A Stochastic Volatility LIBOR Market Model with a closed form solution

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Abstract

Since its initial publication the SABR model has gained widespread use across asset classes and it has now become the standard pricing framework used in the market to quote interest rate products sensitive to the non flat strike-structure of the market implied volatility. While very simple, the model's use has always been based on the original study of its authors who derive a formula for pricing European options through a few approximating assumptions which are at times severely violated in the market. This thesis' main theoretical goal is to set the path for a generalization of the SABR model which possesses a closed form solution free from assumptions about the magnitude of the model's parameters. We propose such model and derive a closed form solution for the particular case in which the underlying forward rate and its volatility are uncorrelated. After using the solution for pricing caplets within a LIBOR Market Model framework we simplify an approximation for the swap rate developed by Piterbarg in order to use the same solution for the pricing of swaptions. We conduct the model's calibration for short maturities using a computationally efficient approach which derives an approximation for the model's implied volatility and uses it to fit the model to market quotes. Finally, we study the properties of the greeks of our model in comparison with those of the classical Black model.

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1 Introduction

While a long time has passed since the introduction of the first interest rate derivatives pricing models in the late 70s, the changes observed in market behavior after the Long Term Capital Management fund crisis and the related Russian default in 1998 teach us two lessons. The first is that a lot still needs to be done in the field and the second is that when one may feel satisfied with a model for a particular period possible radical transformations in the future must be taken into account. From models based on very simple but abstract settings in which the underlying variable is an idealized "infinitesimal spot rate" the mid 1990s have seen a shift towards what are now addressed to as "market models" which have the benefit that the main sources of randomness are clearly observable market variables and are thus much easier to calibrate to factual prices. Since then the main research activity of practitioners and similarly minded academics has revolved around the incorporation of the smile effect accentuated by the aforementioned crisis. A variety of approaches have been attempted, some far more complicated than others, of which it now seems clear that the stochastic volatility one has triumphed as the best fit. Problems remain as most often such pricing frameworks have an inbuilt assumption that the process driving market volatility is characterized by mean reversion. While this is certainly appealing as it causes the variance to "cluster" and mean revert as it is actually observed to, it causes long maturity options to have no smile whatsoever: the implied volatility obtained from all such models has a flat structure for long termed contingent claims. In some markets, like equities for instance, this may not be such a big problem but in interest rates a considerable amount of the structured products traded have very long maturities and the problem thus becomes more important. Indeed, in the past few years a model known as SABR has become the standard used by brokers and broker-dealers to quote interest rate products. The model is able to fit the market quite accurately and describes correctly the dynamics of the implied volatility smile. One major flaw in

its derivation is the use of approximation methods based on the assumption that the volatility-of-volatility is small. This assumption has been put under examinations in recent years given large swings in market volatility, and a generalization of the SABR model which does not make this assumption is thus required. The main theoretical goal of this thesis is to modify the presently widely used stochastic volatility LIBOR market models to create a generalization of a modified SABR model. We do this and analyze the specific case in which the underlying and the volatility are uncorrelated. Most importantly, the analysis we propose does not require any form of assumption regarding the so called "vol-of-vol", indeed in the case of caplets we do not make any assumption whatsoever and a complete closed form is derived using techniques developed by Lewis in [LEWI00]. We hence propose a forward libor model which is able to fit the smile and give correct dynamics for the implied volatility while taking into account the non flat strike-volatility structure of prices of long maturity options. Although in order to obtain a good fit to the market smile, the correlation between the underlying and its volatility is very important, the proposed analysis we conduct opens the door for future research in the correlated case.

Furthermore, in calibrating any type of interest rate market model one is faced with the problem of the existence of two separate, but highly interlinked, markets: the swaps market and the forwards market. These give rise to two possible candidates for building a model but cause difficulty in deriving the dynamics of one market when one starts with the other. On the other hand modeling the two separately gives rise to arbitrage and thus inconsistencies in the prices one obtains. Some work has been carried out in the past two years in trying to bridge this gap by deriving an approximating stochastic differential equation for the swap rate when starting to model the interest rates market through the forward rate. The first notable suggestion about the way that such issue can be solved is in Piterbarg [PITERB03]. This, though, is highly heuristic and can by no means be considered a proper derivation. Further work is under way, see for instance Antonov

and Misirpashaev [ANTO06]. Theoretically the problem is thus still slightly open but experimentally market prices do point out that such approximations exists, see Piterbarg [PITERB03]. Given that the focus of this thesis is in proposing a general model and providing a pricing framework free from the assumptions of small volatility-of-volatility expansions we will not dedicate too much effort on such issue. Instead, we propose a simplification of Piterbarg's heuristic to put us in a plausible position to adopt the same methodology for pricing swaptions within the forward LIBOR model used for caplets. Greeks for such plain vanillas are then produced. Once that is done, the implied volatility of our model is derived and shown to be consistent with the market in the sense of Hagan et al. [HAGAN02]. A calibration methodology based on our study of the implied volatility is developed in the final chapter and applied to the proposed model and a comparison with the SABR formula is carried out.

The next chapter begins with a description of some of the more popular single factor short rate models. Possible variations that one may get from integrating multiple factors in the picture are mentioned. After going through such "affine yield" frameworks, the HJM model is presented and the resulting arbitrage constraints are discussed. The practice of simple compounding of non-instantaneous rates by market participants will lead us into the LIBOR and Swap market models. The inconsistencies between the two models are also mentioned and we then quickly move to review the alternative ways to picture the volatility smile/skew effect within the Libor Market Model to end with the stochastic volatility approach.

2 Literature Review

2.1 Review of Term Structure Models

2.1.1 Short Rate Models

The oldest type of models used to price bonds and fixed income derivatives has historically been the class of frameworks that specify a single process for a so called "instantaneous" short rate which is usually taken to be the shortest tenored yield of the term structure. Although these are evidently problematic since it is not clear that all the rates in the market are necessarily functionally dependent on a single *short* interest rate, their simple structure helps in providing an intuitive understanding of the issues in pricing, so we will start our discussion with them. One common theme is the definition of the Stochastic Differential Equation for the short rate. Let us assume that the short rate $R(t)$ follows

$$\mathrm{dR}(t) = \beta(t, R(t))\mathrm{dt} + \gamma(t, R(t))d\widetilde{W}(t) \tag{2.1}$$

where $\widetilde{W}(t)$ is a Brownian motion in a risk neutral measure. We then define the discount process

$$D(t) = \exp\left\{ -\int_0^t R(s)\,\mathrm{ds} \right\} \tag{2.2}$$

and a zero coupon bond as a contract paying a notional of 1 at maturity T. From the risk neutral pricing formula we obtain

$$D(t)B(t,T) = \widetilde{\mathbb{E}}[D(T) \mid \mathfrak{F}(t)] \Rightarrow B(t,T) = \widetilde{\mathbb{E}}\left[\exp\left\{ -\int_t^T R(s)\,\mathrm{ds} \right\} \mid \mathfrak{F}(t) \right]. \tag{2.3}$$

Finally, we define the yield $Y(t,T)$ as

$$B(t,T) = \exp\{-Y(t,T)(T-t)\} \Rightarrow Y(t,T) = -[1/(T-t)]\log(B(t,T)). \tag{2.4}$$

Since $R(t)$ is the solution to a Stochastic Differential Equation it is a Markov process and we can define $f(t,R(t)) = D(t)B(t,T)$ for some f and given that the discounted bond price is a martingale we can take the differential of f and set the drift term to zero. The result is the Feynman-Kac PDE for Bond prices. There can thus be as many single factor models as one likes. Historically though, we can discern two main currents termed the equilibrium and the no arbitrage streams. Models of the former type start from the very macro level of economic theory and take off by specifying conditions on macro-economic variables from which a stochastic differential equation for the short rate is derived. Of these, three have been popular: the Rendleman and Bartter[RENDLE80], the Vasicek[VASIC77] and the Cox-Ingersoll-Ross model [COX85]. Since there is no difference between the Rendleman and Bartter and the Black-Scholes model except for the fact that interest rates are not tradeable assets, we will not dwell into such model. It suffices to say that the framework assumes that the short rate follows a lognormal process, which is a good starting point for simplicity reasons, but does not take into account the fact that interest rates mean revert to a long term average. Vasicek [VASIC77] incorporates mean reversion by getting to the following SDE for the short rate $R(t)$:

$$\mathrm{dR}(t) = (a - \mathrm{bR}(t))\mathrm{dt} + \sigma \mathrm{dW}(t) \quad \forall\, a, b, \sigma \in \mathbb{R}. \tag{2.5}$$

The Stochastic Differential Equation can be solved by taking the differential of $\exp\left\{\int_0^t c(t)\mathrm{dt}\right\}R(t)$ and finding that

$$R(t) = e^{-\mathrm{bt}}R(0) + \frac{a}{b}(1 - e^{-\mathrm{bt}}) + \sigma e^{-\mathrm{bt}}\int_0^t e^{\mathrm{bs}}\mathrm{dW}(s). \tag{2.6}$$

Using the result that the Ito integral of a deterministic function such as $\int_0^t e^{\mathrm{bs}}\mathrm{dW}(s)$ is normally distributed with mean zero and variance $\int_0^t e^{2\mathrm{bs}}ds$ we can calculate prices of bonds or any type of derivative security by risk neutral pricing. The mean reversion advantage of the Vasicek model can be seen from the drift term: if $R(t) > \frac{a}{b}$, the drift is negative and $R(t)$ is pushed back towards $\frac{a}{b}$, if $R(t) < \frac{a}{b}$then the drift is positive and $R(t)$ is pulled back towards $\frac{a}{b}$ again. Further, if $R(0) = \frac{a}{b}$ then $\mathbb{E}[R(t)] = \frac{a}{b}$ for all $t > 0$. If $R(0) \neq \frac{a}{b}$ we have $\lim_{t\to\infty}\mathbb{E}[R(t)] = \frac{a}{b}$ hence in all cases the process reverts back to its mean $\frac{a}{b}$. We need to be cautious though, as can be seen from the SDE this model permits the rates to become negative: a highly unrealistic case for non deflationary rates markets such as most non Japanese ones.

The first form of correction comes by considering the Cox-Ingersoll-Ross (CIR) model [COX85]. The stochastic differential equation in this case is given by:

$$\mathrm{dR}(t) = (a - \mathrm{bR}(t))\mathrm{dt} + \sigma\sqrt{R(t)}\widetilde{dW}(t). \tag{2.7}$$

For bond prices $B(t, T, R)$ after deriving the Feynman-Kac PDE we look for solutions of affine form:

$$B(t, T, R) = \exp\{-R\,C(t, T) - A(t, T)\}\,. \tag{2.8}$$

Substituting (2.8) in the partial differential equation corresponding to (2.7) we find:

$$\left[(-C^{'}(t, T) + \mathrm{bC}(t, T) + \frac{1}{2}\sigma^2 C^2(t, T) - 1)R - A^{'}(t, T) - \mathrm{aC}(t, T)\right] = 0. \tag{2.9}$$

As this must hold for all R, the only way the above can happen is if the term multiplying

R and the second term are both zero, hence:

$$(-C^{'}(t,T) + \mathrm{bC}(t,T) + \frac{1}{2}\sigma^2 C^2(t,T) - 1) = 0 \tag{2.10}$$

and

$$A^{'}(t,T) = -\mathrm{aC}(t,T) \tag{2.11}$$

The boundary conditions are $A(T,T) = C(T,T) = 0$, and we can thus solve the equations and obtain Bond prices. Similar procedures can be carried out for options, where one may need to make use of numerical methods. Also note that because of the $\sqrt{R(t)}$ term in the defining SDE of term (2.7) the rate will never be negative.

The Cox-Ingersoll-Ross model is the pinnacle of equilibrium models: the rate is not negative, mean reversion is taken into account and prices can be derived quite easily. In the discussion above though, one should note that the current state of the entire yield curve does not enter the picture at all. The calibration of such models never fits the real world exactly and according to some authors, in some cases, even a reasonable approximation is unavailable [HULL06]. This problem gave rise to what are called no-arbitrage models. The main difference is that the parameters in the Stochastic Differential Equation defining the interest rate dynamics are allowed to vary with time. In these, the starting point is not the macro economic environment but rather the current term structure which is used to fit the time varying parameters of the assumed form for the SDE. Many people have come up with a variety of models, the first is the binomial Ho-Lee approach[HOLEE86], whose continuous time limit gives an SDE of the form

$$\mathrm{dR}(t) = \theta(t)\mathrm{dt} + \sigma\mathrm{dW}(t). \tag{2.12}$$

Further models of this type are the Hull-White[HULL90], the Black-Karasinski[BLACK91], and the Black-Derman-Toy[BLACK90]. In particular Maghsoodi [MAGHSOODI96] deals

with a no arbitrage type extension of the CIR. Since the discussion is completely parallel to the above, the sole difference being the initial Stochastic Differential Equation, we will not focus more on them. Instead, we will mention the most obvious generalization which is to have the short rate $R(t)$ be driven by a variety of factors such that:

$$R(t) = \sum_{i=0}^{n} \epsilon_i(t) X_i(t),\ \epsilon_i : \mathbb{R} \to \mathbb{R} \tag{2.13}$$

with

$$\mathrm{dX}_i(t) = \mu_i(t, X_i(t))\mathrm{dt} + \sigma_i(t, X_i(t))d\widetilde{W}_i. \tag{2.14}$$

Note that the factors X_i are "abstract" and unobservable, furthermore the short rate is not a tradeable asset so we cannot directly obtain the market price of risk (that is the amount by which μ_i supercedes the risk neutral rate divided by the volatility -also known as the Sharpe ratio). Hence, the only way to guarantee no arbitrage is to start by assuming the existence of the risk neutral measure and write X_i with respect to it. The factors, it turns out are not completely independent, one is able to get rid of many by writing the vector of processes of (2.14) in a "canonical form" as is done for example by Dai and Singleton[DAI00]. The bond prices may be obtained by the PDE approach where the martingale

$$B(t,T) = \widetilde{\mathbb{E}}\Big[e^{-\int_t^T R(u)\mathrm{du}} \mid \mathfrak{F}(t)\Big], 0 \le t \le T \tag{2.15}$$

is written as a function of all factors. In the case for two:

$$B(t,T) = f(t, X_1(t),\ X_2(t)). \tag{2.16}$$

This can be done because the $X's$ are solutions to a Stochastic Differential Equation and hence satisfy the Markov property, whose multidimensional version gives the above.

Taking the differential of $D(t)\,f$ by using the multidimensional version of Ito's lemma and setting the drift to zero gives the PDE satisfied by the prices. Solutions are then sought by assuming an affine yield solution of the form

$$f(t, x_1, x_2) = e^{-x_1 C_1(T-t) - x_2 C_2(T-t) - A(T-t)}. \tag{2.17}$$

The simplest cases are the two factor models with constant coefficients, and within a mean reverting scenario we can discern three main types of them. The first is one where the factors are allowed to become negative and are of the form of the Vasicek SDE:

$$\mathrm{dX}_1(t) = (a_1 - b_{11}X_1(t) - b_{12}X_2(t))\mathrm{dt} + \sigma_1 d\widetilde{W}_1(t), \tag{2.18}$$

$$\mathrm{dX}_2(t) = (a_2 - b_{21}X_1(t) - b_{22}X_2(t))\mathrm{dt} + \sigma_2 d\widetilde{W}_2(t). \tag{2.19}$$

In canonical form these can be transformed to:

$$\mathrm{dY}_1(t) = -\lambda_1 Y_1(t)\mathrm{dt} + d\widetilde{B}_1(t), \tag{2.20}$$

$$\mathrm{dY}_2(t) = [-\lambda_{21}Y_1(t) - \lambda_2 Y_2(t)]\mathrm{dt} + d\widetilde{B}_2(t). \tag{2.21}$$

Where $\widetilde{B}_1$ and $\widetilde{B}_2$ are independent Brownian motions under a Risk Neutral measure. Another possibility is the two factor CIR model where both factors are positive:

$$\mathrm{dX}_1(t) = (a_1 - b_{11}X_1(t) - b_{12}X_2(t))\mathrm{dt} + \sqrt{Y_1(t)}\sigma_1 d\widetilde{W}_1(t), \tag{2.22}$$

$$\mathrm{dX}_2(t) = (a_2 - b_{21}X_1(t) - b_{22}X_2(t))\mathrm{dt} + \sqrt{Y_2(t)}\sigma_2 d\widetilde{W}_2(t). \tag{2.23}$$

Or the mixed where one factor is taken from the former and one from the latter. For a

general discussion of multifactor models see Duffie and Kan [DUFFIE96][DUFFIE94]. In one case the factors can have a non abstract meaning, this is the approach taken by Chen [CHEN96] in which one has, similarly to Heston's approach and the rest of the FX and Equity stochastic volatility models, a factor for the volatility and a factor for the mean. As can be noticed from the above all these models are characterized by the fact that the yield is an affine function and they are thus also called affine yield models. For further discussion, see [PIAZZESI03][BALDUZZI96][AIT96][CHEN92][CHEN93][CHEN95][DUFEE02] [DUFRES01][DUFRES02]. For an empirical comparison see [CHAN92].

Since the presence of many interest rates is not dealt with and calibration tends to be unstable or simply impossible we introduce the widely used forward rate models.

2.1.2 The HJM model

From our discussion so far, one may conclude that short rate models are simple to deal with, at least in theory, in the sense that pricing is not too complicated in procedural structure. One only needs to choose an SDE for the short rate, derive the associated PDE and solve it using an affine function for bond prices. Their main problem though is the fact that they are set by describing only one yield, the shortest tenored one, and attempt to derive the longer ones from it. Markets are characterized by the existence of bonds and interbank rates of various maturity rather than a single macro-economic short rate. Indeed, to be more specific markets are populated by non zero coupon bonds of different maturities from which zero coupon prices can be extracted by adopting the bootstrapping method. This procedure can also be used to define the forward rate: the interest rate that can be locked in now for a period in the future. To formalize this concept we will follow the elegant approach in [SHREVE04], the whole construct is due to Heath, Jarrow and Morton, for references see [HEATH90a][HEATH90b][HEATH92]. The point in the following is not really the development of a specific forward rate model, but rather to

derive the no arbitrage conditions that any such model needs to have. It is a step towards the development of the Libor Market Model of the next section which is the dominant one used by practitioners.

Suppose we are at some time t, with $T \geq t \geq 0$, take $B(t,T)$ to be the price of a default-free T-maturity bond at time t, let $\delta \in \mathbb{R}^+$, and suppose we enter into the following trades:

1. We short sell a T-maturity bond, thus getting $B(t,T)$,
2. With this income we buy $\frac{B(t,T)}{B(t,T+\delta)}$ of $(T+\delta)$-maturity bonds.

after closing down the trades our net expenses are zero, but at time T we need to pay 1 to the buyer of the bond from trade (1). At the later time $T+\delta$ we receive $\frac{B(t,T)}{B(t,T+\delta)}$ from trade (2), overall we thus have lent 1 at time T and received a payback of $\frac{B(t,T)}{B(t,T+\delta)}$ at time $T+\delta$. In a non inverted yield curve bond prices decrease with maturity and we have that $\frac{B(t,T)}{B(t,T+\delta)} \geq 1$ such that we lock in a profit, otherwise we incur a loss. The continuously compounded yield of this transaction is:

$$\frac{1}{\delta}\log\left[\frac{B(t,T)}{B(t,T+\delta)}\right] = -\frac{\log B(t,T+\delta) - \log B(t,T)}{\delta}. \tag{2.24}$$

Notice that even in the case of the non inverted yield curve this is not arbitrage, we need to pay 1 at time T in order to lock in this yield at t, hence it is rather a form of interest rate seen at time t for lending in the period $[T, T+\delta]$. The limit as $\delta \to 0$ is termed as the forward rate $f(t,T)$ at t for investment at T:

$$f(t,T) = \lim_{\delta \downarrow 0}\left[-\frac{\log B(t,T+\delta) - \log B(t,T)}{\delta}\right] = -\frac{\partial}{\partial T}\log B(t,T). \tag{2.25}$$

Note that the instantaneous rate at t is $R(t) = f(t,t)$. Intuitively, this is the instantaneous rate at T observed at time t, and from the discussion we see that if we have $B(t,T)$ we

can derive f and vice versa. Indeed, we have:

$$B(t,T) = \exp\left\{ -\int_t^T f(t,v)\mathrm{dv} \right\} \tag{2.26}$$

and thus conducting a discussion based on forward rates or bond prices is completely parallel but it turns out that it is easier to deal with the forward rate so we will adopt it for our formulation. As usual we model f by defining the SDE that it must obey. The initial condition is the forward curve observed at time zero $f(0,T)$ and

$$d\, f(t,T) = \alpha(t,T)\mathrm{dt} + \sigma(t,T)\mathrm{dW}(t),\ 0 \le t \le T. \tag{2.27}$$

to get the process implied for bond prices we have:

$$d\left(-\int_t^T f(t,v)\mathrm{dv} \right) = f(t,t)\mathrm{dt} - \int_t^T d\, f(t,v)\ \mathrm{dv} = R(t)\mathrm{dt} - \int_t^T [\alpha(t,v)\mathrm{dt} + \sigma(t,v)\mathrm{dW}(t)]\mathrm{dv}. \tag{2.28}$$

Define:

$$\alpha^*(t,T)\ \mathrm{dt} = \int_t^T \alpha(t,v)\ \mathrm{dvdt}, \tag{2.29}$$

and

$$\sigma^*(t,T)\ \mathrm{dW}(t) = \int_t^T \sigma(t,v)\mathrm{dvdW}(t) \tag{2.30}$$

using Ito's Lemma the bond prices follow:

$$\mathrm{dB}(t,T) = B(t,T)\left\{ \left[R(t) - \alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2 \right]\mathrm{dt} - \sigma^*(t,T)\ \mathrm{dW}(t) \right\} \tag{2.31}$$

We now need to ask the question of whether our construct admits arbitrage. From the appendix on stochastic calculus we note that the absence of arbitrage is equivalent to having a risk neutral measure under which discounted prices are martingales. If we let $D(t)$ denote, as we have previously done, the discounting process, the differential of the discounted bond price is:

$$d(D(t)B(t,T)) = D(t)B(t,T)\left[\left(-\alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2\right)\mathrm{dt} - \sigma^*(t,T)\,\mathrm{dW}(t)\right] \quad (2.32)$$

For this to be a martingale under the risk neutral measure $\widetilde{\mathbb{P}}$, we need to be able to define $\Theta(t)$ such that the square brackets term is written as $-\sigma^*(t,T)[\Theta(t)\mathrm{dt} + \mathrm{dW}(t)]$ so that $\widetilde{W}(t) = \int_0^t \Theta(u)\mathrm{du} + W(t)$ is a Brownian motion under $\widetilde{\mathbb{P}}$. Hence we must have:

$$\left(-\alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2\right)\mathrm{dt} - \sigma^*(t,T)\,\mathrm{dW}(t) = -\sigma^*(t,T)[\Theta(t)\mathrm{dt} + \mathrm{dW}(t)] \Rightarrow$$

$$\Rightarrow -\Theta(t)\sigma^*(t,T) = -\alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2 \quad (2.33)$$

Differentiating both sides with respect to T yields:

$$\alpha(t,T) = \sigma(t,T)[\sigma^*(t,T) + \Theta(t)] \Rightarrow \Theta(t) = \frac{\alpha(t,T)}{\sigma(t,T)} - \sigma^*(t,T). \quad (2.34)$$

Thus, as long as $\sigma \neq 0$, the market does not admit arbitrage and is complete. Under $\widetilde{\mathbb{P}}$:

$$d\,f(t,T) = \sigma(t,T)\sigma^*(t,T)\mathrm{dt} + \sigma(t,T)d\widetilde{W}(t), \quad (2.35)$$

$$d\,B(t,T) = R(t)B(t,T)\mathrm{dt} - \sigma^*(t,T)B(t,T)d\widetilde{W}(t). \quad (2.36)$$

Note that since we have also derived prices of bonds in short rate models, we can derive a forward curve from them and check if they satisfy the no arbitrage condition above, it is trivial to show that the Hull-White and CIR models indeed satisfy it. While the HJM approach gives a very powerful no arbitrage condition on the drift and diffusion part of any interest rate model, the approach just seen, by keeping the rate instantaneous, is still unable to give a stable and good fit to market data, this is mainly due to the fact that, in real life, rates are taken for specific discrete periods, see [SHREVE04][BRACE97][BRACE94][JAMSH97][MILTE97]. This issue gives rise to the so called LIBOR model, which also satisfies the HJM conditions.

2.1.3 The Market Models

2.1.3.1 The Forward Libor Market Model

Consider the HJM forward rate in the risk neutral measure:

$$d\, f(t,T) = \sigma(t,T)\sigma^*(t,T)\mathrm{dt} + \sigma(t,T)d\widetilde{W}(t). \tag{2.37}$$

In the market interest rate caps (portfolios of options on the forward rates -more on them in the second section) are quoted according to a lognormal distribution of the forward rate [BLACK76], to replicate such a quoting system we thus need to impose this on the diffusion and drift. This means [SHREVE04][HEATH92]:

$$\sigma^*(T,t) = \int_t^T \sigma(t,v)\mathrm{dv} = \sigma \int_t^T f(t,v)\mathrm{dv} \tag{2.38}$$

and the drift would then be

$$\sigma^2 f(t,T) \int_t^T f(t,v)\mathrm{dv}. \tag{2.39}$$

This becomes problematic as it explodes for some values of t. To see how, we integrate

the Riemann integral part of the SDE and get

$$f \sim \frac{f(0)}{1 - \sigma^2 f(0)t} \tag{2.40}$$

which has a singularity at $t = \frac{1}{\sigma^2 f(0)}$. We need to be able to maintain log-normality, the arbitrage condition and fit market data more properly. It turns out all this can be combined in models where the interest rate is simply compounded. Recall the series of trades we went through to derive the forward curve of (2.25). The simply compounded rate $F(t,T)$ of the transactions satisfies:

$$F(t,T) = \frac{B(t,T) - B(t,T+\delta)}{\delta B(t,T+\delta)}. \tag{2.41}$$

This is called the forward LIBOR (for London Interbank Offered Rate) rate and the model resulting from specifying its dynamics is referred to as the LIBOR Market Model or LMM in short. In the case t=T we call $F(T,T)$ spot LIBOR. Finally δ is referred to as *tenor,* usually taken to be 3 or 6 months.

To proceed we need to describe the change of numeraire theorem.

Proposition 2.1. *(Change of Risk Neutral Measure)[SHREVE04]: Let S(t) and N(t) be the prices of two assets in some currency, and let $\sigma(t)$ and $\nu(t)$ denote their volatility vector. Take N(t) as numeraire, so that S(t) is expressed in units of N(t): $S^{(N)}(t) = \frac{S(t)}{N(t)}$. Let $\left(\Omega, \mathfrak{F}, \widetilde{\mathbb{P}}\right)$ be a risk neutral probability space from which S(t) and N(t) take value. Define :*

$$\overset{\sim}{\mathbb{P}}^{(N)}(A) = \frac{1}{N(0)} \int_A D(T)N(T) d\,\overset{\sim}{\mathbb{P}}, \forall\, A \in \mathfrak{F} \tag{2.42}$$

Then under $\overset{\sim}{\mathbb{P}}^{(N)}$ *the process* $S^N(t)$ *is a martingale. Furthermore,*

$$\mathrm{dS}^{(N)}(t) = S^{(N)}(t)[\sigma(t) - \nu(t)]d\widetilde{W}_N(t) \tag{2.43}$$

where, $\widetilde{W}_N(t) = \widetilde{W}(t) - \int_0^t \nu(u)\mathrm{du}$ *and finally for any* $\mathfrak{F}(t)$*-measurable random variable* X *with* $s<t$ *we have:* $\overset{\sim}{\mathbb{E}}^{(N)}[X \mid \mathfrak{F}(s)] = \frac{1}{D(s)N(s)}\widetilde{\mathbb{E}}[D(t)N(t)X \mid \mathfrak{F}(s)]$.

The proof is done by applying Ito's lemma. Now, consider using the T-forward price for assets, such that any asset $S(t)$ is expressed in terms of the bond numeraire:

$$\text{For }_S(t,T) = \frac{S(t)}{B(t,T)}. \tag{2.44}$$

Let $\widetilde{\mathbb{P}}_{T+\delta}$ be the $T+\delta$-forward risk neutral measure (that is the one for the numeraire $B(t,T+\delta)$ and $\widetilde{W}_{T+\delta}$ a Brownian motion under $\widetilde{\mathbb{P}}_{T+\delta}$, since

$$F(t,T) = \frac{B(t,T) - B(t,T+\delta)}{\delta B(t,T+\delta)} \equiv \frac{S(t)}{B(t,T+\delta)} \tag{2.45}$$

$F(t,T)$ is a martingale under $\widetilde{\mathbb{P}}_{T+\delta}$, and there is thus a process $\gamma(t,T)$ such that:

$$d\,F(t,T) = \gamma(t,T)\,F(t,T)\,d\,\widetilde{W}_{T+\delta}(t) \tag{2.46}$$

We see that we have rid ourselves of the problematic drift term of the instantaneous continuously compounded rate case.

To get the lognormal distribution, we need $\gamma(t,T)$ to be a non random process, so if then prices of caplets are taken from markets where the Black formula is used, we are forced to make this assumption to derive it. Further, the above Stochastic Differential Equation for F is taken with respect to the risk neutral measure for the numeraire $B(t,T+\delta)$ for every T. In practice we do not have a continuum of maturities but rather discrete quantities T_i, $i = 0, ..., n$, and it would be clearer if we could express all the corresponding

LIBOR rates with respect to a single measure. If we choose the final one for $0 \leq t \leq T_j,\ j = 0, ..., n-1$ we get:

$$d\,F(t,T_j) = \gamma(t,T_j)F(t,T_j)\left[-\sum_{i=j+1}^{n-1}\frac{\delta\gamma(t,T_i)F(t,T_i)}{1+\delta F(t,T_i)}\mathrm{dt} + d\,\widetilde{W}_{T_n}(t)\right]. \tag{2.47}$$

From this, given the relationship between F and bond prices in (2.45), one is able to model bonds too. Notice also that because of the 1 in the denominator for positive rates the drift term does not blow up even in this new measure. It turns out that this model can be fitted extremely well to observed market prices [PELSSER00][SHREVE04][BRIGO06][REBONA02]. Due to this, the LMM or its variations are the most widely used in the industry.

2.1.3.2 The Swap Market Model

The forward rates described above are typical of the interbank deposit and lending markets. Another very important class of instruments are interest rate swaps. These are agreements to exchange cash flows for some period based on movements in the rates market. There are two main types of swaps, the payer's swap and the receiver's swap. So let the time now be t and let $t \leq T_i \leq T_{i+1} \leq \cdots \leq T_F$ be future times, let K be a fixed rate and N be a notional value and let τ_i be the time difference in the convention used in the market between times T_{i+1} and T_i. A payer's swap is an agreement entered at time t to pay the fixed rate K on the notional N and receive the variable forward rate set at T_i at time T_{i+1} until time T_F. Hence, letting the forward rate (as defined in the section above) at time t for investing for the period between T_i and T_{i+1} be $F_i(t)$ and the actual simply compounded rate for investing between T_i and T_{i+1} be $F^*(T_i, T_{i+1})$, the payoff at time t of a payer's swap is:

$$\sum_{r=i+1}^{F} D(t,T_{r+1})N\tau_r[F^*(T_r,T_{r+1}) - K] \tag{2.48}$$

Taking expectation the value at time t of the swap is:

$$\text{PFS (t, F, N, K)} = \text{N} \sum_{r=i+1}^{\text{F}} \tau_r B(t, T_r)[F_r(t) - K] =$$

$$= -N\, B(t, T_i) + N\, B(t, T_F) + N\, K \sum_{r=i+1}^{F} \tau_r B(t, T_r). \tag{2.49}$$

The two sides of the cash flow are termed legs, so that the payments of the variable rate is termed the variable leg and the other the fixed leg. The receiver's swap is exactly the same, only the fixed leg is received and the variable rate paid. Finally, the so called swap rate is the value of K in the above that sets the payer's (or receiver's) swap's value to zero. From (2.49) it is thus defined as:

$$S_{i,l}(t) = \frac{B(t, T_i) - B(t, T_l)}{\sum_{r=i+1}^{l} \tau_r B(t, T_r)} \tag{2.50}$$

Swap Market Models start by modeling this quantity as a stochastic process and are used mainly for the pricing of Swaptions type products (more on them in the next section), or options to get into swaps at a given strike of fixed leg. The market quotes plain vanilla payer or receiver swaptions by assuming a lognormal distribution of the swap rate, and hence market consistent models for pricing vanilla swaptions start by assuming the swap rate to be lognormal.

The problem arises since:

$$\frac{B(t, T_a)}{B(t, T_i)} = \prod_{r=i+1}^{a} \frac{B(t, T_r)}{B(t, T_{r-1})} = \prod_{r=i+1}^{a} \frac{1}{1 + \tau_r F_r(t)} \tag{2.51}$$

and dividing equation (2.50) by $B(t, T_i)$ on both sides we get that the swap rate is related to the forward rate by:

$$S_{i,l}(t) = \frac{1 - \prod_{j=i+1}^{l} \frac{1}{1+\tau_j F_j(t)}}{\sum_{r=i+1}^{l} \tau_r \prod_{j=i+1}^{r} \frac{1}{1+\tau_j F_j(t)}}. \tag{2.52}$$

If we start with a model in which the Libor forward rate is lognormal we cannot consistently have a lognormal swap rate. This inconsistency is an important theoretical issue for market models since they are not able to solve them, in practice though the study by Brace, Dun and Barton [BRACE98] shows that under the lognormal forward market model the swap rate is approximately lognormal. The LMM and the LSM are interchanged in the industry depending on the types of products that need to be priced. In our study starting in the next chapter we will see the use of both within a stochastic volatility framework.

2.2 Implied Volatility and Market Models

2.2.1 The Volatility Smile Problem

As mentioned in the previous section, two types of interest rate instrument are predominant in the plain vanilla sector, the caps/floors types and the swaptions type. Consider the payer's swap defined in the previous section, and imagine that each payment at every payment date is executed if and only if its value is positive, so that the payoff would be:

$$\sum_{r=i+1}^{F} D(t,T_{r+1})N\tau_r\left[(F^*(T_r,T_{r+1})-K)^+\right] \tag{2.53}$$

The structure giving this payoff is known as an interest rate cap, and can be thought of as a portfolio of calls on the libor rate, called caplets, each having a different starting date and different maturity but all with the same strike. A similar product is the interest rate floor which is the same as the cap but starting from a receiver's swap. Let us consider the price of a cap. Each caplet has a payoff of:

$$D(t,T_{r+1})N\tau_r\left[(F^*(T_r,T_{r+1})-K)^+\right] \tag{2.54}$$

where $F^*(T_r,T_{r+1})$ is the LIBOR rate (simply compounded by market convention) set at T_r for investing a dollar and receiving a payment at T_{r+1}. Now in the convention we

used in section 2.1.3.1 we let $\delta = \tau_r$ so that $F^*(T_r, T_{r+1}) = F(T,T)$. Thus under the T_{r+1}-measure from equation (2.46) we have:

$$F(T_r, T_{r+1}) = F(T,T) = F(0,T) \exp\left\{ \int_0^T \gamma(t,T)\, d\widetilde{W}_{T_{R+1}} - \frac{1}{2} \int_0^t \gamma^2(t,T)\, \mathrm{dt} \right\} \quad (2.55)$$

The price at time zero in the risk neutral measure (unit price numeraire) is:

$$\widetilde{\mathbb{E}}[D(T_{r+1})(F(T,T)-K)^+] = B(0,T_{r+1})\widetilde{\mathbb{E}}\left[\tfrac{D(T_{r+1})}{B(0,T_{r+1})}(F(T,T)-K)^+\right] =$$

$$= B(0,T_{r+1})\widetilde{\mathbb{E}}^{T_{r+1}}\left[(F(T,T)-K)^+\right] \quad (2.56)$$

Letting $\bar{\gamma}(T) = \sqrt{\frac{1}{T}\int_0^T \gamma^2(t,T)\,\mathrm{dt}}$ and noting that $\int_0^T \gamma(t,T)\, d\widetilde{W}_{T_{r+1}}$ is normally distributed with no mean and standard deviation $\bar{\gamma}(T)\sqrt{T}$ under $\mathbb{P}^{T_{r+1}}$ the expectation is computed to be:

$$B(0,T_{r+1})\widetilde{\mathbb{E}}^{T_{r+1}}\left[(F(T,T)-K)^+\right] = B(0,T_{r+1})\mathrm{BS}(T, F(0,T), K, 0, \bar{\gamma}(T)) \quad (2.57)$$

where BS is the Black-Scholes formula.

Now pay attention to the volatility used to price the caplet, $\bar{\gamma}(T)$, and notice that the strike is not present at all in its definition. Assume that we have measured $\gamma(t,T)$ historically so that we could use it to price any caplet disregarding the strike. Unfortunately though, this does not produce the prices observed in the market: participants quote prices which use different volatilities depending on the strike and hence contradicting the very definition of $\gamma(t,T)$ as the rate of standard deviation of the underlying process depending on time and tenor. If one substitutes the current underlying $F(0,T)$, the maturity T and strike K into equation (2.57) and leaves the volatility parameter $\bar{\gamma}(T)$ undetermined, while setting the model's price equal to the market price and inverts for $\bar{\gamma}(T)$, the result depends on K (while leaving all other parameters unchanged). The resulting $\bar{\gamma}(T)_K$ is

termed the *implied volatility* and forms a curve which varies depending on the interest rate market underlying the caps and it can be skewed and thus called the *volatility skew* or also (and sometimes simultaneously) of a smile shape then called *volatility smile*. In the interest rates market the smile is quite a recent phenomenon. While in the equity markets the smile made itself noticeable after the 1987 crash, the first signs of a slightly similar effect for caps were noticed around 1994 with Japanese JPY rates when they became significantly lower than for most economies [REBONA05]. Before that volatilities gave a flat structure and the first moves were observed in the short-maturity implied volatility of caplets [REBONA05]. The times during 1994 and 1998 are characterized by a similar evolution in other currencies' interest rates where the shape went from flat to monotonically decreasing with the increase of strike. The effect remains limited until the global crisis of the rates markets with the collapse of the Long Term Capital Management hedge fund and from then onwards it becomes a major characteristic for most major currencies. Below is a plot taken from Rebonato [REBONA05] of the volatility surface implied by caplets on DEM rates on 11 May 1998:

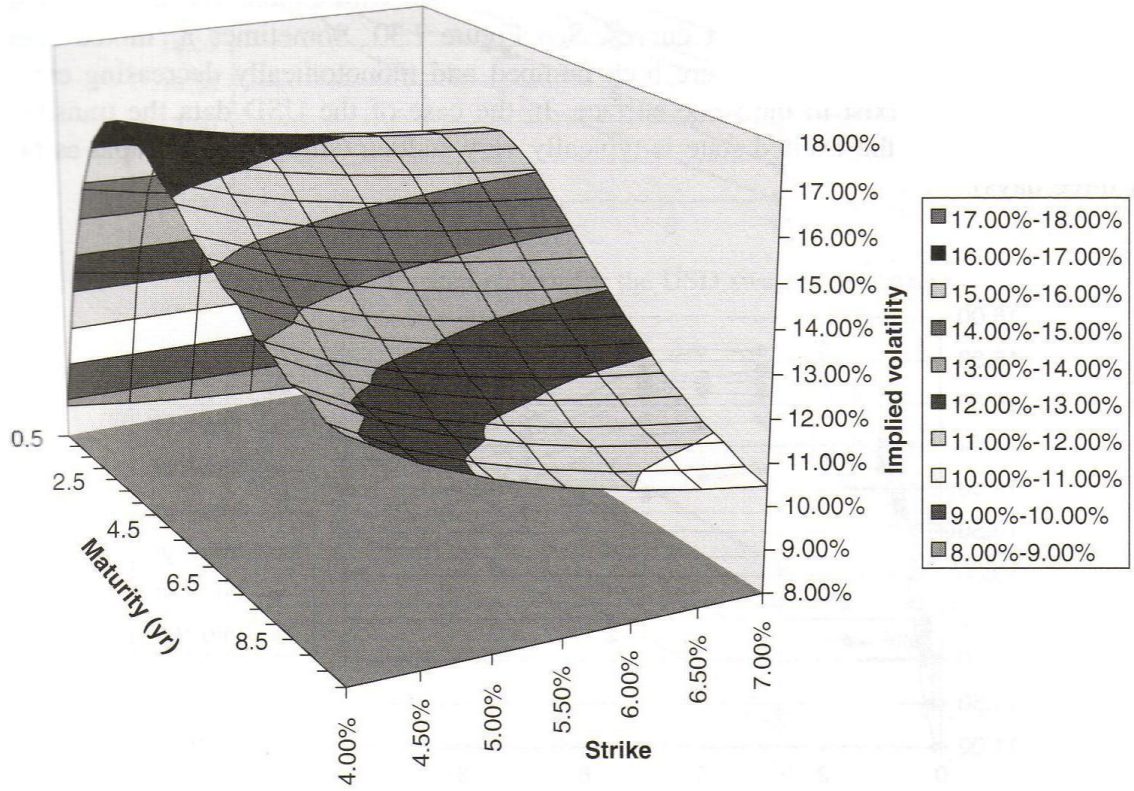


Figure 2.1: DEM Implied Volatility Surface in 1998, taken from [REBONA05].

All of this means that traders simply use the lognormal assumption of the Black formula to translate the price into a volatility which they set according to their expectations of how the real volatility will be in the future time coinciding with the durability of the contract and on the market's supply and demand effect, not as a proper pricing mechanism to get a fair value. For example one possible explanation is that in a market where the main participants, sometimes for regulatory or policy and sometimes for other reasons, are not allowed to go short on bonds, there will be more interest in out of the money floors rather than in the money instruments, thus making such products more expensive and, given the monotonicity of the Black-Scholes formula, have a higher implied volatility. Such difference in supply and demand along the strike can thus be easily viewed as

causing a smile effect in the market. As far as plain vanilla options though, traders may still develop their hedging strategy using the greeks given by a lognormal model and the implied volatility with which one started, taken as the volatility parameter in the formulas for the sensitivities. This is because practitioners tend to view the model as fairly robust for hedging plain vanillas where the hedging is only done through the underlying and the small changes prices of options from different models are not as important when hedging an entire book, for this see Rebonato [REBONA05]. When it comes to exotic products whose payoff is path dependent, the sensitivity to the existence of the smile becomes very important and cannot be disregarded by traders [REBONA05]. For instance there are products that whose payoff depends on the shape of the smile during in the period between the contract's conception and maturiy. What we look for is thus a model whose price, when set equal to the Black's formula and inverted for the volatility function with respect to the strike will give a smile that can be fit as well as possible to the market observed one. To see exactly what needs to be changed, take a caplet with strike K and starting time T_1, maturity T_2 at time zero. In an arbitrage free market its price would be:

$$\text{Cpl}(0, T_1, T_2, K) = B(0, T_2)\tau \overset{\sim}{\mathbb{E}}^{T_2}\left[(F(T_1; T_1, T_2) - K)^+\right] \tag{2.58}$$

Taking the second derivative with respect to the strike of the market caplet prices gives the market implied risk neutral probability measure:

$$\frac{\partial^2 \text{Cpl}(0, T_1, T_2, K)}{\partial K^2} = P(0, T_2)\tau \widetilde{p}(F) \tag{2.59}$$

In a lognormal world $\widetilde{p}$ would not depend on the strike. The above equation permits one to imply a probability distribution from the volatility smile observed in the market. When that is done one notices that option traders do not believe in a lognormal world, a fat tailed probability distribution is implied. This is more consistent with a historical analysis of the rates market than with a lognormal dynamic.

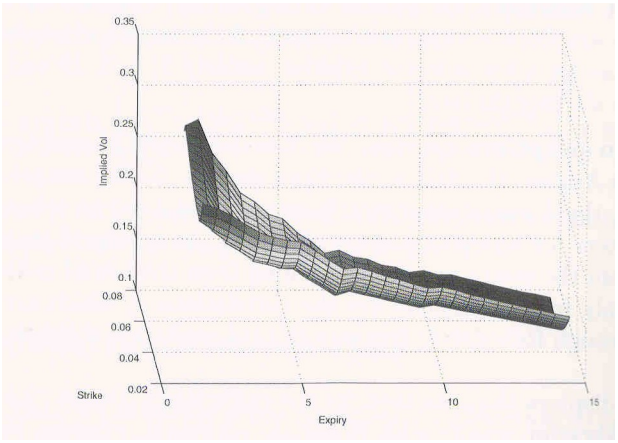


Figure 2.2: Market observed caplets smile, taken from [BRIGO06].

On a first analysis there thus needs to be a way of changing the probability of the forward measure that can satisfy the no arbitrage constraints, which also means that the forward rate must be a martingale under the right forward measure, and that when the prices obtained with it are set equal to the Black's formula, with the strike remaining a variable, the inversion for the volatility gives a smile similar to the one observed in the market. Now consider the forward rate $F_i(t)$ observed at time t for investing between T_i and T_{i+1}, as we saw previously under the T_{i+1}-forward measure, it follows the dynamics:

$$d\,F_i(t) = \gamma(t, T, F_i(t))\,d\,W_t \tag{2.60}$$

where W_t is a Brownian motion in the given measure. The lognormal assumption which gives Black's formula is given by taking $\gamma(t, T, F_i(t)) = \sigma(t, T)F_i(t)$ for some deterministic function $\sigma(t, T)$. One way the issue could change is to take $\gamma(t, T, F_i(t)) = \sigma(t, T, F_i(t))F_i(t)$ for some deterministic function σ. This approach gives rise to the so called local volatility models of which the most common in interest rates are the displaced diffusion model and the constant elasticity of variance ones, subjects of the next section. If on the other hand we have $\sigma(t, T, F_i(t)) = v(t, T)F_i(t)$, where $v(t, T)$ follows a stochastic process, we get the so called stochastic volatility models, the main framework of

this thesis. We will cover these approaches in detail in the next two sections but before beginning, I would like to point out that there are many other frameworks for modeling the smile in interest rates, all of which have their flaws. For instance, one could start by modeling the forward rate not just as a diffusion type process but as a more general jump-diffusion one. The plausibility of these models is very strong given the fact that the tick by tick changes of rates are not continuous and that jumps are observed at times of central bank announcements. Such models though are not efficient in implementation, and furthermore their calibration is unstable with sometimes senseless parameter values [MEIST04]. For a thorough discussion of such issues from an empirical point of view see [MEIST04]; the models by Glasserman et. al. are based mainly on Merton's [MERT76] and can be found in [GLASSER03a] and [GLASSER03b]. Another framework which has gained some popularity is that of the uncertain-parameters models. These can be viewed as elementary forms of the stochastic volatility types, where $\sigma(t, T, F_i(t)) = v(t, T)F_i(t)$ and $v(t, T)$ is not a stochastic process per se, but rather just a random variable which at every instant can take a discrete set of values each with associated probability in the usual sense. Although they are nicely straightforward, we will not delve into them. On the negative side we mention that the smile they derive is not self similar, meaning that when one calibrates the parameters initially to produce the observed smile, the dynamics of this with time do not correspond to the market dynamics. The future implied volatility structure obtained by such models is flat contradicting the fact that the smile is generally speaking similar to what it previously was[MEIST04] and this is not only undesirable from a theoretical view but it would require a trader to recalibrate the model at every hedging point with quite different parameters and hence quite different greeks and could hence lead to losses of money on mishedges. There is a plethora of models which has been tried from this class and this is principally due to the analytical tractability of them, for the most popular see the papers [MERC02][ALEXAN03][GATAREK1][BRIGO04]. Many of the approaches tested do not only take the volatility as uncertain but also unify the char-

acteristics of different models, for instance some may start by using a displaced diffusion model, which we will soon describe at length, and force the displacement parameter to be uncertain. Similarly some begin with a CEV type local volatility framework and force randomness on the volatility parameter instead. Taking rates market models as an example, another framework was developed which starts by modeling the implied volatility term structure as a multidimensional stochastic process, see [SCHOEN99],[LEDOIT98] and [BRACE01] for a particular application to the LMM. The main problems with them is that since volatility is not a tradeable asset, its drift under a risk neutral measure is not determined by no arbitrage arguments and hence they are not simple to calibrate [BRIGO06].

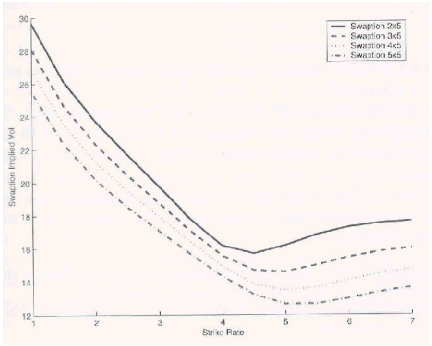


Figure 2.3: Market observed smile for 2y by 5 y, 3x5, 4x5 and 5x5 swaptions taken from [BRIGO06].

Finally, the recent flourishing of applications of more general Levy processes in the study of financial markets has not left this area untouched by them, see [EBERLE05] for a Levy driven LMM. The main drawbacks of these models is that they are very difficult to implement, especially to simulate for pricing exotics. This makes of models with stochastic volatility, which may include hybrids between the pure stochastic volatility approach and the local volatility one (as we will see soon) the best for practical use, especially given that the smile they produce is self similar and that they are fairly tractable analytically speaking and when running simulations. We will begin by reviewing in detail the most

common local volatility constructions and then go into the main stochastic volatility ones.

2.2.2 Local Volatility Models

2.2.2.1 Displaced Diffusion Framework

We will follow the authoritative approach of [BRIGO06]. The simplest possible way of getting a volatility coefficient which depends on the level of the process, in the sense discussed above, is to assume that the process is shifted by some constant factor from a process which follows a pure diffusion in the following sense. Let $F_j(t)$ be the forward rate observed at time t for investing between time T_{j-1} and T_j, assume we are under the T_j-measure. The shifted lognormal model assumes the evolution:

$$\begin{aligned} F_j(t) + a_j &= X_j(t) \\ d\,X_j(t) &= \sigma(t) X_j(t)\, d\,W_t \end{aligned} \tag{2.61}$$

with $a_j \in \mathbb{R}$, σ a non-random function and W a Brownian motion under the given measure. Since $d\,a_j \equiv 0$ we get the evolution of $F_j(t)$:

$$d\,F_j(t) = \sigma(t)(F_j(t) + a_j)\, d\,W_t \tag{2.62}$$

we can thus see a very simple structure for the volatility coefficient in closed form:

$$F_j(t) = -a_j + (F_j(0) + a_j)\exp\left\{ -\frac{1}{2}\int_0^t \sigma^2(u)\, du + \int_0^t \sigma(u)\, d\,W_u \right\}. \tag{2.63}$$

Now the price of a caplet is given by:

$$\mathrm{Cpl}(t, T_{j-1}, T_j, \tau, N, K) = \tau N\, B(t, T_j)\mathbb{E}^{T_j}\Big[(F_j(T_{j-1}) - K)^+ |\, \mathfrak{F}(t) \Big] \tag{2.64}$$

$$= \tau N\, B(t, T_j)\mathbb{E}^{T_j}\Big[(X_j(T_{j-1}) - (K + a_j))^+ |\, \mathfrak{F}(t) \Big]$$

Now note that if we take $K + a_j \geq 0$ then the solution is exactly the same as in Black's lognormal world with time dependent volatility function except that the strike is shifted. To see what type of implied volatility structure we get, we simply set the model's price equal to the Black's price with the volatility taken as a function of the strike and a_j, the wanted implied volatility structure $K \to \hat{\sigma}(K, a_j)$ is given by:

$$\mathrm{Bl}(K, F_j(t), \hat{\sigma}(K, a_j)\sqrt{T_{j-1} - t}) = \mathrm{Bl}\left(K + a_j, F_j(t) + a_j, \sqrt{\int_t^{T_{j-1}} \sigma^2(u)\, d\, u}\right) \tag{2.65}$$

By solving the above, keeping K fixed and differentiating with respect to a_j we see that $\hat{\sigma}(K, a_j)$ increases as a_j decreases, that is the model implied skew shifts up as a_j decreases and down as a_j increases. Although the approach is very simple and does not need any new techniques over the ones already used in the LMM, it is not flexible enough. If one calibrates for a set of strikes and finds a set of parameters a_j then calibration to different strikes needs a completely different set of $a_j's$. On the other hand, calibrating to the whole set of available strikes, leads to smiles which do not resemble actual markets [BRIGO06]. No stable calibration is thus possible [MEIST04].

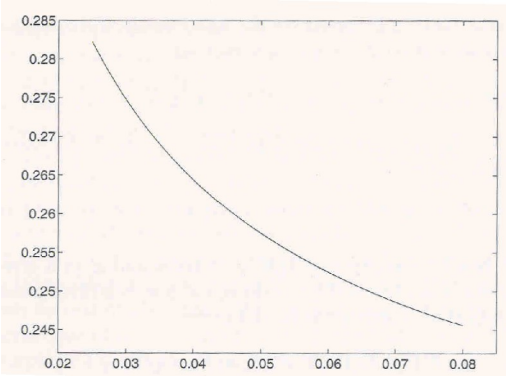


Figure 2.4: Smile generated by drifted lognormal model, taken from [BRIGO06].

2.2.2.2 Constant Elasticity of Variance

Although they are not as simple as the shifted model above, models of this type have been around for a very long time, an instance of them is the CIR short rate model discussed in section 2.1.1. The application to the forward rate as a local volatility LMM is first discussed by Andersen and Andreasen in [ANDER00]. The assumed dynamics are given by:

$$d\,F_j(t) = \sigma_j(t)[F_j(t)]^{\gamma_j} d\,W_t \tag{2.66}$$

with $\gamma_j \in [0,1]$ and $\sigma_j(t)$ a non stochastic function. In the case of $\gamma_j \in \left[0, \frac{1}{2}\right]$ we need to impose a constraint to limit non real rates, without getting into the technicalities we impose that the process has a stopping time at $\tau = \min\{t \in \mathbb{R} \therefore F_j(t) = 0\}$ which in this case is said to be an *absorbing boundary,* so that the process does not take complex values. In order to solve the model we redefine the inner time of the Brownian motion. Recall definition 8.16 of Brownian motion and define the function $v(t,T) = \int_t^T \sigma_j(u)^2\, d\,u$, then imagine making the transformation $t \to v(0,t)$ in the Brownian motion of (2.66); the resulting process is

$$\widetilde{W}(v(0,t)) \triangleq \int_0^t \sigma_j(u)\, d\,W_u \tag{2.67}$$

and we easily see that if σ_j is continuous we have that $d\widetilde{W}_v d\widetilde{W}_v = d\,v$ so that by Levy's theorem 8.20 $\widetilde{W}(v(0,t))$ is a Brownian motion. Define $f_j(v(t)) \triangleq F_j(t)$, then (2.66) becomes:

$$d\,f_j(v) = f_j(v)^{\gamma} d\,\widetilde{W}(v) \tag{2.68}$$

Now let $z_j(v) = \left(\frac{1}{\gamma-1}\right) f_j^{1-\gamma}(v)$ then applying Ito's lemma we obtain:

$$d\, z_j(v) = -\frac{\gamma}{2\, z_j(v)} d\, v + d\, \widetilde{W}(v) \tag{2.69}$$

which is what is called a *Bessel process* of order $\gamma - 1$. The distribution of such a process is well known we will not go into it, for more information see [BOROD96]. The caplet prices are calculated once the distribution is written down, and are found to be [BRIGO06]:

$$\mathrm{Cpl}(t, T_{j-1}, T_j, \tau, N, K) = \tag{2.70}$$

$$= \tau N\, B(t, T_j) \left[F_j(t) \sum_{n=0}^{\infty} g(n+1, u) G\big(c_n, k K^{2(1-\gamma)}\big) - K \sum_{n=0}^{\infty} g(c_n, u) G\big(n+1, k\, K^{2(1-\gamma)}\big) \right]$$

with

$$k = \frac{1}{2v(t,T)(1-\gamma)^2},\ \ u = k[F_j(t)]^{2(1-\gamma)},\ c_n = n + 1 + \frac{1}{2(1-\gamma)} \tag{2.71}$$

$$g(x, y) = \frac{e^{-y} y^{x-1}}{\Gamma(x)} \text{ and } G(x, y) = \int_y^{\infty} g(x, u)\, d\, u.$$

Although slightly complicated, the price is still computable analytically without the need for lengthy simulations. Setting (2.70) equal to the Black's price while setting the volatility as a strike dependent function in the lognormal solution we get the model implied volatility structure. Although more intricate in its formulation the model has exactly the same issues with the shifted lognormal one. We have simply added only one degree of freedom, γ, and the problem of unstable calibration is still present [MEIST04]. As a matter of fact it can be shown, see [MEIST04], that the shifted lognormal and the CEV models are nearly equivalent.

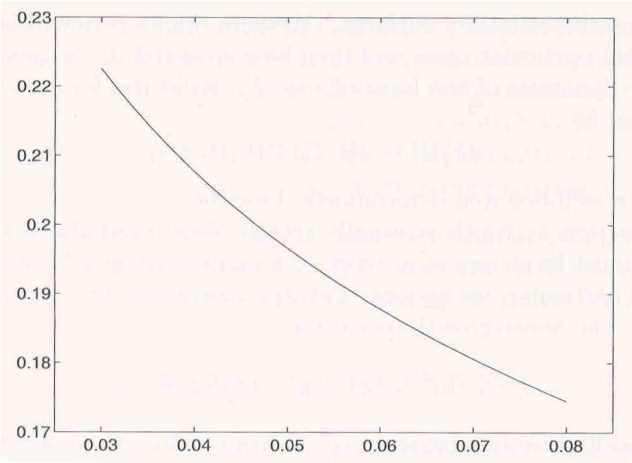


Figure 2.5: Smile generated by CEV model, taken from [BRIGO06].

2.2.2.3 Remarks on LVMs

Although the two approaches presented are not able to fit elaborate market observed smiles different, more complex, constructions within this class have been developed which are able to do that. For instance, Brigo and Mercurio develop a whole class of analytically tractable models which give more flexibility and can be calibrated to market smiles in [BRIGO01], [BRIGO02].

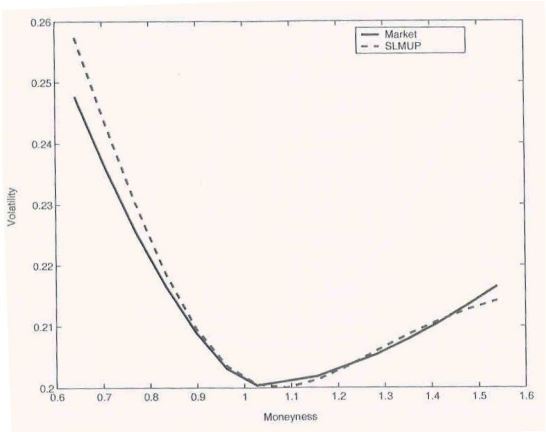


Figure 2.6: Brigo-Mercurio's UVPM fit to market observed smile, taken from [BRIGO06]

The main problem with LVMs is not whether they are able to perfectly fit the observed smile, in fact Dupire in [DUPIRE94] shows that given a volatility smile in the market one can find a local volatility function which gives a perfect fit (and he actually finds the form for the volatility structure). Rather the flaw in such models lies in the way the dynamics of the smile is described. As shown by Hagan et al in [HAGAN02], when presenting their SABR stochastic volatility model (described below), local volatility models force the implied volatility curve $\sigma(K, F)$ to decrease when the underlying F increases and to increase when this decreases. The so called "sticky-moneyness" rule, widely used by market participants where the implied volatility is kept fixed when the quantity $\log(F/K)$ remains unchanged, cannot be derived by local volatility models. As the authors of [HAGAN02] show this causes local volatility models to have hedging parameters (the so called greeks) which perform even worse than the naive Black model even though they can fit the entire volatility curve. Since then, most market participants have been using the stochastic volatility approach which does not have this problem.

2.2.3 Stochastic Volatility LIBOR Market Models

As mentioned above these begin from the observation that the volatility of any financial market is neither constant nor deterministic but rather random and hence describe it as a stochastic process. The general framework is set by specifying:

$$\begin{aligned} d\,F_j(t) &= a_j(t)\varphi(F_j(t))[v(t)]^{\gamma} d\,Z_j(t) \\ d\,v(t) &= b(t,v)d\,t + b(t,v)d\,W(t) \\ d\,W(t)d\,v(t) &= \rho(t)d\,t \end{aligned} \tag{2.72}$$

The type of smile which is recovered depends, mainly, on the specification of $\rho(t)$, $\varphi(F_j(t))$ and γ. Renault and Touzi in [RENAU96] show that if one sets $\rho = 0$ and $\varphi(F_j(t)) = F_j(t)$ with some other rudimentary conditions on the other parameters a smile is obtained with

a zero slope (local minimum) for at the money strikes. Since one often needs to relax this and get non-zero slopes for ATM three mainstreams are adopted. The first is to simply set $\rho \neq 0$, this generally renders pricing analytically challenging, although some tractable models have been developed as we will soon see one particular case. Leaving $\rho = 0$ is attractive because, as shown by Hull and White in [HULL87], the prices of European payoffs can be calculated by first assuming the volatility to be constant and then integrating with a measure given by the volatility's density. Two other main models are thus developed by making the $\rho = 0$ assumption and using approaches known, through LVMs, to produce a good fit to the smile, that is by taking $\varphi(F) = F^m$ or $\varphi(F) = F + a$. Still, although these models are much better at producing good hedging parameters and are consistent with market observations, their analytical tractability is much worse than with LVMs. Each model requires a sophisticated mathematical analysis or approximation method and we will thus not describe the analyses in detail for every single model but will rather limit ourselves to present the results and give an outline of the assumed dynamics.

2.2.3.1 The Andersen, Brotherton-Ratcliffe Model

We present the result of this famous framework solely for the purpose of illustrating the analytical difficulties that one falls into when not carefully choosing a simple model for stochastic volatility. While it first appeared in 2001 it is published in 2005 with [ANDER05] in which the authors start by deriving the following system of Stochastic Differential Equations under the respective forward measure:

$$\begin{aligned} d\,F_j(t) &= \sigma_j(t)\sqrt{V(t)}\varphi(F_j(t))d\,W_j(t) \\ d\,V(t) &= \kappa(\theta - V(t))d\,t + \varepsilon\psi(V(t))\,d\,Z(t) \\ d\,W_j d\,Z_j &= 0 \end{aligned} \tag{2.73}$$

They do an analysis for arbitrary functions φ and ψ, and constants κ, θ and ε, with

the latter in particular taken to be small. After a long asymptotic analysis, they find approximations to the prices of caplets as:

$$\mathrm{Cpl}(t, T_{j-1}, T_j, \tau, K, N) = B(t,T) N \tau g_j(t, F_j(t), x_j(t, V(t))) \tag{2.74}$$

with

$$g_j(t, F, x_j) = F_j(t) N\big(d_+^j\big) - K\, N\big(d_-^j\big)\,, \tag{2.75}$$

$$d_\pm^j = \frac{\ln(F_j/K) \pm \frac{1}{2}\Omega^j(t, F_j, x_j)^2}{\Omega^j(t, F_j, x)}, \tag{2.76}$$

$$\Omega^j(t, F_j, x_j) \;=\; \sum_{i\geq 1} x_j^{i+1/2} \delta_j^{i+1/2} \Omega_i(F_j), \tag{2.77}$$

and

$$\begin{aligned} x_j(t,V) &= \bar{x}_j(t,V) + \varepsilon^2(\alpha_0 + \alpha_1 Y^2) + \varepsilon^4\Big(\beta_0 + \beta_1 Y^2 + \beta_2 Y^4 e^{-\Lambda\varepsilon^2 Y^2}\Big) + O\big(\varepsilon^6\big) \\ \bar{x}_j(t,V) &= \delta_j^{-1} \int_t^{T_{j-1}} \sigma_j(s)^2 \big[\theta + (V(t) - \theta) e^{-\kappa(s-t)}\big]\, d\, s, \\ \Omega_0(F_j) &= \frac{\ln(F/K)}{\int_K^F \varphi(s)^{-1} d\, s}, \end{aligned} \tag{2.78}$$

$N(\cdot)$ being the cumulative normal distribution function, $\delta_j = T_{j-1} - t$, some constant Λ and α_i, β_i functions of ψ, Ω_i and δ_j. Swaptions are easily seen to require even more complication. We mention that the result above is done by carrying out a perturbative expansion which assumes the parameter ε, the volatility-of-volatility, to be small. Sometimes that is absolutely not true in the market and the approximation often becomes highly inaccurate! The authors themselves in the recent version of their work published in [ANDER05] do an analysis for specific forms of the function $\varphi(x) = x$ or $\varphi(x) = \mathrm{a}x + b$

and $\psi(x) = \sqrt{x}$. Another drawback of this modeling approach, as Piterbarg discusses in [PITERB03], is that it is not able to fit the entire swaptions matrix volatility surface for swaptions of different maturity, underlying swap's expiry and strike.

2.2.3.2 The Wu-Zhang Model

In contrast, in their paper [WU06], Wu and Zhang postulate dynamics with correlation between the volatility and forward rates. In the forward measure their model is set by the following Stochastic Differential Equations:

$$\begin{aligned} d\,F_j(t) &= \sqrt{V(t)}F_j(t)\underset{\rightharpoonup_j}{\sigma}(t)\cdot d\,\underset{\rightharpoonup_j}{Z}(t), \\ d\,V(t) &= [\kappa\theta - (\kappa + \varepsilon\xi_j(t))V(t)]d\,t + \varepsilon\sqrt{V(t)}d\,W(t), \\ \xi_j(t) &= \sum_{m=1}^{j}\frac{\tau F_m(t)}{1+\tau F_m(t)}\left|\underset{\rightharpoonup_m}{\sigma}(t)\right|\rho_m(t) \end{aligned} \tag{2.79}$$

where

$$\frac{\left(\underset{\rightharpoonup_j}{\sigma}(t)\cdot d\,\underset{\rightharpoonup}{Z}(t)\right)d\,W(t)}{\left|\underset{\rightharpoonup_j}{\sigma}(t)\right|} = \rho_j(t)\,d\,t \tag{2.80}$$

κ, θ, ε are constants, $\underset{\rightharpoonup}{Z}(t)$ is a vector of independent Brownian motions and $\underset{\rightharpoonup_j}{\sigma}(t)$ a real valued vector function. Besides a small approximation for $\xi_j(t)$, caplet prices are obtained explicitly through the use of moment generating functions for the processes in (2.79). A similar analysis is done in the same paper for the pricing of swaptions. While the prices derived are more efficient than the above model the problem of the fit of the entire swaptions skew is still not solved.

2.2.3.3 The Piterbarg Model

In his influential paper [PITERB03], Piterbarg argues that the models above are not able to calibrate to the entire swaptions volatility-price matrix and that, furthermore, they are not able to produce a volatility smile whose slope for at the money strikes varies

with the underlying swaptions expiry and maturity. He thus starts with a modification of the Andersen and Brotherton-Ratcliffe model described above and chooses for the coefficient function φ a dynamic similar to that obtained by a displaced diffusion LVM. For the volatility-of-volatility instead he chooses $\psi(x) = \sqrt{x}$ so that the forward measure dynamics are given by:

$$\begin{aligned} d\,F_j(t) &= (a\,F_j(t) + (1-a)F_j(0))\sqrt{V(t)}\sigma_j(t)d\,W(t) \\ d\,V(t) &= \kappa(\theta - V(t))d\,t + \varepsilon\sqrt{V(t)}d\,Z(t) \\ d\,Z(t)d\,W(t) &= 0 \end{aligned} \tag{2.81}$$

The author derives approximations for the prices of swaptions without making use of the small volatility-of volatility assumption that all the others do, furthermore he develops an efficient calibration framework. The approach is becoming dominant in practice, since it is the model that is able to fit the market best but it has not yet been able to replace the SABR model's use in the market, mainly due to its simplicity. Another flaw will be described in the next chapter as a motivation for the changes that we introduce to Piterbarg's model.

2.2.3.4 The SABR Model

Ever since its first publication in [HAGAN02] where the authors not only produce the new framework but also study the variety of negative issues found in local volatility models, the SABR model has gained increasing popularity and it is now the most popular stochastic volatility model for forward rates. It is the standard for broker's quotes for exotic rates products. Before we describe it though it is important to note that authors' study is concerned with only one forward asset and not really a whole stochastic volatility LMM; they are only concerned with a single maturity forward rate. The generalization to the development of a whole LMM requires that one defines the correlation structure of the forward rates between themselves and also of the volatility of various rates between

themselves (to make this clear we index the Brownian motion driving the volatility of each rate by j). The dynamics it specifies are given under the forward measure as follows:

$$\begin{aligned} d\,F_j(t) &= V(t)F_j(t)^{\beta} d\,Z_j(t) \\ d\,V(t) &= \varepsilon V(t)\,d\,W_j(t) \\ d\,Z_j(t)\,d\,W(t) &= \rho_j(t)\,d\,t \end{aligned} \tag{2.82}$$

where $\beta \in (0, 1]$, $\varepsilon \geq 0$ and $\rho \in [-1, 1]$. Although one may be tempted to leave the study of parameters until calibration the result by Jourdain in [JOURDA04] shows that the set of SDEs above form a multidimensional martingale when $\beta < 1$ while the case $\beta = 1$ (lognormality) requires the additional condition that $\rho \leq 0$. Caplet prices are obtained by perturbative analysis of the associated pricing PDE and this is where the main flaw of the model lies. The perturbative analysis is based, as for all the models discussed so far, with the sole exception of Piterbarg's, on the assumption that the volatility of volatility parameter, ε, is small. This is not always the case and certainly not in recent market conditions. Solving this problem while maintaining the benefits of the SABR and Piterbarg's model will be the main contribution of this thesis from a theoretical point of view. We will conduct a further discussion of this model in the calibration chapter in which we compare the fit given by the solution of the SABR model of Hagan et al in [HAGAN02] with ours.

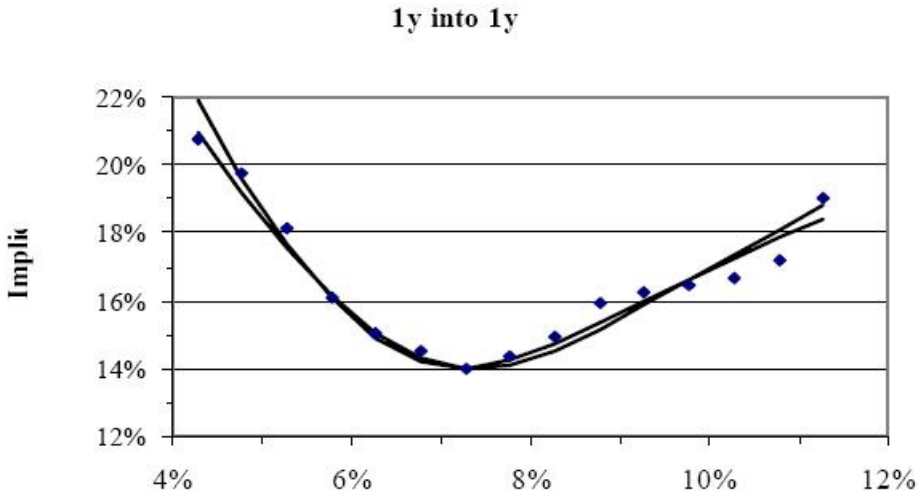


Figure 2.7: SABR model's fit to market observed swaptions smile, taken from [HAGAN02].

Before ending this literature review, for completeness reasons, we mention a model which has gained some support and focuses itself more on the observed historical analysis of forward rates.

2.2.3.5 The Joshi and Rebonato Model

All the models presented so far are mainly concerned with producing a good fit of the observed smile. Although they also take into account a better evolution of the implied volatilities, as opposed to LVMs, they are not constructed with a focus on correctly describing the historically observed evolution of prices (and volatilities). To account for this issue Joshi and Rebonato in [JOSH03a] present an analytically intractable model requiring Monte Carlo simulations for the pricing of payoffs but highly consistent with the swaptions volatility matrix' evolution. They do so by conducting a form of independent component analysis of the rate's covariance matrix and discerning four factors that obey

the following dynamics under the forward measure:

$$\begin{aligned} d\,F_j(t) &= \sigma_j(t)[F_j(t) + \alpha_j]\,d\,Z_j(t) \\ \sigma_j(t) &= [a_1(t) + a_2(t)(T_{j-1} - t)]e^{-a_3(t)(T_{j-1}-t)} + a_4(t) \end{aligned} \tag{2.83}$$

where α_j is the displacement coefficient, and $a_i(t)$ adapted processes independent of F_j and following:

$$d\,a_i(t) = \kappa_i(\theta_i - a_i(t))d\,t + \sigma_i d\,W_i(t), \text{ with } \kappa_i, \theta_i, \sigma_i \in \mathbb{R}. \tag{2.84}$$

2.3 Summary and overture to the next chapter

To summarize, we began the chapter with an introduction to the simplest class of models used to price interest rate products. These were the first in use historically and are set off by conducting a macro-economic analysis to derive a stochastic differential equation satisfied by an instantaneous interest rate. The simplicity of such equilibrium models is discussed and their flaw in not being able to fit the present yield curve is mentioned. This made us move to a short description of no-arbitrage models which are able to fit the observed present state of the yield curve and differ from equilibrium models simply by the fact that the coefficients of the stochastic differential equation used to define the "short rate" are allowed to be time varying. The bond prices derived by such models are not able to describe the evolution of the entire yield curve and a historical solution has been to generalize the short rate models to include several random factors to drive the interest rate. This has thus led us to the multi-factor models, which have the flaw that they are described by completely abstract variables which are still not able to easily fit the market. The market models class was thus introduced with a special focus on the LIBOR Market Model, or LMM for short, which can very easily be calibrated to the market and describe the evolution of the whole term structure of interest rates.

A few years after its introduction the LMM had to be modified in order to account for the changes in the interest rates market which happened with the fall of the Long Term Capital Management fixed income relative value hedge fund. The first approaches to solve this problem are those of the so called Local Volatility Models which are discussed with their failure to describe the dynamics of the implied volatility smile. This justifies the introduction of the stochastic volatility models mentioned towards the end of the chapter. The more popular of these are outlined and each model's assumed dynamics for the forward rate and its volatility is described. Finally, an appendix on stochastic calculus and arbitrage pricing theory is provided to aid the reader with the notions and techniques used throughout. The discussion of the chapter concludes that two models are the most fit when it comes to pricing issues, the SABR and the Piterbarg models. The first has the flaw that the analysis conducted by its authors does not provide an exact closed form solution for the prices, but rather gives an approximation which makes the assumption that the volatility-of-volatility is small. The second model is more robust but, as we will show in the beginning of the next chapter, the assumption it makes of a mean reverting volatility process causes it to underestimate the smile for long maturity options. The goal of this thesis, set and carried out in the next chapter will be to provide a model which is able to maintain the advantages of the Piterbarg model, to adjust the mean reversion flaw and to carry out a closed form solution to the pricing problem which removes the assumption made by the SABR model. This is all carried out in the next chapter.

3 An alternative Stochastic Volatility LMM

3.1 Introduction

Although the Rebonato-Joshi model outlined in section 2.2.3.5 is dynamically more consistent with the swaptions market than all the other ones presented, it remains, computationally speaking, quite demanding given its analytical intractability. The less sophisticated models which have taken the most prominent role in the market are the Piterbarg and the SABR model. The Piterbarg model is widely used in the market because it is able to fit the entire swaptions at the money volatility matrix of tenors and forwards. In his work in [PITERB03] Piterbarg not only proves that his model does indeed provide such fit but also gives a detailed calibration methodology based on effective volatilities in pricing equations. His model as all the other models produced in the literature with the sole exception of the SABR one, rests on the assumption that the volatility follows a mean reverting process. While this is certainly in accordance with the historical properties of the quantity, they result in a strong underestimation of the smile for long maturity options. A solution to this problem can be provided by using local volatility models. As we have discussed in the previous chapter though, these generate greeks that cause severe mishedges due to the fact that they produce a smile which does not obey the so called "sticky moneyness" rule, or the market observed property that the implied volatility remains the same when expressed as a function of a constant moneyness. This property was indeed the main reason why the authors of SABR proposed their model as an alternative to local volatility models in [HAGAN02]. Their model is able to both give a correct fit to the market observed smile, like local volatility models, but, at the same time, it is able to correctly describe its moneyness dynamic so as to produce greeks that give better hedges. Furthermore, the model rests on a very simple approximation derived by its authors which does not require any significant computational power even when compared to the simple Black Scholes framework. All of these properties have made SABR the most commonly

used model in interest rates markets and it is now used as the standard to quote smile sensitive interest rate products, see Brigo [BRIGO06]. On the other hand, the simplicity of the SABR pricing formula overshadows the fact that its authors derive it by making the unfounded assumption that the volatility of volatility is small and that the maturity of the option is not very large. This is definitely not always the case and the pricing formula hence does not translate into the arbitrage free fair value of options in circumstances with large swings in volatility. The model hence lacks a closed form solution which can hold in situations in which the volatility suffers very large and unexpected movements, as has been repeatedly the case in the last decade. A framework which is able to reproduce the positive characteristics of SABR while possessing a closed form solution free of assumptions such that the price derived from it always corresponds to the arbitrage free value is thus needed. This will be the main goal of our thesis and our main contribution will be the proposal of such model along with the derivation of such type of closed form solution for prices of European payoffs where the underlying rate is not correlated to its volatility.

We begin with a detailed and in depth motivation and then move to the model's layout. The prices of European payoffs for the caps and swaptions markets are derived analytically within a unifying methodology which begins by presenting an argument for approximating the swap rate when one starts with a forward LIBOR model. This will be carried out in the next chapter in which we will use a partial differential equations based approach rather than the expectations one typically used in interest rates markets and derive completely exact solutions to our model, which, at least as far as caplets are concerned, will be completely free from ad-hoc assumptions. It is important to point out though that we model the forward rate as a process which is not correlated to its own volatility. This is certainly not fully realistic for long maturity options and produces market fits that are not fully satisfactory. Nevertheless, the fit is better than the SABR model for such case and the point we make is to start the research for the derivation of a closed form solution at least in the uncorrelated case which has not been adopted in

the many years in which the SABR model has now been in use. We leave it for future research to extend our closed form solution to the correlated case.

The model's implied volatility structure will also be derived analytically and plotted with varying parameter values. We also show that the dynamics of the smile is consistent with the market in the sense of the "sticky moneyness" rule. The polynomial derived for the implied volatility will make the foundations for the study of such duynamic and this will also be used in the calibration chapter for fast estimation of parameters. We can now move to the justifications for the choice of processes of our model.

3.2 Motivations and Model Layout

Except for SABR, the most celebrated of the models presented above, i.e. the Piterbarg and Andersen-Brotherton-Ratcliffe models, have a problem in that the absence of correlation between the volatility process and the forward rate in addition to the mean reverting volatility process all work to produce a largely unwanted phenomenon. Indeed, such two characteristics force the implied volatility of any option of long maturity, whether caplet or swaption, to be constant irrespective of strike. Gatarek presents this in [GATAREK03], but, unfortunately, solves the issue by using one of the so called uncertain volatility-parameters models briefly discussed above. The "sticky-moneyness" rule which is one of the pillars of volatility market behaviour is not consistent with such models, besides the fact that they predict a future volatility which is not similar to the present one, also in contrast with the market. Theoretically the main goal is to derive prices without making the, sometimes troublesome, assumption that the volatility-of-volatility parameter is small which is the main bottleneck of the SABR model's closed form solution. Indeed, regardless of the fact that the stochastic volatility approach is, analytically speaking, far more demanding than most uncertain volatility parameters models, we are able to derive closed form solutions for prices without exogenous assumptions for the case of no correlation between the underlying and the volatility. As we will see very clearly in the next

chapter, this yields smiles that do not match the ones present in the market, although when compared to the corresponding uncorrelated SABR ones we obtain a better fit. This motivates possible future research in the study of the correlated case.

Now although the presence of correlation generates very good fits to the market quotes of short and mid maturities all the models discussed above which have a mean reverting process for the volatility and no correlation underestimate the smile for long maturity options. This motivates our choice to stay particularly close to the SABR model, which not only is the dominant one in use in the market but is also free from such flaw. To see the issue at hand we begin with the Hull-White result for pricing in a stochastic volatility model where the volatility process is not correlated with the asset process [HULL87]. We thus divert our attention from the rates world and consider a general two dimensional stochastic volatility model for some asset of the form:

$$\begin{aligned} d\,S(t) &= \phi S(t)\,d\,t + \sigma S(t)\,d\,W(t) \\ d\,V(t) &= \mu(V,S,t)V(t)\,d\,t + \xi(V,S,t)\,V(t)\,d\,Z(t) \\ d\,Z(t) d\,W(t) &= 0 \end{aligned} \tag{3.1}$$

where $V(t) = \sigma^2$ and where ϕ, σ, μ and ξ are functions possibly of any of the stochastic processes involved. Since the model is driven by a Brownian motion we can go to the risk neutral measure (although if we do not assume that $V(t)$ is a tradeable asset the model turns out to be incomplete) and obtain the price of a contingent claim with payoff $f(S(T), \sigma^2(T), T)$ as:

$$f(S(t),\ \sigma^2(t),\ t) = B(t,T) \int f(S(T), \sigma^2(T), T)\, p(S(T) \mid S(t), \sigma^2(t))\, d\,S(T) \tag{3.2}$$

with T the time to maturity, $B(t,T)$ the price of T-maturity bonds, and $p(S(T) \mid S(t), \sigma^2(t))$ the conditional probability of $S(T)$ on knowing $S(t)$ and $\sigma(t)$. Hull and White in [HULL87]

begin by taking any three random variables x, y and z and noting that their conditional probabilities are related by:

$$p(x \mid y) = \int g(x \mid z)h(z \mid y)\,\mathrm{dz} \tag{3.3}$$

Defining the mean variance as:

$$\widetilde{V} = \frac{1}{T-t}\int_t^T \sigma^2(u)\,d\,u \tag{3.4}$$

we can rewrite the conditional probability of the price as:

$$p(S(T) \mid S(t), \sigma^2(t)) = \int g(S(T) \mid S(t), \widetilde{V})h(\widetilde{V} \mid S(t), \sigma^2(t))\,d\,\widetilde{V} \tag{3.5}$$

using this into equation (3.2) we obtain:

$$f(S(t), \sigma^2(t), t) = \int \left[B(t,T)\int f(S(T))g(S(T) \mid S(t), \widetilde{V})\,d\,S(T)\right] h(\widetilde{V} \mid S(t), \sigma(t))\,d\widetilde{V} \tag{3.6}$$

The technical contribution of the two authors in [HULL87] is to note that in the case in which the Brownian motions driving the variance process and the underlying are uncorrelated and the coefficient functions μ and ξ are independent of the underlying then the term in the square brakets is the Black-Scholes price for the payoff given by $f(S(T), \sigma(T), T)$. Thus for any stochastic volatility model where the instantaneous correlation of the asset with its volatility is zero we can write the price of European payoffs as:

$$f(S(t), \sigma^2(t), t) = B(t,T)\int \mathrm{BS}(\,f)\,p(\widetilde{V})\,d\,\widetilde{V} \tag{3.7}$$

where p is the probability density of $\widetilde{V}$. We also see that rather than σ it is its average over the time of maturity that matters when pricing in a stochastic volatility model such

as the one described by equation (3.1).

Now take a long maturity European option, for instance say over 15 years (as is common for many interest rates markets), in such cases we can make the approximation $T \to \infty$, and here is precisely where the problem arises. Gatarek in [GATAREK03] cites a result from stochastic analysis in [BOROV98] which shows that for most stochastic processes $\sigma(t)$ the limit of the average variance $\widetilde{V}$, as defined above, is $\lim_{T\to\infty} \widetilde{V} = \chi(\infty)$ for some random variable $\chi(\infty)$. In the particular case of the class of mean reverting processes the random variable is degenerate and becomes deterministic, so that $\chi(\infty) = m$ and $p(\widetilde{V}) = \delta_m$ where δ_m is the Dirac delta function. Substituting in equation (3.7) we get back to the Black-Scholes price and thus obtain a non strike dependent implied volatility structure for long term options of any type, including swaptions and caplets. The aforementioned author dismisses the stochastic volatility class altogether to develop a simple uncertain volatility parameter model. Since such models have the problem that they are not able to describe the correct evolution of the smile with the changing of the underlying, just as with LVMs, we will not follow Gatarek's solution.

We base our framework on the use of a driftless geometric Brownian motion in place of the mean reverting variance process and account for the lost parameter by displacing the forward rate. We also leave some extra flexibility in the term multiplying $F_j(0)$ in the Piterbarg stochastic differential equation. Furthermore, introducing correlation between the two processes is definitely a realistic need and doing so yields a smile which fits extremely well that observed in the market, as we will see in the calibration chapter. The analytical challenges of incorporating it though are very large and although they may be surmountable a realistic academic study is needed on the simpler, uncorrelated, case first. Indeed, our main goal here will not be to propose the "perfect" model, but rather to have a framework which is able to generalize the SABR model while maintaining its benefits and possessing a closed form solution. We will hence leave the two processes uncorrelated and leave the more challenging case for future reasearch. Given that the SABR model is set

with constant coefficients we will only consider such case. We thus propose the following model for the forward rate $F_j(t)$ and its volatility under the T_{j+1}-forward measure:

$$\begin{aligned} d\,F_j(t) &= \Big[\alpha_j F_j(t) + \beta_j F(0)\Big]\sqrt{z_j(t)}\,d\,U_j(t) \\ d\,z_j(t) &= (\varepsilon_j\, z_j(t) + \lambda_j)\,d\,W(t) \\ d\,W(t)\,d\,U(t) &= \rho\,d\,t \end{aligned} \tag{3.8}$$

for Brownian motions $W(t)$ and $U(t)$, and α_j, β_j, λ_j, ρ_j, $\varepsilon_j \in \mathbb{R}$ and for the purpose of the remainder of our thesis we will take $\rho = 0$, and also we will not need to specify the joint correlation of the various forward rates. A discussion of the fit to market when $\rho \neq 0$ will be provided in the calibration chapter. We mention here that the correlation coefficient significantly enhances the fit but significantly complicates the pricing problem. In this thesis we will focus on solving the non correlated case which already provides a much better fit than its equivalent non correlated SABR model. While the correlated case must also be studied to produce a closed form solution we leave this ambitious goal to future reasearch. Indeed, it is good research practice to first tackle the simpler case and this is what we provide in this thesis.

3.3 Model's Properties

Note that in our model in the equation above we blend local volatility models with stochastic volatility ones through the shifting parameters β and λ which may come in useful by taking into account the lost degrees of freedom from the drift of the mean reverting volatility process in Piterbarg's approach. Also, notice that this model is effectively a generalization of the SABR model's uncorrelated case, especially given, as discussed in [MEIST04], that constant elasticity of variance and shifted lognormal models are equivalent. Hence our model's shifting parameter accounts for the "beta term" for the exponentiation of the forward rate in the SABR model's defining stochastic differential

equation (2.82). This is very interesting as the analysis we will conduct below will derive an exact closed form solution for the prices in our model which hence provide a solution to the SABR model, which has been in use in the market for years with the usage of an unstable approximation. The analysis based on a closed form solution is the most important contribution of this thesis.

Since the volatility does not follow a mean reverting process its limit does not degenerate to a constant number and it is thus free from the flaw presented by Gatarek's argument. To see this we could either calculate prices for some payoff and show the existence of the smile for long maturities, directly calculate the probability density of the mean variance or take the variance of this latter and show that it is not zero. We will follow the latter methodology given its simplicity. Taking the closed form for $z(t)$, the mean variance is:

$$\widetilde{V}(T) = \frac{1}{T-t}\int \left(z(0) - \frac{\lambda}{\varepsilon}\right) e^{(-(\varepsilon^2/2\ )\ u + \varepsilon W(u))} + \frac{\lambda}{\varepsilon} d\, u \tag{3.9}$$

To simplify the notation we define $\widetilde{r}(T) = \widetilde{V}(T) - \frac{\lambda}{\varepsilon}$ so that

$$\widetilde{r}(T) = \frac{1}{T-t}\int \left(z(0) - \frac{\lambda}{\varepsilon}\right) e^{(-(\varepsilon^2/2\ )\ u + \varepsilon W(u))} \tag{3.10}$$

and $\mathrm{Var}(\widetilde{r}(T)) = \mathrm{Var}(\widetilde{V}(T))$. Taking expectations inside the integral the first moment is:

$$\mathbb{E}\Big[\widetilde{r}(T)\Big] = \frac{1}{T-t}\int \left(z(0) - \frac{\lambda}{\varepsilon}\right) e^{-(\varepsilon^2/2)\ u} \mathbb{E} e^{\varepsilon W(u)}\, d\, u \tag{3.11}$$

the expectation term is the moment generating function of an $N(0, u)$ normal random variable:

$$\mathbb{E}\big[e^{\varepsilon W(u)}\big] = e^{\varepsilon^2 u/2} \tag{3.12}$$

So trivially we get:

$$\mathbb{E}\Big[\widetilde{r}(T)\Big] = \left(z(0) - \frac{\lambda}{\varepsilon}\right). \tag{3.13}$$

On the other hand:

$$\widetilde{r}(T)^2 = \frac{1}{(T-t)^2} \iint d\,u\,d\,s\,\left[\left(z(0) - \frac{\lambda}{\varepsilon}\right) e^{-(\varepsilon^2/2)\,u} e^{\varepsilon W(u)}\right] \times \tag{3.14}$$

$$\times \left[\left(z(0) - \frac{\lambda}{\varepsilon}\right) e^{-(\varepsilon^2/2)s} e^{\varepsilon W(s)}\right]$$

integrating after taking expectations where appropriate and noting that if $u > s$:

$$\mathbb{E}\big[e^{x_1 W(u) + x_2 W(s)}\big] = \mathbb{E}\big[e^{x_1(W(u)-W(s)) + (x_1+x_2)W(s)}\big] = \tag{3.15}$$

$$= \mathbb{E}\big[e^{x_1(W(u)-W(s))}\big]\mathbb{E}\big[e^{(x_1+x_2)W(s)}\big] = e^{\frac{1}{2}x_1^2 u + \frac{1}{2}(x_2^2 + 2x_1x_2)s}$$

because time increments of Brownian motions are independent. On the other hand if $s > u$ we obtain:

$$\mathbb{E}\big[e^{x_1 W(u) + x_2 W(s)}\big] = e^{\frac{1}{2}x_1^2 s + \frac{1}{2}(x_2^2 + 2x_1x_2)u} \tag{3.16}$$

In other words for our case:

$$\mathbb{E}\big[e^{\varepsilon W(u) + \varepsilon W(s)}\big] = e^{\frac{1}{2}\varepsilon^2 \max(u,s) + \frac{3}{2}\varepsilon^2 \min(u,s)} \tag{3.17}$$

taking expectation and substituting the above yields:

$$\mathbb{E}\Big[\widetilde{r}(T)^2\Big] = \frac{\left(z(0) - \frac{\lambda}{\varepsilon}\right)^2}{(T-t)^2} \iint \Big[d\,u\,d\,s\,e^{\varepsilon^2 s + \frac{\varepsilon^2}{2}\max(0,s-u) + \frac{3\varepsilon^2}{2}\min(0,u-s)}\Big] = \tag{3.18}$$

$$= \frac{2\big(z(0) - \frac{\lambda}{\varepsilon}\big)^2}{\varepsilon^2(T-t)^2}\left[\frac{e^{\varepsilon^2(T-t)}}{\varepsilon^2} - \frac{1}{\varepsilon^2} - (T-t)\right] \tag{3.19}$$

From equations (3.11) and (3.18) we calculate the limit of the variance of $\widetilde{V}(T)$ as:

$$\lim_{T\to\infty} \mathrm{Var}\Big[\widetilde{V}(T)\Big] = \lim_{T\to\infty} \mathbb{E}\Big[\widetilde{V}(T)^2\Big] - \mathbb{E}\Big[\widetilde{V}(T)\Big]^2 =$$

$$= \lim_{T\to\infty} \frac{2\big(z(0) + \frac{\lambda}{\varepsilon}\big)^2}{\varepsilon^4(T-t)^2} e^{\varepsilon^2(T-t)} = \infty. \tag{3.20}$$

The variance of $\widetilde{V}(T)$ does not go to zero as the time to maturity increases and thus it does not degenerate into a non-random variable. From the Hull-White formula (3.7) we can see that the proposed model will give non-flat implied volatility smiles for long maturity options. Finally, we note that given that the model proposed is hybrid in nature between local volatility models (given the $\beta F(0)$ shifting parameter) and stochastic volatility ones, it could inherit the flaw of local volatility models in respect of incorrect smile dynamics. Hagan et al in [HAGAN02] though demonstrate that the beta parameter does not give a better fit to the market independently of whether it is zero or one, we can thus take it to be null and compare our results below with the formula for prices in the SABR model. The comparison will be the main subject of the calibration chapter. We will show towards the end of this chapter that when $\beta F(0) = 0$ the correct dynamic for the implied volatility is derived and this, along with what has just been stated motivates the decision in our calibration chapter of dropping this term and comparing our results directly with those of Hagan et al.

The model proposed has the advantage that, as opposed to uncertain volatility parameter models (UVPMs) and local volatility setups, it produces a smile which moves in accordance to the market. This will be proved in the final section of this chapter. Furthermore, as opposed to mean reverting models, it is able to maintain a smile for long maturity options, something which we have just proven theoretically and which we

will show in a more practical setting in the next chapter where calibration is dealt with. Finally, it has a closed form solution for plain vanilla options, which we will derive in the remainder of this chapter. This will be our most significant contribution, especially given that the SABR model is now the standard used in the market and that our model provides a generalization for it in the case of uncorrelated underlying and volatility. Indeed, the closed form we derive will tackle one of the major drawbacks of the analysis carried out by the originators of the SABR model, the fact that prices are derived assuming a very small volatility-of-volatility parameters, which is surely not always true. Before getting overly satisfied though, it is important to realize that this simple framework, as all others, has its flaws. First, the lack of correlation between the underlying and its volatility produce smiles that are not able to fit the ones observed in the market in a satisfactory manner. Furthermore, in nearly all markets the volatility is observed to cluster around periods, it may be small for an extensive period of time and then burst across some events. For example, the Federal Reserve Board's Monetary Policy Committee in its report submitted to Congress on February 11, 2003 [FED03] records the following dynamics for the implied volatility of short term interest rates:



Figure 3.1: 3-month Eurodollar rate implied volatility.

Such behaviour is not quite the same as the one which is obtained through the diffusion process which we use. A sample path of a shifted lognormal process as the one for the volatility in our model with $\lambda = 0.12$, $z(0) = 0.45$ and $\varepsilon = 0.07$ is the following:

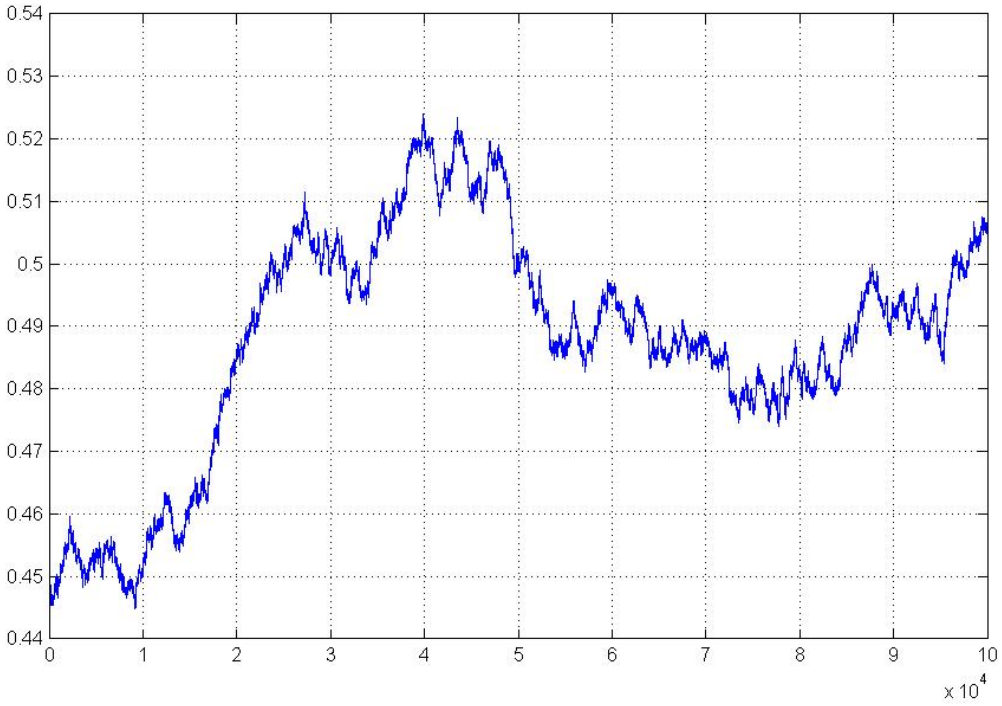


Figure 3.2: Shifted lognormal sample path.

If, instead, one simulates a mean reverting model the pattern is far more similar to figure 3.1. From a dynamical point of view the processes used by other models for the volatility are more consistent with the market in this respect (they are not when it comes to the strike structure of option prices). For instance, Fouque et al. [FOQUE00] are able to reproduce a pattern very similar to the one in figure 3.1 by taking a mean reverting process of the form $d\,y(t) = \alpha(m - y(t))\,d\,t + \beta\,d\,W(t)$ and defining the function $f(y) = 0.35\,(\arctan(y + \pi/2))/\pi + 0.05$. Changing the α parameter then yields the following sample paths for the function (taken from Fouque et al. [FOQUE00]):

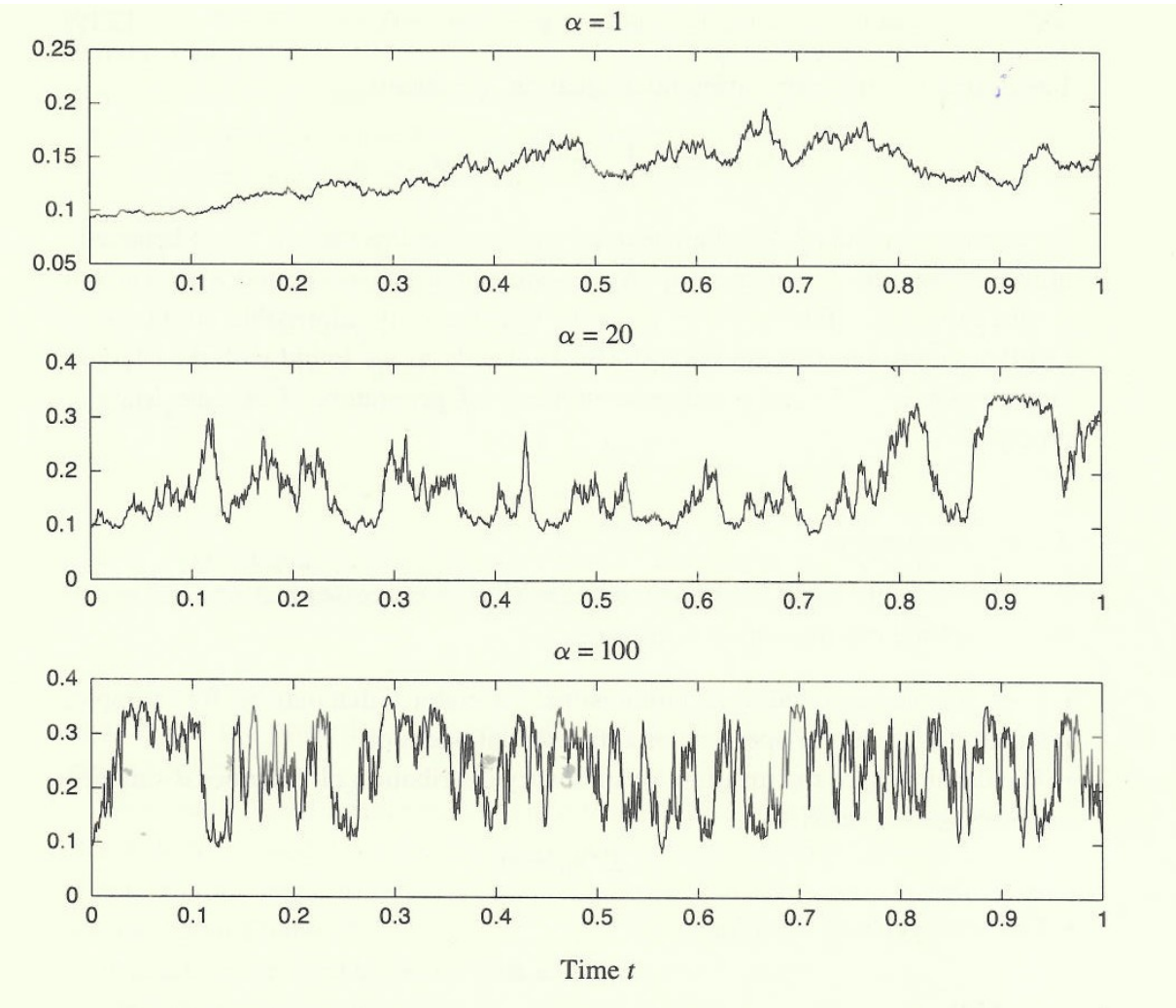


Figure 3.3: Mean reverting process.

We see that mean reverting models can give a much more similar pattern to that observed in practice. Nevertheless, given the positive aspects outlined above, the model proposed can provide an alternative, especially given that the volatility paths are surely more realistic than other popular models such as UVPMs. We thus lie, as far as dynamics are concerned, in between UVPMs and mean reverting models. The first have the flaw that they describe volatility movements in a completely different manner from the market. They also predict a future volatility smile which does not resemble the present one and

the rule of "sticky moneyness" for the implied volatility is not taken into account. The latter models have the correct dynamics for mid maturity options' volatilities but misprice the long maturities. In our case we present an analytically solvable model which is able to fit long maturites, has a more plausible dynamics than UVPMs and does not predict a future volatility which may be completely different from today's. Last but not least we generalize the SABR model and conduct an analysis which is able to produce closed form solutions without assumptions on the volatility-of-volatility parameter. Finally, we note that one could add further flexibility by keeping a drift function to the volatility to obtain, say, a geometric brownian motion. Although being analytically very challening the model would be solvable. For our application though we decide to keep away from a drift term. To see why we omit such term in the defining stochastic differential equations for our model (3.8), we only need to take the expected value of $z(t)$ which is $\mathbb{E}[z(t)] = z(0)$. In the case of a geometric Brownian motion one gets $\mathbb{E}[z(t)] = z(0)\exp(\mu t)$, where μ is the drift coefficient, and we see that this becomes infinite as the time grows when $\mu > 0$ and zero when $\mu < 0$. Both scenarios are highly unrealistic for long maturities.

In order to complete our discussion and prove the added value of our model and analysis we thus need to show that our model is able to imply a smile dynamic which is the same as the one observed in the market and pointed out by the authors of the SABR model. This will be carried out in the next section, the final of this chapter, in which we derive, approximately, a formula for the smile implied by our model, determine its dynamics and show its properties for short maturity options.

3.4 Properties of the Model's Implied Volatility

As mentioned previously, our model lies between local and stochastic volatility models in that it contains a shifting parameter for the forward rate. We know from our discussion of local volatility models that they predict an incorrect dynamic for the implied volatility smile. On the other hand, Hagan et al. in [HAGAN02] show that the SABR model is

able to fit interest rates markets independently of whether the exponentiation parameter is one or zero. As this parameter can be translated into a shift in the forward rate we can suppress the $\beta F(0)$ term in the SDE for the forward rate in (3.8) in order to see if the remaining part of the model derives the correct smile movements when the underlying changes. We will show the differences in fit to the market with and without the shifting parameter in the calibration chapter of the thesis. This will permit us to test whether any additional degrees of freedom to the ones we already employ for the volatility process are needed to fit the plain vanilla market. In order to derive an approximation for the implied volatility of our model, we will apply to our case a set of techniques which are now common for derivatives in the equities and foreign exchange markets that originate from the work carried out by Taylor and Xu in [TAYLOR94] and also described in Lewis [LEWI00]. While we could have taken the same approach as the one done by Hagan et al. in [HAGAN02] by deriving an effective local volatility we find the approach described below more direct and simple. The analysis is carried out for short maturities and will permit us in chapter 5 to fit our model to the market smile with minimal computation.

Consider the Hull-White formula (3.7) presented at the beginning of this chapter in section 3.2. Given that for the model proposed in (3.8) there is no correlation between the underlying and the volatility processes we can apply it to derive an approximation for the model implied volatility. Let $c(F, v)$ be the Black-Scholes formula for prices of European call options and note that c is a continuous function of the underlying F and the volatility v. We can thus Taylor expand c on the variable v around some value v_0:

$$c(F, v) = c(F, v_0) + \frac{\partial c(F, v_0)}{\partial v}(v - v_0) + \frac{1}{2!}(v - v_0)^2 \frac{\partial^2 c(F, v_0)}{\partial v^2} + ... \tag{3.21}$$

Now take v to be the time averaged volatility which we described at the beginning of this

chapter and let v_0 be its mean so that:

$$v = \frac{\alpha^2}{T}\int_0^T z(u)\, d\, u \quad \text{and} \quad v_0 = \mathbb{E}[v] \tag{3.22}$$

Taking expectations on both sides of (3.21) and denoting $\sigma^n = \mathbb{E}[(v - v_0)^n]$ we obtain:

$$\mathbb{E}[c(F, v)] = c(F, v_0) + \frac{1}{2!}\sigma^2 \frac{\partial^2 c(F, v_0)}{\partial v^2} + ... + \frac{1}{n!}\sigma^n \frac{\partial^n c(F, v_0)}{\partial v^n} + ... \tag{3.23}$$

But now when we apply the Hull-White result (3.7) to our specific model, the price of a call under stochastic volatility in the uncorrelated case is $C(F, z) = \mathbb{E}[c(F, v, K - \frac{\beta F(0)}{\alpha})]$ and so in our case we must have that the European call option equals the expression on the right hand side of (3.23). Also the implied volatility of a model is that which when the model's price is evaluated at yields the Black-Scholes price so that

$$C(F, z, K) \equiv c(F, \sigma(T, K)_{\text{imp}}, K).$$

Carrying out an expansion of the right hand side centred at v_0 we obtain:

$$C(F, \sigma(T, z, K) = c(F, v_0, K) + (\sigma(T, K)_{\text{imp}} - v_0)\frac{\partial c(F, v_0, K)}{\partial v} + ... \tag{3.24}$$

Equating (3.23) and (3.24) and setting $\beta F(0) = 0$, as discussed, we obtain an equation which must be satisfied by $\sigma(T, K)_{\text{imp}}$:

$$c(F, v_0, K) + (\sigma(T, K)_{\text{imp}} - v_0)\frac{\partial c(F, v_0, K)}{\partial v} + ... = \tag{3.25}$$

$$= c(F, v, K - \frac{\beta F(0)}{\alpha}) + \frac{1}{2!}\sigma^2 \frac{\partial^2 c(F, v_0, K - \frac{\beta F(0)}{\alpha})}{\partial v^2} + ...$$

Since we are looking for an approximation we look for solutions of the implied volatility not

too far from at-the-money strikes. We define the variable "moneyness" $X = \log(F/K)$, so, for example, for an out-the-money call option with strike $K = 100$ and underlying 150, $X \simeq 0.4055$. For such levels $X^5 \simeq 0.011$ and $X^6 \simeq 0.0044$ and if we were to seek a solution of the implied volatility in terms of moneyness as:

$$\sigma(T,K)_{\text{imp}} = \sum_i \alpha_i(T) X^i \tag{3.26}$$

we could drop high orders of the index i in the sum given that the quoted bid and ask spreads on implied volatility in the market are never more precise that a few percentages in level. We can drop any terms of order 5 or larger for appropriate moneyness levels. Indeed, liquidity drops heavily for deeply in and out of the money options so market prices tend to take this into account even more than arbitrage pricing issues. We will thus stay concerned mainly with options of moneyness levels of up to about 150% for which $X^5 \simeq 0.01$. In the case of out of the money calls or in the money puts a level of moneyness of 50% causes a slightly larger absolute value for the higher order terms but also an alternating sign. So let the implied volatility be approximately given by:

$$\sigma(T,K)_{\text{imp}} \simeq \alpha_0(t,T) + \alpha_1(t,T)X + \alpha_2(t,T)X^2 + \alpha_3(t,T)X^3 + \alpha_4(t,T)X^4 + ... \tag{3.27}$$

we are concerned with finding the coefficients $\alpha_i(t,T)$.

Substituting the expression of (3.27) into (3.25) and dividing both sides of the resulting equation by $\frac{\partial c}{\partial v}$ we obtain:

$$\left[\alpha_0(t,T) - v_0 + \alpha_1(t,T)X + \alpha_2(t,T)X^2 + \alpha_3(t,T)X^3 + \alpha_4(t,T)X^4 + ...\right] + \frac{1}{2}\times$$

$$\times\big[\alpha_0(t,T) - v_0 + \alpha_1(t,T)X + \alpha_2(t,T)X^2 + \alpha_3(t,T)X^3 + \alpha_4(t,T)X^4 + ..\big]^2 R^{(2,0)}$$

$$+.. = \frac{1}{2}\sigma^2 R^{(2,0)} + \frac{1}{3!}\sigma^3 R^{(3,0)} + \frac{1}{4!}\sigma^4 R^{(4,0)} + ... \tag{3.28}$$

where we are using the convention: $R^{(n,m)} = \big[\big(\frac{\partial}{\partial v}\big)^n \big(F\frac{\partial}{\partial F}\big)^m c\big]/\frac{\partial c}{\partial v}$. To proceed we will need to know the moments of the volatility and the $R^{(n,m)}$. We will need the identities for the volatility moments σ^n and $R^{(n,m)}$ in tables 8.1 and 8.2 in appendix 8.2. Matching the coefficients of X^n terms and using identities for $R^{(n,m)}$ we can obtain expressions for the $\alpha_i's$.

To begin, notice that the volatility does not depend on the underlying or the strike of a particular option hence volatility moments do not include any X term. Also note from table 8.1 that the coefficients $R^{(n,0)}$ do not include any odd powers of moneyness. The right hand side of (3.28) thus does not contain any odd powers of X and we conclude that the same must be for the implied volatility on the left hand side. We are only left with three coefficients which are obtained by setting the coefficients of the different powers of X on both sides of (3.28) equal to each other. Letting $R^{(n,0)}_{X^i}$ be the coefficient of X^i in $R^{(n,m)}$ the constant term yields:

$$(\alpha_0(t,T) - v_0) + \gamma(T-t) = -\frac{1}{2}\sigma^2\left(\frac{1}{2v_0(T-t)} + \frac{1}{8}\right)(T-t) + \eta(T-t) \tag{3.29}$$

with

$$\eta(T-t) = \sum_{n\geq 3}\frac{1}{n!}\sigma^n R^{(n,0)}_{X^0} \quad \text{and} \quad \gamma(T-t) = \sum_{n\geq 2}\frac{1}{n!}(\alpha_0 - v_0)\, R^{(n,0)}_{X^0}$$

Table 8.1 shows that the coefficient of smallest order of $\tau \equiv T-t$ in $R^{(n,0)}_{X^0}$ is the term

constant in X, A_0^n/z, also table 8.2 shows that the moment $\sigma^n$for $n \geq 3$ is of order $O(\tau^2)$ hence $\eta(T-t)$ is of order $O(\tau^2)$. Assume $\gamma(T-t)$ is also of order $O(\tau^2)$ then we have:

$$\alpha_0(t,T) = z(0) - \frac{1}{12}\frac{(\varepsilon z(0)+\lambda)^2}{z(0)}\tau + O(\tau^2) \tag{3.30}$$

so that our assumption on $\gamma(T-t)$ is correct and the solution for α_0 correct.

The procedure can be carried out in exactly the same manner for the missing parameters α_2 and α_4 by using the tables provided in the appendix. In order not to repeat calculations already present in the literature we will state the final result for our specific model:

$$\sigma(T,K)_{\text{imp}} = z(0) + \left(\frac{1}{12}\frac{(\varepsilon z(0)+\lambda)^2}{z(0)^2}\right)X^2 + \left[\frac{1}{60}\left(\frac{\varepsilon(\varepsilon z(0)+\lambda)^3}{z(0)^4}\right) - \right. \tag{3.31}$$

$$\left. - \frac{1}{48}\left(\frac{(\varepsilon z(0)+\lambda)^4}{z(0)^5}\right)\right] X^4 + O(X^6) + O(T-t)$$

as can be seen the odd powers of X cancel. See figure 3.4 below for the resulting shape when $\sigma_{\text{ATM}} = 0.1104, \varepsilon = 2.2968, \lambda = -0.03599$ and $F = 0.046$.

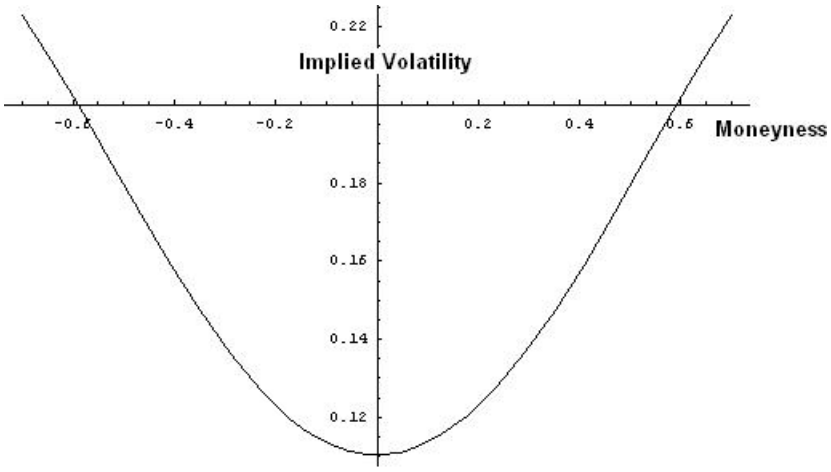


Figure 3.4: Model Implied volatility.

Finally, if we set $F = K$ so that the implied volatility corresponds to the at the money strike substituting the result we obtain:

$$\sigma(T,K)_{\text{imp}} = \sigma_{\text{ATM}} + \left(\frac{1}{12}\frac{(\varepsilon\sigma_{\text{ATM}}+\lambda)^2}{\sigma_{\text{ATM}}^2}\right)X^2 + \left[\frac{1}{60}\left(\frac{\varepsilon(\varepsilon\sigma_{\text{ATM}}+\lambda)^3}{\sigma_{\text{ATM}}^4}\right)- \right. \tag{3.32}$$

$$\left. -\frac{1}{48}\left(\frac{(\varepsilon\sigma_{\text{ATM}}+\lambda)^4}{\sigma_{\text{ATM}}^5}\right)\right]X^4 + O(X^6) + O(T-t)$$

Note that our analysis does not include the shifting parameter for the underlying in our model, that is the $\beta_j F_j(0)\sqrt{z(t)}$ term in the first stochastic differential equation of (3.8). Indeed, from our discussion in section 2.2.2.1 of chapter 2 we know such term increases the flexibility in the variety of implied volatility smiles that can be obtained from our model. The important aspect that we notice when dropping the shifting term $\beta_j F_j(0)\sqrt{z(t)}$ is the "sticky moneyness" rule. This means that the implied volatility of an option depends only on the ratio of the underlying and the strike. Hence, if at one point the underlying is at 100 and the 110 strike call is quoted at a volatility of 10% when the underlying moves

to 110 the 121 struck option also has a 10% volatility. This is important as it is one of the pillars of the market behaviour of the smile and any model must hence predict the right dynamics as those discussed in the paper by Hagan et al. [HAGAN02] in order to produce robust hedging parameters. We confirm that our model carries out what we were seeking it to do while also possessing a closed form solution for prices of plain vanilla European options which the SABR original analysis was unable to produce. The analysis resulting in such solution is the topic of the next chapter.

3.5 Summary

In this chapter we have motivated the setting of our particular model. While we could have made our research only about the investigation of a closed form solution for a model equivalent to the SABR one, we have not taken that route. The SABR model per se does have its flaws, and these are not only related to the approximation for the solution that its authors propose. Indeed, as we have discussed in depth volatility in the market is observed to cluster and mean revert and none of these phenomena are reproduced by the dynamics of SABR. On the other hand one needs to strike a balance between reasonable assumptions and analytical tractability and the SABR model has triumphed in the market because it is able to reproduce most of the market observed features while still not being too complicated for pricing issues. Also, the mean reversion of the other stochastic volatility models that have been proposed generates smiles for long maturities that are largely underestimated. All of these issues motivate our usage of a model which is equivalent to SABR. We have layed out the model and showed that it solves the long maturity smiles issue while still possessing the sticky strike rule needed for correct smile dynamics which is not present in local volatility models. The biggest task is now to derive a closed form solution for European options prices to surmount the main issue of the SABR analysis, which is the fact that the solution provided by its authors does not correspond to an accurate approximation to the real arbitrage free solution when the volatility of

volatility or the maturity are large. In doing this though it is important to note that the analysis we carry out on our model forces no correlation coefficient between the volatility and the underlying rate. While this generates fits that are not fully satisfactory, as we will show in the calibration chapter, it will permit us to derive the closed form which we are seeking. As mentioned previously it is good research practice to deal with this simpler case first and leave the correlated case to subsequent research. We can now move to our pticing solution.

4 Pricing of Caps and Swaptions

4.1 Introduction

As mentioned previously the most liquidly traded options in the interest rates markets are caps/floors and swaptions. Most hedges of derivatives trading books or single exotic products are conducted by trading through interest rate forwards, swaps, bonds and such types of plain vanilla options. Also, implied volatilities in the market are given only for caps and swaptions and they hence form the backbone for calibrating any form of model to the interest rates markets. Calibration may be done in several ways. For instance one may derive the values of the model's coefficients by running several Monte Carlo simulations or by employing discretization methods to solve the pricing partial differential equation associated with the model's defining SDE and constantly changing the parameters to best fit the market. Another possible way is to try and derive statistical variables through the parameters of the model which can then be estimated from historical data. The first alternative may be, computationally speaking, highly inefficient and the second is nearly always not guaranteed to provide a complete fit to the market. One alternative, when closed forms solutions or good appromixations for option prices are available, is to use an optimization algorithm to invert such pricing formulas and find optimal parameters to fit the market. It is often suggested that having a closed form solution can carry out faster calibrations [JAVAHE02]. This though is not the reason for us seeking such solution. Indeed, although finding such solution has intrinsic academic value the practical aspect of a faster calibration using it is not clear, at least in the form that we derive. The practical added value is that the solution provided by Hagan et al. in their approximation to the SABR pricing problem does not correspond to the arbitrage free price in all the market cases. Their solution assumes both that the maturity of the options is not too large and that the volatility of volatility is small. Both such assumptions are very often inapplicable and in such sitatuions their solution does converge to the arbitrage free price and hence

does not give the correct theoretical value that the model would imply. In our model a closed form equation for such a payoff is readily available, most importantly we provide a robust solution which, especially in the case of caplets, is free from assumptions regarding the magnitude of the parameters. The derivations will be the main topic of this chapter where we will tackle the cases of both caplets and swaptions. We note, again, that we do not provide a solution the correlated case which gives a nearly perfect fit to the market but we carry out the solution to the uncorrelated one in the hopes of being able to use it to ultimate the work on the complete case with further research. Furthermore, the uncorrelated case provides a beter fit than the equivalent SABR one.

Finally, we note that because our model sets off by describing the dynamics for the forward rate rather than swap rates in order to use the same pricing techniques for swaptions as the ones used for caps we will propose an approximation for the swap rate which simplifies and eliminates some assumptions from an ad-hoc argument presented in the existing literature by Piterbarg [PITERB03]. Once that is done we will be able to derive solutions to swaptions without repeating the work done for caps. We begin our discussion with our model's solution for caplets. As discussed above, a cap may be viewed as a portfolio of caplets with different expiries and its value is thus obtained by simply adding the prices of the caplets which it contains.

4.2 Caplets

Given that the model proposed in (3.8) is driven by a vector of Brownian motions, it satisfies the Markovian property so that there is a function $f(t, F, z)$ satisfying:

$$f(t, F, z) = \mathbb{E}^{T_{j+1}}\left[(F_j(T) - K)^+ | \, \mathfrak{F}(t)\right] \tag{4.1}$$

Consider the price of a caplet with strike K:

$$\text{Cplt}(F, z, t, K, T_{j+1}) = B(0, T_{j+1}) N \tau \mathbb{E}^{T_{j+1}}\left[(F_j(T) - K)^+ | \, \mathfrak{F}(t)\right] = \tag{4.2}$$

$$= B(0, T_{j+1}) N \tau f(t, F, z)$$

By the law of iterated expectations, conditional expectations are martingales and so $f(F, z, t)$ as defined above is a martingale whose drift must be zero. Taking the differential of the martingale above using the multidimensional version of Ito's theorem we get:

$$f_t + \frac{1}{2} f_{\mathrm{FF}} z(t) (\alpha F(t) + \beta F(0))^2 + \frac{1}{2} f_{\mathrm{zz}} (\varepsilon z + \lambda)^2 = 0 \tag{4.3}$$

Now at maturity the value of the option must be its payoff, also note that $-\beta F(0)/\alpha$ is a value for which the forward rate loses dynamics and stays constant, the value of the option then becomes its current exercise value. Furthermore, if the volatility is $-\lambda/\varepsilon$ we see from the stochastic differential equation governing the dynamics of z that it remains at such level and the value of the option must hence be given by the Black-Scholes solution. Also, if the forward rate goes to infinity the option will be exercised regardless of time and volatility so its value would tend to be $(F - K)$. Finally, the price of a call option is always bounded by the price of the underlying and increases as the volatility increases, thus as the volatility goes to infinity the price must go to F. We thus seek a solution to the partial differential equation (4.3) with the boundary conditions:

$$f(T, F, z, K) = (F - K)^+, \; \forall \, F \geq 0, \; z \geq 0; \tag{4.4}$$

$$f(t, F = -\beta F(0)/\alpha, z, K) = (-\beta F(0)/\alpha - K)^+, \; \forall \; 0 \leq t \leq T, z \geq 0; \tag{4.5}$$

$$f(t, F, -\lambda/\varepsilon, K) = \mathrm{Bl}\left(F + \beta F(0)/\alpha, K + \beta F(0)/\alpha, \alpha \sqrt{-\frac{\lambda}{\varepsilon}(t - T)} \right), \tag{4.6}$$

$\forall\ 0 \leq t \leq T, F \geq 0;$ and where $\mathrm{Bl}(F, K, \sigma)$ is the Black formula with volatility σ;

$$\lim_{F \to \infty} \frac{f(t, F, z, K)}{F - K} = 1, \forall\, 0 \leq t \leq T, z \geq 0; \tag{4.7}$$

$$\lim_{z \to \infty} f(t, F, z) = F, \forall\, 0 \leq t \leq T, F \geq 0. \tag{4.8}$$

These ensure that our problem is well defined and we will very briefly discuss their validity below.

Now consider the change of variable $x = \ln(\alpha F(t) + \beta F(0))$ in (4.3), the second parial differential of F becomes:

$$\frac{\partial^2}{\partial F^2} = \frac{\partial}{\partial F}\left[\frac{\mathrm{dx}}{\mathrm{dF}}\frac{\partial}{\partial x}\right] = \left[\frac{-\alpha^2}{(F(t)+\beta)^2}\frac{\partial}{\partial x} + \frac{\alpha^2}{(F(t)+\beta)^2}\frac{\partial^2}{\partial x^2}\right] \tag{4.9}$$

Substituting into equation (4.3) yields:

$$f_t - \frac{z(t)\alpha^2}{2} f_x + \frac{z(t)\alpha^2}{2} f_{\mathrm{xx}} + \frac{(\varepsilon z + \lambda)^2}{2} f_{\mathrm{zz}} = 0 \tag{4.10}$$

Define the Fourier transform of f as:

$$\widehat{f}(t, k, z) = \int_{-\infty}^{+\infty} e^{\mathrm{ikx}} f(t, F, z)\, \mathrm{dx} \tag{4.11}$$

by Fourier inversion we can write f as:

$$f(t, x, z) = \frac{1}{2\pi} \int_S e^{-\mathrm{ikx}} \widehat{f}(t, k, z)\, \mathrm{dk} \tag{4.12}$$

with S being the appropriate strip in the complex plane in which the transform is analytic.

This will depend on the type of option we are trying to price. For instance in the case of a call the payoff is $C(T) = (F(T) - K)^+ = (e^x - \beta - K)^+$ if we carry out (4.11) we have:

$$\widehat{f}(T,k,z) = \int_{\ln(\alpha K+\beta F(0))}^{\infty} e^{i\,k\,x}\left(\frac{e^x - \beta F(0)}{\alpha} - K\right) d\,x = \tag{4.13}$$

$$= \left(\frac{e^{(i\,k+1)x}}{\alpha(i\,k+1)} - \frac{(K+\beta F(0)/\alpha)e^{i\,\mathrm{kx}}}{i\,k}\right)_{x=\ln(\alpha K+\beta F(0))}^{x=\infty}$$

The upper limit becomes singular unless k has an imaginary part bigger than 1 to account for the term e^x. Hence for a call it is required that the strip S lies so that $\mathrm{Im}\,k > 1$. For a put a similar analysis shows that one requires $\mathrm{Im}\,k < 0$. Imposing such constraints the upper limit of the integral vanishes and (4.13) yields (the result is the same in the case of a put):

$$\widehat{f}(T,k,z) = -\frac{(\alpha K + \beta F(0))^{i\,k+1}}{\alpha(k^2 - i\,k)} \tag{4.14}$$

Noting that the solution does not depend on T and z it thus means that we can write the Fourier transform as:

$$\widehat{f}(t,k,z) = \widehat{f}(T,k,z)h(t,k,z) \tag{4.15}$$

with $h(T,k,z) = 1$. The price of the caplet would then be:

$$C(t,z,F) = -\frac{B(t,T)N\tau}{2\pi}\int_{i\,k_1-\infty}^{i\,k_1+\infty}(\alpha F(t) + \beta F(0))^{-i\,k}\frac{(\alpha K + \beta F(0))^{i\,k+1}}{\alpha(k^2 - i\,k)}\,h(t,k,z)\,d\,k \tag{4.16}$$

with $k_1 > 1$. The result though is not unique in terms of the strip of integration. We can, for instance, start from puts and forwards and derive transforms for calls using put call

parity. To see this we start from the put-call parity relationship:

$$C(t,z,F)-P(t,z,F)=F-K\,B(t,T)\Rightarrow C(t,z,F)=F-[K\,B(t,T)-P(t,z,F)] \quad (4.17)$$

at maturity the term in square brackets is $\min(F,K)=K+\min(F-K,0)$. Its transform is:

$$\widehat{f}(T,k,z)=\int_{-\infty}^{+\infty}e^{i\,k\,x}\left(K+\min(\frac{e^x-\beta F(0)}{\alpha}-K,0)\right)d\,x=\frac{(\alpha K+\beta F(0))^{i\,k+1}}{\alpha(i\,k-k^2)} \quad (4.18)$$

and it is defined for $0<\mathrm{Im}\,k<1$ which means that we can have a different representation of the caplet's price as:

$$\frac{\widetilde{C}(t,z,F)}{N\tau}=B(t,T)\left[F-\int_{i\,k_1-\infty}^{i\,k_1+\infty}\left(\frac{\alpha F(t)+\beta F(0)}{\alpha K+\beta F(0)}\right)^{-i\,k}\frac{\alpha K+\beta F(0)h(t,k,z)}{2\pi\alpha(k^2-i\,k)}\,d\,k\right] \quad (4.19)$$

for some $0<k_1<1$ so that the integrand is analytic. Another form could be to keep the exponential and write:

$$\widetilde{C}(t,z,F)=N\tau B(t,T)\left[F-(\alpha K+\beta F(0))\int_{i\,k_1-\infty}^{i\,k_1+\infty}\frac{e^{-i\,k\,X}}{2\pi}\frac{h(t,k,z)}{\alpha(k^2-i\,k)}\,d\,k\right] \quad (4.20)$$

where $X=\ln[(\alpha F(t)+\beta F(0))/(\alpha K+\beta F(0))\,]$. This form may be useful for checking some boundary conditions.

It can be shown [LEWI00] that the two solutions are related and become identical in the cases in which the fundamental transform satisfies $h(t,k=i,z)=1$ for an analytic

extension for values slightly outside of the strip described. When this is not the case the no arbitrage price of a call is the one given by (4.19). The same happens to put prices, the ones derived from the put-call parity and the ones derived by taking the transform of the payoff equal each other if and only if $h(t, k = 0, z) = 1$. We will not delve too much in this issue as it has already been done for the most general case by Lewis [LEWI00].

We can now rewrite equation (4.10) in Fock space to get:

$$\widehat{f}_t + \frac{(\varepsilon z + \lambda)^2}{2}\widehat{f}_{zz} - z\, c(k)\widehat{f} = 0 \tag{4.21}$$

where $c(k) = \frac{\alpha^2(k^2 - i\, k)}{2}$. This is a simple linear second order partial differential equation with non-constant coefficients. To solve it we do a series of changes of variables, the first being $x = \varepsilon z + \lambda$ which, after eliminating the final T term, yields:

$$h_t + \frac{\varepsilon^2 x^2}{2} h_{xx} - \frac{1}{\varepsilon}(x - \lambda)c(k)h = 0 \tag{4.22}$$

now defining $\iota = \frac{\varepsilon^2}{2}(T - t)$ and $y = \frac{\sqrt{x}}{\varepsilon}$ and noting that $h_{xx} = h_{yy}\frac{1}{4\varepsilon^4 y^2} - h_y \frac{1}{4\varepsilon^4 y^3}$ we finally obtain:

$$-h_\iota + \frac{1}{4}y^2 h_{yy} - \frac{1}{4}y\, h_y - \frac{2}{\varepsilon^3}(\varepsilon^2 y^2 - \lambda)c(k)h = 0 \tag{4.23}$$

Given that the coeficients are not constant the method of separation of variables will not provide us with the most general solution and we thus have to revert to the Laplace transform:

$$L(y, s) = \int_0^\infty e^{-s\tau} h(y, \tau)\, d\tau$$

$$h(y, \tau) = \frac{1}{2\pi i}\int_{S_2} e^{s\tau} L(y, s)\, d\, s$$

with the last integral taken in an appropriate strip in the complex plane. Changing

parameters $c'(k) = \frac{c(k)}{\varepsilon}$ and $s' = s - c'(k)\frac{2\lambda}{\varepsilon^2}$ (4.23) is mapped into the following ordinary differential equation:

$$y^2 L_{yy} - y\, L_y - \left(8c'y^2 + 4s'\right)L = -4 \tag{4.24}$$

Given the non-homogeneity we can solve this by first looking at the homogeneous case which can be reconducted to the transformed Bessel equation of Bowman [BOWMAN38]:

$$y^2 L_{yy} - y\,(2\alpha - 1)L_y + \left(\beta^2\gamma^2 y^{2\gamma} + \alpha^2 - n^2\gamma^2\right)L = 0 \tag{4.25}$$

where $\alpha = \gamma = 1$, $\beta^2 = -8c'$ and $n^2 = 4s'$. Its solution is known as a sum of Bessel functions :

$$L(y, s) = y^{\alpha}[C_1 J_n(\beta y^{\gamma}) + C_2 J_{-n}(\beta y^{\gamma})] \tag{4.26}$$

where $J_n(z)$ is the Bessel function of order n and constants C_1 and C_2.The non-homogeneous case can then be solved by using the following theorem.

Theorem 4.1. *Consider the ordinary differential equation:*

$$y_{xx} + a(x)y_x + b(x)y = c(x) \tag{4.27}$$

where the coefficient functions $a(x)$, $b(x)$ and $c(x)$ are continuous on some open interval L, and let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous case of (4.27) *then a particular solution is:*

$$Y(x) = -y_1(x)\int_{x_0}^{x} \frac{y_2(t)c(t)}{W(y_1, y_2)(t)}d\,t + y_2(x)\int_{x_0}^{x} \frac{y_1(t)c(t)}{W(y_1, y_2)(t)}d\,t \tag{4.28}$$

where $W(y_1, y_2)(t) = y_1 y_2' - y_1' y_2$ is the Wronskian determinant which is precisely zero only when the two solution of the homogeneous case are linearly dependent. Finally, the

general solution to (4.27) *is thus given by:*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + Y(x). \tag{4.29}$$

The integrals are calculated by Lewis [LEWI98] in solving a differential equation for a credit risk problem. He also mentions his solution in [LEWI00] for a general geometric Brownian motion model but, unfortunately, never expands on such solution and does not apply it to the SABR model which has been in use in the market for several years without a known closed form solution. His solution is for the model given by $\lambda = 0$ and $c'(k) = c(k)$. Doing the appropriate transormation of variables and shifting the transform from Lewis's solution by $c'(k)\frac{2\lambda}{\varepsilon^2}$ we get to the closed form solution for our generalization of the SABR model:

$$h(\tau, k, z) = y\, e^{c'(k)\frac{2\lambda}{\varepsilon^2}\tau}\left\{ K_1(y) + \frac{1}{4\pi^2}\times \right. \tag{4.30}$$

$$\left. \times \int_0^\infty \left| \Gamma\left(\frac{-1+i\nu}{2}\right)\right|^2 (\nu \sinh(\nu\pi)) K_{i\nu}(y)\, e^{-\frac{1}{8}\varepsilon^2\tau(1+\nu^2)} d\nu \right\}$$

where $\tau = T - t$, $y = 2\sqrt{2c'(k)(\varepsilon z + \lambda)}/\varepsilon$, Γ the Gamma function and K_j the modified Bessel function of the second kind.

Going back to our discussion above we need to see the behaviour of (4.30) on the strip given by $0 < \operatorname{Im} k < 1$. Doing this analytically may prove impossible and we thus revert to computing tools by plotting the fundamental transform for various values of $\varepsilon, \lambda, z, t, T$ and k. Matlab can compute the integrals quite easily, although the algorithms used for producing the modified Bessel function of complex order are not entirely satisfactory and miss out on part of the plot. We thus make use of Mathematica. The result for parameter values of$\varepsilon = 0.4, \lambda = 0.0001, \alpha = 1, z = 0.2, \tau = 1$ yields in figures 4.1 and 4.2:

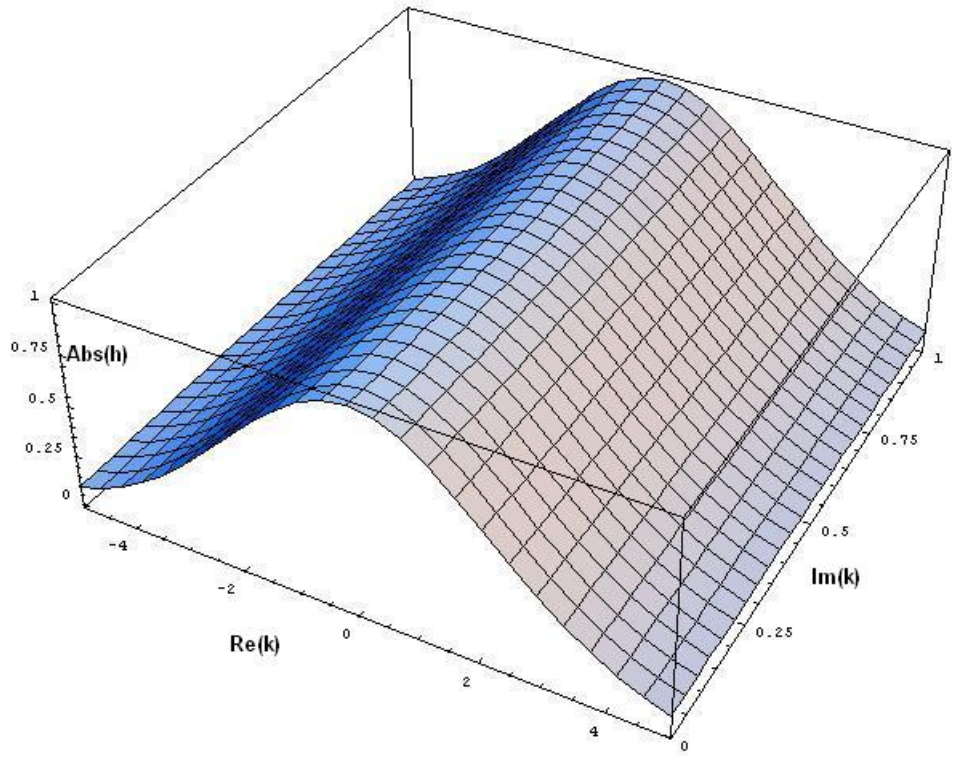


Figure 4.1: Plot of solution's transform h with $-5 < \mathrm{Re}(k) < 5$.

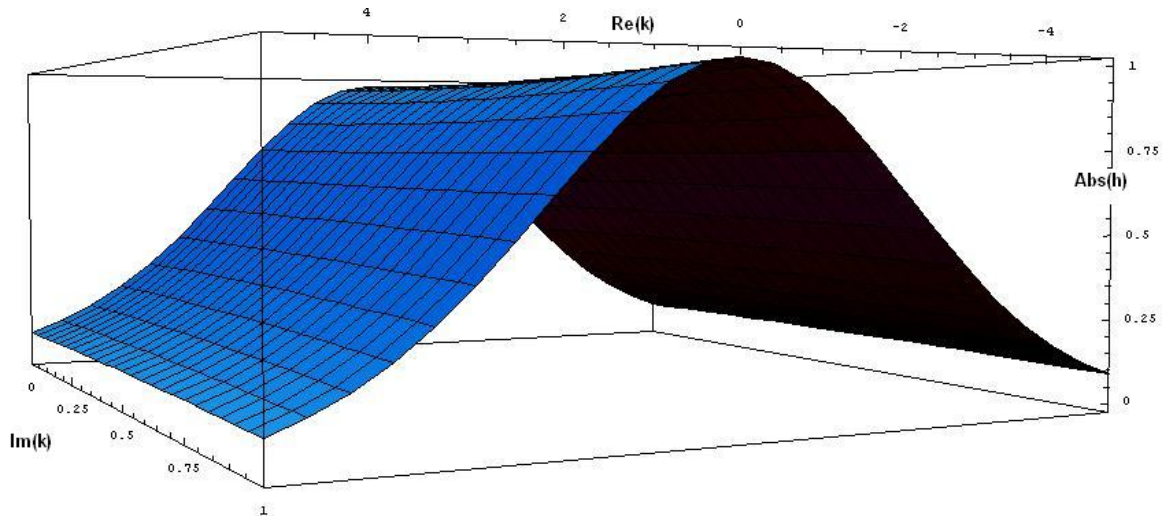


Figure 4.2: Plot of solution's transform h with $-5 < \mathrm{Re}(k) < 5$.

As can be seen the transform is continuous on the strip under concern, and plots for different values of the variables above are also well behaved. We can see that convergence to very small values is extremely fast and this will thus help us truncating the integral in (4.30) for fast calibration in the next chapter. Indeed, enlarging the range of the real part (in the Re(k) axis of the two figures above) we see that the value of abs(h) becomes extremely small very quickly:

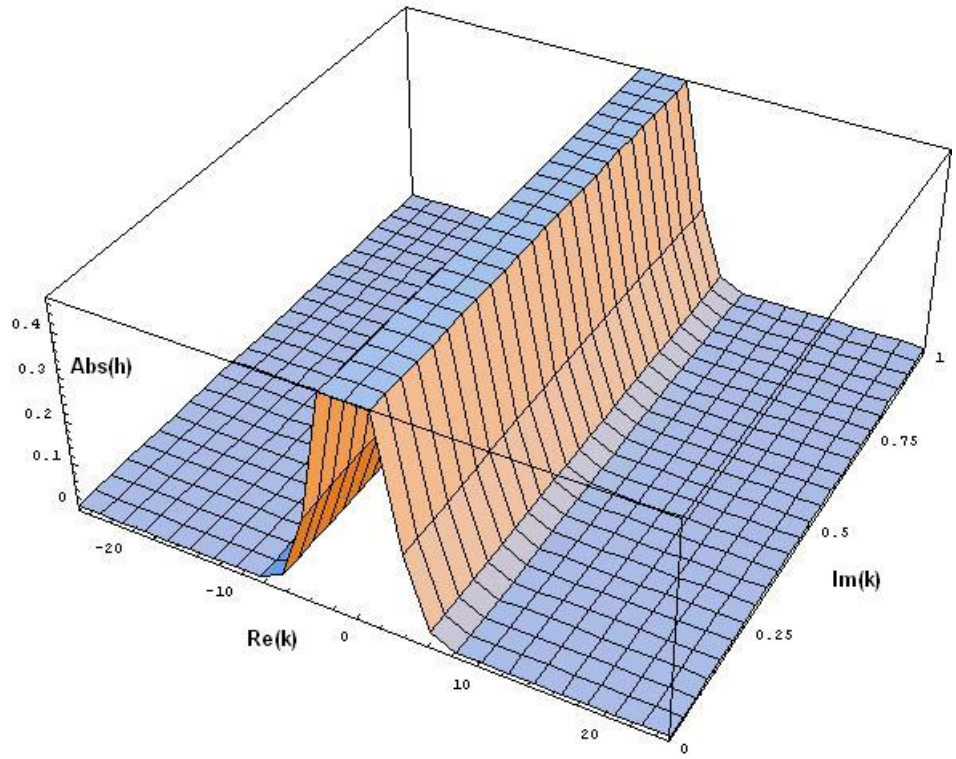


Figure 4.3: Fundamental Transform for $0 < \mathrm{Im}(k) < 1$ and $-25 < \mathrm{Re}(k) < 25$.

We thus choose $\mathrm{Im}(k) = 0.5$ to get the prices of caplets:

$$\widetilde{C}(t,z,F) = N\tau B(t,T)\left[F - (\alpha K + \beta F(0)) \int_{i/2-\infty}^{i/2+\infty} \frac{e^{-i\,k\,X}}{2\pi} \frac{h(\tau,k,z)}{\alpha(k^2 - i\,k)}\, d\,k\right] \tag{4.31}$$

with $X = \ln[(\alpha F(t) + \beta F(0))/(\alpha K + \beta F(0))\,]$. Note that for $k \to i$ or 0 we have $y \to 0$

and $K_1(y) \to 1/y$, also $c(k) \to 0$ and $K_{iv}(y) \to 0.5\Gamma(iv)y^{-iv}$ (see Stegun [STEGUN72]). The latter is of norm $0.5|\Gamma(iv)|$ and hence in the limits $k \to i$ or 0 the integral part of the transform goes to zero when multiplied by y while the rest sums up to 1 and so $\lim_{k \to i \text{ or } 0} h(k, \tau, z) = 1$. We need to check this given what was noted earlier and proved in Lewis [LEWI00] that the prices derived using the put call parity and those derived from starting with the payoff's transform are equal.

Finally, one can check that the boundary conditions are satisfied through the use of numerical packages. The first one, equation (4.4), is obviously satisfied by construction and can be checked by simply plotting the fundamental transform for $\tau = 0$ and seeing that it is 1. For the other conditions numerical methods are needed, the reader may easily check that they are satisfied by using the code provided in the calibration chapter.

4.3 Swaptions

At time t consider the value of a payer's swap paying the difference between LIBOR and K at the set of times $T_\alpha, ..., T_\beta$ on a notional N. By our discussion in Chapter 2 this is given by equation (2.49):

$$\frac{\text{PFS}(t, T_\alpha, ..., T_\beta, K)}{N} = B(t, T_\alpha) - B(t, T_\beta) - K \sum_{i=\alpha+1}^{\beta} [\tau_i B(t, T_i)] = \tag{4.32}$$

$$= (S_{\alpha,\beta}(t) - K) \sum_{i=\alpha+1}^{\beta} [\tau_i B(t, T_i)]$$

where $S_{\alpha,\beta}(t)$ is the swap rate as defined in equation (2.50). A swaption gives the holder the right to enter into a swap at time T_α maturing at time T_β, from above the payoff at maturity is thus:

$$N(S_{\alpha,\beta}(T_\alpha) - K)^+ \sum_{i=\alpha+1}^{\beta} [\tau_i B(T_\alpha, T_i)] \tag{4.33}$$

Letting $C_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} [\tau_i B(t,T_i)]$ and taking expectations the price is thus given by:

$$\text{Swptn}(S,T_\alpha,T_\beta,t,N,K,\tau_i) = N\,\mathbb{E}\big[D(t,T_\alpha)C_{\alpha,\beta}(T_\alpha)(S_{\alpha,\beta}(T_\alpha)-K)^+ \mid \mathfrak{F}(t)\big] = \quad (4.34)$$

$$= C_{\alpha,\beta}(t)N\mathbb{E}^{\alpha,\beta}\big[(S_{\alpha,\beta}(T_\alpha)-K)^+ \mid \mathfrak{F}(t)\big]$$

where the first expectation is taken in the risk neutral measure and the second in the measure associated with the numeraire $C_{\alpha,\beta}(t)$. The result following from the Change of Numeraire theorem 2.1. Given the swap rate's definition from equation (2.50) we see that $S_{\alpha,\beta}(t)$ is a martingale under such measure and if $S_{\alpha,\beta}(t)$ followed a process of similar form to the forward rate in (3.8) we could use our caplet price derivation above to write the price of a swaption in our model as:

$$\text{Swptn}(S,T_\gamma,T_\phi,t,N,K,\tau_i) = N\,C_{\gamma,\phi}(t)\,[S_{\gamma,\phi}(t) - \quad (4.35)$$

$$-(\alpha K+\beta S_{\gamma,\phi}(0))\int_{i/2-\infty}^{i/2+\infty}\frac{e^{-i\,k\,X}}{2\pi}\frac{h(\tau,k,z)}{\alpha(k^2-i\,k)}\,d\,k\Bigg]$$

and $X = \ln[(\alpha S_{\gamma,\phi}(t)+\beta S_{\gamma,\phi}(0))/(\alpha K+\beta S_{\gamma,\phi}(0))\,]$. In order for (4.35) to make sense we need to derive a form for $S_{\alpha,\beta}(t)$ which is of the same type as the forward rate, in the following sense.

The model we propose starts with the forward LIBOR rate rather than the swap rate and so in order to price swaptions a formula for the latter needs to be derived. Doing so analytically may prove impossible. In order to ease the calculation we could proceed in two ways. One possible way is to develop two models, each forcing the forward rates and the swap rates to follow the same type of stochastic differential equations independently

of whether that has any consistency. In doing this we run into possible arbitrage between the caps and swaptions markets as the two rates are related by non linear relations which do not give them the same form [BRIGO06]. Another way is to start in one market and derive approximations for the stochastic differential equation of the other corresponding market. Thus, one may start with the forward rate and derive the swap rate. In doing this one could also try to approximate the derived stochastic differential equations to a form similar to the original forward rate's one so that, once caps are priced, only a change of parameters is necessary to price swaptions. This is the sense in which equation (4.35) holds.

To show that the needed approximation could exist we present a heuristic argument which starts under an ad-hoc assumption similar to what Piterbarg does in [PITERB03]. Our approach though is different not only given that our model is not the same but also because we make only one unmotivated assumption. In contrast to Piterbarg we do not discard second order derivatives terms in the calculations and we try to stay slightly more rigorous without imposing exogenous least-squares approaches as Piterbarg does. The values obtained for the coefficients are not intended to be good enough for use, although as Piterbarg shows they could perfectly well be. The intent here is to show that it is indeed possible to have simultaneously similar forms of stochastic differential equations for both the swap rate and the forward Libor rate, so that we can use the equation derived above in (4.35) with the same transform used for caplets in (4.30) except that the parameters in the stochastic differential equation defining F in (3.8) are substituted by the correct ones.

Consider starting with the forward Libor market model given by (3.8) under the T_{j+1}-forward measure. Let $\mathbb{P}^{i,l}$ be the risk-neutral measure associated with the numeraire

$$\sum_{r=i+1}^{l} \tau_r B(t, T_r). \tag{4.36}$$

From the definition of the swap rate $S_{i,l}(t)$ in equation (2.50) we see that this must be

a martingale in such measure and thus, by the martingale representation theorem 8.23, it cannot have drift under $\mathbb{P}^{i,l}$. Also, since measure changes do not affect the diffusion terms of SDEs we can use the forms of $F_j(t)$ in their form in (3.8) and disregard the drift changes when going to the swap measure. Thus taking the swap rate as a function of $F_j(t)$ and setting any drift term resulting from Ito's lemma to zero we are left with:

$$d\,S_{i,l}(t) = \sum_{j=i}^{l-1} \frac{\partial S_{i,l}(t)}{\partial F_j(t)} d\,F_j(t) = \tag{4.37}$$

$$= \sum_{j=i}^{l-1} \frac{\partial S_{i,l}(t)}{\partial F_j(t)} \{[\alpha_j F_j(t) + \beta_j F_j(0)]\upsilon(t)\}\, d\,W^{i,l}(t)$$

where $W^{i,l}(t)$ is a Brownian motion under $\mathbb{P}^{i,l}$ and $\upsilon(t) = \sqrt{z(t)}$. In order to speed up the pricing of swaptions when we start with the forward rate we will try to find an approximation to $S_{i,l}(t)$ which similar to the original process, similar to what Piterbarg does for his model in the forementioned paper, we thus want to impose the following form as an approximation:

$$d\,S_{i,l}(t) = \{\left[\widetilde{\alpha}_{i,l} S_{i,l}(t) + \widetilde{\beta}_{i,l} S_{i,l}(0)\right]\upsilon(t)\}\, d\,W^{i,l}(t) \tag{4.38}$$

We see that with such a form equation (4.35) would hold. To relate the new parameters to the original ones from the forward rate we begin by defining the two functions:

$$\begin{aligned} f(S,F,t,\upsilon) &= \sum_{j=i}^{l-1} \frac{\partial S_{i,l}(t)}{\partial F_j(t)} \{[\alpha_j F_j(t) + \beta_j) F_j(0)]\upsilon(t)\} \\ g(S,F,t,\upsilon) &= \{\left[\widetilde{\alpha}_{i,l} S_{i,l}(t) + \widetilde{\beta}_{i,l}) S_{i,l}(0)\right]\upsilon(t)\} \end{aligned} \tag{4.39}$$

If (4.38) were to hold exactly for all t, F, υ and S then the two must coincide and so must all of their derivatives. Since there are only three parameters that need to be expressed in the new SDE in terms of those in (3.8) we may get into inconsistencies if we apply all such constraints. We are after an approximation and thus do not make all such assumptions.

One reasonable assumption to make though is that for small perturbations in the swap rate and its volatility process from some constant path $S(t) \triangleq S(t=0)$ and $F(t) \triangleq F(t=0)$, at which we assume the relationship to hold, the SDE for the swap rate will not change. Since the SDE is defined in a path independent manner if we can find a relationship between the parameters on such paths this must hold on all paths, always under the assumption that small changes will not affect the form of the SDE. Doing a Taylor expansion and disredarding terms of the order $\mathbb{O}(\Delta S \Delta S, \Delta S \Delta \upsilon, \Delta \upsilon \Delta \upsilon)$ or higher we get:

$$\begin{aligned} f(S,F,0,\upsilon) &= g(S,F,0,\upsilon), \\ \frac{\partial f}{\partial S}(0) &= \frac{\partial g}{\partial S}(0), \\ \frac{\partial f}{\partial \upsilon}(0) &= \frac{\partial g}{\partial \upsilon}(0). \end{aligned} \tag{4.40}$$

The simplest relationship to start with is the second. Differentiating with respect to $S(0)$ both equations in (4.39) at $t=0$ and setting them equal we obtain:

$$\widetilde{\alpha}_{i,l}\upsilon(0) = \sum_{j=i}^{l-1} \frac{\partial S_{i,l}(0)}{\partial F_j(0)} \left[\alpha_j \frac{\partial F_j(0)}{\partial S_{i,l}(0)}\right] \upsilon(0) \Rightarrow \widetilde{\alpha}_{i,l} = \sum_{j=i}^{l-1} \alpha_j \tag{4.41}$$

Notice that we did not need to run into least squares arguments for equations which are clearly not solvable as Piterbarg does in [PITERB03]. The first equation in (4.40) instead yields:

$$\left(\widetilde{\alpha}_{i,l} S_{i,l}(0) + \widetilde{\beta}_{i,l}\right)\upsilon(0) = \sum_{j=i}^{l-1} \frac{\partial S_{i,l}(0)}{\partial F_j(0)} \{[\alpha_j F_j(0) + \beta_j]\upsilon(0)\} \tag{4.42}$$

whereas the third condition:

$$\widetilde{\alpha}_{i,l} \frac{\partial S_{i,l}(0)}{\partial z(0)} \upsilon(0) + \widetilde{\alpha}_{i,l} S_{i,l}(0) + \widetilde{\beta}_{i,l} = \tag{4.43}$$

$$= \sum_{j=i}^{l-1} \frac{\partial S_{i,l}(0)}{\partial F_j(0)}[\alpha_j F_j(0) + \beta_j] + \sum_{j=i}^{l-1} \frac{\partial^2 S_{i,l}(0)}{\partial \upsilon(0) \partial F_j(0)}[(\alpha_j F_j(0) + \beta_j)\, \upsilon(0)]$$

so that substituting the result for $\widetilde{\alpha}_{i,l}$ yields:

$$\widetilde{\beta}_{i,l} = -\sum\nolimits_{j=i}^{l-1} \alpha_j \left(\frac{\partial S_{i,l}(0)}{\partial \upsilon(0)} \upsilon(0) - S_{i,l}(0) \right) + \tag{4.44}$$

$$+ \sum_{j=i}^{l-1} \frac{\partial S_{i,l}(0)}{\partial F_j(0)}[\alpha_j F_j(0) + \beta_j] + \sum_{j=i}^{l-1} \frac{\partial^2 S_{i,l}(0)}{\partial \upsilon(0) \partial F_j(0)}[(\alpha_j F_j(0) + \beta_j)\, \upsilon(0)]$$

Piterbarg discards second order derivative terms and thus gets simpler forms for his second factor, the model just presented is slightly more complicated. Although we could also simplify more by adopting his assumption, we find this to be unjustified.

We have thus recovered the coefficients of the swap rate under one condition, that the approximation holds on the constant paths defined above. Of course, we do not know whether this is a reasonable assumption, the only reason we can cite is that the result, as pointed by Piterbarg [PITERB03] is experimentally confirmed. More rigourous work has been done this year by Antonov and Misipashaev some of which can be found in [ANTO06]. This makes it easier as we can now use simultaneously both forms when pricing without being too concerned about possible arbitrages and inconsistencies between the swaptions and caps markets. Concluding, with the approximating parameters obtained from equations (4.41)-(4.44) the resulting stochastic differential equation in (4.38) can be used instead of the one for the forward rate so that the transform obtained in (4.30) can also be used to price swaptions in equation (4.35).

4.4 The Greeks

Now that we have prices in closed form we need to derive the hedging parameters, the "greeks", of our model. This shows the possible advantages of using the type of analysis we carry out rather than the ones typically done with the SABR model in which approximations of prices using expansion methods do not yield exact greek parameters, or with more complicated models which require time consuming Monte Carlo methodologies. Although it may prove difficult to construct fast and precise algorithms for calculating the integral in $h(\tau, k, z)$ the analytical derivation of the hedging parameters is very simple. We will see this more in depth in the next chapter were once the calibration of the model is carried out, we will also study the properties and dependencies of these hedging parameters and compare them to the non stochastic volatility case. Given the similarity in the equations for the prices of caplets and swaptions we will derive greeks only for the former derivatives. The "major greeks" (delta, gamma, vega and theta) are all trivial to calculate. The delta, giving the sensitivity to changes in the underlying is:

$$\Delta(F,\ \tau,\ z, K) = \frac{\partial \widetilde{C}}{\partial F} = N\tau B(t,T)\left[1 + i\int_{i/2-\infty}^{i/2+\infty} \frac{h(\tau,k,z)e^{-(ik+1)\ X}}{2\pi(k-i\ )} d\,k\right] \tag{4.45}$$

The Gamma, the sensitivity of the delta with respect to the underlying on the other hand is found to be:

$$\Gamma(F,\tau,z,K) = \frac{\partial^2 \widetilde{C}}{\partial F^2} = N\tau B(t,T)\alpha \int_{i/2-\infty}^{i/2+\infty} \frac{e^{-(ik+2)X}}{2\pi\left(\alpha K + \beta F(0)\right)} h(\tau,k,z)\, d\, k \tag{4.46}$$

For the Theta, the sensitivity of the price with respect to the flow of time, we note that when keeping the forward rates constant bond prices are also constant and so the result

is:

$$\Theta(F,\ \tau,\ z, K) = -\frac{\widetilde{C}}{\tau} - N\tau B(t,T)(\alpha K + \beta F(0)) \int_{i/2-\infty}^{i/2+\infty} \frac{e^{-i\,k\,X}}{2\pi} \frac{h_\tau(\tau,k,z)}{\alpha(k^2 - i\,k)} dk \tag{4.47}$$

where

$$h_\tau = -c'(k)\frac{2\lambda}{\varepsilon^2} h + y\, e^{c'(k)\frac{2\lambda}{\varepsilon^2}\tau} \Bigg\{ \frac{1}{4\pi^2} \times$$

$$\times \int_0^\infty \left| \Gamma\left(\frac{-1+i\nu}{2}\right) \right|^2 \left(\frac{\varepsilon^2}{8} \nu(1+\nu^2)\sinh(\nu\pi) \right) K_{i\nu}(y)\, e^{-\frac{1}{8}\varepsilon^2\tau(1+\nu^2)} d\nu \Bigg\}$$

Finally, the vega, the sensitivity with respect to changes in volatility, is:

$$\mathfrak{V} = \frac{\partial \widetilde{C}}{\partial z} = -N\tau B(t,T)(\alpha K + \beta F(0)) \int_{i/2-\infty}^{i/2+\infty} \frac{e^{-i\,k\,X}}{2\pi} \frac{h_z(\tau,k,z)}{\alpha(k^2 - i\,k)} dk \tag{4.48}$$

with:

$$h_z(\tau,k,z) = \frac{4c'(k)}{\varepsilon\, y} e^{c'(k)\frac{2\lambda}{\varepsilon^2}\tau} \Bigg\{ \Bigg[2K_1(y) - y\, K_2(y) + \frac{1}{4\pi^2} \times \tag{4.49}$$

$$\times \int_0^\infty \left| \Gamma\left(\frac{-1+i\nu}{2}\right) \right|^2 ((1+i\nu)\nu\sinh(\nu\pi)) K_{i\nu}(y)\, e^{-\frac{1}{8}\varepsilon^2\tau(1+\nu^2)} d\nu \Bigg] +$$

$$-y\frac{1}{4\pi^2} \int_0^\infty \left| \Gamma\left(\frac{-1+i\nu}{2}\right) \right|^2 (\nu\sinh(\nu\pi))\, e^{-\frac{1}{8}\varepsilon^2\tau(1+\nu^2)}\, K_{i\nu+1}(y)\, d\,v$$

We will analyze the empirical performance of these in the final chapter. The existence of the closed form solutions for the prices and hedging parameters just derived show that our model has sufficient analytical properties to ehance its implementation in practice. Indeed, our model is more encompassing than the standard SABR model used in the

market, which can be seen to be equivalent to our framework by setting $\beta = \lambda = 0$ and $z = V^2$ in equations (2.82) and (3.8), yet the analysis we carry out provides a fully exact pricing formula for European options on both the forward rate and the swap rate. As a result of this, the hedging parameters are also exact and obtained very easily. All this is achieved without having to resort to types of approximations as the volatility-of-volatility expansion which are used by the authors of the SABR model in [HAGAN02] and which are certainly not justified in market periods, such as the ones of recent years, where market volatility has been highly volatile itself.

4.5 Summary and Overture to the next chapter

In this chapter we carried out an analysis which was able to derive a closed form solution for the prices of European payoffs. This is probably the most important contribution of the thesis as the SABR model, which presents the same benefits and limits of our model and is equivalent to our framework with the correct choice of parameters, has always been analysed using an approximation that assumes the volatility-of-volatility to be a small parameter. This is certainly not true in many market conditions, and certainly not in current ones. A closed form solution for the SABR model was, to our best knowledge, never known in the literature although work useful in deriving it had been carried out in the analysis of a credit risk problem by Lewis in [LEWI98]. We make use of such work to derive a solution for pricing caplets which is completely free of assumptions on the size of the parameters. The solution derived could also be the one used for swaptions if our model were a swap rate based one. Market models in general though are characterized by the fact that they cannot account for the distinction in the market between swap rates and forward rates. Indeed, the market quotes implied volatilities for caps and swaptions using a Black equation for both rates while market models do not imply that the two have the same distributions. While theoretically speaking this difference is obviously not solvable, in practice the difference is too small to be observed. There has hence been a search for an

approximation for the swap rate when one starts with a model for the forward rate which gives the former a stochastic differential equation which resembles the one the model sets out for the forward rate. Initial work was done by Piterbarg in [PITERB03], and we simplify his argument and rid of some exogenous assumptions to derive an approximating SDE for the swap rate which we then use to price swaptions in exaclty the same manner as for caps. Once pricing is carried out we derive the "greeks", or hedging parameters, for our model. This is one of the great benefits of having a closed form solution because when this is not known lengthy Monte Carlo simulations are usually employed to hedge trading books which are usually very slow. While computationally speaking the pricing formulas we derive are still not efficient they permit one to conduct a more thorough analysis to be able to determine what can or cannot be approximated. Also a solution of the integral in the transform solution of equation (4.30) may be carried out as a future research project.

Now that the analytical properties of our model have been laid out and studied in detail we need to see whether it is indeed able to fit real market quotes. This will be the topic of the next chapter in which we will carry out the calibration of our model to the first caplet strip of caps accross maturities and discuss the properties of the hedging parameters derived above. In doing the former the main line of thought will be to conduct a comparative analysis with the SABR model, especially given that our framework's main goal is to provide a closed form solution for it. The hedging parameters will be compared to the Black's model in order to show the properties compared to the standard benchmark for the greeks.

5 Calibration and Empirical Fit of Prices

5.1 Introduction to methodology and data

Whereas the previous chapter focused on presenting the theoretical and analytical properties of our proposed model we will now shift our emphasis to its empirical characteristics. In order to do so the first issue we will need to tackle is the estimation of the parameters that we imposed in the defining stochastic differential equation (3.8). In doing this there are two main streams that can be followed. The first is the historical estimation technique which makes no usage of market option prices, but is instead based on the time series of the forward and swap rates. This would begin by discretizing (3.8) to form a time series model defined by the difference equations:

$$\begin{aligned} F_{t+1} - F_t &= (\alpha F_t + \beta F(0))\sqrt{z_t}\epsilon_t^1 \\ z_{t+1} - z_t &= (\varepsilon z_t + \lambda)\epsilon_t^2 \\ \text{cov}[\epsilon^1, \epsilon^2] &= 0 \end{aligned} \tag{5.1}$$

where ϵ^1 and ϵ^2 are white noise random variables and F and z discrete time series processes taking values at $\{t_0 = 0, t_1, t_2, ...\}$. The four model parameters would then be estimated using filtering techniques on the historical series of F. This has the advantage that one can carry out an estimation without having to resort to deriving option prices. Also, the underlying's price history can be easily obtainained as opposed to option prices. Finally, in some cases, the process, in many cases, does not require elaborate lengthy computations.

Historical price estimation can be used to derive prices of options once pricing methodologies are set. The prices derived though, are not guaranteed to match the ones present in the market as this would not only require that the model correctly decribes the behavior of the underlying, but it would also require that the market correctly reflected such dynamics in the prices for options. Also, the estimation will have to be done in the real

world measure rather than the forward measure in which equation (5.1) is defined. The parameters used to go to the forward measure, the variables in the market price of risk, are themselves most likely dependent on the period of estimation. One way to remove such assumptions and be certain that the estimation will yield the best possible fit that a model can give to market prices is by making an inference starting directly from option prices. In cases in which option prices are readily available this procedure is also simpler than the former one described. This is the approach that we will follow. We hence begin by considering the model's price we have above for caplets or swaptions with strike K and setting the term $\beta F(0) \equiv \beta'$ in equation (3.8), and we will also from now on drop the primed notation. The original form was useful for deriving the swap rate's stochastic differential equation but is of no use for our ongoing discussion. We could do the same with the α parameter as it could be absorbed in the volatility process by multiplying the volatility and the volatility-of-volatility ε parameter by α^2. We will not do this as it turns out that computing time is far less if the term is maintained when using the closed form solution provided.

A traditional way of estimating parameters from option prices would be the following. Consider the price of a caplet and let $C(t, z, F, \alpha, \beta/F(0), \lambda, K, \varepsilon)$ be the function obtained by taking the parameters in (4.31) as variables and dividing the result by $N\tau B(t,T)$ so that:

$$C_i(z_i, \alpha, \beta, \lambda, K_i, \varepsilon) = \left[F - (\alpha K + \beta) \int_{i\,/2-\infty}^{i\,/2+\infty} \frac{e^{-i\,k\,X}}{2\pi} \frac{h(\tau, k, z)}{\alpha(k^2 - i\,k)}\, d\,k \right] \tag{5.2}$$

where X and $h(\tau, k, z)$ are the same as in equation (4.31). From the market prices of caps we can infer the same quantity so that when setting the correct implied volatility, forward rate and maturity we obtain a corresponding $C_i^{\text{mrkt}}(z_i, K_i)$. Going through the different

strikes accross the volatility smile we can define the error function:

$$f(z_i, \alpha, \beta, \lambda, K_i, \varepsilon) = \sum_i \big(C_i(z_i, \alpha, \beta, \lambda, K_i, \varepsilon) - C_i^{\text{mrkt}}(z_i, K_i)\big)^2 \tag{5.3}$$

The purpose of a calibration could thus be to find the values of α, β, ε and λ that minimize f. Alternative error functions may also be used. On the contrary, we will not be undertaking such task. The study by Hagan et al. in [HAGAN02] shows that the SABR model with the exponent β in equation (2.82) gives the same market fit to all the interest rates smiles that they examine irrespective of whether its value is zero or one. Our own $\beta F(0)$ term was added in light of the work of many, such as those of Rebonato in [**?**], and Joshi in [JOSH03b] or the more empricial one by Meister in [MEIST04] who show that Constant Elasticity of Variance and Displaced Diffusion models are equivalent. Hence our model in (3.8) is equivalent to the SABR model with zero correlation between the underlying and the volatility for the case $\lambda = 0$. In this sense, as previously stated, our model is a generalization to the SABR one and given the importance of this latter in the market it is more crucial for us to conduct a comparative analysis between our pricing solution and the one given by Hagan et al. in [HAGAN02] rather than only calibrating our model to as many maturities and tenors in the market as possible with all the possible parameters maintained. The SABR model is able to fit interest rates smiles with three parameters, the correlation coefficient between the underlying and the volatility, the initial level of the volatility and the volatility-of-volatility parameter. Since we do not include the correlation factor in our analysis we will compare the two with zero correlation. As mentioned before, the smiles obtained in doing so are not satisfactory, and we will show that the incorporation of the correlation does solve the issue in both cases. In this sense a contribution of our analysis is in opening the door for the subsequent study of a closed form in the correlated case. We will maintain two non-redundant parameters, the volatility-of-volatility ε and the shifting parameter λ. Given that the exponentiation parameter in the SABR model does not have any effect we will set $\beta = 0$ in our model

and the corresponding exponentiaton to one in SABR. This additional variable may come in useful in cases when one is pricing more exotic structures but as far as the plain vanilla markets that we are focusing on we do not have any need to maintaining it. In the first case the remaining problem would be to find the parameters α, λ and ε such that the quantity $f(z_i, \alpha, 0, \lambda, K_i, \varepsilon)$ of equation (5.3) is minimized. Once that is done we would need to compare the result with the same respective quantity for the SABR model with $\beta = 1$ in equation (2.82). This though can only be done in the cases for which the SABR's pricing equation holds; as its authors show explicitly in their derivation in [HAGAN02] that the approximation is precise only for short maturities, we will focus only on these. The resulting SABR pricing eqution for caplets is then given by:

$$\mathrm{Cpl}(0, F_j, \tau, K) = \tau N\, B(0, T_j)[F_j(0)N(d_+) - K\, N(d_-)] \tag{5.4}$$

$$d_\pm = \frac{\ln(F_j(0)/K) \pm \frac{1}{2}\sigma_{\text{eff}}^2 T_{j-1}}{\sigma_{\text{eff}}\sqrt{T_{j-1}}}$$

and σ_{eff} being an effective volatility given by:

$$\sigma_{\text{eff}} = \frac{\varepsilon \ln(F/K)}{\ln\left(\frac{z-\rho+\sqrt{1-2\rho z+z^2}}{1-\rho}\right)}\left[1 + \left(\frac{\rho\varepsilon V(0)}{4} + \varepsilon^2\frac{2-3\rho^2}{24}\right)T_{j-1}\right] \tag{5.5}$$

where $z = \frac{\varepsilon}{V(0)}\ln(F/K)$ and, very importantly, $\rho\, \epsilon\, [-1, 0]$ otherwise the processes defining the model do not yield a martingale for the forward rate under the forward measure and the pricing equation does not hold (as pointed out in the discussion of the SABR model in section 2.2.3.4). We will take $\rho = 0$.

The methodology presented above of minimizing the quantity in equation (5.3) requires fast pricing formulas. Unfortunately, while the closed form we present opens possibilities for further analytical enhancements in the study of the integral defining the transform in equation (4.30), its current form is too computationally demanding for calibration by

an optimization requiring repetitive price calculation. For instance, when our framework is implemented with the Mathematica code provided above the calculations may require from less than three to over thirty seconds for each price depending on the parameters chosen. This is obviously not feasible and forces us to look for alternative ways to calibrate our model. Since our comparison must be done in the regime in which both models hold we can conduct a calibration of the SABR model through the minimization of the type just described, and on the other hand estimate our model's parameters by the use of the implied volatility polynomial derived in the previous chapter in equation (3.32). The calibration is then translated into the problem of finding the coefficients of the polynomial that give the best fit to the observed market smile. This will also be done in the same manner as the SABR calibration by minimizing an error function of the type discussed above, with the only difference that instead of the prices we will use the implied volatilities. Furthermore, a calibration of the SABR model by a least squares approach would translate into finding the right parameters in the σ_{eff} quantity of equation (5.5) so that it fits the implied volatility. Hence, we will need to only fit the respective model implied volatilities to the ones observed in the market rather than first fitting the prices and then rederiving the implied volatilities as equation (5.3) suggested. It is also very important to note that as regards to our model the variable z cannot be replaced with the volatility but rather by its squared divided by α.

As far as the data is concerned, we will be using the caps/floors market skew supplied by the interdealer broker ICAP which may be found on Bloomberg or Reuters. We note that implied volatilities quotes are given for the whole structure, meaning that if a cap is quoted at an implied volatility of 17% all the caplets contained therein are quoted at such level regardless of their start and settlement dates. Furthermore, it is important to notice that the tenor of the underlying forward rate varies depending on which market quotes we are looking at. For caps of one year, 18 months and two years maturities the underlying quoted for EUR and USD rates is the three months forward LIBOR, whereas

for maturities longer than two years the underlying rate is the 6 months one. In this language a three years EUR cap entered into on 25 September 2007 will start setting the size of the payoffs on 26 March 2008 and end on 27 September 2010 with the payments executed six months after each setting. The structure hence contains five caplets. The convention is that an at the money cap is struck at the swap rate equivalent to its period, in the sense that a one year ATM cap will be struck at the one year swap rate. We will be focusing on EUR rates for our calibration and we will be using the first strip of caplets in a variety of smile structures for caps, ranging from short maturity caps to very long ones. This will illustrate a small inconsistency in the caps/floors market in which caplets with very small difference in strikes may be quoted at very different volatilities because of their presence in different caps.

We will have two different maturities for the caplets we will be studying. The first will be a three months one and is the one obtained by the one year, 18 months and two years caps and the second of six months obtained from caps of maturities longer than two years. The implied volatility structures selected are for the maturities of one year, eighteen months, two, three, six, twelve and twenty five years. The Reuters ICAP page gives the following implied volatilities for EUR rates caps/floors on Monday 15 October 2007:

Table 5.1: EUR Caps/Floors Implied Volatilities, taken from ICAP's Reuters page

M	200	225	250	300	350	400	450	500	550	600	700	800	1000
1y	25.9	23.6	22.0	19.5	15.8	12.7	10.7	10.3	11.1	12.3	14.7	16.7	20.1
18m	26.2	24.2	22.3	19.3	16.3	13.6	11.5	10.9	11.0	11.4	12.8	14.1	16.4
2y	26.2	24.2	22.4	19.2	16.4	13.9	11.9	11.2	11.1	11.4	12.4	13.6	15.6
3y	23.3	21.8	20.5	18.1	16.0	14.1	12.6	11.8	11.5	11.5	11.9	12.4	13.6
6y	21.7	20.4	19.2	17.2	15.6	14.1	13.0	12.4	12.2	12.3	12.7	13.4	14.4
12y	19.5	18.4	17.4	15.8	14.5	13.3	12.4	11.8	11.5	11.4	11.6	11.9	12.6
25y	17.4	16.4	15.6	14.1	13.0	12.1	11.3	10.8	10.5	10.3	10.4	10.8	11.5

Table 5.2: EUR Caps/Floors ATM levels and volatilities

M	1y	18m	2y	3y	6y	12y	25y
STK	440	440	440	449	453	469	478
ATM	10.83	11.68	12.06	12.64	12.95	12.16	10.99

In table 5.1 the numbers in the first row indicate the rates of the strikes in terms of basis points, while the first column indicated the maturity of the cap. The second table on the other hand gives additional quotes for the implied volatility by mapping the strike corresponding to the at the money cap, which as described above is set by the swap rate with the corresponding maturity, with the corresponding volatility. This does not require that the caplets contained therein are at-the-money. Finally, the levels for the forward rates are found to be 465 bps for the three months tenored three months forward, 460.8

bps for the three months tenored six months forward and 466 bps for the six months tenored six months forward rate.

With regards to swaptions the availability of data is more scarce. One can easily obtain ATM volatilities for the entire swaption matrix of tenors and forwards but skew data for out-of-the-money and in-the-money options is restricted to dealers in the market. Given that the properties are exactly the same as in the caplets case we will not be conducting calibrations on swaptions quotes. We can now proceed to the actual calibration.

5.2 Caplets Calibration and comparison with SABR

We will conduct a total of seven calibrations for each model and given that the least squares methodology described above is readily available in Mathematica we will be using the FindFit[] function implemented in such software packages. In order to see whether the incorporation of a correlation factor enhances the fit we will conduct a separate calibration of our model with a correlation factor by using the equation corresponding to (3.32) obtained by Lewis in [LEWI00]:

$$\sigma(T,K)_{\text{imp}} = \sigma_{\text{ATM}} - \frac{\rho\left(\varepsilon\sigma_{\text{ATM}}+\lambda\right)}{2\sqrt{\sigma_{\text{ATM}}}}X + \left(\left(\frac{1}{12}-\frac{11}{48}\rho^2\right)\frac{\left(\varepsilon\sigma_{\text{ATM}}+\lambda\right)^2}{\sigma_{\text{ATM}}^2}+ \right. \tag{5.6}$$

$$\left. +\frac{\rho^2\varepsilon\left(\varepsilon\sigma_{\text{ATM}}+\lambda\right)}{6\sigma_{\text{ATM}}}\right)X^2 + O(\varepsilon^3, X^3)$$

The first case we consider is the implied volatility structure of the first caplet in the one year cap for which the forward rate, as described above, is found to be 465bps. The SABR fit with zero correlation is very poor compared to our model:

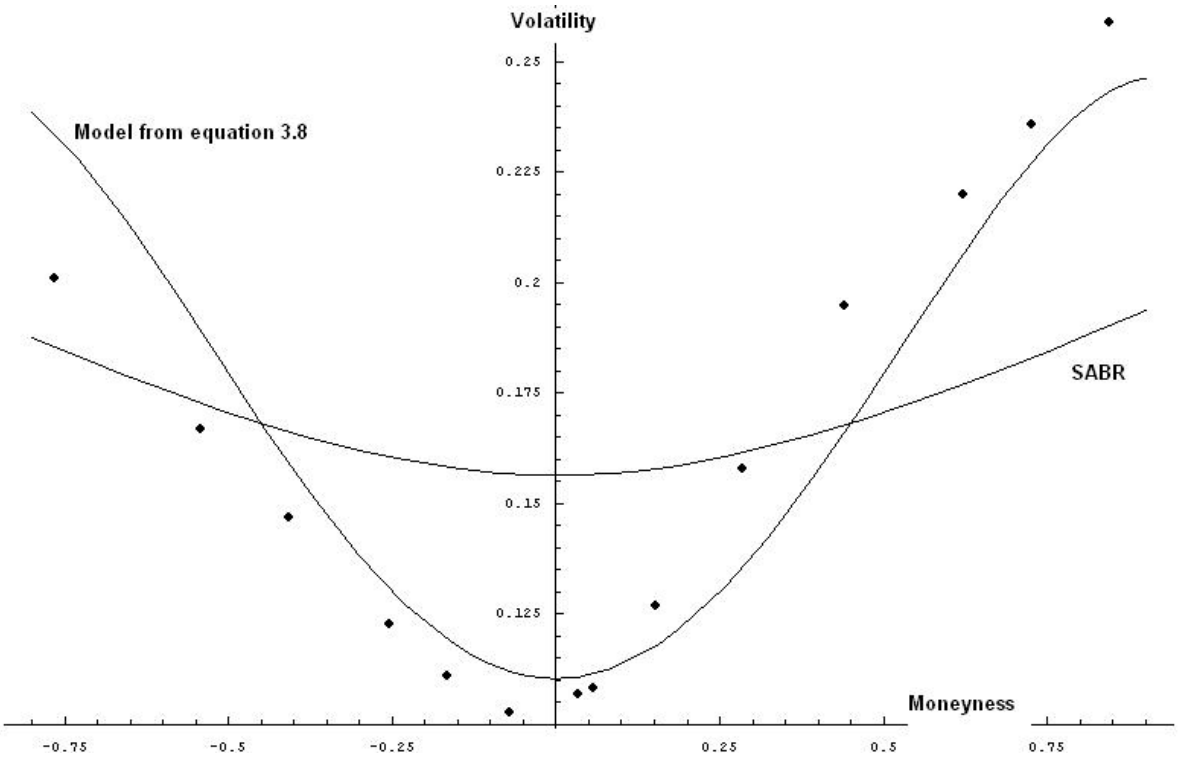


Figure 5.1: Fit to three months caplets on the three months tenored three months forward with smile taken from 1 year caps and no correlation coefficient.

The optimal parameters obtained are $\{V(0) = \sigma_{\text{ATM}} = 0.15839,\ \varepsilon = 0.2171\}$ for the SABR model and $\{\sigma_{\text{ATM}} = 0.1104, \varepsilon = 2.2968, \lambda = -0.03599\}$ for our model. Notice that althought it may not seem perfect the bid-ask spreads on implied volatility are typically of at least two percentage points hence we are very close to that range. The fit is thus acceptable and we can obtain a further improvement when including the correlation coefficient:

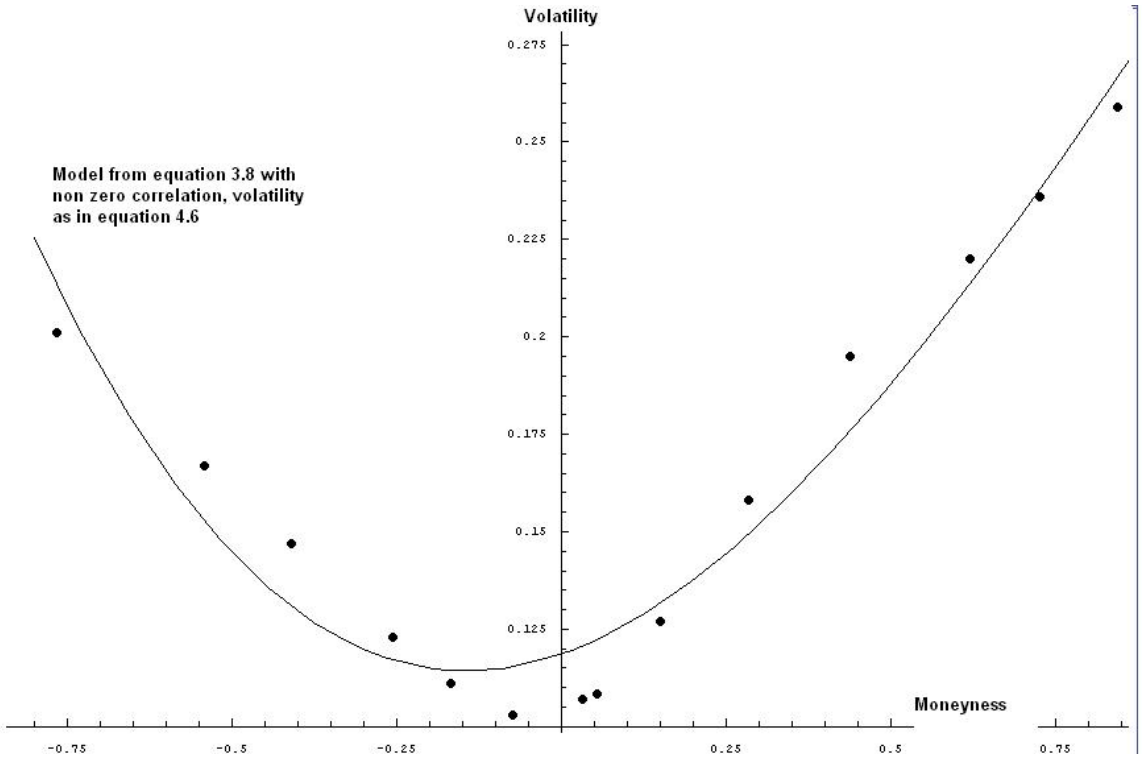


Figure 5.2: Fit to three months caplets on the three months tenored three months forward with smile taken from 1 year caps with correlation coefficient, obtained using equation (5.6).

In this case the coefficients are chosen so that the resulting polynomial from (5.6) results in $0.118655 + 0.0587516\,X + 0.190675\,X^2 - 0.062072\,X^3$. The fit is easily seen to be radically improved. As Hagan et. al show in their paper [HAGAN02] SABR is also able to give a very good fit when the correlation coefficient is used.

The structure of the smile observed in interest rates market ceases to be as symmetric with the increasing in maturity of the cap from which the caplets are extracted. The same is true for swaptions. We can already see this in the smile structure obtained from the 18 months cap, the fit of the three months caplet on the three months tenored three months forward rate is the following:

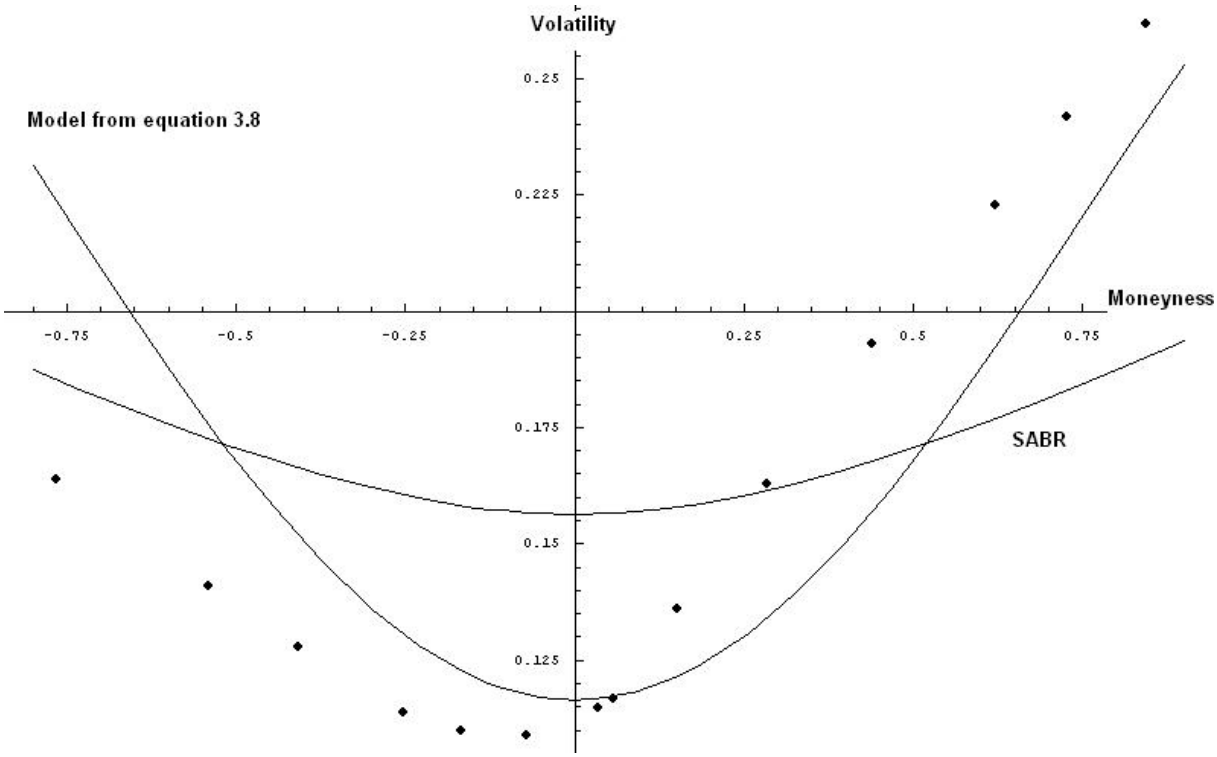


Figure 5.3: Fit to three months caplets on the three months tenored three months forward with smile taken from 18 months caps and no correlation coefficient.

This translates to parameters values of $\sigma = 0.11654$, $\varepsilon = 1.932$ and $\lambda = -0.03511$ for our model and $\sigma = 0.15623$ and $\varepsilon = 0.2480$ for SABR. Again the fit is not particularly impressive but given the bid asks spreads it is still acceptable. Notice, that the volatility shifting parameter is not strongly affected by the change in maturity of the cap containing the caplet. As before, the fit improves dramatically with a correlation coefficient:

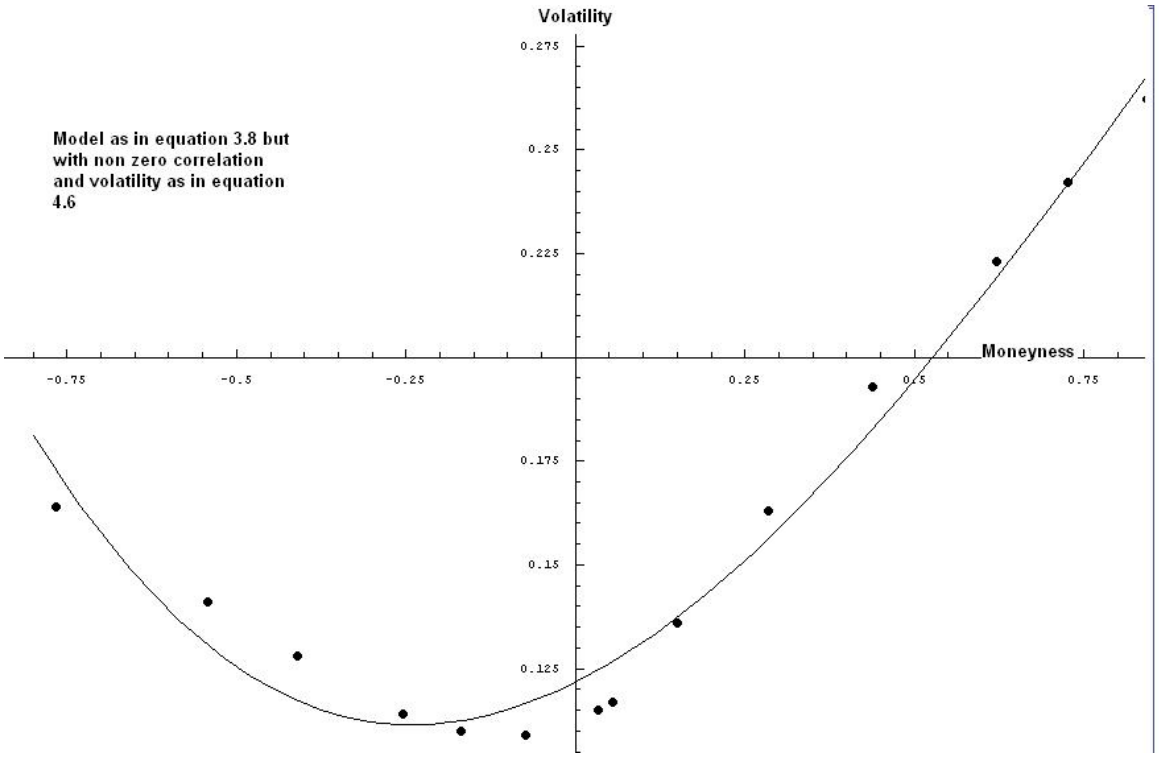


Figure 5.4: Fit to three months caplets on the three months tenored three months forward with smile taken from 18 months caps with correlation coefficient, obtained using equation (5.6).

The polynomial corresponding to the implied volatility of equation (5.6) which results in above plot is $0.1217 + 0.0825\,X + 0.1531X^2 - 0.0539X^3$.

Moving on to the first caplet of the two years cap we notice that the so called call skew (term used to indicate an accentuated volatility towards in-the-money calls) become more noticeable. The result is similar to the above and the SABR implied volatility is not able to fit the market as well when correlation is not taken into account:

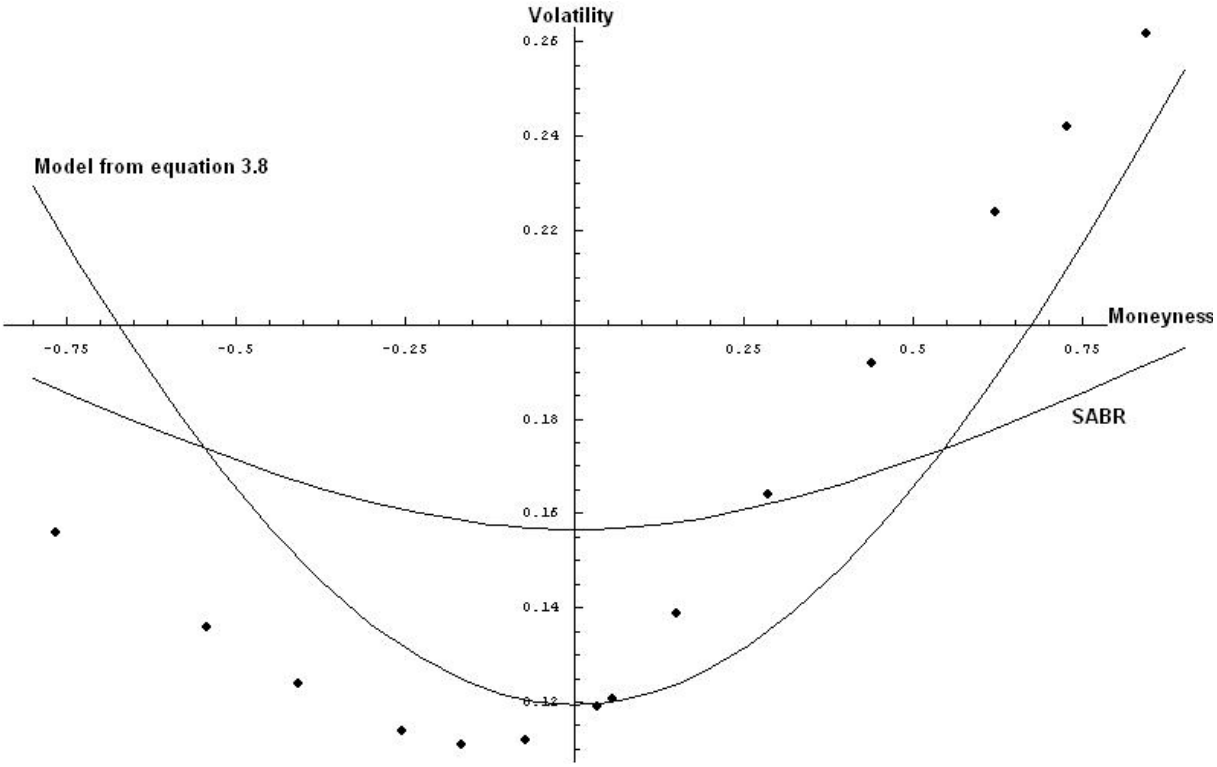


Figure 5.5: Fit to three months caplets on the three months tenored three months forward with smile taken from 2 years caps and no correlation coefficient.

The resulting parameters for this case are $\sigma = 0.11949$, $\varepsilon = 1.83027$ and $\lambda = -0.03761$ for our model whereas $\sigma = 0.15637$ and $\varepsilon = 0.25409$ for SABR. We note that the parameters do not change significantly with the varying of maturity even for this case. The fit begins to worsen for out-of-the-money calls as the smile becomes less symmetric to imply a returns distribution skewed to the left (in the sense discussed previously in section 2.2.1). The errors in fit though are most pronounced for the extreme values of moneyness, and we may think, from our discussion in section 3.4, that this is because the approximation of the implied volatility obtained in equation (3.32) is not very accurate for moneyness away from the ∓ 0.5 range. The correlation case is still not faced with such difficulty though even for such values of moneyness and this argues against this justification:

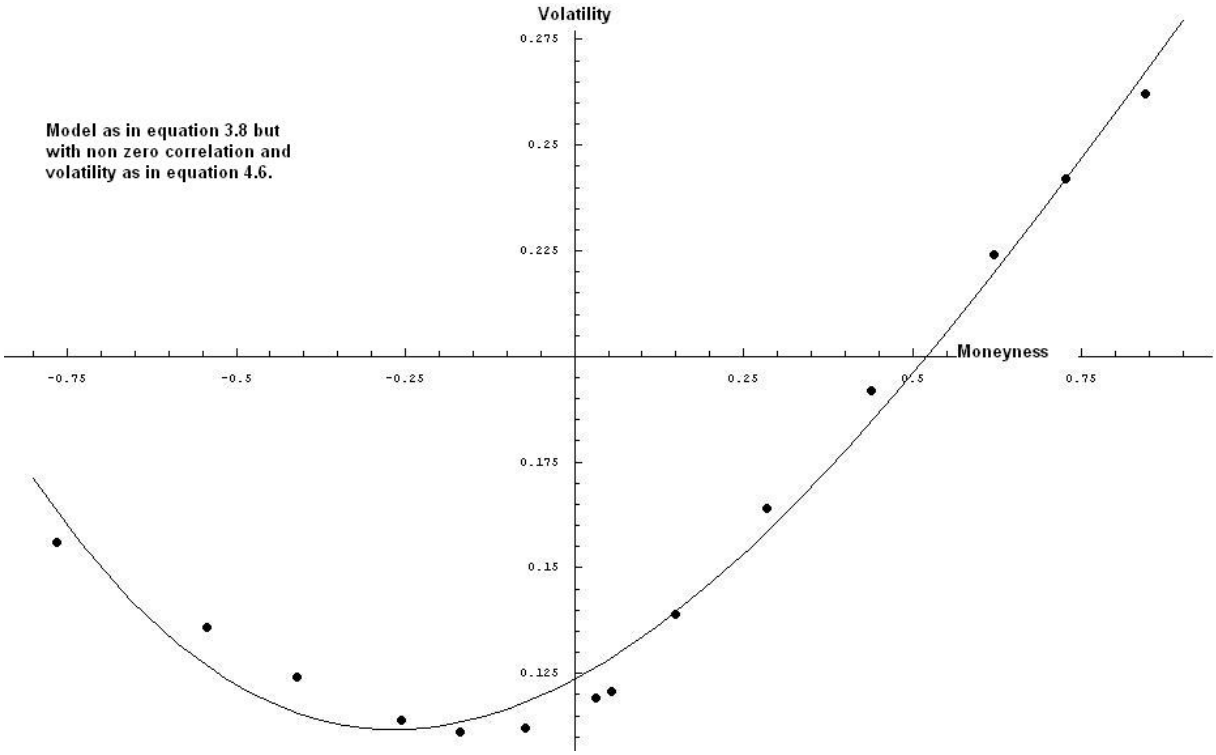


Figure 5.6: Fit to three months caplets on the three months tenored three months forward with smile taken from two years caps with correlation coefficient, obtained using equation (5.6).

In this case the resulting polynomial is $0.12370 \ + 0.08611\, X + 0.14159\, X^2 - 0.0500 X^3$. Note that the parameters of the correlated case are also little changed for the different maturities. In all the cases to come the lack of correlation becomes a much more seirous drawback, especially for out-the-money calls.

The three years cap gives the following result:

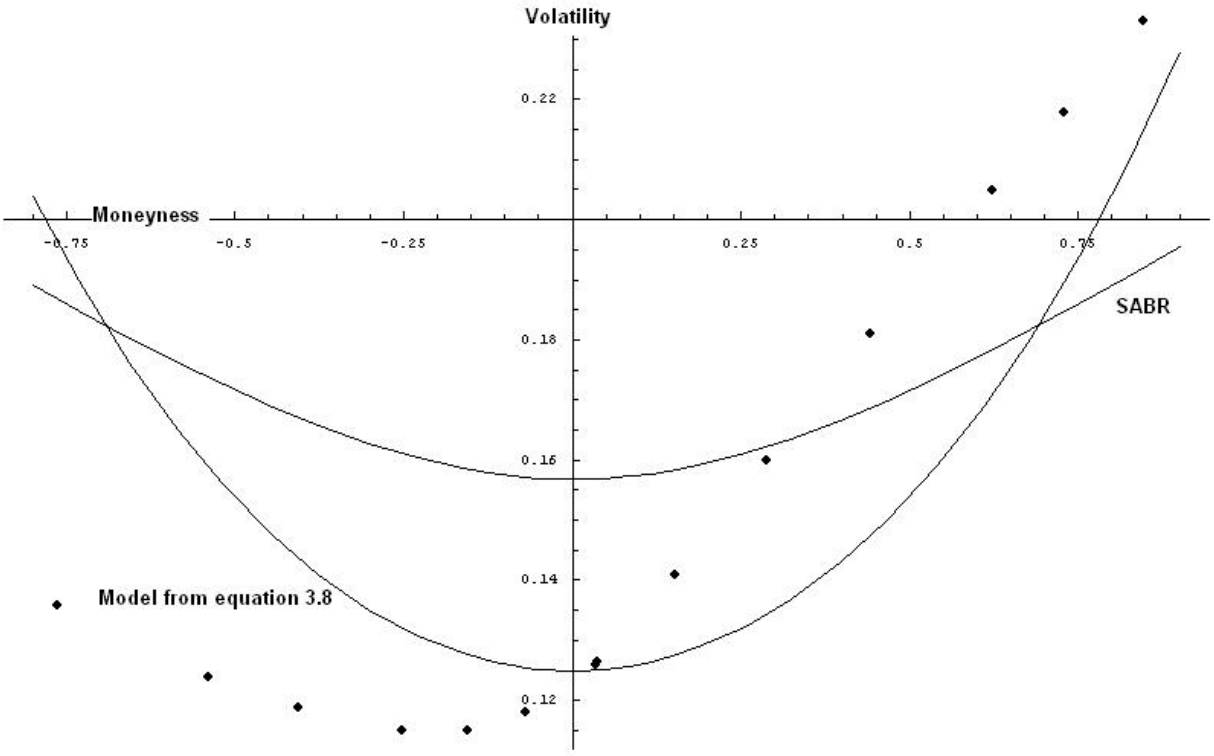


Figure 5.7: Fit to six months caplets on the six months tenored six months forward with smile taken from 3 years caps and no correlation coefficient .

The parameters obtained from such fit are $\sigma = 0.12494$, $\varepsilon = 1.5395$ and $\lambda = -0.04866$ for our model and $\{\sigma = 0.1488, \varepsilon = 0.24263\}$ for SABR. One can also notice that the fit of the two models is becoming of a similar shape. The correlated case is still able to give a very good fit:

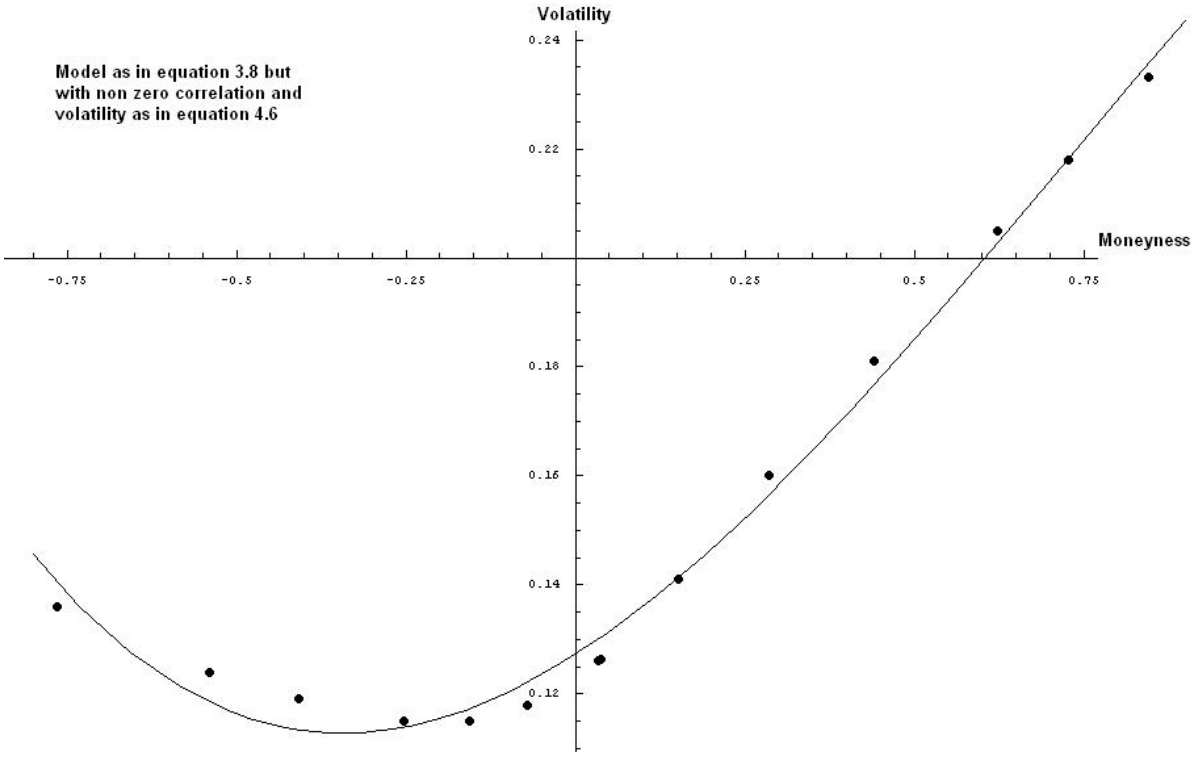


Figure 5.8: Fit to six months caplets on the six months tenored six months forward with smile taken from three years caps with correlation coefficient, obtained using equation (5.6).

The polynomial in this case is $0.127264 \ + 0.079238\, X + 0.093702 X^2 - 0.042492\, X^3$.

The next case is the smile obtained from six years caps:

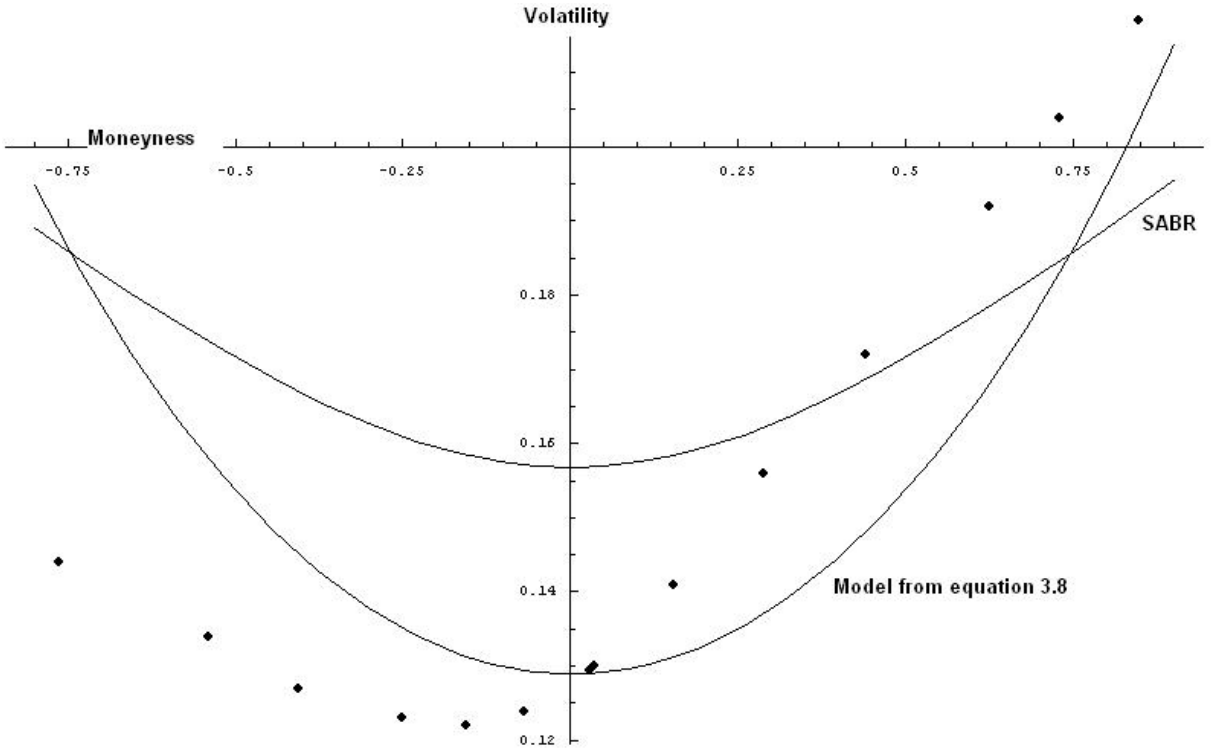


Figure 5.9: Fit to six months caplets on the six months tenored six months forward with smile taken from 6 years caps and no correlation coefficient .

The parameter values obtain for this case are $\sigma = 0.12897$, $\varepsilon = 1.4078$, $\lambda = -0.043076$ for our model and $\{\sigma = 0.14772,\ \varepsilon = 0.22128\}$ for SABR. Notice that the two fits continue to get closer to each other, but the SABR's fit is nevertheless far less satisfactory. The correlated case remains unchanged:

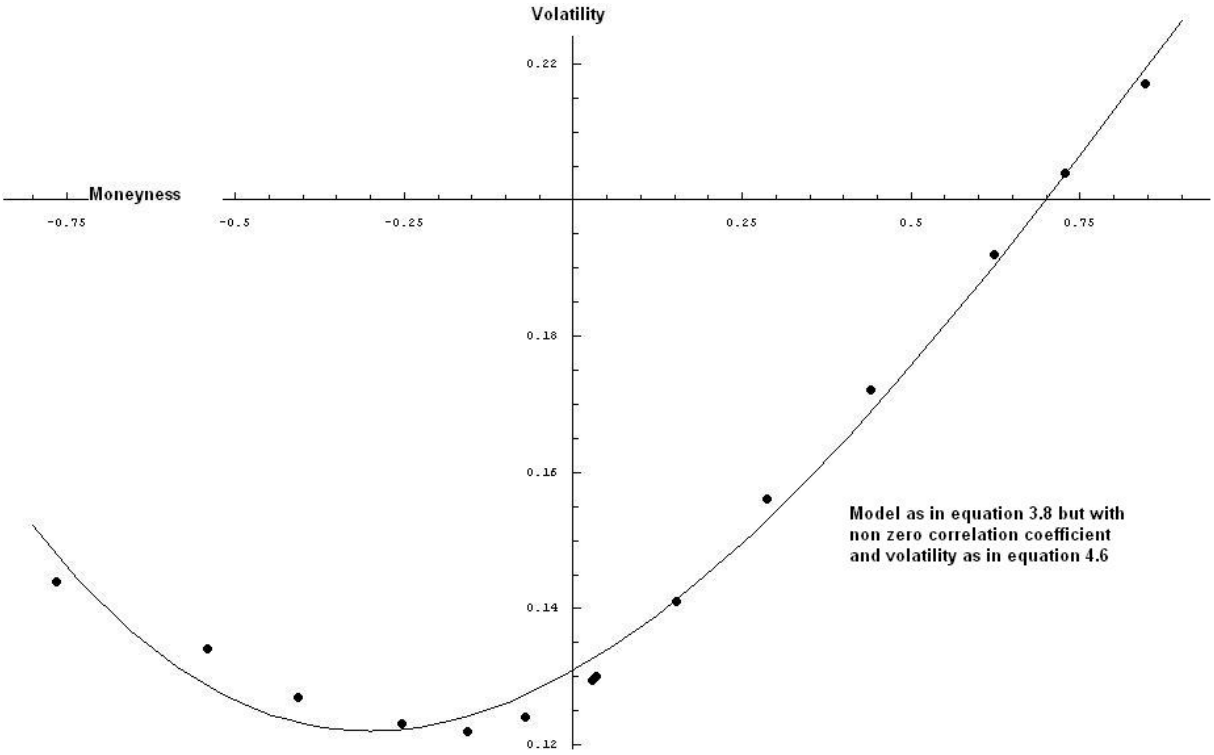


Figure 5.10: Fit to six months caplets on the six months tenored six months forward with smile taken from six years caps with correlation coefficient, obtained using equation (5.6).

Equation (5.6) would be $0.1309 + 0.0564X + 0.0809X^2 - 0.0285X^3$ for the above fit.

The twelve year case diminishes the concavity of the fit:

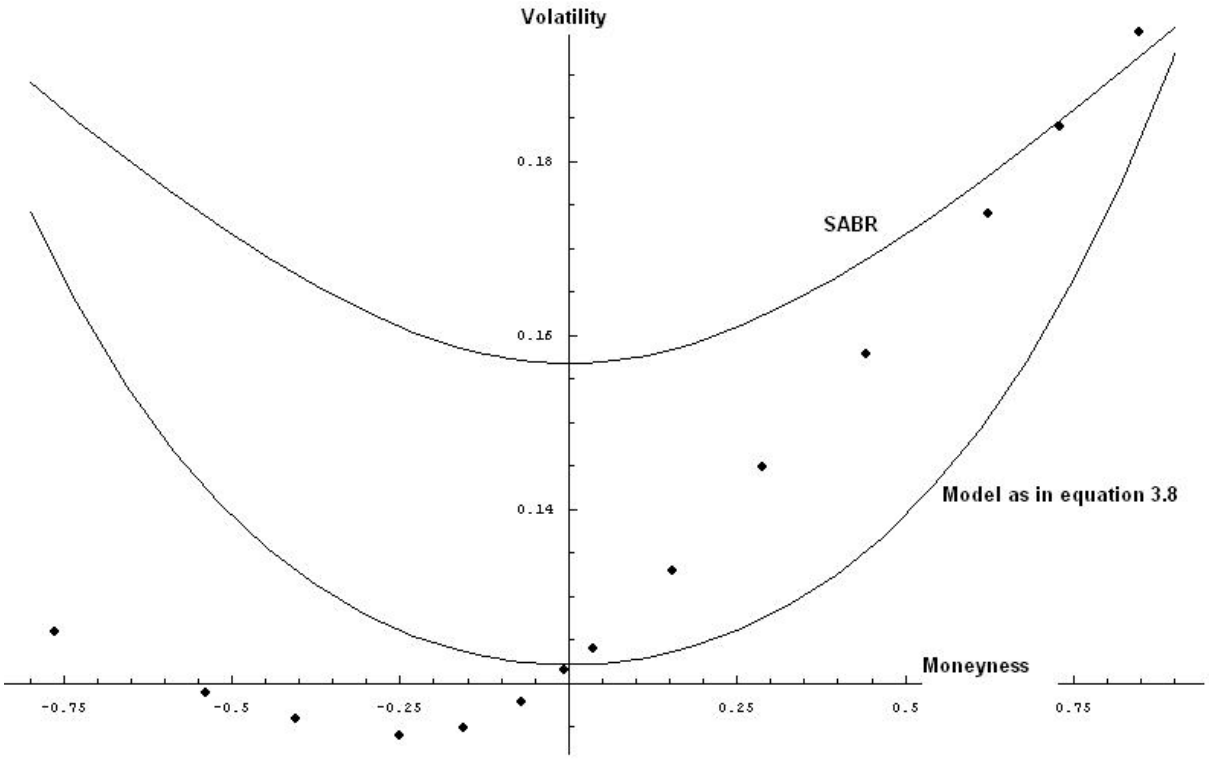


Figure 5.11: Fit to six months caplets on the six months tenored six months forward with smile taken from 12 years caps and no correlation coefficient.

With parameters $\{\sigma = 0.12217,\ \varepsilon = 1.4525,\ \lambda = -0.073519\}$ for our model while for SABR we have $\{\sigma = 0.13574,\ \varepsilon = 0.20625\}$. The correlated case on the other hand yields:

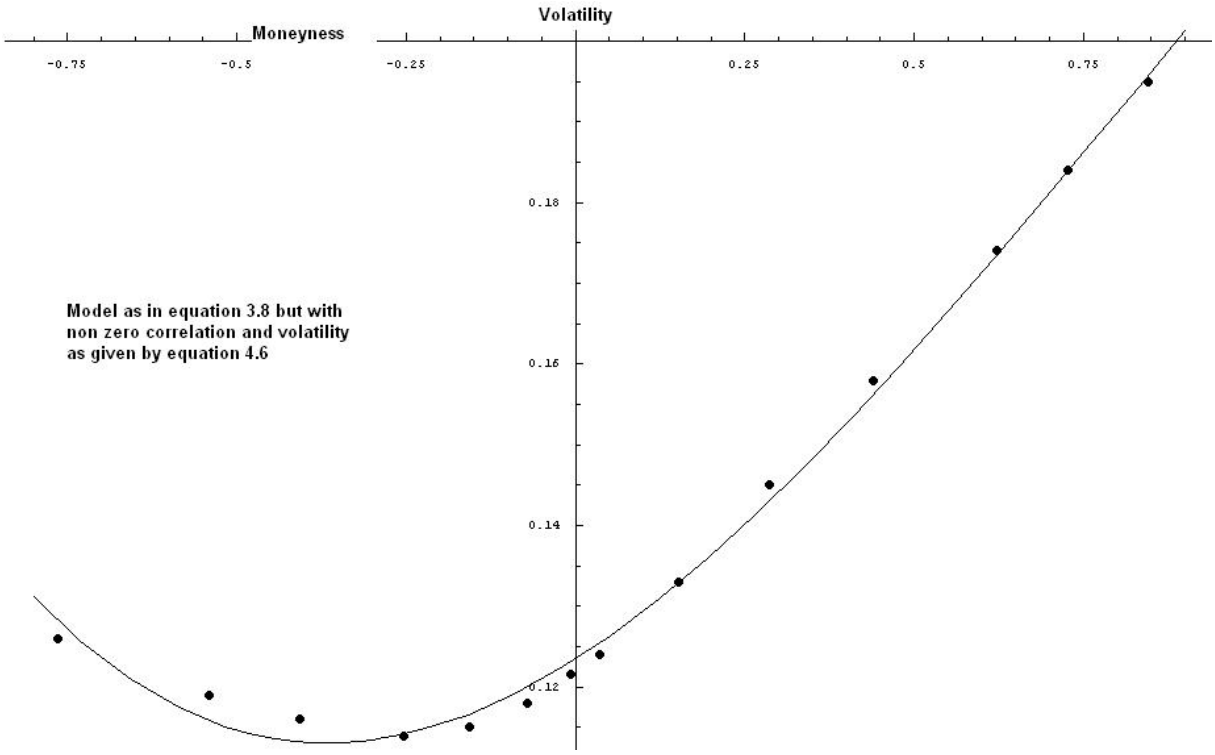


Figure 5.12: Fit to six months caplets on the six months tenored six months forward with smile taken from twelve years caps with correlation coefficient, obtained using equation (5.6).

with the fit corresponding to $0.12349 \ + 0.05326\, X + 0.05903 X^2 - 0.02458 X^3$ from equation (5.6).

Finally the smile of the 25 years cap is considered:

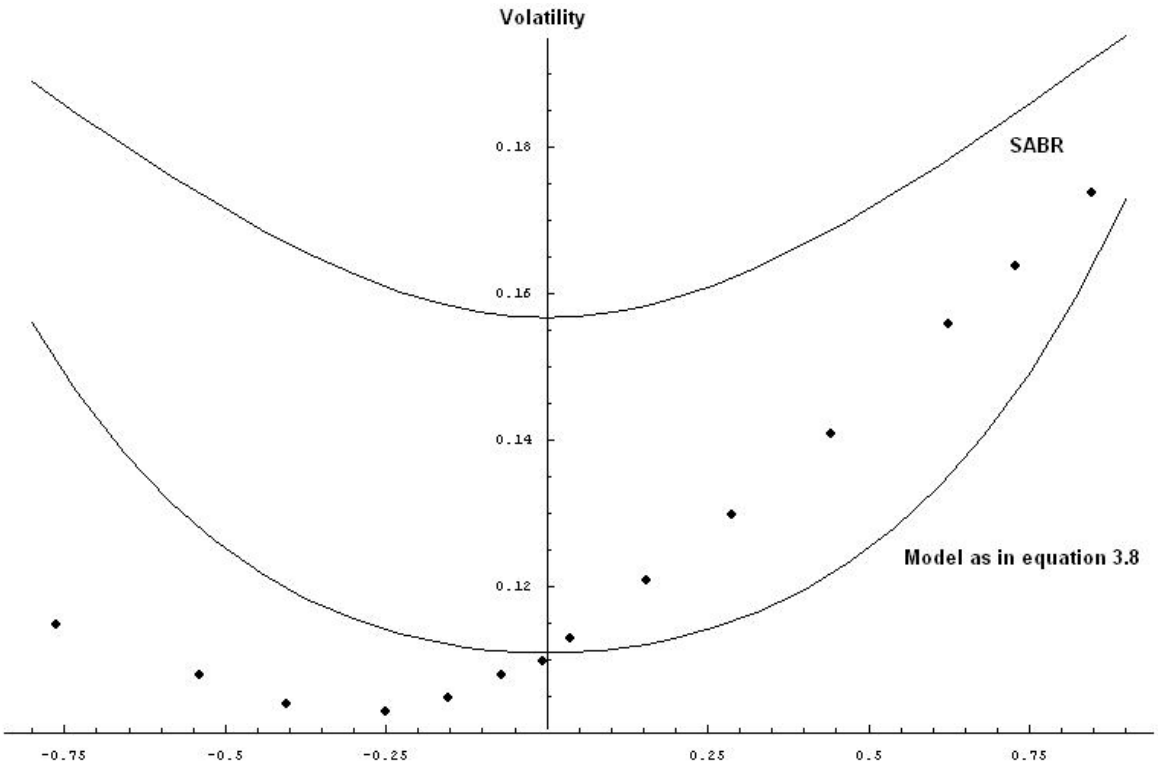


Figure 5.13: Fit to six months caplets on the six months tenored six months forward with smile taken from 25 years caps and no correlation coefficient.

The fit translates into the parameters $\{\sigma = 0.11097,\ \varepsilon = 1.4836,\ \lambda = -0.08036\}$ for our model and $\{\sigma = 0.1223, \varepsilon = 0.18359\}$ for the SABR model.

Finally, the correlated case (which still gives an excellent fit):

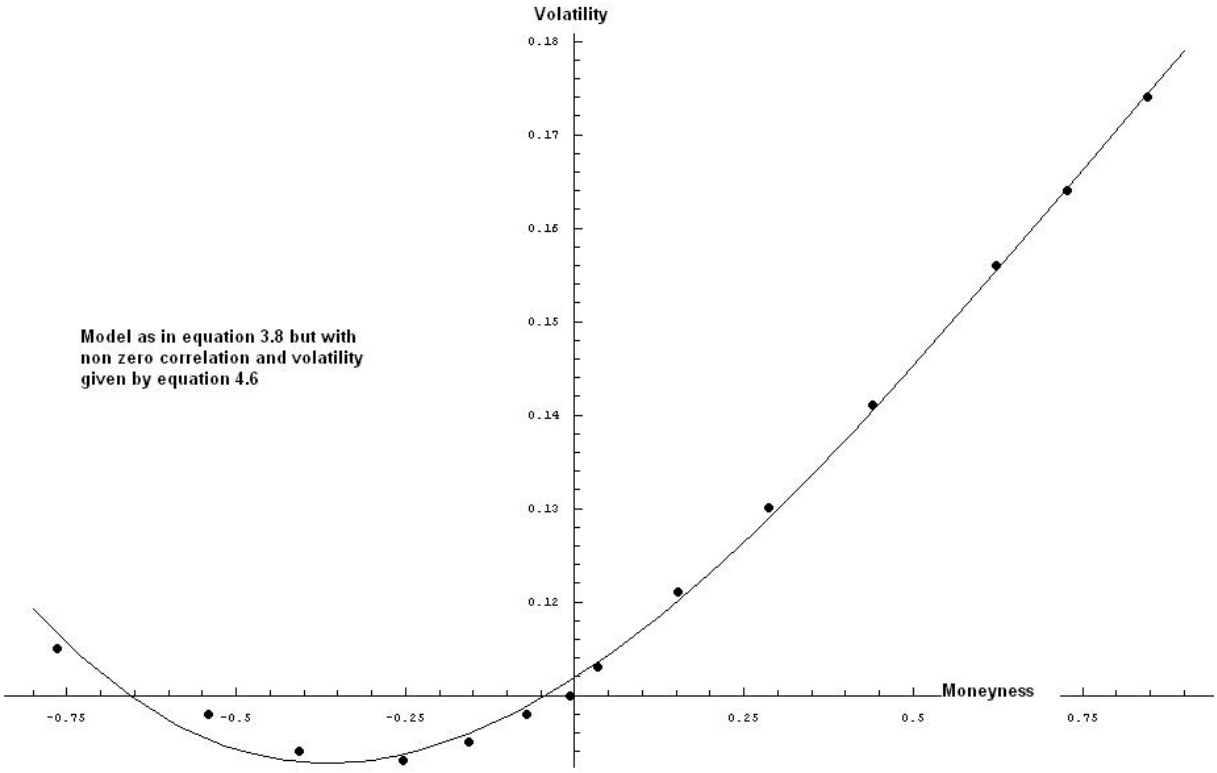


Figure 5.14: Fit to six months caplets on the six months tenored six months forward with smile taken from 25 years caps with correlation coefficient, obtained using equation (5.6).

The polynomial is instead $0.11185 \ + 0.046784\, X + 0.051597 X^2 - 0.02305 X^3$.

It is interesting to note that except for the case of the six years cap in which two of the data points of implied volatility nearly coincide the parameters give rise to interesting patterns when plotted against the maturity of the underlying smile. For instance, the ε fits give rise to the following pattern:

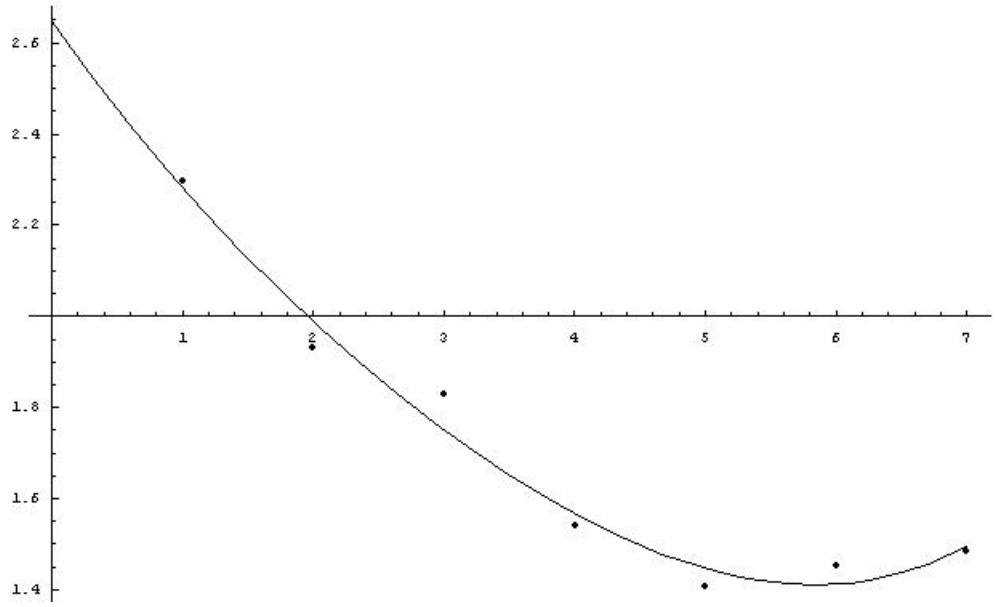


Figure 5.15: ε values from the calibrations above along with fourth order polynomial fit.

The λ fit is more sensitive to the data point in the 6 years cap although a pattern is also observable. This points to the fact that our model can be specified with the same parameters across forward rates so long as the constant coefficients of equation (3.8) are turned into time dependent functions.

This section has demonstrated that although our model can be significantly enhanced by including a correlation between the underlying and the volatility, its ability to fit the market in the case studied is definitely stronger than SABR's. The presence of a closed form solution opens the door for much more precise studies on the asymptotics of the model with the varying of the parameters and variables. Furthermore, knowing the exact form of the solution can, if needed, permit one to approximate the solution with forms that are faster in computation without the danger of making assumptions about its behavior as in the derivation of the SABR model. Now as the above calibrations show the fit for many cases is within the market uncertainty defined by the bid ask spread in volatility. This in turn results in an even smaller difference in fit when it is translated in premium. The issue of fitting the market is important but whereas many models are able to do this,

the most important and discerning factor when considering a model is the performance of its hedging parameters, the so called "Greeks". We will thus move on to conducting an analysis of the properties of the Greeks given by our model in section 4.4 as compared to those of the standard Black type model for the parameters obtained in the calibration above.

5.3 Summary

In this chapter we have carried out an innovative type of calibration that does not require extensive computation. The methodology does not entail neither filtering time series techniques that do not necessarily reproduce the observed option prices nor computation of prices and a minimization of the squared error of prices fit. The least squares approach is used but only to find optimal parameters of a simple polynomial for a model and a well defined function for SABR. The optimal parameters are generated in fractions of a second using fitting functions readily available in the Mathematica software package. While we see that our model yields a far superior fit to the one of the SABR model when correlation is suppressed both frameworks do not provide a satisfactory fit when compared to the correlated case. This confirms our initial considerations when we layed out our model that the correlation factor would be very important and realistic for non extremely short maturities. Indeed, the longer the maturity the more the misfit becomes dramatic. Nevertheless, our objective is to solve the simpler non correlated case and we leave the more demanding task for future research. The smiles for short maturity options are generally very symmetric and the correlation between the volatility and the underlying is not decisive so for such cases our model can already provide a reasonable fit especially when taking into account bid ask spreads. Finally, our analysis also provides insight on the time dependence of the parameters as we are able to find a well defined pattern with the varying of maturity.

6 Comparative Analysis with Black's Model

6.1 Introduction

While in the previous chapters we have dedicated ourselves more to proposing a model which could stand up to the challenge of replacing the SABR model, we now diverge slightly from this task and conduct a different type of analysis. The current literature on stochastic volatility models tends to always focus on conducting a study similar to what we have done so far, that is to propose a model with features that are better than the current ones and attempt solving the pricing problem. Day to day, on a trading desk, people use sophisticated models on an ad hoc basis depending on the type of product and market conditions. The Black Scholes framework is still most often extensively employed as a quick indicator to the choice of action. It is thus more important to know what general features a model generates when compared to the Black Scholes framework with which traders typically develop their intuition rather than comparing a sophisticated model like ours with another with similar sophistication such as the SABR one. Indeed, when one sets up an options trading strategy the typical question, besides the arbitrage free price of the options, is how the prices at which one has invested would change when the markets change and time passes. We thus dedicate this chapter to see how the sensitivities of the options prices derived by our model compare to those of the equivalent Black Scholes framework. Furthermore, the Greeks obtained by the Black Scholes framework, while possessing many flaws, are generally accepted as a robust approximation, especially those concerned with changes in the underlying rather than volatility. If our model were to produce vast divergences and very different properties from the Greeks generated by Black Scholes model we should be concerned with the correctness of our analysis. Comparing our Greeks to those of the Black Scholes framework thus provides with a sound test for the validity of our pricing solution and the Greeks parameters derived from it.

6.2 Prices Comparison

The first question that we would want to tackle is the comparison of the prices generated by our model with those of the classical Black Model. The general shape remains the same as far as the underlying is concerned:

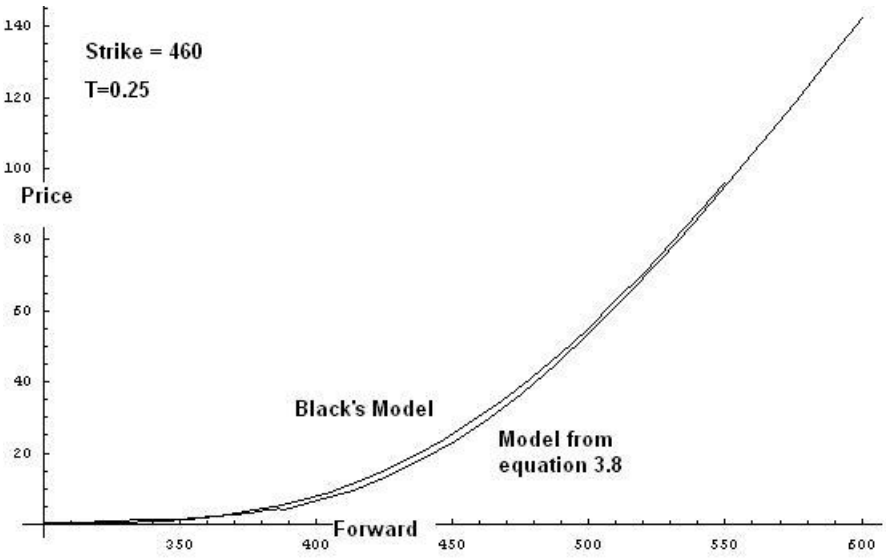


Figure 6.1: Plot of our model's call price vs. underlying in comparison with Black's model.

The general structure is similar although our model gives slightly lower prices for options close to the money but higher prices for deeply in the money and out of the money options. The parameters used are the same as above for the case $T = 0.25$, with z in equation (3.8) taken to equal the square of the Black volatility, taken to be 0.33 throughout, so that $\varepsilon = 2.296$ and $\lambda = -0.036$. The same properties can be noticed when the plot is conducted on the strike price:

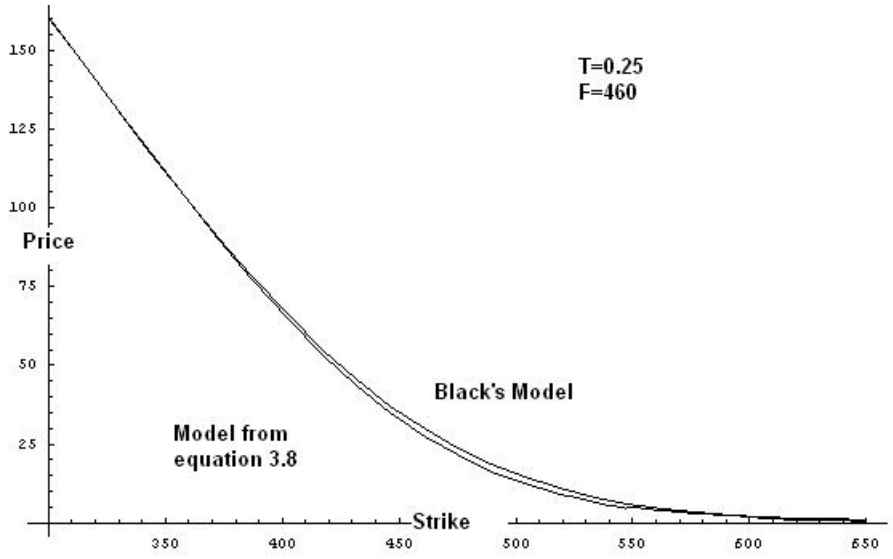


Figure 6.2: Plot of our model's call price vs. strike in comparison with Black's model.

The difference in price for at the money strike while varying the volatility is also very small:

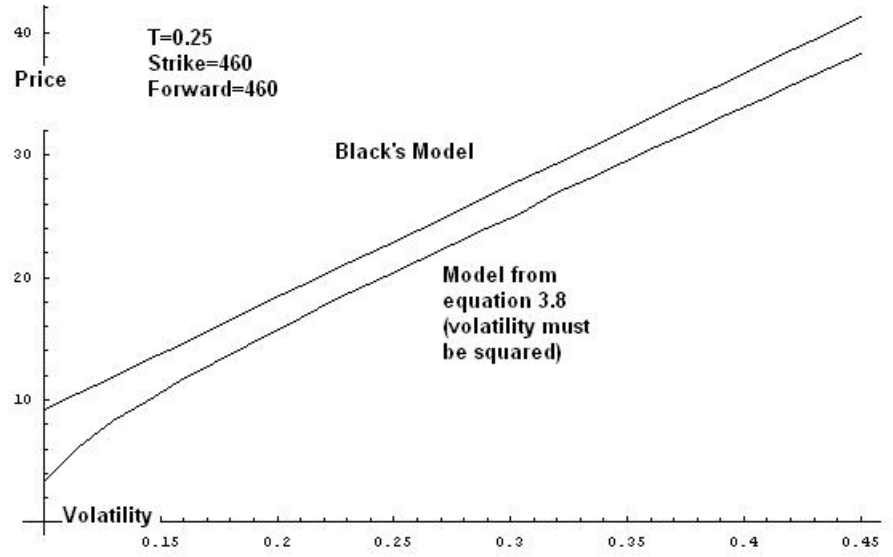


Figure 6.3: Plot of our model's call price vs. volatilty in comparison with Black's model for at the money strikes.

In the case of in the money calls the calculation in Mathematica has a slight instability

due to the algorithms in place for calculating the integrals, but, nevertheless, it is good enough to see the important properties: the price given by our model tends to be less for values of volatility above 25% but larger for smaller values as the probability of a sudden large increase is amplified by the stochasticity.

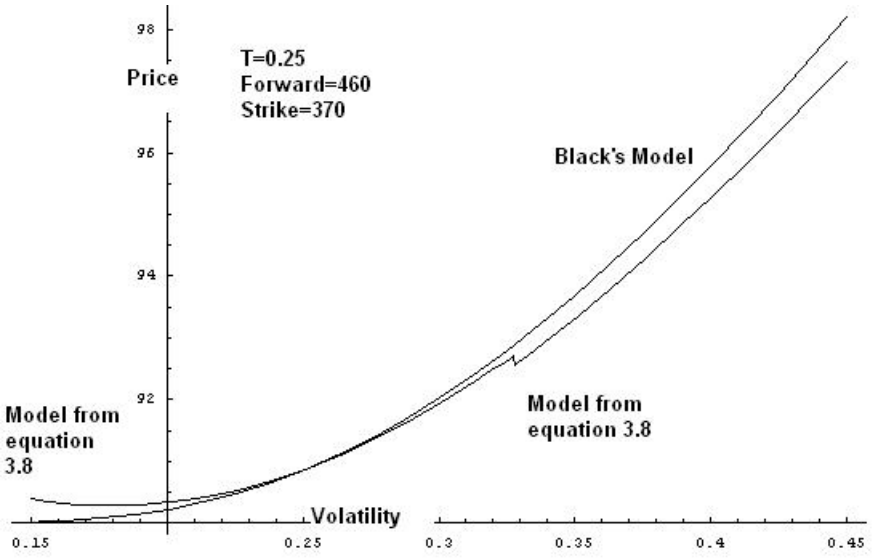


Figure 6.4: Plot of our model's call price vs. volatilty in comparison with Black's model for in the money strikes.

Exactly the same can be seen in the out of the money case:

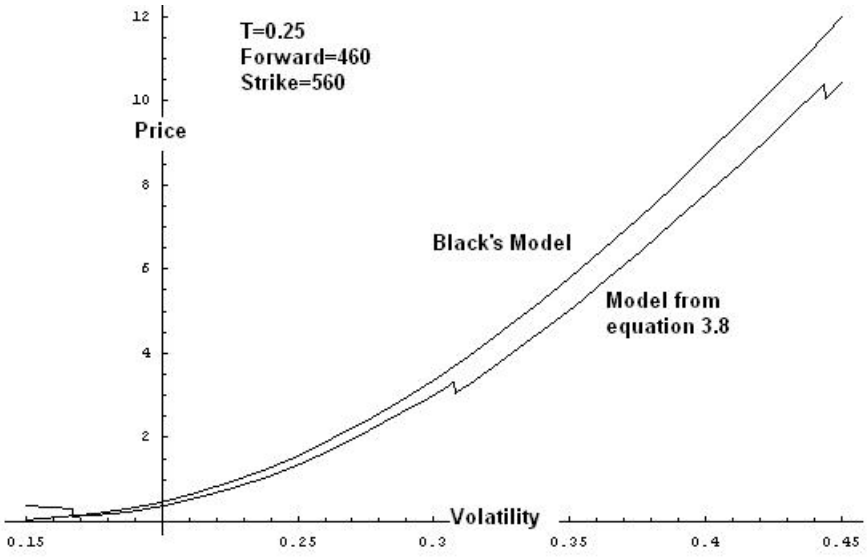


Figure 6.5: Plot of our model's call price vs. volatilty in comparison with Black's model for out of the money strikes.

From the above graphs we can conclude that overall the properties of the prices derived by our model are very similar to those of the classical Black Scholes framework. Indeed, most models have such properties, what really distiguishes a model from the rest is typically the perfomance of its hedging parameters. We will thus dedicate the final part of this thesis to the comparative analysis of our model's greeks with those of the Black model.

6.3 Delta

In comparing the greeks we begin with our model's delta, which we derived in equation (4.45). When plotted against the underlying for the same parameters as before with strike 460 we have the following pattern:

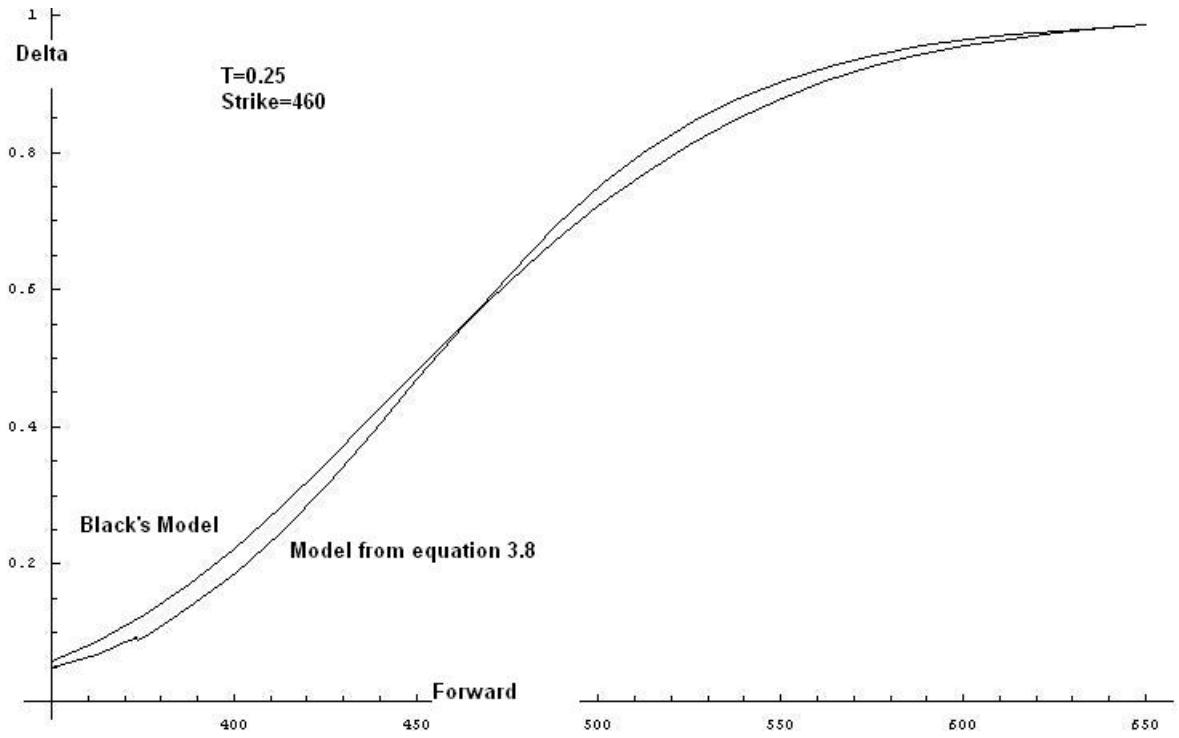


Figure 6.6: Plot of our model's delta vs. underlying in comparison with Black's model.

As can be seen, the difference in delta for nearly at the money levels is very small whereas our model generates lower levels of delta for out of the money calls and higher ones for in the money scenarios. The same property can be seen when the plot is done by varying the strike:

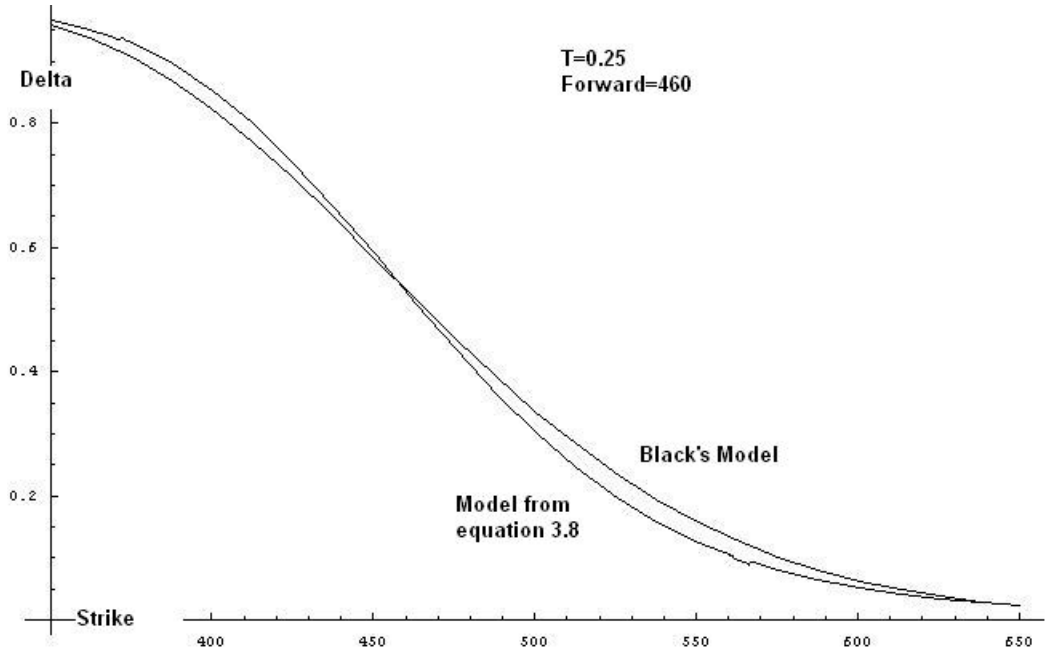


Figure 6.7: Plot of our model's delta vs. strike in comparison with Black's model.

When varying the volatility the differences depend on the moneyness. For out of the money strikes the delta generated by our model is lower than that of Black's model:

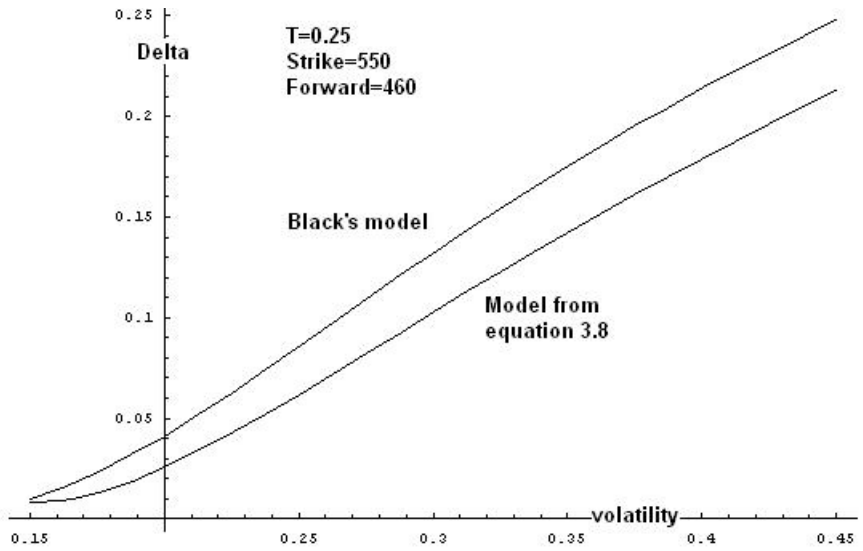


Figure 6.8: Plot of our model's delta vs. volatility in comparison with Black's model for out of the money strikes.

The at the money case generates a similar pattern:

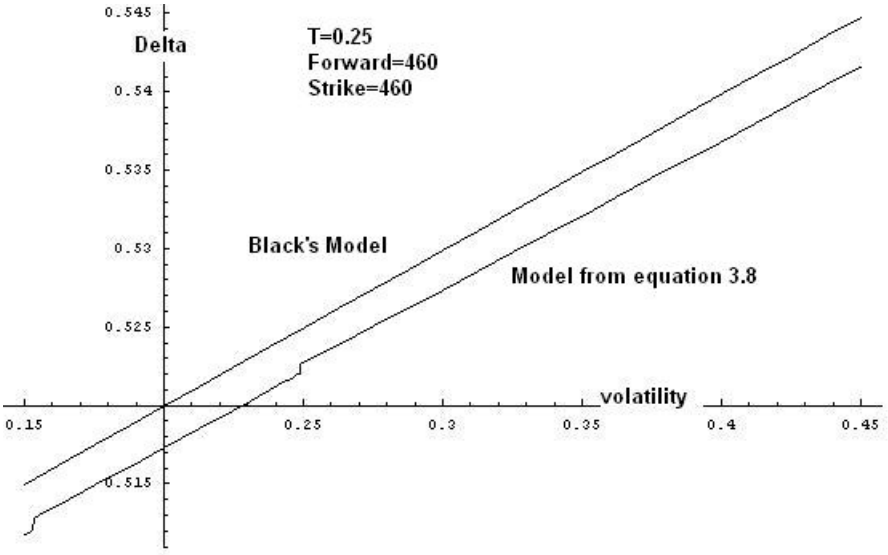


Figure 6.9: Plot of our model's delta vs. volatility in comparison with Black's model for at the money strikes.

On the other hand the in the money case gives the opposite:

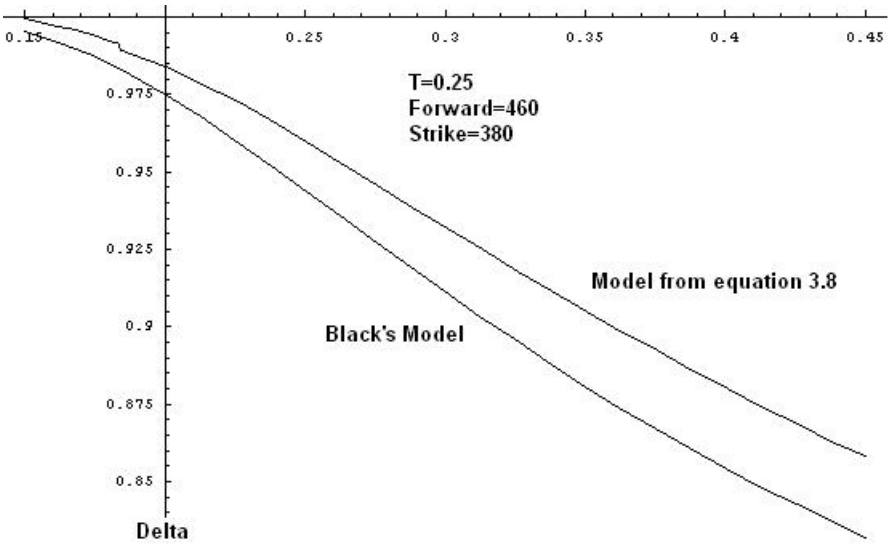


Figure 6.10: Plot of our model's delta vs. volatility in comparison with Black's model for in of the money strikes.

6.4 Gamma

The differences in Gamma are more pronounced. We begin with the case obtained by varying the forward rate when the option's strike is 460:

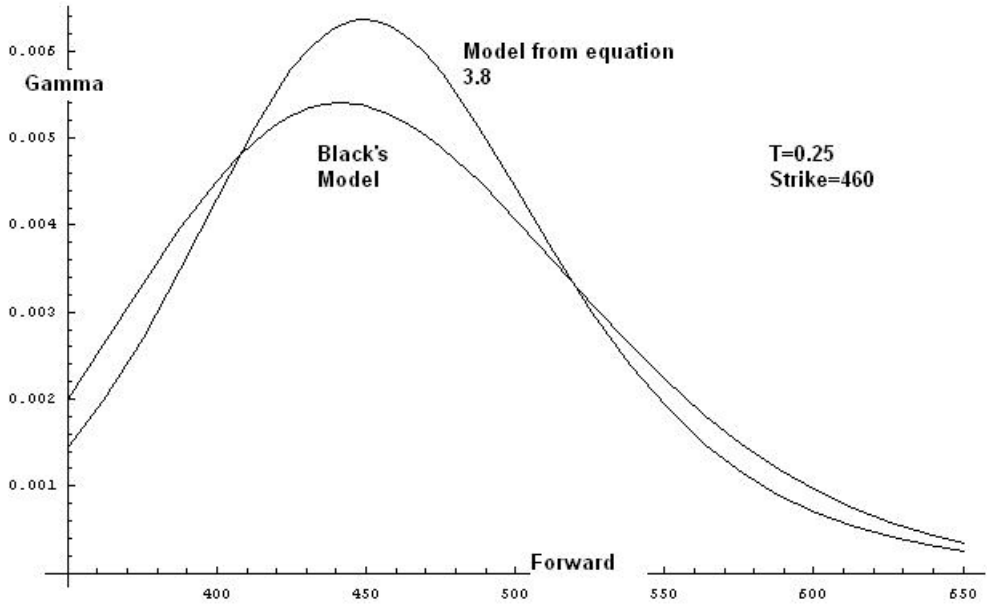


Figure 6.11: Plot of our model's Gamma vs. underlying in comparison with Black's model.

We see that the stochasticity of the volatility adds to the typical "danger of the options' dealer", the so called exposure of an option trader to very large swings in delta when the options are close to the money. On the other the gamma risk given by our model is significantly smaller when the options move deeply in and out of the money. Exactly the same property is observed when the strike is varied:

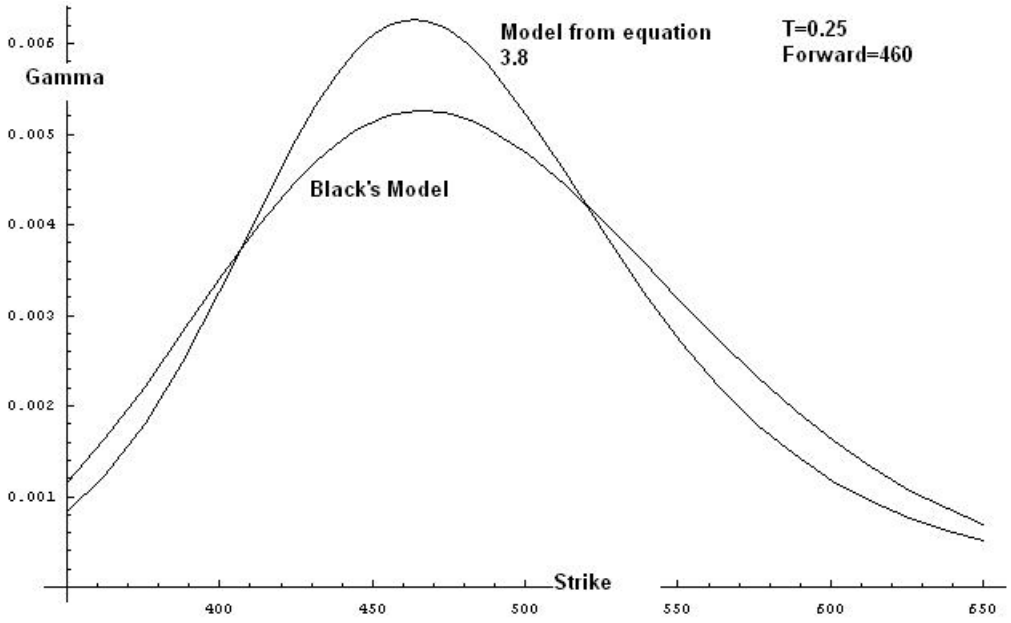


Figure 6.12: Plot of our model's Gamma vs. strike in comparison with Black's model.

The behavior of our model's Gamma under changes of the volatility depends on the moneyness (as would be expected), and does not alter the above properties. For instance the at the money Gamma remains higher for our model irrespective of the volatility although the difference becomes more significant for lower values in volatility.

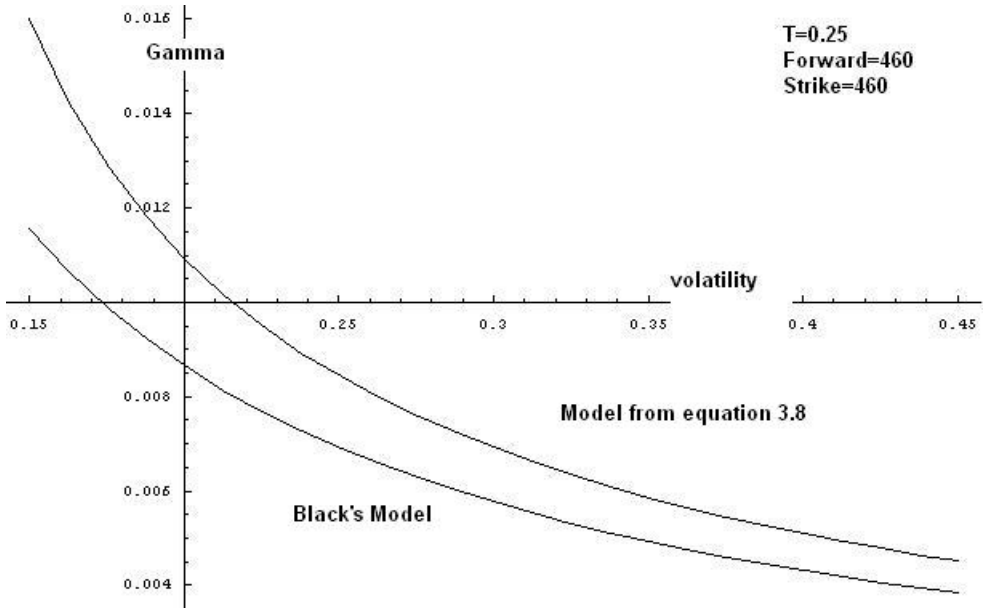


Figure 6.13: Plot of our model's Gamma vs. volatility in comparison with Black's model for at the money options.

For in the money options the difference becomes smaller for both extremes of the volatility:

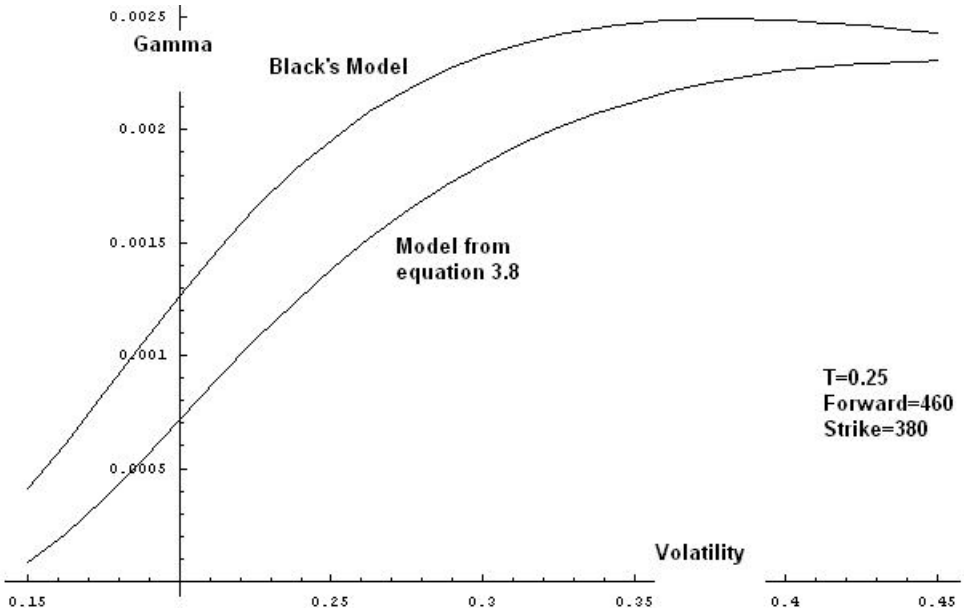


Figure 6.14: Plot of our model's Gamma vs. volatility in comparison with Black's model for in the money options.

Finally, the out of the money case resembles very much the in the money one except the convergence of the two is more accentuated for the high volatility case than the small one:

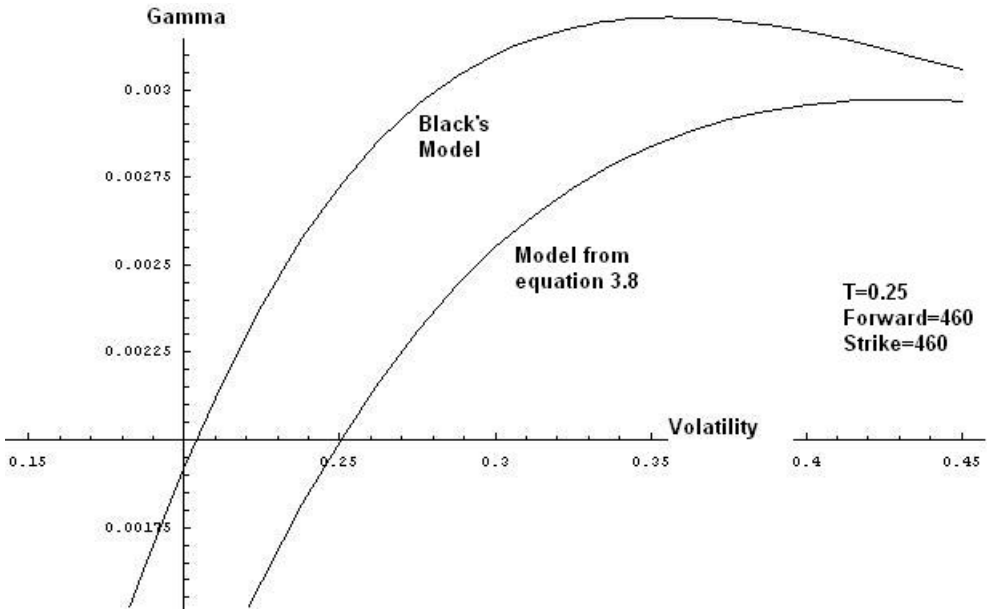


Figure 6.15: Plot of our model's Gamma vs. volatility in comparison with Black's model for out of the money options.

6.5 Theta

The Theta risk requires far more computational power given the current form of our solution. The amount of computations needed to generate plots similar to those above becomes much less for the case in which $\lambda = 0$. Given that the discrepancies are very small when λ is included (besides the parameter being very small for the calibration cases tha we have considered) we will carry out the plots with the same parameters as above expect with $\lambda = 0$. Also, we note that our plots are for the Theta of a call option and not for a caplet which would require the additional term of $-C/\tau$ in equation (4.48). When varying the strike and keep the forward rate at 460 bps we obtain a structure in which our model generates a higher theta for at the money options but lower ones for more deeply in and out of the money ones:

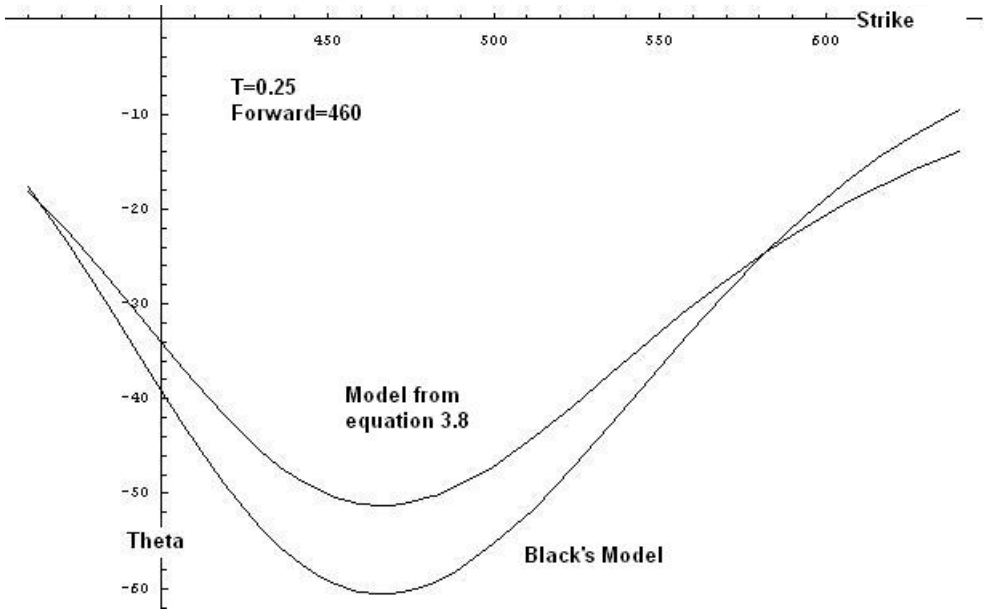


Figure 6.16: Plot of our model's Theta vs. strike in comparison with Black's model.

The structure is the same when the forward rate is varied rather than the strike:

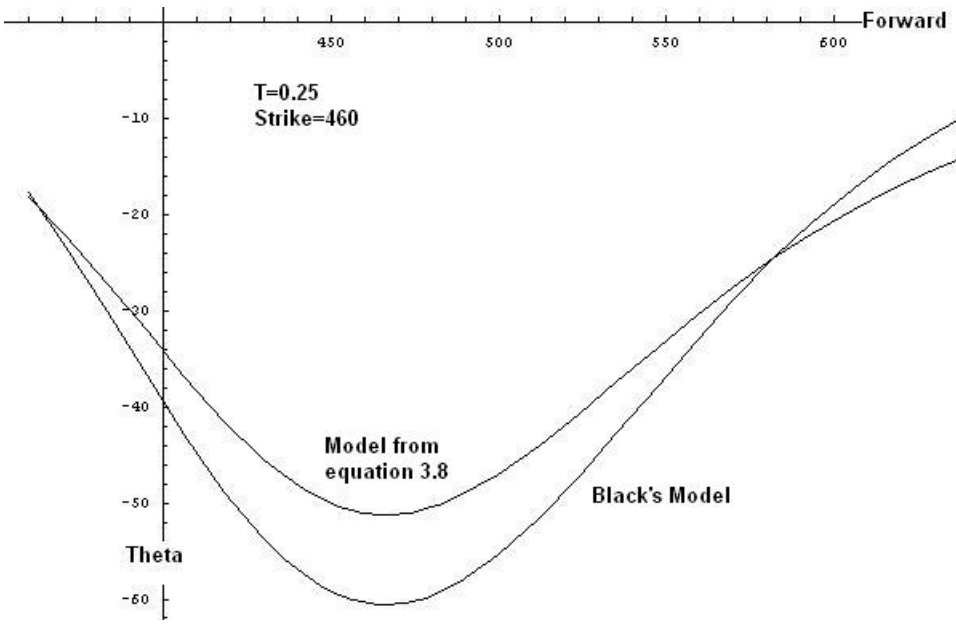


Figure 6.17: Plot of our model's Theta vs. forward in comparison with Black's model.

This is maintained when one stays within the usual levels of volatility but the difference

tends to become smaller for small values of volatility, as can be seen from the following plot for at the money options:

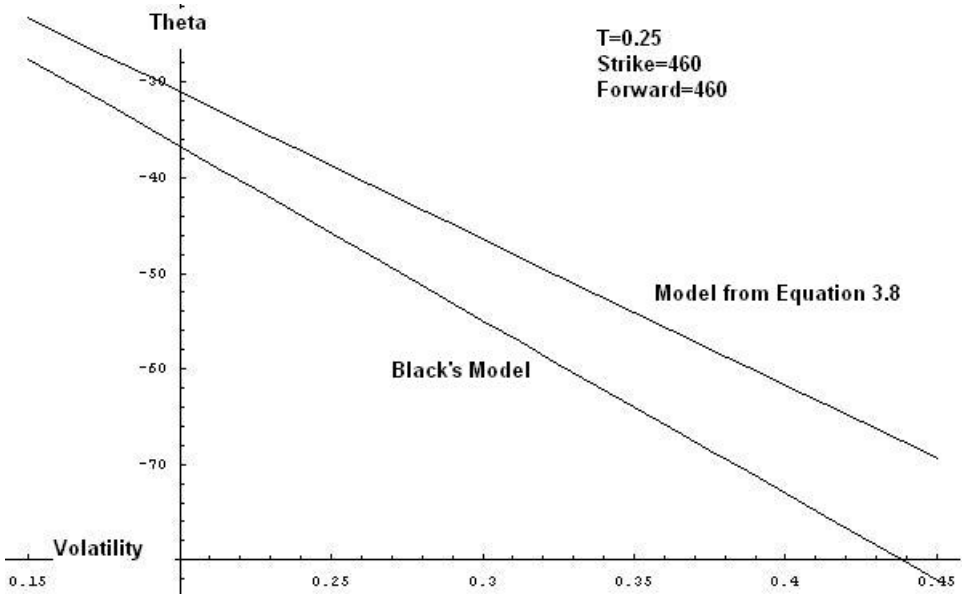


Figure 6.18: Plot of our model's Theta vs. volatility in comparison with Black's model for at the money options.

For deeply out of the money options the difference becomes smaller as volatility increases:

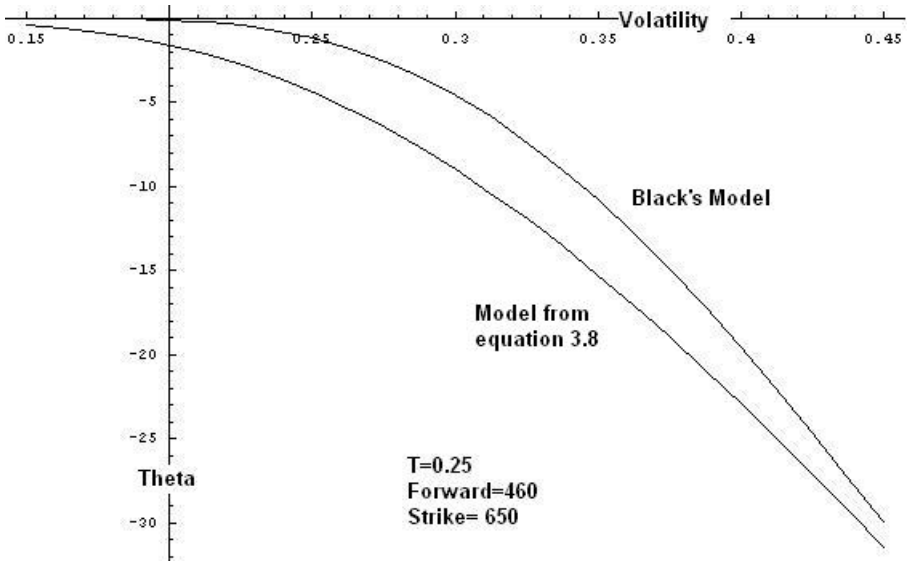


Figure 6.19: Plot of our model's Theta vs. volatility in comparison with Black's model for out the money options.

Finally, for in the money options the convergence is slower and happens for small values of volatility where the two models both give a vanishing Theta:

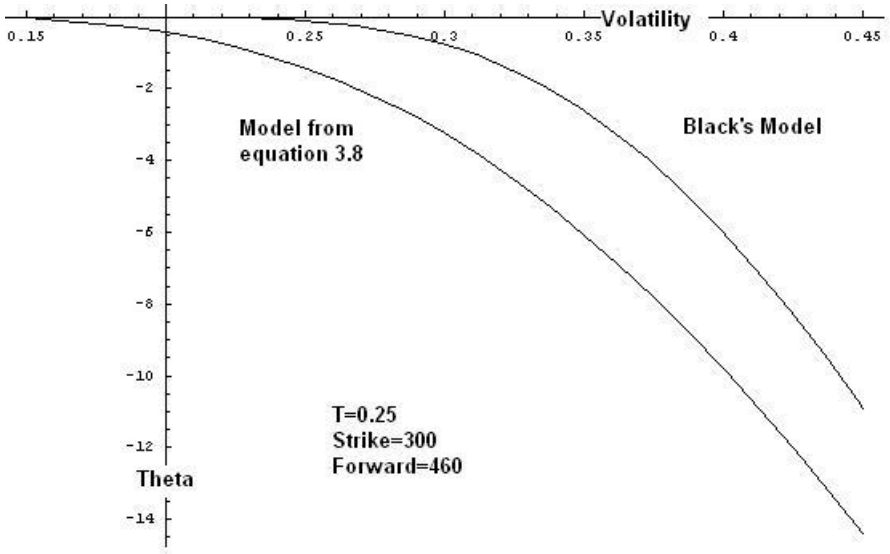


Figure 6.20: Plot of our model's Theta vs. volatility in comparison with Black's model for in the money options.

6.6 Vega

The last of the major greeks is the vega, the sensitivity of the price of an option with respect to changes in volatility. This is probably the most important one when dealing with a stochastic volatility model as the corresponding greek derived from the Black Scholes framework starts with the assumption that the volatility is constant and yet varies carrying out the calculation. Varying the strike results in the following graph for the vega:

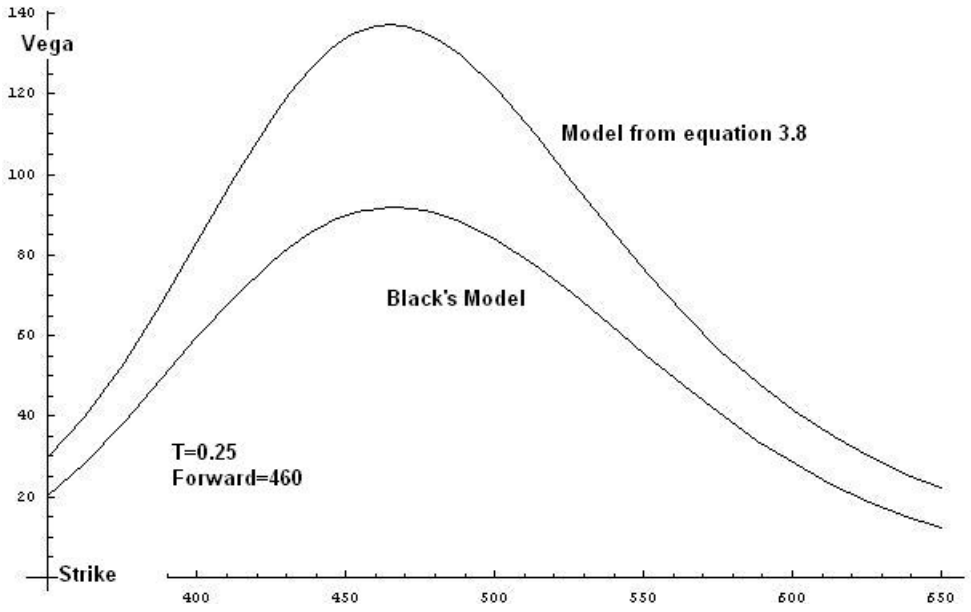


Figure 6.21: Plot of our model's Vega vs. strike in comparison with Black's model.

As can be seen Black's model significantly underestimates volatility risk. The same can be seen when the underlying is varied:

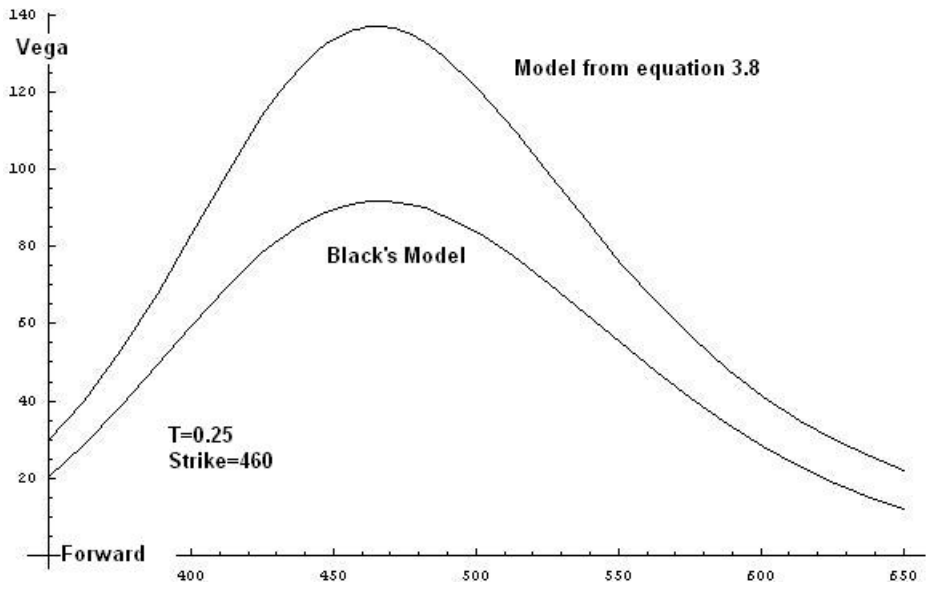


Figure 6.22: Plot of our model's Vega vs. forward in comparison with Black's model.

Now with the parameters we have obtained from the calibration note that the volatility of volatility is quite large and that we can hence have very large movements in volatility even expressed in non percentage terms. This means that when the volatility is small the percentage changes that it may suffer under the obtained parameters are very large, meaning that the volatility risk becomes very significant; we note this phenomenon in the plot for at the money strikes:

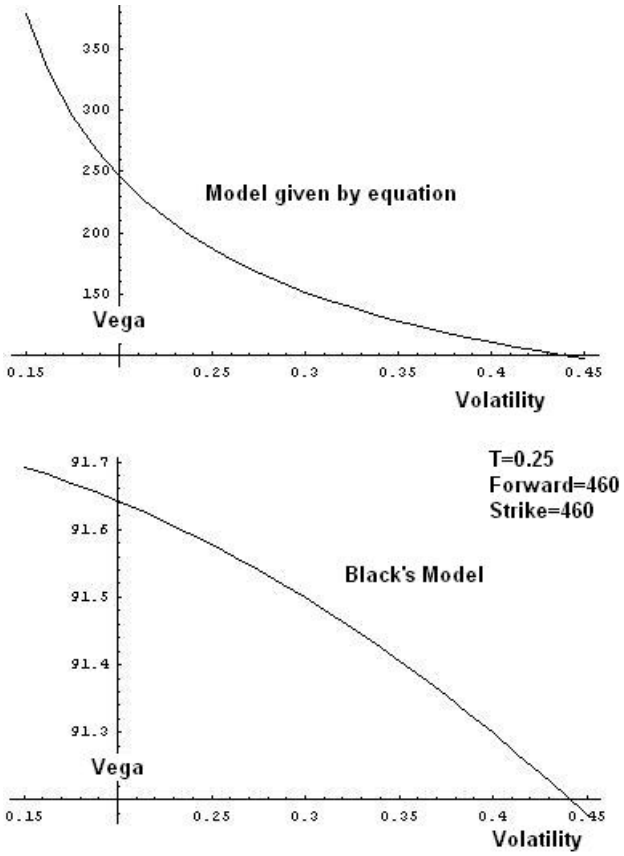


Figure 6.23: Plot of our model's Vega vs. volatility in comparison with Black's model for at the money options.

The difference is far less dramatic when dealing with out of the money and in the money options. For in the money calls we have the following:

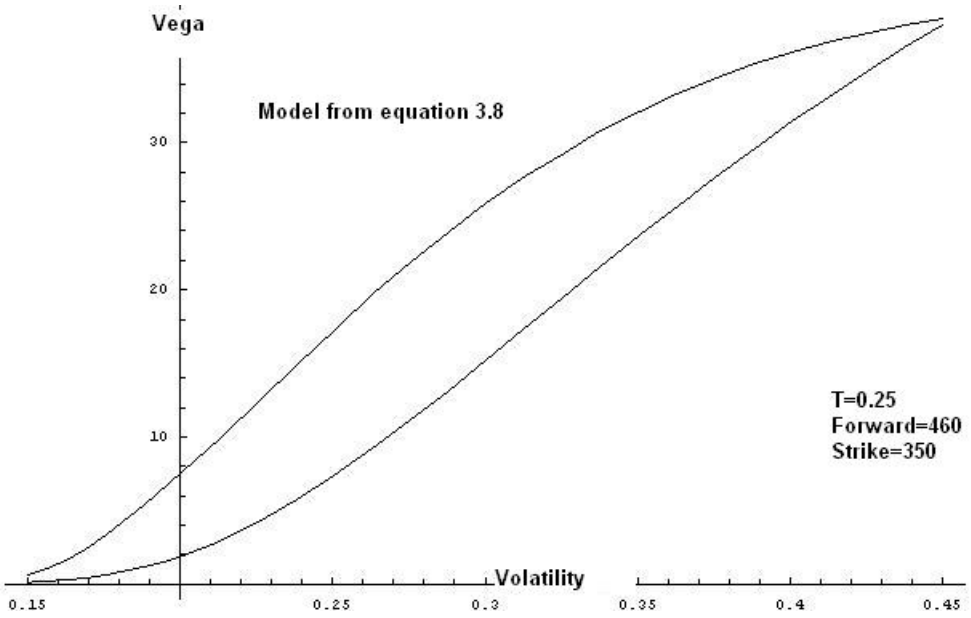


Figure 6.24: Plot of our model's Vega vs. volatility in comparison with Black's model for in the money options.

For out of the money we have the same type of structure and we note that our model always generates a higher vega:

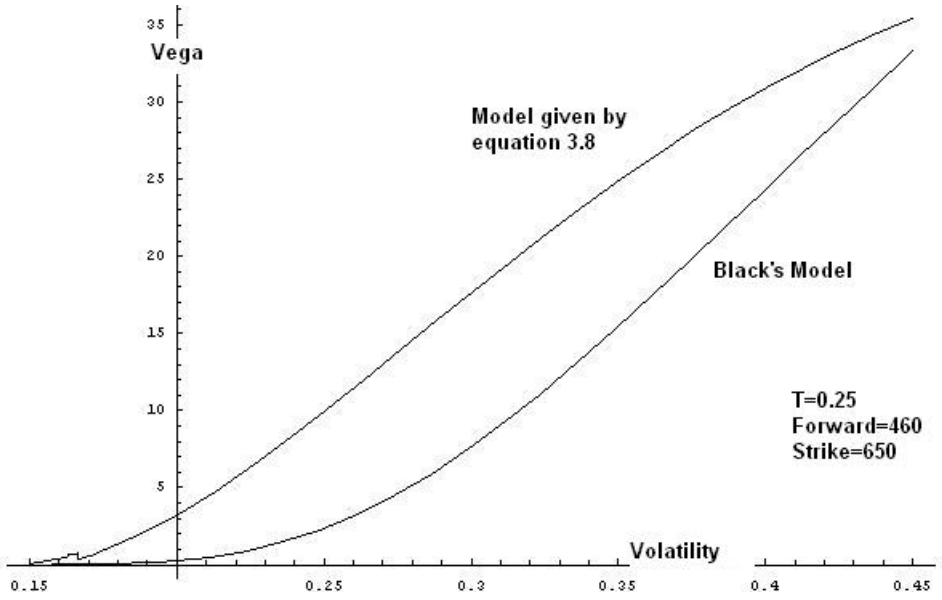


Figure 6.25: Plot of our model's Vega vs. volatility in comparison with Black's model for out of the money options.

6.7 Summary

In this chapter we have carried out a comparative analysis of our model's prices and greeks with those generated by the traders' benchmark given by the Black's 1976 model. We noted that in general the prices were very close when plotted with respect to strike and underlying but the difference, as one would expect, is more accentuated when compared against the variation of volatility. Nevertheless the difference in price is overall very small and does follow the same general features of Black's Model. As far as the delta is concerned, we note a slightly higher one for our model for in the money calls and a slightly lower one for out of the money calls. Again, the variation is more pronounced when observed against volatility with a divergence in the cases for increasing volatility. The gamma is much more pronounced in our model for nearly at the money strikes and the maximum is slightly shifted when compared to the Black one. On the other hand we see a lower gamma for in and out of the money calls. The divergence is in this case

more pronouced for low values of volatility. The time decay obtained by theta is on the other hand less pronounced for at the money strikes but higher in absolute value for tailed values of strikes. The divergences are higher for high values of volatility in the cases of in and out of the money options but smaller for at the money ones. The vega risk, as one would have expected, is much higher for our model throughout the strikes but tends to converge to the Black's model value for small and large levels of volatility. Concluding, we note that the overall behaviour of the plots is close to that of the Black's model giving us further confidence in our derivations.

7 Conclusion

7.1 Summary of Results

This thesis has undertaken the goal of establishing a model which could be held to be equivalent to the SABR one, while possessing the analytical properties necessary to be able to obtain a closed form solution free of assumptions on the scales of its parameters. We began by providing an extensive overview of interest rate models and have taken a systematic approach to presenting the different models, their advantages and disadvantages and possible generalizations. From short rate models unable to account for the joint evolution of the entire term structure of interest rates we moved on to the widely acclaimed Libor Market Model. We then described the smile problem, its origin, causes and the different forms of solutions that the recent history of options pricing has produced. After concluding that in the interest rate markets the most favored and fit approach developed so far is stochastic volatility, we discussed the different models that are commonly used in this category.

Two types of models, the SABR and the Piterbarg, are now the most widely used and we have taken the time to discuss their flaws. Indeed, the Piterbarg model is able to fit the entire swaptions grid but it assumes that volatility follows a mean reverting process. This is certainly appealing as it recovers the historically observed features of volatility, but, as we show, causes the smile for long maturity options to be underestimated. On the other hand, the SABR model is free from such problem but does not possess a closed form solution. Its authors derive prices for European options by assuming that the volatility of volatility is small and that does not always hold in the market. This means that in such market conditions where severe events take place, the price derived by the authors of the model does not translate into the arbitrage free one. We hence proposed a model which is equivalent to SABR and close to Piterbarg's but with slight differences which permit us to use results from other markets to derive a closed form solution free from assumptions

regarding the scale of the model's parameters for the special case in which the underlying forward rate is not correlated to its volatility. While making such assumptions on the correlation is certainly not necessary, slightly unrealistic and produces a worse fit than the correlated factor, it permits us to derive the closed form solution which we seek. Indeed, the calibration we conduct shows that the correlated model is able to produce better fits to the market but it also shows that when both the SABR model and ours are taken with zero correlation our model's fit is easily seen to be more realistic. The solution we provide follows techniques typically used in foreign exchange and equity markets rather than in LIBOR ones. We define a partial differential equation which is satisfied by the caplet price using the standard martingale argument and translate this through Fourier and Laplace transforms into an ordinary differential equation which, in its homogeneous form, is a particular case of Bowman's Bessel Equation. The non homogenous case can then be solved by a known theorem in the theory of ordinary differential equations.

Another feature of our study is the modification of an argument by Piterbarg for the derivation of the swap rates' stochastic differential equation in a forward rate model which has the same form as the original model's dynamic for the forward rate. We do this so that we are able to price swaptions using the same methodology as for caplets. We simplify the argument in a manner that keeps only one assumption: that the approximation which we are trying to derive holds for paths of the rates in which they are constant. We do not provide a theoretical justification of this assumption but, from Piterbarg, we point out that the approximation is backed by experiment.

The final theoretical study that we carry out is the investigation of the dynamic of our model's implied volatility for short maturities. We do this using a standard approach and find that, when the shifting parameter for the forward rate in the stochastic differential equation defining our model (equation (3.8)) is set to zero, our model is indeed able to reproduce the so called sticky moneyness rule used by options traders. This is the rule according to which the implied volatility does not change when the moneyness, the ratio

of the underlying to an option's strike, is kept constant.

The formula we derive for the implied volatility is not only important from a theoretical purpose but it also allows us to conduct the model's calibration on short maturity smiles very efficiently and without lengthy computations. This way the model's parameters are derived simply by fitting a polynomial to the market observed quotes. This is done in microseconds compared to the several hours that it would take when using the present form of our closed form or lengthy simulation techniques. Furthermore, the derivation of the implied volatility that we carry out can be generalized to the correlated case, as Lewis does in [LEWI00]. This can hence be used to see whether adopting a non zero correlation factor can enhance our model's fit, and the calibrations carried out do indeed back such result. Nevertheless, we conclude that when the correlation is taken to be zero our model provides a better fit than SABR.

The final study that we conduct is the comparison of our model's prices and Greeks to those of Black's model. We do this not only because the SABR's solution does not equate to the arbitrage free one but also because benchmarking our model against the Black 1976 one jointly carries out two purposes. The first is that the latter is now the benchmark used by traders when laying out options strategies or when deciding on the course of action to take in order to hedge an options trading book. Comparing our model's Greeks in this sense shows if any significant changes need be made to the type of action to take to, say, hedging theta or gamma risk. Furthermore, our comparison provides us with a more general description of the changes to prices and sensitivities that including stochastic volatility brings in relation to the non stochastic case. In this sense the comparison with the SABR model would not have been able to illustrate this. The comparison also makes us confident that our derivations are indeed correct as the general features of call prices and Greeks with the varying of parameters such as strike and volatility are still observed.

7.2 Suggestions For Further Research

We note that our model is capable of further improvements. First a study of the more complicated model incorporating the correlation factor is needed. While the work of Lewis in [LEWI98] can be easily applied to derive a closed form solution for the correlated case, such solution is extremely complicated and requires far more computational power than one would expect from it.

We began the study of the non correlated case not only to possibly open the door for subsequent academic study on the more general framework, but also in the hopes that the already complex solution that we provide may be simplified by additional analysis. Before solving the more general problem it would be interesting to examine more deeply the closed form we provide. Indeed, all the functions which define our fundamental transform in equation (4.30) have been well known for decades, it would thus be instructive and may reveal important properties if the solution could be reduced to a simpler or more analytical form.

Furthermore, the present form of the solution may also be used to study asymptotics of price behaviour. Accurate approximations may also be derived without a priori assumptions on the form of the solution, as the volatility-of-volatility expansion method typically does.

Finally, as the calibration conducted in Chapter 5 demonstrates, the market fit of all the different maturities may be unified by a single model by making the coefficients of the stochastic differential equation in (3.8) time dependent. This may prove an easier task for future research, one which would probably go hand in hand with that of simplifying the closed form solution provided. We believe this should be the starting point for future research.

8 Appendices

8.1 Appendix to Chapter 2 (Basics of Stochastics and Pricing Theory)

Ever since its first financial application in 1969 for the study of portfolio selection [MERT69], Stochastic Calculus has become the standard language of Mathematical Finance. Indeed, all of the models discussed in this chapter are set within the no arbitrage pricing framework and we must hence complement our literature review by quickly covering the fundamentals needed for our discussion. The basic results of Stochastic Calculus are taken from Shreve's two volumes book [SHREVE04].

Definition 8.1. *(σ-algebra) [SHREVE04]: Let Ω be a non empty set, and let $\mathfrak{F}$ be a collection of subsets of Ω. We say that $\mathfrak{F}$ is a σ-algebra (field) if:*

1. *if Δ belongs to $\mathfrak{F}$ then its complement Δ^c is also in $\mathfrak{F}$, and*
2. *if the sequence $\{\Delta_n\} \in \mathfrak{F}$, $\forall n \in \mathbb{N}$,*

then $\bigcup_{n=0}^{\infty} \Delta_n \in \mathfrak{F}$.

On the real line the so called Borel σ-algebra, $\mathcal{B}$, is that obtained by taking the closed subintervals of [0,1] and all other necessary sets. The elements of the algebra are then known as Borel sets. Functions with the property that the inverse image of a Borel set is also a Borel set are said to be Borel measurable.

Definition 8.2. *(Probability space)[SHREVE04]: Let Ω be a non empty set, and let $\mathfrak{F}$ be a σ-algebra of subsets of Ω. A probability measure $\mathbb{P} : \mathfrak{F} \to [0, 1]$ is a map such that:*

1. *$\mathbb{P}(\Omega) = 1$, and*
2. *(countable additivity) if $\{\Delta_n\} \in \mathfrak{F}$, $\forall n \in \mathbb{N}$, is a sequence of disjoint sets,*

then $\mathbb{P}\left(\bigcup_{n=0}^{\infty}\Delta_n\right) = \sum_{n=0}^{\infty}\mathbb{P}(\Delta_n)$.

$\mathbb{P}(\Delta)$ *is called the probability of* Δ*, and the triplet* $(\Omega, \mathfrak{F}, \mathbb{P})$ *a probability space. Furthermore, if* $\mathbb{P}(\Delta) = 1$ *we say that* Δ *happens almost surely.*

Definition 8.3. *(Random Variable)[SHREVE04]: Let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space. We say that the map* $X : \Omega \to \mathbb{R}$ *is a random variable if for every Borel subset* B *of* $\mathbb{R}$ *the inverse image of* B *under* X *is in the* σ*-algebra* $\mathfrak{F}$*, i.e.*

$$\{X \in B\} \equiv \{\omega \in \Omega;\ X(\omega) \in B\} \in \mathfrak{F}.$$

It turns out that the collection of subsets of Ω of the type $\{X \in B\}$, with B ranging over the real line, form a σ-algebra denoted by $\sigma(X)$ and addressed to as the σ- algebra generated by X. Further, if $\mathfrak{G}$ is a σ-algebra of subsets of Ω such that $\sigma(X) \subset \mathfrak{G}$, we say that X is $\mathfrak{G}$-measurable.

Definition 8.4. *(Distribution Measure)[SHREVE04]: Let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space and* X *a random variable defined on such space. We call the distribution measure of* X *the probability measure* μ_x *defined on* $\mathcal{B}$ *that assigns to every Borel set* $B \in \mathcal{B}$ *the probability of its inverse image under* X*, that is:* $\mu_x(B) = \mathbb{P}\{X \in B\}$.

Definition 8.5. *(Distribution functions)[SHREVE04]: Let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space and* X *a random variable defined on such space. We call the cumulative distribution function (cdf) the function* $F : \mathbb{R} \to \mathbb{R}$ *defined by:* $F(x) = \mathbb{P}\{X < x\}$.

Without getting into the analytical conditions for the existence, two types of random variables can be discerned. Heuristically speaking, one in which the map takes values continuously on the real line, and one in which only a countable set of values are possible. In the first case we can define a *distribution function* $f : \mathbb{R} \to \mathbb{R}$ as $\mu_x[a, b] \equiv \mathbb{P}\{a \leq$

$X \leq b\} \equiv \int_a^b f(x)\,\mathrm{dx}$. The latter case is characterized by the so called *probability mass function* $p : \mathbb{R} \to \mathbb{R}$ defined by $p_i = \mathbb{P}\{X = x_i\}$ such that $\mu_x(B) = \sum_{\{i;x_i \in B\}} p_i\ ,\ B \in \mathcal{B}$.

We can now define the *expectation* $\mathbb{E}[X] : \mathfrak{F} \to \mathbb{R}$ of the random variable as the lesbegue integral where the measure is taken to be the probability measure.

Definition 8.6. *(Expectation)[SHREVE04]: Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and X a random variable defined on it with:*

$$\int_\Omega |X(\omega)|\, d\mathbb{P}(\omega) \ < \ \infty \qquad \textit{(integrability condition)}$$

We define the expectation of X over $\Delta \in \mathfrak{F}$ as:

$$\mathbb{E}[X] = \int_\Delta X(\delta)\, d\mathbb{P}(\delta)$$

Proposition 8.7. *(Computation of Expectation)[SHREVE04]: Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and let X be a random variable defined on it. Further let g :$\mathbb{R} \to \mathbb{R}$ be a Borel-measurable function. Then*

$$\mathbb{E}[|g(X)|] = \int_{\mathbb{R}} |g(x)|\, d\mu_x(x)$$

and if the quantity is finite, then

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x)\, d\mu_x(x)\ .$$

Proposition 8.8. *(Change of Measure)[SHREVE04]: Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability place and let Z be an almost surely nonnegative random variable with $\mathbb{E}[Z] = 1$. For $\Delta \in \mathfrak{F}$ define*

$$\widetilde{\mathbb{P}}(\Delta) = \int_\Delta Z(\delta)\, \mathrm{d}\mathbb{P}(\delta)$$

Then $\widetilde{\mathbb{P}}$ is a probability measure and for any random variable X, $\widetilde{\mathbb{E}}[X] = \mathbb{E}[X\,Z]$, *and* $\mathbb{E}[X] = \widetilde{\mathbb{E}}\big[\frac{X}{Z}\big]$.

If two measures are defined on the same space they are said to be equivalent provided that their null spaces are the same. The random variable Z which relates them in this case is known as the Radon-Nikodym derivative and we write $Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$. Its existence is known as the Radon-Nikodym theorem.

Definition 8.9. *(Filtration)[SHREVE04]: Let $\Omega \neq \varnothing, T \in \mathbb{R}^+$ and assume that $\forall\, t \in [0,T]\, \exists\, \sigma$-algebra $\mathfrak{F}(t)$ of Ω such that $\forall\, s \leq t$, $\mathfrak{F}(s) \subset \mathfrak{F}(t)$. Then the collection $\mathfrak{F}(t), 0 \leq t \leq T$ is called a filtration.*

Note that the filtration can thus be seen as the flow through time of information regarding a random space.

Definition 8.10. *(Adapted Stochastic Process)[SHREVE04]: Let $\Omega \neq \varnothing$, and $\mathfrak{F}(t)$, $0 \leq t \leq T$ a filtration of σ-algebras of it. Let $X(t)$ be a collection of random variables parameterized by $t \in [0,T]$. We say that the collection $X(t)$ is an adapted stochastic process if $X(t)$ is $\mathfrak{F}(t)$-measurable $\forall\, t \in [0,T]$.*

Definition 8.11. *(Independence)[SHREVE04]: Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, and let $\mathfrak{G}$ and $\mathfrak{H}$ be two sub-σ-algebras of $\mathfrak{F}$. We say that $\mathfrak{G}$ and $\mathfrak{H}$ are independents if given any sets $G \in \mathfrak{G}$ and $H \in \mathfrak{H}$:*

$$\mathbb{P}(G \bigcap H) = \mathbb{P}(G) \cdot \mathbb{P}(H)$$

In the case of random variables we say that they are independent if the σ-algebras that they generate are independent.

Definition 8.12. *(Conditional Expectation)[SHREVE04]: Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, let $\mathfrak{G}$ be a sub-σ-algebra of $\mathfrak{F}$, and X an integrable random variable. We call*

the conditional expectation of X given $\mathfrak{G}$, $\mathbb{E}[X \mid \mathfrak{G}]$, any random variable satisfying the following two conditions:

In case $\mathfrak{G}$ is generated by a random variable Y we may write $\mathbb{E}[X \mid Y]$ instead. Computation is taken care of by the following theorem:

Proposition 8.13. *(Computation of Conditional Expectation)[SHREVE04]: Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, $\mathfrak{G}$ a sub-σ-algebra of $\mathfrak{F}$, $\{X_n\}_{n=1}^{k>1}$ be a sequence of $\mathfrak{G}$-measurable random variables, and $\{Y_l\}_{l=1}^{L>1}$ a sequence of $\mathfrak{G}$-independent random variables. For any function $f(x_1, x_2, ..., x_k, y_1, y_2, ..., y_L)$ define:*

$$g(x_1, x_2, ..., x_k) = \mathbb{E}[f(x_1, x_2, ..., x_k, Y_1, Y_2, ..., Y_L)]$$

in the sense that the expectation is only taken with respect to the Y's, then:

$$\mathbb{E}[f(X_1, X_2, ..., X_k, Y_1, Y_2, ..., Y_L) \mid \mathfrak{G}] = g(x_1, x_2, ..., x_k).$$

We now have the language necessary to define the two most important constructs in the applications of stochastic calculus to finance, the concept of a martingale and of Markov processes.

Definition 8.14. *(Martingale) [SHREVE04]: Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, $T \in \mathbb{R}^+$, and let $\mathfrak{F}(t)$, $0 \leq t \leq T$, be a filtration of sub-σ-algebras of $\mathfrak{F}$. An $\mathfrak{F}(t)$ adapted stochastic process $M(t)$, $0 \leq t \leq T$, is said to be a martingale if:*

$$\mathbb{E}[\,\mathrm{M(t)} \mid\, \mathfrak{F}(\mathrm{s})\,] = \mathrm{M(s)} \quad \forall\, 0 \leq \mathrm{s} \leq \mathrm{t} \leq \mathrm{T}.$$

Definition 8.15. *(Markov Process)[SHREVE04]: Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, $T \in \mathbb{R}^+$, and let $\mathfrak{F}(t)$, $0 \leq t \leq T$, be a filtration of sub-σ-algebras of $\mathfrak{F}$. An $\mathfrak{F}(t)$ adapted stochastic process $P(t)$, $0 \leq t \leq T$, is said to be a Markov process if for any borel measurable function $f : \mathbb{R} \to \mathbb{R}$ and any $0 \leq \mathrm{s} \leq \mathrm{t} \leq \mathrm{T}$, there is another borel measurable function $g : \mathbb{R} \to \mathbb{R}$ such that:*

$$\mathbb{E}[f(P(t)) \mid \mathfrak{F}(s)] = g(P(s)).$$

In simple terms a Markov process is one where the expected value of the function $f(P(t))$ given our knowledge of the world at time s, only depends on the value of P at s.

Since Brownian motions have become the standard processes in finance we will now go through a quick description.

Definition 8.16. *(Brownian Motion)[SHREVE04]:* Let $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space. And suppose that* $\forall\, \omega \in \Omega, \exists\, W(t)$ *continuous with* $t > 0$ *and* $W(0) = 0$. *We say that* $W(t)$ *is a Brownian Motion provided for all* $0 = t_0 < t_1 < ... < t_n$ *the increments*

$$W(t_1) - W(t_0),\ W(t_2) - W(t_1), ..., W(t_n) - W(t_{n-1})$$

are independent and normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0,$$

$$\mathrm{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i.$$

Definition 8.17. *(Filtration for Brownian motion)[SHREVE04]:* Let $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space and* $W(t)$ *be a Brownian motion defined on it. A filtration for the Brownian motion is a filtration in the above sense with the following extra conditions*

1. *(Adaptivity)* $W(t)$ *is* $\mathfrak{F}(t)$*-measurable* $\forall\, t \in \mathbb{R}^+$, *in the sense that the Brownian motion is a stochastic process adapted to* $\mathfrak{F}(t)$,

2. *(Independence of future increments)* $\forall\, 0 \leq t \leq u$ *the increment* $W(u) - W(t)$ *is independent of* $\mathfrak{F}(u)$. *That is the information available at time* $t \leq u$ *is of no use for predicting the behavior of the Brownian motion after* t.

This said it is trivial to see that Brownian motions are martingales with respect to the appropriate filtration. Furthermore, from Proposition 8.13 it is easy to see that such

processes satisfy the Markov property. One additional property particularly important for finance is the quadratic variation.

Definition 8.18. *(Quadratic Variation)[SHREVE04]: Let $f(t)$ be a function defined for $0 \leq t \leq T$. The quadratic variation of f to time T is*

$$[f, f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2,$$

where $\Pi = \{t_0, t_1, ..., t_n\}$ and $0 = t_0 \leq t_1 \leq ... \leq t_n = T$.

Proposition 8.19. *[SHREVE04]: Let W be a Brownian Motion. Then $[W, W] = T$ for all $T \geq 0$ almost surely. We write $d\,W\,d\,W = d\,T$. Further $d\,t\,d\,W = d\,t\,d\,t = 0$.*

In dealing with time transformations, specially in local volatility models, we will need to be able to detect a brownian motion, we do this through a very important theorem by Levy:

Theorem 8.20. *(Levy) [SHREVE04]: Assume that $M(\sigma(t))$, $\sigma(t) \geq 0\, \forall t \in \mathbb{R}$, is a continuous martingale and that $[M, M](\sigma(t)) = \sigma(t)\, \forall \sigma(t) \in \mathbb{R}$ then $M(\sigma(t))$ is a Brownian motion.*

At this stage we will spare ourselves the construction of the Ito integral, that is an integral where the integrand is taken with respect to a Brownian motion. We will just simply state Ito's Lemma, and note here that integrals with respect a Brownian motion (Ito integrals) are martingales.

Theorem 8.21. *[SHREVE04]: Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a differentiable function, and let $W(t)$ be a Brownian motion. Then for every $T \geq 0$,*

$$f(T, W(T)) = f(0, W(0)) + \int_0^T f_t(t, W(t))\, \mathrm{dt} + \int_0^T f_w(t, W(t))\, \mathrm{dW}(t) + \tfrac{1}{2} \int_0^T f_{\mathrm{ww}}(t, W(t))\, \mathrm{dt}$$

A mnemonic for remembering this is to write it in differential form $d\,f = f_t \mathrm{dt} + f_w \mathrm{dW} +$

$\frac{1}{2}f_{\text{ww}}\text{dt}$ and to note that if we exchange the last dt term by dWdW then we have the two dimensional version of Taylor's theorem where all the other terms in the series vanish due to the quadratic variations from Proposition 2.19. The theorem can be generalized to the case of a more general continuous process $R(t)$ with independent and normally distributed increments simply by substituting all dW and dWdW terms by the corresponding dR and dRdR. Further, as we previously noted, integrals with respect to a Brownian motion are martingales, hence all processes f with differential form of the type $d\,f = \sigma\,\text{dW}$ for some function σ are martingales. Now assume that a process $R(t, W(t))$ is composed by the sum of an Ito integral and a Riemann integral such that in differential form we may write, $d\,R = \mu(t, R(t))\text{dt} + \sigma(t, R(t))\text{dW}$ we say that the dt part is the drift and the dW part is the diffusion part. In this language any driftless, or pure diffusion process, is a martingale. We have seen that the definition of a Brownian motion entails that the process has increments which are normally distributed. This, means that a process which is a Brownian motion under some specific probability measure may not be so under an equivalent measure. Similarly, a process which is not a Brownian motion under a specific measure may be turned into one under another, this is the result of the so called Girsanov's Theorem.

Theorem 8.22. *(Girsanov)[SHREVE04]: Let $W(t)$, $0 \leq t \leq T$, be a Brownian Motion on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, and let $\mathfrak{F}(t)$, $0 \leq t \leq T$, be a filtration for $W(t)$. Let $\Theta(t)$, $0 \leq t \leq T$, be an adapted process. Define*

$$Z(t) = \exp\Big\{-\int_0^t \Theta(u)\text{dW}(u) \; - \tfrac{1}{2}\int_0^t \Theta^2(u)\text{du}\Big\},\; \widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\,\text{du},$$

and that $\mathbb{E}\Big[\int_o^T \Theta^2(u)Z^2(u)\text{du}\Big] < \infty$. Set $Z = Z(T)$, then $\mathbb{E}[Z] = 1$ and under the probability measure $\widetilde{\mathbb{P}}$ given by $\widetilde{\mathbb{P}}(\Delta) = \int_{\Delta} Z(\delta)d\mathbb{P}(\delta) \forall\, \delta \in \mathfrak{F}$, the process $\widetilde{W}(t)$, $0 \leq t \leq T$, is a Brownian motion.

Assume that the price of some tradeable asset follows a stochastic process known

as generalized Brownian motion, which may be written in differential form as $d\,S(t) = \alpha(t)S(t)\mathrm{dt} + \sigma(t)S(t)d\,W(t)$. Intuitively, if $\sigma = 0$ we see that the term $\alpha\mathrm{dt}$ represents the mean return of the asset over some small time interval dt. We thus note that the diffusion term represents "noise" which drives the return of the asset away from its mean. The σ term as a matter of fact is the volatility of the asset, which is related to the variance of its return. From the theorem above we see that the drift term is dependent on the measure we use, and the $\alpha(t)$ is said to be taken in the *Real World* measure. It is straightforward to see that there may be problems in estimating $\alpha(t)$. The above theorem helps us deduce that there are ways to change this term so that it becomes easier to estimate under a different measure. Indeed, if we take $\Theta(t) = \frac{\alpha(t)-R(t)}{\sigma(t)}$, where $R(t)$ is the risk-free interest rate, we see that under the new measure the process becomes $\mathrm{dS}(t) = R(t)S(t)\mathrm{dt} + \sigma(t)S(t)\mathrm{dW}(t)$ where the mean return is now the short term interest rate. Such measure is known as the Martingale or Risk-Neutral or even Pricing measure.

Having said that diffusion processes are martingales with respect to the filtration of the Brownian motion, we may ask ourselves the inverse question, if a process is a martingale with respect to the filtration of a Brownian motion, is it the case that it is an Ito integral ? The following theorem gives the affirmative answer.

Theorem 8.23. *(Martingale Representation)[SHREVE04]: Let $W(t)$, $0 \leq t \leq T$, be a Brownian Motion on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$,and let $\mathfrak{F}(t)$, $0 \leq t \leq T$, be a filtration for $W(t)$. Let $M(t)$, $0 \leq t \leq T$, be a martingale with respect to $\mathfrak{F}(t)$, then there is an adapted process $\Gamma(t)$, $0 \leq t \leq T$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u)\mathrm{dW}(u),\ 0 \leq t \leq T.$$

If we know the filtration of our space is one for a Brownian motion then all we need to know to check that a process is a martingale is to make sure that its drift is zero. Equivalently, if we are told that a process has no drift then we know it is a Martingale. Assume

that the discounting process for asset prices is of the form $D(t) = \exp\left\{-\int_0^t R(s)\mathrm{ds}\right\}$, a simple application of Ito's lemma shows that the discounted price of any tradeable asset following a generalized Brownian motion has no drift under the Risk-Neutral measure and is hence a martingale. The discussion for more general asset dynamics is more difficult, but the principles are the same, and most importantly it turns out that the existence of the pricing measure (one in which the discounted price of a portfolio of assets is a martingale) guarantees, in all cases, the absence of arbitrage in the market modeled. Furthermore, if we define a complete market to be one where we can replicate the payoff of any derivative by trading in the underlying and the money market, the Risk Neutral measure gives a further characterization: markets are complete if and only if there exists a unique martingale measure. This fact is known as the fundamental theorem of arbitrage, and is not trivial to prove for the most general case.

We will be using the dynamic hedging pricing mechanism whereby the price of a derivative is assumed to be equal to the price of a dynamic portfolio of underlying and bonds which will replicate the payoff of the derivative. Thus, assume that we have a portfolio $X(t)$ of underlying securities and bonds, and a derivative security whose value is $P(t)$. Suppose further that at maturity T, $X(T) = P(T)$. From our discussion the discounted value of the portfolio, $D(t)X(t)$ is a martingale under the risk neutral measure, hence absence of arbitrage gives:

$$D(t)V(t) = D(t)X(t) = \widetilde{\mathbb{E}}[D(T)X(T) \mid \mathfrak{F}(t)] = \widetilde{\mathbb{E}}[D(T)V(T) \mid \mathfrak{F}(t)] \Rightarrow$$

$$\Rightarrow V(t) = \widetilde{\mathbb{E}}\left[\exp\left\{-\int_t^T R(s)\mathrm{ds}\right\}V(T) \mid \mathfrak{F}(t)\right]$$

This permits us to price any derivative security in a market with no arbitrage. Sometimes though an alternative route permits one to price claims more efficiently by relying on the construction of a partial differential equation (PDE) to solve the problem. In the

discussion above we haven't been too clear about the differential forms in which a process can be specified. In the case where a process is taken to be a Riemann integral and an Ito one the most general case is given by: $\mathrm{dX}(t) = \beta(t, X(t))\mathrm{dt} + \gamma(t, X(t))\mathrm{dW}(t)$. This is what is called a stochastic differential equation (SDE) and as before the Ito term is known as the diffusion and the Riemann term the drift. Finding a process which satisfies an SDE for some given initial conditions can sometimes be a daunting or impossible experience. Further, as shown by the following lemma, to every SDE there is associated with it a PDE.

Proposition 8.24. *(Feynman-Kac)[SHREVE04]: Consider the stochastic differential equation* $\mathrm{dX}(t) = \beta(t, X(t))\mathrm{dt} + \gamma(t, X(t))\mathrm{dW}(t)$. *Let* $h(x)$ *be a Borel measurable function* $r, T \in \mathbb{R}, T > 0$ *and* $t \in [0, T]$. *Further define the function*

$$f(t,x) = \mathbb{E}^{t,x}\left[\exp\left\{-\int_t^T R(t)\mathrm{dt}\right\}h(X(T))\right].$$

Then $f(t,x)$ *satisfies the partial differential equation:*

$$f_t(t,x) + \beta(t,x)f_x(t,x) + \frac{1}{2}\gamma^2(t,x)f_{\mathrm{xx}}(t,x) = R\,f(t,x),\ f(T,x) = h(x)\ \forall\, x$$

The principle behind this theorem is that the function $f(t, X(t))$ is not a martingale but $e^{-\int_0^t R\,\mathrm{dt}} f$ is, we thus take its differential and set the drift term to zero. The result is the above PDE. From this and the above discussion about risk neutral pricing, we see that the prices of European style derivatives on an underlying following the SDE described must satisfy a respective PDE.

8.2 Appendix to Chapter 3

With $R^{(n,m)} = \left[\left(\frac{\partial}{\partial v}\right)^n \left(F\frac{\partial}{\partial F}\right)^m c\right]/\frac{\partial c}{\partial v}$, $X = \ln(F/K)$, $\tau = T - t$ and a driftless volatility process we have:

Table 8.1: $R^{(n,m)}$ for a driftless volatility z, source [LEWI00]

$n,\ m$	$R^{(n,m)}$
$2,\ 0$	$\tau\left[\frac{1}{2}\left(\frac{X}{z\tau}\right)^2 - \frac{1}{2z\tau} - \frac{1}{8}\right]$
$n,\ 0$	$\tau^{n-1}\left\{\left(\frac{X}{z\tau}\right)^{2n-2} A^n_{2n-2} + .. + \frac{X^{2j}\left[A^n_{2j}+O(z\tau)\right]}{(z\tau)^{n-1+j}} + .. + \frac{X^2\left[A^n_2+O(z\tau)\right]}{(z\tau)^n} + \frac{\left[A^n_0+O(z\tau)\right]}{(z\tau)^{n-1}}\right\}$

where $A^{n+1}_{2j} = -\left[n + j - \frac{1}{2}\right]A^n_{2j} + \frac{1}{2}A^n_{2j-2}$.

For the moments, letting the volatility follow a generic process of the form $d\,z = a(z)d\,W$ for some function $a(z)$ and $z_0 = z(0)$ while the rest as defined in section 3.4 we have:

Table 8.2: Volatility Moments, source [LEWI00]

n	σ^n
2	$\frac{1}{3}a(z_0)^2\tau + O(\tau^2)$
3	$\frac{2}{5}a'(z_0)a(z_0)^3\tau^2 + O(\tau^3)$
4	$\frac{1}{3}a(z_0)^4\tau^2 + O(\tau^3)$
$n-$ even	$\left(\frac{1}{3}\right)^{n/2}((n-1)!)!a(z_0)^n\tau^{n/2}(1+O(\tau))$
$n-$ odd	$\frac{1}{5}\left(\frac{1}{3}\right)^{(n-1)/2}(n-1)(n!)!a'(z_0)a(z_0)^m\tau^{(m+1)/2}(1+O(\tau))$

8.3 Appendix to Chapter 4 (Implementation of Prices and Greeks)

We will now provide with the implementation in Mathematica of our model's price and Greeks. These will be used extensively in the final chapter and may help the reader in reproducing our results.

Price Implementation

MATHEMATICA IMPLEMENTATION OF EQUATION (4.31)

```
Clear[H1];
H1[z_, V0_, xi_, T_, lamb_, alph_] :=
Module[{y, mu = -1, ans},
cprime = (alph^2)*(z^2 - I z)/xi;
y = 2 Sqrt[cprime*(xi*V0 + lamb)]/xi;
ans = If[y == 0, Re[Limit[y1*E^(cprime*lamb*T/(xi^2))BesselK[1, y1], y1 -> 0]],
Re[y*E^(cprime*lamb*T/(xi^2))BesselK[1, y]]];
Return[N[ans]];
]
Clear[H2];
H2[T_, z_, V0_, xi_, lamb_, alph_, pflag_] :=
Module[{y, mu = -1, ingr, ans, numax = 10 /(xi Sqrt[T]), nu1},
cprime = (alph^2)*(z^2 - I z)/xi;
y = 2 Sqrt[cprime*(xi*V0 + lamb)]/xi;
ingr[nu_] := Re[y*E^(cprime*lamb*T/(xi^2))*(Abs[Gamma[(I nu - 1)/2]])^2 nu BesselK[I nu, y]
*E^(Log[Sinh[nu Pi]] - (1 + nu^2)xi^2 T/8)];
nu1 = 5;
While[ 0.5(Abs[ingr[nu1]] + Abs[ingr[nu1 + 0.5]]) > 10^(-4),
nu1 += 5];
```

```
numax = nu1;
ans = NIntegrate[ingr[nu], {nu, 0, 10, numax}];
ans = 1/(4 Pi^2)ans;
Return[ans];
]
Clear[HNodrift];
HNoDrift[T_, z_, V0_, xi_, lamb_, alph_, pflag_] :=
Module[{h1, h2, tot},
h1 = H1[z, V0, xi, T, lamb, alph];
h2 = H2[T, z, V0, xi, lamb, alph, pflag];
tot = h1 + h2;
Return[tot];
]
Clear[Caprice];
Caprice[Fwd_, Strike_, T_, V0_, xi_, lamb_, bet_, alph_] :=
Module[{ans, cut, bigx, ingrand},
bigx = Log[(alph*Fwd + bet)/(alph*Strike + bet)];
ingrand[k_?NumericQ] := Re[E^(-I*(k + I*0.5)*bigx)*HNoDrift[T, k +
I*0.5, V0, xi, lamb, alph, 0]/(2*Pi*alph*((k + I*0.5)^2 - I*(k +
I*0.5)))];
cut = 10;
While[ 0.5(Abs[ingrand[cut]] + Abs[ingrand[cut + 0.5]]) > 10^(-4),
cut += 5];
ans = Fwd - 2*(alph*Strike + bet)*NIntegrate[ingrand[
k], {k, 0, 10, cut}];
Return[ans];
]
```

Delta's Implementation

MATHEMATICA IMPLEMENTATION OF EQUATION (4.45)

```
Clear[H1];
H1[z_,V0_,xi_,T_,lamb_,alph_]:=
Module[{y,mu=-1,ans},
cprime=(alphˆ2)*(zˆ2-I z)/xi;
y=2 Sqrt[cprime*(xi*V0+lamb)]/xi;
ans=
If[y==0,
Re[Limit[y1*Eˆ(cprime*lamb*T/(xiˆ2))BesselK[1,y1],y1->0]],
Re[y*Eˆ(cprime*lamb*T/(xiˆ2))BesselK[1,y]]];
Return[N[ans]];
]

Clear[H2];
H2[T_,z_,V0_,xi_,lamb_,alph_,pflag_]:=
Module[{y,mu=-1,ingr,ans,numax=10 /(xi Sqrt[T]),nu1},
cprime=(alphˆ2)*(zˆ2-I z)/xi;
y=2 Sqrt[cprime*(xi*V0+lamb)]/xi;
ingr[nu_]:=
Re[y*Eˆ(cprime*lamb*T/(xiˆ2))*(Abs[Gamma[(I nu-1)/2]])ˆ2 nu BesselK[
I nu,y] *Eˆ(Log[Sinh[nu Pi]]-(1+nuˆ2)xiˆ2 T/8)];

nu1=5;
While[ 0.5(Abs[ingr[nu1]]+Abs[ingr[nu1+0.5]])>10ˆ(-4),
nu1+=5];
```

```
numax=nu1;
ans=NIntegrate[ingr[nu],{nu,0,10,numax}];
ans=1/(4 Pi^2)ans;
Return[ans];
]

Clear[HNodrift];
HNoDrift[T_,z_,V0_,xi_,lamb_,alph_,pflag_]:=
Module[{h1,h2,tot},
h1 = H1[z,V0,xi,T,lamb,alph];
h2 = H2[T,z,V0,xi,lamb,alph,pflag];
tot=h1+h2;
Return[tot];
]
Clear[Delta];
Delta[Fwd_,Strike_,T_,V0_,xi_,lamb_,bet_,alph_]:=
Module[{ans, cut,bigx,ingrand},
bigx=Log[(alph*Fwd+bet)/(alph*Strike+bet)];
ingrand[k_?NumericQ]:=
Re[E^(-(I*k+0.5)*bigx)*
HNoDrift[T,k+I*0.5,V0,xi,lamb,alph,0]/(2*Pi*(I*k+0.5))];
cut=10;
While[ 0.5(Abs[ingrand[cut]]+Abs[ingrand[cut+0.5]])>10^(-4),
cut+=5];
ans=NIntegrate[ingrand[k],{k,-cut,10,cut}];
ans=1-ans;
Return[ans];
```

```
]
```

Gamma's Implementation

MATHEMATICA IMPLEMENTATION OF EQUATION (4.46)

```
Clear[H1];
H1[z_, V0_, xi_, T_, lamb_, alph_] :=
Module[{y, mu = -1, ans},
cprime = (alph^2)*(z^2 - I z)/xi;
y = 2 Sqrt[cprime*(xi*V0 + lamb)]/xi;
ans = If[
y == 0, Re[Limit[y1*E^(cprime*lamb*T/(xi^2))BesselK[1, y1], y1 -> 0]],
Re[y*E^(cprime*lamb*T/(xi^2))BesselK[1, y]]];
Return[N[ans]];
]


Clear[H2];
H2[T_, z_, V0_, xi_, lamb_, alph_, pflag_] :=
Module[{y, mu = -1, ingr, ans, numax = 10 /(xi Sqrt[T]), nu1},
cprime = (alph^2)*(z^2 - I z)/xi;
y = 2 Sqrt[cprime*(xi*V0 + lamb)]/xi;
ingr[nu_] := Re[y*E^(cprime*lamb*T/(xi^2))*(Abs[Gamma[(I nu -
1)/2]])^2 nu BesselK[I nu, y] *E^(Log[Sinh[nu Pi]] - (1 + nu^2)
xi^2 T/8)];


nu1 = 5;
```

```
While[ 0.5(Abs[ingr[nu1]] + Abs[ingr[nu1 + 0.5]]) > 10^(-4),
nu1 += 5];
numax = nu1;
ans = NIntegrate[ingr[nu], {nu, 0, 10, numax}];
ans = 1/(4 Pi^2)ans;
Return[ans];
]

Clear[HNodrift];
HNoDrift[T_, z_, V0_, xi_, lamb_, alph_, pflag_] :=
Module[{h1, h2, tot},
h1 = H1[z, V0, xi, T, lamb, alph];
h2 = H2[T, z, V0, xi, lamb, alph, pflag];
tot = h1 + h2;
Return[tot];
]
Clear[Gammaca];
Gammaca[Fwd_, Strike_, T_, V0_, xi_, lamb_, bet_, alph_] :=
Module[{ans, cut, bigx, ingrand},
bigx = Log[(alph*Fwd + bet)/(alph*Strike + bet)];
ingrand[k_?NumericQ] := Re[E^(-(I*k + 1.5)*bigx)*HNoDrift[T, k +
I*0.5, V0, xi, lamb, alph, 0]];
cut = 10;
While[ 0.5(Abs[ingrand[cut]] + Abs[ingrand[cut + 0.5]]) > 10^(-5),
cut += 5];
ans = NIntegrate[ingrand[k], {k, -cut, 10, cut}];
Return[ans*alph/(2*Pi*(alph*Strike + bet))];
```

]

Theta's Implementation

MATHEMATICA IMPLEMENTATION OF EQUATION (4.47)

```
Clear[H1];
H1[z_, V0_, xi_, T_, lamb_, alph_] :=
Module[{y, mu = -1, ans},
cprime = (alph^2)*(z^2 - I z)/(2*xi);
y = 2 Sqrt[2*cprime*(xi*V0 + lamb)]/xi;
ans =
If[y == 0,
Re[Limit[y1*E^(cprime*2*lamb*T/(xi^2))BesselK[1, y1], y1 -> 0]],
Re[y*E^(cprime*2*lamb*T/(xi^2))BesselK[1, y]]];
Return[N[ans]];
]

Clear[H2];
H2[T_, z_, V0_, xi_, lamb_, alph_, pflag_] :=
Module[{y, mu = -1, ingr, ans, numax = 10 /(xi Sqrt[T]), nu1},
cprime = (alph^2)*(z^2 - I z)/(2*xi);
y = 2 Sqrt[2*cprime*(xi*V0 + lamb)]/xi;
ingr[nu_] := Re[y*E^(cprime*2*lamb*T/(
xi^2))*(Abs[Gamma[(I nu - 1)/2]])^2 nu BesselK[I nu, y]
*E^(Log[Sinh[nu Pi]] - (1 + nu^2)(xi^2 )T/8)];
```

```
nu1 = 5;
While[ 0.5(Abs[ingr[nu1]] + Abs[ingr[nu1 + 0.5]]) > 10^(-5),
nu1 += 5];
numax = nu1;
ans = NIntegrate[ingr[nu], {nu, 0, 10, numax}];
ans = 1/(4 Pi^2)ans;
Return[ans];
]

Clear[HNodrift];
HNoDrift[T_, z_, V0_, xi_, lamb_, alph_, pflag_] :=
Module[{h1, h2, tot},
h1 = H1[z, V0, xi, T, lamb, alph];
h2 = H2[T, z, V0, xi, lamb, alph, pflag];
tot = h1 + h2;
Return[tot];
]

Clear[Htcont];
Htcont[T_, z_, V0_, xi_, lamb_, alph_, pflag_] :=
Module[{y, mu = -1, ingr, ans, numax = 10 /(xi Sqrt[T]), nu1},
cprime = (alph^2)*(z^2 - I z)/(2*xi);
y = 2 Sqrt[2*cprime*(xi*V0 + lamb)]/xi;
ingr[nu_] := Re[y*E^(cprime*2*lamb*T/(
xi^2))*((Abs[Gamma[(I nu - 1)/2]])^2) nu* ((1 +
nu^2)(xi^2 )/8)* BesselK[I nu, y] *E^(Log[Sinh[
nu Pi]] - (1 + nu^2)(xi^2 )T/8)];
```

```
nu1 = 5;
While[ 0.5(Abs[ingr[nu1]] + Abs[ingr[nu1 + 0.5]]) > 10^(-4),
nu1 += 5];
numax = nu1;
ans = NIntegrate[ingr[nu], {nu, 0, 10, numax}];
ans = 1/(4 Pi^2)ans;
Return[ans];
]
Clear[Hto];
Hto[T_, z_, V0_, xi_, lamb_, alph_, pflag_] :=
Module[{ans, hti},
hti = 0;
ans = -cprime*2*(lamb/xi^2)*HNoDrift[T, z, V0, xi,
lamb, alph, pflag] + Htcont[T, z, V0, xi, lamb, alph, pflag];
Return[ans];
]


Clear[Seeta];
Seeta[Fwd_, Strike_, T_, V0_, xi_, lamb_, bet_, alph_] :=
Module[{ans, cut, bigx, ingrande},
bigx = Log[(alph*Fwd + bet)/(alph*Strike + bet)];
ingrande[k_?NumericQ] := Re[E^(-I*(k + I*0.5)*bigx)*Hto[T, k + I*0.5, V0,
xi, lamb, alph, 0]/(2*Pi*alph*((k + I*0.5)^2 - I*(k + I*0.5)))];
cut = 10;
While[ 0.5(Abs[ingrande[cut]] + Abs[ingrande[cut + 0.5]]) > 10^(-5),
```

```
cut += 5];
Off[NIntegrate::slwcon];
ans = -2*(alph*Strike + bet)*NIntegrate[ingrande[k], {k, 0, 10, cut}];
Return[ans];
]
```

Vega's Implementation

MATHEMATICA IMPLEMENTATION OF EQUATION (4.48)

```
Clear[H1];
H1[z_, V0_, xi_, T_, lamb_, alph_] :=
Module[{y, mu = -1, ans},
cprime = (alph^2)*(z^2 - I z)/(2*xi);
y = 2 Sqrt[2*cprime*(xi*V0 + lamb)]/xi;
ans = (4*cprime/(xi*y))*
E^(cprime*2*lamb*T/(xi^2))(2*BesselK[1, y] - y*BesselK[2, y]);
Return[N[ans]];
]

Clear[H2];
H2[T_, z_, V0_, xi_, lamb_, alph_, pflag_] :=
Module[{y, mu = -1, ingr, ans, numax = 10 /(xi Sqrt[T]), nu1},
cprime = (alph^2)*(z^2 - I z)/(2*xi);
y = 2 Sqrt[2*cprime*(xi*V0 + lamb)]/xi;
ingr[nu_] := (4*cprime/(xi*y))*Re[E^(
cprime*2*lamb*T/(xi^2))*(Abs[Gamma[(I
nu - 1)/2]])^2 nu (BesselK[I nu, y]*(1 + I nu) - y*BesselK[I nu +
```

```
1, y])*E^(Log[Sinh[nu Pi]] - (1 + nu^2)(xi^2 )T/8)];


nu1 = 5;

While[ 0.5(Abs[ingr[nu1]] + Abs[ingr[nu1 + 0.5]]) > 10^(-4),

nu1 += 5];

numax = nu1;

ans = NIntegrate[ingr[nu], {nu, 0, 10, numax}];

ans = 1/(4 Pi^2)ans;

Return[ans];

]

Clear[Hto];

Hto[T_, z_, V0_, xi_, lamb_, alph_, pflag_] :=

Module[{ans, c},

c = 0;

ans = Re[H1[z, V0, xi, T, lamb,

alph] + H2[T, z, V0, xi, lamb, alph, pflag]];

Return[ans];

]

Clear[Veega];

Veega[Fwd_, Strike_, T_, V0_, xi_, lamb_, bet_, alph_] :=

Module[{ans, cut, bigx, ingrande},

bigx = Log[(alph*Fwd + bet)/(alph*Strike + bet)];

ingrande[k_?NumericQ] := Re[E^(-I*(k + I*0.5)*bigx)*Hto[T, k +

I*0.5, V0, xi, lamb, alph,

0]/(2*Pi*alph*((k + I*0.5)^2 - I*(k + I*0.5)))];

cut = 10;

While[ 0.5(Abs[ingrande[cut]] + Abs[ingrande[cut + 0.5]]) > 10^(-5),
```

```
cut += 5];
Off[NIntegrate::slwcon];
ans = -(alph*Strike + bet)*NIntegrate[ingrande[k], {k, -cut, 10, cut}];
Return[ans];
]
```

9 References

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