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High-temperature limit of Breit–Wheeler pair production in a black-body field

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ABSTRACT

This paper presents an analytic expression for the high-temperature limit of Breit–Wheeler pair production in a black-body field to lowest order in perturbation theory, of interest in early-universe cosmology. The limit is found to be a good approximation for temperatures above about three times the electron rest energy. It is also found that coupling to low-energy processes remains important at arbitrarily high temperatures, due to the exchange of a low-energy virtual fermion near the mass shell. This appears mathematically in the rate as a logarithmic factor of the photon temperature divided by the electron rest mass.

Weaver [1] expressed the rate of pair production by the Breit–Wheeler process in a Black-Body field as a sum of single integrals over special functions in 1976, which reduced to a simple expression in the low-temperature limit $k_B T_\gamma \ll mc^2$, where T_γ is the photon temperature and m is the electron rest mass. Such a quantity, and Weaver's limit, was developed for examining high-energy astrophysical phenomena, such as supernovas [2] and galactic nuclei [3], and has also more recently found application in the study of potential burning plasmas in the laboratory [4–6]. The physical situation assumed here is that the photon field is in equilibrium, while the electron/positron field is not. In the small temperature limit this is clearly a common physical situation, but a sufficiently rapid heating of the photon field can produce it even in the high-temperature limit, where $k_B T_\gamma \gg mc^2$, which has relevance in early universe cosmology. The pair production rate can then be used to study the process of equilibration between the fields. It would also be useful in situations where disequilibrium is maintained by some other mechanism, such as by the imposition of an external field that sweeps created pairs away.

In this paper we present the Breit–Wheeler pair production rate in this high-temperature limit. Weaver [1] gives the rate of two-photon interaction for a black-body radiation field as

$$R_{\gamma\gamma'} = \frac{4c}{\pi^4}\left(\frac{k_B T_\gamma}{c\hbar}\right)^6 \sum_{n,l=1}^{\infty} \frac{1}{\sqrt{nl}} \int_0^\infty d\xi\ \sigma \xi^4 K_1(2\sqrt{nl}\xi) \tag{1}$$

where K_n is the modified Bessel function of the second kind and

$$\xi = p_\gamma^* c/(k_B T_\gamma), \tag{2}$$

where p_γ^* is the centre-of-momentum energy of the colliding photons. The expression originates from an integration over two black-body photon distributions in momentum space, with the sum over n, l corresponding to a sum over the photon occupation numbers of each mode. Jauch & Rohlich [7] give the Breit–Wheeler cross-section as

$$\sigma(\phi) := \begin{cases} \pi r_0^2 \phi^2 \left[(2+2\phi^2-\phi^4)\cosh^{-1}(\phi^{-1}) \right. \\ \qquad \left. -(1+\phi^2)(1-\phi^2)^{1/2} \right], \qquad \phi < 1 \\ 0, \qquad \phi \geq 1. \end{cases}, \tag{3}$$

$$\phi := \frac{mc}{p_\gamma^*} = \beta\xi^{-1}, \quad \beta := \frac{mc^2}{k_B T_\gamma}. \tag{4}$$

Define

$$\sigma_0(\phi) := \pi r_0^2 \phi^2 \left[2\log(2\phi^{-1}) - 1\right] \tag{5}$$

$$\sigma_1 := \sigma - \sigma_0. \tag{6}$$

$\sigma \sim \sigma_0$ in the high-energy limit, and we will show that σ_0 acts as the effective cross-section in the high-temperature limit. σ_1 obeys

$$\sigma_1(\phi) = \mathcal{O}(\phi^4 \log(\phi)), \quad \phi \to 0^+. \tag{7}$$

We can write the dimensionless integral we need to approximate as

$$I := r_0^{-2} \int_\beta^\infty d\xi\ \sigma\xi^4 K_1(2\rho\xi), \tag{8}$$

where $\rho := \sqrt{nl}$. Divide I into two parts,

$$I = I_0 + I_1 \tag{9}$$

$$I_{0,1} := r_0^{-2} \int_\beta^\infty d\xi\ \sigma_{0,1}(\beta\xi^{-1})\xi^4 K_1(2\rho\xi) \tag{10}$$

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We have, first,

$$I_0 = \pi\beta^2 \int_\beta^\infty d\xi \, [2\log\left(\frac{2\xi}{\beta}\right) - 1]\xi^2 K_1(2\rho\xi)$$
$$= \frac{\pi\beta^2}{(2\rho)^3}\left[(-2\log(\rho\beta) - 1)\int_\beta^\infty dy \; y^2 K_1(y) + 2\int_\beta^\infty dy \; \log(y) y^2 K_1(y)\right], \tag{11a}$$

where we have used the substitution $y = 2\rho\xi$. Using

$$\int_\beta^\infty dy \; y^2 K_1(y) = 2 + \mathcal{O}(\beta^2 \log(\beta)) \tag{12a}$$

$$\int_\beta^\infty dy \; y^2 \log(y) K_1(y) = 1 - 2\gamma + 2\log(2) + \mathcal{O}(\beta^2 \log(\beta)), \tag{12b}$$

where γ is the Euler–Mascheroni constant, this becomes

$$I_0 = \frac{\pi}{2\rho^3}\beta^2\left(-\log(\beta) + \log(\frac{2}{\rho}) - \gamma\right) + \mathcal{O}(\beta^4 \log(\beta)) \tag{13}$$

Next,

$$|I_1| \le r_0^{-2}\int_\beta^\infty d\xi \left|\sigma_1(\beta\xi^{-1})\xi^4 K_1(2\rho\xi)\right|$$
$$\le \frac{\beta^3}{r_0^2(2\rho)^2}\mathrm{Max}_{\phi\in(0,1)}\left(|\sigma_1(\phi)\phi^{-3}|\right)\int_0^\infty dy \; yK_1(y) \tag{14}$$
$$= \frac{\beta^3}{r_0^2(2\rho)^2}\mathrm{Max}_{\phi\in(0,1)}\left(|\sigma_1(\phi)\phi^{-3}|\right)\frac{\pi}{2} = \mathcal{O}(\beta^3),$$

where we know $\mathrm{Max}_{\phi\in(0,1)}\left(|\sigma_1(\phi)\phi^{-3}|\right)$ is finite from Eq. (7). Therefore $I = I_0 + \mathcal{O}(\beta^3)$ and hence

$$R_{\gamma\gamma'} = \sum_{n,l=1}^\infty \frac{1}{(nl)^2}\frac{2r_0^2 c}{\pi^3}\left(\frac{mc}{\hbar}\right)^2\left(\frac{k_B T_\gamma}{\hbar c}\right)^4\left[\log\left(\frac{k_B T_\gamma}{mc^2}\right) + \log\left(\frac{2}{\sqrt{nl}}\right) - \gamma + \mathcal{O}\left(\frac{mc^2}{k_B T_\gamma}\right)\right] \tag{15}$$

Using

$$\sum_{n,l=1}^\infty \frac{1}{(nl)^2} = \frac{\pi^4}{36}, \tag{16a}$$

$$\sum_{n,l=1}^\infty \frac{\ln(nl)}{(nl)^2} = -\frac{\pi^4}{18}\left(-12\log(A) + \gamma + \log(2\pi)\right) \tag{16b}$$

and defining

$$\chi := \log(4\pi) - 12\log(A) \approx -0.45403, \tag{17}$$

where A is the Glaisher–Kinkelin constant, this can be written

$$R_{\gamma\gamma'} = \frac{\pi\alpha^2 c}{18}\left(\frac{k_B T_\gamma}{\hbar c}\right)^4\left[\log\left(\frac{k_B T_\gamma}{mc^2}\right) + \chi + \mathcal{O}\left(\frac{mc^2}{k_B T_\gamma}\right)\right]. \tag{18}$$

This approximation is plotted against a numerical calculation of the exact rate (1) in Fig. 1, where it can be seen to be good for $k_B T > 3mc^2$. Assuming a constant rate of pair production, no backwards rate, and free-field equilibrium density [8], this predicts equilibration of the fermion field in $\sim 2\times 10^4 \, \hbar/(k_B T_\gamma[\log(mc^2/(k_B T_\gamma)) + \chi])$. Since $k_B T/\hbar$ is the frequency scale of most particle reactants, this predicts equilibration over long timescales compared to the quantum processes.

The logarithmic term might be surprising. The naïve expectation, on dimensional grounds, is that the thermally averaged cross-section for a two-particle collision at $k_B T \gg mc^2$ obeys $\langle\sigma\rangle \propto T^{-2}$ [8], which would give $R_{\gamma\gamma'} \propto T^4$. This is based on the logic that, at temperature scales where the mass becomes irrelevant, temperature is the only appropriate energy-scale that can be chosen. But instead, as $m \to 0^+$, the rate diverges logarithmically. The high-energy process remains irrevocably coupled to the low-energy regime. To understand why, consider that the logarithmic divergence is inherited directly from the two-photon

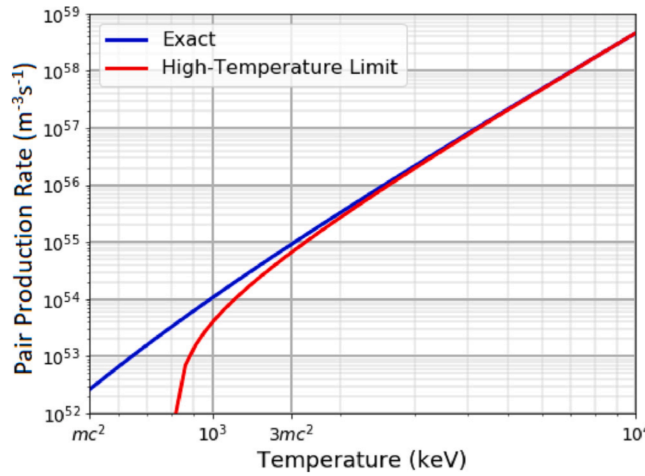

Fig. 1. Approximation of thermal Breit–Wheeler pair creation rate plotting against numerical calculation, with n, l in Eq. (1) summed from 1 to 20, at which there is numerical convergence to precision visible on the graph. The approximation is seen to rapidly approach the exact result from below for $k_B T > 3mc^2$.

cross section, where it appears as a divergence in the virtual fermion propagator [7]. Specifically, the divergence is due to the possibility of Breit–Wheeler being mediated by the exchange of a real fermion of vanishing energy and momentum. The physics behind this is intuitive: in the zero-mass limit, a photon transforming into an electron of the same energy and momentum does not violate energy or momentum conservation. Therefore Breit–Wheeler needs involve only the exchange of a virtual fermion of vanishing energy and momentum. But in the zero-mass limit, this virtual fermion will be on the mass shell. Processes being mediated by real particles correspond to physical processes that can occur between widely-separated particles. We therefore have a clear physical picture for how the mass-scale remains relevant at high temperatures: photons in the thermal gas can interact to create electron–positron pairs which are separated by the length-scale of the inverse of the electron mass. This is in contrast to the "hard thermal loop" (HTL) paradigm [9,10], where the dominant contribution to thermal quantities comes from the exchange of excitations with "hard" momenta $p \sim k_B T_\gamma/c$. This is because we are dealing with hard *external* momenta, while HTL assumes external momenta are soft, and because our virtual fermion propagator is not a thermal propagator, since we are formally examining a situation in which there is not a thermal fermion background.

This absence of a fermion background is a major approximation made by Eq. (18) as a calculation of the physical pair production rate. If we are concerned with the equilibration of the fermion field with a rapidly heated photon field, then it will only hold good for a finite period of time. The other major approximation is that it is a lowest-order perturbative process. This could be problematic because we know that in a thermal context the perturbation expansion might not produce adequate results [9]. In general, in the theory of linear excitations about thermal equilibrium, we expect the perturbative expansion to produce reasonable approximations in the regime of hard external momenta. Since the dominant contributions to the total Breit–Wheeler rate are the production of hard fermions by hard photons, we have some reason to think of it as a meaningful quantity, though it might well underestimate the production of soft fermions.

This is complicated, though, by the fact that we have just shown that the Breit–Wheeler is peculiarly coupled to the low-energy regime. To get a qualitative idea of the impact of the effects neglected, consider that one of the most important non-perturbative effects of a thermal background is to introduce a "thermal mass" to the constituent particles, $m_{\mathrm{th}} := \mu k_B T_\gamma/c^2$, where $\mu \sim \sqrt{\alpha}$ [9,11,12]. (Taking this to be the only thermal adjustment of the dispersion relations becomes a good approximation in the hard momentum regime.) Introducing this thermal mass as a correction to the fermion mass would induce us to replace Eq. (18) as $\mu k_B T \gg mc^2$ with

$$R_{\gamma\gamma'} = \frac{\pi\alpha^2 c}{18}\left(\frac{k_B T_\gamma}{\hbar c}\right)^4 \left[-\log\mu + \chi + \mathcal{O}(\mu) + \mathcal{O}\left(\frac{mc^2}{\mu k_B T_\gamma}\right)\right]. \tag{19}$$

As could be appreciated physically, it is the logarithm, which couples the process to the low-energy regime, which makes the thermal mass a leading-order effect. Corrections to external fermion propagators are sub-leading, as we would expect the corrections to the external photon propagators to be also. It is possible that the effects of a growing thermal fermion background in the process of equilibration could be included naturally in this formalism, by making the fermion thermal mass dependent on the background fermion density. Of course a much more substantial treatment would be needed both to rigorously justify this expression, since it is unclear whether the introduction of a thermal mass is really adequate to handle the interaction between the virtual fermion and the thermal background, and to use it, since the literature results for dispersion relations in a thermal background are restricted to the case of the fermion and photon field in equilibrium, where the notion of a particle creation rate has little meaning.

Quantities of a similar form have been found by other authors for similar quantities in high-temperature limits: the reverse process, i.e. the rate of pair annihilation in electrons and positrons in Maxwell–Jüttner distributions [13–15]; the rate of positron production in a plasma from electron and ion collisions [15]; the mean path of a high-energy non-thermal photon in a thermal bath [16]; and various quantities in an optically-thin relativistic plasma of finite size [17].

CRediT authorship contribution statement

J.J. Beesley: Formal analysis, Writing, Visualisation. **S.J. Rose:** Conceptualisation, Formal analysis, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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