FAULT DETECTION AND ISOLATION FOR LINEAR DYNAMIC SYSTEMS

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A report submitted in fulfilment of requirements for the Doctor of Philosophy at the Department of Electrical and Electronic Engineering, Imperial College London.

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OCTOBER 2011

Abstract

As modern control systems and engineering processes become increasingly more complex and integrated, the consequences of system failures and faults can be disastrous environmentally and economically. This thesis considers the fault detection and isolation (FDI) problem for linear time-invariant (LTI) systems subject to faults, disturbances and model uncertainties.

Firstly, a novel on-line approach to the robust FDI problem for linear discrete-time systems is proposed by using input/output measurement analysis over a finite estimation horizon. Upper and lower bounds on the fault signal are computed at each sampling instant so that a fault is detected and isolated when its upper bound is smaller than zero or its lower bound is larger than zero. Moreover, a subsequent-state-estimation technique, together with an estimation horizon update procedure are given to allow the on-line FDI process to be repeated in a moving horizon scheme.

Secondly, an optimal solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection (FD) problem is given for linear time-invariant systems subject to faults, disturbances and model uncertainties by using an observer-based approach. A new performance index is developed to capture both fault detection and disturbance rejection requirements which is particularly suitable for handling model uncertainties. A class of optimal solutions to the problem is then given in the form of simple linear matrix inequalities (LMI) with two degrees of freedom. By appropriately choosing these degrees of freedom, fault isolation can also be achieved.

Thirdly, in order to improve the FD performance and remove restrictive rank assumptions, routinely made in the literature, observer-based FD problems are investigated at a single frequency and over a finite frequency range, respectively. An optimal solution is derived such that, at a given frequency, the static observer generates a residual signal which minimizes the sensitivity of the residual to disturbances while maintaining a minimum level of sensitivity to faults. Then, an initial investigation is carried out for the FD problem over a finite frequency range. A solution is derived in the form of an LMI optimization by using the generalized KYP lemma followed by a linearization procedure. Conditions under which this solution is optimal are also derived.

Fully worked out numerical examples, mostly from the literature, are given to illustrate the effectiveness of all the proposed schemes.

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Acknowledgements

A thousand words of my English can only convey a grey and weak gratitude to my PhD supervisor Dr. Imad Jaimoukha. This fact makes me recall one story.

Well, some one thousand and three hundred years ago, a speechless Chinese poet's deep appreciation of his friend inspired him to write a simple but memorable line. At this moment, the very last stage of my PhD, I, with the same speechless feeling, would like to borrow that poem line and the poet's one thousand years sincere appreciation and send them to my supervisor Dr. Imad Jaimoukha.

桃花潭水深千尺，不及汪伦送我情。

(Translation: Peach Blossom Pool is a thousand feet deep, yet not so deep, as my thank for you and your favour for me.)

Thank you, Imad.

I would like to express my genuine gratitude to Prof. Richard Vinter for his priceless help and suggestions during my PhD study and job searching.

I am sincerely indebted to all my families and friends who share my memorable joy and pain for the last four years. Thank you all.

To my father and my mother

List of Publications

The following publications have been written during the course of this work:

- Z. Li, E. Mazars, Z. Zhang and I. Jaimoukha, "State-space solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ fault-detection problem," *International Journal of Robust and Nonlinear Control*, Jan. 2011.
- Z. Zhang and I.M. Jaimoukha, "An optimal solution to an $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem," *In Proceedings of the 50th IEEE Conference on Decision and Control*, Orlando, America, Dec. 2011.
- Z. Zhang and I.M. Jaimoukha, "Optimal state space solution to the fault detection problem at single frequency," *In Proceedings of the 18th World Congress of the International Federation of Automatic Control*, Milan, Italy, Aug. 2011.
- Z. Zhang and I.M. Jaimoukha, "A measurement analysis approach to online fault detection and isolation for linear discrete-time uncertain systems," *In Proceedings of the 49th IEEE Conference on Decision and Control*, Atlanta, America, Dec. 2010.
- Z. Zhang and I.M. Jaimoukha, "Optimal state space solution to the fault detection problem at DC," *In Proceedings of Conference on Control and Fault Tolerant Systems*, Nice, France, Oct. 2011.
- Z. Zhang and I.M. Jaimoukha, "Fault detection and isolation for linear discrete-time systems using input/output measurement analysis," *In Proceedings of the 48th IEEE Conference on Decision and Control*, Shanghai, China, Dec. 2009.

Submitted:

- Z. Zhang and I.M. Jaimoukha, "An approach to on-line robust fault detection and isolation for linear discrete-time uncertain systems," *Automatica*, 2011, under review.
- Z. Zhang and I.M. Jaimoukha, "An optimal solution to an $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem," *System and Control Letters*, 2010, under review.
- Z. Zhang and I. M. Jaimoukha, "A finite frequency domain approach to the fault detection problem for linear time-invariant systems," *IEEE Transactions on Automatic Control*, 2011, under review.
- Z. Zhang, Y. Yuan, "Decentralized consensus value computation over network with uncertainties," *IEEE Transactions on Automatic Control*, 2011, under review.

Chapter 1

Introduction

Control systems, which are nowadays exploited to improve the overall performance of industrial processes, involve both sophisticated digital system design techniques and complex hardware (sensors, actuators, components and processing units). Such complexity results in an increased probability of failure, consequently, control systems require automatic supervision of the closed-loop operation for detecting and isolating malfunctions as early as possible. Hence, the problems of FDI and diagnosis in dynamic processes have attracted great attention in both theoretical research and industrial applications.

1.1 Background

1.1.1 Importance of fault detection

As many control systems and engineering processes become more and more complex, widespread and integrated, the effects of system failures can be simply devastating to the infrastructure of modern society for extensive commercial, industrial and safety reasons. Therefore, fault detection and diagnosis is necessary in many control engineering applications such as: safety-critical systems (nuclear reactors, aircraft, missile guidance systems), cost-critical systems (large space structures, space vehicles, autonomous underwater vehicles).

As an example of safety-critical systems, in nuclear power plants, the occurrence of a fault may lead to extreme human and environmental problems. A proper control action could be a safe shut-down for the plant in the presence of emergency. In cost-critical systems, the occurrence of a fault may cause a tremendous economical loss. For example, autonomous underwater and unmanned space vehicles must be able to cope with unexpected events or malfunctions in a predictable man-

ner. In the worst case, the vehicle should be able to return a safe condition without necessarily completing the mission. In industrial applications or manufacture processes, minor faults could cause a shutdown of production systems or even a damage to equipment which could result in a large loss of profit.

Among many examples of catastrophic faulty situations, two cases are cited here:

1. On 26th of April 1986, there occurred near the Ukrainian town of Chornobyl a tremendous explosion at a huge nuclear power plant, followed by a gradual meltdown of the reactor No. 4. The main cause for this tragedy was the faulty outdated technology and the lack of a fault handling mechanism [96].

2. The Ariane 5 rocket exploded on June 4th, 1996 thirty seven seconds after lift-off. The reason was a software exception in the Inertial Reference Unit (IRU) that provides attitude and trajectory information for the control system. The exception caused the normal attitude information to be replaced by some diagnostic information that the control system was not designed to understand [97].

The catastrophes of these faulty situations could have been avoided or at least alleviated if the control system is designed with certain abilities of fault detection and diagnosis. Hence, since the early 1970s, the areas of fault detection, diagnosis and tolerance has become increasingly important and been actively studied due to reliability, security and automatic self-supervision requirements.

1.1.2 Definitions

It should be remarked that there is no consistent terminology in the field of FDI, which causes difficulties for engineers and researchers to understand the contributions and to compare different approaches. To avoid these misunderstandings, the IFAC Technical Committee *SAFEPROCESS* (Fault Detection, Supervision and Safety for Technical Processes) has initiated a common terminology related to the field of fault diagnosis. Further discussions regarding terminology can be found in [78]. The terminology used in this thesis is listed below and is consistent with that defined by the committee *SAFEPROCESS*.

- **Fault:** An unpermitted deviation of at least one characteristic property or parameter of the system from the acceptable, usual or standard condition.
- **Failure:** A permanent interruption of a system's ability to perform a required function under specified operating conditions.

Note that the difference between *fault* and *failure* is that the term *fault* is to represent a malfunction in the process of system operation while *failure* denotes the situation that system functions suffer a complete breakdown. A fault may be tolerable via a proper system design. However, it must be detected and diagnosed as soon as possible even though it has been tolerated at its early stage in order to prevent any serious or dangerous consequences in the future.

The terminology of tasks of a monitoring system for the purpose of fault detection and diagnosis is defined as follows:

- **Fault detection:** to determine whether there is an occurrence of a fault and the time of its occurrence.
- **Fault isolation:** to determine the kind, location (which process, sensor or actuator) and time of detection of a fault, following fault detection.
- **Fault identification:** to determine the size and time-variant behaviour of a fault. Follows fault detection and isolation.
- **Fault diagnosis:** includes fault detection, isolation and identification.
- **Monitoring:** a continuous real-time task of determining the conditions of a physical system, by recording information, recognizing and indicating anomalies in the system behaviour.
- **Supervision:** monitoring a physical system and taking appropriate actions to maintain the operation in the case of fault.

These tasks can be chosen to satisfy the special safety and reliability requirements of modern control systems, however fault detection is absolutely necessary for any practical system.

1.2 A brief history of fault detection and diagnosis

Research on fault diagnosis [11] has been carried out both in theory and application since the early 1970s. The well-known failure detection filter approach for linear systems was proposed in [3] and the application of identification methods to the fault detection of jet engines was considered in [68]. Correlation methods were applied to a practical problem of leak detection in [82].

Since the 1980s, model-based methods for fault detection and diagnosis have attracted extensive attention due to the increasing requirements of reliability and security. Model-based fault detection and diagnosis with specific application to chemical processes was proposed in [43] and sensor failure detection based on the inherent analytical redundancy of multiple observers was given in [11].

The use of parameter estimation techniques for fault detection of technical systems was given in [28] and the development of process fault detection methods based on modelling and parameter estimation was summarized in [45]. Parity equation-based methods were proposed by [10] and then further developed in [63]. Some contributions solve FD problem by using probabilistic approaches [17, 79]. It is noticed that one important approach, namely, the observer–based state estimation approach, has been actively researched over the last two decades and many contributions have been made to research the design of fault detection observers [19, 32, 35, 66, 65, 64, 77]. Also, some important work proposed different approaches to achieve FD for nonlinear systems. See [15, 42, 104] for details.

Recently, FD in a finite frequency setting has gained increasing attention from the theoretical research community. The sensitivity optimization problem over a finite frequency range for fault detection is studied to further improve the performance of fault detection or relax certain assumptions in the problem formulation [90, 92, 58, 30].

The development of $\mathcal{H}_\infty$ filtering techniques provides potential solutions to FD problems [86, 99]. Recently developed linear matrix inequality (LMI) [6, 26] approaches offer promising and numerically attractive techniques for formulating robust FDI decoupling problems. Hou and Patton gave a basic realization of fault detection observer design based on the bounded real lemma [44]. Zhong

et al. proposed a new performance index by introducing a reference residual model, which can be formulated using LMI techniques [111].

1.3 Outline of the thesis

The focus of this thesis is on the FDI problem for linear dynamic systems subject to faults, disturbances and model uncertainties. Three approaches are proposed to solve the problem. The first uses measurement analysis to detect and isolate a fault by evaluating its upper and lower bounds given all available information, while the other two achieve FD by designing static observers to generate fault indicators, namely, the residual signals; the first over the whole frequency range while the second over a finite range. The work in this thesis is only one step towards practiced FD systems. It is also needed to consider thresholds for residual signals, etc.

The thesis is organized as follows. The rest of this chapter will define the notation and introduce related background material on matrix calculations.

Chapter 2 introduces the concept and techniques of analytical redundancy and also reviews the main residual generation techniques in general model-based FD schemes such as parity-space and observer-based approaches.

Chapter 3 proposes a new on-line approach to solve the robust FDI problem for linear discrete-time systems subject to faults, disturbances and norm-bounded uncertainties by using input/output measurement analysis. FDI is achieved by computing upper and lower bounds on the fault signal such that a fault is regarded as having occurred when its upper bound is smaller than zero or lower bound is larger than zero. LMI optimization techniques are employed to obtain the bounds. Furthermore, a subsequent-state-estimation technique, together with an estimation horizon update step are proposed to allow the on-line FDI process to be repeated in a moving horizon procedure. Also, it is shown that the proposed approach reduces to the solution of linear programs when there are no uncertainties in the model description.

In Chapter 4, an optimal state-space solution is given to the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem for linear time-invariant systems subject to faults and disturbances by designing a static observer. A novel performance index is developed such that both the requirements of fault detection and disturbance

rejection are captured, and this index, unlike corresponding ones in the literature, is particularly suitable for handling model uncertainties. Then, a class of solutions to the FD problem is given in the form of simple LMIs with two degrees of freedom. The optimal solution and further specifications such as fault isolation can be achieved by appropriately choosing these degrees of freedom. Furthermore, it is shown that the FD performance may be improved and certain assumptions can be relaxed by applying a weighting filter.

In order to remove a widely made restrictive rank assumption on fault dynamics and to further improve the FD performance, Chapter 5 first considers the FD problem at a given frequency. A simple parameterization of all optimal solutions is derived with two degrees of freedom which can be used to achieve further specifications. In particular, at DC (when $\omega = 0$), the resulting degrees of freedom allow the FD observer to be used simultaneously for control system design. Moreover, an initial research is carried out for the FD problem over a finite frequency range. A solution is derived in the form of a BMI optimization using the generalized KYP lemma followed by a linearization procedure. Also, conditions under which this solution is optimal are derived.

Chapter 6 summarizes the results in this thesis, highlights the contribution and considers possible improvements and extensions to the proposed approaches.

1.4 Notation and preliminaries

This section gives the notation as well as some preliminary results that will be used later in the thesis. The notation used is fairly standard and is summarized as follows.

The set of real (complex) $n \times m$ matrices is denoted by $\mathcal{R}^{n\times m}$ ($\mathcal{C}^{n\times m}$). For $A \in \mathcal{C}^{n\times m}$ we use the notation A^T and A^* to denote the transpose and complex conjugate transpose, respectively. For $A \in \mathcal{C}^{n\times n}$, $\lambda_i(A)$ denotes the i-th eigenvalue of A. For $A = A^* \in \mathcal{C}^{n\times n}$, $A \succeq 0$ ($A \succ 0$) denotes that A is positive semidefinite (positive definite), that is, $\lambda_i(A) \geq 0$ ($\lambda_i(A) > 0$) for all i. Similar definitions apply to $A \preceq 0$ and $A \prec 0$. For $A = A^* \in \mathcal{C}^{n\times n}$, $\bar{\lambda}(A)$ denotes the largest and $\underline{\lambda}(A)$ the smallest eigenvalue of A, respectively. For $A \in \mathcal{C}^{n\times m}$, $\bar{\sigma}(A) = \sqrt{\bar{\lambda}(AA^*)}$ denotes the largest, and $\underline{\sigma}(A) = \sqrt{\underline{\lambda}(AA^*)}$ the smallest singular value of A, respectively.

The $n \times n$ identity matrix is denoted as I_n and the $n \times m$ null matrix is denoted as $0_{n,m}$ with the

subscripts occasionally dropped if they can be inferred from context.

The open left half of the complex plane is denoted by $\mathcal{C}_-$ and the closed right half of the complex plane is denoted by $\bar{\mathcal{C}}_+$. The notation $A = \mathrm{diag}\,(a_1, \ldots, a_n)$ denotes that A is a diagonal matrix with diagonal entries $a_1, \ldots, a_n$.

Continuous and discrete time state–space descriptions of a linear time–invariant dynamic system are given as follows, respectively.

- Continuous–time state–space model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

- Discrete–time state–space model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k), \end{aligned}$$

where $x(t), x(k) \in \mathcal{R}^n$, $u(t), u(k) \in \mathcal{R}^{n_u}$ and $y(t), y(k) \in \mathcal{R}^{n_y}$ are the state, input and output vectors, respectively, and where the real matrices $A,\ B,\ C$ and D have appropriate dimensions. Taking Laplace transform, assuming zero initial conditions, the input/output description is given by $y(s) = G(s)u(s)$ where $G(s) = D + C(sI - A)^{-1}B$ is the system transfer matrix function.

Also, use the notation that $G(s) \stackrel{s}{=} (A, B, C, D)$ and

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

If $G_1(s) \stackrel{s}{=} (A, B_1, C, D_1)$ and $G_2(s) \stackrel{s}{=} (A, B_2, C, D_2)$, it is written as

$$\left[\begin{array}{cc} G_1(s) & G_2(s) \end{array}\right] \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D_1 & D_2 \end{array}\right].$$

The matrix $A \in \mathcal{R}^{n\times n}$ is called stable if $\lambda_i(A) \in \mathcal{C}_-,\ \forall\, i$. The pair (A, C) is called detectable if there exists a real matrix L such that $(A + LC)$ is stable. The pair (A, C) is detectable if and only if the matrix pencil

$$\left[\begin{array}{c} A - \mu I \\ C \end{array}\right]$$

has full column rank for all $\mu \in \bar{\mathcal{C}}_+$.

The set $\mathcal{R}(s)^{m\times p}$ denotes the set of all $m \times p$ proper, real–rational matrix functions of s. $\mathcal{L}_\infty^{m\times p}$ denotes the space of $m \times p$ matrix functions with entries bounded on the extended imaginary axis. The subspace $\mathcal{H}_\infty^{m\times p} \subset \mathcal{L}_\infty^{m\times p}$ denotes matrix functions analytic in the closed right–half of the complex plane. A prefix $\mathcal{R}$ denotes a real–rational function, so that $\mathcal{RL}_\infty^{m\times p}$ denotes the space of $m \times p$ real rational matrix functions with entries bounded on the extended imaginary axis $j\mathcal{R}_e$. The subspace $\mathcal{RH}_\infty^{m\times p} \subset \mathcal{RL}_\infty^{m\times p}$ denotes real rational matrix functions analytic in the closed right–half of the complex plane, i.e., all $m \times p$ stable real–rational matrix functions of s. For $G(s) \in \mathcal{R}(s)^{m\times p}$ dependence on s is occasionally suppressed when it can be inferred by the context. For the spaces $\mathcal{R}(s)^{m\times p}$, $\mathcal{H}_\infty^{m\times p}$, $\mathcal{L}_\infty^{m\times p}$, etc., the dimensions are occasionally suppressed if they can be inferred by the context.

For $G(s) \in \mathcal{RH}_\infty^{m\times p}$, define

$$\|G\|_\infty = \sup_{\omega\in\mathcal{R}} \bar{\sigma}\left(G(j\omega)\right)$$

and

$$\|G\|_- = \inf_{\omega\in\mathcal{R}} \underline{\sigma}\left(G(j\omega)\right).$$

Note that for $\gamma > 0$,

$$\|G\|_\infty \leq \gamma \Longleftrightarrow \gamma^2 I - G^*(j\omega)G(j\omega) \succeq 0,\ \forall\, \omega \in \mathcal{R} \tag{1.1}$$

and

$$\|G\|_- \geq \gamma \Longleftrightarrow G^*(j\omega)G(j\omega) - \gamma^2 I \succeq 0,\ \forall\, \omega \in \mathcal{R}. \tag{1.2}$$

Using these definitions and the singular value inequality $\bar{\sigma}(AB) \geq \underline{\sigma}(A)\bar{\sigma}(B)$ [16], it is straightforward to show that

$$\|G_1G_2\|_\infty \geq \|G_1\|_-\|G_2\|_\infty \tag{1.3}$$

for stable G_1 and G_2 of compatible dimensions.

Remark 1.4.1. *The $\mathcal{H}_-$-norm $\|\bullet\|_-$ for FDI problems was initially introduced by Hou and Patton [44]. Note that $\|G\|_-$ is defined to characterise (1.2) and is not a real norm. However, the author follows convention in the literature and calls it a norm.* □

For $G(s) \stackrel{s}{=} (A, B, C, D) \in \mathcal{RL}_\infty^{m\times p}$, define

$$G^*(s) = G^T(-s) \stackrel{s}{=} (-A^T, -C^T, B^T, D^T)$$

to be the para–Hermitian complex conjugate transpose of $G(s)$. A square matrix function $G(s) \in \mathcal{RH}_\infty^{m\times m}$ is called inner if $G^*(s)G(s) = I_m$ and is called outer if $G^{-1}(s) \in \mathcal{RH}_\infty^{m\times m}$. If $G(s) \stackrel{s}{=} (A, B, C, D) \in \mathcal{R}(s)^{m\times p}$ then $G(s)$ is called co–outer if $m \geq p$ and if $G(s)$ has no zeros [7] in $\bar{\mathcal{C}}_+$, that is, if the matrix pencil

$$\begin{bmatrix} A - \mu I & B \\ C & D \end{bmatrix}$$

has full column rank for all $\mu \in \bar{C}_+$ (equivalently, if $G(s)$ has a left inverse in $\mathcal{RH}_\infty^{p\times m}$ [31]).

The following result considers outer-inner factorizations.

Lemma 1.4.1. *[23, 31] Any matrix transfer function $G(s) \in \mathcal{RH}_\infty^{m\times m}$ has an outer-inner factorization (OIF) $G(s) = G_o(s)G_i(s)$, where $G_i(s) \in \mathcal{RH}_\infty^{m\times m}$, satisfying $G_i^*(s)G_i(s) = I$, is called inner and $G_o(s) \in \mathcal{RH}_\infty^{m\times m}$, called outer, has full rank for every $s \in \bar{\mathcal{C}}_+$.*

The following result gives an LMI formulation for $\mathcal{H}_\infty$-norm based synthesis and design. It is a statement of the Bounded Real Lemma as derived in Theorem 2.41 in [75] for the strict inequality case.

Lemma 1.4.2. *[75] Let $G(s) = D + C(sI - A)^{-1}B$ be real–rational and let $\gamma > 0$. Then A is stable and $G^\sim(s)G(s) - \gamma^2 I < 0$ on the $j\omega$–axis if and only if there exists $P = P^T$ such that*

$$\begin{gathered} P \succ 0 \\ \begin{bmatrix} A^TP + PA + C^TC & PB + C^TD \\ (PB + C^TD)^T & D^TD - \gamma^2 I \end{bmatrix} \prec 0. \end{gathered} \tag{1.4}$$

The bounded real lemma has the following equivalent form.

Lemma 1.4.3. *[75, 76]. Suppose that $G(s) \stackrel{s}{=} (A, B, C, D)$ where $A \in \mathcal{R}^{n\times n}$ and let $\gamma > 0$ be given . Then A is stable and $\|G\|_\infty < \gamma$ if and only if there exists $P = P^T \in \mathcal{R}^{n\times n}$ such that*

$$\begin{gathered} P \succ 0 \\ \begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \prec 0. \end{gathered} \tag{1.5}$$

□

The following lemma can be found in [86]. It is widely used in control theory and in connection with trust region methods in optimization [84].

Lemma 1.4.1. *[39] Let $\mathbf{\Delta}$ be as described as follows.*

$$\begin{aligned}\mathbf{\Delta}: \;=\; & \{diag(\delta_1 I_{n_{\Delta_1}}, \ldots, \delta_l I_{n_{\Delta_l}}, \Delta_{l+1}, \ldots, \Delta_{l+r}) : \delta_i \in \mathcal{R}, |\delta_i| \leq 1, \\ & \Delta_{l+i} \in \mathcal{R}^{n_{\Delta_{l+i}} \times n_{\Delta_{l+i}}}, \|\Delta_{l+i}\| \leq 1\} \subset \mathcal{R}^{n_\Delta \times n_\Delta}.\end{aligned}$$

and define the subspaces

$$\begin{aligned}\mathbf{\Sigma} &= \{diag(S_1, \ldots, S_l, \lambda_1 I_{ql+1}, \ldots, \lambda_s I_{ql+f}) : S_i = S_i^T \in \mathcal{R}^{q_i \times q_i}, \lambda_j \in \mathcal{R}\}, \\ \mathbf{\Gamma} &= \{diag(G_1, \ldots, G_l, 0_{ql+1}, \ldots, 0_{ql+f}) : G_i = -G_i^T \in \mathcal{R}^{q_i \times q_i}\}.\end{aligned}$$

Let $R = R^T$, F, E, H be matrices with appropriate dimensions. We have $det(I - H\Delta) \neq 0$ and $R + F\Delta(I - H\Delta)^{-1}E + E^T(I - \Delta^T H^T)^{-1}\Delta^T F^T \prec 0$ for every $\Delta \in \mathbf{\Delta}$ if there exist $S \in \mathbf{\Sigma}$ and $G \in \mathbf{\Gamma}$ such that $S \succ 0$ and

$$\left[\begin{array}{cc} R + E^T S E & F + E^T S H + E^T G \\ F^T + H^T S E + G^T E & H^T S H + H^T G + G^T H - S \end{array} \right] \prec 0. \tag{1.6}$$

If $\mathbf{\Delta}$ is unstructured (that is, if $\mathbf{\Delta} = \{\Delta \in \mathcal{R}^{n_\Delta \times n_\Delta} : \|\Delta\| \leq 1\}$), then (1.6) becomes

$$\left[\begin{array}{cc} R + \lambda E^T E & F + \lambda E^T H \\ F^T + \lambda H^T E & \lambda(H^T H - I) \end{array} \right] \prec 0 \tag{1.7}$$

for some scalar $\lambda > 0$. In this case, condition (1.7) is both necessary and sufficient.

The finite frequency KYP lemma [47] is stated as follows.

Lemma 1.4.4. *Given matrices A, B and a Hermitian matrix Θ, the following finite frequency domain inequality*

$$\left[\begin{array}{c} (j\omega I - A)^{-1} B \\ I \end{array} \right]^* \Theta \left[\begin{array}{c} (j\omega I - A)^{-1} B \\ I \end{array} \right] \prec 0 \tag{1.8}$$

holds if and only if there exist hermitian matrix Π such that the following LMI

$$\left[\begin{array}{cc} A & B \\ I & 0 \end{array} \right]^* \Pi \left[\begin{array}{cc} A & B \\ I & 0 \end{array} \right] + \Theta \prec 0 \tag{1.9}$$

is satisfied, where Π as shown below are specified for the cases of low, middle and high frequency ranges.

Low frequency range $|\omega| \leq \omega_l$: $\Pi = \left[\begin{array}{cc} -Q & P \\ P & \omega_l^2 \end{array} \right]$,

Middle frequency range $\omega_1 \leq \omega \leq \omega_2$*:* $\Pi = \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix}$, $\omega_c = (\omega_1 + \omega_2)/2$,

High freqency range $|\omega| \geq \omega_h$*:* $\Pi = \begin{bmatrix} Q & P \\ P & -\omega_h^2 Q \end{bmatrix}$,

where $Q \succ 0$ *and* P *are Hermitian matrices and* ω_l*,* ω_1*,* ω_2 *and* ω_h *are all non-negative scalars.*

Chapter 2

Model-based Fault Detection

In the last two decades, the FD problem for dynamic systems has attracted great attention and various FD approaches have been proposed and applied in practice. Approaches which are based on mathematical models of the processes or systems under investigation are usually referred to as Model-based FD.

Generally, model-based FD schemes exploiting analytic redundancy use residuals to detect an occurring fault and indicate its type and location. This chapter gives an overview on the basic principles of model-based FD and illustrates different methods of generating residual signals. Particular attention is given to the methods related to the contributions in this thesis.

Section 2.1 firstly introduces the definition and key points of analytical redundancy. Then Section 2.2 illustrates the modelling of a system with disturbances and faults which are to be detected. Section 2.3 provides a basic introduction to the different methods of residual generation and Section 2.4 illustrates the issues of residual evaluation. Finally, Section 2.5 considers the robustness issue in model-based FD schemes.

2.1 Analytical redundancy

A traditional approach to FD in practical applications is based on hardware or physical redundancy methods which use multiple sets of sensors, actuators and components to measure and control a particular variable. Generally, a voting technique is applied to the hardware redundant system to determine whether a fault has occurred as well as its type and location among all the redundant system components. Obviously, the major disadvantage of hardware redundancy schemes is that they introduce the extra cost of maintenance and require additional space to accommodate the

hardware and equipment [45, 46].

In order to increase the reliability and safety and to avoid the additional cost of redundant hardware, the so-called analytical or functional redundancy approach are proposed. It employs the values of the variables from an analytical or mathematical model of the system to cross-check with the real measured values generated from the monitored system, that is, the consistency checking [64, 8, 33]. The resulting difference generated from the checking is called a residual signal. Ideally, the residual should be zero when the system is in normal operation and should be different from zero when a fault has occurred. This property of the residual is used to determine if there is an occurrence of a fault [8, 62].

The major advantage of analytical redundancy schemes is that no additional hardware components are required to implement an FD algorithm. Analytical redundancy can be implemented via software on a process control computer. In many cases, the measurements which are used to control the process are also sufficient for the FD algorithm so that no additional sensors have to be installed [45, 8, 50].

Generally, FD approaches frequently used in practice monitor the level of the residual signal and the occurrence of a fault is detected when the residual reaches a given level, that is, the threshold. This method is easy to implement but has obvious drawbacks. The most serious one is that the presence of disturbances, input variations and change of operating points of the monitored systems can cause false alarms.

Analytical redundancy employs a mathematical model of the monitored system, hence it is often referred as the model-based approach to FD.

2.2 Modelling of faulty systems

The first step in model-based approaches to FD is to provide a mathematical description of the system under investigation which represents all faulty scenarios. Firstly, consider a typical system

dynamics without faults given by

$$\dot{x}(t) = Ax(t) + B_u u_R(t),$$
$$y_R(t) = Cx(t) + D_u u_R(t),$$

where u_R and y_R are the effective inputs and outputs, that is, the post-actuator inputs and pre-sensor outputs. Concerning the locations of different faults, the system under investigation can be separated into the following different parts that can be affected by faults

- Actuators,
- Process or system components,
- Sensors.

The fault scenarios are illustrated in Figure 2.1.

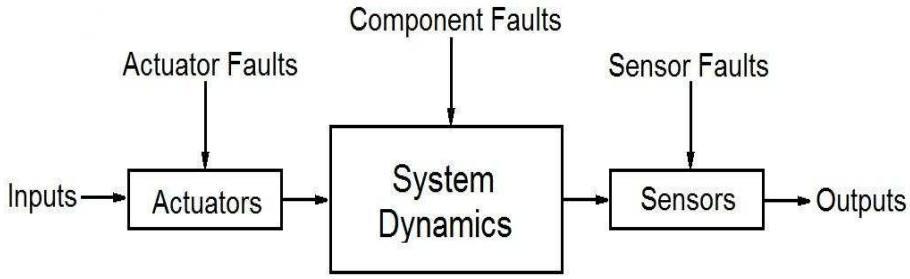


Figure 2.1: An open loop system with faults

Note that the control action is usually performed by a digital computer or controller while the actuator is considered as a part or a component of the whole system, therefore, in model-based FD schemes, faults are usually assumed to occur in actuators while the controller can be regarded as fault-free and neglected. By assuming linearity, the system dynamics can be described by the following time-invariant state-space form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u\hat{u}(t) + B_u f_a(t) + W_1 f_c(t), \\ y(t) &= Cx(t) + D_u\hat{u}(t) + W_2 f_y(t), \\ \hat{u}(t) &= u(t) + W_3 f_u(t) \end{aligned} \tag{2.1}$$

where A, B_u, C and D_u are system matrices with appropriate dimensions and $x(t) \in \mathcal{R}^n$ and $y(t) \in \mathcal{R}^{n_y}$ are the system state vector and output vector, respectively, while $u(t)$ is the input

signal driven by the actuators but not directly readable. f_a, f_c, f_u and f_y denote the actuator, component, input sensor and output sensor faults, respectively. The matrices W_1, W_2 and W_3 are known as the entry matrices of faults which describe how the faults enter the system [64].

By taking $f(t) = \begin{bmatrix} f_a^T & f_u^T & f_c^T & f_y^T \end{bmatrix}^T \in \mathcal{R}^{n_f}$, the state-space model of a faulty system in (2.1) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_f f(t), \\ y(t) &= Cx(t) + D_u u(t) + D_f f(t), \end{aligned}$$

where B_f and D_f are known as fault distribution matrices with appropriate dimensions. Since practical systems are always affected by disturbances, for generality, the state-space model of a linear time-invariant system with faults and disturbances can be described as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_f f(t) + B_d d(t), \\ y(t) &= Cx(t) + D_u u(t) + D_f f(t) + D_d d(t), \end{aligned} \tag{2.2}$$

where B_d and D_d are also known as the disturbance distribution matrices [64].

Since the state-space description provides a general and mathematically rigorous tool for system modelling and residual generation, it has been widely used in the design of FD schemes. Also, the input-output model of a system is frequently applied since it directly shows the influence of faults and disturbances on the system outputs.

By taking Laplace transforms, the system input/output behavior can be described as

$$y(s) = G_u(s)u(s) + G_d(s)d(s) + G_f(s)f(s), \tag{2.3}$$

where

$$\begin{aligned} G_u(s) &= D_u + C(sI - A)^{-1}B_u \in \mathcal{RL}_\infty^{n_y \times n_u}, \\ G_d(s) &= D_d + C(sI - A)^{-1}B_d \in \mathcal{RL}_\infty^{n_y \times n_d}, \end{aligned}$$

and

$$G_f(s) = D_f + C(sI - A)^{-1}B_f \in \mathcal{RL}_\infty^{n \times n_f}$$

are the process, disturbance and fault transfer matrices, respectively [64].

As shown above, general schemes of model-based FD use the model given in (2.2) and (2.3) and are based on the availability of the input and output signals $u(t)$ and $y(t)$, respectively.

2.3 Residual generation techniques

As mentioned in Section 2.1, model-based FD schemes employ residuals as fault indicators. Therefore, the main issue in model-based FD schemes is the generation of residual signals. Various approaches of designing residual generators have been proposed in the literature [37, 18, 20, 29, 36]. This section introduces the two most common approaches which are both widely studied and applied in academia and industry, namely, the parity space and observer-based approaches.

2.3.1 Residual generation via the parity space approach

The parity space approach applies to FD designs for linear discrete-time systems [101]. The basic idea is to carry out a proper check of the parity or consistency between the input and output measurements collected from the monitored system to obtain the redundancy. Then, a parity vector or matrix is introduced to cancel the effects of the system state and to generate the residual signals which are sensitive to faults and insensitive to disturbances and system errors.

In the early stage of research on parity-space-based FD, this approach was applied to static or parallel redundancy schemes, in which input and output measurements could be directly obtained from hardware (hardware redundancy) or analytical relations (analytical redundancy) [70, 8]. In the case of hardware redundancy, two methods can be used to obtain the redundant signals. One requires several sets of sensors with identical or similar functions to measure the same variable while the other approach employs different sets of sensor to measure different variables but with their outputs being related to each other. In the rest of this section, the attention is focused on analytical forms of redundancy.

In analytical schemes, parity space approaches are used to implement the consistency check based

on state-space models. Consider a linear discrete-time system

$$\begin{aligned} x_{k+1} &= Ax_k + B_d d_k + B_f f_k + B_u u_k, \\ y_k &= Cx_k + D_d d_k + D_f f_k + D_u u_k, \end{aligned} \tag{2.4}$$

where $x_k \in \mathcal{R}^n$, $u_k \in \mathcal{R}^{n_u}$ and $y_k \in \mathcal{R}^{n_y}$ are the state, input and output vectors, respectively, and $d_k \in \mathcal{R}^{n_d}$ and $f_k \in \mathcal{R}^{n_f}$ are the disturbance and fault vectors, respectively. Here, $B_f \in \mathcal{R}^{n \times n_f}$ and $D_f \in \mathcal{R}^{n_y \times n_f}$ are the component and instrument fault distribution matrices, respectively, while $B_d \in \mathcal{R}^{n \times n_d}$ and $D_d \in \mathcal{R}^{n_y \times n_d}$ are the corresponding disturbance distribution matrices [32, 35].

Let s be an integer denoting the length of a moving horizon $\left[k-s+1, \quad k\right]$. Input measurement vector u, disturbance vector d and fault vector f for the estimation horizon can then be expressed as

$$u(k) = \begin{bmatrix} u_{k-s+1} \\ u_{k-s+2} \\ \vdots \\ u_k \end{bmatrix}, \; y(k) = \begin{bmatrix} y_{k-s+1} \\ y_{k-s+2} \\ \vdots \\ y_k \end{bmatrix}, \; d(k) = \begin{bmatrix} d_{k-s+1} \\ d_{k-s+2} \\ \vdots \\ d_k \end{bmatrix}, \; f(k) = \begin{bmatrix} f_{k-s+1} \\ f_{k-s+2} \\ \vdots \\ f_k \end{bmatrix}.$$

Using an iterative computation, the output measurements over the horizon from $k-s+1$ to k can be expressed as

$$\begin{aligned} y(k) = &\overbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{s-1} \end{bmatrix}}^{\mathcal{C}_{y0}} x_{k-s+1} + \overbrace{\begin{bmatrix} D_u & 0 & \cdots & 0 \\ CB_u & D_u & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{s-2}B_u & CA^{s-3}B_u & \cdots & D_u \end{bmatrix}}^{\mathcal{D}_{yu}} u \\ &+ \overbrace{\begin{bmatrix} D_d & 0 & \cdots & 0 \\ CB_d & D_d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{s-2}B_d & CA^{s-3}B_d & \cdots & D_d \end{bmatrix}}^{\mathcal{D}_{yd}} d + \overbrace{\begin{bmatrix} D_f & 0 & \cdots & 0 \\ CB_f & D_f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{s-2}B_f & CA^{s-3}B_f & \cdots & D_f \end{bmatrix}}^{\mathcal{D}_{yf}} f \end{aligned} \tag{2.5}$$

In order to achieve FD effects, a parity (row) vector v_s is introduced so that the dynamics of the residual can be described by

$$r(k) = v_s \left[y(k) - \mathcal{D}_{yu} u(k) \right] \tag{2.6}$$

where the design parameter v_s is used to modulate the residual dynamics. Usually, it is required that $v_s\ \mathcal{C}_{y0} = 0$ to eliminate the influence of the initial state. In some contributions, a parity matrix V_s is introduced and designed to achieve FD instead of using a parity vector to improve the sensitivity of residuals to faults.

As shown above, the parity space approach transforms the FD problem to an optimization problem such that v_s is designed to increase the sensitivity of the residual to faults and the robustness to disturbances. Also, it can be seen that if and only if the condition

$$rank\left[\begin{array}{ccc}\mathcal{C}_{y0} & \mathcal{D}_{yd} & \mathcal{D}_{yf}\end{array}\right] > rank\left[\begin{array}{cc}\mathcal{C}_{y0} & \mathcal{D}_{yd}\end{array}\right] \tag{2.7}$$

is satisfied, then a full decoupling from both the initial state and disturbances can always be achieved by choosing a v_s, which means the following condition is satisfied

$$v_s\left[\begin{array}{cc}\mathcal{C}_{y0} & \mathcal{D}_{yd}\end{array}\right] = 0, \qquad v_s\mathcal{D}_{yf} \neq 0 \tag{2.8}$$

In other words, v_s should be chosen to lie in the intersection between the left null space of $\mathcal{C}_{y0}$ and $\mathcal{D}_{yd}$. However, a full-decoupling from the initial state and disturbances is usually not possible in the sense of achieving optimal FD performance. Therefore, the FD problem in parity-space-based schemes is often formulated as the following optimization problem

$$\max_{v_s\mathcal{C}_{y0}=0} g(v_s), \qquad g(v_s) = \frac{v_s\mathcal{D}_{yf}\mathcal{D}_{yf}^T v_s^T}{v_s\mathcal{D}_{yd}\mathcal{D}_{yd}^T v_s^T} \tag{2.9}$$

Generally, a high dimensional v_s will improve the FD performance index $J_s(v_s)$ at the expense of increasing the burden of on-line computational effort. Many contributions have been devoted to the design of the parity vector v_s in the literature. See [100, 103, 21, 110] for details.

Since parity-spaces-based FD design for linear discrete-time systems is characterized by simple mathematical computations and only employ relatively easy matrix or vector valued operations, then this approach attracts wide attention from industry for practical applications.

2.3.2 Residual generation via the observer-based approach

In the last two decades, observer-based FD for complex dynamic systems exploiting analytic redundancy have received increasing attention both in literature and applications [112, 49, 54, 58]. This approach involves the design of a dedicated observer to effectively cancel the nominal process dynamics and generate residual signals to provide fault signatures.

Basically, there are two kinds of observer-based FD approaches for achieving this objective, namely, perfect and approximate disturbance decoupling. In the former, the aim is to decouple the residual signal from disturbances exactly, while the latter requires the transfer matrix from disturbances to the residual signal to be small in either the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm sense. Both approaches have been widely studied and implemented using several methods [9, 65, 33].

Although FD schemes using perfect disturbance decoupling can achieve reasonable FD performance, its detection ability is often restricted and, in most cases of achieving perfect decoupling, solvability conditions are generically difficult to be satisfied. Therefore, much research effort and many contributions have been devoted to approximate the disturbance decoupling method [34]. In [49], a matrix factorization method is developed to obtain an optimal fault detection filter. Sadrnia utilized a Riccati equation iteration method to construct an extended $\mathcal{H}_\infty$ filter [72]. See [27, 113, 54, 48] for more details.

Generally, there are two ways of constructing observers to generate residual signals, that is, static and dynamic observers. The aim of both observers is not to estimate the state of the system, but rather to respond promptly to the occurrence of a fault via residual signals. In this thesis, we only use FD schemes based on static observers. In the following, the mathematical model of static observer-based FD is presented and the dynamics of residuals are formulated in a state-space form.

The idea of static observer-based FD is to use the duplication of the system to cancel the input system dynamics and use the degrees of freedom in the observer to optimize the dynamics of residuals for FD specifications. For the FD dynamics described in (2.2), the static FD observer is

described as:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + B_u u(t) - L\left(y(t) - C\hat{x}(t)\right), \\ r(t) &= H(y(t) - C\hat{x}(t)), \end{aligned} \tag{2.10}$$

assuming $D_u = 0$, where $\hat{x}(t) \in \mathcal{R}^n$ is the observer state and $r(t) \in \mathcal{R}^{n_f}$ is the residual signal. Here $L \in \mathcal{R}^{n \times n_y}$ and $H \in \mathcal{R}^{n_f \times n_y}$ are the observer and residual gain matrices, respectively, and are to be determined. The static FD observer which is employed in this thesis, is illustrated in Figure 2.2.

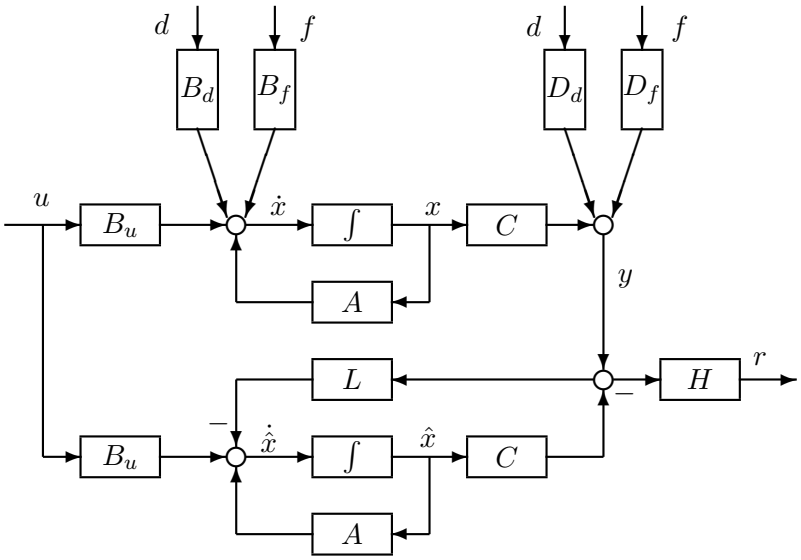


Figure 2.2: Residual generation using a static observer

Define the state estimation error signal as

$$e(t) = x(t) - \hat{x}(t).$$

It follows that the residual dynamics are given by

$$\begin{aligned} \dot{e}(t) &= (A + LC)e(t) + (B_d + LD_d)d(t) + (B_f + LD_f)f(t), \\ r(t) &= HCe(t) + HD_d d(t) + HD_f f(t). \end{aligned} \tag{2.11}$$

By taking Laplace transforms to 2.11, the closed-loop form of the residual dynamics is hence given by

$$r(s) = T_{rf}(s)f(s) + T_{rd}(s)d(s), \tag{2.12}$$

where

$$\left[\begin{array}{cc} T_{rd}(s) & T_{rf}(s) \end{array}\right] \stackrel{s}{=} \left[\begin{array}{c|cc} A+LC & B_d+LD_d & B_f+LD_f \\ \hline HC & HD_d & HD_f \end{array}\right] \in \mathcal{R}(s)^{n_f \times (n_d+n_f)}, \tag{2.13}$$

are the transfer matrices from faults and disturbances to residuals, respectively. Then a simple calculation shows that

$$r(s) = F(s)[G_d(s)d(s) + G_f(s)f(s)],$$

where

$$F(s) \stackrel{s}{=} \left[\begin{array}{c|c} A+LC & L \\ \hline HC & H \end{array}\right] \in \mathcal{R}(s)^{n_f \times n_y}, \tag{2.14}$$

so that

$$\left[\begin{array}{cc} T_{rd}(s) & T_{rf}(s) \end{array}\right] = F(s) \left[\begin{array}{cc} G_d(s) & G_f(s) \end{array}\right].$$

with G_d and G_f defined in Section 2.2. Figure 2.3 shows the structure of a general static observer–based FD residual generator in the frequency domain.

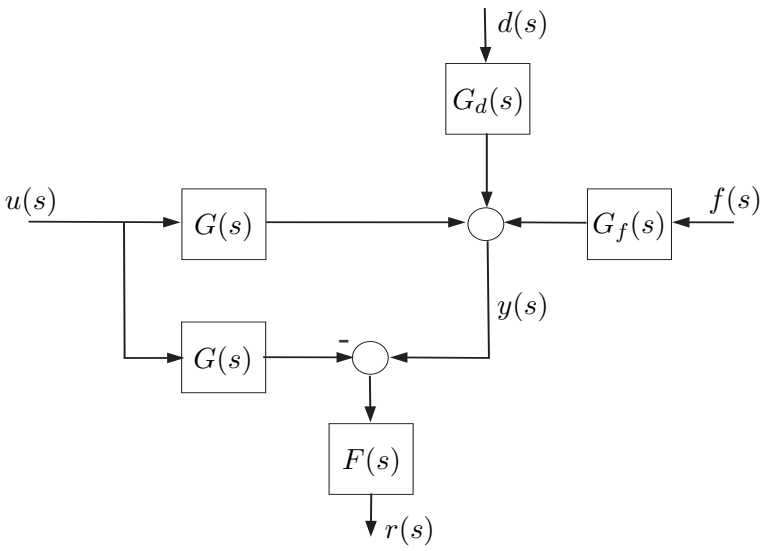


Figure 2.3: Observer-based FD in frequency domain

2.4 Residual evaluation

From the discussion above, the FD problem using a static observer is then transformed to solve a sensitivity optimization problem, which seeks to increase the sensitivity of the residual to faults and simultaneously to reduce the sensitivity to disturbances. In an ideal but not usual situation

[88] when, over the imaginary axis,

$$rank\ [G_f(s)\ \ G_d(s)] = rank\ G_f(s) + rank\ G_d(s)$$
$$rank\ G_f(s)\ =\ n_f\ =\ number\ of\ faults$$

faults can be completely decoupled from disturbances, which means that residuals are only sensitive to faults and completely insensitive to disturbances.

However, the complete elimination of the effects of the disturbances on residuals may not be possible due to the lack of degrees of freedom in the observer design. Hence, an appropriate criterion is needed to examine the residuals in order to determine whether any faults have occurred and where the faults are present.

Usually, the decision process for detecting a fault consists of a threshold test on a residual evaluation function. Define $J(r)$ to be the residual evaluation function, ideally, $J(r)$ should be zero for fault free cases and non-zero for faulty cases. However, due to the presence of disturbances and other uncertainties in the system, a perfect residual is almost impossible to obtain in practice. Therefore, for the aim of fault detection, the residual evaluation function is compared with a threshold J_{th} such that

$$J(r) < J_{th}\ for\ f(t) = 0$$
$$J(r) > J_{th}\ for\ f(t) \neq 0$$

which is illustrated in Figure 2.4.

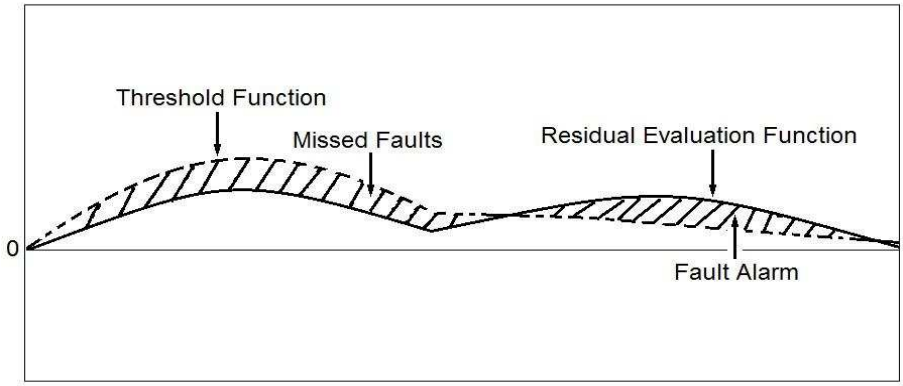


Figure 2.4: An illustration of the threshold function

In many cases [95, 94], the threshold is obtained by recording the value of residuals in the situation that no faults have occurred. For example, an on-line time-domain evaluation function [35] which is formed by $r(t)$ is given as follows

$$J(r) = \sqrt{\frac{1}{\tau}\int_0^\tau r^T(t)r(t)dt},$$

where τ is a user-defined time constant.

Many contributions [35, 59, 102] define thresholds using on-line or off-line functions which, generally, are the energy of the unknown signals (or disturbances) since it is reasonable to assume that bounds on the unknown inputs such as disturbances are available. The selection of thresholds often involves a trade-off between false alarm rates and missed faults so that the thresholding highly depends on the safety requirements on the plant and there are no general selection procedures which can be adopted universally. For example, in [93] the threshold is taken to be

$$J_{th} = \sqrt{\frac{1}{\tau}\int_0^\tau (FG_d d(t))^T FG_d d(t)dt}.$$

If $J(r) > J_{th}$ at some time interval, the occurrence of a fault is indicated.

An important requirement of residual generation is to achieve fault isolation, that is, to indicate the position of the fault corresponding to some sensor or component [81, 80, 14]. One approach to fault isolation [32, 98] is the so called dedicated observer scheme, which uses a bank of residual signals where each of the residuals is designed to be sensitive to a specific fault while insensitive to the other potential faults. The decision logic of residual evaluation is

$$J(r_i) < J_{th_i} \; for \; f_i(t) = 0$$
$$J(r_i) > J_{th_i} \; for \; f_i(t) \neq 0$$

In this thesis, time-domain based residual evaluation and thresholding for either fault detection or isolation is employed in the examples given in subsequent chapters.

2.5 Robustness in model-based FD

Although model-based approaches to FD have been accepted as an effective technique for detecting and even isolating faults, a critical problem of reliability due to unavoidable modeling

uncertainties is still a challenge that remains unresolved [24, 22, 25].

In practice, model uncertainties can be introduced by process noises, parameter variations and nonlinear dynamics. On the other hand, the mathematical models used in model-based FD schemes never perfectly describe the supervised system. The mismatch between the actual system and its mathematical model can also be regarded as model uncertainties. These uncertainties can cause serious technical difficulties and lead to false alarms or missed faults which severely deteriorate the performance and reliability of the FD designs.

To overcome the problem of uncertainties, it is necessary to consider the robustness issue in a model-based FD scheme. The objective is then to increase insensitivity to model uncertainties and disturbances while maintaining a certain level of sensitivity to faults. In the last decade, the robust FD problem has been widely studied by both academia and industry and a number of methods have been proposed. See [111, 61, 92, 53] for details.

The rest of this section describes the mathematical representation of model uncertainties which are usually considered in the literature. Basically, there are two kinds of well-defined model uncertainties, that is, norm-bounded and polytopic uncertainties. The former is firstly illustrated.

Let

$$\mathcal{M} = \begin{bmatrix} A & B & B_d & B_f \\ C & D & D_d & D_f \end{bmatrix}$$

denote the set of system matrices of an uncertain system. In the case of norm-bounded uncertainties, the Linear Fractional Transformation (LFT) representation of uncertainty [115] is described as follows. Here,

$$\mathcal{M} \in \{ \overbrace{\begin{bmatrix} A^o & B^o & B_d^o & B_f^o \\ C^o & D^o & D_d^o & D_f^o \end{bmatrix}}^{\mathcal{M}^o} + \begin{bmatrix} F_A \\ F_C \end{bmatrix} \Delta_H \begin{bmatrix} E_A & E_B & E_d & E_f \end{bmatrix} : \Delta \in \mathbf{\Delta} \} =: \boldsymbol{\mathcal{M}_{\mathbf{\Delta}}},$$

where $\mathcal{M}^o$ represents the nominal model, $\Delta_H = \Delta(I - H\Delta)^{-1}$, where

$$\begin{aligned} \mathbf{\Delta} := \{ \Delta = diag(\delta_1 I_{q1}, \ldots, \delta_l I_{ql}, \Delta_{l+1}, \ldots, \Delta_{l+f}) : \|\Delta\| \leq 1, \\ \delta_i \in \mathcal{R}, \Delta_i \in \mathcal{R}^{q_i \times q_i} \} \subset \mathcal{R}^{n_\Delta \times n_\Delta} \end{aligned} \tag{2.15}$$

and where $F_A, F_C, E_A, E_B, E_d, E_f$ and H are known and constant matrices with appropriate dimensions and where it is assumed that $\mathbf{\Delta}$ is well-posed, that is, $det(I - H\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}$. The LFT representation of norm-bounded uncertainties has great generality and is widely used in control theory [115]. The robustness issue of FD involved in this thesis mainly considers norm-bounded uncertainties.

Chapter 3

Online fault detection and isolation for linear discrete-time uncertain systems

In this chapter, a new on-line approach is proposed to solve robust FDI problems for linear discrete-time systems subject to faults, bounded additive disturbances and norm-bounded uncertainties [105, 106]. This scheme uses a dynamic system model as well as input/output measurements over a finite estimation horizon to compute upper and lower bounds on the fault signals at each sampling instant. Then a fault is regarded as having ocurred when its upper bound is smaller than zero or lower bound is larger than zero. LMI optimization techniques are employed to obtain the bounds. Furthermore, a subsequent-state-estimation technique together with an estimation horizon update procedure is proposed, which allows the on-line FDI process to be repeated in a moving horizon procedure.

One feature of the proposed approach is that the upper and lower bounds on the initial state, if available, can be used and the initial state is not necessarily decoupled. This is in contrast to the parity space approach by which the effect of the initial state has to be removed. Furthermore, this approach to the robust FDI problem is achieved by directly computing upper and lower bounds on the fault signals rather than designing a residual generator as in observer-based and parity-space-based schemes.

Moreover, it is shown that the bounds on the faults can be obtained by solving optimization problems subject to the constraints of the bounds on the initial state, disturbances and norm-bounded uncertainties by using the system model and input/output measurements over a finite estimation horizon. This approach can therefore be regarded as a dual to model predictive control (MPC)

schemes which use the system model to solve an on-line optimization problem to determine an optimal control strategy subject to hard constraints at each sampling instant, usually over a finite horizon [69, 60]. Also, the proposed approach reduces to the solution of a set of linear programs when there are no uncertainties in the model description.

This chapter is organized as follows. After defining the notation, Section 3.1 defines the problem settings. Then Section 3.2 gives the problem formulation and derives upper and lower bounds on the fault signals in the form of solutions to LMI problems. In Section 3.4, a subsequent-state-estimation as well as an estimation horizon update procedures are described, which complete the description of the FDI algorithm. Examples are given to demonstrate the effectiveness of the proposed scheme in Section 3.5. Finally, Section 3.6 summarizes this work.

3.1 FDI problem setting

Consider a linear time-invariant (LTI) discrete-time system subject to disturbances, process, actuator and sensor faults and uncertainties of the form

$$\left[\begin{array}{c} x_{k+1} \\ \hline y_k \end{array}\right] = \left[\begin{array}{c|ccc} A+\Delta A & B_d+\Delta B_d & B_f+\Delta B_f & B_u+\Delta B_u \\ \hline C_y+\Delta C_y & D_{yd}+\Delta D_{yd} & D_{yf}+\Delta D_{yf} & D_{yu}+\Delta D_{yu} \end{array}\right] \left[\begin{array}{c} x_k \\ \hline d_k \\ f_k \\ u_k \end{array}\right] \tag{3.1}$$

for $k \in \mathcal{N}$, where $\mathcal{N} := \{0, 1, \ldots, N-1\}$ is the estimation horizon, $x_k \in \mathcal{R}^n$, $u_k \in \mathcal{R}^{n_u}$ and $y_k \in \mathcal{R}^{n_y}$ are the state, input and output vectors, respectively, and $d_k \in \mathcal{R}^{n_d}$ and $f_k \in \mathcal{R}^{n_f}$ are the disturbance and fault vectors, respectively. Here, $B_f \in \mathcal{R}^{n \times n_f}$ and $D_f \in \mathcal{R}^{n_y \times n_f}$ are the component and instrument fault distribution matrices, respectively, while $B_d \in \mathcal{R}^{n \times n_d}$ and $D_d \in \mathcal{R}^{n_y \times n_d}$ are the corresponding disturbance distribution matrices [35].

It is assumed the norm-bounded uncertainty matrices in (3.1) are of form given by [61]:

$$\left[\begin{array}{cccc} \Delta A & \Delta B_d & \Delta B_f & \Delta B_u \\ \Delta C_y & \Delta D_{yd} & \Delta D_{yf} & \Delta D_{yu} \end{array}\right] = \left[\begin{array}{c} B_p \\ D_{yp} \end{array}\right] \Delta \left[\begin{array}{cccc} C_q & D_{qd} & D_{qf} & D_{qu} \end{array}\right],$$

where B_p, D_{yp}, C_q, D_{qd}, D_{qf} and D_{qu} are known and Δ is an $n_\Delta \times n_\Delta$ unknown matrix with $\Delta \in \mathbf{\Delta}$ where

$$\begin{aligned} \mathbf{\Delta} : \; = \; & \{diag(\delta_1 I_{n_{\Delta_1}}, \ldots, \delta_l I_{n_{\Delta_l}}, \Delta_{l+1}, \ldots, \Delta_{l+r}) : \delta_i \in \mathcal{R}, |\delta_i| \leq 1, i = 1, \ldots, l, \\ & \Delta_{l+j} \in \mathcal{R}^{n_{\Delta_{l+j}} \times n_{\Delta_{l+j}}}, \|\Delta_{l+j}\| \leq 1, j = 1, \ldots, r\} \subset \mathcal{R}^{n_\Delta \times n_\Delta}. \end{aligned} \tag{3.2}$$

Note that Δ has l repeated diagonal blocks and r full block diagonal blocks so that $\sum_{i=1}^{l+r} n_{\Delta_i} = n_\Delta$.

Remark 3.1.1. *Most contributions to the problem of robust FDI consider the uncertainty matrix without a specific structure, that it, $\Delta \in \mathcal{R}^{n_\Delta \times n_\Delta}$ [111, 92, 37, 73]. In this work, the structure of the uncertainty matrix has a more general form with both diagonal and full blocks, which is more suitable and flexible for dealing with specific forms of system uncertainties in practice.*

By introducing the new variables p_k and q_k, the state space equations (3.1) can be rewritten as:

$$\begin{aligned} \begin{bmatrix} x_{k+1} \\ y_k \\ q_k \end{bmatrix} &= \begin{bmatrix} A & B_d & B_f & B_u & B_p \\ C_y & D_{yd} & D_{yf} & D_{yu} & D_{yp} \\ C_q & D_{qd} & D_{qf} & D_{qu} & 0_{n_\Delta} \end{bmatrix} \begin{bmatrix} x_k \\ d_k \\ f_k \\ u_k \\ p_k \end{bmatrix}, \\ p_k &= \Delta q_k. \end{aligned} \tag{3.3}$$

Assume that upper and lower bounds $\bar{x}_0$, $\underline{x}_0$ and $\bar{d}_k$, $\underline{d}_k$ on the initial state and disturbances, respectively, are available and have of the form

$$\underline{x}_0 \leq x_0 \leq \bar{x}_0, \qquad \underline{d}_k \leq d_k \leq \bar{d}_k, \; \forall k \in \mathcal{N}.$$

Also assume that measurements of the input and output signals u_k and y_k are available for all $k \in \mathcal{N}$. For the estimation horizon, let

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \; \underline{d} = \begin{bmatrix} \underline{d}_0 \\ \underline{d}_1 \\ \vdots \\ \underline{d}_{N-1} \end{bmatrix}, \; \bar{d} = \begin{bmatrix} \bar{d}_0 \\ \bar{d}_1 \\ \vdots \\ \bar{d}_{N-1} \end{bmatrix}, \; \xi = \begin{bmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{N-1} \end{bmatrix} \in \mathcal{R}^{N_\xi} \tag{3.4}$$

where $N_\xi = N n_\xi$ with ξ standing for d, f, u and y. It is noted from (3.3) that

$$p = \Delta_a q, \qquad \Delta_a \in \mathbf{\Delta}_a := \{diag(\overbrace{\Delta, \ldots, \Delta}^{N \text{ terms}}) : \Delta \in \mathbf{\Delta}\} \subset \mathcal{R}^{N_\Delta \times N_\Delta} \tag{3.5}$$

where $N_\Delta = N \times n_\Delta$. Using an iterative computation, the system algebraic formulation over the estimation horizon is described as

$$\begin{bmatrix} x \\ y \\ q \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}_d & \mathcal{B}_f & \mathcal{B}_u & \mathcal{B}_p \\ \mathcal{C}_y & \mathcal{D}_{yd} & \mathcal{D}_{yf} & \mathcal{D}_{yu} & \mathcal{D}_{yp} \\ \mathcal{C}_q & \mathcal{D}_{qd} & \mathcal{D}_{qf} & \mathcal{D}_{qu} & \mathcal{D}_{qp} \end{bmatrix} \begin{bmatrix} x_0 \\ d \\ f \\ u \\ p \end{bmatrix}. \tag{3.6}$$

where

$$\mathcal{A}=\begin{bmatrix} I \\ A \\ \vdots \\ A^{N-1} \\ A^{N} \end{bmatrix},\quad \mathcal{C}_{y0}=\begin{bmatrix} C_y \\ C_yA \\ \vdots \\ C_yA^{N-2} \\ C_yA^{N-1} \end{bmatrix},\quad \mathcal{B}_{\zeta}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ B_{\zeta} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-2}B_{\zeta} & A^{N-3}B_{\zeta} & \cdots & 0 \\ A^{N-1}B_{\zeta} & A^{N-2}B_{\zeta} & \cdots & B_{\zeta} \end{bmatrix}$$

$$\mathcal{C}_{q0}=\begin{bmatrix} C_q \\ C_qA \\ \vdots \\ C_qA^{N-2} \\ C_qA^{N-1} \end{bmatrix},\quad \mathcal{D}_{q\zeta}=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ C_qB_{\zeta} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_qA^{N-3}B_{\zeta} & C_qA^{N-4}B_{\zeta} & \cdots & 0 & 0 \\ C_qA^{N-2}B_{\zeta} & C_qA^{N-3}B_{u} & \cdots & C_qB_{\zeta} & 0 \end{bmatrix},$$

$$\mathcal{D}_{y\zeta}=\begin{bmatrix} D_{y\zeta} & 0 & \cdots & 0 & 0 \\ C_yB_{\zeta} & D_{y\zeta} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_yA^{N-3}B_{\zeta} & C_yA^{N-4}B_{\zeta} & \cdots & D_{y\zeta} & 0 \\ C_yA^{N-2}B_{\zeta} & C_yA^{N-3}B_{\zeta} & \cdots & C_yB_{\zeta} & D_{y\zeta} \end{bmatrix},$$

and ζ stands for u, d, f and p.

To simplify the notation, define

$$\begin{aligned} w &= \begin{bmatrix} x_0 \\ d \end{bmatrix}, \qquad \underline{w}=\begin{bmatrix} \underline{x}_0 \\ \underline{d} \end{bmatrix}, \qquad \bar{w}=\begin{bmatrix} \bar{x}_0 \\ \bar{d} \end{bmatrix}, \\ \mathcal{D}_{yw} &= \begin{bmatrix} \mathcal{C}_{y0} & \mathcal{D}_{yd} \end{bmatrix}, \mathcal{D}_{qw}=\begin{bmatrix} \mathcal{C}_{q0} & \mathcal{D}_{qd} \end{bmatrix}, N_w=n+N_d. \end{aligned} \tag{3.7}$$

Applying some matrix manipulations to (3.5) and (3.6), the output over the estimation horizon is given as

$$y = (\mathcal{D}_{yw}+\Delta_{\mathcal{D}_{yw}})w+(\mathcal{D}_{yf}+\Delta_{\mathcal{D}_{yf}})f+(\mathcal{D}_{yu}+\Delta_{\mathcal{D}_{yu}})u \tag{3.8}$$

where $\underline{w} \le w \le \bar{w}$ and

$$\begin{bmatrix} \Delta_{\mathcal{D}_{yw}} & \Delta_{\mathcal{D}_{yf}} & \Delta_{\mathcal{D}_{yu}} \end{bmatrix} = \mathcal{D}_{yp}\Delta_a(I-\mathcal{D}_{qp}\Delta_a)^{-1}\begin{bmatrix} \mathcal{D}_{qw} & \mathcal{D}_{qf} & \mathcal{D}_{qu} \end{bmatrix},\quad \Delta_a \in \mathbf{\Delta}_a. \tag{3.9}$$

For notational convenience, assume that the uncertainty data are re-sequenced so that the uncertainty set has the form

$$\begin{aligned} \mathbf{\Delta}_a := \{&\mathrm{diag}(I_N\otimes\delta_1 I_{n_{\Delta_1}},\ldots,I_N\otimes\delta_l I_{n_{\Delta_l}},I_N\otimes\Delta_{l+1},\ldots,I_N\otimes\Delta_{l+r}) : \delta_i\in\mathcal{R}, \\ &|\delta_i|\le 1, i=1,\ldots,l, \Delta_{l+j}\in\mathcal{R}^{n_{\Delta_{l+j}}\times n_{\Delta_{l+j}}}, \|\Delta_{l+j}\|\le 1, j=1,\ldots,r\}\subset\mathcal{R}^{N_\Delta\times N_\Delta}. \end{aligned} \tag{3.10}$$

Note that this re-sequencing can be simply affected by an appropriate permutation of the variables. Note also that the uncertainty description involves repeated diagonal blocks where the repeated units are full blocks.

3.2 Fault signal bounds

As mentioned in the last section, FDI in this chapter is achieved by obtaining upper and lower bounds on the fault signals over the estimate horizon. This section gives the formulation of the problem of obtaining these upper and lower bounds. Then, an approach for computing these bounds is proposed by solving an LMI optimization. The cases for upper and lower bounds are dealt with separately.

Firstly, the FDI problem for linear discrete-time uncertain systems is described as follows.

Problem 3.2.1. *Let all data and variables be as given above. Then for $i=1,2,\ldots,N_f$ find*

$$\begin{array}{rcl} \bar{\phi}_i := & \max & e_i^T f \\ & \underline{w} \le w \le \bar{w} & \\ & y = (\mathcal{D}_{yw}+\Delta_{\mathcal{D}_{yw}})w+(\mathcal{D}_{yf}+\Delta_{\mathcal{D}_{yf}})f+(\mathcal{D}_{yu}+\Delta_{\mathcal{D}_{yu}})u & \\ & \Delta_{\mathcal{D}_{yf}},\ \Delta_{\mathcal{D}_{yu}},\ \Delta_{\mathcal{D}_{yw}} \text{ given by (3.9) and (3.10)} & \\ & y,\ u \text{ given} & \end{array} \tag{3.11}$$

$$\begin{array}{rcl} \underline{\phi}_i := & \min & e_i^T f \\ & \underline{w} \le w \le \bar{w} & \\ & y = (\mathcal{D}_{yw}+\Delta_{\mathcal{D}_{yw}})w+(\mathcal{D}_{yf}+\Delta_{\mathcal{D}_{yf}})f+(\mathcal{D}_{yu}+\Delta_{\mathcal{D}_{yu}})u & \\ & \Delta_{\mathcal{D}_{yf}},\ \Delta_{\mathcal{D}_{yu}},\ \Delta_{\mathcal{D}_{yw}} \text{ given by (3.9) and (3.10)} & \\ & y,\ u \text{ given} & \end{array} \tag{3.12}$$

where e_i is the ith column of I_{N_f}.

Note that the interval $[\underline{\phi}_i,\ \bar{\phi}_i]$ is our best estimate for $e_i^T f$ given all the information about the system.

Remark 3.2.1. *It is possible to combine our method and parity space methods by decoupling x_0 and then carrying out the on-line optimization proposed above, but nothing, as far as accuracy, can be gained since our solution is optimal (subject to relaxation error), however, it might reduce the on-line computation burden by performing offline computations.* □

The following theorem shows that evaluating an upper bound on each $e_i^T f$ can be obtained by solving an LMI problem.

Theorem 3.2.1. *Let all data be as defined previously. Let $\mathcal{D}^m$ denote the set of all real $m \times m$ diagonal matrices and define*

$$\mathcal{S} := \{\mathrm{diag}(S_1, \ldots, S_l, S_{l+1} \otimes I_{n_{\Delta_{l+1}}}, \ldots, S_{l+r} \otimes I_{n_{\Delta_{l+r}}}) : 0 \preceq S_i = S_i^T \in \mathcal{R}^{N_{\Delta_i} \times N_{\Delta_i}},$$
$$i = 1, \ldots, l, 0 \preceq S_{l+j} = S_{l+j}^T \in \mathcal{R}^{N \times N}, j = 1, \ldots, r\} \subset \mathcal{R}^{N_\Delta \times N_\Delta}$$
$$\mathcal{G} := \{diag(G_1, \ldots, G_l, 0_{N_{\Delta_{l+1}}}, \ldots, 0_{N_{\Delta_{l+r}}}) : G_i = -G_i^T \in \mathcal{R}^{N_{\Delta_i} \times N_{\Delta_i}}, i = 1, \ldots, l\} \subset \mathcal{R}^{N_\Delta \times N_\Delta}.$$

Then for $i = 1, \ldots, N_f$,

$$e_i^T f \leq \bar{\phi}_i \leq \bar{\bar{\phi}}_i := \min_{\substack{\mathcal{L}_i(\bar{f}_i, \bar{D}_i, \bar{\tau}_i, \bar{S}_i, \bar{G}_i) \succeq 0 \\ 0 \prec \bar{D}_i \in \mathcal{D}^{N_w} \\ \bar{S}_i \in \mathcal{S} \\ \bar{G}_i \in \mathcal{G} \\ \bar{\tau}_i \in \mathcal{R}}} \bar{f}_i \tag{3.13}$$

where

$$\mathcal{L}_i(\bar{f}_i, \bar{D}_i, \bar{\tau}_i, \bar{S}_i, \bar{G}_i) = \begin{bmatrix} \mathcal{L}_{11}^i & \star & \star & \star \\ \mathcal{L}_{21}^i & \mathcal{L}_{22}^i & \star & \star \\ \mathcal{L}_{31}^i & \mathcal{L}_{32}^i & \mathcal{L}_{33}^i & \star \\ \mathcal{L}_{41}^i & \mathcal{L}_{42}^i & \mathcal{L}_{43}^i & \mathcal{L}_{44}^i \end{bmatrix},$$

and

$$\begin{aligned}
\mathcal{L}_{11}^i &= \bar{D}_i - \bar{\tau}_i \mathcal{D}_{yw}^T \mathcal{D}_{yw} - \mathcal{D}_{qw}^T \bar{S}_i \mathcal{D}_{qw}, \\
\mathcal{L}_{21}^i &= -\bar{\tau}_i \mathcal{D}_{yf}^T \mathcal{D}_{yw} - \mathcal{D}_{qf}^T \bar{S}_i \mathcal{D}_{qw}, \\
\mathcal{L}_{22}^i &= -\bar{\tau}_i \mathcal{D}_{yf}^T \mathcal{D}_{yf} - \mathcal{D}_{qf}^T \bar{S}_i \mathcal{D}_{qf}, \\
\mathcal{L}_{31}^i &= -\bar{\tau}_i \mathcal{D}_{yp}^T \mathcal{D}_{yw} - \mathcal{D}_{qp}^T \bar{S}_i \mathcal{D}_{qw} - \bar{G}_i \mathcal{D}_{qw}, \\
\mathcal{L}_{32}^i &= -\bar{\tau}_i \mathcal{D}_{yp}^T \mathcal{D}_{yf} - \mathcal{D}_{qp}^T \bar{S}_i \mathcal{D}_{qf} - \bar{G}_i \mathcal{D}_{qf}, \\
\mathcal{L}_{33}^i &= -\bar{\tau}_i \mathcal{D}_{yp}^T \mathcal{D}_{yp} + \bar{S}_i - \mathcal{D}_{qp}^T \bar{S}_i \mathcal{D}_{qp} - \bar{G}_i \mathcal{D}_{qp} + \mathcal{D}_{qp}^T \bar{G}_i, \\
\mathcal{L}_{41}^i &= -\tfrac{1}{2}(\bar{w} + \underline{w})^T \bar{D}_i + \bar{\tau}_i (y - \mathcal{D}_{yu} u)^T \mathcal{D}_{yw} - u^T \mathcal{D}_{qu}^T \bar{S}_i \mathcal{D}_{qw}, \\
\mathcal{L}_{42}^i &= \bar{\tau}_i (y - \mathcal{D}_{yu} u)^T \mathcal{D}_{yf} - u^T \mathcal{D}_{qu}^T \bar{S}_i \mathcal{D}_{qf} - \tfrac{1}{2} e_i^T, \\
\mathcal{L}_{43}^i &= \bar{\tau}_i (y - \mathcal{D}_{yu} u)^T \mathcal{D}_{yp} - u^T \mathcal{D}_{qu}^T \bar{S}_i \mathcal{D}_{qp} + u^T \mathcal{D}_{qu}^T \bar{G}_i, \\
\mathcal{L}_{44}^i &= \bar{w}^T \bar{D}_i \underline{w} - \bar{\tau}_i (y - \mathcal{D}_{yu} u)^T (y - \mathcal{D}_{yu} u) - u^T \mathcal{D}_{qu}^T \bar{S}_i \mathcal{D}_{qu} u + \bar{f}_i,
\end{aligned} \tag{3.14}$$

and where the dependence of the $\mathcal{L}_{jk}^i$s on the variables is suppressed for convenience.

Proof. Take $p = \Delta_a (I - \mathcal{D}_{qp} \Delta_a)^{-1} \begin{bmatrix} \mathcal{D}_{qw} w & \mathcal{D}_{qf} f & \mathcal{D}_{qu} u \end{bmatrix}$, then (3.8) can be represented as the following equalities,

$$\begin{aligned}
y &= \mathcal{D}_{yw} w + \mathcal{D}_{yf} f + \mathcal{D}_{yu} u + \mathcal{D}_{yp} p, \\
p &= \Delta_a (\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p).
\end{aligned}$$

It follows that the problem formulation for obtaining the upper bound on f_i in (3.11) can be rewritten as

$$\begin{array}{rcc} \bar{\phi}_i = & \min & e_i^T f \\ & \underline{w} \leq w \leq \bar{w} & \\ & y = \mathcal{D}_{yw} w + \mathcal{D}_{yf} f + \mathcal{D}_{yu} u + \mathcal{D}_{yp} p & \\ & p = \Delta_a (\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p) & \\ & y,\ u\ given,\ \Delta_a \in \mathbf{\Delta}_a & \end{array} \tag{3.15}$$

Define $\mathcal{Q} := \{Q : Q\Delta_a = \Delta_a Q, \forall \Delta_a \in \mathbf{\Delta}_a\}$ which is the set of matrices which commute with all matrices in $\mathbf{\Delta}_a$. Note that since $q = \mathcal{D}_{yw} w + \mathcal{D}_{yf} f + \mathcal{D}_{yu} u + \mathcal{D}_{yp} p$ then $\bar{Q}_i p = \Delta_a \bar{Q}_i q$ for any $\bar{Q}_i \in \mathcal{Q}$. According to the specific structure of Δ_a given in (3.10), it can be verified that

$$\begin{aligned} \mathcal{Q} : \ = \ & \{\mathrm{diag}(Q_1, \ldots, Q_l, Q_{l+1} \otimes I_{n_{\Delta_{l+1}}}, \ldots, Q_{l+r} \otimes I_{n_{\Delta_{l+r}}}) : \\ & Q_i \in \mathcal{R}^{N_{\Delta_i} \times N_{\Delta_i}}, i = 1, \ldots, l \ \ Q_{l+j} \in \mathcal{R}^{N \times N}, j = 1, \ldots, r\} \subset \mathcal{R}^{N_\Delta \times N_\Delta}. \end{aligned}$$

Note that the constraints in (3.15) imply that $p^T \bar{Q}_i^T \bar{Q}_i p \leq q^T \bar{Q}_i^T \bar{Q}_i q, \forall \bar{Q}_i \in \mathcal{Q}$. Also, note that $\bar{G}_i$ is skew-symmetric with the structure as defined above, then a computation shows that $p^T \bar{G}_i q = 0$.

Then, a manipulation verifies that the identity

$$\begin{aligned} e_i^T f - \bar{f}_i \ = \ & -(\bar{w} - w)^T \bar{D}_i (w - \underline{w}) + p^T \bar{Q}_i^T \bar{Q}_i p \\ & -(\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p)^T \bar{Q}_i^T \bar{Q}_i (\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p) \\ & -\bar{\tau}_i (y - \mathcal{D}_{yw} w - \mathcal{D}_{yf} f - \mathcal{D}_{yu} u - \mathcal{D}_{yp} p)^T (y - \mathcal{D}_{yw} w - \mathcal{D}_{yf} f - \mathcal{D}_{yu} u - \mathcal{D}_{yp} p) \\ & -p^T \bar{G}_i (\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p) + (\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p)^T \bar{G}_i p \\ & - \begin{bmatrix} w^T & f^T & p^T & 1 \end{bmatrix} \mathcal{L}_i(\bar{f}_i, \bar{D}_i, \bar{\tau}_i, \bar{Q}_i, \bar{G}_i) \begin{bmatrix} w \\ f \\ p \\ 1 \end{bmatrix} \end{aligned} \tag{3.16}$$

is valid for all $\bar{f}_i \in \mathcal{R}$, $\bar{D}_i \in \mathcal{R}^{N_w \times N_w}$, $\bar{\tau}_i \in \mathcal{R}$, $\bar{Q}_i \in \mathcal{R}^{N_\Delta \times N_\Delta}$ and $\bar{G}_i \in \mathcal{R}^{N_\Delta \times N_\Delta}$. From (3.16), it can be seen that $\bar{f}_i$ is an upper bound on $e_i^T f$ if there exist $0 \preceq \bar{D}_i \in \mathcal{D}^{N_w}$, $\bar{\tau}_i \in \mathcal{R}$, $\bar{Q}_i \in \mathcal{Q}$ and $\bar{G}_i \in \mathcal{G}$ such that $\mathcal{L}_i(\bar{f}_i, \bar{D}_i, \bar{\tau}_i, \bar{Q}_i, \bar{G}_i) \succeq 0$. An inspection of the sets $\mathcal{Q}$ and $\mathcal{S}$ verifies that $\bar{Q}_i \in \mathcal{Q}$ if and only if $\bar{S}_i := \bar{Q}_i^T \bar{Q}_i \in \mathcal{S}$ and so an upper bound on $e_i^T f$ is given by solving the LMI optimization in (3.13). □

Remark 3.2.2. *In the case of unstructured uncertainty on the data, the uncertainty set in (3.2) is simply* $\mathbf{\Delta} = \mathcal{R}^{n_\Delta \times n_\Delta}$. *However, defining the variables in (3.4) over the horizon gives the uncertainty set over the horizon as*

$$\mathbf{\Delta}_a := \{I_N \otimes \Delta : \Delta \in \mathcal{R}^{n_\Delta \times n_\Delta}, \|\Delta\| \leq 1\}.$$

It follows that in Theorem 3.2.1 we have

$$\mathcal{S} = \{S \otimes I_{n_\Delta} : 0 \preceq S = S^T \in \mathcal{R}^{N \times N}\}, \qquad \mathcal{G} = 0.$$

It follows that even in the case of unstructured uncertainty description in the original problem data, the uncertainty description over the horizon is structured. Thus the uncertainty description in Theorem 3.2.1 for both structured and unstructured uncertainties is more general than the one usually used in the literature [111, 92, 37, 73]. □

Similarly, the next theorem derives lower bounds on the faults. The proof is omitted since it follows similar lines to that of Theorem 3.2.1.

Theorem 3.2.2. *Let all data and variables as defined in Theorem 3.2.1. Then for* $i = 1, \ldots, N_f$,

$$e_i^T f \geq \underline{\phi}_i \geq \underline{\underline{\phi}}_i := \max_{\substack{\mathcal{L}_i(\underline{f}_i, \underline{D}_i, \underline{\tau}_i, \underline{S}_i, \underline{G}_i) \preceq 0 \\ 0 \succeq \underline{D}_i \in \mathcal{D}^{N_w} \\ -\underline{S}_i \in \mathcal{S} \\ \underline{G}_i \in \mathcal{G} \\ \underline{\tau}_i \in \mathcal{R}}} \underline{f}_i \tag{3.17}$$

where $\mathcal{L}_i(\underline{f}_i, \underline{D}_i, \underline{\tau}_i, \underline{S}_i, \underline{G}_i)$ *is as defined in (3.14).*

Remark 3.2.3. *From Theorems 3.2.1 and 3.2.2, it can be seen that the upper and lower bounds on* $e_i^T f$ *are given in the form of the LMI optimizations (3.13) and (3.17), respectively. Moreover, since LMI problems are relatively easy to solve, it is possible to implement the proposed method on-line.* □

Remark 3.2.4. *Similar to the way of dealing with constraints in MPC, in our approach, the bounds on the disturbances are explicitly included in the optimization. This is in contrast to observer-based methods which deal with the energy transfer levels from disturbances/faults to residuals.* □

Remark 3.2.5. *It can be seen that our approach uses the same information as the parity space approach to achieve FD except that the approach in this work can exploit any available bounds on the initial state as a constraint rather than decouple the state completely.* □

Remark 3.2.6. *Since, by this proposed approach, lower and upper bounds on $e_i^T f$ are evaluated, then provided that the assumptions are satisfied, that is, the bounds on the initial state and disturbances are satisfied and the actual model falls within the description of the uncertainty, then the algorithm in this work is immune to false alarms. However, it is pointed out that the proposed approach may fail to identify a fault in the following sections.* □

3.3 Fault signal energy bounds

The previous section gives a solution to the discrete-time FDI problem by providing upper and lower bounds on each component of the fault signal at each time instant. In the case that the interest is only in fault detection, then it is only needed to decide whether the energy of fault signal $f^T f > 0$. This section gives the formulation of the problem of obtaining a lower bound on $f^T f$ over the estimation horizon, then an approach is proposed to evaluate the bound by using an LMI optimization. Then a fault is regarded as having occurred if the lower bound on $f^T f > 0$. Firstly, the problem is described as follows.

Problem 3.3.1. *Let all data and variables be as given above. Find*

$$\underline{\eta} := \begin{array}{c} \min \\ \underline{w} \le w \le \bar{w} \\ y = (\mathcal{D}_{yw}+\Delta_{\mathcal{D}_{yw}})w+(\mathcal{D}_{yf}+\Delta_{\mathcal{D}_{yf}})f+(\mathcal{D}_{yu}+\Delta_{\mathcal{D}_{yu}})u \\ \Delta_{\mathcal{D}_{yf}},\ \Delta_{\mathcal{D}_{yu}},\ \Delta_{\mathcal{D}_{yw}} \text{ given by (3.9) and (3.10)} \\ y,\ u \text{ given} \end{array} f^T f \tag{3.18}$$

The theorem below is proposed to obtain the lower bound on $f^T f$. Since the proof is similar to that of Theorem 3.2.1 it is omitted.

Theorem 3.3.1. *Let all data and variables as defined in Theorem 3.2.2. Then*

$$f^T f \ge \underline{\eta} \ge \underline{\underline{\eta}} := \begin{array}{c} \min \\ \mathcal{L}(\gamma, D, \tau, S, G) \preceq 0 \\ 0 \preceq D \in \mathcal{D}^{N_w} \\ -S \in \mathcal{S} \\ G \in \mathcal{G} \\ \tau \in \mathcal{R} \end{array} \gamma$$

where

$$\mathcal{L}(\gamma, D, T, S, G) = \begin{bmatrix} \mathcal{L}_{11} & \star & \star & \star \\ \mathcal{L}_{21} & \mathcal{M}_{22} & \star & \star \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} & \star \\ \mathcal{L}_{41} & \mathcal{M}_{42} & \mathcal{L}_{43} & \mathcal{L}_{44} \end{bmatrix}, \tag{3.19}$$

with

$\mathcal{M}_{22} = -\tau \mathcal{D}_{yf}^T \mathcal{D}_{yf} - \mathcal{D}_{qf}^T S \mathcal{D}_{qf} - I_{N_f}$,
$\mathcal{M}_{42} = \tau (y - \mathcal{D}_{yu} u)^T \mathcal{D}_{yf} - u^T \mathcal{D}_{qu}^T S \mathcal{D}_{qf}$,

where the entries denoted by $\mathcal{L}_{ij}$ are as defined in (3.14).

3.4 Subsequent state estimation

As shown in Section 3.1, the initial state x_0 is assumed to be in a given interval. After obtaining the bounds on each f_i within the estimation horizon from 0 to $N-1$, it is required to predict the interval of the next state x_1 to carry on the proposed method for the next horizon which is from 1 to N. In this section, a modified version of the method proposed in the last section is given to obtain upper and lower bounds on the subsequent state x_1.

Next, the problem of obtaining upper and lower bounds on x_1 is formulated, given all information during the estimation horizon.

Problem 3.4.1. *Let all data be as defined previously. Then for $i = 1, 2, \ldots, n$ find*

$$\bar{\chi}_i := \begin{array}{c} \max \\ \underline{w} \le w \le \bar{w} \\ y = (\mathcal{D}_{yw} + \Delta_{\mathcal{D}_{yw}}) w + (\mathcal{D}_{yf} + \Delta_{\mathcal{D}_{yf}}) f + (\mathcal{D}_{yu} + \Delta_{\mathcal{D}_{yu}}) u \\ \Delta_{\mathcal{D}_{yf}},\ \Delta_{\mathcal{D}_{yu}},\ \Delta_{\mathcal{D}_{yw}} \text{ given by (3.9) and (3.10)} \\ y,\ u \text{ given} \end{array} e_i^T x_1 \tag{3.20}$$

$$\underline{\chi}_i := \begin{array}{c} \min \\ \underline{w} \le w \le \bar{w} \\ y = (\mathcal{D}_{yw} + \Delta_{\mathcal{D}_{yw}}) w + (\mathcal{D}_{yf} + \Delta_{\mathcal{D}_{yf}}) f + (\mathcal{D}_{yu} + \Delta_{\mathcal{D}_{yu}}) u \\ \Delta_{\mathcal{D}_{yf}},\ \Delta_{\mathcal{D}_{yu}},\ \Delta_{\mathcal{D}_{yw}} \text{ given by (3.9) and (3.10)} \\ y,\ u \text{ given} \end{array} e_i^T x_1 \tag{3.21}$$

where e_i is the ith column of I_n.

Next, the upper and lower bounds on x_1 are obtained separately. Firstly, the upper bound on each element of x_1 can be evaluated by solving the LMI problem in the following theorem.

Theorem 3.4.1. *Let all data and variables as defined in Theorem 3.2.1. Then for $i = 1, \ldots, n$,*

$$e_i^T x_1 \le \bar{\chi}_i \le \bar{\bar{\chi}}_i := \begin{array}{c} \min \\ \mathcal{K}_i(\gamma, \bar{D}_i, \bar{\tau}_i, \bar{S}_i, \bar{G}_i) \succeq 0 \\ 0 \preceq \bar{D}_i \in \mathcal{D}^{N_w} \\ \bar{S}_i \in \mathcal{S} \\ \bar{G}_i \in \mathcal{G} \\ \bar{\tau}_i \in \mathcal{R} \end{array} \gamma$$

where

$$\mathcal{K}_i(\gamma, \bar{D}_i, \bar{\tau}_i, \bar{S}_i, \bar{G}_i) = \begin{bmatrix} \mathcal{L}_{11} & \star & \star & \star \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \star & \star \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} & \star \\ \mathcal{K}^i_{41} & \mathcal{K}^i_{42} & \mathcal{K}^i_{43} & \mathcal{K}^i_{44} \end{bmatrix}, \tag{3.22}$$

with

$$\begin{aligned}
\mathcal{K}^i_{41} &= -\tfrac{1}{2}(\bar{w} + \underline{w})^T \bar{D}_i + \bar{\tau}_i (y - \mathcal{D}_{yu} u)^T \mathcal{D}_{yw} - u^T \mathcal{D}_{qu}^T \bar{S}_i \mathcal{D}_{qw} - \tfrac{1}{2} e_i^T \begin{bmatrix} A & \mathcal{B}_d^u \mathcal{I}_d \end{bmatrix}, \\
\mathcal{K}^i_{42} &= \bar{\tau}_i (y - \mathcal{D}_{yu} u)^T \mathcal{D}_{yf} - u^T \mathcal{D}_{qu}^T \bar{S}_i \mathcal{D}_{qf} - \tfrac{1}{2} e_i^T \mathcal{B}_f^u \mathcal{I}_f, \\
\mathcal{K}^i_{43} &= \bar{\tau}_i (y - \mathcal{D}_{yu} u)^T \mathcal{D}_{yp} - u^T \mathcal{D}_{qu}^T \bar{S}_i \mathcal{D}_{qp} + u^T \mathcal{D}_{qu}^T \bar{G}_i - \tfrac{1}{2} e_i^T \mathcal{B}_p^u \mathcal{I}_p, \\
\mathcal{K}^i_{44} &= \bar{w}^T \bar{D}_i \underline{w} - \bar{\tau}_i (y - \mathcal{D}_{yu} u)^T (y - \mathcal{D}_{yu} u) - u^T \mathcal{D}_{qu}^T \bar{S}_i \mathcal{D}_{qu} u - e_i^T \mathcal{B}_u^u \mathcal{I}_u u + \gamma.
\end{aligned}$$

where $\mathcal{B}^u_\psi = \begin{bmatrix} B^u_\psi & AB^u_\psi & \dots & A^{N-2}B^u_\psi & A^{N-1}B^u_\psi \end{bmatrix}$, $\mathcal{I}_\psi = \begin{bmatrix} I_{n_\psi} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$

and where ψ *stands for* u, d, f *and* p. *Note that the entries denoted by* $\mathcal{L}_{ij}$ *are as defined in (3.14).*

Proof. Similar to (3.15), the problem formulation for obtain the upper bound on $e_i^T x_1$ can be rewritten as

$$\begin{aligned}
\bar{\chi}_i = \quad & \min_{\substack{\underline{w} \le w \le \bar{w} \\ y = \mathcal{D}_{yw} w + \mathcal{D}_{yf} f + \mathcal{D}_{yu} u + \mathcal{D}_{yp} p \\ p = \Delta_a (\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p) \\ y,\ u\ given,\ \Delta_a \in \mathbf{\Delta}_a}} e_i^T x_1
\end{aligned} \tag{3.23}$$

From (3.1), it can be seen that

$$e_i^T x_1 - \gamma \quad = \quad e_i^T A x_0 + e_i^T \mathcal{B}_d^u \mathcal{I}_d d + e_i^T \mathcal{B}_f^u \mathcal{I}_f f + e_i^T \mathcal{B}_u^u \mathcal{I}_u u + e_i^T \mathcal{B}_p^u \mathcal{I}_p p - \gamma$$

A manipulation verifies that the identity

$$\begin{aligned}
e_i^T x_1 - \gamma \ = \ & -(\bar{w} - w)^T \bar{D}_i (w - \underline{w}) + p^T \bar{Q}_i^T \bar{Q}_i p \\
& -(\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p)^T \bar{Q}_i^T \bar{Q}_i (\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p) \\
& -\bar{\tau}_i (y - \mathcal{D}_{yw} w - \mathcal{D}_{yf} f - \mathcal{D}_{yu} u - \mathcal{D}_{yp} p)^T (y - \mathcal{D}_{yw} w - \mathcal{D}_{yf} f - \mathcal{D}_{yu} u - \mathcal{D}_{yp} p) \\
& -p^T \bar{G}_i (\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p) + (\mathcal{D}_{qw} w + \mathcal{D}_{qf} f + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p)^T \bar{G}_i p \\
& - \begin{bmatrix} w^T & f^T & p^T & 1 \end{bmatrix} \mathcal{K}_i(\gamma, \bar{D}_i, \bar{\tau}_i, \bar{Q}_i, \bar{G}_i) \begin{bmatrix} w \\ f \\ p \\ 1 \end{bmatrix}
\end{aligned} \tag{3.24}$$

is valid for all $\gamma \in \mathcal{R}$, $\bar{D}_i \in \mathcal{R}^{N_w \times N_w}$, $\bar{\tau}_i \in \mathcal{R}$, $\bar{Q}_i \in \mathcal{R}^{N_\Delta \times N_\Delta}$ and $\bar{G}_i \in \mathcal{R}^{N_\Delta \times N_\Delta}$. From (3.24), it can be seen that γ is an upper bound on $e_i^T x_1$ if there exist $0 \preceq \bar{D}_i \in \mathcal{D}^{N_w}$, $\bar{Q}_i \in \mathcal{Q}$ and $\bar{G}_i \in \mathcal{G}$ such that $\mathcal{K}_i(\gamma, \bar{D}_i, \bar{\tau}_i, \bar{Q}_i, \bar{G}_i) \succeq 0$. It can be verified that $\bar{Q}_i \in \mathcal{Q}$ if and only if $\bar{S}_i := \bar{Q}_i^T \bar{Q}_i \in \mathcal{S}$ and therefore an upper bound on $e_i^T x_1$ is given by solving the LMI optimization in (3.23). □

Similarly, it is proposed the following theorem which solves the lower bound on each element of x_1.

Theorem 3.4.2. *Let all data and variables as defined in Theorem 3.2.2. Then for $i = 1, \ldots, n$,*

$$e_i^T x_1 \geq \underline{\chi}_i \geq \underline{\underline{\chi}}_i := \begin{array}{c} \max \\ \mathcal{K}_i(\gamma, \underline{D}_i, \underline{\tau}_i, \underline{S}_i, \underline{G}_i) \preceq 0 \\ 0 \succeq \underline{D}_i \in \mathcal{D}^{N_w} \\ -\underline{S}_i \in \mathcal{S} \\ \underline{G}_i \in \mathcal{G} \\ \underline{\tau}_i \in \mathcal{R} \end{array} \gamma$$

where $\mathcal{K}_i(\gamma, \underline{D}_i, \underline{\tau}_i, \underline{S}_i, \underline{G}_i)$ is as defined in (3.22).

The proof is omitted since it can be obtained just by following the lines in the proof of Theorem 3.4.1.

Remark 3.4.1. *Note that there is no observer or residual generator design involved in the proposed approach, and FDI is achieved by directly computing the upper and lower bounds on the fault signal. This implies that FDI could be implemented on-line based on the input and output data, system model and uncertainty description.* □

Remark 3.4.2. *Observer-based FD methods generate residuals which seek to indicate faults for all plants within the uncertainty set [111, 61], while the approach in this work is proposed for the actual system with its particular uncertainty since it is implemented by using on-line input and output data of the system. It follows that our approach can in fact be used to identify the uncertainty of a system model. This is described in more detail in Section 6.2.5.* □

Remark 3.4.3. *Since the optimization problem to obtain the bounds is solved on-line, it could be computationally expensive, hence our approach can only be used to monitor sufficiently slow systems. To some extent, this can be ameliorated by varying the horizon length but at the expense of precision.* □

As illustrated above, the upper and lower bounds on the fault signal and subsequent state can simply be obtained by solving LMI problems. Next, a procedure is given to illustrate the steps by which the proposed scheme is applied to real systems.

Algorithm 3.4.1.

1. *Set* $i = 0$.
2. *Obtain the input and output data* u_i *and* y_i.
3. *Set* $i = i + 1$, *go to step 2 until* $i = N - 1$.
4. *Compute the upper and lower bounds on* $f_0, \ldots, f_{N-1}$.
5. *Set* $j = 1$.
6. *Compute upper and lower bounds on* x_j.
7. *Obtain* u_{j+N-1}, *and* y_{j+N-1}, *compute upper and lower bounds on* $f_j, f_{j+1}, \ldots, f_{j+N-1}$, *and* x_{j+1}.
8. *Set* $j = j + 1$ *and go to Step 6.*

Note that steps 1 and 2 only involve data collections: the computations start after the estimation horizon N is reached. If fault estimate before the horizon is reached are required, the algorithm can be easily modified so that the faults can be estimated using an increasing horizon length.

By applying the procedure, the upper and lower bounds on faults can be computed over the moving horizon. A fault is regarded as having occurred at some sampling instant when its upper and lower bounds obtained are both either greater or smaller than zero.

Remark 3.4.4. *In some sense, the threshold for FDI in this work is implicitly built into the algorithm since the upper and lower bounds on the fault signal are computed. The need for threshold in the proposed approach only arises if the actual system and disturbances fall outside our model description.* □

Remark 3.4.5. *On the other hand, if the upper bound on* $e_i^T f$ *is greater than zero and the lower bound is less than zero, then there is insufficient information to determine whether a fault has occurred and faults may be missed, although the availability of the lower and upper bounds on the faults may still be useful.* □

Remark 3.4.6. *Although the procedure presented above is for linear time-invariant systems, they can be easily modified for linear time-varying systems.* □

Remark 3.4.7. *Similarly, the scheme proposed in this work can be used to solve FDI problems for linear discrete-time systems without model uncertainties. Optimal upper and lower bounds on the fault signal and the subsequent state can be easily obtained by solving the following Linear Programming problems:*

$$\begin{array}{rcl} \bar{\phi}_i/\underline{\phi}_i = & \max / \min & e_i^T f \\ & \underline{x}_0 \leq x_0 \leq \bar{x}_0 & \\ & \underline{d} \leq d \leq \bar{d} & \\ & y = \mathcal{C}_{y0}x_0 + \mathcal{D}_{yd}d + \mathcal{D}_{yf}f + \mathcal{D}_{yu}u & \\ & y,\ u\ given & \\ \bar{\chi}_i/\underline{\chi}_i = & \max / \min & e_i^T x_1 \\ & \underline{x}_0 \leq x_0 \leq \bar{x}_0 & \\ & \underline{d} \leq d \leq \bar{d} & \\ & y = \mathcal{C}_{y0}x_0 + \mathcal{D}_{yd}d + \mathcal{D}_{yf}f + \mathcal{D}_{yu}u & \\ & y,\ u\ given & \end{array} \tag{3.25}$$

where $x_1 = Ax_0 + \mathcal{B}_d^u\mathcal{I}_d d + \mathcal{B}_u^u\mathcal{I}_u u + \mathcal{B}_f^u\mathcal{I}_f f$*, while the problem of obtaining a lower bound on the fault energy is solved by solving the convex Quadratic Programming problem*

$$\begin{array}{rcl} \underline{\eta} := & \min & f^T f \\ & \underline{x}_0 \leq x_0 \leq \bar{x}_0 & \\ & \underline{d} \leq d \leq \bar{d} & \\ & y = \mathcal{C}_{y0}x_0 + \mathcal{D}_{yd}d + \mathcal{D}_{yf}f + \mathcal{D}_{yu}u & \\ & y,\ u\ given & \end{array}$$

On-line FDI can be implemented by following the procedure given above. □

Remark 3.4.8. *Many observer-based FDI schemes [49, 59, 53] are based on the assumption that* D_f *or* D_d *has full column rank, which could be restrictive in some circumstances. In this work, this rank assumption is not necessary for applying the proposed FDI approach. However, loss of column rank on* D_f *could make the upper and lower bounds on the fault signal at the last sampling instant of each horizon not reliable.* □

3.5 Numerical examples

In this section, two examples are given to illustrate the effectiveness of the proposed scheme.

3.5.1 Example 1

Firstly, consider a discrete-time model (with no uncertainties) of a vertical takeoff and landing aircraft in the vertical plane used in [90], where the states are the horizontal velocity (knot), vertical

velocity (knot), pitch rate (degree/second) and pitch angle (degree), respectively. The actuator inputs are collective pitch control and longitudinal cyclic pitch control, respectively. The parameters of the linearized model are given as

$$
\begin{aligned}
A &= \begin{bmatrix} 0.9828 & 0.0083 & -0.0454 & -0.2461 \\ 0.0117 & 0.5813 & -0.3898 & -1.6662 \\ 0.0458 & 0.1274 & 0.8230 & 0.4803 \\ 0.0117 & 0.0358 & 0.4433 & 1.1361 \end{bmatrix}, \\
B_u &= \begin{bmatrix} 0.2666 & 0.0365 \\ 1.7629 & -3.2664 \\ -2.3152 & 1.7209 \\ -0.6083 & 0.4660 \end{bmatrix}, B_f = \begin{bmatrix} 0.2666 & 0.0365 \\ 1.7629 & -3.2664 \\ -2.3152 & 1.7209 \\ -0.6083 & 0.4660 \end{bmatrix}, \\
C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \qquad B_d = \begin{bmatrix} -0.0069 & 0.0026 \\ -0.0688 & 0.3896 \\ 0.4433 & 0.0358 \\ 0.1144 & 0.0063 \end{bmatrix}, \\
D_d &= \begin{bmatrix} 0 & 0.2 \\ 0 & 0.1 \\ 0.3 & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_u = 0_{4\times 2}, \qquad D_f = 0_{4\times 2}.
\end{aligned}
$$

The same faults and disturbances are used as given in [90]: $f(k) = \begin{bmatrix} 1.2 & 0.8 \end{bmatrix}^T$, for $k \geq 20$ and $f(k) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ elsewhere. Also, $d(k) = \begin{bmatrix} 0.5\cos(k)e^{-0.05k} & 0.5\sin(k) \end{bmatrix}^T$ for all k and take the horizon length $N = 10$. Generally, FDI schemes give better results as the number of outputs increases. However, to demonstrate the effectiveness of the proposed method, the system model is modified by removing the first output. Applying the LP optimization given in (3.25), the following figures are obtained by using the last three outputs showing the upper and lower bounds on the fault signal from sampling instants 12 to 21 and 15 to 24, respectively.

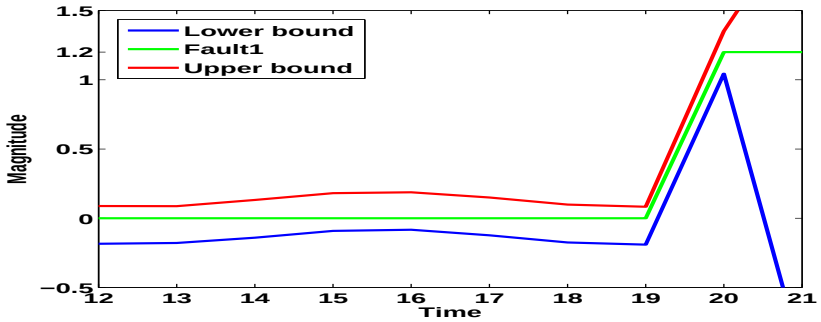


Figure 3.1: Fault 1 from the instant 12 to 21 using three outputs

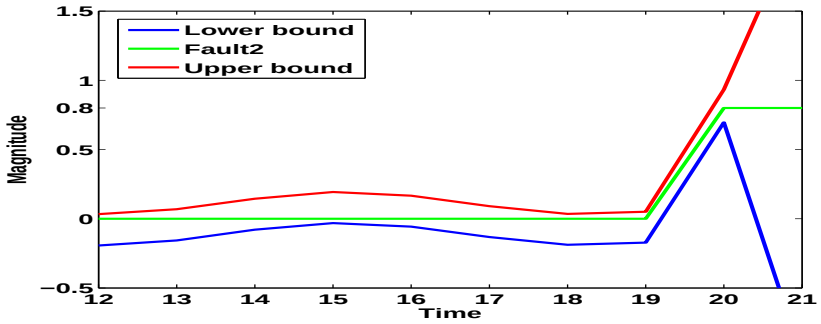


Figure 3.2: Fault 2 from the instant 12 to 21 using three outputs

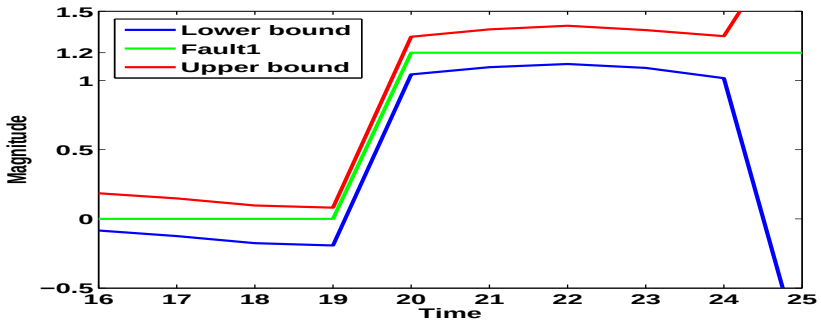


Figure 3.3: Fault 1 from the instant 16 to 25 using three outputs

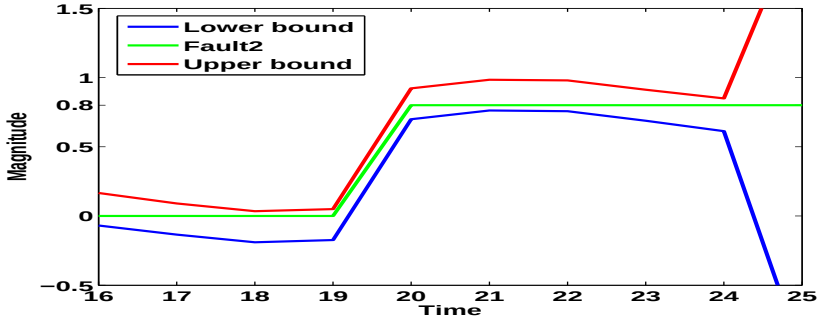


Figure 3.4: Fault 2 from the instant 16 to 25 using three outputs

From the graphs shown in Figures 3.1, 3.3, 3.2 and 3.4 and the values of the bounds, it can be seen at the 21st sampling instant that both the upper and lower bounds were greater than zero at the 20th sampling instant, which indicates fault occurrence in the system. The band between the upper and lower bounds on the fault signal is narrow and the fault is well-located. Note that since, in this example $D_f = 0$, the bounds on the faults at the last sampling instant of each estimation horizon are not reliable. Also, using Theorem 3.4.1 and 3.4.2, it can be shown that state x_{16} is

within the lower and upper bound estimates given by (3.25):

$$\underline{x}_{16} = \begin{bmatrix} -0.9649 & 0.7665 & -0.1595 & -0.3022 \end{bmatrix}^T$$

$$x_{16} = \begin{bmatrix} 0.2508 & 0.8626 & -0.1283 & -0.2579 \end{bmatrix}^T$$

$$\overline{x}_{16} = \begin{bmatrix} 1.4179 & 0.8627 & -0.0081 & -0.2064 \end{bmatrix}^T$$

Furthermore, the proposed method can give even better results when it is applied using all four outputs as in [90]. Figures 3.5−3.8 illustrate the FDI results with the same faults and disturbances. It shows that both the upper and lower bounds are tight so that the faults are accurately detected at the 21st sampling instant.

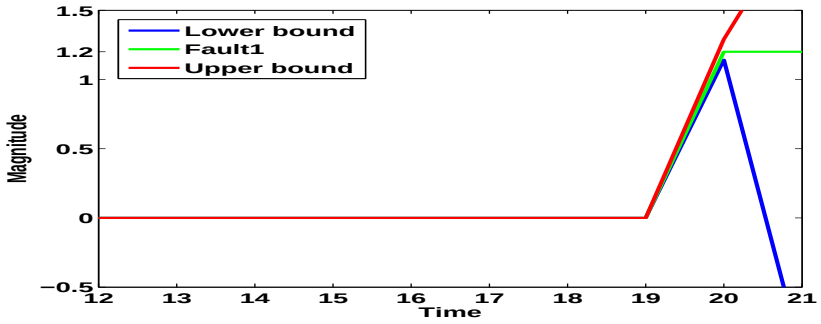


Figure 3.5: Fault 1 from the instant 12 to 21 using all outputs

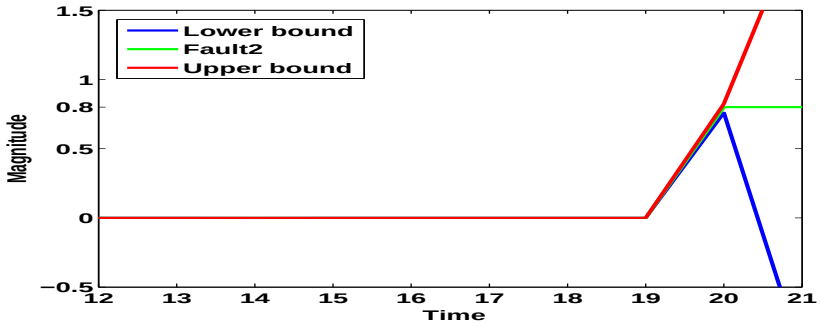


Figure 3.6: Fault 2 from the instant 12 to 21 using all outputs

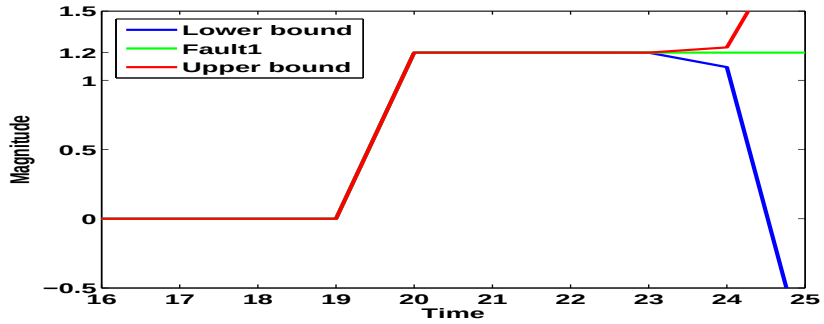


Figure 3.7: Fault 1 from the instant 16 to 25 using all outputs

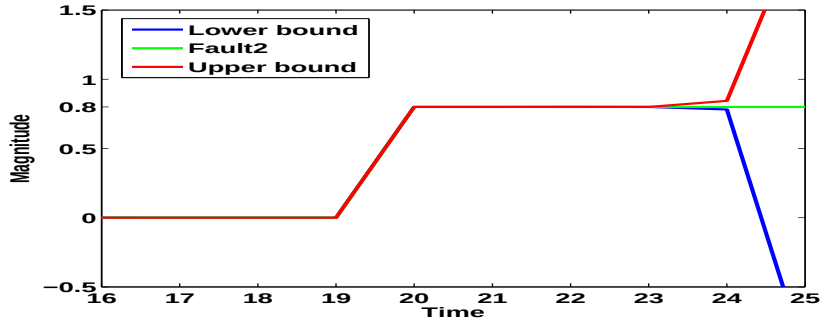


Figure 3.8: Fault 2 from the instant 16 to 25 using all outputs

3.5.2 Example 2

To demonstrate the effectiveness of the proposed robust FDI scheme in a real system with uncertainties, a jet engine example is considered in this section. By using zero order hold equivalent method [2] with a sample period $T = 0.1$s, the discrete-time state-space model of the GE-21 jet engine [48, 67, 89] is given as follows.

$$
\begin{aligned}
A &= \begin{bmatrix} 0.7119 & 0.1257 \\ -0.0250 & 0.8252 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0.0525 & -0.1127 & 0.1072 \\ 0.0365 & 0.1037 & 0.0109 \end{bmatrix}, \\
B_d &= \begin{bmatrix} -0.0252 & 0.0027 \\ 0.0013 & -0.0166 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0.0525 & -0.1127 \\ 0.0365 & 0.1037 \end{bmatrix}, \\
C_y &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.731 & 0.786 \end{bmatrix}, \quad D_{yu} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.267 & -0.025 & -0.146 \end{bmatrix}, \\
D_{yd} &= \begin{bmatrix} -0.0008 & 0.0006 \\ 0.0008 & 0.0007 \\ -0.0005 & -0.0008 \end{bmatrix}, D_{yf} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Since the example in [67, 89] does not involve disturbances, the matrices B_d and D_{yd} have been

assumed as above. Two fault signals are considered in this example. $f_1(k)$ and $f_2(k)$ are simulated as positive and negative step signals with magnitude 1 which are connected from $k = 7$ and 9, respectively. The disturbances are assumed to be given as $d(k) = \begin{bmatrix} 0.5\cos(k)e^{-0.05k} & 0.5\sin(k) \end{bmatrix}^T$ and the inputs are taken as unit step signals starting from the initial sampling instant. Also, assume that the upper and lower bounds on the initial state x_0 and disturbances are 0.5 and -0.5, respectively, and take the estimation horizon length $N = 5$.

Assuming no uncertainties, by applying the algorithm given in (3.25), upper and lower bounds on the fault signals are obtained as follows.

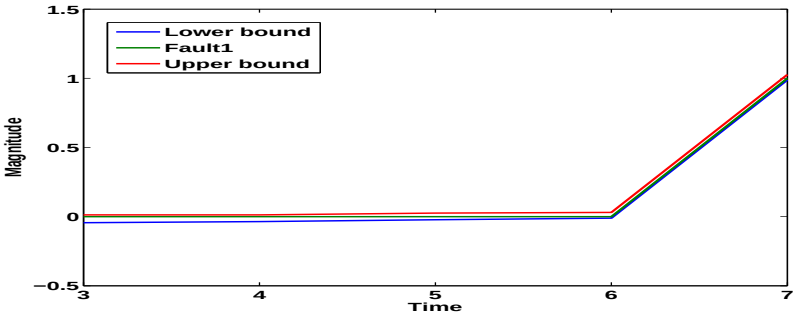


Figure 3.9: Fault 1 from the instant 3 to 7 without uncertainties

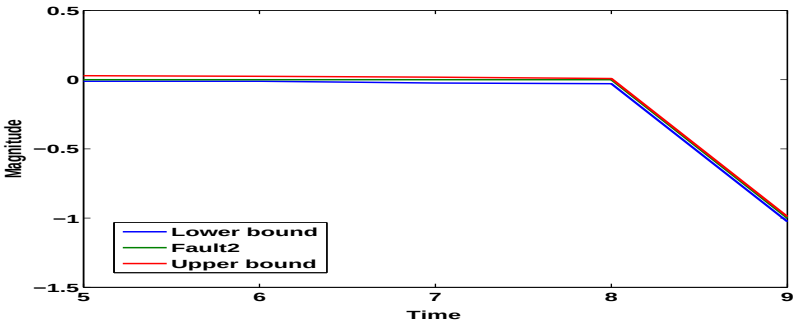


Figure 3.10: Fault 2 from the instant 5 to 9 without uncertainties

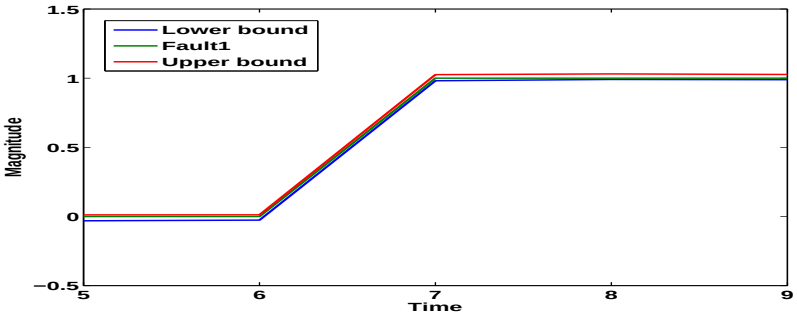


Figure 3.11: Fault 1 from the instant 5 to 9 without uncertainties

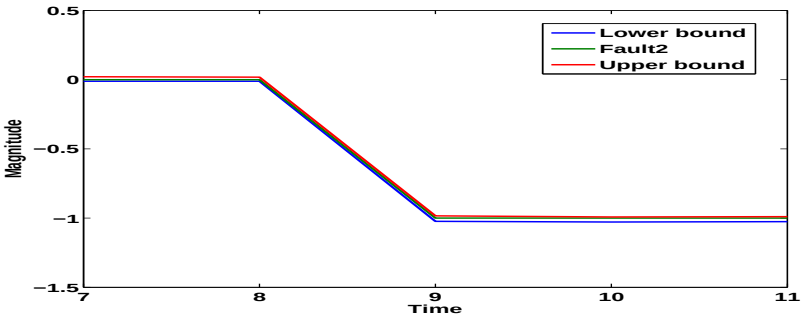


Figure 3.12: Fault 2 from the instant 7 to 11 without uncertainties

According to the decision logic of the proposed FDI scheme, it can be concluded that there are two faults connected from the 7th and 9th sampling instant, respectively. Moreover, as can be seen from Figure 3.9−3.12, the band between the upper and lower bounds on each fault signal is narrow and its corresponding fault is well-located in it.

In order to illustrate the robustness of the FDI approached given in Theorems 3.2.1 and 3.2.2, assume that uncertainty parameters are introduced into the same system model. Here consider an unstructured norm-bounded uncertainty with the parameters as follows.

$$
\begin{aligned}
\Delta &= \begin{bmatrix} 0.8063 & -0.8364 \\ 0.1002 & -0.6527 \end{bmatrix}, B_p = \begin{bmatrix} -0.0330 & -0.0211 \\ -0.0520 & 0.0570 \end{bmatrix}, \\
C_q &= \begin{bmatrix} -0.020 & 0.026 \\ -0.029 & -0.032 \end{bmatrix}, \quad D_{qu} = \begin{bmatrix} 0.0124 & -0.0233 & 0.0106 \\ -0.063 & -0.0123 & -0.011 \end{bmatrix}, \\
D_{qf} &= \begin{bmatrix} 0.038 & 0.011 \\ 0.0084 & -0.05 \end{bmatrix}, \quad D_{qd} = \begin{bmatrix} -0.015 & -0.016 \\ 0.011 & 0.023 \end{bmatrix} \\
D_p &= \begin{bmatrix} -0.1012 & 0.0550 \\ 0.0231 & -0.1100 \\ -0.0501 & 0.0282 \end{bmatrix}.
\end{aligned}
$$

By solving the LMIs in (3.14) and (3.17), the figures and values of the bounds are obtained as follows. It can be concluded that Fault 1 and 2 are well detected and isolated at the 7th and 9th sampling instants, respectively. Note that since there exist uncertainties in the system model, the bounds shown in Figure 3.13−3.16 are less tight than the ones shown above when no uncertainties are considered.

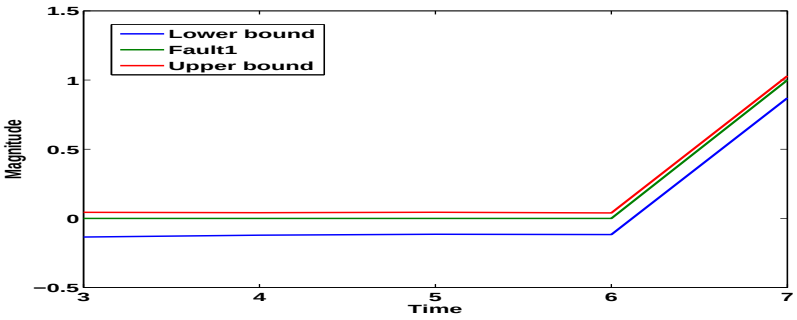


Figure 3.13: Fault 1 from the instant 3 to 7 with uncertainties

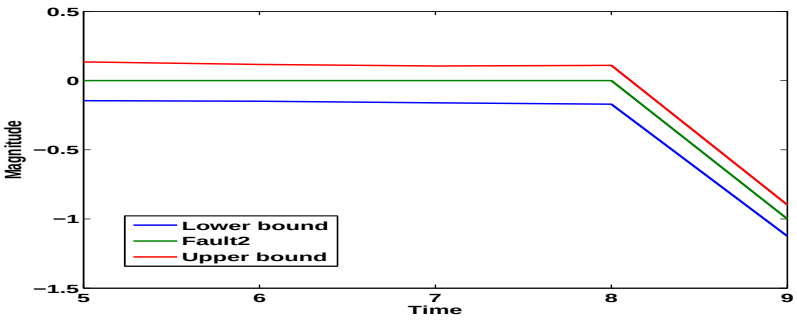


Figure 3.14: Fault 2 from the instant 5 to 9 with uncertainties

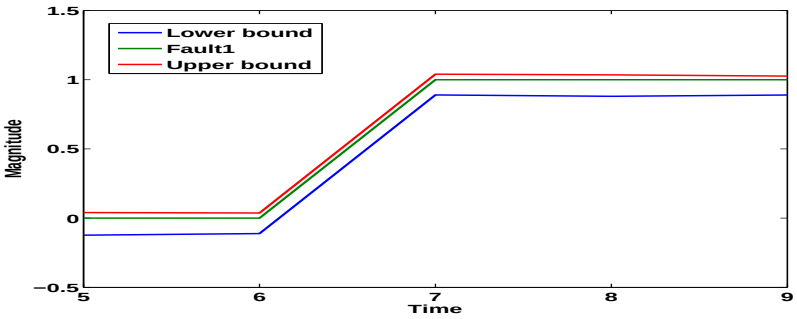


Figure 3.15: Fault 1 from the instant 5 to 9 with uncertainties

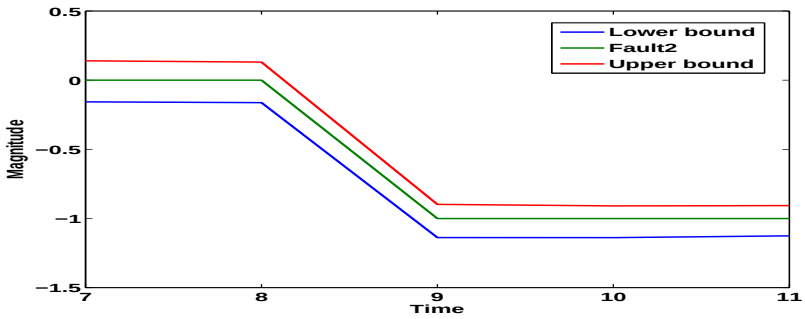


Figure 3.16: Fault 2 from the instant 7 to 11 with uncertainties

Also, using Theorem 3.4.1 and 3.4.2, the lower and upper bounds on the subsequent state x_{21} are solved and given as follows. It shows that x_{21} is well-estimated and within its lower and upper bounds.

$$\underline{x}_{21} = \begin{bmatrix} 0.5592 & 0.9404 \end{bmatrix}^T$$
$$x_{21} = \begin{bmatrix} 0.6050 & 1.0182 \end{bmatrix}^T$$
$$\bar{x}_{21} = \begin{bmatrix} 0.6812 & 1.0622 \end{bmatrix}^T$$

It is noted that, although the band between the upper and lower bounds is narrow, there is still the possibility that the fault is missed if its magnitude is very small. The following simulation shows that when there is only an abrupt pulse fault at 7s with value 0.1, the lower and upper bound on Fault 1 is negative and positive, respectively at 7s. Hence, no conclusion can be drawn on the result of FDI according to Theorem 3.2.1 and 3.2.2.

In this case, the method in Theorem 3.3.1 is used such that the objective is only to achieve fault detection. By solving (3.19), the lower bounds on $f^T f$ over the estimation horizon from 3 to 7 is 0.0044 which is larger than zero. Therefore, a conclusion can be drawn that a fault occurred during the estimation horizon from 3s to 7s. Thus Theorem 3.3.1 provides a better reliability for fault detection or, in other words, fault detection is relatively easier to achieve than fault isolation.

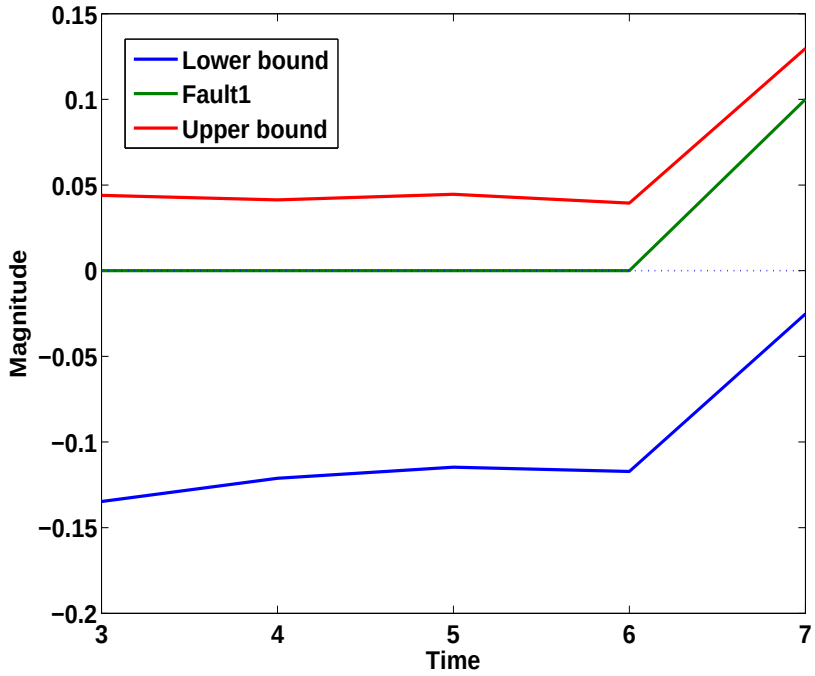


Figure 3.17: Fault 1 from the instant 3 to 7

3.6 Concluding remarks

This chapter considers the FDI problem for linear discrete-time uncertain systems subject to faults, bounded additive disturbances and norm-bounded uncertainties. A novel approach is proposed to solve the FDI problems by using a system model, together with input/output measurements over a finite estimation horizon. First, the proposed method is used to compute upper and lower bounds on the fault signals on-line. Then a fault is regarded as detected and isolated if, at some sampling instant, its upper bound is lower than zero or the lower bound is larger than zero. LMI optimization techniques are used to obtain the bounds. Similarly, FD can be achieved by solving the lower bounds on the fault energy $f^T f$.

Then, a subsequent-state-estimation technique together with an estimation horizon update procedure are given to allow the on-line FDI process to be repeated in a moving horizon procedure. Also, the proposed approach reduces to the solution of a set of linear programs when no model uncertainties are considered.

In this work, the bounds on faults are obtained by solving optimization problems subject to the constraints of the bounds on the initial state, disturbances and norm-bounded uncertainties by using system model and input/output measurements over a finite estimation horizon. Therefore, it can be regarded as a dual to model predictive control (MPC) schemes which use the system model to solve an on-line optimization problem to determine an optimal control strategy subject to hard constraints at each sampling instant, usually over a finite horizon.

One advantage of the proposed approach over the parity space approaches is that in this work the upper and lower bounds on the initial state, if available, can be used and the initial state is not necessarily decoupled, which can give more information in computing the bounds on faults. Furthermore, by applying the proposed approach, FDI is achieved by computing upper and lower bounds on the actual fault signals rather than producing a residual signal. So that the occurrence of a fault is detected if its upper bound is lower than zero or the lower bound is larger than zero. Hence, this approach is free of false alarms.

Chapter 4

An optimal solution to a robust $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem

As introduced in Chapter 2, the observer-based FD problem is essentially a sensitivity optimization problem which is to reduce the sensitivity of the residual to disturbances and increase the sensitivity to faults. Recently, mixed-norm FD problems have attracted a great deal of attention in the literature [57] since these schemes increase the insensitivity to disturbances while maintain a certain level of sensitivity to faults to ensure the effectiveness of FD. Various schemes have been proposed in many contributions.

In [93], an $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem is investigated and a suboptimal solution is given, which is due to the fact that a bilinear matrix inequality formulation is employed so that some variables are needed to be pre-defined to solve the optimization problem. Wang *et al* [91] consider the $\mathcal{H}_\infty/\mathcal{H}_\infty$ FD problem and give a solution which is suboptimal due to the fact that a common Lyapunov variable is used to solve the matrix inequalities, which results in a conservative FD observer design. In [87], both $\mathcal{H}_\infty/\mathcal{H}_\infty$ and $\mathcal{H}_-/\mathcal{H}_\infty$ FD problems are considered and a suboptimal solution is given since the FD problem is formulated as a quasi-linear matrix inequality problem that necessitates an iterative method to approximate the optimal solution. In [49, 57, 54] an optimal solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ is presented. However, the solution depends on the availability of an exact model of the system and therefore may not have good robustness properties against model uncertainties.

This work gives an optimal solution to the robust $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem for linear time-invariant systems subject to faults and disturbances. First, a new performance index is developed to capture the requirements of fault detection and disturbance rejection which is particularly suitable for

dealing with model uncertainties. The disturbance rejection performance is measured, using the $\mathcal{H}_\infty$ norm, by the size of the disturbance to residual dynamics. Using a static observer, a class of solutions to a novel FD problem is given in the form of a simple LMI with two degrees of freedom. Then, the degrees of freedom in this class of solutions are used to derive the optimal filter to the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem. Moreover, it is shown that, under certain conditions, fault isolation can also be achieved. Finally, it is shown that, with the use of weighting filters, the results can be improved and certain, commonly held, rank assumptions can be relaxed.

The structure of this chapter is organized as follows. A novel $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem is formulated in Section 4.1. Section 4.2 gives a class of solutions to the FD problem. In Section 4.3, the optimal filter design for the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem is given and further specifications are investigated. Section 4.4.1 gives a technique to improve the results through the use of weighting filters, while Section 4.4.2 considers the removal of certain common rank assumptions on the fault dynamics. Section 4.5 extends the scheme to uncertain systems. Finally, numerical examples are presented in Section 4.6 to validate the proposed approaches, and Section 4.7 summarizes the results in this chapter.

4.1 Fault detection problem formulation

Consider a linear time-invariant dynamic system subject to disturbances, modeling errors and process, sensor and actuator faults modeled as

$$\begin{aligned} \dot{x}(t) &= Ax(t)+B_u u(t)+B_f f(t)+B_d d(t), \\ y(t) &= Cx(t)+D_u u(t)+D_f f(t)+D_d d(t), \end{aligned} \tag{4.1}$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^{n_u}$ and $y(t) \in \mathcal{R}^{n_y}$ are the process state, input and output vectors, respectively, and where $f(t) \in \mathcal{R}^{n_f}$ and $d(t) \in \mathcal{R}^{n_d}$ are the fault and disturbance vectors, respectively. Here, $B_f \in \mathcal{R}^{n\times n_f}$ and $D_f \in \mathcal{R}^{n_y\times n_f}$ are the component and instrument fault distribution matrices, respectively, while $B_d \in \mathcal{R}^{n\times n_d}$ and $D_d \in \mathcal{R}^{n_y\times n_d}$ are the corresponding disturbance distribution matrices [35]. As mentioned in Chapter 2, a residual signal in an FD system represents the inconsistency between the system variables and the mathematical model. Here, consider a residual generator using a static observer of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t)+B_u u(t)-L\left(y(t)-C\hat{x}(t)-D_u u(t)\right), \\ r(t) &= H(y(t)-C\hat{x}(t)-D_u u(t)), \end{aligned} \tag{4.2}$$

where $\hat{x}(t) \in \mathcal{R}^n$ is the observer state and $r(t) \in \mathcal{R}^{n_r}$ is the residual signal. Here $L \in \mathcal{R}^{n \times n_y}$ and $H \in \mathcal{R}^{n_r \times n_y}$ are the observer and residual gain matrices, respectively, and are to be determined. Define the state estimation error signal as $e(t) = x(t) - \hat{x}(t)$. It follows that the residual dynamics are given by

$$\begin{aligned} \dot{e}(t) &= (A+LC)e(t) + (B_f+LD_f)f(t) + (B_d+LD_d)d(t), \\ r(t) &= HCe(t) + HD_f f(t) + HD_d d(t). \end{aligned}$$

By taking Laplace transforms, $r(s) = T_{rf}(s)f(s) + T_{rd}(s)d(s)$, where

$$\begin{aligned} \left[\begin{array}{cc} T_{rf}(s) & T_{rd}(s) \end{array}\right] &\stackrel{s}{=} \left[\begin{array}{c|cc} A+LC & B_f+LD_f & B_d+LD_d \\ \hline HC & HD_f & HD_d \end{array}\right] \\ &= F(s)\left[\begin{array}{cc} G_f(s) & G_d(s) \end{array}\right] \end{aligned} \tag{4.3}$$

are the transfer matrices from faults and disturbances to the residual, respectively, and where

$$F(s) \stackrel{s}{=} \left[\begin{array}{c|c} A+LC & L \\ \hline HC & H \end{array}\right], \quad \left[\begin{array}{cc} G_f(s) & G_d(s) \end{array}\right] \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_f & B_d \\ \hline C & D_f & D_d \end{array}\right] \tag{4.4}$$

are the FD filter and the fault and disturbance dynamics, respectively.

Remark 4.1.1. *Greater flexibility can be introduced to the design of the FD schemes based on residual generation by considering a dynamic observer. However, this causes a couple of difficulties. See comments in Section 6.2.2.* □

With the preliminaries above, the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem under investigation is formulated as follows.

Problem 4.1.1. *With all variables as defined above, assume that:*

A1. The pair (A, C) is detectable (see Remark 4.1.3.)

A2. $G_f(s)$ has full column rank over the extended imaginary axis (see Remarks 4.1.4 and Section 4.4.)

A3. $n_y \geq n_f$ (see Remark 4.1.6.)

A4. $n_r = n_f$ (see Remark 4.1.7.)

Find

$$\gamma_0 := \min \left\{ \frac{\|T_{rd}\|_\infty}{\|T_{rf}\|_-} : A + LC \text{ is stable} \right\}.$$

Remark 4.1.2. *An interpretation of Problem 4.1.1 is that $\bar{\sigma}(T_{rd}(j\omega))$ is required to be smaller than or equal to $\gamma_0\underline{\sigma}(T_{rf}(j\omega))$ for all $\omega \in \mathcal{R}$. This captures the requirement of ensuring insensitivity to the disturbance and sensitivity to faults.* □

Since

$$\left\|T_{rf}^{-1}T_{rd}\right\|_\infty \leq \left\|T_{rf}^{-1}\right\|_\infty \|T_{rd}\|_\infty = \|T_{rd}\|_\infty / \|T_{rf}\|_-,$$

then,

$$\gamma_0 \geq \min\ \{\left\|T_{rf}^{-1}T_{rd}\right\|_\infty : A + LC \text{ is stable}\} =: \gamma_1. \tag{4.5}$$

It follows that γ_1 is a lower bound on γ_0, which motivates the following problem:

Problem 4.1.2. *Let all variables and assumptions be as given in Problem 4.1.1. For a given $\gamma > 0$ find L and H, if they exist, such that $A + LC$ is stable and*

$$\left\|T_{rf}^{-1}T_{rd}\right\|_\infty < \gamma. \tag{4.6}$$

Remark 4.1.3. *Assumption A1 is needed to guarantee the existence of at least one L such that $A + LC$ is stable.* □

Remark 4.1.4. *The cost function in (4.6) can be expressed as $T_{rf}^{-1}(s)T_{rd}(s)T_{rd}^*(s)T_{rf}^{-*}(s) \prec \gamma^2 I$ or, equivalently, as*

$$T_{rd}(s)T_{rd}^*(s) \prec \gamma^2 T_{rf}(s)T_{rf}^*(s), \tag{4.7}$$

If $G_f(s)$ loses rank over the extended imaginary axis, then so does $T_{rf}(s) = F(s)G_f(s)$ and (4.7) cannot be satisfied for any $\gamma > 0$. In particular, the assumption implies that D_f has full column rank which is necessary for $T_{rf}^{-1}(s)$ to have a proper state-space realization. This assumption can be (somewhat) relaxed if the cost function in (4.7) is changed to $T_{rd}(s)T_{rd}^*(s) - \gamma^2 T_{rf}(s)T_{rf}^*(s) \preceq 0$ and if $G_d(s)$ loses rank over the imaginary axis whenever $G_f(s)$ does. Dealing with this situation will necessitate an intricate analysis of spectral factorizations with imaginary axis zeros [13], which is outside the scope of this work. In Section 4.4, this assumption is relaxed by modifying the cost function in (4.6) and introducing weighting functions. □

Remark 4.1.5. *An alternative approach in the case that $G_f(s)$ loses rank over the external imaginary axis is to design the FD scheme over a specific finite frequency range inside which $G_f(s)$ has full rank [90, 92, 58], although the solution involves non-convex optimization.* □

Remark 4.1.6. *Suppose that $n_y < n_f$, then assumptions A1 and A2, together with Theorem 13.32 in [114] imply that $G_f(s) = \hat{G}_f(s)G_i(s)$ where $\hat{G}_f(s) \in \mathcal{RL}_\infty^{n_y \times n_y}$ and $G_i(s) \in \mathcal{RL}_\infty^{n_y \times n_f}$ such that $G_i(s)G_i^*(s) = I_{n_y}$. Furthermore, $\hat{G}_f(s)$ can be chosen to have the same A and C matrices as $G_f(s)$. Using (4.4) and (4.7) it gives*

$$\begin{aligned} T_{rd}(s)T_{rd}^*(s) \prec \gamma^2 T_{rf}(s)T_{rf}^*(s) \quad &\Leftrightarrow \quad F(s)G_d(s)G_d^*(s)F^*(s) \prec \gamma^2 F(s)G_f(s)G_f^*(s)F^*(s) \\ &\Leftrightarrow \quad F(s)G_d(s)G_d^*(s)F^*(s) \prec \gamma^2 F(s)\hat{G}_f(s)\hat{G}_f^*(s)F^*(s). \end{aligned}$$

Thus $G_f(s)$ may be replaced by $\hat{G}_f(s)$ and so there is no loss of generality in assuming that $n_f \leq n_y$. □

Remark 4.1.7. *$n_r = n_f$ is chosen for the following reasons:*

1. *There is no need for $n_r > n_f$ since the interest of this work is in increasing the sensitivity of the residual to faults by increasing the singular values of $T_{rf}(s)$ relative to those of $T_{rd}(s)$, and there are at most n_f singular values of $T_{rf}(s)$ since $n_y \geq n_f$.*

2. *Since the set of all optimal filters when $n_r < n_f$ is a subset of those when $n_r = n_f$, maximal sensitivity to faults is obtained when $n_r = n_f$.*

3. *This will prove useful when fault isolation is considered in Section 4.3 below.* □

4.2 A class of optimal FD filters for Problem 4.1.2

In this section, a set of (not necessarily stable) filters of the form given in (4.4) is characterized such that (4.6) is satisfied. Then, the approach is to use the bounded real lemma to derive necessary and sufficient conditions for (4.6) in the form of a (nonlinear) matrix inequality. Then, a change of variables is introduced to derive an equivalent LMI.

Since $D_f \in \mathcal{R}^{n_y \times n_f}$ has full column rank, there exist $D_f^\dagger \in \mathcal{R}^{n_f \times n_y}$ and $D_f^\perp \in \mathcal{R}^{(n_y - n_f) \times n_y}$ such that

$$\begin{bmatrix} D_f^\dagger \\ D_f^\perp \end{bmatrix} D_f = \begin{bmatrix} I_{n_f} \\ 0 \end{bmatrix}, \qquad \operatorname{rank}\left(\begin{bmatrix} D_f^\dagger \\ D_f^\perp \end{bmatrix} \right) = n_y.$$

Let

$$H = U_1 D_f^\dagger + U_2 D_f^\perp, \qquad L = (Z_1 - B_f)D_f^\dagger + Z_2 D_f^\perp \tag{4.8}$$

where $U_1 \in \mathcal{R}^{n_f \times n_f}$, $U_2 \in \mathcal{R}^{n_f \times (n_y - n_f)}$, $Z_1 \in \mathcal{R}^{n \times n_f}$ and $Z_2 \in \mathcal{R}^{n \times (n_y - n_f)}$ are free parameters, with U_1 nonsingular to ensure that HD_f is invertible. Then

$$HD_f = U_1, \qquad B_f + LD_f = Z_1, \qquad (HD_f)^{-1}H = D_f^{\dagger} + U_0 D_f^{\perp}.$$

where $U_0 = U_1^{-1}U_2$. By taking some matrix manipulations, it shows that

$$T_{rf}^{-1}(s) \stackrel{s}{=} \begin{bmatrix} A + LC - (B_f + LD_f)(HD_f)^{-1}HC & -(B_f + LD_f)(HD_f)^{-1} \\ (HD_f)^{-1}HC & (HD_f)^{-1} \end{bmatrix}.$$

Then

$$T_{rf}^{-1}(s)T_{rd}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{4.9}$$

where

$$\begin{aligned} A_c &= A + LC - (B_f + LD_f)(HD_f)^{-1}HC, \\ B_c &= (B_d + LD_d) - (B_f + LD_f)(HD_f)^{-1}(HD_d), \\ C_c &= (HD_f)^{-1}HC, \\ D_c &= (HD_f)^{-1}(HD_d). \end{aligned}$$

By using the rearrangements of the variables as given in (4.9), it follows that

$$T_{rf}^{-1}(s)T_{rd}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} A_1 + Z_0C_2 & B_{d,1} + Z_0D_{d,2} \\ \hline C_1 + U_0C_2 & D_{d,1} + U_0D_{d,2} \end{array}\right], \tag{4.10}$$

where

$$\left[\begin{array}{c|c} A_1 & B_{d,1} \\ \hline C_1 & D_{d,1} \\ C_2 & D_{d,2} \end{array}\right] = \left[\begin{array}{c|c} A - B_f D_f^{\dagger} C & B_d - B_f D_f^{\dagger} D_d \\ \hline D_f^{\dagger} C & D_f^{\dagger} D_d \\ D_f^{\perp} C & D_f^{\perp} D_d \end{array}\right] \tag{4.11}$$

and $Z_0 = Z_2 - Z_1U_0$. The next result gives the solution of Problem 4.1.2.

Theorem 4.2.1. *Let all variables and assumptions be as given in Problem 4.1.2. For some $\gamma > 0$, there exist L and H such that $A + LC$ is stable and (4.6) is satisfied if and only if there exist $P = P^T$, Q and U_0 such that the following LMI*

$$\begin{bmatrix} A_1^T P + C_2^T Q^T + P A_1 + Q C_2 & \star & \star \\ B_{d,1}^T P + D_{d,2}^T Q^T & -\gamma I & \star \\ C_1 + U_0 C_2 & D_{d,1} + U_0 D_{d,2} & -\gamma I \end{bmatrix} \prec 0, \tag{4.12}$$

is satisfied, where $\star$ represents terms readily inferred from symmetry.

Proof. From a generalized bounded real lemma in [52], there exist L and H such that (4.6) is satisfied if and only if there exists $P = P^T$ (not necessarily positive definite) such that

$$\begin{bmatrix} A_c^T P + P A_c & P B_c & C_c^T \\ B_c^T P & -\gamma I & D_c^T \\ C_c & D_c & -\gamma I \end{bmatrix} \prec 0. \tag{4.13}$$

Substituting (4.10) in (4.13) and then making the change of variables $Q = PZ_0$ shows that there exist L and H such that (4.6) is satisfied if and only if there exist $P = P^T,\ Q$ and U_0 such that (4.12) is satisfied. Furthermore, if (4.12) has feasible solutions $P,\ Q = PZ_0$ and U_0 and defining

$$H_0 = D_f^{\dagger} + U_0 D_f^{\perp}, \qquad L_0 = -B_f D_f^{\dagger} + Z_0 D_f^{\perp}, \tag{4.14}$$

so that L and H in (4.8) can be written as

$$L = L_0 + Z_1 H_0, \qquad H = U_1 H_0, \tag{4.15}$$

then a class of (not necessarily stable) FD filters $F(s)$ of the form (4.4) is parameterized by free variables U_1 and Z_1, subject to U_1 nonsingular. It can be seen that $A + LC = (A + L_0 C) + Z_1(H_0 C)$. Then it remains to prove the stability of the filter $F(s)$. The proof below shows that it is always possible to stabilize $A + LC$ by appropriately choosing Z_1. Consider the following three cases:

1. If $A + L_0 C$ is stable then choose $Z_1 = 0$.

2. If the pair $(A + L_0 C, H_0 C)$ is detectable, then choose Z_1 such that $A + L_0 C + Z_1 H_0 C$ is stable.

3. Suppose that the pair $(A + L_0 C, H_0 C)$ is not detectable. Then, by introducing a similarity transformation if necessary, it follows that

$$\begin{bmatrix} A + L_0 C \\ H_0 C \end{bmatrix} = \left[\begin{array}{cc} A_{11} + L_{0,1} C_{11} & A_{12} + L_{0,1} C_{12} \\ 0 & A_{22} + L_{0,2} C_{12} \\ \hline 0 & H_0 C_{12} \end{array}\right]$$

   where $A_{11} + L_{0,1} C_{11}$ is anti-stable (that is, all its poles are in the open right half-plane), the pair $(A_{22} + L_{0,2} C_{12}, H_0 C_{12})$ is detectable, $H_0 C_{11} = 0$ and $A_{21} + L_{0,2} C_{11} = 0$ and where A, L_0 and C have been appropriately partitioned. Next, it is proved that the pair (A_{11}, C_{11}) is detectable. Since it is assumed that the pair (A, C) is detectable, it follows that

$$\begin{bmatrix} A_{11} - sI & A_{12} \\ A_{21} & A_{22} - sI \\ C_{11} & C_{12} \end{bmatrix}$$

has full column rank for all s such that $s + s' > 0$. Using the fact that $A_{21} = -L_{0,2}C_{11}$, it follows that

$$\begin{bmatrix} A_{11} - sI & A_{12} \\ 0 & A_{22} + L_{0,2}C_{12} - sI \\ C_{11} & C_{12} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & L_{0,2} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{11} - sI & A_{12} \\ A_{21} & A_{22} - sI \\ C_{11} & C_{12} \end{bmatrix}$$

has full column rank for all s such that $s + s' > 0$. Thus the pair (A_{11}, C_{11}) is detectable. It follows that there exists $\tilde{L}_{0,1}$ such that $A_{11} + \tilde{L}_{0,1}C_{11}$ is stable. Define $\tilde{L}_0 = [\tilde{L}_{0,1}^T \;\; L_{0,2}^T]^T$. Then

$$\begin{bmatrix} A+\tilde{L}_0C \\ H_0C \end{bmatrix} = \left[\begin{array}{cc} A_{11}+\tilde{L}_{0,1}C_{11} & A_{12}+\tilde{L}_{0,1}C_{12} \\ 0 & A_{22}+L_{0,2}C_{12} \\ \hline 0 & H_0C_{12} \end{array}\right]$$

and so $(A + \tilde{L}_0C, H_0C)$ is detectable. Define $\tilde{L} = \tilde{L}_0 + Z_1H_0$ and $\tilde{F}(s) \stackrel{s}{=} (A + \tilde{L}C, \tilde{L}, HC, H)$. Then a calculation shows that $\tilde{F}(s) = F(s)$. It follows that $\left\|(\tilde{F}G_f)^{-1}(\tilde{F}G_d)\right\|_\infty < \gamma$. Redefining $\tilde{L}_0$ as L_0 and $\tilde{F}(s)$ as $F(s)$, then since $(A + L_0C, H_0C)$ is detectable, choose Z_1 such that $A+L_0C+Z_1H_0C$ is stable. □

Remark 4.2.1. *If P is singular, then, because of the strict inequality in (4.12), we can perturb P such that the perturbed P is nonsingular and inequality (4.12) is still satisfied. This shows that we can always recover Z from P and Q.* □

4.3 Design of an optimal $\mathcal{H}_-/\mathcal{H}_\infty$ FD filter

In the previous section, a set of FD filters is derived and residual dynamics are parameterized by free Z_1 and U_1, with U_1 nonsingular that solve Problem 4.1.2. This section shows that, through a suitable choice of Z_1 and U_1, the optimal $\mathcal{H}_-/\mathcal{H}_\infty$ FD filter for Problem 4.1.1 is obtained and further specifications such as fault isolation can also be achieved in some circumstances.

Theorem 4.3.1. *Let all variables and assumptions be as given in Problem 4.1.1 and Theorem 4.2.1 and define $F(s)$ using (4.14), (4.15) and (4.4). Then*

1. *If $A + L_0C$ is stable, let $Z_1 = 0$ and $U_1 = I_{n_f}$. Then $T_{rf}(s) = I_{n_f}$ so that $F(s)$ is a solution of Problem 4.1.1 and furthermore achieves fault isolation.*

2. *If $A + L_0C$ is unstable, then there always exist a $Z_1 \in \mathcal{R}^{n\times n_f}$ and a unitary $U_1 \in \mathcal{R}^{n_f\times n_f}$ such that $F(s)$ is stable and $T_{rf}^*(s)T_{rf}(s) = I_{n_f}$. It follows that $F(s)$ solves Problem 4.1.1.*

Proof. Firstly, note that since $B_f + L_0 D_f = 0$ and $H_0 D_f = I_{n_f}$ from (4.14), then

$$T_{rf}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A+L_0C+Z_1H_0C & Z_1 \\ \hline U_1H_0C & U_1 \end{array}\right] \tag{4.16}$$

from (4.3) and (4.15). To ensure fault isolability for all faults, it is required that the transfer matrix $T_{rf}(s)$ is diagonal. For example, if $T_{rf}(s) = I_{n_f}$ (ignoring disturbances), $r = f$ and each individual fault only affects its corresponding residual signal.

1. . If $A + L_0C$ is stable, we can obtain $T_{rf}(s) = I_{n_f}$ by choosing $Z_1 = 0$ and $U_1 = I_{n_f}$ so that the filter achieves optimal FDI.

2. If $A + L_0C$ is not stable we choose suitable Z_1 and U_1 such that $F(s)$ is stable and $T^*_{rf}(s)T_{rf}(s) = I_{n_f}$. A simple calculation using (4.10) and (4.14) shows that

$$A + L_0C = A_c. \tag{4.17}$$

It follows from the (1,1) block of (4.13) and the assumed nonsingularity of P that $A + L_0C$ has no eigenvalues on the imaginary axis. It follows that a similarity transformation can be effected on the data, if necessary, such that

$$\left[\begin{array}{c} A + L_0C \\ H_0C \end{array}\right] = \left[\begin{array}{cc} A_s & 0 \\ 0 & A_a \\ \hline C_s & C_a \end{array}\right]$$

where A_s and $-A_a$ are stable. Then (4.16) implies that

$$T_{rf}(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_s & 0 & 0 \\ Z_aC_s & A_a+Z_aC_a & Z_a \\ \hline U_1C_s & U_1C_a & U_1 \end{array}\right] \tag{4.18}$$

where set $Z_1 = [\, 0 \quad Z_a^T \,]^T$. Let U_1 be any orthogonal matrix and

$$Z_a = -P_a^{-1}C_a^T, \tag{4.19}$$

where $P_a = P_a^T \succ 0$ is the unique solution of the Lyapunov equation

$$A_a^T P_a + P_a A_a - C_a C_a^T = 0. \tag{4.20}$$

Note that $P_a \succ 0$ follows from the facts that $-A_a$ is stable and $(A_a,\ C_a)$ is observable, which in turn follows from the fact that $(A + L_0C,\ H_0C)$ is detectable. Then (4.18) implies

that

$$\begin{aligned} T_{rf}(s) &\overset{s}{=} \left[\begin{array}{c|c} A_a + Z_a C_a & Z_a \\ \hline U_1 C_a & U_1 \end{array}\right] \\ &= \left[\begin{array}{c|c} A_a - P_a^{-1} C_a^T C_a & -P_a^{-1} C_a^T \\ \hline U_1 C_a & U_1 \end{array}\right]. \end{aligned}$$

Hence, $T_{rf}(s)$ and so that $T_{rd}(s)$ are stable, which follows from $(A_a+Z_aC_a)^T P_a+P_a(A_a+Z_aC_a)+C_a^T C_a = 0$ and $P_a \succ 0$. Since $U_1^T U_1 = I$ and given (4.20) is satisfied , by taking matrix calculations, it shows that $T_{rf}^*(s)T_{rf}(s) = I$.

As demonstrated above, it is always possible to construct $F(s)$ such that $T_{rf}^*(s)T_{rf}(s) = I$ which gives $\|T_{rf}\|_- = 1$ and

$$\frac{\|T_{rd}\|_\infty}{\|T_{rf}\|_-} = \|T_{rd}\|_\infty = \left\|T_{rf}^{-1} T_{rd}\right\|_\infty < \gamma. \tag{4.21}$$

Therefore, the above result shows that a suitable choice of Z_1 and U_1 ensures $T_{rf}^*(s)T_{rf}(s) = I$ and then $\gamma_0 = \gamma_1$ in (4.5). Therefore, this proves that our scheme gives the optimal solution of Problem 4.1.1. □

Remark 4.3.1. *Note that if $A + L_0C$ is not stable then due to the specific choice of Z_1 in (4.19) to stabilize $F(s)$, $T_{rf}(s)$ is no longer diagonal, which implies that it is not possible to achieve fault isolation. However, it is always possible to choose the free variable U_1 suitably such that $F(s)$ achieves fault isolation at DC ($\omega = 0$). Set $U_1 = \left(I_{n_f} - C_a(A_a + Z_aC_a)^{-1}Z_a\right)^T$, then $T_{rf}(0) = U_1 - U_1C_a(A_a + Z_aC_a)^{-1}Z_a = I_{n_f}$, which ensures optimal FDI at DC.* □

Remark 4.3.2. *Note that if $A + L_0C$ stable, the optimal FD filter for the $\mathcal{H}_-/\mathcal{H}_\infty$ problem also achieves fault isolation as shown in Part 1 of Theorem 4.3.1. Now suppose in the LMI in (4.13), the constraint $P = P^T \succ 0$ is added in instead of just $P = P^T$. This will, in general, result in a larger optimal γ but will ensure that $A + L_0C$ is stable from (4.17). Hence, it follows from the preceding argument that the fault isolation capability of the filter can be improved at the expense of its fault detection ability.* □

4.4 Extensions of the fault detection problem

In the previous sections, an optimal FD scheme with some fault isolation properties is presented to solve Problem 4.1.1. In this section, two extensions are briefly outlined to the proposed scheme.

First, a frequency weighting filter is introduced to improve the fault detection effectiveness and also to remove assumption $A2$ in Problem 4.1.1. Then, an approach is given to extend the scheme in this work to uncertain systems.

4.4.1 Further improvement in FD performance

In the fault detection scheme developed in the previous section, the performance of the FD filter depends on the value of γ. If $\gamma < 1$, then the smallest singular value of $T_{rf}(s)$ is larger than the largest singular value of $T_{rd}(s)$ over all frequencies and the performance of the FD scheme is expected to be satisfactory. On the other hand, if $\gamma >> 1$, then the fault detection may not be effective.

The problem is in our requirement that

$$\bar{\sigma}(T_{rf}(j\omega)^{-1}T_{rd}(j\omega)) < \gamma \tag{4.22}$$

for all frequencies $\omega \in \mathcal{R}$. In practice, for the purpose of fault detection, it is only needed that (4.22) is achieved over a limited frequency range, or even at a single frequency. Let W_{ω_0} be a stable and proper minimum-phase single-input single-output bandpass filter with a peak gain of one at frequency $\omega_0 \in \mathcal{R}$. Suppose the cost function in (4.6) is replaced by $\left\|(FG_fW_{\omega_0}^{-1})^{-1}(FG_dW_{\omega_0})\right\|_\infty < \gamma_{\omega_0}$. This can be recast as Problem 4.1.2 by replacing $[G_f(s)\ \ G_d(s)]$ by $\left[G_fW_{\omega_0}^{-1}(s)\ \ G_dW_{\omega_0}(s)\right]$, which, because of our assumptions on W_{ω_0}, inherits the assumptions of Problem 4.1.2.

Suppose further that we enforce that $FG_fW_{\omega_0}^{-1}(s)$ is inner (see Section 4.3). Then $\sigma_i(T_{rf}(j\omega)) = |W_{\omega_0}(j\omega)|$ and $\bar{\sigma}(T_{rd}(j\omega)) < \gamma_{\omega_0}|W_{\omega_0}^{-1}(j\omega)|$, where $\sigma_i(\cdot)$ denotes the ith singular value. It follows that $\bar{\sigma}(T_{rd}(j\omega_0)) < \gamma_{\omega_0}\underline{\sigma}(T_{rf}(j\omega_0))$. Thus γ_{ω_0} can be calculated for several values of ω_0 in a suitable frequency range $[\underline{\omega},\ \bar{\omega}]$ to minimize γ_{ω_0}. Since $\gamma_{\omega_0} \leq \gamma$ for all ω_0, the fault detection properties could be improved at a single frequency ω_0. The fault detection filter $F(s)$ may now be followed by a second filter $W_{\omega_0}(s)$ to shape the residual signal r.

4.4.2 Removal of assumption $A2$ in Problem 4.1.1

A major assumption that is widely made is that $G_f(s)$ has no zeros over the extended imaginary axis, and in particular that D_f has full column rank. This excludes important situations such as no sensor faults ($D_f = 0$). The problem is caused by the choice of the cost function in (4.7):

in the case that $G_f(j\omega_1)$ loses rank at $\omega_1 \in \mathcal{R}$, then so does $T_{rf}(j\omega_1)$, so that (4.7) cannot in general be satisfied at $j\omega_1$. In the following, a discussion is given for relaxing Assumption A2 and ameliorating the problem of loss of rank of $G_f(s)$.

Suppose that $G_f(s)$ has zeros on the extended imaginary axis. Then $G_f(s)$ can be written as $G_f(s) = \hat{G}_f(s)G_z(s)$ where $\hat{G}_f(s)$ has no zeros over the extended imaginary axis and $G_z(s)$ has all its zeros on the extended imaginary axis [54]. Suppose now that the cost function in (4.6) is replaced by $\left\|\hat{T}_{rf}^{-1}T_{rd}\right\|_\infty < \gamma$ where $\hat{T}_{rf}(s) = F(s)\hat{G}_f(s)$ and the we enforce that $\hat{T}_{rf}(s)$ is inner as shown in Section 4.3. Then $\sigma_i(T_{rf}(j\omega)) = \sigma_i(\hat{T}_{rf}(j\omega)G_z(j\omega)) = \sigma_i(G_z(j\omega))$, since $\hat{T}_{rf}(s)$ is inner, and $\bar{\sigma}(T_{rd}(j\omega)) < \gamma$. Note that the singular values of $T_{rf}(j\omega)$ are set equal to those of $G_z(j\omega)$ instead of being set equal to 1 in the original scheme, therefore γ is the fault detection level relative to $\sigma_i(G_z(j\omega))$ rather than 1 as given in (4.21).

4.5 Fault detection for uncertain systems

This section shows that the FD scheme developed in the chapter can also be applied to uncertain systems. Two different type of uncertainties, i.e. unstructured and structured uncertainties, are dealt with separately. For each case, a suboptimal solution is derived in the form of bilinear matrix inequality (BMI).

Consider the following uncertain system:

$$\begin{array}{rcl} \dot{x}(t) & = & (A+\Delta A)x(t) + (B_u+\Delta B_u)u(t) + B_d d(t) + B_f f(t), \\ y(t) & = & (C+\Delta C)x(t) + (D_u+\Delta D_u)u(t) + D_d d(t) + D_f f(t), \end{array} \tag{4.23}$$

The matrices $\Delta A, \Delta B_u, \Delta C, \Delta D_u$ represent modeling errors and are given by:

$$\left[\begin{array}{cc} \Delta A & \Delta B_u \\ \Delta C & \Delta D_u \end{array}\right] = \left[\begin{array}{c} F_a \\ F_c \end{array}\right] \Delta_H \left[\begin{array}{cc} E_a & E_b \end{array}\right],$$

where F_a, F_c, E_a, E_b, H are known matrices of compatible dimensions and $\Delta_H = \Delta(I - H\Delta)^{-1}$, with $\Delta \in \mathbf{\Delta}$ where

$$\mathbf{\Delta} := \{diag(\delta_1 I_{q1}, \ldots, \delta_l I_{ql}, \Delta_{l+1}, \ldots, \Delta_{l+f}) : \Delta \leq 1, \delta_i \in \mathcal{R}, \Delta_i \in \mathcal{R}^{q_i \times q_i}\} \subset \mathcal{R}^{n_\Delta \times n_\Delta}. \tag{4.24}$$

This LFT representation of uncertainty, which is assumed to be well-posed over $\mathbf{\Delta}$ (that is, $det(I - H\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}$), has great generality and is widely used in control theory [115]. The same static observer scheme of Section 4.1 is used to get:

$$r(s) \quad = \quad T_{rf}(s)f(s) + T_{rdu}(s) \begin{bmatrix} d(s) \\ u(s) \end{bmatrix},$$

where

$$[T_{rf}(s) \mid\mid T_{rdu}(s)] \stackrel{s}{=} \left[\begin{array}{cc|c||cc} A+LC & \Delta A+L\Delta C & B_f+LD_f & B_d+LD_d & \Delta B_u+L\Delta D_u \\ 0 & A+\Delta A & B_f & B_d & B_u+\Delta B_u \\ \hline HC & H\Delta C & HD_f & HD_d & H\Delta D_u \end{array}\right].$$

and $T_{rdu}(s)$ is the transfer function from the disturbances and input to the residual.

Remark 4.5.1. *Note that the residual signal in the case of uncertain systems depends on the input signal u and so the observer does not cancel the system dynamics. This makes the FD problem much harder than the case when there are no uncertainties and the solution will involve BMI optimizations.* □

By following a similar derivation to (4.9) and separating the terms involving modeling uncertainties from the other terms, the state space representation of $T_{rf}^{-1}(s)T_{rdu}(s)$ is given as follows.

$$T_{rf}^{-1}(s)T_{rdu}(s) \stackrel{s}{=} (A_c^o + A_c^\Delta, B_c^o + B_c^\Delta, C_c^o + C_c^\Delta, D_c^o + D_c^\Delta) \tag{4.25}$$

where

$$\left[\begin{array}{c|c} A_c^o & B_c^o \\ \hline C_c^o & D_c^o \end{array}\right] = \left[\begin{array}{cc|cc} A_1 + Z_0C_2 & 0 & B_{d,1} + Z_0D_{d,2} & 0 \\ -B_f(C_1 + U_0C_2) & A & B_d - B_f(D_{d,1} + U_0D_{d,2}) & B_u \\ \hline C_1 + U_0C_2 & 0 & D_{d,1} + U_0D_{d,2} & 0 \end{array}\right].$$

and where

$$\left[\begin{array}{c|c} A_c^\Delta & B_c^\Delta \\ \hline C_c^\Delta & D_c^\Delta \end{array}\right] = \begin{bmatrix} F_{r1} \\ F_{r2} \end{bmatrix} \Delta_H \begin{bmatrix} E_{r1} & E_{r2} \end{bmatrix} = \left[\begin{array}{c} F_a + Z_0D_f^\perp F_c - B_f D_f^\dagger F_c \\ F_a - B_f(D_f^\dagger F_c + U_0 D_f^\perp F_c) \\ \hline D_f^\dagger F_c + U_0 D_f^\perp F_c \end{array}\right] \Delta_H \left[\begin{array}{cc|cc} 0 & E_a & 0 & E_b \end{array}\right].$$

and where the changes of variables in (4.8), (4.9) and (4.11) were applied. It follows from the generalized bounded real lemma in [52] that there exists a (not necessarily stable) filter of the form given in (4.4) such that

$$\left\| T_{rf}^{-1} T_{rdu} \right\|_\infty < \gamma$$

is satisfied if and only if there exists $P_c = P_c^T$ such that

$$T_c^o + T_c^\Delta \prec 0, \tag{4.26}$$

where

$$\begin{aligned} T_c^o &= \begin{bmatrix} (A_c^o)^T P_c + P_c A_c^o & P_c B_c^o & (C_c^o)^T \\ (B_c^o)^T P_c & -\gamma I & (D_c^o)^T \\ C_c^o & D_c^o & -\gamma I \end{bmatrix} \\ T_c^\Delta &= \begin{bmatrix} (A_c^\Delta)^T P_c + P_c A_c^\Delta & P_c B_c^\Delta & (C_c^\Delta)^T \\ (B_c^\Delta)^T P_c & 0 & (D_c^\Delta)^T \\ C_c^\Delta & D_c^\Delta & 0 \end{bmatrix}. \end{aligned}$$

A calculation shows that $T_c^\Delta = \tilde{F}\Delta_H\tilde{E} + \tilde{E}^T{\Delta_H}^T\tilde{F}^T$ where

$$\tilde{F} = \begin{bmatrix} P_c F_{r1} \\ 0 \\ F_{r2} \end{bmatrix}, \qquad \tilde{E} = \begin{bmatrix} E_{r1} & E_{r2} & 0 \end{bmatrix}. \tag{4.27}$$

The next two subsections derive the solution to the FD problem with unstructured and structured uncertainties, respectively.

4.5.1 Solution with unstructured uncertainties

When the system uncertainty is unstructured, by applying Lemma 1.4.1 from [86], it follows that

$$\begin{aligned} (4.26) \quad &\Leftrightarrow \quad T_c^o + \tilde{F}\Delta_H\tilde{E} + \tilde{E}^T{\Delta_H}^T\tilde{F}^T \prec 0 \\ &\Leftrightarrow \quad \begin{bmatrix} T_c^o + \lambda\tilde{E}^T\tilde{E} & \tilde{F} + \lambda\tilde{E}^T H \\ \tilde{F}^T + \lambda H^T\tilde{E} & \lambda(H^T H - I) \end{bmatrix} \prec 0 \\ &\Leftrightarrow \quad \begin{bmatrix} T_c^o & \tilde{F} \\ \tilde{F}^T & -\lambda I \end{bmatrix} - \begin{bmatrix} \lambda\tilde{E}^T \\ \lambda H^T \end{bmatrix} (-\lambda^{-1} I) \begin{bmatrix} \lambda\tilde{E} & \lambda H \end{bmatrix} \prec 0 \end{aligned} \tag{4.28}$$

for some $\lambda > 0$. Using a Schur complement argument in (4.28), it is obtained that

$$(4.26) \Leftrightarrow \begin{bmatrix} (A_c^o)^T P_c + * & P_c B_c^o & (C_c^o)^T & P_c F_{r1} & \lambda E_{r1}^T \\ * & -\gamma I & (D_c^o)^T & 0 & \lambda E_{r2}^T \\ * & * & -\gamma I & F_{r2} & 0 \\ * & * & * & -\lambda I & \lambda H^T \\ * & * & * & * & -\lambda I \end{bmatrix} \prec 0,$$

Then the following lemma is obtained to solve the FD problem for systems with unstructured uncertainties.

Lemma 4.5.1. *Let all variables be as in (4.23) and (4.25) and given that* $\Delta = \{\Delta \in \mathcal{R}^{n_\Delta \times n_\Delta} : \|\Delta\| \leq 1\}$. *For given* $\gamma > 0$, $\left\|T_{rf}^{-1} T_{rdu}\right\|_\infty < \gamma$ *if and only if there exist* $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = P^T$ *and* U_0 *such that* $\mathcal{M} \prec 0$ *is satisfied, where*

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \mathcal{M}_{14} & \mathcal{M}_{15} \\ \star & -\gamma I & \mathcal{M}_{23} & 0 & \mathcal{M}_{25} \\ \star & \star & -\gamma I & \mathcal{M}_{34} & 0 \\ \star & \star & \star & -\lambda I & \lambda H^T \\ \star & \star & \star & \star & -\lambda I \end{bmatrix}$$

and

$$\begin{aligned}
\mathcal{M}_{11} &= \begin{bmatrix} P_{11}(A_1 + Z_0 C_2) - P_{12} B_f (C_1 + U_0 C_2) + \star & \star \\ \left((A_1 + Z_0 C_2)^T P_{12} - (C_1 + U_0 C_2)^T B_f^T P_{22}\right)^T & P_{22} A + \star \end{bmatrix}, \\
\mathcal{M}_{12} &= \begin{bmatrix} P_{11}(B_{d,1} + Z_0 D_{d,2}) + P_{12}\left(B_d - B_f(D_{d,1} + U_0 D_{d,2})\right) & P_{12} B_u \\ P_{12}^T(B_{d,1} + Z_0 D_{d,2}) + P_{22}\left(B_d - B_f(D_{d,1} + U_0 D_{d,2})\right) & P_{22} B_u \end{bmatrix}, \\
\mathcal{M}_{13} &= \begin{bmatrix} (C_1 + U_0 C_2)^T \\ 0 \end{bmatrix}, \mathcal{M}_{23} = \begin{bmatrix} (D_{d,1} + U_0 D_{d,2})^T \\ 0 \end{bmatrix}, \mathcal{M}_{15} = \begin{bmatrix} 0 \\ \lambda E_a^T \end{bmatrix}, \\
\mathcal{M}_{14} &= \begin{bmatrix} P_{11}(F_a + Z_0 F_{c,2} - B_f F_{c,1}) + P_{12}(F_a - B_f(F_{c,1} + U_0 F_{c,2})) \\ P_{12}^T(F_a + Z_0 F_{c,2} - B_f F_{c,1}) + P_{22}(F_a - B_f(F_{c,1} + U_0 F_{c,2})) \end{bmatrix}, \\
\mathcal{M}_{34} &= \begin{bmatrix} F_{c,1} + U_0 F_{c,2} \end{bmatrix}, \quad \mathcal{M}_{25} = \begin{bmatrix} 0 \\ \lambda E_b^T \end{bmatrix}.
\end{aligned}$$

□

It follows that a solution can be obtained by solving a BMI using the global optimaization algorithms given in [40, 99], as well as local numerical searching algorithms that converge much faster [38, 5]. A related discussion of the solution techniques for BMIs can be found in [71].

However it is possible to obtain a suboptimal solution that requires the solution of a quasi-LMI. If set $P_{12} = 0$, then by pre- and post-multiplying $\mathcal{M}$ by $T = diag(I, P_{22}^{-1}, I)$, and setting $Q_1 = P_{11} Z_0$ and $Q_2 = P_{22}^{-1}$ it gives that

$$\left\|T_{rf}^{-1} T_{rdu}\right\|_\infty < \gamma \Leftarrow \exists \lambda > 0,\ \exists P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} = P^T \text{ such that } \mathcal{M}^{sub} \prec 0,$$

where

$$\mathcal{M}^{sub} = \begin{bmatrix} \mathcal{M}_{11}^{sub} & \mathcal{M}_{12}^{sub} & \mathcal{M}_{13}^{sub} & \mathcal{M}_{14}^{sub} & \mathcal{M}_{15}^{sub} \\ \star & -\gamma I & \mathcal{M}_{23} & \mathcal{M}_{24} & \mathcal{M}_{25} \\ \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & -\lambda I & \lambda H^T \\ \star & \star & \star & \star & -\lambda I \end{bmatrix},$$

and where

$$\begin{aligned}
\mathcal{M}_{11}^{sub} &= \begin{bmatrix} P_{11}A_1+Q_1C_2+\star & -(C_1+U_0C_2)^T B_f^T \\ \star & AQ_2+\star \end{bmatrix}, \\
\mathcal{M}_{12}^{sub} &= \begin{bmatrix} P_{11}B_{d,1}+Q_1D_{d,2} & 0 \\ B_d-B_f(D_{d,1}+U_0D_{d,2}) & B_u \end{bmatrix}, \quad \mathcal{M}_{13}^{sub} = \begin{bmatrix} (C_1+U_0C_2)^T \\ 0 \end{bmatrix}, \\
\mathcal{M}_{14}^{sub} &= \begin{bmatrix} P_{11}(F_a-B_fF_{c,1})+Q_1F_{c,2} \\ F_a-B_f(F_{c,1}+U_0F_{c,2}) \end{bmatrix}, \quad \mathcal{M}_{15}^{sub} = \begin{bmatrix} 0 \\ \lambda Q_2E_a^T \end{bmatrix}.
\end{aligned}$$

The solution can be retrieved by noting that $Z_0 = P_{11}^{-1}Q_1$. Although $\mathcal{M}_{15}^{sub}$ contains a nonlinear term, it is possible to get the optimal solution since it is linear for a fixed λ. So that $\lambda \in [0, +\infty)$ is tuned until the minimum value of γ is found. It follows that a class of filters that solves Problem 4.1.1 is obtained for uncertain models. Z_1 and U_1 can then be chosen to get a stable filter as has been done in the case of no model uncertainty.

Remark 4.5.2. *Note that the results given in [56, 55] are optimal, however, not suitable for models with uncertainties since the requirement that the transfer matrix $T_{rf}(s) = I$ is essential, which cannot be achieved for uncertain system model.* □

4.5.2 Solution with structured uncertainties

When Δ is structured as in (4.24) we proceed as follows. By using (1.6) from Lemma 1.4.1, we have

$$\begin{aligned}
(4.26) \;&\Leftrightarrow\; T_c^o+\tilde{F}\Delta_H\tilde{E}+\tilde{E}^T{\Delta_H}^T\tilde{F}^T \prec 0 \\
&\Leftarrow\; \exists S \in \Sigma \;\&\; G \in \Gamma \text{ s.t. } S \succ 0 \;\&\; T_{SG} \prec 0,
\end{aligned}$$

where

$$T_{SG} = \begin{bmatrix} T_c^o & \tilde{F}+\tilde{E}^TG \\ \tilde{F}^T+G^T\tilde{E} & H^TG+G^TH-S \end{bmatrix} + \begin{bmatrix} \tilde{E}^TS \\ H^TS \end{bmatrix} S^{-1} \begin{bmatrix} S\tilde{E} & SH \end{bmatrix}. \tag{4.29}$$

By taking a Schur complement argument to (4.29), then

$$T_{SG} \prec 0 \Leftrightarrow \bar{T}_{SG} := \begin{bmatrix} (A_c^o)^TP_c+* & P_cB_c^o & (C_c^o)^T & P_cF_{r1}+E_{r1}^TG & E_{r1}^TS \\ * & -\gamma I & (D_c^o)^T & E_{r2}^TG & E_{r2}^TS \\ * & * & -\gamma I & F_{r2} & 0 \\ * & * & * & H^TG+G^TH-S & H^TS \\ * & * & * & * & -S \end{bmatrix} \prec 0,$$

where $\tilde{E}$ and $\tilde{F}$ are as defined in (4.27). The following lemma derives a solution for the FD

problem for systems with structured uncertainties.

Lemma 4.5.2. *Let all variables be as in (4.23) and (4.25) and given that $\Delta \in \mathbf{\Delta}$. For some $\gamma > 0$, $\left\| T_{rf}^{-1} T_{rdu} \right\|_\infty < \gamma$ if and only if there exist $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = P^T$ and U_0 such that $\mathcal{L} \prec 0$ is satisfied, where*

$$\mathcal{L} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \mathcal{L}_{14} & \mathcal{L}_{15} \\ * & -\gamma I & \mathcal{M}_{23} & \mathcal{L}_{24} & \mathcal{L}_{25} \\ * & * & -\gamma I & \mathcal{L}_{34} & 0 \\ * & * & * & H^T G + G^T H - S & H^T S \\ * & * & * & * & -S \end{bmatrix} \prec 0. \tag{4.30}$$

and

$$\begin{aligned} \mathcal{L}_{14} &= \begin{bmatrix} P_{11}(F_a + Z_0 F_{c,2} - B_f F_{c,1}) + P_{12}(F_a - B_f(F_{c,1} + U_0 F_{c,2})) \\ P_{12}^T(F_a + Z_0 F_{c,2} - B_f F_{c,1}) + P_{22}(F_a - B_f(F_{c,1} + U_0 F_{c,2})) + E_a^T G \end{bmatrix}, \\ \mathcal{L}_{15} &= \begin{bmatrix} 0 \\ E_a^T S \end{bmatrix}, \mathcal{L}_{24} = \begin{bmatrix} 0 \\ E_b^T G \end{bmatrix}, \mathcal{L}_{25} = \begin{bmatrix} 0 \\ E_b^T S \end{bmatrix}, \mathcal{L}_{34} = \begin{bmatrix} F_{c,1} + U_0 F_{c,2} \end{bmatrix}. \end{aligned}$$

□

Similar to the case of unstructured uncertainties, the optimal solution is obtained by solving a BMI by using the algorithms guaranteeing global convergence given in [40] and [99].

4.6 Numerical examples

In this section, two examples are considered to illustrate the effectiveness of the proposed FD scheme. The first one, from the literature, compares our FD filter with other design techniques. The second example is randomly generated, and illustrates the extensions given in Section 4.3 and Section 4.4.

4.6.1 Example from the literature

Consider example 2 given in [91] and also considered in [87]. Both approaches employed an $\mathcal{H}_\infty/\mathcal{H}_\infty$ FD formulation. The state-space model of the LTI system is described as follows.

$$\left[\begin{array}{c|c|c} A & B_d & B_f \\ \hline C & D_d & D_f \end{array}\right] = \left[\begin{array}{cccc|cc|c} -10 & 0 & 5 & 0 & 0.8 & 0.04 & 4 \\ 0 & -5 & 0 & 2.5 & -2.4 & 0.08 & 4 \\ 0 & 0 & -2.5 & 0 & 1.6 & 0.08 & 8 \\ 0 & 5 & 0 & -3.75 & 0.8 & 0.08 & -8 \\ \hline 1 & 0 & 0 & 1 & 0.2 & 0.04 & 2 \\ 1 & 0 & 1 & 1 & 0.4 & 0.06 & -1 \end{array}\right].$$

By following the proposed algorithm, we get $\gamma_0 = 0.2392$ and a stable $A + L_0C$. Set $Z_1 = 0$ and $U_1 = 2.6I$ so that

$$L = \begin{bmatrix} -2.0034 & -1.0735 \\ 0.7778 & 5.4963 \\ -4.8810 & -1.6664 \\ 2.1436 & -2.8805 \end{bmatrix}, \quad H = \begin{bmatrix} 1.0385 & -0.5231 \end{bmatrix}.$$

This setting is optimal for the $\mathcal{H}_-/\mathcal{H}_\infty$ FDI following Remark 4.3.2. Next, consider two faulty scenarios for the example

Scenario 1:

As given in [91]: the disturbance is taken as $d(t) = [1.2\sin(2t)e^{-0.1t} \;\; 1.5\cos(2t)e^{-0.1t}]^T$ and the fault f_1 is defined as $f_1(t) = 0.5,\ 5s \leq t \leq 10s$ and $f_1(t) = 0$ elsewhere. Figure 4.1 shows the residuals and highlights the fact that our residual based on $\mathcal{H}_-/\mathcal{H}_\infty$ is more robust to disturbances.

Also, note that the residuals in [91] and [87] are of dimension 2 since in their scheme the dimension of the residual is the same as the number of outputs.

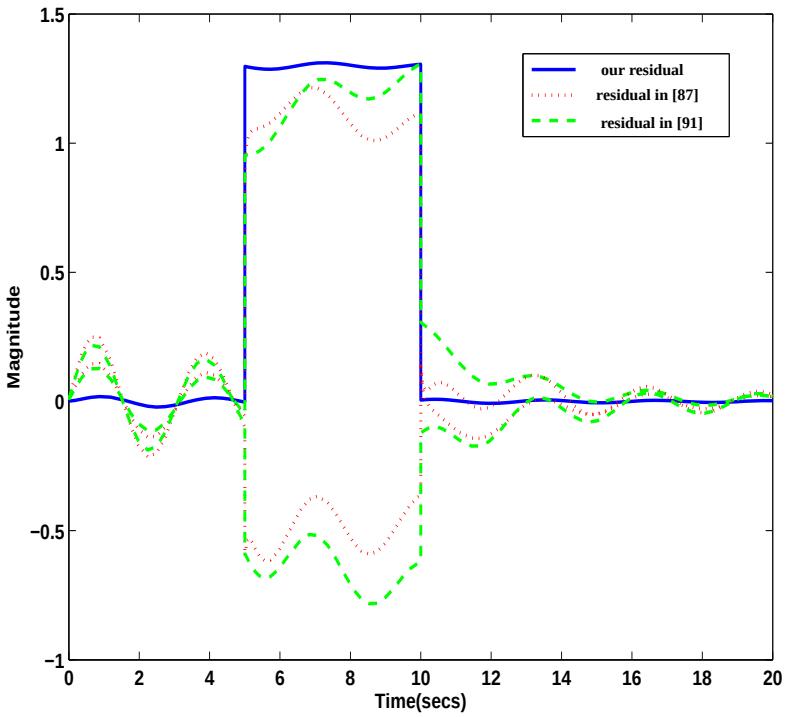


Figure 4.1: Residual time responses for Scenario 1

Scenario 2:

Next, consider the same system but this time subject to two white noise disturbances of mean zero and covariance 7 connected at $t = 0$. A fault is simulated by a unit positive jump connected at $t = 22s$. Figure 4.2 shows the residuals. This example justifies the use of $\mathcal{H}_-/\mathcal{H}_\infty$ schemes over $\mathcal{H}_\infty/\mathcal{H}_\infty$.

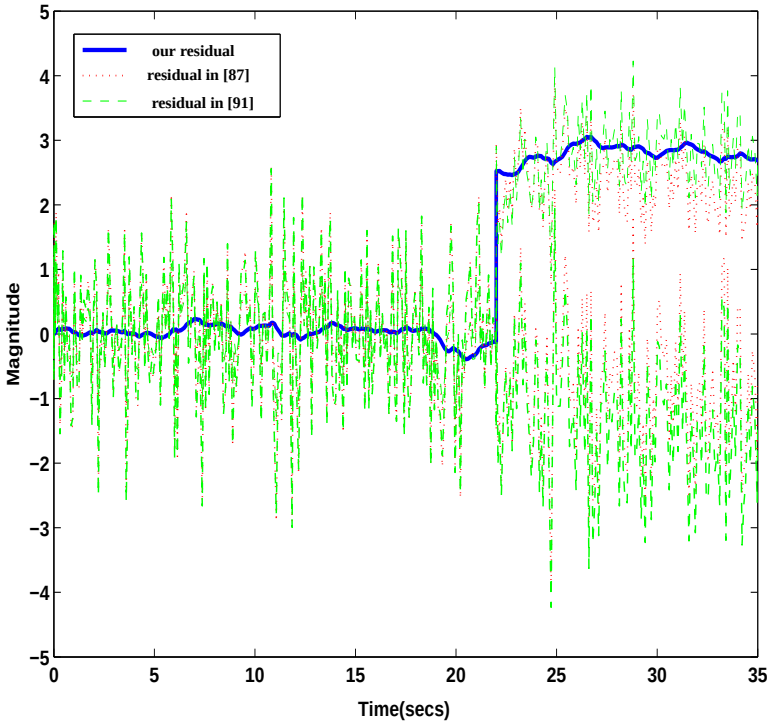


Figure 4.2: Residual time responses for Scenario 2

4.6.2 Randomly generated example

Here, consider the following randomly generated data:

$$A=\begin{bmatrix} -30.4438 & 4.8715 & -0.3643 & -23.6674 & -27.0508 \\ 9.8449 & -7.1563 & 6.7607 & 5.6055 & 1.1667 \\ -0.4460 & -2.0473 & -5.2344 & 0.0314 & 0.5579 \\ 23.0620 & 3.3600 & 5.1258 & 19.5067 & 27.8028 \\ -2.9175 & -8.1727 & -6.1518 & -4.5557 & -10.6214 \end{bmatrix},$$

$$C=\begin{bmatrix} -0.9800 & -2.1080 & -0.6891 & -0.4471 & 0.3988 \\ -0.3647 & -0.8006 & 0.4282 & 0.2585 & 1.4254 \\ 0.9081 & 0.5181 & -1.1179 & 0.2686 & 0.9972 \end{bmatrix},$$

$$B_f=\begin{bmatrix} -0.8999 & 1.7625 \\ 0.1478 & 0.1409 \\ -1.3598 & 1.1176 \\ -1.3020 & 0.7315 \\ 0.8723 & -0.4119 \end{bmatrix},\ D_f=\begin{bmatrix} -0.8032 & -0.1423 \\ -0.9889 & 1.4751 \\ -1.0297 & -0.4817 \end{bmatrix},$$

$$B_d=\begin{bmatrix} 0.0880 & 1.2801 \\ -0.9907 & 0.2294 \\ 1.3235 & 0.2285 \\ 1.1952 & 0.5443 \\ -0.9682 & 0.9220 \end{bmatrix},\ D_d=\begin{bmatrix} 0.0905 & -0.3783 \\ 0.8423 & -1.8179 \\ 0.4173 & 1.1086 \end{bmatrix}.$$

Our algorithm gives $\gamma = 0.8170$ and an unstable $A + L_0C$. It follows that fault isolation is not possible; however, from section 4.3, the optimal solution and fault isolation at DC are both achieved by setting

$$Z_1=\begin{bmatrix} 0.9353 & 0.2914 \\ -0.3267 & -0.1976 \\ -0.1134 & -0.0265 \\ -2.3823 & -0.9739 \\ 1.1268 & 0.6113 \end{bmatrix},\quad U_1=\begin{bmatrix} 0.9970 & -0.0778 \\ 0.0778 & 0.9970 \end{bmatrix}.$$

Suppose that the system is subject to a disturbance, which is a white noise with mean zero and standard deviation 1. Two faults are simulated by a unit negative and positive jump respectively, connected from the 12^{th} and 35^{th} second. Figure 4.3 shows the residuals. It is clear that fault isolation is reached at DC and the impact of disturbances on the residuals has been attenuated.

Next, by applying the results of Section 4.4, a bandpass filter is chosen as

$$W_{\omega_0}(s) = \frac{0.516s^2 + 26.83s + 51.6}{s^2 + 26.83s + 100},$$

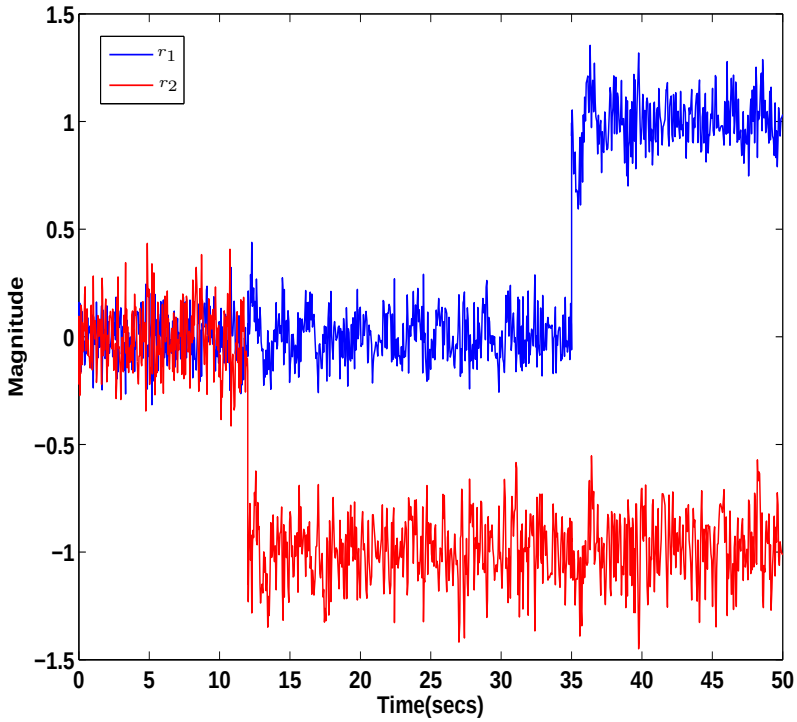


Figure 4.3: residual time responses for Example 4.6.2

centered around a frequency $\omega_0 = 10$ rad/s is used. The optimal value of γ is now $\gamma = 0.5898$. The modified residual responses are also provided in Figure 4.4. The result validates the employment of a weighting filter. Note that with the design incorporating the weights, it is easier to determine a residual threshold for fault detection.

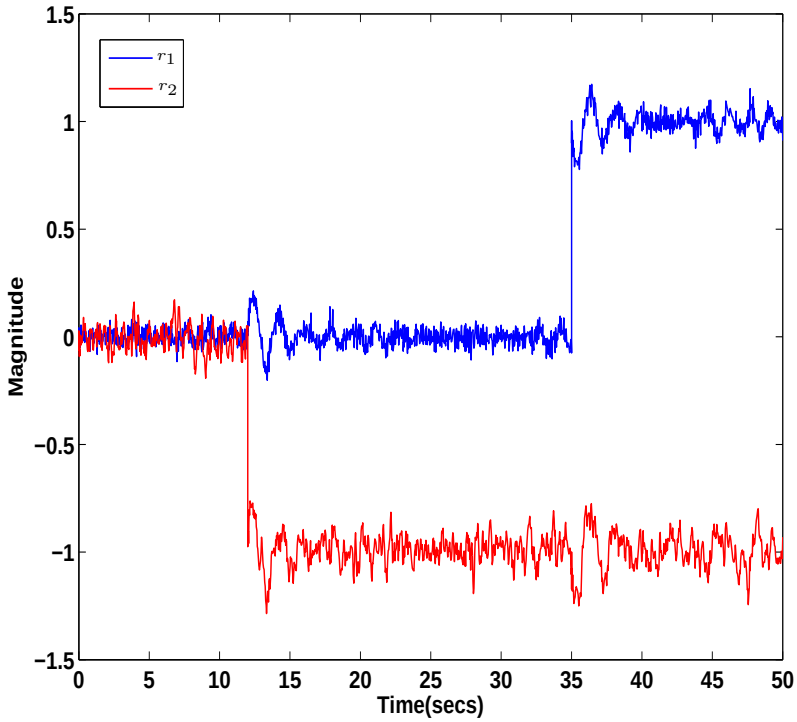


Figure 4.4: residual time responses with weights

4.7 Concluding remarks

In this chapter, the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem for linear time-invariant systems is considered subject to faults and disturbances. A new performance index is proposed to capture both performance requirements, namely, fault detection and disturbance rejection. The disturbance rejection performance is measured by the size of the disturbance to residual dynamics where the $\mathcal{H}_\infty$ norm is used as the measure. Then, by using a static observer, a class of solutions to a FD problem is given in the form of a simple LMI with two degrees of freedom.

It is shown that the degrees of freedom in this class of solutions can be used to derive the optimal filter to the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem. Under certain conditions, further specifications such as fault isolation can also be achieved. Moreover, a technique is given to improve the results and remove certain assumptions with the use of weighting filters and it is shown that the proposed scheme can be extended to uncertain systems, although the resulting solutions are suboptimal.

Chapter 5

A finite frequency domain approach to the fault detection problem

As mentioned in Chapter 4, many contributions [49, 54, 92, 58, 48] implemented FD approaches to carry out the sensitivity optimization over the whole frequency range . However, a major drawback of these schemes is that a restrictive rank assumption on the fault dynamics is needed to ensure a nonzero level of sensitivity to faults. A second drawback is that, for many systems, the resulting sensitivity to disturbances is still large compared with the sensitivity to faults.

As a way of ameliorating these problems, more recent FD schemes carry out the sensitivity optimization over a finite frequency range within which the rank assumption on the fault dynamics is satisfied [90, 92, 58]. However, the major difficulty of these schemes is that the solution algorithms are difficult to implement because the optimization problems are nonlinear which may result in suboptimal solutions.

In principle, for the purpose of fault detection, it is only needed that the sensitivity to disturbances is small compared with the sensitivity to faults over a small frequency range or even at a single frequency. This chapter considers the observer-based FD problem for linear time invariant systems at a single frequency and over a finite frequency range. For the FD problem at a single frequency, an optimal solution is derived such that the FD observer generates a residual signal which minimizes the sensitivity of the residual to disturbances while maintaining a minimum level of sensitivity to faults at the given frequency. LMI optimization techniques are used to obtain the optimal solution. Moreover, we derive a simple parameterization of all optimal solutions which can be used to achieve other specifications. In particular, it is shown that, when $\omega = 0$, the resulting degrees of

freedom allow the FD observer to be used simultaneously for control system design. Conversely, this work demonstrates that observers used for control system design can be also employed to implement fault detection schemes at DC with minor modifications.

For the FD problem over a frequency range, an initial research is carried out such that a novel problem formulation is developed, by which the FD problem is transformed to a sensitivity optimization problem over a finite frequency range. Then, a solution is derived in the form of an BMI optimization using the generalized KYP lemma followed by a linearization procedure. Also, conditions under which this solution is optimal are also derived. However, the resulting observer parameters may be complex so that they cannot be directly used in practice.

This chapter is organized as follows. Section 5.1 considers the fault detection problem at a single frequency. The optimal LMI solution to the single frequency FD problem is given and further results are derived for the FD problem at DC. Examples are given to demonstrate the effectiveness of the proposed scheme. In Section 5.2, an initial study is given for the FD problem over a frequency range. Finally, Section 5.3 summarizes this work.

5.1 Fault detection problem at a single frequency

It is well known that FD at DC is a very important specification which is usually required for practical systems, therefore this section first gives a general investigation for the FD problem at a single frequency and then specializes the result to FD at DC. It also shows that, at DC, further specifications such as control system design can be achieved while maintaining optimal FD properties.

5.1.1 Preliminaries and problem formulation

Consider a linear time-invariant (LTI) dynamic system subject to disturbances and process, sensor and actuator faults modeled as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_f f(t) + B_d d(t), \\ y(t) &= Cx(t) + D_u u(t) + D_f f(t) + D_d d(t), \end{aligned} \tag{5.1}$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^{n_u}$ and $y(t) \in \mathcal{R}^{n_y}$ are the process state, input and output vectors, respectively, and where $f(t) \in \mathcal{R}^{n_f}$ and $d(t) \in \mathcal{R}^{n_d}$ are the fault and disturbance vectors, re-

spectively. Here, $B_u \in \mathcal{R}^{n\times n_u}$ and $D_u \in \mathcal{R}^{n_y\times n_u}$ are input distribution matrices, respectively. $B_f \in \mathcal{R}^{n\times n_f}$ and $D_f \in \mathcal{R}^{n_y\times n_f}$ are the component and instrument fault distribution matrices, respectively, while $B_d \in \mathcal{R}^{n\times n_d}$ and $D_d \in \mathcal{R}^{n_y\times n_d}$ are the corresponding disturbance distribution matrices [35]. Taking Laplace transforms, the system input/output behavior can be described by

$$y(s) = G_u(s)u(s) + G_d(s)d(s) + G_f(s)f(s),$$

where

$$\begin{array}{c} \\ n_y \end{array} \begin{array}{c} \begin{array}{ccc} n_u & n_f & n_d \end{array} \\ \left[\begin{array}{ccc} G_u(s) & G_f(s) & G_d(s) \end{array}\right] \end{array} \stackrel{s}{=} \left[\begin{array}{c|ccc} A & B_u & B_f & B_d \\ \hline C & D_u & D_f & D_d \end{array}\right] \tag{5.2}$$

are the system process, disturbance and fault transfer matrices, respectively.

In this chapter, the residual signal generator is given using a static observer of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + B_u u(t) - L\left(y(t) - C\hat{x}(t) - D_u u(t)\right), \\ r(t) &= H(y(t) - C\hat{x}(t) - D_u u(t)), \end{aligned} \tag{5.3}$$

where $\hat{x}(t) \in \mathcal{R}^n$ is the observer state and $r(t) \in \mathcal{R}^{n_f}$ is the residual signal. Here $L \in \mathcal{R}^{n\times n_y}$ and $H \in \mathcal{R}^{n_f\times n_y}$ are the observer and residual gain matrices, respectively, and are to be determined. Define the state estimation error signal as $e(t) = x(t) - \hat{x}(t)$. It follows that the residual dynamics are given by

$$\begin{aligned} \dot{e}(t) &= (A + LC)e(t) + (B_f + LD_f)f(t) + (B_d + LD_d)d(t), \\ r(t) &= HCe(t) + HD_f f(t) + HD_d d(t). \end{aligned}$$

Taking Laplace transforms, $r(s) = T_{rf}(s)f(s) + T_{rd}(s)d(s)$, where

$$\begin{aligned} \left[\begin{array}{cc} T_{rf}(s) & T_{rd}(s) \end{array}\right] &\stackrel{s}{=} \left[\begin{array}{c|cc} A + LC & B_f + LD_f & B_d + LD_d \\ \hline HC & HD_f & HD_d \end{array}\right] \\ &= F(s)\left[\begin{array}{cc} G_f(s) & G_d(s) \end{array}\right] \end{aligned} \tag{5.4}$$

are the transfer matrices from disturbances and faults to the residual, respectively, and where

$$F(s) \stackrel{s}{=} \left[\begin{array}{c|c} A + LC & L \\ \hline HC & H \end{array}\right] \tag{5.5}$$

is the state space of the FD filter.

With these preliminaries, the FD problem at a single frequency is described as follows.

Problem 5.1.1. *Let the system and filter dynamics be as defined in (5.1) and (5.5) and let $\omega > 0$ be given. Assume that:*

A1. The pair (A, C) is detectable (see Remark 5.1.1.)

A2. $G_f(j\omega)$ has full column rank.

A3. $n_y \geq n_f$ (see Remark 5.1.2.)

Find a stable filter $F(s)$, of the form given in (5.5) such that the optimum

$$\gamma_o = \inf_{\substack{\underline{\sigma}\left(F(j\omega)G_f(j\omega)\right)\geq 1 \\ F(s)\in\mathcal{H}_\infty^{n_f\times n_y}}} \|F(j\omega)G_d(j\omega)\|, \tag{5.6}$$

is achieved.

Remark 5.1.1. *Assumption A1 is needed to guarantee there exists at least one L such that $A+LC$ is stable.* □

Remark 5.1.2. *In the case that $n_y < n_f$, a bank of observers may be used such that for each observer $n_y \geq n_f$ is satisfied. In particular, in the case that $n_f > 1$, the banks of observers can also be used to isolate faults, namely, each observer is dedicated to detect and isolate one of the faults while the other faults are regarded as disturbances for that observer. See [1, 34, 12] for more details.* □

Remark 5.1.3. *The KYP lemma [47] shows that given system matrices (A, B, C, D), then $\|G(j\omega)\| < \gamma$ for a single frequency ω if and only if there exist Hermitian matrices P, $Q \in \mathcal{C}^{n\times n}$ such that $Q \succ 0$*

$$\begin{bmatrix} -A^TQA - j\omega QA + j\omega A^TQ - \omega^2 Q + PA + A^TP & \star & \star \\ -B^TQA + B^TP + j\omega B^TQ & -B^TQB - \gamma I & \star \\ C & D & -\gamma I \end{bmatrix} \prec 0 \tag{5.7}$$

By rearranging the (1,1) entry, the LMI in (5.7) can be equivalently expressed as

$$\begin{bmatrix} -(-j\omega I - A)^TQ(j\omega I - A) - (-j\omega I - A)^TP - P(j\omega - A) & \star & \star \\ B^TQ(j\omega I - A) + B^TP & -B^TQB - \gamma I & \star \\ C & D & -\gamma I \end{bmatrix} \prec 0 \tag{5.8}$$

Take a congruence transformation by post and pre multiplying T and T^* to (5.8), where

$$T = \begin{bmatrix} (j\omega I - A)^{-1} & (j\omega I - A)^{-1}B & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

it is obtained that

$$\begin{bmatrix} -Q-(-j\omega I-A)^{-T}P-P(j\omega I-A)^{-1} & \star & \star \\ -B^T(-j\omega I-A)^{-T}P & -\gamma I & \star \\ C(j\omega I-A)^{-1} & D+C(j\omega I-A)^{-1}B & -\gamma I \end{bmatrix} \prec 0. \tag{5.9}$$

Since both P and Q are free parameters, by following a Schur complement [6] it is easy to see that

$$\begin{bmatrix} -\gamma I & G^T(-j\omega) \\ G(j\omega) & -\gamma I \end{bmatrix} = \begin{bmatrix} -\gamma I & D^T+B^T(-j\omega I-A)^{-T}C^T \\ D+C(j\omega I-A)^{-1}B & -\gamma I \end{bmatrix} \prec 0 \tag{5.10}$$

is necessary and sufficient to guarantee (5.7) is satisfied. Hence, in the following sections only (5.10) is used to achieve the infimum of the norm of a transfer matrix rather than by using the GKYP lemma since it gives relatively easy formulation and avoids unnecessary computation of the variables P and Q. □

5.1.2 Equivalent problem reformulation

This section derives equivalent problem formulations to Problem 5.1.1 which are used to remove or relax constraints and to give a simple LMI solution to the FD filter design at a single frequency. First, an equivalent formulation to Problem 5.1.1 is given for the purpose of removing the constraint on the stability of $F(s)$. It shows that if there exists an $F(s) \in \mathcal{RL}_\infty^{n_f \times n_y}$ that achieves the minimum of the cost function, it is always possible to find a left coprime factorization $F(s) = (M(s))^{-1}\tilde{F}(s)$ such that $M(s) \in \mathcal{RH}_\infty^{n_f \times n_f}$ is inner and $\tilde{F}(s) \in \mathcal{RH}_\infty^{n_f \times n_y}$.

Lemma 5.1.1. *Let all variables be as defined in Problem 5.1.1. Then*

$$\begin{aligned} \gamma_o \;&:=\; \inf\{\|\tilde{F}(j\omega)G_d(j\omega)\| : \underline{\sigma}(\tilde{F}(j\omega)G_f(j\omega)) \geq 1, \\ &\qquad \tilde{F}(s) \overset{s}{=} (A+\tilde{L}C, \tilde{L}, \tilde{H}C, \tilde{H}) \in \mathcal{RH}_\infty^{n_f \times n_y}\} \\ &=\; \inf\{\|F(j\omega)G_d(j\omega)\| : \underline{\sigma}(F(j\omega)G_f(j\omega)) \geq 1, \\ &\qquad F(s) \overset{s}{=} (A+LC, L, HC, H) \in \mathcal{RL}_\infty^{n_f \times n_y}\} =: \gamma \end{aligned} \tag{5.11}$$

Proof. Note first that $\gamma_o \geq \gamma$ since $\mathcal{RH}_\infty^{n_f \times n_y} \subset \mathcal{RL}_\infty^{n_f \times n_y}$. Let $F(s) \overset{s}{=} (A+LC, L, HC, H) \in \mathcal{RL}_\infty^{n_f \times n_y}$ be a minimizer for (5.11) so that

$$\|F(j\omega)G_d(j\omega)\| = \gamma, \qquad \underline{\sigma}(F(j\omega)G_f(j\omega)) \geq 1.$$

To prove $\gamma_o = \gamma$ we construct $\tilde{F}(s) \overset{s}{=} (A+\tilde{L}C, \tilde{L}, \tilde{H}C, \tilde{H}) \in \mathcal{RH}_\infty^{n_f \times n_y}$ such that $\underline{\sigma}(\tilde{F}(j\omega)G_f(j\omega)) \geq 1$ and $\|\tilde{F}(j\omega)G_d(j\omega)\| = \gamma$. The rest of the proof depends on the detectability of the pair $(A+LC, HC)$.

(a) Suppose that the pair $(A + LC, HC)$ is detectable. Then it follows from Theorem 13.34 in [115] that there exists a left coprime factorization $F(s) = (M(s))^{-1}\tilde{F}(s)$ such that $M(s) \in \mathcal{RH}_\infty^{n_f \times n_f}$ is inner and $\tilde{F}(s) \in \mathcal{RH}_\infty^{n_f \times n_y}$ where

$$\begin{bmatrix} M(s) & \tilde{F}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A+LC+L_0HC & L_0 & L+L_0H \\ \hline HC & I_{n_f} & H \end{array}\right] = \left[\begin{array}{c|cc} A+\tilde{L}C & L_0 & \tilde{L} \\ \hline HC & I_{n_f} & H \end{array}\right]$$

where

$$\tilde{L} = L + L_0H, \qquad L_0 = -XC^TH^T \tag{5.12}$$

and where $X = X^T \in \mathcal{R}^{n\times n}$ is the stabilizing solution of the algebraic Riccati equation

$$(A + LC)X + X(A + LC)^T - XC^TH^THCX = 0. \tag{5.13}$$

Then $\|\tilde{F}(j\omega)G_d(j\omega)\| = \|M(j\omega)F(j\omega)G_d(j\omega)\| = \gamma$ and $\underline{\sigma}(\tilde{F}(j\omega)G_f(j\omega)) = \underline{\sigma}(M(j\omega)F(j\omega)G_f(j\omega)) \geq 1$ since $M(s)$ is inner. Thus $\gamma_o = \gamma$.

(b) Suppose that the pair $(A + LC, HC)$ is not detectable. Then, introducing a similarity transformation on the data if necessary,

$$F(s) \stackrel{s}{=} \left[\begin{array}{c|c} A+LC & L \\ \hline HC & H \end{array}\right] = \left[\begin{array}{cc|c} A_{11}+L_1C_1 & A_{12}+L_1C_2 & L_1 \\ 0 & A_{22}+L_2C_2 & L_2 \\ \hline 0 & HC_2 & H \end{array}\right] \tag{5.14}$$

where $A_{11} + L_1C_1$ is antistable (that is, all its poles are in the open right half-plane), the pair $(A_{22} + L_2C_2, HC_2)$ is detectable, $HC_1 = 0$ and $A_{21} + L_2C_1 = 0$ and where the partitioning of A and C is induced by the partitioning of the realization of $F(s)$ in (5.14). Next we prove that the pair (A_{11}, C_1) is detectable. Since we assume that the pair (A, C) is detectable, it follows that

$$\begin{bmatrix} A_{11} - sI & A_{12} \\ A_{21} & A_{22} - sI \\ C_1 & C_2 \end{bmatrix}$$

has full column rank for all s such that $s + s^* > 0$. Using the fact that $A_{21} = -L_2C_1$, it follows that

$$\begin{bmatrix} A_{11} - sI & A_{12} \\ 0 & A_{22} + L_2C_2 - sI \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & L_2 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{11} - sI & A_{12} \\ A_{21} & A_{22} - sI \\ C_1 & C_2 \end{bmatrix}$$

has full column rank for all s such that $s + s^* > 0$. Thus the pair (A_{11}, C_1) is detectable. It follows that there exists $\tilde{L}_1$ such that $A_{11} + \tilde{L}_1 C_1$ is stable. Define $\tilde{F}(s) \stackrel{s}{=} (A + \tilde{L}C, \tilde{L}, HC, H)$ where $\tilde{L} = [\tilde{L}_1^T \;\; L_2^T]^T$. Then

$$\tilde{F}(s) \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} + \tilde{L}_1 C_1 & A_{12} + \tilde{L}_1 C_2 & \tilde{L}_1 \\ 0 & A_{22} + L_2 C_2 & L_2 \\ \hline 0 & HC_2 & H \end{array}\right]$$

is detectable. Furthermore, removing a nonminimal part of the realizations of $F(s)$ and $\tilde{F}(s)$ we get $\tilde{F}(s) = F(s)$. Then $\|\tilde{F}(j\omega)G_d(j\omega)\| = \gamma$ and $\underline{\sigma}(\tilde{F}(j\omega)G_f(j\omega)) \geq 1$. Thus it follows from Part (a) that $\gamma_o = \gamma$. □

The next result shows that we may restrict $F(s)$ such that $F(j\omega)$ is the set of all left–inverses of $G_f(j\omega)$.

Lemma 5.1.2. *Let all variables be as defined in Problem 5.1.1. Then*

$$\begin{aligned} \gamma_o &= \inf\{\|\tilde{F}(j\omega)G_d(j\omega)\| : \underline{\sigma}(\tilde{F}(j\omega)G_f(j\omega)) \geq 1, \\ &\qquad \tilde{F}(s) \stackrel{s}{=} (A + \tilde{L}C, \tilde{L}, \tilde{H}C, \tilde{H}) \in \mathcal{RL}_\infty^{n_f \times n_y}\} \\ &= \inf\{\|F(j\omega)G_d(j\omega)\| : F(j\omega)G_f(j\omega) = I_{n_f}, \\ &\qquad F(s) \stackrel{s}{=} (A + LC, L, HC, H) \in \mathcal{RL}_\infty^{n_f \times n_y}\} =: \gamma \end{aligned} \tag{5.15}$$

Proof. Note first that $\gamma_o \leq \gamma$ since $\underline{\sigma}(I_{n_f}) \geq 1$. Let $\tilde{F}(s) \stackrel{s}{=} (A + \tilde{L}C, \tilde{L}, \tilde{H}C, \tilde{H})$ be a minimizer for (5.15) and define $Q(s) = \tilde{F}(s)G_f(s)$ so that $\underline{\sigma}(Q(j\omega)) \geq 1$ and $\|\tilde{F}(j\omega)G_d(j\omega)\| = \gamma_o$. Then $Q^{-1}(s) \in \mathcal{RL}_\infty^{n_f \times n_f}$ and $\|Q^{-1}(j\omega)\| \leq 1$. Let $F(s) = (Q(s))^{-1}\tilde{F}(s) \in \mathcal{RL}_\infty^{n_f \times n_f}$. Then a state–space manipulation verifies that $F(s) \stackrel{s}{=} (A + LC, L, HC, H)$ where $L = \tilde{L} - (B_f + \tilde{L}D_f)(\tilde{H}D_f)^{-1}\tilde{H}$ and $H = (\tilde{H}D_f)^{-1}\tilde{H}$. Furthermore, $F(j\omega)G_f(j\omega) = I_{n_f}$. Now, $\|F(j\omega)G_d(j\omega)\| = \|Q^{-1}(j\omega)\tilde{F}(j\omega)G_d(j\omega)\| \leq \|\tilde{F}(j\omega)G_d(j\omega)\| = \gamma_0$ since $\|Q^{-1}(j\omega)\| \leq 1$. This proves that $\gamma \leq \gamma_0$ and since $\gamma \geq \gamma_o$ it follows that $\gamma_o = \gamma$. □

Remark 5.1.4. *As can be concluded from Lemma 5.1.2 that the optimal cost function is obtained by achieving the minimum of $\|F(j\omega)G_d(j\omega)\|$ where $F(s)$ is such that $F(j\omega)$ is the set of all left inverses of $G_f(j\omega)$. Hence, the constraint on $F(j\omega)$ can be expressed as an equality constraint such that $F(j\omega)G_f(j\omega) = I_{n_f}$.* □

Remark 5.1.5. *Note that the equality constraint $F(j\omega)G(j\omega) = I$ where $F(s) \in \mathcal{RL}_\infty^{n_f \times n_y}$ is the condition to ensure fault isolation at ω, which constructs an identity transfer matrix from faults to residuals. It means that if the obtained $F(s)$ is stable, optimal fault detection filter also achieves*

fault isolation. In the case that $F(s)$ is not stable, by using the techniques in Lemma 5.1.1, it is always possible to find an $\tilde{F}(s)$ that achieves optimal fault detection, however $\tilde{F}(j\omega)G_f(j\omega) \neq I_{n_f}$ but is unitary. □

5.1.3 LMI solution to Problem 5.1.1

As stated in Lemma 5.1.2, the FD problem is reduced to the solution to the optimization problem in (5.15). In this section, a parameterization of all L and H satisfying the equality constraint $F(j\omega)G_f(j\omega) = I_{n_f}$ is derived. Then, Problem 5.1.1 is solved by using LMI optimization techniques in which stability constraints are not imposed.

Firstly, let all variables and assumptions be as given in Problem 5.1.1 and define

$$Y = (j\omega I - A - LC)^{-1}L. \tag{5.16}$$

By following some matrix manipulation, it can be seen that

$$\begin{aligned}
& (j\omega I - A - LC)^{-1} \\
= \; & -(A - j\omega I)^{-1}\left[I + LC(A - j\omega I)^{-1}\right]^{-1} \\
= \; & -(A - j\omega I)^{-1}\left[I - L(I + C(A - j\omega I)^{-1}L)^{-1}C(A - j\omega I)^{-1}\right] \\
= \; & -(A - j\omega I)^{-1}\left[I - (I + LC(A - j\omega I)^{-1})^{-1}LC(A - j\omega I)^{-1}\right] \\
= \; & (j\omega I - A)^{-1} + (j\omega I - A - LC)^{-1}LC(j\omega I - A)^{-1} \\
= \; & (j\omega I - A)^{-1} + YC(j\omega I - A)^{-1}
\end{aligned} \tag{5.17}$$

Next, let $F(s)$ and $G_f(s)$ be defined as in (5.4) and (5.5). By using (5.16) and (5.17), $F(j\omega)G_f(j\omega)$ can be expressed as

$$\begin{aligned}
F(j\omega)G_f(j\omega) &= HD_f + HC(j\omega I - A - LC)^{-1}(B_f + LD_f) \\
&= HD_f + HC(j\omega I - A - LC)^{-1}B_f + HCYD_f \\
&= H(I + CY)(D_f + C(j\omega I - A)^{-1}B_f) \\
&= H(I + CY)G_f(j\omega)
\end{aligned} \tag{5.18}$$

Since it is assumed that $G_f(j\omega)$ has full column rank, there exist $G_f^{\dagger} \in \mathcal{C}^{n_f \times n_y}$ and $G_f^{\perp} \in \mathcal{C}^{(n_y - n_f)\times n_y}$ which can be obtained from a singular value decomposition of $G_f(j\omega)$, such that

$$\begin{bmatrix} G_f^{\dagger} \\ G_f^{\perp} \end{bmatrix} G_f(j\omega) = \begin{bmatrix} I_{n_f} \\ 0 \end{bmatrix}, \quad \operatorname{rank}\left(\begin{bmatrix} G_f^{\dagger} \\ G_f^{\perp} \end{bmatrix}\right) = n_y. \tag{5.19}$$

It follows that all $H(I+CY)$ satisfying the constraint $F(j\omega)G_f(j\omega) = I_{n_f}$ can be parameterized by

$$H(I + CY) = G_f^{\dagger} + SG_f^{\perp} \tag{5.20}$$

where $S \in \mathcal{C}^{n_f \times (n_y - n_f)}$ is a free parameter and $S = S_1 + jS_2$ where $S_1, S_2 \in \mathcal{R}^{n_f \times (n_y - n_f)}$. The next theorem gives an LMI solution to the optimization problem proposed in Lemma 5.1.2 by rearranging the variables as given in (5.16) and (5.17).

Theorem 5.1.1. *Let all variables be as defined in Problem 5.1.1. Then*

$$\gamma_o \;=\; \inf\{\gamma : S_1, S_2 \in \mathcal{R}^{n_f \times (n_y - n_f)}, \begin{bmatrix} W_1 & W_2 \\ -W_2 & W_1 \end{bmatrix} \prec 0\} \tag{5.21}$$

where

$$\begin{aligned} W_1 &= \begin{bmatrix} -\gamma I & \bar{G}_{f1}^T + \tilde{G}_{f1}^T S_1^T - \tilde{G}_{f2}^T S_2^T \\ \bar{G}_{f1} + S_1\tilde{G}_{f1} - S_2\tilde{G}_{f2} & -\gamma I \end{bmatrix}, \\ W_2 &= \begin{bmatrix} 0 & -\bar{G}_{f2}^T - \tilde{G}_{f2}^T S_1^T - \tilde{G}_{f1}^T S_2^T \\ \bar{G}_{f2} + S_1\tilde{G}_{f2} + S_2\tilde{G}_{f1} & 0 \end{bmatrix}, \end{aligned}$$

$\bar{G}_{f1}$ and $\bar{G}_{f2}$ are the real and imaginary part of $G_f^{\dagger}(D_d + C(j\omega I - A)^{-1}B_d)$, respectively while $\tilde{G}_{f1}$ and $\tilde{G}_{f2}$ are the real and imaginary part of $G_f^{\perp}(D_d + C(j\omega I - A)^{-1}B_d)$, respectively.

Proof. As shown in (5.10), $\|F(j\omega)G_d(j\omega)\| < \gamma$ if and only if

$$\begin{bmatrix} -\gamma I & G_d^T(-j\omega)F^T(-j\omega) \\ F(j\omega)G_d(j\omega) & -\gamma I \end{bmatrix} \prec 0 \tag{5.22}$$

where $F(j\omega)G_d(j\omega) = HD_d + HC(j\omega I - A - LC)^{-1}(B_d + LD_d)$. By using (5.16) and (5.17), the nonlinear matrix inequality in (5.22) can be written as

$$\begin{bmatrix} -\gamma I & G_d^T(-j\omega)(I + CY)^* H^T \\ H(I + CY)G_d(j\omega) & -\gamma I \end{bmatrix} \prec 0 \tag{5.23}$$

To ensure $F(j\omega)G_f(j\omega) = I_{n_f}$, the parameterization given in (5.20) is used in the matrix inequality to give

$$\begin{bmatrix} -\gamma I & G_d^*(j\omega)\left[G_f^{\dagger} + SG_f^{\perp}\right]^* \\ \left[G_f^{\dagger} + SG_f^{\perp}\right] G_d(j\omega) & -\gamma I \end{bmatrix} \prec 0 \tag{5.24}$$

It can be seen that (5.24) is linear but complex. To numerically solve this LMI problem, it is

necessary to convert the complex LMI to a real LMI with larger dimensions by noting that if $W = W_1 + jW_2$ where W_1 and W_2 are real, then

$$W \prec 0 \Leftrightarrow \begin{bmatrix} W_1 & W_2 \\ -W_2 & W_1 \end{bmatrix} \prec 0. \tag{5.25}$$

□

Since S can be obtained by solving the LMI optimazation problem in (5.21), then by following (5.20) the relation between L and H can be expressed as

$$H = Z(I + CY)^{-1} = Z[I + C(j\omega I - A - LC)^{-1}L]^{-1} \tag{5.26}$$

where $Z = G_f^{\dagger} + SG_f^{\perp}$. Hence, L can be chosen first and then H is computed by solving (5.26). However, H may be a complex matrix. Next, a parameterization of L is derived such that H is real.

Take Z_1 and Z_2 to be the real and imaginary parts of $G_f^{\dagger} + SG_f^{\perp}$, respectively, where Z_1, $Z_2 \in \mathcal{R}^{n_f \times n_y}$. Also, set

$$E = (Z_2CA + \omega Z_1 C)(A^2 + \omega^2 I)^{-1}. \tag{5.27}$$

Then

$$\begin{aligned} H &= Z[I + C(j\omega I - A - LC)^{-1}L]^{-1} \\ &= Z[I + C(j\omega I - A)^{-1}(I - LC(j\omega I - A)^{-1})^{-1}L]^{-1} \\ &= Z[I + C(j\omega I - A)^{-1}L(I - C(j\omega I - A)^{-1}L)^{-1}]^{-1} \\ &= Z[I - C(j\omega I - A)^{-1}L] \\ &= (Z_1 + jZ_2)[I + C(A^2 + \omega^2 I)^{-1}(j\omega I + A)L]. \end{aligned} \tag{5.28}$$

It follows that L should be chosen such that the imaginary part of H is zero. Hence, L should satisfy

$$(Z_2CA + \omega Z_1 C)(A^2 + \omega^2 I)^{-1}L = EL = -Z_2 \tag{5.29}$$

By taking a singular value decomposition of E, it follows that all L satisfying (5.29) is of the form

$$L = -E^{\dagger}Z_2 + E^{\perp}K \tag{5.30}$$

where $K \in \mathcal{R}^{(n-n_f)\times n_y}$ is free. $E^{\dagger} \in \mathcal{R}^{n \times n_f}$ and $E^{\perp} \in \mathcal{R}^{n \times (n-n_f)}$ are obtained from a singular value decomposition of E such that

$$E \begin{bmatrix} E^{\dagger} & E^{\perp} \end{bmatrix} = \begin{bmatrix} I_{n_f} & 0 \end{bmatrix}, \quad \operatorname{rank} \begin{bmatrix} E^{\dagger} & E^{\perp} \end{bmatrix} = n. \tag{5.31}$$

Using this parameterization of L it follows that H is given by

$$H = Z_1 + (Z_1CA - \omega Z_2C)(A^2 + \omega^2 I)^{-1}L \tag{5.32}$$

To summarize the results obtained above, a procedure is given to illustrate the steps to solve Problem (5.1.1).

Algorithm 5.1.1. *Let $G_d(s) \in \mathcal{RL}_\infty^{n_y \times n_d}$ and $G_f(s) \in \mathcal{RL}_\infty^{n_y \times n_f}$ be as given in (5.2) and let $\omega > 0$ be given. Assume that $n_y \geq n_f$, (A, C) is detectable and that $G_f(j\omega)$ has full rank.*

1. *Define $G_f^\dagger$ and $G_f^\perp$ such that (5.19) is satisfied.*
2. *Compute $\bar{G}_{f1}$, $\bar{G}_{f2}$, $\tilde{G}_{f1}$ and $\tilde{G}_{f2}$ from (5.21).*
3. *Find γ_o and S that solve the LMI optimization in Theorem (5.1.1).*
4. *Define Z_1, Z_2 and compute E by following (5.27).*
5. *Parameterize L as shown in (5.30) and choose K, say $K = 0$, find L from (5.30) and compute H from (5.32).*
   (a) *If $A + LC$ is stable, we are done.*
   (b) *Else, if the pair $(A + LC, HC)$ is detectable, let $X = X^T \in \mathcal{R}^{n \times n}$ be the stabilizing solution of the algebraic Riccati equation (5.13) and redefine $L := L - XC^T H^T H$.*
   (c) *Else, redefine L using the procedure in part (2) of the proof of Lemma 5.1.1 so that the pair $(A + LC, HC)$ is detectable and go to Step 5(b).*
6. *The optimal FD filter is given by $F(s) \stackrel{s}{=} (A + LC, L, HC, H) \in \mathcal{RH}_\infty^{n_f \times n_y}$.*

Remark 5.1.6. *Since L can be parameterized as shown in (5.30), it follows that there remains the degrees of freedom in K to ensure other design specifications. For example, to achieve fault isolation requires the stability of $A + LC$ (see Remark 5.1.5). In other words, it is required to find a K such that $A + LC = A - E^\dagger Z_2 C + E^\perp KC$ is stable. However, this is equivalent to the problem of stabilization using static output feedback, which is known to be a difficult problem. See [83] for a recent technique for solving this problem.* □

5.1.4 FD problem at DC

This section considers the special case when $\omega = 0$, that is, we consider the FD problem at DC. In order to exploit the available degrees of freedom, $\|FG_d\|_\infty$ is minimized rather than $\|F(0)G_d(0)\|$.

Problem 5.1.2. *Let the variables and assumptions as defined in Problem (5.1.1). Find a stable filter $F(s)$ such that the optimum*

$$\gamma_o = \inf_{\substack{F(0)G_f(0)=I_{n_f} \\ F(s)\in\mathcal{H}_\infty^{n_f\times n_y}}} \|FG_d\|_\infty, \tag{5.33}$$

is achieved, where $G_f(0) = D_f - CA^{-1}B_f$

Next, a parameterization of L and H is given such that the equality constraint in Problem 5.1.2 is always satisfied. Let $Y = (A+LC)^{-1}L$, then $(A+LC)^{-1} = A^{-1} - YCA^{-1}$. By setting $w = 0$ and following similar lines of the derivations in Section 5.1.3, it follows that

$$H(I - CY)G_f(0) = I_{n_f} \tag{5.34}$$

where $G_f(0) = D_f - CA^{-1}B_f$. Take $\tilde{H} = H(I - CY)$. Next, by using a singular value decomposition of $G_f(0)$ find $G_f^\dagger \in \mathcal{R}^{n_f\times n_y}$ and $G_f^\perp \in \mathcal{R}^{(n_y-n_f)\times n_y}$,

$$\begin{bmatrix} G_f^\dagger \\ G_f^\perp \end{bmatrix} G_f(0) = \begin{bmatrix} I_{n_f} \\ 0 \end{bmatrix}, \operatorname{rank}\left(\begin{bmatrix} G_f^\dagger \\ G_f^\perp \end{bmatrix}\right) = n_y, \tag{5.35}$$

$\tilde{H}$ can be parameterized such that all $\tilde{H}$ satisfying (5.34) are of the form

$$\tilde{H} = H(I - CY) = G_f^\dagger + S_0 G_f^\perp \tag{5.36}$$

where $G_f^\dagger \in \mathcal{R}^{n_f\times n_y}$ and $G_f^\perp \in \mathcal{R}^{(n_y-n_f)\times n_y}$.

Note that $S_0 \in \mathcal{R}^{n_f\times(n_y-n_f)}$ is a free parameter. Following from the application of a congruence transformation to the bounded real lemma [6], the theorem below gives an LMI solution to the optimization problem in (5.33) without imposing stability constraint.

Theorem 5.1.2. *Let all variables be as defined in Problem 5.1.2. Then*

$$\gamma_o = \inf\{\gamma : R\in\mathcal{R}^{n\times n_y}, S_0\in\mathcal{R}^{n_f\times(n_y-n_f)}, P = P^T\in\mathcal{R}^{n\times n}, W_0 \prec 0\} \tag{5.37}$$

where

$$W_0 = \begin{bmatrix} PA^{-1} + A^{-T}P - RCA^{-1} - A^{-T}C^TR^T & \star & \star \\ -B_d^TA^{-T}P - {G_d}^T(0)R^T & -\gamma I & \star \\ (G_f^\dagger + S_0G_f^\perp)CA^{-1} & (G_f^\dagger + S_0G_f^\perp)G_d(0) & -\gamma I \end{bmatrix} \tag{5.38}$$

and $G_d(0) = D_d - CA^{-1}B_d$.

Proof. Using the Bounded Real Lemma [6], $\|FG_d\|_\infty < \gamma$ if and only if there exists a symmetric P such that

$$\begin{bmatrix} P(A+LC) + (A+LC)^TP & P(B+LD_d) & (HC)^T \\ (B+LD_d)^TP & -\gamma I & (HD_d)^T \\ HC & HD_d & -\gamma I \end{bmatrix} \prec 0 \tag{5.39}$$

Take a congruence transformation by post and pre multiplying T and its transpose to (5.39), where

$$T = \begin{bmatrix} (A+LC)^{-1} & -(A+LC)^{-1}(B_d+LD_d) & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

This gives

$$\begin{bmatrix} (A+LC)^{-T}P + P(A+LC)^{-1} & \star & \star \\ -(B_d+LD_d)^T(A+LC)^{-T}P & -\gamma I & \star \\ HC(A+LC)^{-1} & H[D_d - C(A+LC)^{-1}(B_d+LD_d)] & -\gamma I \end{bmatrix} \prec 0 \tag{5.40}$$

Redefine $(A+LC)^{-1}L = Y$ and $(A+LC)^{-1} = A^{-1} - YCA^{-1}$ and apply the parameterization given in (5.36), then the nonlinear matrix inequality in (5.40) is equivalent to the LMI in (5.37).□

Remark 5.1.7. *As noted, to achieve optimal fault detection, the stability condition $P \succ 0$ is not imposed as in [52]. This implies that the design variable L in (5.37) may not be stabilizing. However, by following similar procedures as given in Lemma 5.1.1, the filter can be stabilized without losing any optimality.* □

Remark 5.1.8. *For a special case of Theorem 5.1.1, namely, at DC, the variable S can be solved from the LMI optimization in (5.21) by setting $\omega = 0$. However, by following (5.20), only the product $H(I - C(A+LC)^{-1}L)$ should be fixed for achieving the optimal solution. Hence, the filter parameter L can be freely chosen firstly and H can be then obtained from L. It follows that the observer parameter L can be used to implement other control specifications such as LQG control or Kalman filter [115] first, then with little extra effort, an H from (5.26) can be implemented and optimal fault detection at DC can also be achieved.* □

5.1.5 Numerical examples

To demonstrate the effectiveness of the proposed fault detection filter scheme in a real system, a jet engine example is considered in this section. The state-space model of the GE-21 jet engine [48, 67] is given as

$$\begin{aligned} A &= \begin{bmatrix} -3.370 & 1.636 \\ -0.325 & -1.896 \end{bmatrix}, \; B_u = \begin{bmatrix} 0.586 & -1.419 & 1.252 \\ 0.410 & 1.118 & 0.139 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.731 & 0.786 \end{bmatrix}, \quad D_u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.267 & -0.025 & -0.146 \end{bmatrix}. \end{aligned}$$

The system has $n = 2$ and $n_y = n_u = 3$. Suppose that this system is subject to two potential faults and two disturbances ($n_f = 2$ and $n_d = 2$). The distribution matrix of faults are given by

$$B_f = \begin{bmatrix} 0.586 & -1.419 \\ 0.410 & 1.118 \end{bmatrix}, \quad D_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

Since the example in [67] does not involve disturbances, B_d and D_d are randomly generated as follows:

$$B_d = \begin{bmatrix} -0.2973 & 0.0467 \\ 0.0094 & -0.1814 \end{bmatrix}, \quad D_d = \begin{bmatrix} 0.0865 & 0.1177 \\ 0.3331 & -0.4366 \\ -0.0251 & 0.0273 \end{bmatrix}.$$

Suppose we consider the fault detection problem at $w = 1$. Since $n_f < n_y$, we apply Algorithm 5.1.1 in Section 5.1.3. By solving (5.21) in Theorem 5.1.1, we obtain $S = \begin{bmatrix} -0.0420 + j0.1717 & 1.6657 + j0.9250 \end{bmatrix}^T$ and the corresponding optimal $\gamma_o = 0.2669$. By using (5.30) and (5.32), the observer gains are given as follows

$$L = \begin{bmatrix} -0.5860 & 1.4190 & 13.1302 \\ -0.4100 & -1.1180 & -11.3298 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 0 & -0.0040 \\ 0 & 1 & 10.3635 \end{bmatrix}.$$

Note that since $A + LC$ is stable, then $F(j\omega)G_f(j\omega) = I_{n_f}$ and the given filter achieves fault isolation at $w = 1$.

Suppose that disturbance 1 is a band-limited white noise with power 0.2 (zeroth-order-hold with sampling time 0.1s) and disturbance 2 is a constant biases of amplitude 1 occurred from 2

second. The two faults are assumed to be $f_1 = 0.5sin(t)$ and $f_2 = sin(t)$ connected from the 10th and 15th seconds, respectively. Figure 1 gives the residual responds before the filter, while Figure 2 shows the filtered residual responds. The example demonstrates that the designed filter gives satisfactory fault detection and isolation performance and also shows its robustness against disturbances.

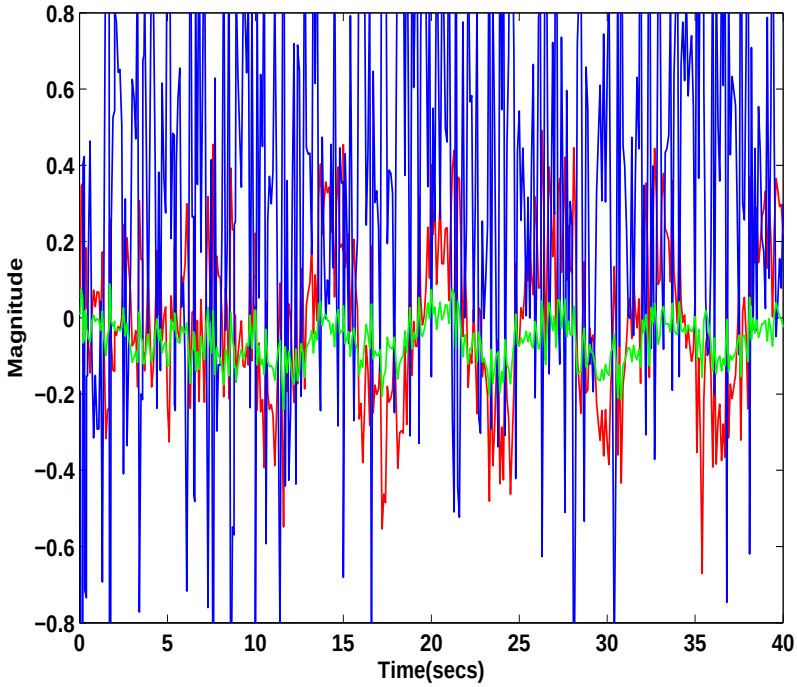


Figure 5.1: Residual time responds before the filter

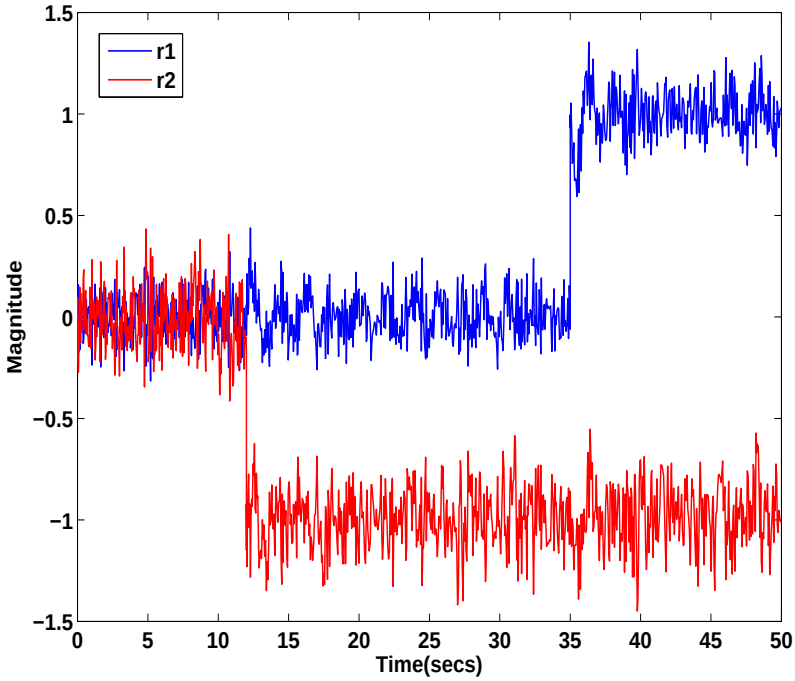


Figure 5.2: Residual time responds after the filter

5.2 Fault detection problem over a frequency range

This section gives an initial research on the FD problem over a frequency range. By applying the result in Chapter 4, a novel problem formulation is proposed, which transforms this problem to an optimization problem which is to, over a frequency range, minimize the sensitivity to disturbances with respect to the sensitivity to faults. Then by using a generalized GKYP lemma, a solution is derived in the form of an LMI optimization followed by a linearization procedure.

5.2.1 Problem formulation

Consider a linear time-invariant dynamic system subject to disturbances and faults as given in (4.1) and suppose the residual generator using a static observer be of the form in (4.2). For $G(s) \in \mathcal{RH}_\infty^{m\times p}$, define

$$\|G\|_\omega = \sup_{|\omega_l|<\omega} \bar{\sigma}\left(G(j\omega_l)\right). \tag{5.41}$$

Then the FD problem over a frequency range is formulated as follows.

Problem 5.2.1. *Let all variables and assumptions be as given in Problem 4.1.2. For a given*

$\gamma > 0$, find L and H, if they exist, such that $A + LC$ is stable and

$$\left\| T_{rf}^{-1} T_{rd} \right\|_{\omega} < \gamma_o. \tag{5.42}$$

where

$$T_{rf}^{-1}(s) T_{rd}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_1 + Z_0 C_2 & B_{d,1} + Z_0 D_{d,2} \\ \hline C_1 + U_0 C_2 & D_{d,1} + U_0 D_{d,2} \end{array}\right], \tag{5.43}$$

and where A_1, C_1, C_2, $B_{d,1}$, $D_{d,1}$, $D_{d,2}$, Z_0 and U_0 are as defined in (4.11).

Remark 5.2.1. *The major difference of this work from Problem 4.1.2 is that the cost function (5.42) which combines both fault detection and disturbance attenuation requirements is minimized only over a finite frequency range such that the 'ratio' between these two objectives is optimized over a limited frequency domain rather than over the whole frequency range.* □

5.2.2 LMI solution to Problem 5.2.1

The previous section proposes a single cost function which captures both fault detection and disturbance attenuation, then the FD problem is transformed to a minimization problem using the norm as a measure as defined in (5.41). In this section, an LMI solution to the problem is given by using the following generalized KYP lemma [47].

Firstly, let $Y_\omega = (j\omega I - A_1 - Z_0 C_2)^{-1} Z_0$, by taking similar matrix manipulations as in (5.17) and (5.18), we have

$$\begin{aligned} &(j\omega I - A_1 - Z_0 C_2)^{-1} = (j\omega I - A_1)^{-1} + Y_\omega C_2 (j\omega I - A_1)^{-1} \\ &(j\omega I - A_1 - Z_0 C_2)^{-1}(B_{d,1} + Z_0 D_{d,2}) = (j\omega I - A_1)^{-1} B_{d,1} + Y_\omega \left[D_{d,2} + C_2 (j\omega I - A_1)^{-1} B_{d,1} \right] \end{aligned} \tag{5.44}$$

Also, take $\tilde{H} = U_0(I + C_2 Y_\omega) + C_1 Y_\omega$. Then

$$\begin{aligned} &(C_1 + U_0 C_2)(j\omega I - A_1 - Z_0 C_2)^{-1} = C_1 (j\omega I - A_1)^{-1} + \tilde{H} C_2 (j\omega I - A_1)^{-1}, \\ &(D_{d,1} + U_0 D_{d,2}) + (C_1 + U_0 C_2)(j\omega I - A_1 - Z_0 C_2)^{-1}(B_{d,1} + Z_0 D_{d,2}) = \\ &\qquad \left[D_{d,1} + C_1 (j\omega I - A_1)^{-1} B_{d,1} \right] + \tilde{H} \left[D_{d,2} + C_2 (j\omega I - A_1)^{-1} B_{d,1} \right]. \end{aligned} \tag{5.45}$$

To simply the notations, in the following we take $(j\omega I - A_1)^{-1} = A_1^r + j A_1^i$ where $A_1^r, A_1^i \in \mathcal{R}^{n \times n}$. The next result shows that the FD problem over a frequency range can be solved by using the following form of the generalized KYP lemma [47].

Lemma 5.2.1. *Given matrices (A, B, C, D), for $\forall\ |\omega_l| < \omega$, the following finite frequency domain inequality*

$$\left\|D + C(j\omega_l I - A)^{-1}B\right\| < \gamma_o \tag{5.46}$$

holds if and only if there exist hermitian matrices $Q \succ 0$ and P such that the following LMI

$$\begin{bmatrix} -A^TQA + PA + A^TP + \omega^2 Q & \star & \star \\ -B^TQA + B^TP & -B^TQB - \gamma_o I & \star \\ C & D & -\gamma_o I \end{bmatrix} \prec 0 \tag{5.47}$$

□

The next theorem gives a solution to Problem 5.2.1 using Lemma 5.2.1 followed by a series of linearizations.

Theorem 5.2.1. *Let all variables be as defined in Problem 5.1.1. Then*

$$\begin{aligned} \gamma_o = \inf\{\gamma : X_1, X_2 \in \mathcal{R}^{n\times n}, Z_1, Z_2 \in \mathcal{R}^{n\times(n_y-n_f)}, \\ \tilde{H}_1, \tilde{H}_2 \in \mathcal{R}^{n_f\times(n_y-n_f)}, M = \begin{bmatrix} M^r & M^{iT} \\ M^i & M^r \end{bmatrix} \prec 0\} \end{aligned} \tag{5.48}$$

where

$$M^r = \begin{bmatrix} M^r_{11} & \star & \star \\ M^r_{21} & -\gamma_o I & \star \\ M^r_{31} & M^r_{32} & -\gamma_o I \end{bmatrix},\ M^i = \begin{bmatrix} M^i_{11} & -{M^i_{21}}^T & -{M^i_{31}}^T \\ M^i_{21} & 0 & -{M^i_{32}}^T \\ M^i_{31} & M^i_{32} & 0 \end{bmatrix}.$$

and where

$$\begin{aligned} M^r_{11} &= \frac{X_2 + X_2^T}{2\omega} - (X_1A^r_1 - X_2A^i_1) - (X_1A^r_1 - X_2A^i_1)^T - (Z_1C_2A^r_1 - Z_2C_2A^i_1) \\ &\quad -(Z_1C_2A^r_1 - Z_2C_2A^i_1)^T, \\ M^r_{21} &= -(X_1A^r_1B_{d,1} - X_2A^i_1B_{d,1})^T - [Z_1(D_{d,2} + C_2A^r_1B_{d,1}) - Z_2C_2A^i_1B_{d,1}]^T \\ M^r_{31} &= \tilde{H}_1C_2A^r_1 - \tilde{H}_2C_2A^i_1 + C_2A^r_1, \\ M^r_{32} &= \tilde{H}_1(D_{d,2} + C_2A^r_1B_{d,1}) - \tilde{H}_2C_2A^i_1B_{d,1} + (D_{d,2} + C_2A^r_1B_{d,1}), \\ M^i_{11} &= \frac{X_1^T - X_1}{2\omega} - (X_2A^r_1 + X_1A^i_1) + (X_2A^r_1 + X_1A^i_1)^T - (Z_1C_2A^i_1 + Z_2C_2A^r_1), \\ &\quad +(Z_1C_2A^i_1 + Z_2C_2A^r_1)^T, \\ M^i_{21} &= (X_2A^r_1B_{d,1} + X_1A^i_1B_{d,1})^T + [Z_2(D_{d,2} + C_2A^r_1B_{d,1}) + Z_1C_2A^i_1B_{d,1}]^T, \\ M^i_{31} &= \tilde{H}_1C_2A^i_1 + \tilde{H}_2C_2A^r_1 + C_2A^i_1, \\ M^i_{32} &= \tilde{H}_2(D_{d,2} + C_2A^r_1B_{d,1}) + \tilde{H}_1C_2A^i_1B_{d,1} + C_2A^i_1B_{d,1}. \end{aligned}$$

Proof. Firstly, by rearranging the entries, the LMI in (5.47), applied to the realisation in (5.43), can be equivalently expressed as

$$\begin{bmatrix} -(-j\omega I-A)^TQ(j\omega I-A)-(-j\omega I-A)^T(P-j\omega Q)-(P+j\omega Q)(j\omega I-A) & \star & \star \\ B^TQ(j\omega I-A)+B^T(P-j\omega Q) & -B^TQB-\gamma I & \star \\ C & D & -\gamma I \end{bmatrix} \prec 0$$

Apply a congruence transformation by post- and pre-multiplying (5.39) by T and its Hermitian where

$$T=\begin{bmatrix} (j\omega I-A)^{-1} & (j\omega I-A)^{-1}B & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Then,

$$\begin{bmatrix} -Q-(-j\omega I-A^T)^{-1}(P-j\omega Q)-(P+j\omega Q)(j\omega I-A)^{-1} & \star & \star \\ B^T(j\omega I-A^T)^{-1}(P+j\omega Q) & -\gamma I & \star \\ C(j\omega I-A)^{-1} & D+C(j\omega I-A)^{-1}B & -\gamma I \end{bmatrix} \prec 0$$

By substituting the state space model formulated in (5.43) into (5.49), a nonlinear matrix inequality is obtained as follows.

$$M = \begin{bmatrix} M_{11} & \star & \star \\ M_{21} & -\gamma I & \star \\ M_{31} & M_{32} & -\gamma I \end{bmatrix} \prec 0, \tag{5.49}$$

where

$$\begin{aligned} M_{11} &= -Q-(-j\omega I-A_1^T-C_2^TZ_0^T)^{-1}(P-j\omega Q)-(P+j\omega Q)(j\omega I-A_1-Z_0C_2)^{-1}, \\ M_{21} &= B^T(j\omega I-A_1^T-C_2^TZ_0^T)^{-1}(P+j\omega Q), \\ M_{31} &= (C_1+U_0C_2)(j\omega I-(A_1+Z_0C_2))^{-1}, \\ M_{32} &= (D_{d,1}+U_0D_{d,2})+(C_1+U_0C_2)(j\omega I-A_1-Z_0C_2)^{-1}(B_{d,1}+Z_0D_{d,2}). \end{aligned}$$

Take $X = P - j\omega Q$ and $Z_0 = XY_\omega$, $X_1 = P_1 + \omega Q_2$, $X_2 = P_2 - \omega Q_1$ and by rearranging the variables as shown in (5.44) and (5.45), the matrix inequality in (5.49) is linearized as follows.

$$\begin{aligned} M_{11} &= \frac{X-X^*}{2\omega j}-X(j\omega I-A_1)^{-1}-Z_0C_2(j\omega I-A_1)^{-1}-(-j\omega I-A_1^T)^{-1}X^*-(-j\omega I-A_1^T)^{-1}C_2^TZ_0^*, \\ M_{21} &= B_{d,1}^T(-j\omega I-A_1^T)^{-1}X^*+\left[D_{d,2}^T+B_{d,1}^T(-j\omega I-A_1^T)^{-1}C_2^T\right]Z_0^*, \\ M_{31} &= C_1(j\omega I-A_1)^{-1}+\tilde{H}C_2(j\omega I-A_1)^{-1}, \\ M_{32} &= \left[D_{d,1}+C_1(-j\omega I-A_1)^{-1}B_{d,1}\right]+\tilde{H}\left[D_{d,2}+C_2(-j\omega I-A_1)^{-1}B_{d,1}\right]. \end{aligned}$$

As can be seen from (5.49), the LMI is complex. To numerically solve this LMI problem, it is necessary to transform the complex LMI to a real LMI but with larger dimensions as stated in (5.25). Take $M = M^r + jM^i$ where M^r and M^i are real, then

$$M \prec 0 \Leftrightarrow \begin{bmatrix} M^r & M^{iT} \\ M^i & M^r \end{bmatrix} \prec 0.$$

□

By solving the LMI in (5.48), it is obtained that $X = X_1 + jX_2$, $Z = Z_1 + jZ_2$ and $\tilde{H} = \tilde{H}_1 + j\tilde{H}_2$ so that $Y = X^{-1}Z$. By using (5.44), it follows that

$$Z_0 = (j\omega I - A_1)Y_\omega(I + C_2Y_\omega)^{-1}, \qquad U_0 = (\tilde{H} - C_1Y_\omega)(I + C_2Y_\omega)^{-1}. \tag{5.50}$$

So that by following the arrangements defined in (4.14) and (4.15), the observer parameters L and H satisfying (5.48) are calculated as follows.

$$L = -B_f D_f^\dagger + Z_0 D_f^\perp + Z_1 D_f^\dagger + Z_1 U_0 D_f^\perp, \qquad H = U_1 D_f^\dagger + U_1 U_0 D_f^\perp; \tag{5.51}$$

It can be seen that L and H from (5.51) may be complex since Z_0 and U_0 calculated from (5.50) may not be real so that the observer is not practically implementable. Take

$$Z_0 = Z_0^r + jZ_0^i, \qquad U_0 = U_0^r + jU_0^i. \tag{5.52}$$

Although there are remain degrees of freedom in $Z_1 \in \mathcal{R}^{n \times n_f}$ and $U_1 \in \mathcal{R}^{n_f \times n_f}$ in (5.51) which can be chosen to achieve

$$Z_0^i + Z_1 U_0^i = 0, \qquad U_1 U_0^i = 0 \tag{5.53}$$

to make the design parameters L and H real, however, $U_1 = HD_f$ means that U_1 should always have full rank, which implies there may not always exist a full rank U_1 such that (5.53) is satisfied. Therefore, the optimal pair (L, H) from (5.51) could be complex and the resulting filter is not applicable. Further research is needed to find a real and optimal pair (L, H) or to find a systematic algorithm to get a suboptimal solution to guarantee (L, H) can be always real.

5.3 Concluding remarks

In this chapter, the observer-based FD problem for linear time invariant systems at a single frequency and over a finite frequency range are investigated, respectively. For the FD problem at a single frequency, an optimal solution is derived such that at a given frequency, the residual signal minimizes the sensitivity of the residual to disturbances while maintaining a minimum level of

sensitivity to faults. LMI optimization techniques are used to obtain the optimal solution. Moreover, a simple parameterization of all optimal solutions is derived such that other specifications can also be achieved by choosing the degrees of freedom remaining in the parameterization. In particular, it is shown that, when $\omega = 0$, the resulting degrees of freedom allow the FD observer to be used simultaneously for control system design. Conversely, our work demonstrates that observers used for control system design can be also employed to implement fault detection schemes at DC with minor modifications.

Moreover, an initial investigation is carried out on the FD problem over a finite frequency range whose solution turns out to involve a difficult nonlinear optimization. By applying the Generalized KYP lemma and using the results given in Chapter 4, a novel problem formulation is proposed such that the FD problem is transformed to a sensitivity optimization problem over a finite frequency range, where the ratio of the sensitivity of disturbances over the sensitivity of faults is minimized. By following a linearization procedure, a state space solution is obtained by solving an LMI optimization and conditions under which this solution is optimal are also derived. However, this problem is still not fully resolved since the resulting observer parameters could be complex and may not be usable in practice. Further work is needed to find an algorithm to obtain real and optimal solutions.

Chapter 6

Conclusion

6.1 Contributions

This thesis presents research work on the problem of model-based FDI for linear dynamic systems subject to faults, disturbances and model uncertainties. It contains three different approaches to solve the FDI problem: an on-line moving horizon estimation procedure using measurement analysis, a robust static observer design for a mixed $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem and a FD procedure for a finite frequency range etc.

The three approaches are summarized as follows:

- First, in Chapter 3 a novel on-line approach to the robust FDI problem is proposed for linear discrete-time systems subject to faults, disturbances and norm-bounded uncertainties. The upper and lower bounds on the fault signal are computed at each sampling instant. Then a fault is regarded as having occurred when its upper bound is smaller than zero or lower bound is larger than zero. LMI optimization techniques are employed to obtain the bounds. Furthermore, a subsequent-state-estimation technique together with an estimation horizon update procedure is proposed, which allows the on-line fault detection and isolation process to be repeated in a moving horizon. Also, it is shown that the proposed approach reduces to the solution of linear programs when no uncertainties are considered in the model description. The details of this approach have been published in [105, 106].

- Chapter 4 gives an optimal state-space solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem for linear time-invariant systems subject to faults, disturbances and model uncertainties. First, a novel performance index is developed to capture both the requirements of fault detection and disturbance rejection which is particularly suitable for handling model uncertainties. Then,

by using a static observer, a class of solutions to the FD problem is given in the form of a simple LMI with two degrees of freedom. These degrees of freedom are then used to derive the optimal filter to the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem and achieve further specifications such as fault isolation. Moreover, it is shown that by introducing a weighting filter, the results are further improved and certain widely held assumptions can be removed. Finally, the proposed approach is extended to implement FD for uncertain systems. The details of this approach have been published in [108].

- Chapter 5 first considers the FD problem at a single frequency. A simple parameterization of all optimal solutions is derived such that other specifications such as fault isolation can also be achieved by choosing the degrees of freedom remaining in the parameterization. LMI optimization techniques are used to obtain the optimal solution. In particular, when $\omega = 0$, the degrees of freedom allow the FD observer to be used simultaneously for control system design. Then, an initial research is carried out on the FD problem over a finite frequency range whose solution turns out to involve a difficult nonlinear optimization. By following a linearization procedure, a solution is derived in the form of an LMI optimization using the generalized KYP lemma. Also, conditions under which this solution is optimal are also derived. However, this problem is not fully solved since the resulting design parameters could be complex so that the design may not be usable in practice. The results in this chapter have been published in [107, 109].

Moreover, a further study and result related to the on-line FDI approach in Chapter 3 are given at the end of this chapter. It provides a new idea of reconfiguring system models and estimating model uncertainties by using only input and output measurements. This result has already been applied to the area of networked system control as stated in [116].

6.2 Future Research

6.2.1 Flexible horizon length

The approach in Chapter 3 achieves FDI by computing upper and lower bounds on faults over a finite estimation horizon for linear discrete-time systems with model uncertainties. However, one main drawback is that solving LMIs over a finite estimation horizon of a certain length could be unnecessary and computationally expensive, which may limit its application to fast systems. Hence, it would be useful to propose an algorithm to choose a flexible horizon length to save

computational resources and apply this approach more efficiently.

6.2.2 Comparison between static and dynamic observers

The approach proposed in Chapter 4 is based on static observer design. In fact, in the literature there is another kind of observer , that is, the dynamic observer which has already been used to achieve FDI [61]. Generally, a dynamic observer can be described as the following form:

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} A_k & B_{ku} & B_{ky} \\ C_k & D_{ku} & D_{ky} \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ u(t) \\ y(t) \end{bmatrix}, \tag{6.1}$$

where $\hat{x}(t) \in \mathcal{R}^{n_k}$ is the observer state and $r(t) \in \mathcal{R}^{n_f}$ is the residual signal. Figure 6.1 shows a dynamic observer in the residual generation scheme. By defining the following augmented state

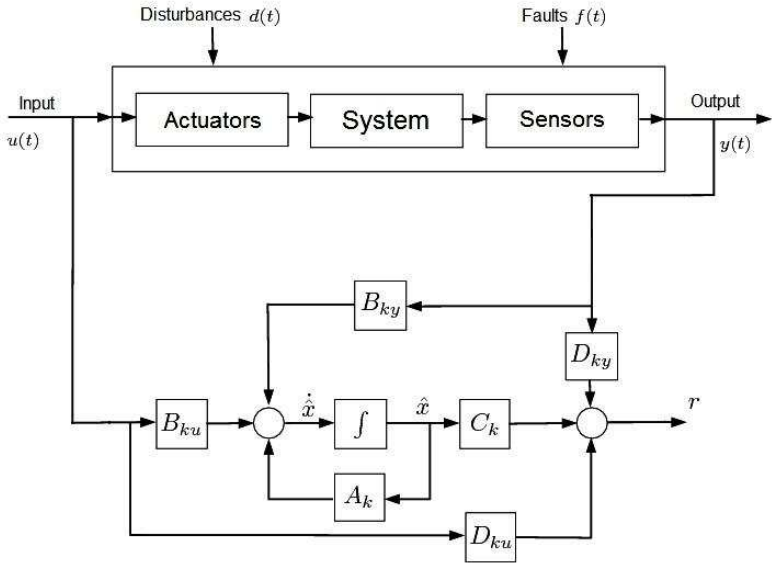


Figure 6.1: Residual generation using a dynamic observer

$$z(t) = \begin{bmatrix} x(t)^T & \tilde{x}(t)^T \end{bmatrix}^T,$$

the residual dynamics are given by

$$\begin{bmatrix} \dot{z}(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B}_f & \hat{B}_d & \hat{B}_u \\ \hat{C} & \hat{D}_f & \hat{D}_d & \hat{D}_u \end{bmatrix} \begin{bmatrix} z(t) \\ f(t) \\ d(t) \\ u(t) \end{bmatrix}$$

and by taking Laplace transform, it follows that

$$\begin{bmatrix} T_{rf}(s) & T_{rd}(s) & T_{ru}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|ccc} \hat{A} & \hat{B}_f & \hat{B}_d & \hat{B}_u \\ \hline \hat{C} & \hat{D}_f & \hat{D}_d & \hat{D}_u \end{array}\right]$$
$$= \left[\begin{array}{cc|ccc} A & 0 & B_f & B_d & B_u \\ B_{ky}C & A_k & B_{ky}D_f & B_{ky}D_d & B_{ku}+B_{ky}D_u \\ \hline D_{ky}C & C_k & D_{ky}D_f & D_{ky}D_d & D_{ku}+D_{ky}D_u \end{array}\right], \quad (6.2)$$

where $T_{rf}(s), T_{rd}(s)$ and $T_{ru}(s)$ denote the dynamics from faults, disturbances and inputs to the residual, respectively. It follows that

$$r(s) = T_{rf}(s)f(s) + T_{rd}(s)d(s) + T_{ru}(s)u(s). \quad (6.3)$$

Since dynamic observer-based FDI schemes have more degrees of freedom in the observer parameters, it may be possible to obtain better performance for fault detection and disturbance attenuation. Therefore, it is of interest to use a dynamic observer and derive an optimal solution to the FD problem. Also, it would be useful to carry out a comparison of FD performance between the static and dynamic observer-based approaches.

6.2.3 Comparison between the approach in Chapter 3 and other FD approaches

Chapter 3 gives a novel idea for FDI by evaluating the fault signal bounds while most contributions in the literature usually achieve FD by generating the residuals. Also, there are also approaches from the model control literature [85] concerning state constraints which are also considered in Chapter 3. Hence, it is needed and of interest to compare the on-line FDI results proposed in Chapter 3 with the results by using other FD approaches from the literature. Further improvement on the proposed FDI approach may be achieved from the comparison.

6.2.4 FD over a finite frequency range

FD problems over a finite frequency range have received increasing attention from the literature [90, 92, 58] since they can relax the restrictive rank assumptions on fault dynamics and improve the FD performance. However, many contributions give sub-optimal solutions and are difficult to implement due to the involved nonlinear optimization. In Chapter 5, an initial research is presented for this problem over a low frequency range $|\omega| < \omega_l$ where a solution is given in the form of LMIs and conditions under which the solution is optimal are also derived. However, the resulting parameters may be complex and not usable in practice. Hence, it is of interest to find a systematic algorithm to obtain real and optimal solutions to the problem. Moreover, it would

also be useful to investigate the FD problem over the middle ($\omega_1 < |\omega| < \omega_2$) and high frequency ranges ($|\omega| > \omega_h$).

6.2.5 Uncertainty estimation

In practice, system model uncertainties can be inevitably introduced by process noises, parameter variations and non-linear dynamics. A precise uncertainty estimation and description is important for designing control systems and achieving other specifications such as FD. Recently, much research effort has been devoted to the uncertainty estimation problem for linear systems. See [51, 74, 41, 4] for details.

In this section, a novel approach is proposed to solve the uncertainty estimation problem for linear discrete-time systems subject to bounded additive disturbances and norm-bounded uncertainties. Similar to the FDI scheme in given Chapter 3, this approach achieves uncertainty estimation by computing upper and lower bounds on the entries of the uncertainty matrix using a dynamic system model as well as input/output measurements over a finite horizon.

Problem formulation and solution

Consider an LTI discrete-time system subject to disturbances and uncertainties of the form

$$\left[\begin{array}{c} x_{k+1} \\ \hline y_k \end{array}\right] = \left[\begin{array}{c|cc} A+\Delta A & B_d+\Delta B_d & B_u+\Delta B_u \\ \hline C_y+\Delta C_y & D_{yd}+\Delta D_{yd} & D_{yu}+\Delta D_{yu} \end{array}\right] \left[\begin{array}{c} x_k \\ \hline d_k \\ u_k \end{array}\right] \tag{6.4}$$

for $k \in \mathcal{N}$, where $\mathcal{N} := \{0, 1, \ldots, N-1\}$ is the estimation horizon. It is assumed the norm-bounded uncertainty matrices in (6.4) are of form given by [61]:

$$\left[\begin{array}{ccc} \Delta A & \Delta B_d & \Delta B_u \\ \Delta C_y & \Delta D_{yd} & \Delta D_{yu} \end{array}\right] = \left[\begin{array}{c} B_p \\ D_{yp} \end{array}\right] \Delta \left[\begin{array}{ccc} C_q & D_{qd} & D_{qu} \end{array}\right],$$

where B_p, D_{yp}, C_q, D_{qd}, and D_{qu} are known and Δ is an $n_\Delta \times n_\Delta$ unknown matrix with $\Delta \in \mathbf{\Delta}$ where

$$\mathbf{\Delta} : \quad = \quad \{diag(\delta_1 I_{n_{\Delta_1}}, \ldots, \delta_l I_{n_{\Delta_l}} : \delta_i \in \mathcal{R}, \underline{\delta}_i^0 \leq \delta_i \leq \bar{\delta}_i^0, i = 1, \ldots, l\} \subset \mathcal{R}^{n_\Delta \times n_\Delta}.$$

In this description $\underline{\delta}_i^0$ and $\bar{\delta}_i^0$ represent initial bounds on the uncertainty. In the following the problem of refining the lower and upper bounds for this simple uncertainty set is presented while the problem for more complex uncertainty descriptions is left for future research.

Note that Δ has l repeated diagonal blocks. Assume that upper and lower bounds $\bar{x}_0$, $\underline{x}_0$ and $\bar{d}_k$, $\underline{d}_k$ on the initial state and disturbances, respectively, are available and define

$$w = \begin{bmatrix} x_0 \\ d \end{bmatrix}, \underline{w} = \begin{bmatrix} \underline{x}_0 \\ \underline{d} \end{bmatrix}, \bar{w} = \begin{bmatrix} \bar{x}_0 \\ \bar{d} \end{bmatrix}, \mathcal{D}_{yw} = \begin{bmatrix} \mathcal{C}_{y0} & \mathcal{D}_{yd} \end{bmatrix}, N_w = n + N_d.$$

Similarly to (3.10), the uncertainty data are re-sequenced so that the uncertainty set has the form:

$$\mathbf{\Delta}_e := \{\mathrm{diag}(I_N \otimes \delta_1 I_{n_{\Delta_1}}, \ldots, I_N \otimes \delta_l I_{n_{\Delta_l}}, \underline{\delta}_i^0 \leq \delta_i \leq \bar{\delta}_i^0, i = 1, \ldots, l, \} \subset \mathcal{R}^{N_\Delta \times N_\Delta}. \quad (6.5)$$

Using an iterative computation, the system algebraic formulation over the estimation horizon N is described as:

$$y \quad = \quad (\mathcal{D}_{yw} + \Delta_{\mathcal{D}_{yw}})w + (\mathcal{D}_{yu} + \Delta_{\mathcal{D}_{yu}})u \quad (6.6)$$

where $\underline{w} \leq w \leq \bar{w}$ and

$$\begin{bmatrix} \Delta_{\mathcal{D}_{yw}} & \Delta_{\mathcal{D}_{yu}} \end{bmatrix} = \mathcal{D}_{yp}\Delta_e(I - \mathcal{D}_{qp}\Delta_e)^{-1} \begin{bmatrix} \mathcal{D}_{qw} & \mathcal{D}_{qu} \end{bmatrix}, \quad \Delta_e \in \mathbf{\Delta}_e. \quad (6.7)$$

Then, the uncertainty estimation problem for linear discrete-time systems is described as follows.

Problem 6.2.1. *Let all data and variables be as given above. Then for $i = 1, 2, \ldots, l$ find*

$$\bar{\varphi}_i := \max_{\substack{\underline{w} \leq w \leq \bar{w} \\ y = (\mathcal{D}_{yw} + \Delta_{\mathcal{D}_{yw}})w + (\mathcal{D}_{yu} + \Delta_{\mathcal{D}_{yu}})u \\ \Delta_{\mathcal{D}_{yu}},\ \Delta_{\mathcal{D}_{yw}} \text{ given by (6.5) and (6.6)} \\ y,\ u \text{ given}}} e_i^T \delta \quad (6.8)$$

$$\underline{\varphi}_i := \min_{\substack{\underline{w} \leq w \leq \bar{w} \\ y = (\mathcal{D}_{yw} + \Delta_{\mathcal{D}_{yw}})w + (\mathcal{D}_{yu} + \Delta_{\mathcal{D}_{yu}})u \\ \Delta_{\mathcal{D}_{yu}},\ \Delta_{\mathcal{D}_{yw}} \text{ given by (6.5) and (6.6)} \\ y,\ u \text{ given}}} e_i^T \delta \quad (6.9)$$

where $\delta = \begin{bmatrix} \delta_1 & \delta_2 & \ldots & \delta_l \end{bmatrix}^T$ and e_i is the ith column of I_l.

Note that the interval $[\underline{\varphi}_i,\ \bar{\varphi}_i]$ is our best estimate for δ_i given all the information about the system. The following theorem shows that evaluating an upper bound on each $e_i^T\delta$ can be obtained by solving an LMI problem.

Theorem 6.2.1. *Let all data be as defined previously. Let $\mathcal{D}^m$ denote the set of all real $m \times m$ diagonal matrices and define*

$$\begin{aligned}
\mathcal{S} &:= \{\mathrm{diag}(S_1,\ldots,S_l) : 0 \preceq S_i = S_i^T \in \mathcal{R}^{N_{\Delta_i}\times N_{\Delta_i}}, i=1,\ldots,l\} \subset \mathcal{R}^{N_\Delta \times N_\Delta} \\
\mathcal{G} &:= \{diag(G_1,\ldots,G_l) \ : \ G_i = -G_i^T \in \mathcal{R}^{N_{\Delta_i}\times N_{\Delta_i}}, i=1,\ldots,l\} \subset \mathcal{R}^{N_\Delta \times N_\Delta} \\
\mathcal{K} &:= \{\mathrm{diag}(\kappa_1,\ldots,\kappa_l) : \kappa_i \in \mathcal{R}^{N_{\Delta_i}}, i=1,\ldots,l\} \subset \mathcal{R}^{N_\Delta \times l}.
\end{aligned}$$

Take $M \subset \mathcal{R}^l$ with all its entries equal to 1. Then for $i = 1,\ldots,l$,

$$e_i^T\delta \le \bar{\varphi}_i \le \bar{\bar{\varphi}}_i := \begin{array}{c} \min \quad \bar{\delta}_i \\ \mathcal{L}_i(\bar{\delta}_i,\bar{K}_i,\bar{E}_i,\bar{D}_i,\bar{\tau}_i,\bar{S}_i,\bar{G}_i) \succeq 0 \\ 0 \preceq \bar{D}_i \in \mathcal{D}^{N_w}, 0 \preceq \bar{E}_i \in \mathcal{D}^l \\ \bar{S}_i \in \mathcal{S}, \bar{K}_i \in \mathcal{K} \\ \bar{G}_i \in \mathcal{G} \\ \bar{\tau}_i \in \mathcal{R} \end{array} \tag{6.10}$$

where

$$\mathcal{L}_i(\bar{\delta}_i,\bar{K}_i,\bar{E}_i,\bar{D}_i,\bar{\tau}_i,\bar{Q}_i,\bar{G}_i) = \begin{bmatrix} \mathcal{L}_{11}^i & \star & \star & \star \\ \mathcal{L}_{21}^i & \mathcal{L}_{22}^i & \star & \star \\ \mathcal{L}_{31}^i & \mathcal{L}_{32}^i & \mathcal{L}_{33}^i & \star \\ \mathcal{L}_{41}^i & \mathcal{L}_{42}^i & \mathcal{L}_{43}^i & \mathcal{L}_{44}^i \end{bmatrix}, \tag{6.11}$$

and

$$\begin{aligned}
&\mathcal{L}_{11}^i = \bar{D}_i - \bar{\tau}_i\mathcal{D}_{yw}^T\mathcal{D}_{yw} - \mathcal{D}_{qw}^T\bar{S}_i\mathcal{D}_{qw},\ \mathcal{L}_{21}^i = \tfrac{1}{2}\bar{K}_i^T\mathcal{D}_{qw},\ \mathcal{L}_{22}^i = \bar{E}_i, \\
&\mathcal{L}_{31}^i = -\bar{\tau}_i\mathcal{D}_{yp}^T\mathcal{D}_{yw} - \mathcal{D}_{qp}^T\bar{S}_i\mathcal{D}_{qw} - \bar{G}_i\mathcal{D}_{qw},\ \mathcal{L}_{32}^i = \tfrac{1}{2}\mathcal{D}_{qp}^T\bar{K}_i, \\
&\mathcal{L}_{33}^i = -\bar{\tau}_i\mathcal{D}_{yp}^T\mathcal{D}_{yp} + \bar{S}_i - \mathcal{D}_{qp}^T\bar{S}_i\mathcal{D}_{qp} - \bar{G}_i\mathcal{D}_{qp} + \mathcal{D}_{qp}^T\bar{G}_i, \\
&\mathcal{L}_{41}^i = -\tfrac{1}{2}(\bar{w}+\underline{w})^T\bar{D}_i + \bar{\tau}_i(y-\mathcal{D}_{yu}u)^T\mathcal{D}_{yw} - u^T\mathcal{D}_{qu}^T\bar{S}_i\mathcal{D}_{qw}, \\
&\mathcal{L}_{42}^i = \tfrac{1}{2}u^T\mathcal{D}_{qu}^T\bar{K}_i - \tfrac{1}{2}(\bar{\delta}^0+\underline{\delta}^0)^T\bar{E}_i - \tfrac{1}{2}e_i^T, \\
&\mathcal{L}_{43}^i = \bar{\tau}_i(y-\mathcal{D}_{yu}u)^T\mathcal{D}_{yp} - u^T\mathcal{D}_{qu}^T\bar{S}_i\mathcal{D}_{qp} + u^T\mathcal{D}_{qu}^T\bar{G}_i - \tfrac{1}{2}M^T\bar{K}_i^T, \\
&\mathcal{L}_{44}^i = \bar{w}^T\bar{D}_i\underline{w} - \bar{\tau}_i(y-\mathcal{D}_{yu}u)^T(y-\mathcal{D}_{yu}u) - u^T\mathcal{D}_{qu}^T\bar{S}_i\mathcal{D}_{qu}u + (\bar{\delta}^0)^T\bar{E}_i\underline{\delta}^0 + \bar{\delta}_i
\end{aligned}$$

and where $\underline{\delta}^0 = \begin{bmatrix} \underline{\delta}_1^0 & \underline{\delta}_2^0 & \ldots & \underline{\delta}_l^0 \end{bmatrix}^T$ and $\bar{\delta}^0 = \begin{bmatrix} \bar{\delta}_1^0 & \bar{\delta}_2^0 & \ldots & \bar{\delta}_l^0 \end{bmatrix}^T$.

Proof. Take $p = \Delta_e(I - \mathcal{D}_{qp}\Delta_e)^{-1}\begin{bmatrix} \mathcal{D}_{qw}w & \mathcal{D}_{qu}u \end{bmatrix}$, then (6.6) can be represented as the following equalities,

$$\begin{aligned}
y &= \mathcal{D}_{yw}w + \mathcal{D}_{yu}u + \mathcal{D}_{yp}p, \\
p &= \Delta_e(\mathcal{D}_{qw}w + \mathcal{D}_{qu}u + \mathcal{D}_{qp}p).
\end{aligned}$$

It follows that the problem formulation for obtaining the upper bound on δ_i in (6.8) can be rewritten

as

$$\bar{\varphi}_i = \min_{\substack{\underline{w} \le w \le \bar{w} \\ y = \mathcal{D}_{yw} w + \mathcal{D}_{yu} u + \mathcal{D}_{yp} p \\ p = \Delta_e (\mathcal{D}_{qw} w + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p) \\ y,\ u\ given,\ \Delta_e \in \mathbf{\Delta}_e}} e_i^T \delta. \tag{6.12}$$

Define $\mathcal{Q} := \{Q : Q\Delta_e = \Delta_e Q, \forall \Delta_e \in \mathbf{\Delta}_e\}$ which is the set of matrices which commute with all matrices in $\mathbf{\Delta}_e$. Note that since $q = \mathcal{D}_{yw} w + \mathcal{D}_{yu} u + \mathcal{D}_{yp} p$ then $\bar{Q}_i p = \Delta_e \bar{Q}_i q$ for any $\bar{Q}_i \in \mathcal{Q}$. According to the specific structure of Δ_e given in (3.10), it can be verified that

$$\mathcal{Q} := \{\mathrm{diag}(Q_1, \ldots, Q_l) : Q_i \in \mathcal{R}^{N_{\Delta_i} \times N_{\Delta_i}}, i = 1, \ldots, l\} \subset \mathcal{R}^{N_\Delta \times N_\Delta}.$$

Note that the constraints in (6.12) imply that $p^T \bar{Q}_i^T \bar{Q}_i p \le q^T \bar{Q}_i^T \bar{Q}_i q, \forall \bar{Q}_i \in \mathcal{Q}$. Also, note that $\bar{G}_i$ is skew-symmetric with the structure as defined above, then a computation shows that $p^T \bar{G}_i q = 0$.

Then, a manipulation verifies that the identity

$$\begin{aligned} e_i^T \delta - \bar{\delta}_i \;=\; & -(\bar{w} - w)^T \bar{D}_i (w - \underline{w}) - \left(\bar{\delta}^0 - \delta\right)^T \bar{E}_i \left(\delta - \underline{\delta}^0\right) \\ & + p^T \bar{Q}_i^T \bar{Q}_i p - (\mathcal{D}_{qw} w + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p)^T \bar{Q}_i^T \bar{Q}_i (\mathcal{D}_{qw} w + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p) \\ & - \bar{\tau}_i (y - \mathcal{D}_{yw} w - \mathcal{D}_{yu} u - \mathcal{D}_{yp} p)^T (y - \mathcal{D}_{yw} w - \mathcal{D}_{yu} u - \mathcal{D}_{yp} p) \\ & - p^T \bar{G}_i (\mathcal{D}_{qw} w + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p) + (\mathcal{D}_{qw} w + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p)^T \bar{G}_i p \\ & - \bar{K}_i^T [p - \Delta_e (\mathcal{D}_{qw} w + \mathcal{D}_{qu} u + \mathcal{D}_{qp} p)] \\ & - \begin{bmatrix} w^T & \delta^T & p^T & 1 \end{bmatrix} \mathcal{L}_i(\bar{\delta}_i, \bar{K}_i, \bar{E}_i, \bar{D}_i, \bar{\tau}_i, \bar{Q}_i, \bar{G}_i) \begin{bmatrix} w \\ \delta \\ p \\ 1 \end{bmatrix} \end{aligned} \tag{6.13}$$

is valid for all $\bar{\delta}_i \in \mathcal{R}$, $\bar{K}_i \in \mathcal{R}^{N_\Delta \times l}$, $\bar{D}_i \in \mathcal{R}^{N_w \times N_w}$, $\bar{E}_i \in \mathcal{R}^{l \times l}$, $\bar{\tau}_i \in \mathcal{R}$, $\bar{Q}_i \in \mathcal{R}^{N_\Delta \times N_\Delta}$ and $\bar{G}_i \in \mathcal{R}^{N_\Delta \times N_\Delta}$. From (6.13), it can be seen that $\bar{\delta}_i$ is an upper bound on $e_i^T \delta$ if there exist $0 \preceq \bar{D}_i \in \mathcal{D}^{N_w}$, $0 \preceq \bar{E}_i \in \mathcal{D}^l$, $\bar{K}_i \in \mathcal{K}$, $\bar{\tau}_i \in \mathcal{R}$, $\bar{Q}_i \in \mathcal{Q}$ and $\bar{G}_i \in \mathcal{G}$ such that $\mathcal{L}_i(\bar{\delta}_i, \bar{K}_i, \bar{R}_i, \bar{D}_i, \bar{\tau}_i, \bar{Q}_i, \bar{G}_i) \succeq 0$. An inspection of the sets $\mathcal{Q}$ and $\mathcal{S}$ verifies that $\bar{Q}_i \in \mathcal{Q}$ if and only if $\bar{S}_i := \bar{Q}_i^T \bar{Q}_i \in \mathcal{S}$ and so an upper bound on $e_i^T \delta$ is given by solving the LMI optimization in (6.10). □

Similarly, the next theorem derives lower bounds on the faults. The proof is omitted since it follows similar lines to that of Theorem 6.2.1.

Theorem 6.2.2. *Let all data and variables as defined in Theorem 6.2.1. Then for* $i = 1, \ldots, n_l$,

$$e_i^T \delta \geq \underline{\varphi}_i \geq \underline{\underline{\varphi}}_i := \begin{array}{c} \max \\ \mathcal{L}_i(\underline{\delta}_i, \underline{K}_i, \underline{E}_i, \underline{D}_i, \underline{\tau}_i, \underline{S}_i, \underline{G}_i) \preceq 0 \\ 0 \succeq \underline{D}_i \in \mathcal{D}^{N_w}, 0 \succeq \underline{E}_i \in \mathcal{D}^l \\ \bar{S}_i \in \mathcal{S}, \underline{K}_i \in \mathcal{K} \\ \underline{G}_i \in \mathcal{G} \\ \underline{\tau}_i \in \mathcal{R} \end{array} \underline{\delta}_i \tag{6.14}$$

where $\mathcal{L}_i(\underline{\delta}_i, \underline{K}_i, \underline{E}_i, \underline{D}_i, \underline{\tau}_i, \underline{S}_i, \underline{G}_i) \preceq 0$ *is as defined in (6.11).*

Example

To demonstrate the effectiveness of the proposed approach, Example 2, given in Chapter 3, is considered in this section. Here, assume that the system is fault-free and the disturbances are uniformly distributed random numbers between -1 and 1. Also assume that the upper and lower bounds on the initial state x_0 are 1 and -1, respectively, and the initial upper and lower bounds $\bar{\delta}^0$ and $\underline{\delta}^0$ on δ are 10 and -10, respectively.

Here, the uncertainty matrix in the system model is assumed to be

$$\Delta = \begin{bmatrix} 0.1363 & 0 \\ 0 & -0.5327 \end{bmatrix}$$

so that $\delta = \begin{bmatrix} 0.1363 & -0.5327 \end{bmatrix}$. Take the estimation horizon length $N = 5$. By applying Theorems 6.2.1 and 6.2.2, the upper and lower bounds on δ are solved and given as follows.

$$\underline{\delta} = \begin{bmatrix} 0.1286 & -0.5332 \end{bmatrix}^T$$

$$\delta = \begin{bmatrix} 0.1363 & -0.5327 \end{bmatrix}^T$$

$$\bar{\delta} = \begin{bmatrix} 0.1440 & -0.5204 \end{bmatrix}^T$$

It shows that the difference between the upper and lower bounds on δ are small, which means δ is well-estimated.

Bibliography

[1] E. Alcorta-Garcia and P.M. Frank. Analysis of a class of dedicated observer schemes to sensor fault isolation. In *UKACC International Conference on Control '96*, pages 60–65, 1996.

[2] K.J. Astrom and B. Wittenmark. *Computer-Controlled Systems: Theory and Design, Second ed.* Prentice-Hall, Inc., 1990.

[3] R.V. Beard. *Failure accommodation in linear systems through self–reorganization*. PhD thesis, Massachusetts Institute of Technology, 1971.

[4] G. Belforte, B. Bona, and V. Cerone. Parameter estimation algorithms for a set-membership description of uncertainty. *Automatica*, 26:887–898, 1990.

[5] E. Beran and K. Grigoriadis. A combined alternating projections and semidefinite programming algorithm for low-order control design. In *In Proc. Prep. 13th IFAC World Congr.*, pages 85–90, San Francisco, CA, 1996.

[6] S.P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*. SIAM Publications, Philadelphia, PA, 1994.

[7] F.M. Callier and C.A. Desoer. *Linear System Theory*. Springer–Verlag, New York, 1991.

[8] J. Chen and R. J. Patton. *Robust model–based fault diagnosis for dynamic systems*. Boston: Kluwer Academic Publishers, 1999.

[9] J. Chen, R. J. Patton, and H.Y. Zhang. Design of unknown input observers and robust fault detection filters. *Int. J. Control*, 63(1):85–105, 1996.

[10] E. Y. Chow and A. S. Willsky. Analytical redundancy and the design of robust failure detection systems. *IEEE Trans. Automatic Control*, 29(7):603–614, 1984.

[11] R.N. Clark. Instrument fault detection. *IEEE trans. aero. and electronic Systems*, 14(3), 1978.

[12] R.N. Clark, D.C. Fosth, and V.M. Walton. Detecting instrument malfunctions in control systems. *IEEE Trans. on Aerosp. and Elec. Sys.*, AES–11(4):465–473, 1975.

[13] D.J. Clements, B.D.O. Anderson, A.J. Laub, and J.B. Matson. Spectral factorization with imaginary–axis zeros. *Lin. Alg. Appl.*, 250:225–252, 1997.

[14] Tramone D. Curry and Emmanuel G Collins. Robust fault detection and isolation using robust l1 estimation. *Journal of Guidance, Control and Dynamics*, 28(6):1131–1139, 2005.

[15] C. De Persis and A. Isidori. A geometric approach to nonlinear fault detection and isolation. *IEEE Trans. Automatic Control*, 46(6):853–865, Jun. 2001.

[16] S. X. Ding, T. Jeinsch, P.M. Frank, and E. L. Ding. A unified approach to the optimization of fault detection systems. *Int. J. Adapt. Control Signal Process*, 14:725–745, 2000.

[17] S. X. Ding, P. Zhang, and P. M. Frank. Application of probabilistic robustness technique to the fault detection system design. In *Proc. IEEE Conf. Dec. & Control*, pages 972–977, 2003.

[18] X. Ding and P.M. Frank. An approach to robust residual generation and evaluation. In *Proc. IEEE Conf. Dec. & Control*, pages 656–661, Brighton, England, 1990.

[19] X. Ding and P.M. Frank. Fault detection via factorization approach. *Systems & Control Letts.*, 14:431–436, 1990.

[20] X. Ding, P.M. Frank, and L. Guo. An approach to residual generator and evaluator design and synthesis. In *Proc. of IFAC World Congress*, Sydney, Australia, 1993.

[21] X. Ding, L. Guo, and P. Shi. A characterization of parity space and its application to robust fault detection. *IEEE Trans. Automatic Control*, 44(2):337–343, 1999.

[22] J.C. Doyle. Structured uncertainty in control system design. *IEE Proceedings*, 129-D(6):242–250, 1982.

[23] J.C. Doyle. *Lecture notes in advances in multivariable control*. ONR/Honeywell Workshop, 1984.

[24] J.C. Doyle, K. Glover, P. Khargonekar, and B. Francis. State space solutions to standard $\mathcal{H}^2$ and $\mathcal{H}^\infty$ control problems. *IEEE Trans. Automatic Control*, 24(8):731–747, 1989.

[25] A. Emami-Naeini, M.M. Akhter, and S.M. Rock. Effect of model uncertainty on failure detection - the threshold selector. *IEEE Trans. Automatic Control*, 33(2):1106–1115, 1988.

[26] Y.E. Faitakis and J.C. Kantor. Residual generation and fault detection for discrete–time systems using l_∞technique. *Int. J. Control*, pages 155–174, 1996.

[27] H. Fang, H. Ye, and M. Zhong. Fault diagnosis of networked control systems. *Annual Reviews in Control*, 31:55–68, 2007.

[28] D. Filbert and K. Metzger. Quality test of systems by parameter estimation. In *9th IMEKO congress*, Berlin, 1982.

[29] M. Fliess, C. Join, and H. Sira-Ramirez. Robust residual generation for linear fault diagnosis: an algebraic setting with examples. *Int. J. Control*, 77(14):1223–1242, 2004.

[30] K. F. Fong and A. P. Loh. A frequency domain approach for fault detection. In *Proc. IEEE Conf. Dec. & Control*, pages 1023–1028, Orlando, Florida, USA, 2001.

[31] B. A. Francis. *A Course in $\mathcal{H}^\infty$ Control Theory*. Springer-Verlag: Heidelberg, New York, 1987.

[32] P. M. Frank. Fault diagnosis in dynamic systems using analytical and knowledge–based redundancy: A survey and some new results. *Automatica*, 26(3):459–474, 1990.

[33] P. M. Frank. Analytical and qualitative model-based fault diagnosis - a survey and some new results. *European Journal of Control*, 2(1):6–28, 1996.

[34] P. M. Frank and X. Ding. Frequency domain approach to optimally robust residual generation and evaluation for model–based fault diagnosis. *Automatica*, 30(5):789–804, 1994.

[35] P. M. Frank and X. Ding. Survey of robust residual generation and evaluation methods in observer–based fault detection systems. *J. Proc. Cont.*, 7(6):403–424, 1997.

[36] E. Frisk and L. Nielsen. Robust residual generation for diagnosis including a reference model for residual behavior. In *Proceedings of the 14th IFAC World Congress*, pages 55–60, Beijing, China, 1999.

[37] E. Frisk and L. Nielsen. Robust residual generation for diagnosis including a reference model for residual behavior. *Automatica*, 42:437–445, 2006.

[38] J.C. Geromel, P.L.D. Peres, and S.R. de Souza. Output feedback stabilization of uncertain systems through a min/max problem. In *Preprints of Papers, 12th World Congress, I.F.A.C., July, 1993, Volume 6*, pages 35–38. The Inst. of Engineers, Australia, 1993.

[39] L. El Ghaoui and J.P. Folcher. Multi-objective robust control of LTI systems subject to unstructured perturbations. *Systems & Control Letts.*, 28(1):23–30, 1996.

[40] K. C. Goh, M. G. Safonov, and G. P. Papavassilopoulos. A global optimization approach for the bmi problem. In *Proc. IEEE Conf. Dec. & Control*, pages 2009–2014, 1994.

[41] G. C. Goodwin and M. E. Salgado. A stochastic embedding approach for quantifying uncertainty in the estimation of restricted complexity models. *International Journal of Adaptive COntrol and Signal Processing*, 3:161–172, August 2005.

[42] H. Hammouri, M. Kinnaert, and E. Yaagoubi. Observer-based approach to fault detection and isolation for nonlinear systems. *IEEE Trans. Automatic Control*, 44(10):1879–1884, 1999.

[43] D.M. Himmelblau. *Fault diagnosis in chemical and petrochemical processes*. Elsevier, 1978.

[44] M. Hou and R.J. Patton. An LMI approach to $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection observers. In *UKACC International Conference on Control. '96*, pages 305–310, England, 1996.

[45] R. Isermann. Supervision, fault detection and fault diagnosis methods: an introlduction. *Control engineering practice*, 5(5):639–652, 1997.

[46] R. Isermann and P. Balle. Trends in the application of model-based fault detection and diagnosis of technical processes. *Control engineering practice*, 5(5):709–719, 1997.

[47] T. Iwasaki and S. Hara. Generalized kyp lemma: unified frequency domain inequalities with design applications. *IEEE Trans. Automatic Control*, 50(1):41–59, 2005.

[48] I.M. Jaimoukha, Z. Li, and E. Mazars. Filter design to the optimal weighted $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem. In *UKACC International Conference on Control. '06*, Glasgow, UK, Aug 2006.

[49] I.M. Jaimoukha, Z. Li, and V. Papakos. A matrix factorization solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem. *Automatica*, 42(11):1907–1913, 2006.

[50] M. Kinnaert. Design of redundancy relations for failure detection and isolation by constrained optimization. *Int. J. Control*, 63(3):609–622, 1996.

[51] R. Kosut. Adaptive robust control via transfer function uncertainty estimation. In *Proc. Amer. Control Conf.*, pages 349–354, 1988.

[52] A. Laszlo. A generalization of the bounded real lemma. *Lin. Alg. Appl.*, 372:79–103, 2003.

[53] X. Li and K. Zhou. A time domain approach to robust fault detection of linear time-varying systems. *Automatica*, 45:94–102, 2009.

[54] Z. Li and I. M. Jaimoukha. Observer-based fault detection and isolation filter design for linear time-invariant systems. *Int. J. Control*, 82:171–182, 2009.

[55] Z. Li, I. M. Jaimoukha, and E. Mazars. State space solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem. In *Proc. IEEE Conf. Dec. & Control*, San Diego, CA, USA, Dec 2006.

[56] Z. Li, E. Mazars, and I. M. Jaimoukha. State space solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem. *International Journal of Robust and Nonlinear Control*, 2010.

[57] Z. Li, E. Mazars, Z. Zhang, and I. Jaimoukha. Statespace solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ fault-detection problem. *Int. J. Robust & Nonlinear Control*, 2011. accepted.

[58] J. Liu, J. L. Wang, and G. H. Yang. An LMI approach to minimum sensitivity analysis with application to fault detection. *Automatica*, 41:1995–2004, 2005.

[59] N. Liu and K. Zhou. Optimal robust fault detection for linear discrete time systems. *Journal of control science and engineering*, 2008(829459), 2008.

[60] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: stability and optimality. *Automatica*, 36:789–814, 2000.

[61] E. Mazars, I.M. Jaimoukha, and Z. Li. Computation of a reference model for robust fault detection and isolation residual generation. *Journal of control science and engineering*, 2008:1–12, 2008.

[62] R.J. Patton. Fault detection and diagnosis in aerospace systems using analytical redundancy. *Computing and Control Engineering*, 2(3):127–136, 1991.

[63] R.J. Patton and J. Chen. A review of parity space approaches to fault diagnosis for aerospace systems. *AIAA journal of guidance, control and dynamics*, 17(2):278–285, 1994.

[64] R.J. Patton, P.M. Frank, and R.N. Clark (Eds). *Fault diagnosis in dynamic systems, theory and applications*. Prentice Hall, 1989.

[65] R.J. Patton, P.M. Frank, and R.N. Clark (eds). *Issues of Fault Diagnosis for Dynamic Systems*. Springer, 2000.

[66] R.J. Patton and S. M. Kangethe. Robust fault diagnosis using the model–based approach. In *IEE Colloquium on Condition Monitoring and Fault Diagnosis – Part 1*, pages 2/1–2/13, 1988.

[67] M. L. Rank and H. Niemann. Norm based design of fault detectors. *Int. J. Control*, 72(9):773–783, 1999.

[68] A. Rault, A. Richalet, A. Barbot, and J. P. Sergenton. Identification and modelling of a jet engine. In *In IFAC Symposium on Digital Simulation of Continuous Processes*, Gejor, 1971.

[69] J.B. Rawlings and D.Q. Mayne. *Model predictive control: Theory and Design*. Nob Hill Publishing, Madison, 2009.

[70] A. Ray and R. Luck. An introduction to sensor signal validation in redundanct measurement systems. *IEEE Control Systems Magazine*, 11(2):44–49, 1991.

[71] A. Saberi, P. Sannuti, and B. M. Chen. h_2 optimal control system and control engineering. London, UK, 1995.

[72] M.A. Sadrnia, J. Chen, and R.J. Patton. Robust fault diagnosis observer design using $\mathcal{H}_\infty$ optimisation and μ synthesis. *IEE Colloquium on Modelling and Signal Processing for Fault Diagnosis*, pages 9/1–9/16, 1996.

[73] D. Sauter and F. Hamelin. Frequency-domain optimization for robust fault detection and isolation in dynamic systems. *IEEE Trans. Automatic Control*, 44(4):878–882, 1999.

[74] A. V. Savkin and I. R. Petersen. Robust state estimation and model validation for discrete-time uncertain systems with a deterministic description of noise and uncertainty. *Automatica*, 34:333–356, 1989.

[75] C. Scherer. *The Riccati inequality and state–space $\mathcal{H}_\infty$ optimal control*. PhD thesis, Univ. W$\ddot{u}$rzburg, Germany, 1990.

[76] C. Scherer, P. Gahinet, and M. Chilali. Multi-objective output-feedback control via LMI optimization. *IEEE Trans. Automatic Control*, 42(7):896–911, 1997.

[77] D. Van Schrick. Investigation of reliability for instrument fault detection state-estimator scheme. *European Journal of diagnosis and Safety in Automation*, 1(1):63–78, 1991.

[78] D. Van Schrick. Remarks on terminology in the field of supervision, fault detection and diagnosis. In *Proceedings of IFAC Symposium on Dault Detection, Supervision and Safety for Technical Processes*, pages 959–964, SAFEPROCESS'97, Pergamon, UK, 1997.

[79] M. L. Schwall and J. C. Gerdes. A probabilistic approach to residual processing for vehicle fault detection. In *Proc. Amer. Control Conf.*, pages 2552–2557, 2002.

[80] L. Shen, S. Chang, and P. Hsu. Robust fault detection and isoaltion with unstructured uncertainty using eigenstructure assignment. *Journal of guidance control and dynamics*, 21(1):50–57, 1998.

[81] L. Shen and P. Hsu. Robust design of fault isolation observers. *Automatica*, 34(11):1421–1429, 1998.

[82] H. Siebert and R. Isermann. *Fault diagnosis via on-line correlation analysis*. Technical report, 1976.

[83] J. Stefanovski and G. Dimirovski. A new approach to static output feedback stabilization of linear dynamic systems. *International Journal of Systems Science*, 37(9):643–662, 2006.

[84] R. J. Stern and H. Woldkowicz. Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations. *SIAM J. Optimization*, 5:286–313, 1995.

[85] D. Sui, T. Johansen, and L. Feng. Linear moving horizon estimation with pre-estimating observer. *IEEE Trans. Automatic Control*, 55(10):2363–2368, 2010.

[86] K. P. Sun and A. Packard. Robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filters for uncertain lft systems. *IEEE Trans. Automatic Control*, 50(5):715–720, 2005.

[87] F. Tao and Q. Zhao. Fault detection observer design with unknown inputs. In *Proceeding of the conference on control application*, pages 1275–1280, Toronto, Canada, 2005.

[88] M. Vidyasagar. *Control Systems Synthesis: A Factorization Approach*. MIT Press, Cambridge, MA, 1985.

[89] N. Viswanadham, J.H. Taylor, and E.C. Luce. A frequency domain approach to failure detection and isolation with application. *Control–theory and Advanced Technology*, 3:45–72, 1987.

[90] H. Wang and G. H. Yang. A finite frequency domain approach to fault detection for linear discrete-time systems. *Int. J. Control*, 81:1162–1171, 2008.

[91] H. B. Wang, L. Lam, S. X. Ding, and M. Y. Zhong. Iterative linear matrix inequality algorithms for fault detection with unknown inputs. *Journal of Systems and Control Engineering*, 219(2):161–172, August 2005.

[92] H.B. Wang, J.L. Wang, and J. Lam. Robust fault detection observer design: iterative LMI approach. *Journal of Dynamic Systems, Measurements and Control*, 129(1):77–82, 2007.

[93] J. L. Wang, G. H. Yang, and J. Liu. An LMI approach to $\mathcal{H}_-$ index and mixed $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection observer design. *Automatica*, 43:1656–1665, 2007.

[94] Y. Q. Wang, S. X. Ding, Y. Hao, and G. Z. Wang. A new fault detection scheme for networked control systems subject to uncertain time-varying delays. *IEEE Trans. Signal Processing*, 56(10):5258–5268, 2008.

[95] Y. Q. Wang, H. Ye, S. X. Ding, Y. Cheng, P. Zhang, and G. Z. Wang. Fault detection of networked control systems with limited communication. *Int. J. Control*, 82(7):1344–1356, 2009.

[96] Wikipedia. *http://en.wikipedia.org/wiki/Chernobyl_disaster*.

[97] Wikipedia. *http://en.wikipedia.org/wiki/Ariane_5_Flight501*.

[98] J. Wunnenberg. *Observer-based fault detection in dynamic systems*. PhD thesis, University of Duisburg, 1990.

[99] Y. Yamada, S. Hara, and H. Fujioka. Global optimization for constantly scaled h_∞control problem. In *Proc. American Contr. Conf*, pages 427–430, 1995.

[100] H. Ye, G. Wang, and S. X. Ding. A new parity space approach for fault detection based on stationary wavelet transform. *IEEE Trans. Automatic Control*, 49(2):281–287, 2004.

[101] P. Zhang and S. X. Ding. On fault detection in linear discrete-time, periodic, and sampled-data systems. *Journal of control science and engineering*, 2008(849546), 2008.

[102] P. Zhang, S. X. Ding, G. Z. Wang, and D. H. Zhou. Fault detection of linear discrete-time periodic systems. *IEEE Trans. Automatic Control*, 50(2):239–244, 2005.

[103] P. Zhang, H. Ye, S. X. Ding, G. Z. Wang, and D. H. Zhou. On the relationship between parity space and h_2 approaches to fault detection. *Systems & Control Letts.*, 55:94–100, 2006.

[104] X. Zhang, M. Polycarpou, and T. Parisini. A robust detection and isolation scheme for abrupt and incipient faults in nonlinear systems. *IEEE Trans. Automatic Control*, 47(4):576–593, 2002.

[105] Z. Zhang and I.M. Jaimoukha. Fault detection and isolation for linear discrete-time systems using input/output measurement anaysis. In *Proc. IEEE Conf. Dec. & Control*, pages 4908–4913, Shanghai, China, 2009.

[106] Z. Zhang and I.M. Jaimoukha. A measurement analysis approach to online fault detection and isolation for linear discrete-time uncertain systems. In *Proc. IEEE Conf. Dec. & Control*, Atlanta, USA, 2010.

[107] Z. Zhang and I.M. Jaimoukha. Optimal state space solution to the fault detection problem at dc. In *Conference on Control and Fault Tolerant Systems*, Nice, France, 2010.

[108] Z. Zhang and I.M. Jaimoukha. An optimal solution to an $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem. In *Proc. IEEE Conf. Dec. & Control*, Orlando, USA, 2011.

[109] Z. Zhang and I.M. Jaimoukha. Optimal state space solution to the fault detection problem at single frequency. In *Proc. of IFAC World Congress*, Milan, Italy, 2011.

[110] M. Zhong, Q. Ding, and P. Shi. Parity space-based fault detection for markovian jump systems. *International Journal of Systems Science*, 40(4):421–428, 2009.

[111] M. Zhong, S.X. Ding, J. Lam, and H. Wang. An LMI approach to design robust fault detection filter for uncertain LTI systems. *Automatica*, 39(3):543–550, 2003.

[112] M. Zhong, H. Ye, S. X. Ding, and G. Wang. Observer-based fast rate fault detection for a class of multirate sampled-data systems. *IEEE Trans. Automatic Control*, 52(3):520–525, 2007.

[113] D.H. Zhou and P.M. Frank. Fault diagnostics and fault tolerant control. *IEEE Transactions on Aerospace and Electronic Systems*, 34(2):420–427, 1998.

[114] K. Zhou and J. Chen. Performance bounds for coprime factor controller reductions. *Systems & Control Letts.*, 26(2):119–127, 1995.

[115] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall Inc., NJ, 1996.

[116] Z.Zhang and Y. Yuan. Decentralized consensus value computation over network with uncertainties. *IEEE Trans. Automatic Control*, 2011. submitted.