightarrow b_1$, where the $\mathcal{O}(1)$ part is analytic on U_1 . Similarly, as $z \rightarrow b_2$ we have:

$$\Delta_1^{(b_2)}(z) = \frac{(B_1 - B_2)^{1/2}}{16s\sqrt{\zeta_{2,0}w_2}} \left(I + \frac{F}{b_2 - 1} \right) D(\infty)^{-1} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} D(\infty) \left(I - \frac{F}{b_2 - 1} \right) + \mathcal{O}(1), \quad (169)$$

again with $\mathcal{O}(1)$ analytic.

3.6 Small norm RH problem

We define R as follows:

$$R(z) = \begin{cases} SM^{-1} & \text{for } z \in \mathbb{C} \setminus \overline{(\cup_{j=0}^2 U_j)}, \\ SP^{-1} & \text{for } z \in U_0, \\ SP_j^{-1} & \text{for } z \in U_j \text{ where } j = 1, 2. \end{cases} \quad (170)$$

Using standard small norm analysis, it follows from Proposition 3.2, (168)–(169) and the fact that the contour lengths are $\partial U_j = \mathcal{O}(s/n)$ for $j = 1, 2$ as $s/n \rightarrow 0$ and $\partial U_0 = \mathcal{O}(s/n(\log u_0^{-1}))$ as $s/n, u_0 \rightarrow 0$ that given $\epsilon > 0$,

$$R(z) = I + \mathcal{O}(1/n), \quad (171)$$

uniformly for $|z - 1| > \epsilon$.

If $z \in U_0$ then it follows from Proposition 3.2, and (168)-(169) that

$$R(z) = I + R_1(z) + R_2(z) + \mathcal{O}(\|R_1\| \|R_2\| + \tilde{e}u_0^{-2|x|}(u_0 \log u_0^{-1})^3 + \tilde{e}(s^2 u_0^4 (\log u_0^{-1})^2)), \quad (172)$$

where $\|R_j\|$ is the largest element of R_j in absolute value for $j = 1, 2$, and where the matrices R_j are given by

$$\begin{aligned} R_1(z) &= \int_{\partial U_0} \frac{\Delta_1^{(1)}(u)}{(u-z)} \frac{du}{2\pi i} + \sum_{j=1,2} \int_{\partial U_j} \frac{\Delta_1^{(b_j)}(u)}{(u-z)} \frac{du}{2\pi i}, \\ R_2(z) &= \int_{\partial U_0} \frac{R_{1-}(u)\Delta_1^{(1)}(u) + \Delta_2^{(1)}(u) + \Xi^{(1)}(u)}{u-z} \frac{du}{2\pi i} \\ &\quad + \sum_{j=1,2} \int_{\partial U_j} \frac{R_{1-}(u)\Delta_1^{(b_j)}(u) + \Delta_2^{(b_j)}(u)}{u-z} \frac{du}{2\pi i}, \end{aligned} \quad (173)$$

with clockwise orientation taken in the integrals.

4 Asymptotic analysis of the differential identity and correlation functions

4.1 Asymptotics of χ_n

From (42) we have

$$\chi_{n-1}^2 = -(Y_n)_{21}(0). \quad (174)$$

By the transformations $Y = \tilde{g}^{-n\sigma_3} T e^{ng\sigma_3}$ and $T = S = RM$ at $z = 0$, (see (92), (110), (170)) and recalling that χ_n is positive, we find from (174) that

$$|\chi_{n-1}^2 \tilde{g}^{-2n}| = |\tilde{g}^{-n} e^{ng(0)} (R(0)M(0))_{21}|. \quad (175)$$

From the definition of g in (76) and $\tilde{g}$ in (91) it follows by computing $g_1(0)$, $h(0)$ in (55), (68) that

$$|\tilde{g}^{-1} e^{g(0)}| = \left| \frac{1 - \frac{b_1+b_2}{2} + (b_1 b_2)^{1/4} \sqrt{|(1-b_1)(1-b_2)|}}{1 - \frac{b_1^{-1}+b_2^{-1}}{2} + (b_1 b_2)^{-1/4} \sqrt{|(1-b_1)(1-b_2)|}} \right|^{\Omega/2\pi} = 1, \quad (176)$$

so that

$$|\chi_{n-1}^2 \tilde{g}^{-2n}| = |(R(0)M(0))_{21}|. \quad (177)$$

By (171)

$$R(0) = I + \mathcal{O}(1/n),$$

as $n \rightarrow \infty$. Furthermore, we note that $F = \mathcal{O}(s/n)$ and that $\gamma_2(0) = -1 + \mathcal{O}(s/n)$ as $s/n \rightarrow 0$, and substitute this into the definition of M in (112) to find

$$(R(0)M(0))_{21} = -(I + \mathcal{O}(s/n))e^{-\langle \frac{n\Omega}{2\pi} \rangle (h(\infty) + h(0))} = -(I + \mathcal{O}(s/n))|e^{-\langle \frac{n\Omega}{2\pi} \rangle h(\infty)}|^2, \quad (178)$$

as $s/n \rightarrow 0$. Substituting (178) into (177) and recalling the notation (129) it follows that

$$|\tilde{g}^{-2n} \chi_{n-1}^2| = (1 + \mathcal{O}(s/n)) |\tilde{h}|^2, \quad (179)$$

as $s/n \rightarrow 0$. We note that $\tilde{g} = 1 + \mathcal{O}(s/n(\log u_0^{-1}))$ as $s/n(\log u_0^{-1}) \rightarrow 0$, and thus we have

$$|\tilde{g}^{-2n} \chi_n^2| = (1 + \mathcal{O}(s/n)) |\tilde{h}|^2, \quad (180)$$

as $s/n \rightarrow 0$.

4.2 Convergence of correlation functions

Let $H_n(x, y)$ be the kernel built out of the orthogonal polynomials on J

$$H_n(x, y) = \frac{s}{\pi n} \sum_{j=0}^{n-1} \phi_j^{(n)} \left(\exp \left(\frac{2sxi}{n} \right) \right) \overline{\phi_j^{(n)} \left(\exp \left(\frac{2syi}{n} \right) \right)}. \quad (181)$$

By the Christoffel-Darboux formula, H_n also has the following useful form

$$H_n(y_1, y_2) = \frac{s}{\pi n} \frac{z_1^n z_2^{-n} \phi_n^{(n)}(z_2) \overline{\phi_n^{(n)}(z_1)} - \overline{\phi_n^{(n)}(z_2)} \phi_n^{(n)}(z_1)}{1 - z_2^{-1} z_1}, \quad (182)$$

where $z_j = \exp \left(\frac{2sy_j i}{n} \right)$ for $j = 1, 2$. Let $\hat{K}_n$ be defined similarly, but for the special case where $J = C$, namely:

$$\hat{K}_n(y_1, y_2) = \frac{s}{\pi n} \frac{z_1^{n/2} z_2^{-n/2} - z_2^{n/2} z_1^{-n/2}}{1 - z_2^{-1} z_1}. \quad (183)$$

Let $\rho_m^{(n)}$ be the m 'th correlation function of the determinantal point process with correlation kernel $\hat{K}_n$, and let $\rho_m^{(n,A)}$ be the m 'th correlation function of the same process conditioned to have no points in $A = (\alpha, -\nu) \cup (\nu, \beta)$. Then

$$\begin{aligned} \rho_m^{(n)}(x_1, \dots, x_m) &= \det(\hat{K}_n(x_i, x_j))_{i,j=1}^m, \\ \rho_m^{(n,A)}(x_1, \dots, x_m) &= \det(H_n(x_i, x_j))_{i,j=1}^m. \end{aligned} \quad (184)$$

The two correlation functions are also related as follows:

$$\rho_m^{(n,A)}(x_1, \dots, x_m) = \frac{\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_{A^j} \rho_{j+m}^{(n)}(x_1, \dots, x_{j+m}) dx_{m+1} \dots dx_{j+m}}{D_n(J^{(n)})}. \quad (185)$$

Similarly, we can write ρ_m^A in terms of ρ_m (both defined in Remark 1.4):

$$\rho_m^A(x_1, \dots, x_m) = \frac{\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_{A^j} \rho_{j+m}(x_1, \dots, x_{j+m}) dx_{m+1} \dots dx_{j+m}}{\det(I - K_s)_A}. \quad (186)$$

The infinite sums (185) and (186) can be seen to converge for fixed s by Hadamard's inequality. Since

$$\left| \widehat{K}_n(x, y) - K_s(x, y) \right| = \mathcal{O}(1/n), \quad (187)$$

as $n \rightarrow \infty$, it follows by formulas (186) and (185) that

$$|\rho_m^{(n,A)}(x_1, \dots, x_m) - \rho_m^A(x_1, \dots, x_m)| \rightarrow 0, \quad (188)$$

as $n \rightarrow \infty$ for fixed s (similarly to convergence of the determinants (34), see the Appendix).

By the definition of Y in (42) and the formula for H_n in (182) we have

$$H_n(y_1, y_2) = \frac{s\chi_n^2}{\pi n(1 - z_1 z_2^{-1})} (z_1^n z_2^{-n} Y_{11}(z_2) \overline{Y_{11}(z_1)} - \overline{Y_{11}(z_2)} Y_{11}(z_1)), \quad (189)$$

where $z_j = \exp\left(\frac{2sy_j i}{n}\right)$ for $j = 1, 2$. For the asymptotics of the correlation kernel we are less ambitious and choose not to proceed with all the detail in last section, and work with $|x|$ bounded away from $1/2$ as $n \rightarrow \infty$. Since the intention of F and X was to obtain uniform asymptotics up to the points $|x| = 1/2$, we can let $F, X = 0$ in M and P when we consider $|x|$ bounded away from $1/2$. Then, in place of Proposition 3.2, we have $PM^{-1} = I + o(1)$ as $u_0 \rightarrow 0$ and $s, n \rightarrow \infty$ such that $s/n \rightarrow 0$ and $k \in \mathbb{N}, |x| < 1/2$ remain fixed, uniformly on the boundary ∂U_0 . Thus $R = I + o(1)$ in the same limit, and tracing back the transformations $Y \rightarrow T = S = RP$ we have, by (92), (110), (170), that for $z \in J_1$:

$$\begin{aligned} Y_{11}(z) &= \widetilde{g}^{-n} (RP)_{11}(z) e^{ng(z)} \\ &= \widetilde{g}^{-n} z^{n/2} \widetilde{h}^{-1} (\gamma_1 B_{11}(1) \Phi_{11}(\zeta(z)) - \gamma_2 B_{22}(1) \Phi_{21}(\zeta(z))) (1 + o(1)), \end{aligned} \quad (190)$$

where $\widetilde{g}$ is given in (91), and $\widetilde{h}$ in (129). Thus it follows by (118), (180), the fact that $\widehat{g}_1$ is real to the main order and that

$$\gamma_1 \overline{\gamma_2} = -1/2 + o(1) \quad (191)$$

as $s/n \rightarrow 0$, that as $u_0 \rightarrow 0$ and $s, n \rightarrow \infty$ such that $s/n \rightarrow 0$, while $k \in \mathbb{N}$ and $|x| < 1/2$ remain fixed we have

$$H_n(y_1, y_2) = \frac{\kappa_{k-1}}{\kappa_k} \frac{L_k(2\zeta_0 y_1) L_{k-1}(2\zeta_0 y_2) - L_k(2\zeta_0 y_2) L_{k-1}(2\zeta_0 y_1)}{y_1 - y_2} (1 + o(1)). \quad (192)$$

Since $\eta_1 = -2 + \mathcal{O}(u_0)$ and $\eta_2 = 2 + \mathcal{O}(u_0)$ as $u_0 \rightarrow 0$, it follows by continuity of the polynomials that $L_k^{(\eta_1, \eta_2)}$ can be replaced by $L_k^{(-2, 2)}$ in (192) without modifying the error term. Similarly, by (85), $2|\zeta_0|$ can be replaced by $4u_0^{-1}$ without modifying the error terms. Thus, combining (192) and (188), we prove the statement in Remark 1.4.

4.3 Expansion of Differential Identity

In this section we start by writing the differential identity in a more convenient form, and find an expansion for it as $s, n \rightarrow \infty$ and $u_0 \rightarrow 0$ such that $su_0 \rightarrow 0$ and $s/n \rightarrow 0$, before proceeding to integrate it in Section 4.4. Throughout the rest of the paper, the implicit constants in $\mathcal{O}(\dots)$ are independent of s, u_0, n . For example, if we write $\mathcal{O}(\frac{u_0}{s} + u_0^2)$, then in particular it is uniform in n , and if we write $\mathcal{O}(1)$, it means this expression is bounded in the double scaling limit described above.

Write the parametrix P in (126) in U_0 by grouping the factors as follows

$$P(z) = A(z)B(z)C(z)e^{-n(g(z) - \frac{1}{2} \log z)\sigma_3}, \quad (193)$$

where $A(z)$ and $C(z)$ are by

$$\begin{aligned} A(z) &= \left(I + \frac{F}{z-1} \right) \tilde{h}^{-\sigma_3} \begin{pmatrix} \gamma_1(z) & -\gamma_2(z) \\ \gamma_2(z) & \gamma_1(z) \end{pmatrix} \\ C(z) &= \left(I - \frac{X}{\zeta(z)} \right) \Phi(\zeta(z)) \end{aligned} \quad (194)$$

By the transformations $Y = \tilde{g}^{-n\sigma_3} T e^{ng\sigma_3}$ and $T = S = RM$ at $z = 1$, (see (92), (110), (170)) we have for $z \in U_0$

$$Y_{11}(z) = \tilde{g}^{-n} z^{n/2} [RABC]_{11}, \quad (195)$$

where

$$\begin{aligned} A(z) &= A_1(z) + \frac{A_2(z)}{z-1}, \\ A_1(z) &= \tilde{h}^{-\sigma_3} \begin{pmatrix} \gamma_1(z) & -\gamma_2(z) \\ \gamma_2(z) & \gamma_1(z) \end{pmatrix}, \\ A_2(z) &= f \tilde{h}^{-\sigma_3} \begin{pmatrix} \gamma_1(z) + \gamma_2(z)\psi & -\gamma_2(z) + \gamma_1(z)\psi \\ -\gamma_2(z) - \gamma_1(z)/\psi & -\gamma_1(z) + \gamma_2(z)/\psi \end{pmatrix}, \end{aligned} \quad (196)$$

and

$$C(z) = \begin{cases} \begin{pmatrix} \Phi_{11}(\zeta) - \frac{\Phi_{1,12}}{\zeta} \Phi_{21}(\zeta) & * \\ \Phi_{21}(\zeta) & * \end{pmatrix} & \text{for } 0 \leq x < 1/2, \\ \begin{pmatrix} \Phi_{11}(\zeta) & * \\ \Phi_{21}(\zeta) - \frac{\Phi_{1,21}}{\zeta} \Phi_{11}(\zeta) & * \end{pmatrix} & \text{for } -1/2 \leq x < 0. \end{cases} \quad (197)$$

The expression for C in (197) is valid for $k \geq 1$, while for $k = 0$, we have

$$C(z) = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}. \quad (198)$$

It follows that

$$Y'_{11}(z) = \tilde{g}^{-n} z^{n/2} \left[\frac{n}{2z} RABC + R'ABC + RA'BC + RAB'C + RABC' \right]_{11}, \quad (199)$$

where we suppress dependency on the variable z on the right hand side, and where $'$ denotes differentiation with respect to z . Substituting (180), (195), (199) into (43) we find that

$$F(z) = -2|\tilde{h}|^2 \text{Re} \left[z \overline{(RABC)_{11}} (R'ABC + RA'BC + RAB'C + RABC')_{11} \right] (1 + \mathcal{O}(s/n)), \quad (200)$$

for $z \in U_0$. We would now like to evaluate $F(a)$ and $F(\bar{a})$. Since $\zeta(e^{i\theta})$ is real for real θ on U_0 , it follows that $\frac{d}{d\theta} \zeta(e^{i\theta})$ is real. Consider the entries of C and recall that $\Phi_{11}(x)$ is real for $x \in \mathbb{R}$, and that $\Phi_{21}(x)$ and $\Phi_{1,12}$ are purely imaginary. By (197), $C_{11}(e^{i\theta})$ is real and so $z \frac{d}{dz} C_{11}(e^{i\theta})$ is purely imaginary, while $C_{21}(e^{i\theta})$ is purely imaginary and $z \frac{d}{dz} C_{21}(e^{i\theta})$ is real. From (59), we recall that $B_{jj}(e^{i\theta})$ is real for $j = 1, 2$. Thus $\overline{B_{11}} B_{22} = \overline{B_{22}} B_{11} = 1$. From these observations, we find that

$$\begin{aligned} \text{Re} \left[z \overline{(RABC)_{11}} (RABC')_{11} \right] &= z(C_{11}C'_{21} - C'_{11}C_{21}) \text{Re} \left[\overline{(RA)_{11}} (RA)_{12} \right], \\ \text{Re} \left[z \overline{(RABC)_{11}} (RAB'C)_{11} \right] &= z(B_{11}B'_{22} - B'_{11}B_{22}) C_{11}C_{21} \text{Re} \left[\overline{(RA)_{11}} (RA)_{12} \right]. \end{aligned} \quad (201)$$

When $k = 0$, both expressions in (201) are equal to 0.

4.3.1 Evaluation of (201)

Using the expansion for ζ from (83)–(84), and the fact that $\zeta(a) - \zeta(\bar{a}) = 4$ from (82), we find that

$$\begin{aligned} \zeta(e^{\pm i\theta_0}) &= \pm 2 \left(1 \pm \zeta_1 i u_0 + \mathcal{O} \left(u_0 \frac{s}{n} + u_0^2 (\log u_0^{-1})^2 \right) \right), \\ \left. \frac{d}{dz} \zeta(z) \right|_{z=e^{\pm i\theta_0}} &= -\frac{2in}{szu_0} \left(1 \pm 2\zeta_1 i u_0 + \mathcal{O} \left(u_0 \frac{s}{n} + u_0^2 (\log u_0^{-1})^2 \right) \right). \end{aligned} \quad (202)$$

We substitute the expansion of ζ from (83) into (197), and recall (121)-(125), to find

$$\begin{aligned}
C_{11}(e^{\pm i\theta_0}) &= \begin{cases} \Phi_{11}(\zeta(e^{\pm i\theta_0}))^{\frac{k+1}{2k+1}}(1 \pm iu_0\zeta_1 + \mathcal{O}(r_1)) & \text{for } 0 \leq x < 1/2, \\ \Phi_{11}(\zeta(e^{\pm i\theta_0})) & \text{for } -1/2 \leq x < 0, \end{cases} \\
C_{21}(e^{\pm i\theta_0}) &= \begin{cases} \Phi_{21}(\zeta(e^{\pm i\theta_0})) & \text{for } 0 \leq x < 1/2, \\ \Phi_{21}(\zeta(e^{\pm i\theta_0}))^{\frac{k-1}{2k-1}}(1 \pm iu_0\zeta_1 + \mathcal{O}(r_1)) & \text{for } -1/2 \leq x < 0, \end{cases} \\
r_1 &= u_0(\log u_0^{-1})^2/s + (u_0 \log u_0^{-1})^2 + su_0/n.
\end{aligned} \tag{203}$$

Using the expression

$$\Phi'_\zeta(\eta_j) = \begin{pmatrix} (-1)^j \frac{k(k+1)}{4} \Phi_{11}(\eta_j) & * \\ (-1)^j \frac{k(k-1)}{4} \Phi_{21}(\eta_j) & * \end{pmatrix} \quad j = 1, 2. \tag{204}$$

which follows from (125), we compute the following:

$$\begin{aligned}
& z \left(C_{11}(z) \frac{d}{dz} C_{21}(z) - \frac{d}{dz} C_{11}(z) C_{21}(z) \right) \Big|_{z=e^{\pi i\theta_0}} = \\
& = \begin{cases} \frac{2\pi k^2 n}{su_0} \left(\frac{k+1}{2k+1} \pm 2i\zeta_1 u_0 \right) (1 + \mathcal{O}(r_1)) & \text{for } 0 \leq x < 1/2, \\ \frac{2\pi k^2 n}{su_0} \left(\frac{k-1}{2k-1} \pm 2i\zeta_1 u_0 \right) (1 + \mathcal{O}(r_1)) & \text{for } -1/2 \leq x < 0. \end{cases}
\end{aligned} \tag{205}$$

We also have

$$(B_{11}(z)B'_{22}(z) - B'_{11}B_{22}(z)) \Big|_{z=e^{\pm i\theta_0}} = \mathcal{O}(n \log u_0^{-1}/s), \tag{206}$$

where the derivative is taken with respect to z . We now evaluate RA . Let K denote the constant

$$K = \frac{n}{s} \frac{u_1^{-1} - u_2^{-1}}{4}. \tag{207}$$

We have the derivatives of $\gamma_1(e^{i\theta})$ and $\gamma_2(e^{i\theta})$ with respect to θ evaluated at $\theta = 0$:

$$\begin{aligned}
\frac{d}{d\theta}\gamma_1(1) &= -iK\gamma_2(1)(1 + \mathcal{O}(s/n)) \\
\frac{d}{d\theta}\gamma_2(1) &= iK\gamma_1(1)(1 + \mathcal{O}(s/n)) \\
\frac{d^2}{d\theta^2}\gamma_1(1) &= \frac{n^2}{s^2 2^4} (\gamma_1(1)(u_1^{-2} + u_2^{-2} - 2u_1^{-1}u_2^{-1}) - 4i\gamma_2(1)(u_1^{-2} - u_2^{-2})) (1 + \mathcal{O}(s/n)) \\
\frac{d^2}{d\theta^2}\gamma_2(1) &= \frac{n^2}{s^2 2^4} (\gamma_2(1)(u_1^{-2} + u_2^{-2} - 2u_1^{-1}u_2^{-1}) + 4i\gamma_1(1)(u_1^{-2} - u_2^{-2})) (1 + \mathcal{O}(s/n)), \\
\frac{d^3}{d\theta^3}\gamma_1(1) &= \frac{n^3}{s^3} \left(\frac{\gamma_1(1)}{8} (u_1^{-1} - u_2^{-1})(u_1^{-2} - u_2^{-2}) \right. \\
&\quad \left. - \frac{3i}{26}\gamma_2(1)(11(u_1^{-3} - u_2^{-3}) + u_1^{-1}u_2^{-2} - u_2^{-1}u_1^{-2}) \right) (1 + \mathcal{O}(s/n)), \\
\frac{d^3}{d\theta^3}\gamma_2(1) &= \frac{n^3}{s^3} \left(\frac{\gamma_2(1)}{8} (u_1^{-1} - u_2^{-1})(u_1^{-2} - u_2^{-2}) \right. \\
&\quad \left. + \frac{3i}{26}\gamma_1(1)(11(u_1^{-3} - u_2^{-3}) + u_1^{-1}u_2^{-2} - u_2^{-1}u_1^{-2}) \right) (1 + \mathcal{O}(s/n)).
\end{aligned} \tag{208}$$

Let x_1 and x_2 denote the following functions:

$$\begin{aligned}
x_1(z) &= \tilde{h}^{-1}R_{11}(z)\gamma_1(1) + \tilde{h}R_{12}(z)\gamma_2(1), \\
x_2(z) &= -\tilde{h}^{-1}R_{11}(z)\gamma_2(1) + \tilde{h}R_{12}(z)\gamma_1(1).
\end{aligned} \tag{209}$$

Then, using (196), (208) and (209), expand A . When $0 \leq x < 1/2$

$$\begin{aligned}
(RA)_{11}(e^{i\theta}) &= \left[x_1(z) \left(1 - Kf \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) \left(1 + \frac{n\theta}{2s}(u_1^{-1} + u_2^{-1}) \right) \right) + iKx_2(z)\theta \right. \\
&\quad + \frac{n^2\theta^2}{s^2 2^5} \left(x_1(z)(u_1^{-1} - u_2^{-1})^2 - \frac{nf}{4s}x_1(z) \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) (11(u_1^{-3} - u_2^{-3}) + u_1^{-1}u_2^{-2} - u_1^{-2}u_2^{-1}) \right. \\
&\quad \left. \left. + 4ix_2(z)(u_1^{-2} - u_2^{-2}) \right) \right] (1 + \mathcal{O}(s/n)) + \mathcal{O}(|\theta|^3 n^3/s^3), \\
(RA)_{12}(e^{i\theta}) &= \left[\frac{ifx_1(z)}{\theta} \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) + x_2(z) + ix_1(z)K\theta \left(-1 + K\frac{f}{2} \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) \right) \right. \\
&\quad + \frac{n^2\theta^2}{s^2 2^5} \left(x_2(z)(u_1^{-1} - u_2^{-1})^2 - 4ix_1(z)(u_1^{-2} - u_2^{-2}) \right. \\
&\quad \left. \left. - \frac{2fnx_1(z)}{3s} \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) (u_1^{-3} + u_2^{-3} - u_1^{-1}u_2^{-2} - u_1^{-2}u_2^{-1}) \right) \right] (1 + \mathcal{O}(s/n)) + \mathcal{O}(|\theta|^3 n^3/s^3),
\end{aligned} \tag{210}$$

where we denote $\gamma_j = \gamma_j(1)$ for $j = 1, 2$. When $-1/2 \leq x < 0$

$$\begin{aligned}
(RA)_{11}(e^{i\theta}) &= \left[\frac{ifx_2(z)}{\theta} \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) + x_1(z) + ix_2(z)K\theta \left(1 + K\frac{f}{2} \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) \right) \right. \\
&\quad + \frac{n^2\theta^2}{s^22^5} \left(x_1(z)(u_1^{-1} - u_2^{-1})^2 - \frac{2fnx_2(z)}{3s} \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) (u_1^{-3} + u_2^{-3} + u_1^{-1}u_2^{-2} + u_1^{-2}u_2^{-1}) \right. \\
&\quad \left. \left. + 4ix_2(z)(u_1^{-2} - u_2^{-2}) \right) \right] (1 + \mathcal{O}(s/n)) + \mathcal{O}(|\theta|^3 n^3/s^3), \\
(RA)_{12}(e^{i\theta}) &= \left[x_2(z) \left(1 + Kf \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) \left(1 + \frac{n\theta}{2s} (u_1^{-1} + u_2^{-1}) \right) \right) - iKx_1(z)\theta \right. \\
&\quad + \frac{n^2\theta^2}{2^5s^2} \left(x_2(z)(u_1^{-1} - u_2^{-1})^2 + \frac{nf}{4s} x_2(z) \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) (11(u_1^{-3} - u_2^{-3}) + u_1^{-1}u_2^{-2} - u_1^{-2}u_2^{-1}) \right. \\
&\quad \left. \left. - 4ix_1(z)(u_1^{-2} - u_2^{-2}) \right) \right] (1 + \mathcal{O}(s/n)) + \mathcal{O}(|\theta|^3 n^3/s^3),
\end{aligned} \tag{211}$$

We note that

$$\begin{aligned}
|\gamma_1|^2, |\gamma_2|^2 &= \frac{1}{4} \left(\left| \frac{u_1}{u_2} \right|^{1/2} + \left| \frac{u_2}{u_1} \right|^{1/2} \right) (1 + \mathcal{O}(s/n)) \\
\overline{\gamma_1}/\overline{\gamma_2}, \gamma_2/\gamma_1 &= \frac{-2\sqrt{|u_1u_2|} - i(u_1 + u_2)}{u_1 - u_2} (1 + \mathcal{O}(s/n)).
\end{aligned} \tag{212}$$

Recalling (129), (113), (207), it is readily checked that

$$f \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) \in \mathbb{R}, \tag{213}$$

and that as $n, s \rightarrow \infty, s/n \rightarrow 0$,

$$f \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) K = \begin{cases} \frac{\rho}{1+\rho} (1 + \mathcal{O}(s/n)) & \text{for } 0 \leq x < 1/2, \\ \frac{\rho}{1-\rho} (1 + \mathcal{O}(s/n)) & \text{for } -1/2 \leq x < 0. \end{cases} \tag{214}$$

From (210), (211), (213)–(214) it follows that

$$\operatorname{Re} \left[\overline{(RA)_{11}(z)} (RA)_{12}(z) \right] = \operatorname{Re} \left[\overline{x_1(z)} x_2(z) \right] (1 + \mathcal{O}(s/n + u_0^2)). \tag{215}$$

Thus, from (201) and (205),

$$\begin{aligned}
F_0(e^{\pm i\theta_0}) &= \operatorname{Re} \left[\overline{z(RABC)_{11}} (RABC')_{11} \right] = \operatorname{Re} \left[\overline{x_1(e^{\pm i\theta_0})} x_2(e^{\pm i\theta_0}) \right] \times \\
&\quad \times \begin{cases} \frac{2\pi k^2 n}{su_0} \left(\frac{k+1}{2k+1} \pm i\zeta_1 u_0 \right) (1 + \mathcal{O}(r_1)) & \text{for } 0 \leq x < 1/2, \\ \frac{2\pi k^2 n}{su_0} \left(\frac{k-1}{2k-1} \pm i\zeta_1 u_0 \right) (1 + \mathcal{O}(r_1)) & \text{for } -1/2 \leq x < 0, \end{cases}
\end{aligned} \tag{216}$$

where r_1 was defined in (203).

Proposition 4.1. *We have*

$$\operatorname{Re} [\overline{x_1}x_2] = \frac{|\tilde{h}|^{-2}}{2} + \mathcal{O} \left(s/n + \left(\tilde{e}u_0^2s \log u_0^{-1} + \tilde{e}u_0^{1+2|x|} \log u_0^{-1} + s^{-1} \right)^2 \right), \quad (217)$$

and $|\tilde{h}|^{-1} = \mathcal{O}(1)$, where $\tilde{h}$ was given in (129) and $\tilde{e}$ was defined in (131).

The main term of Proposition 4.1 is easy to calculate from (209) and (113), but we defer the rest of the proof to Section 4.5.

From (125), (203) we obtain that

$$C_{11}(e^{\pm i\theta_0}), C_{21}(e^{\pm i\theta_0}) = \mathcal{O}(\sqrt{k}). \quad (218)$$

Recall that $k = \mathcal{O}(s/\log u_0^{-1})$. Combining (201), (206), (215), (218), and using Proposition 4.1 gives us

$$\operatorname{Re} \left[z \overline{(RABC)_{11}} (RAB'C)_{11} \right] = \mathcal{O}(n). \quad (219)$$

4.3.2 Evaluation of (200)

Suppressing z dependence, we write

$$\begin{aligned} \operatorname{Re} \left[z \overline{(RABC)_{11}} (RA'BC)_{11}(z) \right] &= F_1 + F_2 + F_3 + F_4, \\ F_1 &= B_{11}^2 C_{11}^2 \operatorname{Re} \left[z \overline{(RA)_{11}} (RA')_{11} \right], \\ F_2 &= \operatorname{Re} \left[z C_{11} \overline{(RA)_{11}} C_{21} (RA')_{12} \right], \\ F_3 &= -\operatorname{Re} \left[z C_{11} (RA')_{11} C_{21} \overline{(RA)_{12}} \right], \\ F_4 &= -B_{22}^2 C_{21}^2 \operatorname{Re} \left[z \overline{(RA)_{12}} (RA')_{12} \right]. \end{aligned} \quad (220)$$

Recall that $K = \mathcal{O}(n/s)$, and that $\theta_0 = u_0 \frac{s}{n}$. From (210) and (211) we obtain that for $k \geq 1$

$$\begin{aligned} \frac{F_1(e^{\pm i\theta_0})}{B_{11}^2 C_{11}^2} &= \begin{cases} \operatorname{Re} [\overline{x_1}x_2] \left[\frac{K}{1+\rho} \right] + \mathcal{O}(u_0 n/s + 1) & \text{for } 0 \leq x < 1/2, \\ -\operatorname{Re} [\overline{x_1}x_2] \left[\frac{f}{\theta_0^2} \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) \right] + \mathcal{O}(n/s) & \text{for } -1/2 \leq x < 0, \end{cases} \\ F_2(e^{\pm i\theta_0}) + F_3(e^{\pm i\theta_0}) &= \pm i C_{11} C_{21} \operatorname{Re} [\overline{x_1}x_2] \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) \left[\frac{2fK}{\theta_0} + \mathcal{O}(n/s) \right], \\ \frac{F_4(e^{\pm i\theta_0})}{B_{22}^2 C_{21}^2} &= \begin{cases} \operatorname{Re} [\overline{x_1}x_2] \left[\frac{f}{\theta_0^2} \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) \right] + \mathcal{O}(n/s) & \text{for } 0 \leq x < 1/2, \\ \operatorname{Re} [\overline{x_1}x_2] \left[\frac{K}{1-\rho} \right] + \mathcal{O}(u_0 n/s + 1) & \text{for } -1/2 \leq x < 0. \end{cases} \end{aligned} \quad (221)$$

When $k = 0$, we have $F_2, F_3, F_4 = 0$, while F_1 is as in (221) for $0 < x < 1/2$.

From (87) we have that $\Omega = \mathcal{O}\left(\frac{s}{n \log u_0^{-1}}\right)$, and thus substituting $k + x = \frac{n\Omega}{2\pi}$ into (85) we have

$$\frac{8}{u_1^{-1} - u_2^{-1}} e^{-\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} = u_0 \left(1 + \mathcal{O}\left(\frac{s \log u_0^{-1}}{n} + u_0^2 (\log u_0^{-1})^2\right) \right). \quad (222)$$

Recalling the expansion of g_1 in (63)–(64) and the definition of B in (127), and substituting the values of κ_j and Φ from (124)–(125) into the expansion of C from (203), we obtain that for $k \geq 1$,

$$\begin{aligned} (B_{11}^2 C_{11}^2)(e^{\pm i\theta_0}) &= \begin{cases} e^{2x \frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} \frac{(k+1)^2}{4(2k+1)\kappa_k^2} (1 + \mathcal{O}(r_2)) & \text{for } 0 \leq x < 1/2, \\ e^{2x \frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} \frac{(2k+1)}{4\kappa_k^2} (1 + \mathcal{O}(r_2)) & \text{for } -1/2 \leq x < 0, \end{cases} \\ (B_{22}^2 C_{21}^2)(e^{\pm i\theta_0}) &= \begin{cases} -e^{-2x \frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} \pi^2 \kappa_{k-1}^2 (2k-1) (1 + \mathcal{O}(r_2)) & \text{for } 0 \leq x < 1/2, \\ -e^{-2x \frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} \frac{\pi^2 \kappa_{k-1}^2 (k-1)^2}{(2k-1)} (1 + \mathcal{O}(r_2)) & \text{for } -1/2 \leq x < 0, \end{cases} \quad (223) \\ (C_{11} C_{21})(e^{\pm i\theta_0}) &= \begin{cases} \mp \pi i k \frac{k+1}{2k+1} (1 + \mathcal{O}(u_0 \log u_0^{-1} + s/n)) & \text{for } 0 \leq x < 1/2, \\ \mp \pi i k \frac{k-1}{2k-1} (1 + \mathcal{O}(u_0 \log u_0^{-1} + s/n)) & \text{for } -1/2 \leq x < 0, \end{cases} \\ r_2 &= u_0 \log u_0^{-1} + \log u_0^{-1} s/n. \end{aligned}$$

When $k = 0$ (and $0 < x < 1/2$) we have

$$(B_{11}^2 C_{11}^2)(e^{\pm i\theta_0}) = e^{s\sqrt{|u_1 u_2|}} (1 + \mathcal{O}(u_0 \log u_0^{-1} + s^2/n)). \quad (224)$$

Substituting (214), (217), (223) into (221) we find that

$$\begin{aligned} F_1 &= \begin{cases} \frac{|\tilde{h}|^{-2}}{8} \frac{K}{1+\rho} \frac{(k+1)^2}{(2k+1)\kappa_k^2} e^{2x \frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} (1 + \mathcal{O}(r_2 + s^{-2})) & \text{for } 0 \leq x < 1/2, \\ -\frac{|\tilde{h}|^{-2}}{8} \frac{\rho}{\theta_0^2 K(1-\rho)} \frac{(2k+1)}{\kappa_k^2} e^{2x \frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} (1 + \mathcal{O}(r_2 + s^{-2})) & \text{for } -1/2 \leq x < 0, \end{cases} \\ F_2 + F_3 &= \begin{cases} |\tilde{h}|^{-2} \frac{\rho}{1+\rho} \frac{1}{\theta_0} \frac{\pi k(k+1)}{2k+1} (1 + \mathcal{O}(u_0 \log u_0^{-1} + \frac{s}{n} + u_0^{2|x|} + s^{-2})) & \text{for } 0 \leq x < 1/2, \\ |\tilde{h}|^{-2} \frac{\rho}{1-\rho} \frac{1}{\theta_0} \frac{\pi k(k-1)}{2k-1} (1 + \mathcal{O}(u_0 \log u_0^{-1} + \frac{s}{n} + u_0^{2|x|} + s^{-2})) & \text{for } -1/2 \leq x < 0, \end{cases} \\ F_4 &= \begin{cases} -\frac{|\tilde{h}|^{-2} \rho \pi^2 \kappa_{k-1}^2 (2k-1)}{2(1+\rho) K \theta_0^2} e^{-2x \frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} (1 + \mathcal{O}(r_2 + s^{-2})) & \text{for } 0 \leq x < 1/2, \\ -\frac{|\tilde{h}|^{-2} K \pi^2 \kappa_{k-1}^2 (k-1)^2}{2(1-\rho) (2k-1)} e^{-2x \frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} (1 + \mathcal{O}(r_2 + s^{-2})) & \text{for } -1/2 \leq x < 0, \end{cases} \quad (225) \end{aligned}$$

when evaluated at $e^{\pm i\theta_0}$. When $k = 0$ (and $0 < x < 1/2$), we have $F_2, F_3, F_4 = 0$, while F_1 is given by

$$F_1(e^{\pm i\theta_0}) = \frac{|\tilde{h}|^{-2} K}{2(1+\rho)} e^{s\sqrt{|u_1 u_2|}} (1 + \mathcal{O}(u_0 \log u_0^{-1} + s^2/n + s^{-2})). \quad (226)$$

Finally we find the order of the term which includes R' in (200). Using the equation for R_1 in (173), it is readily seen that

$$\begin{aligned} \left| \frac{d}{dz} R(e^{\pm i\theta_0}) \right| &\leq \left(\sup_{u \in \partial U_0} |\Delta_1^{(1)}(u)| \right) \int_{\partial U_0} \frac{du}{2\pi|u - e^{\pm i\theta_0}|^2} \\ &+ \sum_{j=1,2} \left(\sup_{u \in \partial U_j} |\Delta_1^{(b_j)}(u)| \right) \int_{\partial U_j} \frac{du}{2\pi|u - e^{\pm i\theta_0}|^2}. \end{aligned} \quad (227)$$

We recall that the radius of U_0 is of size $\mathcal{O}\left(\frac{s}{n \log u_0^{-1}}\right)$, and that the radius of U_j is of size $\mathcal{O}(s/n)$ for $j = 1, 2$. Thus

$$\int_{\partial U_0} \frac{du}{2\pi|u - e^{\pm i\theta_0}|^2} = \mathcal{O}\left(\frac{n}{s \log u_0^{-1}}\right), \quad \int_{\partial U_j} \frac{du}{2\pi|u - e^{\pm i\theta_0}|^2} = \mathcal{O}(n/s) \quad \text{for } j = 1, 2.$$

Substituting the asymptotics for $\Delta_1^{(1)}$ from (132) and $\Delta_1^{(b_1)}, \Delta_1^{(b_2)}$ from (167) into (227), it follows that

$$\frac{d}{dz} R(e^{\pm i\theta_0}) = \mathcal{O}(\tilde{e} u_0^{1+2|x|} (\log u_0^{-1})^2 n/s + \tilde{e} n (u_0 \log u_0^{-1})^2 + n/s^2). \quad (228)$$

Thus, we have from (209), since $\tilde{h}, \tilde{h}^{-1}, \gamma_j(1) = \mathcal{O}(1)$, that also

$$\frac{d}{dz} x_1(e^{\pm i\theta_0}) = \mathcal{O}(\tilde{e} u_0^{1+2|x|} (\log u_0^{-1})^2 n/s + \tilde{e} n (u_0 \log u_0^{-1})^2 + n/s^2). \quad (229)$$

The formula for $R'A$ is given by (210), (211) but with x_1, x_2 replaced by the derivatives x'_1, x'_2 . Recall that

$$\begin{aligned} C_{11}, C_{21} &= \mathcal{O}(\sqrt{k}) = \mathcal{O}\left(\sqrt{s/\log u_0^{-1}}\right), \\ B_{11} &= \mathcal{O}(u_0^{-x}), \quad B_{22} = \mathcal{O}(u_0^x), \quad K = \mathcal{O}(n/s), \quad f = \mathcal{O}\left(\frac{s}{n} u_0^{1-2|x|}\right). \end{aligned} \quad (230)$$

From (229) we obtain that

$$\begin{aligned} \text{Re} \left[\left(\overline{(RABC)_{11}} (R'ABC)_{11} \right) (e^{\pm i\theta_0}) \right] \\ = \mathcal{O}(\tilde{e} n u_0 \log u_0^{-1} + \tilde{e} n s u_0^{2-2|x|} \log u_0^{-1} + n u_0^{-2|x|} / (s \log u_0^{-1})). \end{aligned} \quad (231)$$

By substituting (216), (219), (220), (231) into the definition of $F(z)$ from (200), and substituting the resulting expression into the expression for the differential identity (43), we obtain the following proposition.

Proposition 4.2. *We have the following asymptotics for $\log D_n(J)$, as $u_0 \rightarrow 0$ and $s, n \rightarrow \infty$ such that $su_0 \log u_0^{-1} \rightarrow 0$ and $s/n \rightarrow 0$:*

$$\begin{aligned} \log D_n(J) = \log D_n(J_2) + \frac{|\tilde{h}|^2}{\pi} \int_0^{\theta_0} \left[\left(\sum_{j=0}^4 F_j(e^{i\theta}) + F_j(e^{-i\theta}) \right) (1 + \mathcal{O}(s/n)) \right. \\ \left. + \mathcal{O}(n + ns(1 + u^{1-2|x|} \log u^{-1})^2 u^{2-2|x|} \log u^{-1} + nu^{-2|x|}/(s \log u^{-1})) \right] d\theta, \end{aligned} \quad (232)$$

where the integration variable $\theta = \frac{s}{n}u$, and where the asymptotics of $F_0(z)$ are given in (216) and the asymptotics of F_1, F_2, F_3, F_4 are given in (225).

4.4 Integration of Differential Identity

We evaluate the integral in formula (232) asymptotically to prove Theorem 1.2.

Using (222), (129) we find that

$$\begin{aligned} \theta_0 = \theta_0(k; x) &= \frac{su_0}{n} = \frac{8s}{(u_1^{-1} - u_2^{-1})n} e^{-\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}} \left(1 + \mathcal{O}\left(\frac{s}{n} \log u_0^{-1} + u_0^2 (\log u_0^{-1})^2\right) \right) \\ \frac{d\theta_0}{dx} &= \frac{s\theta_0 \sqrt{|u_1 u_2|}}{2(k+x)^2} \left(1 + \mathcal{O}\left(\frac{s}{n} \log u_0^{-1} + u_0^2 (\log u_0^{-1})^2\right) \right), \\ \rho &= \begin{cases} \frac{1}{2\pi\kappa_k^2} e^{\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}(-1+2x)} \left(1 + \mathcal{O}\left(\frac{s}{n} \log u_0^{-1}\right) \right) & \text{for } 0 \leq x < 1/2 \\ -2\pi\kappa_{k-1}^2 e^{\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}(-1-2x)} \left(1 + \mathcal{O}\left(\frac{s}{n} \log u_0^{-1}\right) \right) & \text{for } -1/2 \leq x < 0. \end{cases} \end{aligned} \quad (233)$$

Letting k in the expression for θ_0 in (233) be fixed, we integrate in θ_0 , denoting

$$\int_{\theta_0(k, -1/2)}^{\theta_0(k, 1/2)} * d\theta = \int_{x=-1/2}^{x=1/2} * d\theta_0. \quad (234)$$

Note that by (124) with $\eta_2 - \eta_1 = 4$,

$$\frac{\kappa_{k-1}^2}{\kappa_k^2} = \frac{4k^2}{(2k+1)(2k-1)}. \quad (235)$$

We integrate F_1 from (225), changing the variable of integration using (233) and recalling K from (207) and ρ from (233) to find that for $k \geq 1$,

$$\begin{aligned}
& \frac{|\tilde{h}|^2}{\pi} \int_{x=-1/2}^{x=1/2} F_1(e^{\pm i\theta_0}) d\theta_0 \\
&= \frac{(k+1)^2}{2(2k+1)} \int_0^{1/2} \left(\frac{1}{2\pi\kappa_k^2} e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} \right) \frac{s\sqrt{|u_1u_2|}}{2(k+x)^2} \frac{(1+\mathcal{O}(r_2+s^{-2}))dx}{1+\frac{1}{2\pi\kappa_k^2}e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}} \\
&+ \frac{(2k+1)\kappa_{k-1}^2}{8\kappa_k^2} \int_{-1/2}^0 \frac{s\sqrt{|u_1u_2|}}{2(k+x)^2} \left(1 - \frac{2\pi\kappa_{k-1}^2 e^{-(2x+1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}}{1+2\pi\kappa_{k-1}^2 e^{-(2x+1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}}} \right) (1+\mathcal{O}(r_2+s^{-2}))dx \\
&= \left[\frac{(k+1)^2}{2(2k+1)^2} \log \left(1 + \frac{1}{2\pi\kappa_k^2} e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} \right) \right]_{x=0}^{1/2} + \left[\frac{k^2}{2(2k-1)^2} \times \right. \\
&\times \log \left(1 + 2\pi\kappa_{k-1}^2 e^{-(2x+1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} \right) - \frac{sk^2\sqrt{|u_1u_2|}}{4(2k-1)(k+x)} \left. \right]_{x=-1/2}^0 \\
&+ \mathcal{O} \left(\max_{x \in [-1/2, 1/2]} [r_2 \log u_0^{-1} + s^{-2} \log u_0^{-1}] \right) \\
&= \frac{(k+1)^2}{2(2k+1)^2} \log \left(1 + \frac{1}{2\pi\kappa_k^2} \right) - \frac{k^2}{2(2k-1)^2} \log (1 + 2\pi\kappa_{k-1}^2) + s\sqrt{|u_1u_2|} \frac{k}{4(2k-1)^2} \\
&+ \mathcal{O} \left(\max_{x \in [-1/2, 1/2]} [r_2 \log u_0^{-1} + s^{-2} \log u_0^{-1}] \right), \tag{236}
\end{aligned}$$

where r_2 is given in (223), $u_0 = u_0(k, x)$.

When $k = 0$ we have for $x \in [0, 1/2]$

$$\begin{aligned}
& \frac{|\tilde{h}|^2}{\pi} \int_{x=0}^{x=x} F_1(e^{\pm i\theta_0}) d\theta_0 = \frac{1}{2} \log \left(1 + \frac{2}{\pi} e^{-(2x+1)\frac{s\sqrt{|u_1u_2|}}{2x}} \right) + \mathcal{O} \left(su_0^{1-2x} (u_0 \log u_0^{-1} + s^2/n + s^{-2}) \right), \tag{237}
\end{aligned}$$

where, in the $\mathcal{O}$ error term, $u_0 = u_0(k=0, x)$. Similarly, we integrate F_4 for $k \geq 1$:

$$\begin{aligned}
& \frac{|\tilde{h}|^2}{\pi} \int_{-1/2}^{1/2} F_4(e^{\pm i\theta_0}) d\theta_0 \\
&= \left[\frac{(k-1)^2}{2(2k-1)^2} \log \left(1 + 2\pi\kappa_{k-1}^2 e^{(-2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} \right) \right]_{x=-1/2}^0 \\
&+ \left[\frac{k^2}{2(2k+1)^2} \log \left(1 + \frac{1}{2\pi\kappa_k^2} e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} \right) + \frac{sk^2\sqrt{|u_1u_2|}}{4(2k+1)(k+x)} \right]_{x=0}^{1/2} \\
&+ \mathcal{O} \left(\max_{x \in [-1/2, 1/2)} [r_2 \log u_0^{-1} + s^{-2} \log u_0^{-1}] \right) \\
&= -\frac{(k-1)^2}{2(2k-1)^2} \log(1 + 2\pi\kappa_{k-1}^2) + \frac{k^2 \log \left(1 + \frac{1}{2\pi\kappa_k^2} \right)}{2(2k+1)^2} - \frac{sk\sqrt{|u_1u_2|}}{4(2k+1)^2} \\
&+ \mathcal{O} \left(\max_{x \in [-1/2, 1/2)} [r_2 \log u_0^{-1} + s^{-2} \log u_0^{-1}] \right). \tag{238}
\end{aligned}$$

When $k=0$ and $x \in [0, 1/2)$, we have $F_4 = 0$. Thus, for $k \geq 1$,

$$\begin{aligned}
& \frac{|\tilde{h}|^2}{\pi} \int_{-1/2}^{1/2} (F_1 + F_4)(e^{\pm i\theta_0}) d\theta_0 = -\frac{k^2 + (k-1)^2}{2(2k-1)^2} \log(1 + 2\pi\kappa_{k-1}^2) \\
&+ \frac{k^2 + (k+1)^2}{2(2k+1)^2} \log \left(1 + \frac{1}{2\pi\kappa_k^2} \right) - \frac{sk\sqrt{|u_1u_2|}}{4} \left(\frac{1}{(2k+1)^2} - \frac{1}{(2k-1)^2} \right) \\
&+ \mathcal{O} \left(\max_{x \in [-1/2, 1/2)} \left[u_0(\log u_0^{-1})^2 + \frac{s}{n}(\log u_0^{-1})^2 + s^{-2} \log u_0^{-1} \right] \right). \tag{239}
\end{aligned}$$

If $-1/2 < x < 1/2$, then for $k \geq 1$

$$\begin{aligned}
& \frac{|\tilde{h}|^2}{\pi} \int_{-1/2}^x (F_1 + F_4)(e^{\pm i\theta_0}) d\theta_0 \\
&= -\frac{k^2 + (k-1)^2}{2(2k-1)^2} \log(1 + 2\pi\kappa_{k-1}^2) - \frac{s}{4} \sqrt{|u_1u_2|} w_1(x) + r_3, \quad r_3 = o(1) \tag{240} \\
&w_1(x) = \begin{cases} -\frac{k}{(2k-1)^2} + \frac{k}{2k+1} \frac{x}{k+x} & \text{for } 0 \leq x < 1/2, \\ -\frac{k^2}{(2k-1)^2} \frac{1+2x}{k+x} & \text{for } -1/2 \leq x < 0. \end{cases}
\end{aligned}$$

We keep the term r_3 in (240) as it is not uniform in x , and is not small as x approaches $\pm 1/2$.

$$\begin{aligned}
r_3 &= \frac{k^2 + (k+1)^2}{2(2k+1)^2} \log \left(1 + \frac{1}{2\pi\kappa_k^2} e^{(2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} \right) + \\
&+ \frac{k^2 + (k-1)^2}{2(2k-1)^2} \log \left(1 + 2\pi\kappa_{k-1}^2 e^{(-2x-1)\frac{s\sqrt{|u_1u_2|}}{2(k+x)}} \right) \\
&+ \mathcal{O} \left(\max_{x \in [-1/2, 1/2)} \left[u_0(\log u_0^{-1})^2 + \frac{s}{n}(\log u_0^{-1})^2 + s^{-2} \log u_0^{-1} \right] \right). \tag{241}
\end{aligned}$$

Similarly but simpler, using (225) and (233), we find that for $k \geq 1$,

$$\begin{aligned} & \frac{|\tilde{h}|^2}{\pi} \int_{-1/2}^x (F_2 + F_3)(e^{\pm i\theta_0}) d\theta_0 \\ &= \begin{cases} -\frac{k(k-1)}{(2k-1)^2} \log(1 + 2\pi\kappa_{k-1}^2) + r_4 & \text{for } |x| < 1/2, \\ -\frac{k(k-1)}{(2k-1)^2} \log(1 + 2\pi\kappa_{k-1}^2) + \frac{k(k+1)}{(2k+1)^2} \log\left(1 + \frac{1}{2\pi\kappa_k^2}\right) \\ \quad + \mathcal{O}\left(\max_{x \in [-1/2, 1/2)} \left[u_0 \log u_0^{-1} + \frac{1}{s} + \frac{s}{n}\right]\right) & \text{for } x = 1/2, \end{cases} \end{aligned} \quad (242)$$

where $r_4 = o(1)$. When $|x| < 1/2$ we again keep track of the error term

$$\begin{aligned} r_4 &= \frac{k(k+1)}{(2k+1)^2} \log\left(1 + \frac{1}{2\pi\kappa_k^2} e^{(2x-1)\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}}\right) \\ &+ \frac{k(k-1)}{(2k-1)^2} \log\left(1 + 2\pi\kappa_{k-1}^2 e^{(-2x-1)\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}}\right) + \mathcal{O}\left(\max_{x \in [-1/2, 1/2)} \left[u_0 \log u_0^{-1} + \frac{1}{s} + \frac{s}{n}\right]\right). \end{aligned} \quad (243)$$

When $k = 0$, we have $F_2, F_3, F_4 = 0$, and thus the integral of $F_1 + F_2 + F_3 + F_4$ over $[x = 0, x = x_0 < 1/2]$ for $k = 0$ is given by (237). For any $k \geq 1$ and $-1/2 \leq x < 1/2$, combining (237), (239), (240), (242)

$$\begin{aligned} & \frac{|\tilde{h}|^2}{\pi} \int_0^{\theta_0} (F_1 + F_2 + F_3 + F_4)(e^{\pm i\theta}) d\theta = -\frac{s}{4} \sqrt{|u_1 u_2|} w_1(x) + \frac{1}{2} \delta_k(x) \\ & - \frac{1}{2} \sum_{j=0}^{k-1} \left(\log 2\pi\kappa_j^2 + \frac{s}{2} \sqrt{|u_1 u_2|} \left(\frac{j}{(2j+1)^2} - \frac{j}{(2j-1)^2} \right) \right) \\ & + \mathcal{O}\left(su_0 \log(u_0)^{-1} + s^3/n + 1/\log(u_0)^{-1} + 1/s\right), \\ & \delta_k(x) = \begin{cases} \log(1 + 2\pi\kappa_{k-1}^2) & \text{for } x = -1/2 \\ o(1) & \text{for } |x| < 1/2. \end{cases} \end{aligned} \quad (244)$$

By (241) and (243), it follows that for $|x| < 1/2$, $\delta_k(x)$ is given explicitly as

$$\delta_k(x) = \log\left(1 + 2\pi\kappa_{k-1}^2 e^{-(2x+1)\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}}\right) + \log\left(1 + (2\pi\kappa_k^2)^{-1} e^{(2x-1)\frac{s\sqrt{|u_1 u_2|}}{2(k+x)}}\right). \quad (245)$$

Furthermore, by (216) (which holds for all $k \geq 0$) and Proposition 4.1, we find that

$$\begin{aligned} & \frac{|\tilde{h}|^2}{\pi} \int_0^{\theta_0} (F_0(e^{i\theta}) + F_0(e^{-i\theta})) d\theta \\ &= s\sqrt{|u_1 u_2|} \left(w_2(x) + \sum_{j=1}^{k-1} \frac{j(j+1)}{(2j+1)^2} + \frac{j(j-1)}{(2j-1)^2} \right) + \mathcal{O}\left(\frac{s^3}{n} + su_0 \log u_0^{-1} + \frac{1}{\log u_0^{-1}}\right) \quad (246) \\ & w_2(x) = \begin{cases} \frac{k(k-1)}{(2k-1)^2} + \frac{k(k+1)}{2k+1} \frac{x}{k+x} & \text{for } 0 \leq x < 1/2 \\ \frac{k^2(k-1)}{(2k-1)^2} \frac{1+2x}{k+x} & \text{for } -1/2 \leq x < 0, \end{cases} \end{aligned}$$

for $k \geq 1$ and $F_0 = 0$ for $k = 0$.

4.4.1 Proof of Theorem 1.2

We sum together (244), (246), and substitute the result into (232), to find that

$$\begin{aligned} \log D_n(J) &= \log D_n(J_2) + s\sqrt{|u_1 u_2|} \left(w_3(x) + \sum_{j=1}^{k-1} \frac{2j^2}{4j^2 - 1} \right) - \sum_{j=0}^{k-1} \log 2\pi \kappa_j^2 \\ &\quad + \delta_k(x) + \mathcal{O}(u_0 + su_0 \log u_0^{-1} + s^3/n + 1/\log u_0^{-1}). \end{aligned} \quad (247)$$

$$w_3 = -\frac{w_1}{2} + w_2 = \frac{k^2}{2(2k-1)} \frac{1+2x}{k+x},$$

where $\sum_{j=1}^0 = 0$. The first sum can be evaluated by noting that

$$\sum_{j=1}^{k-1} \frac{2j^2}{4j^2 - 1} = \frac{k(k-1)}{2k-1}, \quad (248)$$

and, it follows that

$$s\sqrt{|u_1 u_2|} \left(w_3 + \sum_{j=1}^{k-1} \frac{2j^2}{4j^2 - 1} \right) = \frac{s}{2} \sqrt{|u_1 u_2|} \left(\omega - \frac{x^2}{\omega} \right), \quad \omega = k + x. \quad (249)$$

The sum with the leading coefficients κ_j is given by

$$\prod_{j=0}^{k-1} \kappa_j^2 = 4^{-k^2} \frac{G(2k+1)}{G(k+1)^4}, \quad (250)$$

where G is the Barnes G-function. By substituting (249), (250) into (247) we find that

$$\begin{aligned} \log D_n(J(u_0)) &= \log D_n(J_2) + \frac{s}{2} \sqrt{|u_1 u_2|} \left(\omega - \frac{x^2}{\omega} \right) + c(k) + \delta_k(x) \\ &\quad + \mathcal{O}(s^3/n + su_0 \log u_0^{-1} + 1/\log u_0^{-1}). \end{aligned} \quad (251)$$

Now define $\alpha = u_2/2$, $\beta = u_1/2$, and

$$\nu = \frac{8}{\beta^{-1} - \alpha^{-1}} e^{-\frac{s\sqrt{|\alpha\beta|}}{\omega}}. \quad (252)$$

Then, by (233),

$$\frac{u_0}{2} = \nu \left(1 + \mathcal{O} \left(\frac{s}{n} \log \nu^{-1} + \nu^2 (\log \nu^{-1})^2 \right) \right). \quad (253)$$

It is an easy exercise using (253), the continuity of w, x as functions of u_0 , and (251) to show that

$$\log D_n(\nu) = \log D_n(u_0) + \mathcal{O}(u_0 + su_0 \log u_0^{-1} + s^3/n + 1/\log u_0^{-1}). \quad (254)$$

Substituting the asymptotics from (251) into (254), and using uniformity of the error terms, we prove Theorem 1.2.

4.5 Proof of Proposition 4.1

We consider the small norm matrices, and we prove Proposition 4.1. From (209) it follows that

$$\begin{aligned} \operatorname{Re}[(\overline{x_1}x_2)(e^{\pm i\theta_0})] &= -|\tilde{h}|^{-2} \operatorname{Re}[\overline{\gamma_1}\gamma_2] (1 + 2\operatorname{Re} R_{1,11}(e^{\pm i\theta_0}) + 2\operatorname{Re} R_{2,11}(e^{\pm i\theta_0})) \\ &\quad + \mathcal{O}(|\gamma_1|^2 - |\gamma_2|^2 + (|R_{1,11}| + |R_{12}|)^2 + ||R_3||), \end{aligned} \quad (255)$$

where $||\cdot||$ denotes the value of the largest element of the matrix in absolute value. From (212) we have

$$\begin{aligned} \operatorname{Re}[\overline{\gamma_1}\gamma_2] &= -1/2 + \mathcal{O}(s/n), \\ |\gamma_1|^2 - |\gamma_2|^2 &= \mathcal{O}(s/n). \end{aligned} \quad (256)$$

It follows from Proposition 3.2, (168)–(169), (173) that

$$\begin{aligned} R_1(e^{\pm i\theta_0}) &= \mathcal{O}(s^{-1} + \tilde{e}u_0^{1+2|x|} \log u_0^{-1} + \tilde{e}su_0^2 \log u_0^{-1}), \\ R_2(e^{\pm i\theta_0}) &= \mathcal{O}((s^{-1} + \tilde{e}u_0^{1+2|x|} \log u_0^{-1})^2 + \tilde{e}su_0^2 \log u_0^{-1}), \\ R_3(e^{\pm i\theta_0}) &= \mathcal{O}(|R_1| ||R_2|| + (u_0 \log u_0^{-1})^3). \end{aligned} \quad (257)$$

From (255)–(257), it follows that

$$\begin{aligned} \operatorname{Re}[(\overline{x_1}x_2)(e^{\pm i\theta_0})] &= \frac{1}{2}|\tilde{h}|^{-2} (1 + 2\operatorname{Re} R_{1,11}(e^{\pm i\theta_0}) + 2\operatorname{Re} R_{2,11}(e^{\pm i\theta_0})) \\ &\quad + \mathcal{O}\left(\left(s^{-1} + \tilde{e}u_0^{1+2|x|} \log u_0^{-1} + \tilde{e}su_0^2 \log u_0^{-1}\right)^2 + s/n\right). \end{aligned} \quad (258)$$

We will evaluate $\operatorname{Re} R_{1,11}$ and $\operatorname{Re} R_{2,11}$ to prove Proposition 4.1.

We recall from (173) that R_1 is a sum of 3 terms. The first term is an integral of $\Delta_1^{(1)}$, and the two other terms are integrals of $\Delta_1^{(b_1)}$ and $\Delta_1^{(b_2)}$. We first evaluate the contribution from the terms $\Delta_1^{(b_1)}$ and $\Delta_1^{(b_2)}$.

4.5.1 Contribution to R_1 from $\Delta_1^{(b_1)}$ and $\Delta_1^{(b_2)}$

It follows from (168) and (169) that

$$\frac{\Delta_{1,11}^{(b_1)}(z)}{b_1 - e^{\pm i\theta_0}} = \frac{C_{\Delta,1}}{z - b_1} + \mathcal{O}(1), \quad \frac{\Delta_{2,11}^{(b_1)}(z)}{b_2 - e^{\pm i\theta_0}} = \frac{C_{\Delta,1}}{z - b_2} + \mathcal{O}(1) \quad (259)$$

as $z \rightarrow b_1$ and $z \rightarrow b_2$ respectively, and that the matrices C_{Δ_1} and C_{Δ_2} are given as follows as $s \rightarrow \infty$

$$\begin{aligned} C_{\Delta_1} &= - \frac{\left[i + \frac{fi}{\theta_1}(\psi + \psi^{-1}) + \frac{f^2}{\theta_1^2} \left(2i + \left(\frac{\gamma_1}{\gamma_2} - \frac{\gamma_2}{\gamma_1} \right) \right) \right]}{8s(u_1 \mp u_0) \left(1 - 4 \frac{k+x}{s(u_1-u_2)} \left| \frac{u_1}{u_2} \right|^{1/2} \right)} (1 + \mathcal{O}(s/n)), \\ C_{\Delta_2} &= \frac{\left[-i + \frac{fi}{\theta_1}(\psi + \psi^{-1}) + \frac{f^2}{\theta_1^2} \left(-2i + \left(\frac{\gamma_1}{\gamma_2} - \frac{\gamma_2}{\gamma_1} \right) \right) \right]}{8s(u_2 \mp u_0) \left(1 - 4 \frac{k+x}{s(u_1-u_2)} \left| \frac{u_1}{u_2} \right|^{1/2} \right)} (1 + \mathcal{O}(s/n)). \end{aligned} \quad (260)$$

We recall that f is real to the main order and from (129), (212) we have

$$\text{Im}(\psi + \psi^{-1}) = \mathcal{O}(s/n), \quad \text{Re} \left(\frac{\gamma_1}{\gamma_2} - \frac{\gamma_2}{\gamma_1} \right) = \mathcal{O}(s/n). \quad (261)$$

Since the interior of the bracket $[\]$ in (260) is imaginary to main order, we can calculate the residue in the integral of (259) to find:

$$\text{Re} \left[\sum_{j=1,2} \int_{\partial U_j} \frac{\Delta_{1,11}^{(b_j)}(u)}{u - e^{\pm i\theta_0}} \frac{du}{2\pi i} \right] = \mathcal{O}(s/n). \quad (262)$$

4.5.2 Contribution to R_1 from $\Delta_1^{(1)}$

Denote

$$\begin{aligned} y_0(z) &= -2i(\gamma_1^2 - \gamma_2^2) = -i \left(\left(\frac{z-b_2}{z-b_1} \right)^{1/2} + \left(\frac{z-b_1}{z-b_2} \right)^{1/2} \right), \\ x_0(z) &= -4\gamma_1\gamma_2 = -i \left(\left(\frac{z-b_2}{z-b_1} \right)^{1/2} - \left(\frac{z-b_1}{z-b_2} \right)^{1/2} \right), \end{aligned} \quad (263)$$

with branch cuts on J_2 such that the square root is positive as $z \rightarrow \infty$. Our goal is to evaluate the terms in (138), and given a matrix X we denote

$$(LX)(z) = \tilde{E}(z)B(z)XB^{-1}(z)\tilde{E}^{-1}(z). \quad (264)$$

Define $D(z)$ and $E^{(\pm)}(z)$ by

$$D(z) = L \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E^{(+)}(z) = L \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E^{(-)}(z) = L \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then from (138) it follows that

$$\Delta_{1,11}^{(1)}(z) = \begin{cases} \zeta^{-1}(z) (\Phi_{1,11}D(z) + B_{22}^2(z)\Phi_{1,21}E^{(-)}(z)), & 0 \leq x < 1/2, \\ \zeta^{-1}(z) (\Phi_{1,11}D(z) + B_{11}^2(z)\Phi_{1,12}E^{(+)}(z)), & -1/2 \leq x < 0. \end{cases} \quad (265)$$

Recalling the definition of B in (127), the definition of F in (128), and the definition of $\tilde{E}$ in (136), we find that

$$\begin{aligned} D(z) &= \frac{i}{2}y_0(z) - \frac{x_0(z)f}{2(z-1)}(\psi + \psi^{-1}) - \frac{f^2}{(z-1)^2} \left(iy_0(z) + \frac{x_0(z)}{2}(-\psi + \psi^{-1}) \right), \\ E^{(\pm)}(z) &= \frac{1}{4} \left(x_0(z) + \frac{f}{z-1} (iy_0(z)(\psi + \psi^{-1}) \pm 2(-\psi + \psi^{-1})) \right. \\ &\quad \left. - \frac{f^2}{(z-1)^2} (2x_0(z) + iy_0(z)(\psi - \psi^{-1}) \mp 2(\psi + \psi^{-1})) \right). \end{aligned} \quad (266)$$

We analyze the sign of $\Delta_{1,11}^{(1)}$ in (265). From (129) and (212) we have

$$\operatorname{Im} (\psi + \psi^{-1}) = \mathcal{O}(s/n), \quad \operatorname{Re} (\psi - \psi^{-1}) = \mathcal{O}(s/n). \quad (267)$$

From (263) we see that

$$\begin{aligned} \operatorname{Im} (x_0(e^{i\theta})) &= \mathcal{O}(s/n), & \operatorname{Im} \left(\frac{d^j}{d\theta^j} x_0(e^{i\theta}) \right) &= \mathcal{O}(n^{j-1}/s^{j-1}), \\ \operatorname{Im} (y_0(e^{i\theta})) &= \mathcal{O}(s/n), & \operatorname{Im} \left(\frac{d^j}{d\theta^j} y_0(e^{i\theta}) \right) &= \mathcal{O}(n^{j-1}/s^{j-1}), \end{aligned} \quad (268)$$

for $e^{i\theta} \in U_0 \cap C$. Write (265) in the form

$$\Delta_1^{(1)}(z) = \frac{\Delta_{1,-3}^{(1)}(z)}{(z-1)^3} + \frac{\Delta_{1,-2}^{(1)}(z)}{(z-1)^2} + \frac{\Delta_{1,-1}^{(1)}(z)}{(z-1)}, \quad (269)$$

where $\Delta_{1,-j}^{(1)}$ are analytic functions in z in U_0 . Then a calculation of residues gives the following expansion as $z \rightarrow 1$:

$$\begin{aligned} \int_{\partial U_0} \frac{\Delta_1^{(1)}(u)}{u-z} \frac{du}{2\pi i} &= \frac{1}{6} \frac{d^3}{dz^3} \left(\Delta_{1,-3}^{(1)} \right) (1) + \\ &\quad \frac{1}{2} \frac{d^2}{dz^2} \left(\Delta_{1,-2}^{(1)} \right) (1) + \frac{d}{dz} \left(\Delta_{1,-1}^{(1)} \right) (1) + \mathcal{O}(z-1). \end{aligned} \quad (270)$$

We note that ζ is real on J_1 , and recall the expansion of ζ in (83)–(84). We also note that $\operatorname{Im} f = \mathcal{O}(s^2/n^2)$, and that $\Phi_{1,11}$ is real but that $\Phi_{1,12}$ and $\Phi_{1,21}$ are imaginary. Combining with (265)–(270), we conclude that

$$\operatorname{Re} \left(\int_{\partial U_0} \frac{\Delta_{1,11}^{(1)}(u)}{u - e^{\pm i\theta_0}} \frac{du}{2\pi i} \right) = \mathcal{O}(s/n). \quad (271)$$

As a consequence of (262) and (271) we have

$$\operatorname{Re} R_{1,11}(e^{\pm i\theta_0}) = \mathcal{O}(s/n). \quad (272)$$

4.5.3 Order of $R_2(e^{\pm i\theta_0})$

From (266) and (138) we have

$$\begin{aligned} \Delta_{2,11}^{(1)}(z) &= \begin{cases} \zeta^{-2}(z) \left[\Phi_{2,11} D(z) + B_{11}^2(z) (\Phi_{2,12} - \Phi_{1,12} \Phi_{1,22}) E^{(+)}(z) + \mathcal{O}(1) \right], & 0 \leq x < \frac{1}{2}, \\ \zeta^{-2}(z) \left[\Phi_{2,11} D(z) + B_{22}^2(z) (\Phi_{2,21} - \Phi_{1,21} \Phi_{1,11}) E^{(-)}(z) + \mathcal{O}(1) \right], & -\frac{1}{2} \leq x < 0, \end{cases} \\ \Xi^{(1)}(z) &= \begin{cases} -\zeta^{-3}(z) \left[B_{11}^2(z) \Phi_{1,12} \Phi_{2,22} E^{(+)}(z) + \mathcal{O}(1) \right], & 0 \leq x < \frac{1}{2}, \\ -\zeta^{-3}(z) \left[B_{22}^2(z) \Phi_{1,21} \Phi_{2,11} E^{(-)}(z) + \mathcal{O}(1) \right], & -\frac{1}{2} \leq x < 0. \end{cases} \end{aligned}$$

By inspection of the signs of each element, it follows that

$$\operatorname{Re} \left(\int_{\partial U_0} \frac{\Delta_{2,11}^{(1)}(u) + \Xi_{11}^{(1)}(u)}{u - e^{\pm i\theta_0}} \frac{du}{2\pi i} \right) = \mathcal{O}(s/n). \quad (273)$$

The remaining contributions to $R_{2,11}$, defined in (138), are calculated using rougher estimates from (257). Thus it follows that

$$\operatorname{Re} R_{2,11} = \mathcal{O} \left(s/n + (\tilde{e} u_0^{1+2|x|} \log u_0^{-1} + \tilde{e} s u_0^2 \log u_0^{-1} + s^{-1})^2 \right). \quad (274)$$

Substituting (272) and (274) into (258) yields Proposition 4.1.

5 Connection to the asymptotics of [7]

Consider the Deift-Its-Zhou asymptotics (6) for 2 fixed gaps $A = (\alpha_1, \beta_1) \cup (\alpha_2, \beta_2)$. Without loss of generality, we assume that $\nu \equiv \alpha_2 = -\beta_1$. We also denote $\alpha = \alpha_1$, $\beta = \beta_2$. The following lemma shows that these asymptotics can be extended (with a worse error term) to the region where ν is decreasing at a sufficiently slow rate as $s \rightarrow \infty$. This gives a connection to the asymptotics of Theorem 1.1 (see Remark 1.2 following Theorem 1.1).

Lemma 5.1. *Let $\varepsilon > 0$. As $s \rightarrow \infty$, uniformly for $\nu \in (\nu_1, \nu_2)$ where $\nu_2 > 0$ is fixed and $\nu_1 = \nu_1(s) \rightarrow 0$ s.t.*

$$s\nu_1^{1/2+\varepsilon} \rightarrow \infty,$$

we have

$$\frac{\partial}{\partial s} \log P_s(A) = -2G_0(\alpha, \beta, \nu)s + \frac{\partial}{\partial s} \log \theta(sV(\alpha, \beta, \nu); \tau(\alpha, \beta, \nu)) + O((s\nu^{2|<sV>|+\varepsilon})^{-1}), \quad (275)$$

where V, τ, G_0 are defined in equations (12), (11) above with $\nu = \alpha_2 = -\beta_1$, $\alpha = \alpha_1$, $\beta = \beta_2$; $\gamma = (\beta^{-1} - \alpha^{-1})/8$, and $\langle x \rangle = x - k$ ($-1/2 < \langle x \rangle \leq 1/2$) with k the closest integer to x .

Proof. Consider the setup of [7] for 2 gaps $(\alpha, -\nu) \cup (\nu, \beta)$. In the notation of [7], $\alpha = a_0$, $\nu = -b_0 = a_1$, $\beta = b_1$, $s = x$. We now verify that, if ν tends to zero at a sufficiently slow rate with $s \rightarrow \infty$, the jump matrices of the R matrix in the Deift-Its-Zhou RH problem remain uniformly close to the identity, and therefore the analysis of [7] is extendable into that region. We encircle the end-points of the gaps by nonintersecting discs. Note that the discs around ν , $-\nu$ will have to contract as ν tends to zero, we choose their radii to be $\nu/3$. For the matching of the local parametrices and the global one on the boundaries of the discs, we need, in particular, the parameter (see (4.100), (4.102), etc in [7])

$$\rho(z) = s\Omega^{(0)}(z) = s \int_{-\nu}^z \frac{q(t)}{\sqrt{p(t)}} dt, \quad (276)$$

to be uniformly large in absolute value on the boundary of the disc around $-\nu$. Here $p(z) = (z - \alpha)(z^2 - \nu^2)(z - \beta)$, $q(z) = z^2 + q_1 z + q_0$, where $q_1 = -(\alpha + \beta)/2$, and the value of the constant q_0 is determined by the equation

$$0 = \int_{\nu}^{\beta} \frac{q(t)}{\sqrt{p(t)}} dt = \int_{\nu}^{\beta} \frac{t^2 + q_1 t}{\sqrt{p(t)}} dt + q_0 \int_{\nu}^{\beta} \frac{dt}{\sqrt{p(t)}} \quad (277)$$

To analyze the integrals in the limit $\nu \rightarrow 0$, we split the interval $(\nu, \beta) = (\nu, \sqrt{\nu}) \cup [\sqrt{\nu}, \beta)$, and change the integration variable $y = t/\sqrt{\nu}$ in the integration over the first one. We then obtain

$$\int_{\nu}^{\beta} \frac{t^2}{\sqrt{|p(t)|}} dt = \sqrt{|\alpha|\beta} + (\beta - |\alpha|) \arctan \sqrt{\frac{\beta}{|\alpha|}} + \frac{\nu^2 \log \nu^{-1}}{2\sqrt{|\alpha|\beta}} + O(\nu^2) \quad (278)$$

$$\int_{\nu}^{\beta} \frac{t}{\sqrt{|p(t)|}} dt = 2 \arctan \sqrt{\frac{\beta}{|\alpha|}} - \frac{\beta - |\alpha|}{8(|\alpha|\beta)^{3/2}} \nu^2 \log \nu^{-1} + O(\nu^2) \quad (279)$$

$$\int_{\nu}^{\beta} \frac{1}{\sqrt{|p(t)|}} dt = \frac{1}{\sqrt{|\alpha|\beta}} \left(\log(\gamma\nu)^{-1} + \frac{1}{16} \left\{ \frac{3}{\alpha^2} + \frac{3}{\beta^2} - \frac{2}{|\alpha|\beta} \right\} \nu^2 \log \nu^{-1} \right) + O(\nu^2), \quad (280)$$

where $\gamma = (\beta^{-1} + |\alpha|^{-1})/8$. And therefore, (277) gives

$$q_0 = -\frac{|\alpha|\beta}{\log(\gamma\nu)^{-1}} (1 + O(\nu^2 \log \nu^{-1})), \quad \nu \rightarrow 0. \quad (281)$$

Substituting this expansion into (276), we obtain that

$$|\rho(z)| \geq c \frac{s}{\log(\gamma\nu)^{-1}}, \quad c > 0, \quad (282)$$

on the boundary of the disc around $-\nu$ (c is independent of z, ν, s). Similarly, we carry out the analysis around the other end-points of the gaps and obtain that the inequality (282) holds for the relevant quantities on the boundaries of all the disc around the end-points of the intervals. In the notation of [7], this means that

$$v_{p,s} = I + O\left(\frac{\log(\gamma\nu)^{-1}}{s}\right) \quad (283)$$

uniformly on the boundaries of the discs.

To prove the lemma, we need to verify that the jump matrices $v_{R,s}(z)$ in (4.123) in [7] have the form $v_{R,s}(z) = I + o(1)$ on the jump contour Figure 4.122 in [7] in the asymptotic regime of the lemma. First, on the boundaries of the discs (see Figure 4.122 in [7]),

$$v_{R,s}(z) = f^\infty(v_{p,s})^{-1}(f^\infty)^{-1}, \quad (284)$$

where (see (3.42) in [7])

$$f^\infty(z) = \begin{pmatrix} \frac{\theta(u_\infty+d)}{\theta(u_\infty+sV+d)} & 0 \\ 0 & \frac{\theta(u_\infty+d)}{\theta(u_\infty-sV+d)} \end{pmatrix} \begin{pmatrix} \frac{\theta(u(z)+sV+d)}{\theta(u(z)+d)} \frac{\delta+\delta^{-1}}{2} & \frac{\theta(u(z)-sV-d)}{\theta(u(z)-d)} \frac{\delta-\delta^{-1}}{2i} e^{is\Omega_0} \\ \frac{\theta(u(z)+sV-d)}{\theta(u(z)-d)} \frac{\delta-\delta^{-1}}{-2i} e^{-is\Omega_0} & \frac{\theta(u(z)-sV+d)}{\theta(u(z)+d)} \frac{\delta+\delta^{-1}}{2} \end{pmatrix}.$$

Here $\theta(z) = \theta(z; \tau)$ and V, τ are as in (8) and (12),

$$\Omega_0 = 2\alpha + 2 \int_\infty^\alpha \left(\frac{q(x)}{\sqrt{p(x)}} - 1 \right) dx, \quad u(z) = \frac{\int_\alpha^z \frac{dt}{\sqrt{p(t)}}}{2 \int_\nu^\beta \frac{dt}{\sqrt{p(t)}}}, \quad \delta(z) = \left(\frac{(z+\nu)(z-\beta)}{(z-\nu)(z-\alpha)} \right)^{1/4}.$$

The sheet of the Riemann surface $w = p(z)^{1/2}$ is chosen such that $p(z)^{1/2} \rightarrow 1, z \rightarrow \infty$. The constant $u_\infty = u(\infty)$, and d is chosen such that the zero of $\theta(u(z) - d)$ coincides with the zero of $\delta(z) - \delta(z)^{-1}$ (which is inside $(-\nu, \nu)$). Note (see [7]) that $\theta(u(z) - d)$ has no other zeros, and $\theta(u(z) + d)$ has no zeros. Thus $f^\infty(z)$ is analytic outside A and clearly the limit $f^\infty(\infty) = I$. Moreover by standard arguments based on Liouville theorem, $\det f^\infty(z) = 1$ for all z . Furthermore [7], $u_\infty + d \equiv 0$ modulo the lattice $m + n\tau, m, n \in \mathbb{Z}$.

In the limit $\nu \rightarrow 0$, we have the expansions

$$V = \frac{1}{\pi} \int_{-\nu}^\nu \frac{q(t)}{\sqrt{|p(t)|}} dt = -\sqrt{|\alpha|\beta} \left(\frac{1}{\log(\gamma\nu)^{-1}} - \frac{(\alpha+\beta)^2}{16\alpha^2\beta^2} \nu^2 \right) + O(\nu^2 / \log \nu^{-1}), \quad (285)$$

$$\tau = \frac{i\pi}{\log(\gamma\nu)^{-1}} (1 + O(\nu^2 / \log \nu^{-1})), \quad (286)$$

and therefore

$$\kappa = e^{-i\pi/\tau} = (\gamma\nu)^{1+O(\nu^2/\log \nu^{-1})}. \quad (287)$$

Note that, using the inversion formula ($\tau \rightarrow 1/\tau$) for the theta-functions, we can write

$$\theta(z) = \frac{1}{\sqrt{-i\tau}} \sum_k e^{-\frac{i\pi}{\tau}(k-z)^2} = \frac{\kappa^{<z>^2}}{\sqrt{-i\tau}} (1 + O(\kappa^{1-2|<z>|})), \quad (288)$$

where

$$z = j + <z>, \quad -1/2 << \operatorname{Re} z > \leq 1/2, \quad j \in \mathbb{Z}.$$

We can now estimate the matrix elements of f^∞ on the boundaries of the discs. On the boundary of the disc around $-\nu$, we have $u(z) = -1/2 + r(z)$, where $|r(z)| < \varepsilon$ with a suitable $\varepsilon > 0$. Recalling periodicity properties of the theta-function, $\theta(z + n \pm \tau) = e^{\mp 2\pi i z - i\pi \tau} \theta(z)$, we write for some $C > 0$ uniformly on the boundary

$$\begin{aligned} & \left| \frac{\theta(u_\infty + d)}{\theta(u_\infty + sV + d)} \frac{\theta(u(z) + sV + d)}{\theta(u(z) + d)} \frac{\delta(z) + \delta(z)^{-1}}{2} \right| \\ & < C \left| \frac{\theta(0)}{\theta(sV)} \frac{\theta(1/2 + r(z) + sV + d)}{\theta(1/2 + r(z) + d)} \right| = O(\nu^{-(1+2\varepsilon)|<sV>|}). \end{aligned}$$

Similar estimate holds for the other elements of f^∞ (we replace $\delta - \delta^{-1}$ and $\theta(u(z) - d)$ in the off-diagonal elements with their derivatives at their zero). In the same vein, using the behaviour of $u(z)$, one obtains similar estimates on the discs around the other end-points. (Note that, e.g., at α , we can assume $|u(z)| < \varepsilon$). Recalling (283) we thus conclude that uniformly on these boundaries

$$v_{R,s} = I + f^\infty O\left(\frac{\log(\gamma\nu)^{-1}}{s}\right) (f^\infty)^{-1} = I + O\left(\frac{\log(\gamma\nu)^{-1}}{s} (\nu^{-(1+2\varepsilon)|<sV>|})^2\right). \quad (289)$$

Adjusting ε , we can write this estimate as $v_{R,s} = I + O((s\nu^{2|<sV>|+\varepsilon})^{-1})$. The error term here is not small at the point $<sV> = 1/2$, and we analyse the case of $|<sV>|$ close to $1/2$ separately below. Assume for now that $|<sV>| \leq 1/4$. Then (289) is the estimate we need to prove the lemma in this case. It remains, however, to obtain the same, or better, estimate for $v_{R,s}$ on the intervals outside the discs, where (Figure 4.122 in [7]),

$$v_{R,s} = f_+^\infty \begin{pmatrix} 1 & -2e^{2isg_+(z)} \\ 0 & 1 \end{pmatrix} (f_+^\infty)^{-1}, \quad g(z) = z + \int_\infty^z \left(\frac{q(x)}{\sqrt{p(x)}} - 1 \right) dx. \quad (290)$$

Since by definition of $q(z)$ ((1.17) in [7] or (10) in the introduction),

$$0 = \int_\nu^\beta \frac{q(t)}{\sqrt{p(t)}} dt = \int_\alpha^{-\nu} \frac{q(t)}{\sqrt{p(t)}} dt,$$

the estimation of $g_+(z)$ is similar to that of $\Omega^{(0)}(z)$ above, and we obtain that $\operatorname{Re}(ig_+(z)) < 0$ and

$$-\operatorname{Re}(ig_+(z)) \geq \frac{C}{\log(\gamma\nu)^{-1}}, \quad C > 0,$$

on the intervals outside the discs, and so

$$e^{2isg_+(z)} = O\left(e^{-\frac{cs}{\log(\gamma\nu)^{-1}}}\right),$$

with some constant $c > 0$ independent of s, ν, z . Substituting this into (290), we obtain as above,

$$v_{R,s} = I + O\left(\nu^{-1}e^{-\frac{cs}{\log(\gamma\nu)^{-1}}}\right) \quad (291)$$

uniformly on the intervals outside the discs. Combining this result with (289), we see that the estimate

$$v_{R,s} = I + O((s\nu^{2|<sV>|+\varepsilon})^{-1})$$

holds uniformly on the whole contour for R in the asymptotic regime of the lemma, and therefore the lemma is proved in the case $|<sV>| \leq 1/4$ by the arguments of [7].

Now consider the remaining case $1/4 < |<sV>| \leq 1/2$. Let

$$t = <sV> + k/2,$$

where $k = \pm 1$ is chosen so that $t \in (-1/4, 1/4)$. Consider the following function which solves the same jump conditions (given in [7]) as f^∞

$$\tilde{f}^\infty = \frac{1}{\gamma(z_-)} \begin{pmatrix} \frac{\theta(u(z_-)+d')}{\theta(u(z_-)+t+d')} & 0 \\ 0 & \frac{\theta(u(z_-)+d')}{\theta(u(z_-)-t+d')} \end{pmatrix} \begin{pmatrix} \frac{\theta(u(z)+t+d')}{\theta(u(z)+d')} \frac{\gamma+\gamma^{-1}}{2} & \frac{\theta(u(z)-t-d')}{\theta(u(z)-d')} \frac{\gamma-\gamma^{-1}}{2i} e^{is\Omega_0} \\ \frac{\theta(u(z)+t-d')}{\theta(u(z)-d')} \frac{\gamma-\gamma^{-1}}{-2i} e^{-is\Omega_0} & \frac{\theta(u(z)-t+d')}{\theta(u(z)+d')} \frac{\gamma+\gamma^{-1}}{2} \end{pmatrix}.$$

Here

$$\gamma(z) = \nu^{-1/4} \left(\frac{(z+\nu)(z-\nu)}{(z-\alpha)(z-\beta)} \right)^{1/4}.$$

The sheet of the Riemann surface $w = p(z)^{1/2}$ is chosen as before such that $p(z)^{1/2} \rightarrow 1$, $z \rightarrow \infty$. It is easy to verify that $\gamma(z) - \gamma(z)^{-1}$ has 2 zeros z_+, z_- . As $\nu \rightarrow 0$, $z_\pm = \pm\sqrt{\nu|\alpha\beta|}(1+o(1))$. The constant d' is chosen such that the zero of $\theta(u(z) - d')$ coincides with the zero z_+ of $\gamma(z) - \gamma(z)^{-1}$. As in [7], Abel theorem then shows that $u(z_-) + d' \equiv 1/2$ modulo the lattice. Furthermore, $\theta(u(z) - d')$ has no other zeros, and $\theta(u(z) + d')$ has no

zeros. Thus $\tilde{f}^\infty(z)$ is analytic outside A and the limit $\tilde{f}^\infty(z_-) = I$. It follows by standard arguments that $\det \tilde{f}^\infty(z) = 1$ for all z . We also note that the limit

$$\Lambda = \tilde{f}^\infty(\infty)$$

has $\det \Lambda = 1$ but is not the identity as before. By standard uniqueness arguments

$$f^\infty(z) = \Lambda^{-1} \tilde{f}^\infty(z). \quad (292)$$

The construction of local parametrices $\tilde{f}_p$ around the edge points is similar to that in [7]. The definition of the new R -matrix is now as follows: $R(z) = \Lambda f(z) \tilde{f}_p^{-1}(z)$ in the discs around the end-points and $R(z) = \Lambda f(z) (\tilde{f}^\infty(z))^{-1}$ outside. The jump matrices for R at the boundaries of the discs have the same form as before

$$v_{R,s} = I + \tilde{f}^\infty O\left(\frac{\log(\gamma\nu)^{-1}}{s}\right) (\tilde{f}^\infty)^{-1},$$

and a similar (to the one above) examination of the order of $\tilde{f}^\infty$ on the boundaries shows that uniformly

$$v_{R,s} = I + O\left(\frac{1}{s\nu^{1/2+\varepsilon}}\right), \quad |t| < 1/4. \quad (293)$$

As before, a better estimate holds on the rest of the jump contour of R . Thus the asymptotics obtained holds in the regime $s\nu^{1/2+\varepsilon} \rightarrow \infty$, for $|t| < 1/4$. To finish the proof of the lemma it only remains to verify (275) for $|t| < 1/4$. By Equation (3.9) in [7],

$$\frac{\partial}{\partial s} \log P_s(A) = -2G_0(\alpha, \beta, \nu)s + i(f_{1,22} - f_{1,11}),$$

where f_1 is the coefficient in the large z expansion $f(z) = I + f_1/z + O(1/z^2)$. (Below, we also use $f^\infty(z) = I + f_1^\infty/z + O(1/z^2)$, $\tilde{f}^\infty(z) = \Lambda + \tilde{f}_1^\infty/z + O(1/z^2)$.) By our definition of R ,

$$f(z) = \Lambda^{-1} R(z) \tilde{f}^\infty(z) = \Lambda^{-1} \left(I + \frac{R_1}{z} + O\left(\frac{1}{z^2}\right) \right) \left(\Lambda + \frac{\tilde{f}_1^\infty}{z} + O\left(\frac{1}{z^2}\right) \right)$$

and therefore, using also (292),

$$f_1 = \Lambda^{-1} \tilde{f}_1^\infty + \Lambda^{-1} R_1 \Lambda = f_1^\infty + \Lambda^{-1} R_1 \Lambda.$$

We have $\Lambda = O(\nu^{-1/4+|t|})$, and since R_1 has the same order as the error term in (293), $\Lambda^{-1} R_1 \Lambda = O((s\nu^{1-2|t|+\varepsilon})^{-1}) = O((s\nu^{2|sV|+\varepsilon})^{-1})$.

Thus

$$\frac{\partial}{\partial s} \log P_s(A) = -2G_0(\alpha, \beta, \nu)s + i(f_{1,22}^\infty - f_{1,11}^\infty) + O((s\nu^{2|<sV>|+\epsilon})^{-1}),$$

But it was shown in [7] (Equation (3.48)) that $i(f_{1,22}^\infty - f_{1,11}^\infty) = \frac{\partial}{\partial s} \log \theta(sV; \tau)$, and we again obtain (275) now for $1/4 < |<sV>| \leq 1/2$. The lemma is proved. $\square$

Remark 5.1. In the overlap region $(0, \nu_0) \cap (\nu_1, \nu_2)$ of the asymptotics of Theorem 1.1 and the lemma, we can explicitly see, as an exercise, the coincidence of the main terms. Indeed, from (11), with $\alpha = \alpha_1$, $\beta = \beta_2$, $\nu = -\beta_1 = \alpha_2$,

$$G_0 = q_0 + \frac{1}{8}(\beta - \alpha)^2 + \nu^2/2.$$

Substituting here the expansion (281), we obtain

$$G_0 = \frac{1}{8}(\beta - \alpha)^2 - \frac{|\alpha\beta|}{\log(\gamma\nu)^{-1}} + O(\nu^2).$$

Since $s\nu \rightarrow 0$, we see that this gives exactly the main (order s^2) term in (24).

Remark 5.2. Integration of the asymptotics of the lemma is related to the determination of the constant c_1 in (13) which will be addressed in a future publication.

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Appendix

We include a proof of the well-known formula (34), using arguments from [6]. As mentioned in the introduction, the gap probability of m gaps in the bulk scaling limit is given by the sine-kernel Fredholm determinant (2) for a wide class of random matrix ensembles. A particular such ensemble is the Circular Unitary Ensemble (CUE), which is the group of $n \times n$ unitary matrices equipped with the Haar measure. The Haar measure induces a probability measure $p_n(\theta)d^n\theta$ on the eigenvalues of the matrix given by

$$p_n(\theta) = \frac{1}{n!} \left(\frac{1}{2\pi} \right)^n \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^2, \quad \theta = (\theta_j)_{j=1}^n \in [0, 2\pi)^n. \quad (294)$$

From Heine's identity and (294), it follows that the probability that there are no eigenvalues on a set $\Sigma \subset C$, where C is the unit circle, is given by the following

$$D_n(J = C \setminus \Sigma) = \int_{e^{i\theta_j} \in J} p_n(\theta) d^n \theta, \quad (295)$$

where $D_n(J)$ was defined in (14). Denote $J^{(n)} = C \setminus \Sigma^{(n)}$ where

$$\Sigma^{(n)} = \left\{ z \in \mathbb{C} : \arg z \in \left(\frac{2s\alpha}{n}, -\frac{2s\nu}{n} \right) \cup \left(\frac{2s\nu}{n}, \frac{2s\beta}{n} \right) \right\}. \quad (296)$$

Using the definition (294) it is easily seen that

$$p_n(\theta) = \frac{1}{n!} \det \left(\tilde{K}_n(\theta_j, \theta_k) \right)_{j,k=1}^n, \quad (297)$$

where $\tilde{K}_n(x, y) = \frac{1}{2\pi} \sum_{j=0}^{n-1} e^{ji(x-y)}$. Let

$$K_n(x, y) = e^{-i\frac{n-1}{2}(x-y)} \tilde{K}_n(x, y) = \frac{1}{2\pi} \frac{\sin \frac{n}{2}(x-y)}{\sin \frac{1}{2}(x-y)}. \quad (298)$$

It follows that

$$p_n(\theta) = \frac{1}{n!} \det (K_n(\theta_j, \theta_k)). \quad (299)$$

The kernel K_n has the reproducing kernel property, meaning that for $r = 1, \dots, n$

$$\int \det(K_n(\theta_j, \theta_k))_{j,k=1}^n d\theta_{n-r+1} \dots d\theta_n = r! \det(K_n(\theta_j, \theta_k))_{j,k=1}^{n-r}, \quad (300)$$

where

$$\det(K_n(\theta_j, \theta_k))_{j,k=1}^0 \equiv 1. \quad (301)$$

From (295), we see that

$$\begin{aligned} D_n(J) &= \int_{\theta \in (0, 2\pi)^n} \prod_{j=1}^n (1 - \chi_\Sigma(\theta_j)) p_n(\theta) d^n \theta = \int_{\theta \in (0, 2\pi)^n} p_n(\theta) d^n \theta \\ &\quad - n \int_{\theta \in (0, 2\pi)^n} p_n(\theta) \chi_\Sigma(\theta_1) d^n \theta + \binom{n}{2} \int_{\theta \in (0, 2\pi)^n} p_n(\theta) \chi_\Sigma(\theta_1) \chi_\Sigma(\theta_2) d^n \theta \\ &\quad + \dots + (-1)^n \binom{n}{n} \int_{\theta \in (0, 2\pi)^n} p_n(\theta) \prod_{j=1}^n \chi_\Sigma(\theta_j) d^n \theta. \end{aligned} \quad (302)$$

The Fredholm determinant of a trace-class operator K acting on a set S can be represented as

$$\det(I - K)_S = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int_S \det(K(\theta_i, \theta_k))_{i,k=1}^j d^j \theta. \quad (303)$$

For bounded S and K , one may verify that the sum indeed converges using Hadamard's inequality. Let $J^{(n)}$ be given by (32), and $\Sigma^{(n)} = C \setminus J^{(n)}$ be the complement. Recall A from (18). Noting (299), we apply (300) to (302) to find that

$$D_n(J^{(n)}) = \det(I - K_n)_{\Sigma^{(n)}} = \det \left(I - \widehat{K}_n \right)_A, \quad (304)$$

where

$$\widehat{K}_n(x, y) = \frac{s \sin s(x - y)}{\pi n \sin \frac{s(x-y)}{n}}. \quad (305)$$

For fixed s , as $n \rightarrow \infty$, we have

$$\left| \widehat{K}_n(x, y) - K_s(x, y) \right| = \mathcal{O}(1/n). \quad (306)$$

Since the sum (303) converges,

$$\sum_{j=M}^{\infty} \frac{(-1)^j}{j!} \int_A \det(K(\theta_i, \theta_k))_{i,k=1}^j d^j \theta \rightarrow 0 \quad (307)$$

as $M \rightarrow \infty$, for $K = \widehat{K}_n, K_s$, where s remains fixed and uniformly for $n > N$ for some N . From (306), it follows that for fixed but arbitrarily large M ,

$$\left| \sum_{j=1}^M \frac{(-1)^j}{j!} \int_A \det(\widehat{K}_n(\theta_i, \theta_k))_{i,k=1}^j d^j \theta - \sum_{j=1}^M \frac{(-1)^j}{j!} \int_A \det(K_s(\theta_i, \theta_k))_{i,k=1}^j d^j \theta \right| = \mathcal{O}(1/n) \quad (308)$$

as $n \rightarrow \infty$. Thus it follows that

$$\left| D_n(J^{(n)}) - \det(I - K_s)_A \right| \rightarrow 0 \quad (309)$$

as $n \rightarrow \infty$ and s remains fixed.

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