Calogero type bounds in two dimensions

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Abstract

For a Schrödinger operator on the plane $\mathbb{R}^2$ with electric potential V and Aharonov–Bohm magnetic field we obtain an upper bound on the number of its negative eigenvalues in terms of the $L^1(\mathbb{R}^2)$-norm of V. Similar to Calogero's bound in one dimension, the result is true under monotonicity assumptions on V. Our proof method relies on a generalisation of Calogero's bound to operator-valued potentials. We also establish a similar bound for the Schrödinger operator (without magnetic field) on the half-plane when a Dirichlet boundary condition is imposed and on the whole plane when restricted to antisymmetric functions.

Keywords: Calogero's bound, Aharonov–Bohm magnetic field, Cwikel–Lieb–Rozenblum inequality, Hardy inequality

MSC Classification: Primary: 35P15; Secondary: 81Q10

1 Introduction and main results

For a self-adjoint Schrödinger operator $H_0 - V = -\Delta - V$ on $L^2(\mathbb{R}^d)$ with non-negative potential $V \in L^{d/2}(\mathbb{R}^d)$ in dimension $d \geq 3$ the celebrated Cwikel–Lieb–Rozenblum inequality [5, 21, 25] provides an upper bound on the number

$N(H_0 - V)$ of negative eigenvalues. This so-called CLR bound states that

$$N(H_0 - V) \leq C_d \int_{\mathbb{R}^d} V(x)^{d/2} \, \mathrm{d}x \tag{1}$$

with a constant C_d independent of V. In dimensions $d = 1$ and $d = 2$ such a bound cannot hold true since 0 is a resonance state of the spectrum, i.e. for any potential $V \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}^d} V(x)^{d/2} \, \mathrm{d}x > 0$ the Schrödinger operator $H_0 - V$ already has at least one negative eigenvalue. In dimension $d = 2$, the operator $H_0 - V$ may not even be semibounded under the assumption $V \in L^1(\mathbb{R}^2)$. For (1) to be true, repulsive terms are needed, which can be provided through additional electric potentials, magnetic fields, boundary conditions and/or particle statistics.

In dimension $d = 1$ one can for example restrict to $\mathbb{R}_+$ with Dirichlet condition at the origin. If the potential V is furthermore non-increasing, i.e. $V(x) \geq V(x')$ for $x \leq x'$, the well known Calogero bound [3]

$$N(H_0 - V) \leq \frac{2}{\pi} \int_{\mathbb{R}_+} V(x)^{1/2} \, \mathrm{d}x \tag{2}$$

holds true. Our first result can be seen as a generalisation of this bound to dimension $d = 2$. To this end we define the half-plane $\mathbb{R}_+^2 = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \, x_2 \in \mathbb{R}\}$.

Theorem 1 *Let $V \in L^1(\mathbb{R}_+^2), V \geq 0$ be non-increasing in x_1, i.e. $V(x_1, x_2) \geq V(x_1', x_2)$ for $x_1 \leq x_1'$. Then the number of negative eigenvalues of the operator $H_0 - V = -\Delta - V$ with Dirichlet boundary condition on $x_1 = 0$ satisfies the inequality*

$$N(H_0 - V) \leq C \int_{\mathbb{R}_+^2} V(x) \, \mathrm{d}x,$$

where the constant $C \leq 4.32$ is independent of V.

In dimension $d = 2$ the resonance state at zero can also be removed by adding an extra positive "Hardy term" $\frac{b}{x^2}$ with $b > 0$ to the operator. However, for general $V \in L^1(\mathbb{R}^2)$, the operator $H_b - V := -\Delta + \frac{b}{x^2} - V$ may still not be semibounded, requiring further restrictions on the potential. In [14] a CLR bound (1) for $N(H_b - V)$ was proved for a class of potentials $V(x) = V(|x|)$ depending only on $|x|$. In [16] this result was extended to a class of potentials belonging to $L^1(\mathbb{R}_+, r \, \mathrm{d}r; \; L^p(\mathbb{S}^1))$, $1 < p \leq \infty$, still assuming that there is an additional positive Hardy term. In polar coordinates the bound then takes the form

$$N(H_b - V) \leq C_{b,p} \int_{\mathbb{R}_+} \|V(r, \cdot)\|_{L^p(\mathbb{S}^1)} r \, \mathrm{d}r \, .$$

In [1] the authors noticed that the Hardy term could be provided by the Laplacian with an Aharonov–Bohm magnetic vector potential A with non-integer flux Ψ, see [20]. For the operator $H_A - V = (-\mathrm{i}\nabla + A)^2 - V$ with potential $V \in L^1(\mathbb{R}_+, r\,\mathrm{d}r;\ L^\infty(\mathbb{S}^1))$ the authors proved that

$$N(H_A - V) \le C_\Psi \int_{\mathbb{R}_+} \|V(r,\cdot)\|_{L^\infty(\mathbb{S}^1)} r\,\mathrm{d}r\,.$$

Only in the case $V(x) = V(|x|)$ (where the sharp value of the constant C_Ψ was obtained in [15]) the right-hand side coincides with the $L^1(\mathbb{R}^2)$-norm of V as in the CLR bound (1).

In our paper we will present a bound (1) for potentials $V \in L^1(\mathbb{R}^2)$ satisfying a monotonicity condition. To be more precise, we consider the Schrödinger operator with Aharonov–Bohm type magnetic field, i.e. the operator

$$H_A = (-\mathrm{i}\nabla + A(x))^2$$

on $L^2(\mathbb{R}^2)$ with $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined, in polar coordinates, as

$$A(r,\varphi) = \frac{\psi(\varphi)}{r}(\sin\varphi, -\cos\varphi)$$

where $\psi \in L^1(\mathbb{S}^1)$. The flux is given by $\Psi = (2\pi)^{-1}\int_{\mathbb{S}^1}\psi(\varphi)\,\mathrm{d}\varphi$. Our main result is the following.

Theorem 2 *Let $V \in L^1(\mathbb{R}^2), V \ge 0$ be a potential that is non-increasing along any ray from the origin, i.e. in polar coordinates $V(r,\varphi) \ge V(r',\varphi)$ for $r \le r'$. If $\Psi \notin \mathbb{Z}$ then the number of negative eigenvalues of the operator $H_A - V$ satisfies the inequality*

$$N(H_A - V) \le C_\Psi \int_{\mathbb{R}_+} \|V(r,\cdot)\|_{L^1(\mathbb{S}^1)} r\,\mathrm{d}r = C_\Psi \int_{\mathbb{R}^2} V(x)\,\mathrm{d}x$$

where the constant C_Ψ is independent of V.

Remark 1 The results of Theorem 1 and Theorem 2 cannot hold for general integrable potential V even under the additional assumption that V is smooth and has compact support. To see this, one can consider any potential $V \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x > 0$. Since 0 is a resonance state of the Schrödinger operator on $\mathbb{R}^2$ there exists $u \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\langle u, (-\Delta - V)u\rangle_2 < 0$. Simultaneously translating V and u to $\widetilde{V}$ and $\widetilde{u}$ such that both are entirely supported in the half-plane $\mathbb{R}^2_+$ then shows that the operator in Theorem 1 with potential $\widetilde{V}$ also has at least one negative eigenvalue. For the magnetic operator, we can argue as in [1, Section 5] and note that, since $\mathrm{curl} A = 0$ on the simply connected set $\mathbb{R}^2_+$, the Poincaré transformation yields a gauge transform such that H_A and $-\Delta$ are equivalent in $\mathbb{R}^2_+$. This shows that the operator in Theorem 2 with potential $\widetilde{V}$ has at least one negative eigenvalue.

Finally, we note that a positive Hardy term can also be provided by restricting the Laplacian to antisymmetric functions (see for example [9]), i.e. by considering $-\Delta - V$ on the Hilbert space $L^2_{\mathrm{as}}(\mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2) : u(x_1, x_2) = -u(x_2, x_1)\}$.

Theorem 3 *Let $V \in L^1(\mathbb{R}^2), V \geq 0$ be a potential that is non-increasing along any ray from the origin, i.e. in polar coordinates $V(r, \varphi) \geq V(r', \varphi)$ for $r \leq r'$. Then the number of negative eigenvalues of the operator $H_{\mathrm{as}} - V = -\Delta - V$ on the space $L^2_{\mathrm{as}}(\mathbb{R}^2)$ of antisymmetric functions satisfies*

$$N(H_{\mathrm{as}} - V) \leq C \int_{\mathbb{R}^2} V(x)\, \mathrm{d}x,$$

where the constant $C \leq 5.43$ is independent of V.

Substantial efforts were made (so far without any success) in finding necessary and sufficient conditions on a class of potentials that provide a finite number of negative eigenvalues for a two-dimensional Schrödinger operator and also necessary and sufficient conditions for the validity of the Weyl asymptotics. In [2] the authors gave examples of different non-Weyl law formulae under the condition $V \in L^1(\mathbb{R}^2)$. Some upper estimates for $N(H_0 - V)$ in the two-dimensional case were obtained in papers [8, 12, 17, 18, 23, 24, 28]. In [27] the author gives estimates for the number of negative eigenvalues of a two-dimensional Schrödinger operator in terms of $L \log L$ type Orlicz norms of the potential and proves a conjecture by N. N. Khuri, A. Martin and T. T. Wu [13] (see also [4]). The fact that the classes $L \log L$ of potential functions are relevant to estimates for $N(H_0 - V)$ was first discovered by M. Solomyak [26].

The proofs of Theorem 1, Theorem 2 and Theorem 3 all use the same central idea and employ the "lifting argument" presented in [19] (see also [11] and [7]). In each setting a Hardy inequality holds, which allows us to add a Hardy term at the expense of reducing the 'kinetic' part of the operator. Using the structure of the operators involved, all three problems can then be reduced to studying corresponding one-dimensional differential operators with operator-valued potentials. The Hardy term allows us to prove Calogero type bounds for these operators in terms of 1/2 Schatten norms of the operator-valued potentials. Lastly, upper bounds on these norms can be established through Lieb–Thirring type inequalities [22]. To this end, in the case of Theorem 2, we will prove a new inequality of this type for the magnetic operator on $L^2(\mathbb{S}^1)$, following the proof in [29] of a similar bound on $L^2(\mathbb{R})$. Note that for such operators in [6] the authors obtained a range of Keller–Lieb–Thirring inequalities for the lowest eigenvalue. In our proofs we will also show that the operators involved can be defined as semibounded, self-adjoint operators.

As a first step we will consider one-dimensional differential operators with operator-valued potentials in Section 2. Under the assumed presence of an additional Hardy term, we will establish Calogero-type bounds for these operators. As a corollary, we will obtain a generalisation of Calogero's bound (2) to operator-valued potentials. Sections 3, 4 and 5 then contain the proofs of our theorems.

2 Calogero type bounds for 1D operators with operator-valued potentials and Hardy term

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and let $\mathcal{V}(t)$, $t \geq 0$, be an operator-valued function (potential), whose values are compact, non-negative self-adjoint operators in $\mathcal{H}$. By $\mathfrak{S}_p$, $p > 0$, we denote the Schatten class of compact operators whose s-numbers satisfy the inequality

$$\sum_{n=1}^{\infty} s_n^p < \infty.$$

As usual $\mathfrak{S}_\infty$ denotes the class of compact operators. For $f \in \mathcal{C}^\infty(\overline{\mathbb{R}_+})$ with $f(t) > 0, r \in \mathbb{R}_+$ we consider the Hilbert space $L^2(\mathbb{R}_+, f(t)\,\mathrm{d}t;\ \mathcal{H})$. For our applications it will be sufficient to consider $f(t) = 1$ and $f(t) = t$, corresponding to the radial part of the Laplacian in one and two dimensions, respectively. We will prove the results in slightly more generality though, assuming that f is non-decreasing and a polynomial. We expect the results to also hold if the latter assumption is weakened, for example to that of [30, Theorem 15.2] along with the assumption that $\sup_{k\in\mathbb{Z}} f(a^{k+1})/f(a^k) < \infty$ for some $a > 1$. The subspace $\mathcal{C}_0^\infty(\mathbb{R}_+;\ \mathcal{H})$, i.e. the set of functions $U : \mathbb{R}_+ \to \mathcal{H}$ which are smooth (with respect to the topologies on $\mathbb{R}_+$ and $\mathcal{H}$) with compact support in $\mathbb{R}_+$, is dense in $L^2(\mathbb{R}_+, f(t)\,\mathrm{d}t;\ \mathcal{H})$. If $\mathcal{V} \in L^1(\mathbb{R}_+;\ \mathfrak{S}_\infty)$ then, for any $b \geq 0$, the quadratic form

$$\int_{\mathbb{R}_+} \left(\left\| \frac{\mathrm{d}}{\mathrm{d}t} U(t) \right\|_{\mathcal{H}}^2 + \frac{b}{t^2} \|U(t)\|_{\mathcal{H}}^2 - \langle U(t), \mathcal{V}(t) U(t) \rangle_{\mathcal{H}} \right) f(t)\,\mathrm{d}t$$

is well-defined on $\mathcal{C}_0^\infty(\mathbb{R}_+;\ \mathcal{H})$. It corresponds to the symmetric operator

$$H_b - \mathcal{V} = -\frac{1}{f(t)} \frac{\mathrm{d}}{\mathrm{d}t} f(t) \frac{\mathrm{d}}{\mathrm{d}t} \otimes \mathbb{I} + \frac{b}{t^2} \otimes \mathbb{I} - \mathcal{V}\,.$$

If $\mathcal{V} \in L^1(\mathbb{R}_+;\ \mathfrak{S}_\infty)$, then the quadratic form is semibounded. We can thus consider the self-adjoint Friedrichs extension of $H_b - \mathcal{V}$, which we refer to as the operator with Dirichlet boundary condition and which we continue to denote by $H_b - \mathcal{V}$. Finally, we say that the family of operators $\mathcal{V}(t)$ is non-increasing if $\mathcal{V}(t) \geq \mathcal{V}(t')$ for $0 \leq t \leq t'$ in the usual sense of quadratic forms. Namely, for any fixed vector $U \in \mathcal{H}$

$$\langle U, \mathcal{V}(t) U \rangle_{\mathcal{H}} \geq \langle U, \mathcal{V}(t') U \rangle_{\mathcal{H}}, \qquad 0 \leq t \leq t'\,.$$

Proposition 4 *Let $\mathcal{H}$ and f be as above. Furthermore let $\mathcal{V} \in L^1(\mathbb{R}_+;\ \mathfrak{S}_{1/2}), \mathcal{V} \geq 0$ be non-increasing. Then the number of negative eigenvalues of the operator $H_b - \mathcal{V}$ with Dirichlet boundary condition satisfies*

$$N(H_b - \mathcal{V}) \leq C_{b,f} \int_{\mathbb{R}_+} \sqrt{\|\mathcal{V}(t)\|_{\mathfrak{S}_{1/2}}}\,\mathrm{d}t$$

where the constant $C_{b,f}$ is independent of $\mathcal{V}$.

Proof For $U \in \mathcal{C}_0^\infty(\mathbb{R}_+;\ \mathcal{H})$ there exists $k \in \mathbb{N}$ such that $\operatorname{supp}(U) \subset [2^{-k}, 2^k]$ and thus

$$\begin{aligned}
&\langle U, (H_b - \mathcal{V})U\rangle_2 \\
&= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\|U'(t)\|_{\mathcal{H}}^2 + \frac{b}{t^2}\|U(t)\|_{\mathcal{H}}^2 - \langle U(t), \mathcal{V}(t)U(t)\rangle_{\mathcal{H}} \right) f(t)\,\mathrm{d}t \\
&\geq \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\|U'(t)\|_{\mathcal{H}}^2 + \frac{b}{t^2}\|U(t)\|_{\mathcal{H}}^2 - \frac{f(2^{k+1})}{f(2^k)} \Big\langle U(t), \mathcal{V}(2^k)U(t)\Big\rangle_{\mathcal{H}} \right) f(2^k)\,\mathrm{d}t\,.
\end{aligned}$$

Let $B_k := \{t \in \mathbb{R}_+ : t \in (2^k, 2^{k+1})\}$. Each of the quadratic forms above is closed on $H^1(B_k;\ \mathcal{H})$. We can thus consider the corresponding self-adjoint operator h_k. By the variational principle $H_b - \mathcal{V} \geq \bigoplus_{k\in\mathbb{Z}} h_k$.

Now look on each h_k and see

$$\begin{aligned}
N(h_k) &= \sup\dim \Bigg\{ F \subset H^1(B_k) : \\
&\qquad \int_{2^k}^{2^{k+1}} \left(\|U'\|_{\mathcal{H}}^2 + \frac{b}{r^2}\|U\|_{\mathcal{H}}^2 - \frac{f(2^{k+1})}{f(2^k)} \Big\langle U, \mathcal{V}(2^k)U\Big\rangle_{\mathcal{H}} \right) \mathrm{d}t < 0, U \in F \Bigg\} \\
&= \sup\dim \Bigg\{ F \subset H^1(B_0) : \\
&\qquad \int_1^2 \left(\|U'\|_{\mathcal{H}}^2 + \frac{b}{s^2}\|U\|_{\mathcal{H}}^2 - 2^{2k}\frac{f(2^{k+1})}{f(2^k)} \Big\langle U, \mathcal{V}(2^k)U\Big\rangle_{\mathcal{H}} \right) \mathrm{d}s < 0, U \in F \Bigg\} \\
&\leq \sup\dim \Bigg\{ F \subset H^1(B_0) : \\
&\qquad \int_1^2 \left(\|U'\|_{\mathcal{H}}^2 + \frac{b}{4}\|U\|_{\mathcal{H}}^2 - 2^{2k}\frac{f(2^{k+1})}{f(2^k)} \Big\langle U, \mathcal{V}(2^k)U\Big\rangle_{\mathcal{H}} \right) \mathrm{d}s < 0, U \in F \Bigg\}.
\end{aligned}$$

On $L^2((1,2);\ \mathcal{H})$ we now consider the operator with constant coefficients

$$-\frac{\mathrm{d}^2}{\mathrm{d}s^2} \otimes \mathbb{I} + \frac{b}{4} \otimes \mathbb{I} - 2^{2k}\frac{f(2^{k+1})}{f(2^k)}\mathcal{V}(2^k)$$

and Neumann boundary conditions in the eigenbasis of the compact, non-negative operator $\mathcal{V}(2^k)$. Denoting by $\mu_n(2^k) \geq 0$ the eigenvalues of $\mathcal{V}(2^k)$ we find

$$N(h_k) \leq \sum_n \left(\#\left\{ m \in \mathbb{N}_0 : \pi^2 m^2 + \frac{b}{4} < 2^{2k}\frac{f(2^{k+1})}{f(2^k)}\mu_n(2^k) \right\} \right).$$

Each of the summands above can be bounded by $R(b)2^k\sqrt{f(2^{k+1})/f(2^k)}\sqrt{\mu_n(2^k)}$ with some constant $R(b)$ depending only on b (see Remark 2 below for an explicit choice) and thus

$$N(h_k) \leq R(b)2^k\sqrt{\frac{f(2^{k+1})}{f(2^k)}}\sum_n\sqrt{\mu_n(2^k)} \leq R(b)2^k\sqrt{\frac{f(2^{k+1})}{f(2^k)}}\sqrt{\|\mathcal{V}(2^k)\|_{\mathfrak{S}_{1/2}}}.$$

Importantly $\mathcal{V}(t)$ is a non-increasing operator-valued function in t and thus

$$\begin{aligned} N(H_b - \mathcal{V}) &\leq \sum_{k\in\mathbb{Z}} N(h_k) \leq R(b)\sum_{k\in\mathbb{Z}} 2^k\sqrt{\frac{f(2^{k+1})}{f(2^k)}}\sqrt{\|\mathcal{V}(2^k)\|_{\mathfrak{S}_{1/2}}} \\ &\leq 2R(b)\sup_{k\in\mathbb{Z}}\sqrt{\frac{f(2^{k+1})}{f(2^k)}}\int_{\mathbb{R}_+}\sqrt{\|\mathcal{V}(t)\|_{\mathfrak{S}_{1/2}}}\,\mathrm{d}t\,. \end{aligned}$$

□

Remark 2 (Explicit constant) Here we derive an explicit upper bound on $R(b)$: Consider the inequality $(\alpha m^2+\beta)/(m+1)^2 \geq \alpha\beta/(\alpha+\beta)$, for $\alpha,\beta,m\geq 0$. Then for sets of the type above, we find

$$\begin{aligned} \#\{m\in\mathbb{N}_0 :\, \alpha m^2+\beta<\gamma^2\} &\leq \#\{m\in\mathbb{N}_0 :\, (\alpha m^2+\beta)(m+1)^2/(m+1)^2<\gamma^2\} \\ &\leq \#\{m\in\mathbb{N} :\, (\alpha\beta/(\alpha+\beta))m^2<\gamma^2\} \leq \gamma\sqrt{\frac{\alpha+\beta}{\alpha\beta}}. \end{aligned}$$

A direct consequence of this is the upper bound $R(b)\leq\sqrt{\frac{b+4\pi^2}{\pi^2 b}}$. We note that we can consider a more general scheme, where we split $\mathbb{R}_+$ by the intervals (a^k, a^{k+1}), with $a>1$. Applying the above to this general setting, we see that

$$C_{b,f} \leq aR\Big(\frac{4(a-1)^2 b}{a^2}\Big)\sup_{k\in\mathbb{Z}}\sqrt{\frac{f(a^{k+1})}{f(a^k)}}$$

where $C_{b,f}$ is the constant in Proposition 5.

An immediate consequence is a generalisation of Calogero's bound (2) to operator-valued potentials.

Theorem 5 *Let $\mathcal{V}\in L^1(\mathbb{R}_+;\ \mathfrak{S}_{1/2}), \mathcal{V}\geq 0$ be non-increasing. Then the number of negative eigenvalues of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\otimes\mathbb{I}-\mathcal{V}$ with Dirichlet boundary condition satisfies*

$$N\Big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\otimes\mathbb{I}-\mathcal{V}\Big) \leq C\int_{\mathbb{R}_+}\sqrt{\|\mathcal{V}(t)\|_{\mathfrak{S}_{1/2}}}\,\mathrm{d}t,$$

where $C\leq 8.63$ is a constant independent of $\mathcal{V}$.

Proof If $U\in C_0^\infty(\mathbb{R}_+;\ \mathcal{H})$, we can apply the standard Hardy inequality on $\mathbb{R}_+$ and obtain

$$\int_{\mathbb{R}_+}\|U'(t)\|_{\mathcal{H}}^2\,\mathrm{d}t \geq \frac{1}{4}\int_{\mathbb{R}_+}\frac{\|U(t)\|_{\mathcal{H}}^2}{t^2}\,\mathrm{d}t.$$

Let $0 < \vartheta < 1$. Splitting the operator of the second derivative $-\mathrm{d}^2/\mathrm{d}t^2 = -(1-\vartheta)\mathrm{d}^2/\mathrm{d}t^2 - \vartheta\mathrm{d}^2/\mathrm{d}t^2$ and using the Hardy inequality we have (with $f(t) = 1$)

$$N(H_0 - \mathcal{V}) \leq N((1-\vartheta)H_{\vartheta/(1-\vartheta)} - \mathcal{V}),$$

where we used that $\mathcal{C}_0^\infty(\mathbb{R}_+;\ \mathcal{H})$ is a form-core of the operators involved. We can apply the proposition to obtain the desired result with $C = 2R(\vartheta/4(1-\vartheta))(1-\vartheta)^{-1/2}$. □

Remark 3 (Explicit constant) We find the upper bound $C \leq 8.63$ by following Remark 2. Dividing into intervals of order a and optimising over a and ϑ yields the best value when $a = 1.92882$ and $\vartheta = 0.928815$. The best ϑ being close to 1 is, we believe, a consequence of our proof method despite even small ϑ being sufficient to remove the zero-modes in the Neumann bracketing argument.

Open problem. We do not believe the constant 8.63 in Theorem 5 to be optimal and thus it would be interesting to find the sharp constant. If $\mathcal{V}$ is diagonal, then the bound holds with the same sharp constant $2/\pi$ as in the scalar case.

3 Proof of Theorem 1

To prove Theorem 1 we use the "lifting argument" developed in [19]. Noting that $L^2(\mathbb{R}^2_+)$ is isomorphic to $L^2(\mathbb{R}_+;\ L^2(\mathbb{R}))$ and, using the structure of the Laplacian, we write

$$H_0 - V = -\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} \otimes \mathbb{I} - W(x_1),$$

where

$$W(x_1) = \frac{\mathrm{d}^2}{\mathrm{d}x_2^2} + V(x_1, x_2)$$

is an operator-valued potential in $L^2(\mathbb{R})$. By Fubini's theorem $V(x_1, \cdot) \in L^1(\mathbb{R})$ for almost all $x_1 \in \mathbb{R}_+$ and thus $W(x_1)$ is a self-adjoint operator on the domain $H^1(\mathbb{R})$ for almost all $x_1 \in \mathbb{R}_+$. As we will see below, its positive part $W_+(x_1)$ is a compact operator. Furthermore on $\mathcal{C}_0^\infty(\mathbb{R}_+;\ H^1(\mathbb{R})) \subset L^2(\mathbb{R}_+;\ L^2(\mathbb{R}))$ the operator inequality

$$-\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} \otimes \mathbb{I} - W(x_1) \geq -\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} \otimes \mathbb{I} - W_+(x_1)$$

holds. As discussed in Section 2 the operator on the right-hand side is semi-bounded and consequently the same holds true for the former operator on the left-hand side. We can thus consider their respective self-adjoint Friedrichs extensions, which we continue to denote by the same symbols and which still satisfy the operator inequality. This establishes that the operator in Theorem 1 is well-defined. By the variational principle

$$N(H_0 - V) \leq N\left(-\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} \otimes \mathbb{I} - W_+(x_1)\right).$$

Note that $W_+(x_1)$ is a non-increasing operator-valued function. Applying Theorem 5 we find

$$N(H_0 - V) \leq C \int_{\mathbb{R}_+} \sqrt{\|W_+(x_1)\|_{\mathfrak{S}_{1/2}}}\, \mathrm{d}x_1. \tag{3}$$

For each x_1 we use the sharp result on the 1/2 moments for a one-dimensional Schrödinger operator [10] (see also [11]) saying that, for $v \in L^1(\mathbb{R}), v \geq 0$,

$$\operatorname{Tr}\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - v(t)\right)_-^{1/2} \leq \frac{1}{2}\int_{\mathbb{R}} v(t)\, \mathrm{d}t. \tag{4}$$

Applying (4) to the operator $W_+(x_1)$ we have

$$\sqrt{\|W_+(x_1)\|_{\mathfrak{S}_{1/2}}} \leq \frac{1}{2}\int_{\mathbb{R}} V(x_1, x_2)\, \mathrm{d}x_2.$$

Combining the latter inequality with (3) completes the proof of Theorem 1.

Remark 4 (Explicit constant) From Remark 3, the best constant known to us is $C \approx 4.31244$.

4 Proof of Theorem 2

Since the operator with general ψ and flux Ψ is gauge equivalent to the operator with constant $\psi = \Psi$, we assume without loss of generality that $\psi = \Psi$. In polar coordinates the quadratic form of H_A on $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ is given by

$$\langle u, H_A u\rangle_2 = \int_0^\infty \int_{\mathbb{S}^1} \left(\left|\frac{\partial}{\partial r}u(r,\varphi)\right|^2 + \frac{1}{r^2}\left|\mathrm{i}\frac{\partial}{\partial \varphi}u(r,\varphi) + \Psi u(r,\varphi)\right|^2\right) \mathrm{d}\varphi\, r\, \mathrm{d}r$$

and on this space the Hardy inequality [20]

$$\langle u, H_A u\rangle_2 \geq c_\Psi \int_{\mathbb{R}_+}\int_{\mathbb{S}^1} \frac{|u(r,\varphi)|^2}{r^2}\, \mathrm{d}\varphi\, r\, \mathrm{d}r \tag{5}$$

holds with $c_\Psi = \min_{k\in\mathbb{Z}} |\Psi + k|^2$.

Let $0 < \vartheta < 1$. Splitting the operator $H_A = (1-\vartheta)H_A + \vartheta H_A$ and using (5) we have the operator inequality

$$H_A - V \geq \widetilde{H}_A - V,$$

on $\mathcal{C}_0^\infty(\mathbb{R}^2 \setminus \{0\})$ with

$$\widetilde{H}_A = (1-\vartheta)H_A + \vartheta \frac{c_\Psi}{|x|^2}\,.$$

The space $L^2(\mathbb{R}_+ \times \mathbb{S}^1, r\,\mathrm{d}r\,\mathrm{d}\varphi)$ is isomorphic to $L^2(\mathbb{R}_+, r\,\mathrm{d}r;\; L^2(\mathbb{S}^1))$ which allows us to write

$$\widetilde{H}_A - V = -(1-\vartheta)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r} \otimes \mathbb{I} + \vartheta \frac{c_\Psi}{r^2} \otimes \mathbb{I} - W(r) \tag{6}$$

with the operator-valued potential

$$W(r) = -\frac{1-\vartheta}{r^2}\left(\mathrm{i}\frac{\partial}{\partial\varphi} + \Psi\right)^2 + V(r,\varphi)\,.$$

For almost all $r \in \mathbb{R}_+$, the potential is self-adjoint on $H^1(\mathbb{S}^1)$ by the assumptions on V and we can consider its positive part $W_+(r)$, which is compact as we will show below. Arguing similarly as in Section 3 the operators above are all semibounded and we can thus consider their respective Friedrichs extensions. This establishes that the operator $H_A - V$ in Theorem 2 is well-defined. Note that W_+ is a non-increasing operator-valued function in r and thus by Proposition 4 and the variational principle (note that $f(t) = t$ and $\sup_{k\in\mathbb{Z}} f(2^{k+1})/f(2^k) = 2$)

$$N(H_A - V) \leq 2^{3/2} R\Big(\frac{c_\Psi \vartheta}{1-\vartheta}\Big)(1-\vartheta)^{-1/2} \int_{\mathbb{R}+} \sqrt{\|W_+(r)\|_{\mathfrak{S}_{1/2}}}\,\mathrm{d}r\,. \tag{7}$$

To bound the integrand we need a Lieb–Thirring type inequality for the 1/2 moments of a Schrödinger operator $W(r)$ with constant magnetic vector potential on $L^2(\mathbb{S}^1)$, which we will prove in Lemma 1 below. Combining this bound with (7) we obtain the desired inequality

$$N(H_A - V) \leq 2^{3/2} R\Big(\frac{c_\Psi \vartheta}{1-\vartheta}\Big)(1-\vartheta)^{-1} d_\Psi \int_{\mathbb{R}+}\int_{\mathbb{S}^1} V(r,\varphi) r\,\mathrm{d}\varphi\,\mathrm{d}r\,.$$

Lemma 1 *Let $v \in L^1(-\pi,\pi), v \geq 0$ and $\Psi \notin \mathbb{Z}$. Then the operator*

$$h - v := \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi} + \Psi\right)^2 - v$$

with periodic boundary conditions satisfies

$$\mathrm{Tr}\left(\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi}+\Psi\right)^2-v\right)_-^{1/2}\leq d_\Psi\int_{-\pi}^{\pi}v(\varphi)\,\mathrm{d}\varphi$$

with a constant d_Ψ independent of v.

We will see that $d_\Psi\to\infty$ as $\Psi\to k\in\mathbb{Z}$. Our proof will follow [29], where a Lieb–Thirring inequality for the 1/2 moments of a Schrödinger operator on $\mathbb{R}$ (i.e. the bound (4) without the sharp constant) was proved. The main idea of the proof in [29] is to use Neumann bracketing whereby $\mathbb{R}$ is partitioned into disjoint intervals that each support at most one eigenvalue. Importantly the length of each interval (compared to the L^1-norm of the potential on the interval) can be chosen to be uniformly bounded from below. To achieve the latter on the bounded set $(-\pi,\pi)$ we will use that for $\Psi\notin\mathbb{Z}$ the operator h does not have any zero-modes and that $h-v$ does not admit any negative eigenvalues if $\int_{-\pi}^{\pi}v(\varphi)\,\mathrm{d}\varphi$ is small. The desired partition can then be constructed with multiplicity 2. The arguments of [29] all carry over, as we will show in detail below. We start with recalling the following from [29, Lemmata 1 and 2].

Lemma 2 *Let $q\in L^1(0,\ell), q\geq 0$ and consider the operator $-\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}-q$ on $L^2(0,\ell)$ with Neumann boundary conditions.*

(i) With g denoting the inverse of the strictly increasing function $x\tanh x$ on $\mathbb{R}_+$, the lowest eigenvalue λ_1 of the operator is bounded as

$$|\lambda_1|^{1/2}\leq\frac{g(\ell\int_0^\ell q(\varphi)\,\mathrm{d}\varphi)}{\ell}\,.$$

(ii) If

$$\ell\int_0^\ell q(\varphi)\,\mathrm{d}\varphi\leq 3$$

then the operator has a single negative eigenvalue.

We now give the proof of Lemma 1.

Proof of Lemma 1 For $\lambda<\min_{k\in\mathbb{Z}}(k-\Psi)^2$ the resolvent kernel of the free operator $\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi}+\Psi\right)^2$ with periodic boundary conditions is

$$G_\lambda(\eta,\eta')=G_\lambda(\eta-\eta')=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\frac{\mathrm{e}^{-\mathrm{i}n(\eta-\eta')}}{(n+\Psi)^2-\lambda}\,.$$

Using the Poisson summation formula we can compute that

$$G_0(0)=\frac{\pi}{2\sin(\Psi\pi)^2}\,.$$

The number of eigenvalues of $h-v$ below 0 is bounded from above by the trace of the Birman–Schwinger operator $K_0=\sqrt{v}h^{-1}\sqrt{v}$, i.e. by $G_0(0)\int_{-\pi}^{\pi}v(\varphi)\,\mathrm{d}\varphi$.

In the remainder we may thus assume that $\int_{-\pi}^{\pi} v(\varphi)\,\mathrm{d}\varphi \geq 1/G_0(0) = 2\sin(\Psi\pi)^2/\pi$, otherwise there are no negative eigenvalues. We now iteratively decompose $[-\pi,\pi]$ into intervals $I_k = [x_k, x_{k+1}]$ with $x_1 = -\pi$ and

$$|I_k| \int_{I_k} 2v(\varphi)\,\mathrm{d}\varphi = \min(3, 4\pi/G_0(0)) =: \varepsilon\,. \tag{8}$$

Here it is essential that we assumed $2\pi\int_{-\pi}^{\pi} v(\varphi)\,\mathrm{d}\varphi \geq 2\pi/G_0(0)$ as otherwise it would not be possible to define I_1. Since $|I_k| \geq \min(3, 4\pi/G_0(0))/\int_{-\pi}^{\pi} 2v(\varphi)\,\mathrm{d}\varphi$ it will eventually occur that at some n

$$(\pi - x_n)\int_{x_n}^{\pi} 2v(\varphi)\,\mathrm{d}\varphi < \min(3, 4\pi/G_0(0))\,.$$

We then end the construction and define the last interval as $I_n = [y,\pi]$ with $y \leq x_n$ such that (8) holds. We thus have covered $[-\pi,\pi]$ by $n-1$ disjoint intervals I_k and one additional interval I_n that intersects with (only) I_{n-1}. The multiplicity of this covering is at most 2. Importantly (8) holds for all I_k. For $u \in H^1(-\pi,\pi)$ we compute

$$\langle u, (h-v)u\rangle_2 = \int_{-\pi}^{\pi} (|\mathrm{i}u' + \Psi u|^2 - v|u|^2)\,\mathrm{d}\varphi \geq \sum_{k=1}^{n}\int_{I_k}\left(\frac{1}{2}|\mathrm{i}u' + \Psi u|^2 - v|u|^2\right)\mathrm{d}\varphi.$$

Note that each of the quadratic forms in the sum above is closed on $H^1(I_k)$. We now consider an operator $H := \bigoplus_{k=1}^{n} h_k$ on $\mathcal{H} := \bigoplus_{k=1}^{n} L^2(I_k)$. Each h_k is defined as the self-adjoint operator corresponding to the k-th quadratic form in the above sum on $H^1(I_k)$. If $u \in H^1(-\pi,\pi)$ then $\bigoplus_{k=1}^{n} u|_{I_k} \in \bigoplus_{k=1}^{n} H^1(I_k)$ and

$$\frac{1}{2}\left\|\bigoplus_{k=1}^{n} u|_{I_k}\right\|_{\mathcal{H}}^2 \leq \|u\|_2^2 \leq \left\|\bigoplus_{k=1}^{n} u|_{I_k}\right\|_{\mathcal{H}}^2.$$

Using the variational principle and the above bounds, we can conclude that the negative eigenvalues $\lambda_m(h)$ of h satisfy

$$\begin{aligned}\lambda_m(h-v) &\geq \inf_{\substack{F\subset H^1(-\pi,\pi)\\ \dim F = m}} \sup_{u\in F\setminus\{0\}} \frac{-(\langle u, hu\rangle_2)_-}{\|u\|_2^2}\\ &\geq 2\inf_{\substack{F\subset H^1(-\pi,\pi)\\ \dim F = m}} \sup_{u\in F\setminus\{0\}} \frac{-(\langle \bigoplus_{k=1}^{n} u|_{I_k}, H\bigoplus_{k=1}^{n} u|_{I_k}\rangle_{\mathcal{H}})_-}{\|\bigoplus_{k=1}^{n} u|_{I_k}\|_{\mathcal{H}}^2}\end{aligned}$$

If F is an m-dimensional subspace of $L^2(-\pi,\pi)$, then $\{\bigoplus_{k=1}^{n} u|_{I_k} : u \in F\}$ is an m-dimensional subspace of $\mathcal{H}$ and thus

$$\lambda_m(h-v) \geq 2\inf_{\substack{F\subset \bigoplus_{k=1}^{n} H^1(I_k)\\ \dim F = m}} \sup_{U\in F\setminus\{0\}} \frac{-(\langle U, HU\rangle_{\mathcal{H}})_-}{\|U\|_{\mathcal{H}}^2} = 2\lambda_m(H)\,.$$

The negative eigenvalues $\lambda_m(H)$ of H coincide (including multiplicity) with the individual eigenvalues of the operators h_k. Via the transform $u \mapsto \mathrm{e}^{\mathrm{i}\Psi\varphi}u$ each h_k is seen to be unitarily equivalent to the operator

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} - v(\varphi)$$

with Neumann boundary conditions on I_k. Thus, by (8) and Lemma 2, each h_k has a single negative eigenvalue $\lambda_1(h_k)$ which is bounded by

$$|2\lambda_1(h_k)|^{\frac{1}{2}} \leq \frac{g(|I_k|\int_{I_k} 2v(\varphi)\,\mathrm{d}\varphi)}{|I_k|} = \frac{g(\varepsilon)}{|I_k|} = \frac{g(\varepsilon)}{\varepsilon}\int_{I_k} 2v(\varphi)\,\mathrm{d}\varphi\,.$$

We can conclude that

$$\sum_{m\geq 1} |\lambda_m(h-v)|^{\frac{1}{2}} \leq \sum_{k=1}^{n} |2\lambda_1(h_k)|^{\frac{1}{2}} \leq \sum_{k=1}^{n} \frac{g(\varepsilon)}{\varepsilon} \int_{I_k} 2v(\varphi)\,\mathrm{d}\varphi \leq \frac{4g(\varepsilon)}{\varepsilon} \int_{-\pi}^{\pi} v(\varphi)\,\mathrm{d}\varphi$$

which finishes the proof. If $\Psi \to k \in \mathbb{Z}$ then $1/G_0(0) \to 0$ and thus $\varepsilon \to 0$. Since $g(\varepsilon)/\varepsilon \to \infty$ as $\varepsilon \to 0$, we observe that $d_\Psi = 4g(\varepsilon)/\varepsilon \to \infty$. □

Remark 5 (Explicit constant) For particular Ψ, an upper bound on the constant in Theorem 2 can be found following the scheme of Remark 2.

5 Proof of Theorem 3

Finally, consider the operator $H_{\mathrm{as}} - V = -\Delta - V$ on $L^2_{\mathrm{as}}(\mathbb{R}^2)$. It is well known that for antisymmetic functions u, in $H^1(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x \geq \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\,\mathrm{d}x.$$

Repeating the arguments above, we arrive at an equation similar to (6) with $\Psi = 0$, $c_\Psi = 1$ and operator-valued potential

$$W(r) = \frac{1-\vartheta}{r^2}\frac{\partial^2}{\partial\varphi^2} + V(r,\varphi)$$

defined on the domain of periodic functions in $H^1(-\pi,\pi)$ which satisfy $u(\varphi) = -u(-\varphi)$. From Proposition 4 we have (with $f(t) = t$)

$$N(H_{\mathrm{as}} - V) \leq 2^{3/2} R\Big(\frac{\vartheta}{1-\vartheta}\Big)(1-\vartheta)^{-1/2} \int_{\mathbb{R}+} \sqrt{\|W_+(r)\|_{\mathfrak{S}_{1/2}}}\,\mathrm{d}r\,.$$

Since functions in the domain of $W(r)$ vanish at $\pm\pi$ we can extend them onto the whole line by zero and use the standard Lieb–Thirring bound (4) (where V is also extended by zero) to obtain

$$N(H_{\mathrm{as}} - V) \leq \sqrt{2} R\Big(\frac{\vartheta}{1-\vartheta}\Big)(1-\vartheta)^{-1} \int_{\mathbb{R}+}\int_{\mathbb{S}^1} V(r,\varphi) r\,\mathrm{d}\varphi\,\mathrm{d}r\,.$$

Remark 6 (Explicit constant) The constant in this case can be bounded above by 5.42152, following Remark 2.

Acknowlegements

A. Laptev is grateful to M. Z. Solomyak for useful discussions of Calogero inequality for matrix-valued potentials. A. Laptev was partially supported by

RSF grant 18-11-0032. L. Schimmer was supported by the VR grant 2017-04736 at Royal Swedish Academy of Sciences. The authors thank T. Weidl for a helpful discussion.

Declaration

Data sharing is not applicable to this article as no datasets were generated or analysed for this work.

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