Financial Contagion in Network Economies and Asset Prices

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This paper studies intertemporal asset pricing in network economies when distress shocks can propagate through the network, similarly to epidemic outbreaks. Two classes of equilibria exist. In the first, idiosyncratic shocks are diversifiable and don't affect valuations: CCAPM applies. In the second, idiosyncratic shocks generate non-diversifiable long-run cascades of shocks (financial pandemics) that introduce a new risk premium component unexplained by traditional systematic factors. We derive closed-solutions for asset prices as a function of the network properties and discuss their properties. After a structural break (1984), we find evidence of a network risk premium that is statistically and economically significant.

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A central tenet of the macro-finance literature builds on the Lucas (1977) diversification argument. As firm-specific shocks average out in aggregate, they have negligible effects on asset risk premia. This argument plays an important role in the Ross (1976) asset pricing theory (APT), in the Consumption Capital Asset Pricing Model (CCAPM), and in many macro dynamic stochastic general equilibrium (DSGE) models that focus on economies with a representative firm.[1] In this paper, we reconsider this argument in an endowment economy where several firms are connected in a network and their distress shocks can propagate in time and in the cross-section giving rise to a process of economic contagion. Firms' dividend stream is a stochastic process with two components. The first is driven by an exogenous aggregate shock, which is common to all firms; the second is firm-specific and follows a Markov chain whose intensities depend on the state of other (connected) firms. This allows us to study the externality that the state of distress of a firm generates on others in an otherwise traditional DSGE model. The global nature of our network allows

[1] Acemoglu et al. (2012) depart from this tradition and investigate the role of networks in production economies by assuming input-output linkages among firms. Gabaix (2011) emphasizes that granular idiosyncratic shocks of a few large firms can explain aggregate fluctuations because they cannot be fully diversified due to power-law distribution. They prove that sectorial shocks may generate aggregate macroeconomic output fluctuations.

both for propagation and feedback effects in the transmission of shocks. Contagion may emerge, for instance, when a real cash-flow shock to borrowing firms affects the balance sheet of lenders that may, in turn, reduce their ability to extend additional credit to other borrowers.

The key contribution of this paper is the derivation of a new set of asset pricing and risk premia implications that depends on the global properties of the network. We show the existence of two classes of distinct dynamics. In the first class, Lucas' classical diversification argument holds: firm-specific shocks can be diversified away and only aggregate shocks are priced. When preferences are time separable, network topology and firm-to-firm direct interaction are irrelevant for asset prices. We call this class of dynamics *subcritical.* In the second class, however, firm-specific shocks can give rise to aggregate cascades of distress. We call this class of dynamics *supercritical.* They are similar to epidemic states of virus spreading in which a single infected individual can spread the virus through the network connections. In these states, the initially firm-specific risk has the potential to become endemic in the sense that there exists a positive probability that even in the long-run a positive fraction of firms will be infected, irrespective of their initial health. In these supercritical states, diversification of firm-specific shocks cannot be achieved: the chain of firm-to-firm interactions and feedbacks is sufficiently strong that firm-specific distress can give rise to aggregate fluctuations with strictly positive probability, a 'cascade'. These risks are not diversifiable and the mean time of return to steady state diverges as the number of firms in the network increase.[2]

A common challenge in the network literature is the curse of dimensionality that often makes models complex and opaque. Accordingly, after deriving general implications, in Section 2, we propose a modeling approach that captures the first-order properties of the network dynamics and still allows us to obtain closed-form solutions. This rank-one model builds on an optimal approximation of the directed network based on two firm-specific measures, vulnerability and systemicness. A firm is said to be vulnerable if its dynamics depend on the state of distress of other firms in the network. Similarly, a firm is said to be systemic if its distress causes distress of other firms. Vulnerability and systemicness are the solution of a joint system of linear equations and correspond to the elements of principal right and left singular vectors of the Singular Value Decomposition of the network matrix. Their recursive nature captures the externalities induced by each firm on the global propagation of distress through the network. Using this rank-one approach, we derive closed-form solutions for both the threshold that separates supercritical and subcritical dynamics and the long-term probability of distress.

[2] This channel is related to the description of macroeconomic fluctuations discussed by Scheinkman and Woodford (1994) who describe a long-term state where economic fundamentals spontaneously approach a critical level such that micro-fluctuations trigger dynamic fluctuations at the aggregate level.

Three set of empirical implications emerge. Section 3 shows that in supercritical states equilibrium risk premia are affected by two separate components. The first component is the traditional CCAPM risk premium term, which captures conditionally linear exposure to instantaneous market risk (β); the second one is a risk premium proportional to firms' exposure to cascades of distress shocks. In contrast to the traditional CCAPM economy, in supercritical equilibria market β does not fully capture this second exposure. The reason is intuitive. In a supercritical equilibrium, firm-specific events give rise to undiversifiable cascades of distress shocks that lead to an endemic state with lower expected aggregate consumption. The increase in firms' risk premia depends both on their own vulnerability and on the systemicness of the firms currently in distress. These two quantities depend on the global characteristics of the network structure and not just on the local instantaneous exposure of the firm to aggregate endowment. The risk premium is provided in Theorem 2 and it helps to rationalize some empirical difficulties of the CCAPM. Indeed, heterogeneous exposure to systemic network risk generates a cross-section of expected excess return which depends on the firms' vulnerability. This additional premium helps also to explain Campbell et al. (2008)'s distress puzzle. They show that financially distressed firms have delivered low abnormal returns compared to their high standard deviations and betas. In our model, network risk premia are the sum of two components: a distress risk premium, which is positive, and a recovery risk premium, which is negative. For firms with low frequency of firm-specific distress the first term dominates and the systemic network risk premium is positive. However, for firms that spent more time in distress, the recovery risk premium dominates and expected excess returns can be negative, consistent with Campbell et al. (2008).

A second prediction of the model is that, in a super-critical equilibrium, the network risk premium is state-dependent and is a function of both the current state of the firm and the level of conditional systemic network risk which depends on the systemicness of those firms who are in distress. Above the tipping point, firms' risk premia are time-varying and correlated with the dividend-price ratio. In supercritical equilibria greater values of network distress correlate both with greater spreads in the dividend-price ratios spread of cheap (vulnerable) versus expensive (resilient) stocks and larger spreads in risk premia between vulnerable and resilient firms. Asness et al. (2018) refers to episodes of elevated dividend-price ratio spreads as "Deep Value" states and document that in these periods equity risk premia of high dividend-price ratio (value) firms are greater. They find that this additional positive excess return is unexplained by traditional factors. On the other hand, this additional positive excess return is consistent with our model.

Finally, the model generates endogenous economic and financial skewness. In supercritical equilibria, the propagation and amplification of firm-specific shocks shift the left tail of the cross-sectional distribution of firms cash flows as contagions give rise to clusters of distress. This manifests in an increase in cross-sectional skewness, which is greater for more vulnerable firms and sectors. This prediction relates to an important literature in economics that investigates the business cycle dynamics of the distribution of firm-level variables such as sales, profit, inventories, and employment. Salgado et al. (2019) report strong evidence of negative skewness in firm-level economic variables during recessions but not during expansions.

In the second part of the paper, we investigate an empirical application of the model. We consider a panel formed by the CRSP-Compustat universe of firms over the period 1970-2019. We match this panel with the Input-Output Accounts Data available at the sectorial level and provided by the U.S. Bureau of Economic Analysis. This sequence of input-output matrices are used to obtain the singular decomposition of the network to estimate firm systemicness and vulnerability. These measures are used to compute network distress risk as the weighted average of firms' distress. Firm specific probabilities of distress are obtained estimating a logit model as in Campbell et al. (2008); the aggregation weights are proportional to the firm systemicness.

The dynamics of network systemic risk reveals a transition from a subcritical to a supercritical equilibrium in June 1984. After this transition, the state of the economy remains in the vicinity of the tipping point, in the region characterized by the so called self-organized critical behavior, a notion introduced by Scheinkman and Woodford (1994) whose signature is cascades dynamics. The result is consistent with the findings of Giesecke et al. (2011) who document an increase in default frequencies following the Bankruptcy Reform Act of 1978.[3] After June 1984, we find four additional spikes in network systemic risk with the emergence of cascades that change distributional properties of firm distress.

We test the cross-sectional asset pricing implication of the model by estimating a Fama-MacBeth two pass regression including as factors both the market beta and the exposure to network systemic risk. Test assets include 5x5 portfolio sorted with respect to firm size and book-to-value characteristics and decile portfolios sorted with respect to the probability of distress using Campbell et al. (2008) methodology. Consistent with the prediction of the model, we find that the price of systemic network risk prior to the transition is zero and it becomes positive and significant after the transition. This result is robust after controlling for different combinations of the 5 Fama-French traded factors. The size of the systemic network risk premium is economically significant and about twice that of the CAPM market risk premium. We also find that exposures to systemic network risk of the 25 quintile portfolios are decreasing with respect to the size and increasing with respect to the value characteristics, supporting the early conjecture by Fama and French (1993) and Chan and Chen (1991) who argued that these anomalies could partially be compensation for exposure to

[3] See figure 1 in Giesecke et al. (2011).

distress risk. Finally, we find evidence of non-linearity in firms exposure to the systemic network risk premium in a way that is consistent with Campbell et al. (2008) distress premium puzzle.

We study the second implication of the model which relates to Asness et al. (2018)'s "Deep Value" by sorting firms according to their vulnerability and computing the spread in the sales-to-price ratio between high and low vulnerability firms (VmR). Two results emerge. First, consistent with the model, the spread VmR_t is increasing in the level of systemic network risk, with a correlation coefficient equal to 0.51. Second, in supercritical equilibria spikes in VmR_t help predict the future value premium. We interpret this finding as the confirmation that in a supercritical state cascades of distress are priced by investors and that vulnerability, our measure of distress risk exposure, drives also variation in the valuation ratios. Within our framework, the "Deep Value" predictability relationship is a rational signal, which anticipates the recovery effect.

The third implication of the model relates to the link between firm cash-flow skewness and its vulnerability. To test this prediction, we consider economic periods characterized by a large level of network systemic risk. During these periods, we compute the cross-sectional skewness at the sector level using log-sales growth and study the link between cross-sectional skewness and vulnerability. Consistent with the model, we find that in supercritical states sectors with the greatest vulnerability also display the most negative Kelley Skewness. We also test the counterfactual prediction that in subcritical states this link is not present. We cannot reject this null hypothesis. This confirms the importance of both distinguishing the nature of the equilibrium and the role played by the global network structure, which ultimately determines firms' vulnerabilities.

Related Literature. The paper relates to three streams of the literature.

The first stream studies credit risk pricing in the presence of default contagion effects in interacting intensity models (Jarrow et al. (2005), Bai et al. (2015), and Bo and Capponi (2016)). In an influential paper, Bai et al. (2015) argue that credit risk premia cannot be explained by firm specific credit events (jump-to-default). They calibrate a model with a reduced-form contagion channel and show that, because firm-specific credit events can be diversified, contagion risk is a dominant component in credit risk premia. Capponi and Frei (2017) develop a novel calibration procedure and find that systemic dependencies are statistically significant and play an important role to explain the time series of equity and CDS data. Recently, Chen et al. (2020) propose a structural industry equilibrium model with long-term defaultable debt where strategic competition drives feedback effects and contagion. They argue that when distressed firms tend to compete more aggressively, they can also drive competitors to default. Their structural approach provides an important empirical and theoretical microfoundation of the model considered in our framework. In both models, firm interaction is driven by the (left-tail) idiosyncratic jump shocks and this interaction generates novel asset price dynamics. The key contribution of our paper is to propose a structural model in which priced credit contagion can emerge endogenously because of the network connectivity. The propagation of firms' specific shocks produces endogenously countercyclical dynamics for risk prices consistent with the one, assumed exogenously, in Chen et al. (2020). We derive the link between aggregate fluctuations and two structural properties of the network: the vectors of systemicness and vulnerabilities, which are determined by the principal singular vector of the network adjacency matrix and show that in general equilibrium the cross-section of firm risk premia depends on these two properties.

The second stream relates to macroeconomic studies of how firm and/or sectorial specific shocks can give rise to aggregate fluctuations in production, trade, and banking networks. Kiyotaki and Moore (1997) suggests that 'a small, temporary shock to the liquidity of some firms may cause a chain reaction in which other firms get into financial difficulties, thus generating a large, persistent fall in aggregate activity'. Important contributions include Long Jr and Plosser (1983), Horvath (1998), Horvath (2000), Acemoglu et al. (2012), Barrot and Sauvagnat (2016) and Kramarz et al. (2020). Dupor (1999) discuss conditions under which the second moment properties of aggregates in multi-sector models are the same as their single-sector counterparts and shows broad class of input–output structures that generate aggregates with identical second moment properties. However, Herskovic et al. (2020) show that network effects are essential to explaining the joint evolution of the empirical firm size and firm volatility distributions. They use customers-suppliers information obtained from Compustat (see also Cohen and Frazzini (2008)), estimate a network model of firm volatility in which shocks to customers influence their suppliers, and document implications on the distributions of firm volatility, size, and customer concentration. In our empirical application, we follow Herskovic (2018) and explore transmission of shock using input-output matrices aggregated at sectorial level to estimate the network structure and discuss the equilibrium properties. A rank-one description of the network contagion risk shows that it may and does generate aggregate fluctuations that affect idiosyncratic variances and cross-sectional skewness. In the terminology of Scheinkman and Woodford (1994) and Nirei and Scheinkman (2021) we find empirical evidence of a region characterized by "self-organized criticality" and cascades. The emerging cash-flow dynamics provide a potential microfoundation to the sectorial left skewed business cycle dynamics discussed in Salgado et al. (2019) and highlights the relevance of higher order moment in the analysis of macro-financial fluctuations.

Finally, our work relates to the general equilibrium literature that studies asset pricing implications of multiple firms (orchards), e.g. Cochrane et al. (2007) and Martin (2013). In these models, however, direct economic interaction among firms is missing and cross-sectional heterogeneity in asset prices is driven by trees' share relative sizes. In our economy, each firm is infinitesimal and trees share sizes are individually irrelevant. However, firms interact in an economic network and we study the asymptotic diversification properties of seemingly firm-specific shocks for $N \to +\infty$. As in Lucas (1977) and Ross (1976), equilibrium asset price are determined only by fluctuations that survive in the large economy limit. In our model, the properties of the network structure determine the extent to which seemingly firm-specific distress shocks give rise to undiversifiable aggregate fluctuations because of contagion. An important related paper is Jarrow et al. (2005) who discuss conditions for the existence of an equivalent martingale measure in economies with asymptotically large number of defaultable securities. To derive their existence result, they assume conditional diversifiability of individual default events. As a consequence, defaults events do not give rise to aggregate risk. We contribute to this literature by considering more general (see Assumption 2) network structures that explicitly allow for propagation of credit events and contagion. Herskovic (2018) is the first study that analyses the asset pricing implications of a multi-sector economy in which sectors are connected to each other through an input-output network. Changes in the structure of the network are sources of systematic risk reflected in equilibrium asset prices. In our framework, equilibrium prices depend only on the properties of the network that survive in the large economy limit that are described by the systemicness and vulnerability vectors.

1. The Network Economy

We consider an infinite-horizon, pure exchange Lucas economy. The investment opportunity set consists of a locally risk-less security in zero net supply, with a rate of return r_t, and N risky securities in positive net supply, each paying a stochastic dividend stream $Y_t x_t^i$, $i = 1, \ldots, N$. We refer to each i as a 'firm'. The process Y_t plays the role of a common permanent systematic shock

$$\frac{dY_t}{Y_t} = \mu dt + \sigma dW_t \tag{1}$$

and x_t^i is a two-state process with states $x^i(0)$ and $x^i(1)$, $x^i(0) > x^i(1) > 0$. We label $x^i(1)$ the 'distress' state. In the benchmark specification, all firms are identical in terms of their risk exposure to the common log-normal factor Y_t, namely $\beta = 1$. The cash-flow term x_t^i depends on the indicator variable H_t^i by the simple linear relation $x_t^i = x_i(0)(1 - H_t^i) + x_i(1) H_t^i$. An innovation $dH_t^i = 1$ denotes a transition to a distress state of firm i, while $dH_t^i = -1$ denotes a recovery event. The vector $\mathbf{H} = (H^1, \ldots, H^N)'$ specifies the distress state of all firms and identifies uniquely each of the 2^N possible configurations of the economy's state, ranging from the case where no firm is in distress, $H^i = 0$, to one where all firms are in distress, $H^i = 1$, $i = 1, 2, \ldots, N$. The set of all configurations will be denoted by $\mathcal{C}$. Let S denote the subset of vertices of the network corresponding to the cluster of firms that are in distress at time $t = 0$. Then the vector describing the initial distress configuration is denoted by $\mathbf{H}_0^S$ and its elements are $H_0^i = 1$, for $i \in S$, and $H_0^i = 0$, for $i \notin S$. Our goal is to explore how the characteristics of the network economy affect the probability and duration of the contagion process $\mathbf{H}_t^S$ originated from $\mathbf{H}_0^S$. The evolution of $\mathbf{H}_t^S$ is described by a regular continuous-time Markov chain whose properties are given by the elements of the $2^N \times 2^N$ transition rate matrix $\mathbf{A}$ that are uniquely determined[4] by the firm-specific distress and recovery intensities $\lambda^i(\mathbf{H})$ and $\eta^i(\mathbf{H})$; that is, the probability of a negative (positive) dividend jump during the next time instant, provided the firm is not (the firm is) in distress. To model network connectivity in a parsimonious way, we allow for the likelihood of distress during the next infinitesimal time interval to be positively affected by the state of distress of directly connected firms. For this reason, we introduce an adjacency matrix Δ (the network matrix, thereafter), with positive elements $\Delta_{ij} > 0$ if firm i is connected to firm j, $\Delta_{ij} = 0$ otherwise. The network matrix Δ_{ij} enters in the definition of firm i conditional distress intensity:

$$\lambda^i(\mathbf{H}) = \lambda_i + \lambda \sum_{j=1}^{N} \Delta_{ij} H^j. \tag{2}$$

The parameter $\lambda_i > 0$ represents the firm-specific distress transition rate, while the term $\Delta_{ij} H^j$ is the contagion term, which captures the increase in the probability of distress of firm i due to a distress of firm j. This modeling choice is motivated by the work of Jacobson and Von Schedvin (2015) who show that the transition to distress of a trade credit counterparty raises by almost 50% the probability of firm distress. Moreover, Herskovic et al. (2020) show that customer-supplier linkages drive heteroskedasticity of firm-specific volatilities and Chen et al. (2020) incorporate dynamic strategic competition into an industry equilibrium model with distress.

While the idiosyncratic shocks dH_t^i are by definition instantaneously independent across firms, the intensities depend on the state of distress H_{t-}^j, $j \neq i$, of other firms. For simplicity, we set recovery intensity rates to be constant and state-independent:

$$\eta^i(\mathbf{H}) = \eta > 0. \tag{3}$$

Consistent with the macroeconomic literature that investigates how the distribution of the growth rate of firm-level variables (sales, profit, inventories, and employment) changes over the business cycle, we focus on modelling the externality in the transmission of negative shocks.[5] Salgado et al. (2019) find that the cross-sectional skewness becomes more negative during recessions and is close to zero during periods of sustained expansion. Without loss of generality, it is possible to set $\eta = 1$ and rescale the distress transition rates per unit of η. Then, defining $\upsilon_i := \lambda_i/\eta$:

$$\frac{\lambda^i(\mathbf{H})}{\eta} = \upsilon_i + \frac{\lambda}{\eta} \sum_{j=1}^{N} \Delta_{ij} H^j \tag{4}$$

[4] Its exact expression is reported for completeness in Definition EC.1 in the Appendix.

[5] The model can be easily generalized to allow for network dependence $\eta(\Delta)$ in the transmisison of (positive) productivity shocks.

Contagion dynamics are uniquely identified in terms of (i) the firm-specific distress-to-recovery transition rates $\{\upsilon_i\}_{i=1,\dots N}$, $\upsilon_i > 0$ and (ii) the contagion intensity ratios λ/η. Unless otherwise stated, we will assume $\upsilon_i > 0$ and υ will denote $\upsilon := \sup_{i=1,\dots N}\{\upsilon_i\}$. For $\lambda/\eta = 0$, firm-specific transition rates $\{\upsilon_i\}_{i=1,\dots N}$ drive independent idiosyncratic fluctuations that by the law of large numbers, in the large N limit, can be diversified away in a large portfolio. For $\lambda/\eta > 0$ the possibility of direct inter-firm distress propagation renders single firm fluctuations interdependent, making the traditional Law of Large Numbers argument inapplicable. Diversifiability of firm-specific fluctuations depend on the specific nature and intensity of these interactions.[6] While the model formulation is flexible enough to accommodate several extensions, it is worth observing that the benchmark dynamics that are selected by equations (3) and (4) is convenient both for its simplicity and for its empirical relevance. Salgado et al. (2019) observe that the drop in profits is more significant in recessions compared to the growth in expansions. Following this empirical observation, we deliberately allow for asymmetric transition rates, with cascades only arising in recessions. We set the recovery rate independent and homogeneous across firms. This is motivated by the empirical observation that the characteristic time of distress resolution, $T_\eta = \eta^{-1}$ depends mainly on the regulatory and legal framework and is not firm-specific. Note also that we model propagation of firms distress, as opposed to default. For distress we mean the temporary inability of a firm to produce cash flows sufficient to service existing debt. It can be resolved through a restructuring process without necessarily resulting in the liquidation or default of the firm. Consistently, firms do not disappear from the network as this would lead the network topology to become time-varying and stochastic.

1.1. Network Topologies

The specification of the network matrix Δ allows us to study alternative network topologies. A DIRECTED NETWORK $\mathcal{G}$ is defined by the pair of sets $(V^{\mathcal{G}}, E^{\mathcal{G}})$ and by an adjacency matrix of weights $\Delta^{\mathcal{G}}_{ij} \geq 0$. $V^{\mathcal{G}}$ is the set of its vertices, N will denote their number, and $E^{\mathcal{G}}$ is the set of edges, $\Delta^{\mathcal{G}}_{ij} > 0$ is the positive weight assigned to each edge $(j,i) \in E^{\mathcal{G}}$. The degree of a node j is the number of outgoing edges $(j,i) \in E^{\mathcal{G}}$. A NETWORK is said to be (weakly) CONNECTED if for any two vertices i,j there exists a (un)directed path formed by edges in $E^{\mathcal{G}}$ joining i and j. The degree of a vertex i is the number of edges (i,j) such that $\Delta^{\mathcal{G}}_{ij} > 0$. It is said that there is non-zero feedback along an edge joining i and j if $\Delta^{\mathcal{G}}_{ij} > 0$ implies $\Delta^{\mathcal{G}}_{ji} > 0$.

ASSUMPTION 1. *Consider a network $\mathcal{G}$ that is connected, with feedback along each edge, bounded edge weights, and maximum degree.*

[6] Although we gear the specification to study distress dynamics, one can see that the model can be adapted to study innovation adoption and narrative diffusion.

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Assumption 1 impose a minimal set of conditions on the class of network matrices to guarantee that the contagion dynamics does not have a trivial behavior. Indeed, connectedness and feedback along the edges avoid that different regions of the networks behave like 'islands', structurally protected from the propagation of contagion. Boundedness of the network contact matrix and the finite degree avoid that in the large N limit a small number of firms can cluster mechanically generating a 'systemic' firm whose shocks would trivially generate aggregate fluctuations.

Example 1. Figure 1-Panel A shows a simple example of how a directed network topology is conveniently summarized by the specification of Δ. This network describes an economy where Firm 1 is the center of a star and is connected to satellite firms through directed connections with heterogeneous intensities. Firm 6 is connected to the economy only through a directed connection to Firm 4. The specification of the network matrix can be used to describe the intensity of firms interlinkages in the transfer of distress shocks. This network does not satisfy Assumption 1 since it is not weakly connected.

Example 2. Rank One Networks are identified by a network matrix that can be written as: $\Delta_{ij}^{Comp} := n_i^R n_j^L$ for arbitrary $n_i^R \geq 0$, $n_j^L \geq 0$ and $i,j = 1,..,N$. We can distinguish two types of rank one networks. Figure 1-Panel B represents an $N = 8$ star directed network. The star directed network is assumed to describe a situation where the only relevant source of network risk distress is the central firm. If the coordinates of the central node are set equal to $i,j = 1$, then the elements of the adjacency matrix are: $\Delta_{i1}^{SN} = 1$ for $i = 2,...,8$ and $\Delta_{ij}^{SN} = 0$ otherwise. In the special case of $n^L = n^R$, $\Delta_{ij} = \Delta_{ji}$ for any $i,j = 1,...,N$ one obtains a complete undirected network. Figure 1-Panel C represents a complete undirected network with $N = 8$ nodes connected by interactions with uniform intensity $\Delta_{ij}^{CN} = 1$ for any $i,j = 1,...,N$.

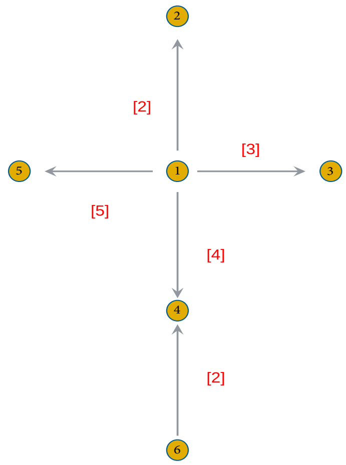
The network Δ^{CN} in Figure 1 Panel C satisfies Assumption 1. On the other hand, both networks in Panel A and B are examples that do not satisfy the condition. Notice, however, that networks that can be approximated as linear combinations of the economy in Panel A and C (i.e. $\varepsilon\Delta^{SN} + (1-\varepsilon)\Delta^{CN}$, with $\varepsilon > 0$) satisfy the Assumption 1.

2. Systemicness and Vulnerability.

In this section, we introduce a reduced-rank specification of the network structure introducing two quantities to capture in a tractable way the *individual* role played by each institution on the *global* network propagation of distress and the collective nature of these dynamics. We refer to these two quantities of firm interaction as vulnerability and systemicness. The systemicness of a firm j

Figure 1 Examples of Network Topologies.

Panel A



$$\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

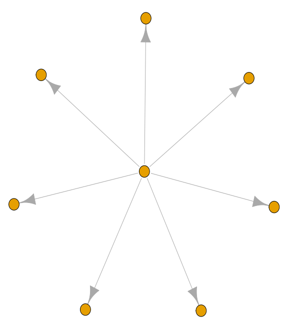
Kleinberg Hub and Authority:

$$\begin{aligned} \alpha^{\mathcal{G}} &= 7.0433 \\ \nu^L &= [0.98, 0, 0, 0, 0, 0.15]\,, \\ \nu^R &= [0, 0.27, 0.40, 0.57, 0.66, 0]\,. \end{aligned}$$

Eigenvector Centrality:

$$\begin{aligned} \alpha_0 &= 0 \\ C &= [1, 0, 0, 0, 0, 0]\,. \end{aligned}$$

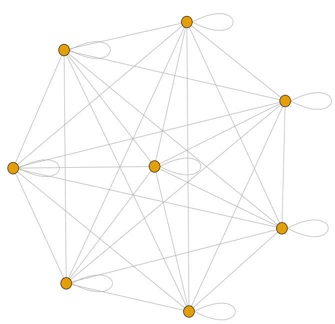
Panel B



$$\Delta^{SN} = \alpha^{\mathcal{G}} \left(\nu^R\right)^T \cdot \nu^L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\alpha^{\mathcal{G}} = 1, \quad \nu^L = [1, 0, ..., 0]\,, \quad \nu^R = [0, 1, ..., 1]\,,$$

Panel C



$$\Delta^{CN} = \alpha^{\mathcal{G}} \left(\nu^R\right)^T \cdot \nu^L = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\alpha^{\mathcal{G}} = 1, \quad \nu^L = \nu^R = [1, 1, ..., 1]\,.$$

Note. Panel A: Generic directed network: Firm 1 is at the centre of the network; firm 6 is a firm at the periphery connected to firm 4. Values in square brackets define the strength of the connection. Panel B: Star directed network. Panel C: Complete undirected network. $[\nu^R, \nu^L, \alpha^{\mathcal{G}}]$ denote respectively the right and left singular vectors and the principal singular value of $\Delta^{\mathcal{G}}$.

describes the extent to which its state affects the state of all other firms i; this differs from their vulnerability, which instead relates to the extent to which its state depends on the state of all other firms. Notice that in a network the two concepts are jointly linked to each other: a firm is more vulnerable if it is strongly linked to systemic ones and a firm is more systemic if it can more strongly influence vulnerable ones.

DEFINITION 1. *Consider the network matrix $\Delta = \{\Delta_{k,l}\}_{k,l=1,\ldots,N}$, the vulnerability ν_i^R (systemicness ν_i^L) of firm i with respect to the network of interactions Δ is given by the (nonnegative) $i-th$ component of the right (left) singular vector associated to the highest singular value of the matrix Δ.*[7]

Let us define ν^L and ν^R the $(N \times 1)$ systemicness and vulnerability vectors. We borrow the concepts of "hub" and "authority" introduced by Kleinberg (1999) in graph theory to extend the notion of centrality to directed networks. To formalize the definitions, one can require that firm j systemicness ν_j^L increases if $\sum_i \nu_i^R \Delta_{i,j}$ increases and, at the same time, that firm i vulnerability ν_i^R increases if $\sum_j \Delta_{i,j} \nu_j^L$ increases. If one restricts the increase to be linear with scale factors c_1 and c_2, we can write $\nu^R = c_1 \Delta \nu^L$ and $\nu^L = c_2 \Delta' \nu^R$. Therefore, after substitution, ν^R and ν^L must solve the following system of equations:

$$v^R = (c_1 c_2) \Delta \Delta' v^R \qquad \text{and} \qquad v^L = (c_1 c_2) \Delta' \Delta v^L.$$

This implies that ν^R and ν^L are the eigenvectors of the symmetric matrices $\Delta\Delta'$ and $\Delta'\Delta$, respectively, also known as the singular values of the non-symmetric matrix Δ (Definition 1).[8] Notice that the vulnerability and systemicness depend on the overall topological network structure and exhibit a mutually reinforcing relationship. In contrast to conventional measures of risk exposure, such as CAPM β, that measure the conditional linear degree of co-movement between single firm cash-flows or prices and aggregate variables, vulnerability measures the exposure to systemic chains of idiosyncratic distress events that propagate through network linkages. Suppose we want to approximate $\Delta^{\mathcal{G}}$ with a lower rank network matrix $\hat{\Delta}^{\mathcal{G}}$. Let $[\nu_i^R, \nu_j^L, \alpha^{\mathcal{G}}]$ be the right and left singular vectors and principal singular values of $\Delta^{\mathcal{G}}$, respectively.[9] Eckart-Young's Theorem shows

[7] Singular components are defined up to a normalization. Unless otherwise stated, we will assume that the vectors ν^R (ν^L) are normalized in such a way that $\nu^L \cdot \mathbf{1} = \nu^R \cdot \mathbf{1} = \mathbf{1}$

[8] In MatLab, one can compute the vectors of vulnerabilities and systemicness computing the singular value decomposition using the routine $SVD(\Delta)$ and the selecting only the principal right and left singular vectors, i.e. those that correspond to the largest singular value.

[9] Singular value decomposition extends standard spectral analysis and is used to provide optimal low rank approximations to network matrix, see e.g. (Golub and Van Loan (2012)). Notice that after the rank reduction the singular components coincide with the principal right and left eigenvectors of the resulting reduced network matrix. The corresponding principal eigenvalue is equal to: $\alpha^{\mathcal{G}} \left(\nu^L \cdot \nu^R\right)$. Higher order factor representations may be relevant to analyze more in detail the community structure generated by the network topology. It is also easy to show that $\alpha^{\mathcal{G}}$ is linked to c_1 and c_2 by the relationship $\left(\alpha^{\mathcal{G}}\right)^2 = 1/(c_1 c_2)$

that the optimal $\hat{\Delta}^{\mathcal{G}}$ that minimizes the Frobenius norm of the difference between $\Delta^{\mathcal{G}}$ and $\hat{\Delta}^{\mathcal{G}}$ can be obtained from the singular value decomposition of $\Delta^{\mathcal{G}}$. In the unit rank case, this is given by $\alpha^{\mathcal{G}}\nu_i^R\nu_j^L$, so that

$$\hat{\Delta}_{i,j}^{\mathcal{G}} = \alpha^{\mathcal{G}}\nu_i^R\nu_j^L \quad i,j = 1,..,N. \tag{5}$$

It is important to notice that the vulnerability and systemicness of each firm i is the same in $\Delta^{\mathcal{G}}$ and $\hat{\Delta}^{\mathcal{G}}$, so that the economic interpretation is preserved in $\hat{\Delta}^{\mathcal{G}}$. Moreover, introducing this reduced form representation relying on $2N+1$ components of $[\nu_i^R, \nu_j^L, \alpha^{\mathcal{G}}]$ provides an approximate description of the N^2 elements of the adjacency matrix $\Delta^{\mathcal{G}}$ of a generic network. The following two examples illustrate the role of systemicness and vulnerability scores in the network examples previously introduced.

Example 1 (Continued). Figure 1-Panel A shows a generic directed network: firm 1 is at the centre of a star and can propagate distress to other firms. The adjacency matrix Δ contains values that are proportional to the strength of the connection. In this economy, the authority of the central firm is $\nu_1^L = 0.9880$ while $\nu_i^L = 0$ for $i = 2,\ 3,\ 4,\ 5$. Satellite firms have a different level of exposure to shocks propagated from the centre depending on the intensity of the connection. These exposure levels are quantified by the vulnerability scores. Firm 1 has zero vulnerability score ($\nu_1^R = 0$) since the central firm is not affected by distress events affecting satellite firms; firms 2, 3, and 4 have increasing vulnerabilities, due to their increasing level of connectivity ($\nu_2^R < ... < \nu_5^R$). Consider now firm 6. This firm can spread shocks to firm 4. For this reason, it has the second largest systemicness, $\nu_6^L = 0.1542$. Nonetheless, it has zero eigenvector centrality. This highlights the limitations of eigenvector centrality to fully capture the roles of nodes outside strongly connected components directed networks. This is particularly important for asset pricing purposes.

Example 2 (Continued). It is easy to verify that for the star directed network, represented in Figure 1-Panel B, the decomposition $\Delta_{ij}^{star} = \alpha^{\mathcal{G}}\nu_i^R\nu_j^L$ holds *exactly* for any N. Indeed, in this network notice that:

$$\alpha^{\mathcal{G}} = 1, \quad \nu^L = [1, 0, ..., 0], \quad \nu^R = [0, 1, ..., 1].$$

The first element of the left singular vector ν^L is equal to 1, while all other terms are equal to 0. Only firm 1, which is the core one, can propagate its shocks to the rest of the network. At the same time, the right singular vector ν^R shows that periphery firms $2-8$ are the most vulnerable. Figure 1-Panel C shows the example of a Complete directed network with $N = 8$ nodes. In this case, we have:

$$\alpha^{\mathcal{G}} = 1, \quad \nu^L = \nu^R = [1, 1, ..., 1].$$

In the uniform complete network, the distance between the left and right singular value reaches its minimum value, $\|\nu^L - \nu^R\| = 0$: no firm is more systemic or vulnerable than any other, due to the existence of perfect feedback effects. The opposite is true for a star network. The role and economic interpretability of these two indicators in relation to the directionality of distress in network dynamics is particularly useful to study contagion across financial and production networks. As observed by Acemoglu et al. (2015), the global effects of institutions in complex networks go above and beyond those to/from their immediate creditors. Therefore, even in the rank-one network reduction, these indicators provide additional information with respect to more traditional measures of factor risk exposure. In a supply-chain, disruption of a producer will reduce production efficiency for the overall chain. A reduction in the degree of vulnerability of peripheral (low systemicness) suppliers also lowers the level of systemicness of core (high degree of systemicness) producers. [10]

2.1. Distress Dynamics in the Rank-one model

One major advantage of the rank-one model is to provide a simple representation of distress transition rates in a generic network. This is best understood recalling first the expression of the transition rate probabilities of a periphery firm in the star directed network considered in Example 2. Let firm 1 be the 'systemic firm' at the center of the star - this information is encoded in the left singular vector $\nu^L = [1, 0, ..., 0]$ - whose dynamics are regulated on the distress state process H_t^1 that is independent of the state of any other firm. The remaining $i = 2, .., N$ firms, as revealed by $\nu^R = [0, 1, ..., 1]$, are identical 'satellite' firms that are vulnerable to the distress of the central systemic firm $i = 1$. For each of these firms, the transition rate matrix conditional on the state of the firm is given by:

$$A^{(i)}_{\nu, H_t^1} = \begin{bmatrix} -\lambda_i - \lambda H_t^1 & \lambda_i + \eta\lambda H_t^1 \\ \eta & -\eta \end{bmatrix} \quad i = 2, ..., N.$$

A similar representation extends to a generic network.

Let us aggregate firm-specific distress to obtain the network systemic risk H_t^ν, which is defined as the weighted average of firm-specific distress H_t^i :

$$H_t^\nu := \frac{\sum_{i=1}^N \nu_i^L H_t^i}{\sum_{i=1}^N \nu_i^L}.$$

Importantly, H_t^ν accounts for the level systemicness ν_i^L of each firm. Thus, the greater H_t^ν, the greater the conditional probability of distress of healthy firms due to the negative externality of firms currently in distress. Substituting the rank-one representation of the network matrix provided

[10] A major limitation of the rank one description of the network is the impossibility to capture multiple layer recursive tree-like topologies that determine a recursive block structure of the network matrix. We leave the treatment of this relevant case to future research.

in equation (5) in the general expression of the transition rate matrix leads to a closed-form representation:

$$A^{(i)}_{\nu^R, H^\nu_t} = \begin{bmatrix} -\lambda_i - \lambda\alpha\nu^R_i H^\nu_t & \lambda_i + \lambda\alpha\nu^R_i H^\nu_t \\ \eta & -\eta \end{bmatrix}.$$

where $\alpha := \alpha^{\mathcal{G}} \sum_{i=1}^{N} \nu^L_i$. Then the inter-firm rank-one transition rate matrix of any firm i equals $\lambda\alpha\nu^R_i H^\nu_t$ and is proportional to the vulnerability of firm i, ν^R_i, multiplied by the level of distress of the representative systemic firm H^ν_t.[11] The transition rate depends also on the product $\alpha\lambda$. An increase in $\alpha\lambda$ is equivalent to an increase of the overall interconnectivity parameter λ by the same amount.

2.2. Supercritical Dynamics and the Tipping point: A Closed-form Solution

The class of rank-one networks introduced in subsection 2.1 allows us to derive a closed-form solution for the tipping point. We follow a procedure proposed in Graham et al. (2009) and consider a sequence of networks that satisfy the following minimal properties:

Assumption 2. *Consider a sequence of connected networks $\mathcal{G}_N$ indexed by the number of firms $N \in \mathbb{N}$ satisfying Assumption 1 and the following properties:*
(i) [Normalization] For any network $\mathcal{G}_N$, the product $\alpha^{\mathcal{G}_N} N$ is equal to a constant finite value L; then $\alpha = \alpha^{\mathcal{G}_N}\left(\sum_{i=1}^N \nu^L_i\right) = L\frac{1}{N}\left(\sum_{i=1}^N \nu^L_i\right)$.
(ii) [Heterogeneity] Any network $\mathcal{G}_N$ is populated by K classes of firms with different finite vulnerability ν^R_k and systemicness ν^L_k. Each class C_k is populated by N_k firms distributing a dividend $x_k\left(H^i_t\right)$, with $k = 1,..,K$. We assume that each class is not empty asymptotically in the sense that each class C_k contains a strictly positive fraction $p_k > 0$ of firms: $\lim_{N\to+\infty} \frac{N_k}{N} = p_k \in (0,1]$, $k = 1,...,K$ with $\sum_{k=1}^K p_k = 1$.
(iii) [Zero firm-specific rates] Firm-specific transition rates are set to zero, $\upsilon_i = 0$, $i = 1,..,N$.

The normalization condition in (i) avoids the possibility of divergence of the principal singular value in large networks. This condition is equivalent to assume a normalization such that L is the amount of liabilities *per firm*. Correspondingly, we introduce the following normalized scalar product to $\mathbf{x}\cdot\mathbf{y} := \frac{1}{N}\sum_{i=1}^N x_i y_i$, so that $\alpha = L\left(\nu^L \cdot \mathbf{1}\right)$. Assumption (ii) assures that the network is not trivially dominated by a unique class of firms in the limit. Assumption (iii) is introduced to simplify the discussion focusing on network contagion effects only. Overall, these assumptions have a financial interpretation: it is possible to aggregate firms in groups having homogeneous network characteristics, namely vulnerability and systemicness.

[11] At first sight, the expressions of the generators for a GENERIC and STAR NETWORK are similar. A crucial difference, however, is that the evolution of the state of distress for the central firm H^1_t in the star network example is exogenously determined, while H^ν_t depends on the dynamics of all the variables H^i_t, hence it is endogenously determined.

2.2.1. Contagion dynamics When firms interaction is sufficiently strong, the long term dynamics may include the emergence of cascades of firm-specific shocks. A cascade is defined as a sequence of distress shocks whose effects are so persistent to alter the long-term behavior even when the initial shock is small:

Definition 2 (Cascade). *Consider a large economy satisfying Assumptions 1 and 2. Let* h_t^k *for* $k = 1, ..., K$ *be the fraction of firms in distress in a large network:* $h_t^k := \lim_{N\to+\infty} (p_k N)^{-1} \left(\sum_{i\in C_k} H_t^i \right)$ *and let* $h_\infty^k := \lim_{t\to+\infty} h_t^k$ *be the long term fraction of distressed firms in class* C_k*. A cascade is a process* $\mathbf{H}_t^S$ *such that, for some* k*,* $h_\infty^k \neq 0$*. The endowment dynamics is said to be supercritical if there's a non-zero probability that a cascade takes place.*

This definition of cascades and of supercritical dynamics parallels the definition of an endemic infection in epidemiology. In that literature, an infection is said to be endemic if a finite fraction of the population is expected to be infected even in the long-time limit. In the language of epidemics, a cascade corresponds to an outbreak whose duration is increasing with the size of the population being affected. In the context of our network economy, it is possible to show that duration of outbreak depends on the spectral properties (eigenvalues) of the generator $\mathbf{A}$ of the Markov process $\mathbf{H}_t$.[12]

The following Theorem shows the conditions under which there exists a tipping point $K(\Delta)$ above which the long-term dynamics includes cascades and endemic distress.

Theorem 1 **(Criticality and Cascades: The Threshold Representation)**. *Consider a large economy satisfying Assumptions 1 and 2. There exists a finite Tipping Point* $0 \leq K(\Delta) < +\infty$ *separating two distinct dynamics:*

- *Subcritical Dynamics. When* $\frac{\lambda}{\eta} < K(\Delta)$ *the probability of occurrence of a cascade is zero and the long term fraction* $h_\infty^k := \lim_{t\to+\infty} h_t^k = 0$ *(No Endemic Distress)*

[12] Drawing on the early contributions of Kemeny and Snell (1976) for ergodic discrete time Markov chains and of Cui and Mao (2010) for the transitory, continuous time case, the mean duration of a contagion process depends on the spectral properties of the generator $\mathbf{A}$. A properly defined time to reach steady state $\mathcal{T}^{\mathbf{A}}$ is independent of the initial condition S and is equal to:

$$\mathcal{T}^{\mathbf{A}} = \sum_{n=2}^{2^N} \frac{1}{\lambda_n^{\mathbf{A}}}, \tag{6}$$

where $\{\lambda_n^{\mathbf{A}}\}_{n=1,...,2^N}$ are the eigenvalues of the transition rate matrix $-\mathbf{A}$ (of the process $\mathbf{H}_t^S$). In particular, since $\lambda_1 = 0$, a sufficient condition for the emergence of a cascade, i.e. $\mathcal{T}^{\mathbf{A}} \to +\infty$, is that in the large economy limit $N \to +\infty$ the spectral gap is vanishing $(\lambda_2 - \lambda_1) = \lambda_2 \overset{N\to+\infty}{\to} 0$. Based on Assumptions 1 and 2, a sufficient condition for the presence of cascades in the large N limit is that the expected time $E[\tau_{\mathbf{H=0}}]$, is diverging as $N \to +\infty$. Indeed, it is well known, see e.g. Draief and Massoulie (2010), that $\lambda_2 = E[\tau_{\mathbf{H=0}}]^{-1}$, hence $E[\tau_{\mathbf{H=0}}] \overset{N\to+\infty}{\to} +\infty$ implies $\lambda_2 = E[\tau_{\mathbf{H=0}}]^{-1} \overset{N\to+\infty}{\to} 0$ and a fortiori $\mathcal{T}^{\mathbf{A}} \overset{N\to+\infty}{\to} +\infty$. Thus the existence of a stable stationary point such that $h_\infty^k > 0$ for some k is a sufficient condition to prove that the dynamics is supercritical and cascades may occur.

- *Supercritical Dynamics. When* $\frac{\lambda}{\eta} > K(\Delta)$*, the distress of any arbitrary small set of firms will trigger cascades. Correspondingly* $h_{\infty}^{k} > 0$ *and they are determined by:*[13]

$$h_{\infty}^{k}\left(h_{\infty}^{\nu}\right) = \frac{\alpha\frac{\lambda}{\eta}\nu_k^R}{1+\alpha\frac{\lambda}{\eta}h_{\infty}^{\nu}\nu_k^R}h_{\infty}^{\nu}, \tag{7}$$

where $h_{\infty}^{\nu} > 0$ *is the unique positive solution to the equation*

$$1 = \sum_{k=1}^{K} p_k \frac{\nu_k^L \nu_k^R \frac{L\lambda}{\eta}}{\frac{L\lambda}{\eta}\nu_k^R h_{\infty}^{\nu}\left(\nu^L\cdot\mathbf{1}\right)+1}. \tag{8}$$

- *The Tipping Point is given by:*

$$K(\Delta) = \frac{1}{L\left(\nu^L\cdot\nu^R\right)}. \tag{9}$$

The existence of a Tipping Point and of cascades is a general property of all networks satisfying Assumptions 1 and 2.

Theorem 1 generates several interesting testable empirical implications. First, the emergence of a tipping point implies that, close to it, an infinitesimal variation in the intensity of inter-firm connection alters the qualitative nature of inter-firm contagion and generates an instability giving rise to cascades of distress shocks that investors are unable to diversify. Second, the exact location of the tipping point depends on the network structure which becomes therefore relevant for macro-financial stability. Indeed, equation (9) shows that the tipping point is determined by the inverse of the inner product between the vulnerability and the systemicness, namely $\left(\nu^L\cdot\nu^R\right)^{-1}$. The intuition is simple: the emergence of cascades relies on the existence of feed-back effects. In order to have cascades, a sufficient number of vulnerable firms needs to be either closely connected to systemic firm in distressed or systemic themselves. The larger the overlap $\left(\nu^L\cdot\nu^R\right)$ between more vulnerable and more systemic firms in the economy, the greater the amplification and, in turn, the lower the tipping point $K(\Delta)$ above which local shocks induce aggregate fluctuations. In the limit of an economy where vulnerable firms have zero systemicness and viceversa, namely $\left(\nu^L\cdot\nu^R\right)\to +0$, the tipping point $K(\Delta)\to+\infty$ simply disappears.

The existence of a supercritical dynamics has an impact on aggregate consumption. The results in Theorem 1, imply the following:

Corollary 1. *Under the same assumptions of Theorem 1, below the tipping point the aggregate consumption* C_t^{LT} *follows a standard 'Lucas Tree' lognormal dynamics determined uniquely by* Y_t*.*

[13] With a slight abuse of notation, we denote with α also its limiting expression for $N\to+\infty$: $\alpha := \lim_{N\to+\infty}\alpha_0^{\mathcal{G}}\sum_{i=1}^{N}\nu_i^L$. In light of Assumption 2, it is also equal to: $\alpha = L\left(\nu^L\cdot\mathbf{1}\right)$

Above the tipping point, for any initial condition $h_0^{\nu}>0$ the presence of distress contagion reduces the level of aggregate consumption by a factor:

$$\frac{C_t^{Super}}{C_t^{LT}} \overset{N\to+\infty}{\to} \sum_{k=1}^{K} p_k\left(\left(1-\left(1-\frac{x^k(1)}{x^k(0)}\right)h_t^k\right)\right)$$

In addition, above the tipping point, the aggregate consumption dynamics includes an additional drift component:

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{i=1}^{N}\frac{dx_i(H_t)}{x(0)} = \sum_{k=1}^{K} p_k\left(1-\frac{x(1)}{x(0)}\right)\left(\left[(-1)\left(1-\nu_k^R\alpha\lambda h_t^{\nu}\right)\nu_k^R\alpha\lambda h_t^{\nu}+\eta\nu_k^R\alpha\lambda h_t^{\nu}\right]dt\right)$$

The corresponding expression for higher consumption moments is reported together with the proof in the Appendix.

Above the tipping point, distress contagion reduces aggregate consumption. The size of the drop depends on the cross-section of firm vulnerabilities ν_k^R and the level of aggregate distress h_t^{ν} through h_t^k, $k=1,..,K$. For a supercritical economy close to the tipping point, i.e. $\lambda/\eta = K_c(1+\varepsilon)$ with $\varepsilon << 1$, the drop in aggregate consumption is equal to:

$$\frac{C_t^{Super}}{C_t^{LT}} \overset{\varepsilon<<1}{\simeq} \sum_{k=1}^{K} p_k \underbrace{\left(1-\left(1-\frac{x^k(1)}{x^k(0)}\right)\nu_k^R\alpha\frac{\lambda}{\eta}h_t^{\nu}\right)}_{\text{drop dividends class } k} < 1.$$

The impact on aggregate consumption is not diversifiable and is equal to the weighted sum of the dividend drop of firms suffering distress in the endemic state. The weight is given by p_k, the fraction of firms in class k, and the expected drop is proportional to the vulnerability level ν_k^R of firms in class k. The greater the $\frac{\lambda}{\eta}h_t^{\nu}$ ratio, the greater the expected consumption loss in the supercritical equilibrium. Notice that h_t^{ν} depends in the global topological properties of the network, captured by ν^L. Although the Corollary provide a result for the first moment, all higher odd moments are affected in the supercritical equilibrium, including skewness (see Appendix). We now turn to the investigation of their equilibrium asset pricing implications.

3. Valuation in a network economy

3.1. Stochastic discounting

Since the focus of our interest is to model cash-flow risks, we assume the simplest preference structure for the representative agent, who maximize a time additive Constant Relative Risk Aversion utility of intertemporal consumption

$$U_0 = \mathbb{E}\left[\int_0^{\infty} e^{-\delta s}\frac{C_s^{1-\gamma}}{1-\gamma}ds\right],$$

where γ and δ are the relative risk aversion and subjective discount rate coefficients, respectively. In order to focus on the network implications, we explicitly avoid more general preference assumptions.

In the benchmark specification all firms are identical and produce a fraction $1/N$ of the total aggregate dividend. This ensures that the β exposure of each firm to the aggregate risk factor is identical and equal to 1. In equilibrium aggregate consumption is equal to total dividends $C_t = Y_t X_t^{(N)}$, where $X_t^{(N)} = \sum_{i=1}^{N} x_t^i(\mathbf{H}_t)$. Hence marginal utility of consumption in each state of nature is given by:

$$\xi_t = e^{-\delta t}\left(Y_t X_t^{(N)}\right)^{-\gamma}.$$

The corresponding dynamic evolution can be written as:[14]

$$\frac{d\xi_t}{\xi_t} = -r_t(\mathbf{H}_t)\,dt - \kappa dW_t + \sum_{j=1}^{N}\left(\theta^j(\mathbf{H}_t) - 1\right) dM_t^j,$$

The parameters r_t and θ^j define the equilibrium risk free rate, the purely diffusive and Sharpe ratio components, respectively. Their explicit expressions are provided in the Appendix (see equations (EC.8, EC.9). It should be noticed that each contribution $\theta^j(\mathbf{H}_t) - 1$, is of order $O(N^{-1})$ and is vanishing as $N \to +\infty$; however, the number of these contributions is also diverging as $N \to +\infty$. Thus, since Theorem 1 shows that firm-specific shocks may generate aggregate fluctuations, the asset pricing implications may be non-trivial. To investigate these properties, first we compute asset prices for a finite N, then we derive the implications for $N \to +\infty$ relying on the results of Theorem 1. The results are summarized by the following:

THEOREM 2 **(Network Relevance in Supercritical Economies and Long-Run Risks)**. *Consider a large economy satisfying Assumptions 1, 2 and assume*[15] $a := \delta - \mu(1-\gamma) - \frac{\sigma^2\gamma(1-\gamma)}{2} > 0$ *. The risk premium of firm i, conditional on its state* H_t^i *and* h_t^ν *is:*

- *(Subcritical equilibrium). When* $\frac{\lambda}{\eta} < K(\Delta)$*, as* $N \to +\infty$ *the risk premium converges to a constant CCAPM risk premium term:*

$$rp^i\left(H_t^i, h_t^\nu\right) = \underbrace{\kappa\sigma}_{CCAPM\ Term}$$

where $\kappa = \gamma\sigma$.

- *(Supercritical equilibrium). When* $\frac{\lambda}{\eta} > K(\Delta)$*, as* $N \to +\infty$ *the instantaneous risk premium of firm i converges to*[16]

$$rp^i\left(H_t^i, h_t^\nu\right) := \underbrace{\kappa\sigma}_{CCAPM\ Term} + \underbrace{\left(1 - H_t^i\right) rp_\lambda^i\left(h_t^\nu\right) + H_t^i rp_\eta^i\left(h_t^\nu\right)}_{\mathcal{NRP}^i\left(h_t^\nu\right)},$$

[14] the process dM_t^j is the martingale component of dH_t^j and is explicitly defined in eq.(EC.4)

[15] The constant a is equal to the equilibrium dividend-price ratio in a CCAPM economy with a single Lucas Tree driven by the lognormal shock Y_t.

[16] The exact expressions of $rp_\lambda^i(h_t^\nu)$ and rp_η^i for any value of the ratio $x^i(1)/x^i(0)$ are given in the Appendix.

$$rp^i_\lambda\left(h^\nu_t\right) := \nu^R_i\left(1-\frac{x^i(1)}{x^i(0)}\right)h^\nu_t\frac{a}{a+\eta}\alpha\lambda+O\left(\frac{x^i(1)}{x^i(0)}\right),$$
$$rp^i_\eta := -\left(1-\frac{x^i(1)}{x^i(0)}\right)\eta a+O\left(\frac{x^i(1)}{x^i(0)}\right).$$

In subcritical equilibria, firm-specific distress shocks have marginal contributions of order $1/N$ to the stochastic discount factor. Long-run fluctuations do not arise, namely $h^i_\infty = h^\nu_\infty = 0$, so that firm specific shocks do not affect the marginal utility as $N \to +\infty$. Valuations are observationally equivalent to those emerging in a classical CCAPM.

In a supercritical state the long-term stable equilibrium levels h^i_∞ and h^ν_∞ are strictly positive (Theorem 1) hence conditional risk premia are affected by a second term, the *Network Risk Premium.* Since firm-specific distress shocks can give rise to cascades, i.e. clusters of distress transitions changing the level of h^ν_t, prices at time t need to adjust accordingly and risk premia are modified in a permanent way.

$\mathcal{NRP}^i\left(h^\nu_t\right)$ is given by the sum of two terms. The first term is non-zero conditional on the firm not being in distress already, namely $H^i_t = 0$, $\mathcal{NRP}^i\left(h^\nu_t\right)$ is positive and proportional to vulnerability ν^R_i, to the dividend lost conditional on distress $\frac{x^i(1)}{x^i(0)}$, and to the level of network systemic risk h^ν_t, that depends on which firms are currently in distress and on their systemicness ν^L_i. Indeed, in presence of a network structure, the degree of systemicness defines the potential role of a firm in spreading long-run distress h^ν_∞. For instance, a firm which is a specialized supplier of many customers may have high systemicness, so that its distress may disrupt production of downstream producers and amplify the propagation of distress along a supply chain.[17] The second term in $\mathcal{NRP}^i\left(h^\nu_t\right)$ is a recovery premium and is negative conditional on firm i being in distress, i.e. $H^i_t = 1$. Once a firm is already in distress, the expected recovery acts as a positive externality on other firms since it reduces their probability of distress, thus reducing the aggregate quantity of risk.

The Theorem implies that cross-sectional differences in firm vulnerability ν^R_i to network systemic risk h^ν_t predict cross-sectional differences in expected excess returns. Notice also that firm-i vulnerability ν^R_i may depend on both individual firm characteristics and also on the characteristics of connectivity of the firm within the network, since ν^R_i is an element of the (right) principal singular component of the *global* adjacency matrix Δ.[18] As a Corollary to the previous theorem, one can derive the average risk premium of a firm in class C_k, conditional on the state of average distress h^k_t of firms in this class and of the systemic network risk h^ν_t:

[17] Notice that in this framework systemicness of a firm can large even in the case of a firm whose size is vanishing in the large N limit. It is worth stressing that this amplification channel is not driven by covariation with aggregate consumption. Hence the mechanics of this channel is distinct from the idiosyncratic small size limit considered in the analysis of the two asset Lucas orchard economy by Martin (2013).

[18] At the same time, firm-j authority ν^L_j depends on the entire vector of other firms exposures ν^R_i according to the relation $\nu^L_j = \frac{1}{\alpha^{\mathcal{G}}}\sum_{i:\ (i,j)\in E^{\mathcal{G}}}\nu^R_i\Delta^{\mathcal{G}}_{i,j}$

COROLLARY 2. *In supercritical equilibria, the average risk premium of firms belonging to class C_k is given by:*

$$\kappa\sigma + \mathcal{NRP}^k\left(h_t^\nu\right), \quad k = 1, \ldots K \tag{10}$$

where[19]

$$\begin{aligned} \mathcal{NRP}^k\left(h_t^\nu\right) \ :&= \left(1 - h_t^k\right) rp_\lambda^k\left(h_t^\nu\right) + \left(h_t^k\right) rp_\eta^k \\ &= \underbrace{rp_\lambda^k\left(h_t^\nu\right)}_{>0} + h_t^k \underbrace{\left[rp_\eta^k - rp_\lambda^k\left(h_t^\nu\right)\right]}_{<0} \end{aligned} \tag{11}$$

Furthermore, the drift of the dividend component distributed by firms belonging to class k equals $-\mathcal{NRP}^k\left(h_t^\nu\right)$.

This Corollary provides a consumption-based interpretation of the network risk premium. At time t, the conditional risk premium is equal to the sum of two terms:

$$\mathcal{RP}_t^k = \beta_k^{Mkt} \times \mathcal{MRP} + \mathcal{NRP}_t^k\left(h_t^\nu\right)$$

The first term is the traditional beta exposure to the market risk premium that corresponds in our economy to $\mathcal{MRP} = \kappa\sigma$ (in this economy, all the firms have unit market risk exposure). The second term is the network risk premium $\mathcal{NRP}_t^k\left(h_t^\nu\right)$, which depends on two terms. The first term is positive and proportional to the firm's network exposure ν_k^R to the network systemic risk h_t^ν. The second is a recovery network discount. Since $h_t^k\left(h_t^\nu\right) \cdot rp_\eta^k\left(h_t^\nu\right)$ in equation (11) is negative, the sign of $\mathcal{NRP}_t^k\left(h_t^\nu\right)$ is positive (negative) when the first (second) term is dominating. This occurs when h_t^k is large, which corresponds to states when large distress has already occurred and firms face the possibility of recovering. The recovery discount is proportional to η, which controls the speed of recovery from distress.

It is convenient to notice that, in the limit of $x^i\left(1\right)/x^i\left(0\right) \to 0$ when $h_t^k \cong \alpha\lambda\nu_k^R h_t^\nu$, the risk premium is a quadratic function of h_t^k and h_t^ν. Let $h^* \equiv (1 - \eta a - \eta^2)$, then:

$$\mathcal{NRP}_t^k = \left[h^* - h_t^k\right] h_t^k \left(1 - \frac{x^k\left(1\right)}{x^k\left(0\right)}\right) \frac{a}{a + \eta}$$

which shows that $\mathcal{NRP}_t^k$ can be expressed as a concave quadratic function reaching its maximum value when $h_t^k = \frac{h^*}{2}$. When h_t^k is in the interval $[0, \frac{h^*}{2}]$, $\mathcal{NRP}_t^k$ is increasing w.r.t. h_t^k. However, for levels of exposure to network distress risk $h_t^k > \frac{h^*}{2}$, $\mathcal{NRP}_t^k$ starts decreasing with h_t^k due to the positive recovery expectation and finally for $h_t^k > h^*$ the risk premium turns to negative values.

[19] The functional forms of the two terms $rp_\lambda^k\left(h_t^\nu\right)$ and rp_η^k are the same as in Theorem 2.

This Corollary generates a number of empirical testable implications. The first relates to the potential resolution of Campbell's "Distress Risk Puzzle". Indeed, $\mathcal{NRP}_t^k$ is consistent, at the same time, with (a) lower unconditional expected excess returns of firms with larger probability of distress and (b) a positive price of distress risk. To appreciate this result, it is convenient to consider a local linearization of the risk exposure $\mathcal{NRP}_t^k$ with respect to h_t^ν, so that $\mathcal{NRP}^k(h^\nu) \simeq \beta_k^{h^\nu} \times h^\nu$:

$$\beta_k^{h^\nu} \simeq \frac{\partial \mathcal{NRP}^k(h^\nu)}{\partial h_t^\nu} = \nu_k^R \alpha\lambda \left[h^* - 2h^k(h^\nu)\right] \left(1 - \frac{x^k(1)}{x^k(0)}\right) \frac{a}{a+\eta},$$

The model predicts that the conditional $\beta_k^{h^\nu}$ switches sign from positive to negative for values of $h^k(h^\nu)$ crossing the threshold level $\frac{h^*}{2}$. Below this threshold, the portfolio beta and the network risk premium are positive and increasing in the vulnerability ν_k^R. However, for large levels of $h^k(h^\nu)$ the exposure of the most vulnerable firms turns negative. The economic intuition is that when firms have already experience large levels of distress, the expected dividend growth turns positive and they face the (negative) risk of recovery. This occurs precisely when the level of distress in the economy is the largest. The negative relationship between $\mathcal{NRP}_t^k$ and the level of individual firms probability of distress h^k has been documented by Campbell et al. (2008) and is generated by the model's procyclical *risk exposure* and countercyclical *price of risk.* When taken together, these two facts can also rationalize the emergence of an unconditional CAPM α_k which is negatively covarying h^k. This is consistent with the empirical results of Jagannathan and Wang (1996) Lewellen and Nagel (2006), who find that when the conditional CAPM model is tested using an unconditional linear regression specification, the model misspecification bias manifests in a value of α_k proportional to the covariance of the firm's beta with the market risk premium. In the context of our model, the model misspecification is due to the endogenous negative correlation between conditional $\beta_k^{h^\nu}$ and distress shocks h^k.

3.2. Predictability

In this section we study the extent to which the information about $\mathcal{NRP}_t^i$ is rationally revealed by valuation ratios such as the dividend-to-price ratio. The following Theorem summarizes the link between $\mathcal{NRP}_t^i$ and $\frac{D(H^i)}{P^i(H^i,h^\nu)}$.

THEOREM 3 **(Cross-Sectional Predictability)**. *Consider a large economy limit of networks satisfying Assumptions 1 and 2. The cross-section of expected risk premium of firms i is given by:*

$$rp^i(H_t^i, h_t^\nu) = \begin{cases} \kappa\sigma & \textit{Subcritical Equilibrium} \\ \kappa\sigma + \left[\frac{D(H_t^i)}{P^i(H_t^i, h_t^\nu)} - a\right]. & \textit{Supercritical Equilibrium} \end{cases} \tag{12}$$

In subcritical equilibria, risk premia are constant both in the time-series and cross-section. In a supercritical equilibrium, equation (12) shows that the cross-section of expected returns is predicted by the cross-section of dividend-price ratios. The parameter a is the reference dividend-price ratio in a conventional Lucas tree economy driven by a purely diffusive aggregate lognormal shock and satisfies the continuous-time Gordon growth formula:

$$\underbrace{a}_{d/p} = \underbrace{\kappa\sigma + r_f}_{rp+r_f} - \underbrace{\mu}_{g}. \tag{13}$$

where $r_f = \delta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2$ is the risk free rate in the standard Lucas Tree economy driven by the log-normal shock.

In the supercritical state, the dividend-price ratio and the risk premium component become state dependent:[20]

$$\frac{D(H_t^i)}{P^i(H_t^i, h_t^\nu)} = rp^i(H_t^i, h_t^\nu) + r_f - \mu. \tag{14}$$

Taking the difference between equations (13) and (14) one obtains (12). Hence, when $h_t^\nu > 0$ deviations of risk premia from the standard Lucas tree value are predictable and proportional to the spread between the observable dividend price ratio $\frac{D(H^i)}{P^i(H^i,h^\nu)}$ and its subcritical level a. The economic intuition is that both the dividend price and the risk premium are a function of the vulnerability ν_i^R and the network systemic risk h_t^ν.

The results summarized in Theorems 2 and 3 can be related to the 'deep value anomaly' emerging during period of significant economic and financial distress, as documented by Asness et al. (2018). They show that the average risk premium increases when the spread between value versus growth premia deepens ('deep value states'). In our network economy, the widening of this spread naturally occurs in super-critical states, when $\frac{\lambda}{\eta} > K(\Delta)$ and network systemic risk increases. Thus, it is a rational equilibrium valuation property.

In the context of our model, one can define the Vulnerability Spread (VmR) as the difference between the mean dividend yield of high vulnerability firms and low vulnerability (i.e. resilient) firms:

$$VmR(h_t^\nu) := Y_t\left[\frac{x(0)}{P^V(0, h_t^\nu)} - \frac{x(0)}{P^G(0, h_t^\nu)}\right].$$

Indeed, value (growth) stocks can be interpreted as firms that are *not yet* in distress ($H_t^i = 0$) but have high (low) vulnerability ν_R^k. In subcritical equilibria, dividend price ratios are constant and homogeneous. In these equilibria, the vulnerability spread $VmR(h_t^\nu) = 0$ and the cross-section of expected returns is degenerate. No predictability exists either in the cross-section or time-series. In supercritical equilibria, however, $h_t^\nu > 0$ and the dividend-price ratio becomes state-dependent and

[20] Expected dividend growth and risk free rate are not affected by the transitory component. Let μ be the mean long-term expected dividend growth:
$\lim_{h\to+\infty} \frac{1}{h} E_{t_0}\left[\int_{t_0}^{t_0+h} \frac{d\left(Y_t x^i(H_t)\right)}{Y_t x^i(H_t)}\right] = \lim_{h\to+\infty} \frac{1}{h} E_{t_0}\left[\int_{t_0}^{t_0+h} \frac{dY_t}{Y_t}\right] = \mu$

increasing in firm's vulnerability: $\frac{dVmR(h^{\nu})}{dh^{\nu}} > 0$. As a consequence, a transition to a supercritical equilibrium with $h_t^{\nu} > 0$ can be revealed by the broadening of $VmR(h_t^{\nu})$.

Since in supercritical equilibria firm-specific conditional risk premia are both increasing in vulnerability, $\frac{drp^i(0,h_t^{\nu})}{d\nu_i^R} > 0$ and increasing in h_t^{ν}, i.e. $\frac{drp^i(0,h_t^{\nu})}{dh_t^{\nu}} > 0$, an implication of Theorem 2 is that in supercritical equilibria the expected excess return (HmL) of a portfolio long value firms and short growth firm earns a positive risk premium. Moreover, Theorem 3 implies that $\frac{dHmL}{dVmR\left(h_t^{\nu}\right)} > 0$: the spread in risk premia is increasing in the Vulnerability Spread $VmR(h_t^{\nu})$. Indeed, equation (12) implies

$$HmL \propto rp^V\left(H_t^i, h_t^{\nu}\right) - rp^G\left(H_t^i, h_t^{\nu}\right) = VmR\left(h_t^{\nu}\right). \tag{15}$$

This lends support to the observation by Asness et al. (2018) that deep value states with large $VmR(h^{\nu})$ also have larger HmL expected excess returns.

An additional implication of Theorem 3 is that the dividend-price ratio is positively correlated with expected returns and uncorrelated with expected dividend growth. This is consistent with dividend-price predictability tests discussed in Cochrane (2011) who argues that most of the variability of the dividend-price ratio is linked to changes in expected returns as opposed to future dividend growth.

The model produces also the additional prediction that VmR is increasing in h_t^{ν}, which relates to conditional tests of the Fama and French factor model showing that long-short value strategies become profitable mainly when the VmR spread is very large (Deep Value states).

4. Empirical Results

In this section we study the empirical implications of the model. We use a monthly panel data-set of firms from Jan 1970 to Dec 2019, which are part of the CRSP-COMPUSTAT dataset restricted to firms with CRSP code 10,11 and 12. We consider firms with at least five years of sales data available. We notice that in our model cash-flows dynamics can be factored as a product $Y_t x_t^i$ where Y_t denotes the systematic component and a single idiosyncratic component x_t^i. Empirical reliability of this reduced form modeling approach is supported by the findings of Herskovic et al. (2016) who show that the sales-growth and return firm-specific volatilities are driven by a common, single component. We replicate and confirm their findings in our extended sample.[21]

We determine time varying measures of vulnerability and systemicness reconstructing the network information from the input-output BEA industry tables. Since 1947, the BEA has provided IO accounts of dollar flows between all producers and purchasers in the U.S. economy. The IO tables are based primarily on data from the Economic Census and are updated every five years with

[21] Results are available upon request

a five-year lag. BEA provides Make-Use tables and we compute the corresponding IO table repeating the construction that Ahern and Harford (2014) use to compute the matrix that is referred as REVSHARE. Hence, we set $\Delta_{ij}^{(t)} = IO_{ij}^{(t)}/IO_{ii}^{(t)}$ as the revenue share that is produced by industry i and consumed by industry $j \neq i$ at year t. Note that statistical reliability of our modeling approach requires the verification of Assumption 1 and, in light of Assumption 2, it is expected to increase with the size of the population of firms with homogeneous levels of vulnerability and systemicness. In light of these modeling assumptions, we consider IO tables aggregated at the coarsest level and isolate 15 groups of firms as defined in Table 1 so that in the application it is possible to associate an industry sector to each class $k = 1, ..., 15$ of firms with homogeneous degree of vulnerability and systemicness. Note that the Fig. 2 provides a graphical illustration of the network connecting these 15 groups as determined by the first 1970 IO-Table and by the last 2015 IO-Table.

Singular value decomposition of the mean Input-Output matrix determines a first singular value that accounts for 85% percent of the total Frobenius norm[22]; the second and third singular value account for 13% and 1%, respectively. In light of this evidence, we restrict the rest of our empirical analysis to the rank one version of the model. For each network matrix $\Delta^{(t)}$, we compute its right and left singular vectors ν^R and ν^L normalized so that the sum of the elements of each vector is one. We recompute these two vectors every 5 years and use their beginning of period value to avoid any look-ahead bias. We thus obtain empirical (sector based) proxies for ν_i^R and ν_i^L for each firm which is associated to its group relying on SIC (prior to 1997) NAICS (post 1997) classification codes.

Considering the sample mean values, we find that "Manufacturing" is the sector with the highest systemicness and lowest vulnerability, with $\nu_i^L = 0.36$ and $\nu_i^R = 0.01$. As one may expect, "Educational services, health care, and social assistance" and "Agriculture, forestry, fishing, and hunting" have low systemicness and high vulnerability, with $\nu_i^L = 0.006$, 0.007, and $\nu_i^R = 0.08$, 0.07, respectively. The financial sector is the second most systemic sector with $\nu_i^L = 0.14$.

Empirical vulnerability and systemicness parameters show substantial time-variation driven by the variation in time of the relative IO-Tables. For example, it is apparent from Fig. 2 that systemicness of group 10, Finance Insurance and Real Estate (FIRE hereafter) firms have dramatically risen over the period 1970-2015. Its initial value determined by the 1970 IO-Table is 0.105, it reaches 0.225 prior to the 2007 crisis then it decreases to the value 0.202 as determined by the last 2015 IO-Table. In turn, vulnerability of this group of firms has remained low and essentially constant across the whole sample. On the contrary, group 5, Manufacturing, has progressively reduced its systemicness from 0.440 down to 0.259 while its vulnerability has tripled raising from 0.0067 up to 0.0200.

[22] Recall that the square of the Frobenius norm equals the sum of the squares of the singular values.

4.1. Network risk dynamics and tipping point

For each firm i and at each time t, we proxy h_t^i with the one-month ex-ante expected probability of distress by estimating a logit model as in Campbell et al. (2008). Details are summarized in the Appendix. We follow the model construction and define a sample proxy of the network systemic risk as the weighted average of h_t^i with weights given by the elements of the vector of systemicness ν_i^L:

$$h_t^\nu := \frac{\sum_{i=1}^N \nu_i^L h_t^i}{\sum_{i=1}^N \nu_i^L}. \tag{16}$$

Figure 3 summarizes the time series properties of h_t^ν. In the model, the dynamics of h_t^ν are determined by Theorem 1. Below the tipping point K_c, the dynamics is subcritical with $h_t^\nu = 0$; above the tipping point $h_t^\nu > 0$. In the earlier part of the sample, we find that h_t^ν is insignificantly different from zero, suggesting a subcritical dynamics. We also find five episodes in which h_t^ν has spiked to significantly high levels (1985-1992, 1999, 2000-2002, 2008-2009, 2017). We run a *Sup-Wald* test that identifies a structural break in the dynamics of h_t^ν in June 1984. The Chow test-statistics executed in correspondence to *June* 1984 produces a test statistics of 147.6404 with a $p\text{-}value = 0.000$. Before this date, the average value of h_t^ν is 0.00021 (Table 2, Panel A). After this date, the mean of h_t^ν is 7.5 time larger. In the four periods in which h_t^ν spikes and goes above its long run mean, both the volatility and third moment of h_t^ν are significantly higher. The volatility is 0.0143, versus 0.00034, and the third moment is 7.49×10^{-5}, versus 3.93×10^{-6}.

This structural transition in the dynamics of h_t^ν is consistent with Giesecke et al. (2011) observation that the introduction of the Bankruptcy Reform Act of 1978 which marked the first major bankruptcy law overhaul in forty years. The new law introduced vast reforms in both liquidation and reorganization proceedings shifting from the previous creditor protection spirit of the Bankruptcy Act of 1898 to a new more debtor friendly approach.[23]

After June 1984, the four additional spikes in h_t^ν suggest the presence of distress cascades which are short-lived but with a significantly higher variance. Indeed, the model suggests that the distributional properties of network systemic risk is different below and above the tipping point. Below the tipping point ($\lambda/\eta << \mathcal{T}_c$), the distribution of h_t^ν should display exponential dampening of fluctuations, in line with the Lucas assumption. In the vicinity of the tipping point ($\lambda/\eta \sim \mathcal{T}_c$) one should expect a Pareto tail in the distribution of h_t^ν:

$$\Pr\left[h_t^\nu > H\right] \simeq H^{-\alpha},$$

with $\alpha = 0.5$. Indeed, in the supercritical equilibrium contagion drives bursts of distress and the distribution should display increasing logarithmic mean and variances. To test for these differences,

[23] See figure 1 in Giesecke et al. (2011).

we estimate α using a Hill estimator before and after the structural transition. Consistent with the model, after June 1984 we find that $\alpha = 0.51$ with a standard deviation equal to 0.03, which is compatible with a Pareto tail. Before June 1984, on the other hand, we can reject the null hypothesis $H_0 : \alpha = 0.5$. A graphical illustration of the distributional properties is reported in Fig. 5.

4.2. The network risk premium

The first empirical asset pricing implication of the model relates to the emergence of a risk premium associated to the network systemic risk tracked by the level of h_t^ν.

To assess the economic importance of the network risk premium, we consider a Fama and MacBeth (1973) two-step multifactor regression (FMB hereafter) methodology and exploit information on risk premia and on tradable portfolios. The cross-section of returns is formed by 45 test assets: (i) 10 value-weighted portfolios sorted with respect to firm-specific probability of distress based on the Campbell et al. (2008) logit estimation approach, (ii) 10 value-weighted portfolios sorted by idiosyncratic return volatility, and (iii) 5×5 portfolios sorted with respect to BE and M/B.

$$\mathcal{R}_{t+1}^{k} - R_t^f = \beta_k^{Mkt} \times \left[\mathcal{R}_t^{Mkt} - R_t^f\right] + \beta_k^{h^\nu} \times \frac{h_t^\nu}{\sigma_{h^\nu}} + \varepsilon_{t+1}^k.$$

where $\sigma_{h^\nu} = stdev(h_t^\nu)$ over the period of time

In the first stage, we estimate β_k^{Mkt} and $\beta_k^{h^\nu}$ by regressing the time-series of the 45 test assets monthly excess returns onto the market excess return and the standardized version of the network systemic risk $\frac{h_t^\nu}{\sigma_{h^\nu}}$. Then, in a second stage, we project the monthly excess returns of the test assets on the vector $[\beta_k^{Mkt}, \beta_k^{h^\nu}]$ to obtain the prices of risks associated to the factors. Table 3 summarizes the results. Column (1) reports the results of a CAPM specification, which excludes $\mathcal{NRP}_t^k$. We find that the monthly market price of risk is equal to 0.683 and statistically significant at the 1% confidence level. However, the intercept is -0.177 and statistically significant, suggesting that the CAPM model is misspecified.

In column (2) we add the exposure to network systemic risk $\mathcal{NRP}_t^k$. It is notable that the statistical significance of a strictly positive $\mathcal{NRP}_t$ can only be consistent with the existence of a supercritical equilibrium in this period. We find that the monthly price of risk of the exposure to network distress is 1.132% and is statistically significant at the 1% confidence level. Moreover, we find that the absolute value of the intercept drops to 0.065 and is no longer significant, suggesting that the new term reduces the misspecification error found in column (1). The evidence of a priced network factor is robust after controlling for different combinations of the Common idiosyncratic volatility factor of Herskovic et al. (2016) and of the five factor Fama-French model (market, size, value, profitability, and investment), as shown in columns (4)-(6). In all these alternative specifications, the slope coefficient for $\beta^{h^{\nu}}$ is statistically significant and its value ranges between 1.165 and 1.45.

The model predicts that there's substantial time variation of risk premia. Hence, we analyze the intermediate Fama-McBeth statistics to study the time variability of the risk premium. First, it is worth observing that before June 1984, the equally weighted average of $\mathcal{NRP}_t^k$, which we refer to as $\mathcal{NRP}_t$, in Table 2 Panel B, is equal to a monthly return of 0.007% with a t-statistics equal to 0.7, which is consistent with a subcritical dynamics; after June 1984, $\mathcal{NRP}_t = 1.49\%$ monthly with a t-statistics equal to 4.81. Figures 6 and 7 summarize the economic significance of the network risk premium relative to the CAPM market risk premium by showing the cumulative return of a portfolio with unit exposure on each of the two sources of systematic risk. They are obtained as the cumulative sum of the slopes from the monthly cross-sectional regressions. These cumulative returns relate to zero-cost portfolios of the 45 test assets with a unit exposure on the corresponding factor and zero on all the others. Temporal variation of the distress risk premium shows, as predicted by the model, a positive and countercyclical increase that is steeper in correspondence to market recessions.

Factor exposures are estimated by running an unrestricted time-series linear regression. However, the model implies a humped shape function of $\beta_k^{h^{\nu}}$ as a function of h_t^k. Table 4 summarizes the results of the first stage regression coefficients. Consistent with the model, we find that for low levels of h_t^k the coefficient $\beta_k^{h^{\nu}}$ is positive, while for larger levels of specific distress risk h_t^k the parameter $\beta_k^{h^{\nu}}$ becomes decreasing in h_t^k and becomes strongly negative and significant for the top distress deciles sorted portfolios. This is important given Campbell et al. (2008) observation of a counterfactual negative premium earned by firms with large firm specific distress risk.

A related interesting empirical confirmation about the relevance of the network distress factors relates to Fama and French (1993) and Chan and Chen (1991) who conjecture that size and book-to-market anomalies could partially be compensation for distress risk. This is indeed consistent with our model. We directly investigate this conjecture and estimate the risk exposure of a cross-section 5×5 sorted portfolios according to size and value characteristics. Confirming Fama and French (1993) and Chan and Chen (1991) conjecture, we find that companies with the largest B/M and smallest size are also those with the highest exposures to network distress risk $\beta_k^{h^{\nu}}$. Table 10 summarizes the results. The portfolio of companies in the bottom size quintile and top size decile has a network beta $\beta_k^{h^{\nu}}$ equal to 0.207; on the other hand, the exposure to network risk decreases for large-growth companies and the $\beta_k^{h^{\nu}}$ of the portfolio of companies in the top size quintile and bottom size decile $\beta_k^{h^{\nu}}$ is equal to -0.432. Results in Table 3 confirm that the traded factors HML_t and SMB_t are redundant after we control for h_t^{ν}.

4.3. The "Deep Value" effect

A second prediction of the model (Theorem 3) is that, in a super-critical equilibrium, the network risk premium $\mathcal{NRP}_t^k$ is state-dependent and a function of both the current state H_t^i of the firm and the level of network systemic risk h_t^ν. The state dependence can be expressed in terms of the observed dividend-yield of the firm. Indeed,

$$\mathcal{NRP}_t^k\left(H_t^i, h_t^\nu\right) \equiv rp^k(H^i, h^\nu) - \kappa\sigma = \left[\frac{Y_t x_k\left(H_t^i\right)}{P\left(H_t^i, h_t^\nu\right)} - a\right]. \tag{17}$$

The parameter a is the permanent component of the dividend-price ratio: it is linked to cash flows and prices according to the Gordon's type of relationship: $a = [\kappa\sigma + r_f] - \mu$. Below the tipping point, $\frac{Y_t x_k\left(H_t^i\right)}{P\left(H_t^i, h_t^\nu\right)}$ is constant and equal to a, so that $rp^k(H^i, h^\nu) = k\sigma$ and firms' risk premia are constant. However, above the tipping point, firms' risk premia are time-varying and correlated with the dividend-price ratio:

$$rp^k\left(H_t^i, h_t^\nu\right) = (\mu - r_f) + \frac{Y_t x_k\left(H_t^i\right)}{P\left(H_t^i, h_t^\nu\right)}.$$

For stocks of firms that are not in distress, greater values of h_t^ν induce both greater risk premia $\mathcal{NRP}_t^k\left(H_t^i, h_t^\nu\right)$ and larger dividend-price ratios $\frac{Y_t x_k\left(H_t^i\right)}{P\left(H_t^i, h_t^\nu\right)}$. Moreover, since $\frac{\partial^2 rp^k\left(H_t^i, h_t^\nu\right)}{\partial \nu_k^R \partial h_t^\nu} > 0$ at $h_t^\nu = 0$, as the equilibrium become supercritical greater values of h_t^ν correlate with a larger vulnerability spread VmR_t, namely a larger spread in the dividend-price ratios between cheap and expensive stocks. Asness et al. (2018) refers to episodes of elevated dividend-price ratios spread as "Deep Value" states and document that in these periods equity risk premia of high dividend-price ratio (value) firms are greater. Moreover, they find that this additional positive excess return is unexplained by traditional factors. Within our framework, we expect that "Deep Value" states corresponds to periods of high VmR_t and in light of eq.(17) we expect VmR_t to be a predictor of larger future excess returns of value relative to growth stocks. We investigate this additional model implication. We sort stocks in terciles portfolios k according to firm vulnerability ν_k^R as determined by the input-output network. We compute the average sale-to-price ratio of each tercile portfolio and define VmR_t as the spread of the mean sale-to-price ratio of the portfolios of firms with the highest vulnerability tercile minus the lowest tercile. [24] We find that the portfolio in the lowest tercile of vulnerability has an average vulnerability ν_k^R equal to 0.01 and sale-to-price ratio equal to 1.82; the portfolio in the top tercile has an average vulnerability ν_k^R equal to 0.07 and sale-to-price ratio equal to 2.7. Table 8 summarizes the results. As predicted by the model, we find that the sale-to-price ratio of the portfolio of firms with the highest vulnerability is higher than that of the portfolio of firm with lower vulnerability (see Figure 8). We also find that, consistent with the

[24] We use price-to-sales ratio instead of price-to-dividend ratio. This choice is internally consistent with the measure of vulnerability to idiosyncratic cash flow shocks that relies on sales data reported in IO tables and on the analysis of skewness that will be discussed later.

model, in the supercritical equilibrium (i.e. 1985-2019) VmR_t is increasing in h_t^ν, with a correlation between h_t^ν and VmR_t is 0.51. Figure 9 shows the joint dynamics of the network systemic risk h_t^ν and of VmR_t. There are four periods in which network systemic risk h_t^ν has increased significantly well above the tipping point of a supercritical equilibrium. In all these four periods, the sales-to-price ratio have spiked in correspondence to spikes in h_t^ν. In these periods, the model predicts that the price of more vulnerable stocks should drop more significantly than resilient firms. Indeed, we find that VmR_t increases during these episodes, as Table 8 and Figure 8 indicate. The average value of VmR_t is 0.865 when the economy is above the tipping point, while it is 0.463 when the economy is below the tipping point.

At the same time, the models predicts that expected excess returns of value firms should increase relative to growth firms, see equation (15). Indeed, in the context of our model, value (growth) stocks can be interpreted as firms that are not yet in distress ($H_t^i = 0$) but have high (low) vulnerability ν^R. These firms have a high dividend-to-price ratio (as proxied by the sales-to-price ratio) and also a higher book-to-market, as shown in Table 8. We investigate this implication by testing whether a widening of the level of network systemic risk h_t^ν ("Deep Value "states) correlates with an increase in the value risk premium. Accordingly, we run a predictive regression of the HmL_{t+1} value portfolio return, which we download from the Ken French website, on h_t^ν and VmR_t:[25]

$$
\begin{aligned}
HmL_{t+1:t+\tau+1} &= a_\tau^{h^\nu} + b_\tau^{h^\nu} h_t^\nu + \varepsilon_{t+1:t+\tau+1}^\tau, \qquad\qquad \tau = 2, ..5 \\
HmL_{t+1,t+\tau+1} &= a_\tau^{VmR} + b_\tau^{VmR} VmR_t + \eta_{t+1:t+\tau+1}^\tau.
\end{aligned} \tag{18}
$$

Table 9 summarizes the results for the period 1985-2020. The *t-statistics* use Newey-West corrected standard deviations to account for autocorrelation and heteroskedasticity in errors due to overlapping data. As predicted by the model, prior to June 1984 (below the tipping point), the slopes of the two predictive regressions are not significant. After June 1984, however, the slope coefficient $b_\tau^{h^\nu}$ of the first regression becomes statistically significant at the 1% confidence level for holding periods returns above 2 years. The slope coefficient b_τ^{VmR} in the second regression is statistically significant at the 10% level above 2 years and at the 1% level for longer maturities. This confirms that both network systemic risk h_t^ν and the VmR_t spread are significant predictors of the dynamics of the HmL factor. It is also interesting to notice that we cannot reject the null hypothesis that $H_0: a = 0$ in either regression, suggesting that these variables can capture a large component of the unconditional value of the HmL factor, which was unexplained before.

[25] We cross-checked the results considering also value portfolios constructed relying on earning-price ratios and results are the same.

4.4. Cash-flow Skewness and Vulnerability

An additional prediction of the model is that, when network systemic risk h_t^{ν} spikes in supercritical equilibria, cascades of cash-flow shocks give rise to clusters of distress transitions. This manifests in more negative cross-sectional skewness; the effect should be greater for more vulnerable firms and sectors.

This prediction relates to an important literature in economics that investigates how the distribution of the growth rate of firm-level variables (sales, profit, inventories, and employment) changes over the business cycle. Salgado et al. (2019) finds that the third moments of this distribution—skewness—is strongly pro cyclical. This happens because the distribution of negative growth rates expands during recessions while the distribution of positive growth rates changes little. They argue that this is the main driver behind the counter cyclicality of dispersion. Figure 10 provides a graphical illustration of the shift in the cross-sectional skewness during two well-known crises: (a) 11 September 2001 (i.e. "9/11") and (b) September 2008 ("Global Financial Crisis").

In our model, this effect is due to the propagation effect that emerges in supercritical equilibria. To test this prediction, we consider economic periods characterized by a large level of network systemic risk h_t^{ν}. During these periods, we compute the cross-sectional skewness KSK_k at the sector level for yearly log-sales growth, a proxy for cash flow growth. We sort sectors in five quintiles based on their network vulnerability ν_k^R, $k = 1, .., 5$ and study the link between KSK_k and ν_k^R. Table 10 summarizes the results.

When we select the top 5% tail events in terms of h_t^{ν}, we find a strong monotonic relationship between KSK_k and ν_k^R. It ranges between $+1.18\%$ for the first quintile group of firms and -15.01% for the most vulnerable quintile. The value of KSK_k for the last two quintile group of firms is significant at the 1% confidence level. A negative value of Kelley Skewness indicates that the left tail accounts for more than one-half of the total dispersion and the distribution is negatively skewed. As predicted by the model, the greater the vulnerability the more negative the Kelley Skewness. The result is robust if one were to restrict the sample to tail events corresponding to even larger levels of aggregate distress. Indeed, in time periods in which h_t^{ν} is in the top 5% of the distribution, the difference in skewness between the top and bottom ν_k^R group of firm exceeds 20%.

An important counterfactual implication of the model is that this link should not be present in subcritical equilibria, due to the absence of cascades. To test this counterfactual, we consider the subcritical subsample prior to the transition of h_t^{ν} above its critical value and repeat the test. Consistent with the model, we cannot reject the null hypothesis of a lack of relationship between KSK_k and ν_k^R. The KSK_k value of the most vulnerable firms is -0.48% versus -2.6% for the least vulnerable firms and neither is statistically significant. This confirms the importance of both distinguishing the nature of the equilibrium and the role played by the global network structure, which ultimately determines firms' vulnerabilities.

5. Conclusion

The introduction of a network structure in a DSGE model shows the existence of two classes of valuation equilibria. In subcritical equilibria Lucas diversification assumption implies that firm-specific distress shocks can be diversified and CCAPM asset price implications applies. In supercritical equilibria, however, seemingly idiosyncratic shocks generate externalities that propagate generating aggregate fluctuations. In these equilibria, these externalities survive in the long-run and must be compensated ex-ante, giving rise to a second distinct source of (network) risk premia. We use the model to study the importance of accounting for the networks dynamics to address four questions bringing them to the data. First, we identify a structural transition between a sub and a supercritical valuation equilibrium showing that a shift in the investor expectations about the aggregate probability of distress drove also a change in the price of risk of network systemic risks determined by cross sectional analysis of the CRSP-Compustat panel. Taking into account the network systemic risk factor improves the cross-sectional pricing performance rationalizing documented distress related anomalies. Transition to a supercritical equilibrium and network systemic risk pricing provides also evidence of a relationship between spikes in distress cascade risk and 'Deep Value' episodes. A Network systemic risk factor, h_t^{ν} and a Vulnerability based spread (VmR_t) have forecasting power and anticipate the increase in the Value premium driven by these episodes. Finally, we show that as predicted by the model, a higher level of firm vulnerability command also a higher drop of cash flows, as proxied by the Kelley Skewness of sales-growth, during high distress intensity, tail events.

There are many possible directions of improvement of this analysis. First, it would be interesting to explore the asset pricing implications of a network observed at a finer scale. This will require the extension of the analysis to a network including recursive directed tree geometries that violate Assumption 1 that are important to characterize transmission of shocks along customer-supplier chains. The model has also relevant implications also for the link between the cross-section of idiosyncratic volatilities and risk adjusted expected returns α_{3FF}. It can potentially rationalize the puzzling inverse relationship between idiosyncratic risk and risk premia has been studied also by Ang et al. (2006). Last but not least[26], a line of research we are currently exploring is the application of this network model to the analysis of systemic risk for financial institutions where the impact of illiquidity and the price expectations of financial institutions feed-back into expectation and may generate macroeconomic instability.

[26] Some results along these lines were already present in previous version of this paper and are part of a new dedicated one that is in preparation.

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6. Tables

Table 1 **Sectoral Mean Systemicness and Vulnerabilities.**

	Sector	ν_L	ν_R
1	Agriculture, forestry, fishing, and hunting	0.0075	0.0674
2	Mining	0.0325	0.0727
3	Utilities	0.0292	0.0520
4	Construction	0.0195	0.1118
5	Manufacturing	0.3679	0.01185
6	Wholesale trade	0.0528	0.0647
7	Retail trade	0.0209	0.0696
8	Transportation and warehousing	0.0557	0.0643
9	Information	0.0469	0.0607
10	Finance, insurance, real estate, rental, and leasing	0.1448	0.0292
11	Professional and business services	0.1344	0.0476
12	Educational services, health care, and social assistance	0.0063	0.0832
13	Arts, entertainment, recreation, accommodation, and food services	0.0203	0.0970
14	Other services, except government	0.0199	0.0896
15	Government	0.0413	0.0784

Systemicness and Vulnerability indicators are computed as the left and right singular vectors for the network matrix Δ_{ij} which is estimated using the input-output two-digit BEA industry tables. BEA provides Make-Use tables and the corresponding IO table is referred by Ahern et al. (2014) as REVSHARE. IO tables are aggregated at their coarsest Sectoral level and set $\Delta_{ij} = IO_{ij}/IO_{ii}$ as the revenue share that is produced in industry i and consumed by industry $j \neq i$. We recompute these two vectors every 5 years and use its beginning of period value to avoid any look-ahead bias. Both Sectoral Systemicness and Vulnerability values are normalized so that $\sum_{i=1}^{N} \nu^{L,R} = 1$.

Table 2 **Structural Break Descriptive Statistics.**

	Date < June 1984	Date >June 1984	$h^{\nu} > Mean\_Post\_1984$	$h^{\nu} < Mean\_Post\_1984$
	Panel A			
Mean h^{ν}	0.00021	0.00157	0.00273	0.00119
Stdev h^{ν}	0.00016	0.00164	0.00143	0.00034
$E[(h^{\nu})^3]$	$4.48e-6$	$9.41e-5$	$7.49e-5$	$3.93e-6$
	Panel B			
Mean $\mathcal{NRP}^h$ (monthly)	0.007	1.49	1.34	1.56
t-stat $\mathcal{NRP}^h$ (monthly)	[0.7]	$[4.81]^{***}$	$[3.33]^{***}$	$[4.80]^{***}$
N. Obs	163	425	148	278

Statistics of the level of network systemic risk indicator and of its price of risk. The first two column report statistics computed on two samples before and after the structural break of June 1984, see also Figure 3. The third (fourth) column reports statistics on the subsamples post transition selected by the condition that the level of network systemic risk is above (below) the post-1984 mean level. Panel A Line 1 reports the unconditional mean. Line 2 (3) report the standard deviation (the third moment) of h_t^{ν}. In Panel B we report in line 1 (2) the best estimate (the t-statistics) of $\mathcal{NRP}^h$. $\mathcal{NRP}^h$ is estimated computing the mean and the standard deviation, over the relevant period, of the first pass slopes obtained in the Fama-MacBeth test reported in Table 3. We consider as reference the benchmark model that includes as factors $R_{mkt} - R_f$ and h_t^{ν}. Expected returns are expressed in percentage terms (1 equals 1%)

Table 3 Fama-McBeth Regression results.

	(1)	(2)	(3)	(4)	(5)	(6)
cons	-0.177**	0.065	0.047	-0.102*	-0.042	-0.066
	(0.079)	(0.069)	(0.07)	(0.061)	(0.072)	(0.074)
β^{mkt}	0.683***	0.648***	0.589***	0.764***	0.663***	0.668***
	(0.201)	(0.204)	(0.207)	(0.2)	(0.196)	(0.206)
$\beta^{h^{\nu}}$		1.132***	1.450***	1.357***	1.165***	1.177***
		(0.222)	(0.234)	(0.228)	(0.199)	(0.229)
β^{CIV}			-0.416***	-0.891***	-0.807***	-0.769***
			(0.117)	(0.131)	(0.128)	(0.144)
β^{smb}				-0.033		
				(0.127)		
β^{hml}				0.106		
				(0.158)		
β^{rmw}					0.701***	0.605***
					(0.143)	(0.15)
β^{cma}					-0.027	
					(0.144)	
N	25055	25055	25055	25055	25055	25055
R^2	0.289	0.405	0.459	0.603	0.59	0.536
RMSE	0.540	0.490	0.486	0.339	0.311	0.293

The set of test assets are 25 portfolios sorted on size (ME) and book-to-market (BM)ratio, 10 portfolios sorted with respect to Idiosyncratic Volatility which is computed following the procedure proposed by Ang et al. (2006) considering as reference model the Fama French 3 factor model and 10 portfolios sorted on the firm-specific probability of distress as computed by Campbell Hilsher and Szilagy (2008) relying on the inhomogeneous grid of percentile cutoffs used in the original reference to provide a better sample of the tails. The estimation sample is 1970.01–2019.12. Column 1 contains the excess market return Column 2 includes as factors the excess market return and the aggregate distress factor h_t^{ν} as defined in the text, considering the systemicness weighted mean of individual firm probability of distress as computed Campbell Hilsher and Szilagy (2008) Column 3 includes the excess market return, the aggregate distress factor h_t^{ν} and common idiosyncratic volatility (CIV) innovation as factors. The model in Column 4 (5) include the excess market return, the aggregate distress factor h_t^{ν} and common idiosyncratic volatility (CIV) innovation as defined in Herskovic et al. (2016) and the SMB HML (RMW CMA) traded factors as dowloaded from the ken French website. The aggregate distress factor h_t^{ν} and common idiosyncratic volatility (CIV) innovation have been standardized to unit volatility. The FMB including all factors has not been included since SMB and HML can be replicated as a linear combination of the factors included in Column 4. The Table reports the risk premia estimates (λ) associated with the factors and their Newey and West standard errors (with 4 lags) from an average of cross sectional regressions of monthly excess portfolio returns on factor exposures which have been computed considering a widening window with a minimum of 60 months including at least 24 observations.The last row reports the mean R^2. $*, **, ***$ denote significance at $10, 5, 1$ percent, respectively.

Table 4 **Risk exposures and Alphas for stocks sorted on the probability of distress.**

	0005	0510	1020	2040	4060	6080	8090	90 95	9599	9900
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Mean Exc Ret	+0.322	+0.207	+0.136	-0.010	+0.026	+0.082	-0.324	-0.953	-1.550	-1.861
β_{mkt}	-0.004	-0.013	-0.026	+0.014	+0.004	+0.142	+0.246	+0.393	+0.446	+0.401
$se_{\beta_{mkt}}$	(0.026)	(0.024)	(0.015)	(0.012)	(0.016)	(0.027)***	(0.036)***	(0.043)***	(0.051)***	(0.069) ***
β_{h^ν}	-0.057	+0.208	+0.040	+0.005	-0.085	-0.179	-0.235	-0.296	-0.657	-0.739
$se_{\beta_{h^\nu}}$	(0.110)	(0.104)**	(0.065)	(0.053)	(0.073)	(0.122)	(0.160)	(0.175)*	(0.221)***	(0.294)***
α	+0.379	+0.019	+0.115	-0.024	+0.102	+0.166	-0.247	-0.920	-1.223	-1.407
se_{α}	(0.154)***	(0.146)	(0.091)	(0.074)	(0.101)	(0.167)	(0.219)	(0.257)***	(0.309)***	(0.415)***

We sort all stocks based on the predicted 1-month probability of failure and divide them into 10 portfolios based on percentile cutoffs, for example, 0 to 5th percentile (0005) and from the 99th to 100th percentile (9900) of the h_i distribution. For each portfolio we report the value-weighted mean excess return over the market. We regress the time series over the period 1970-2019 of excess returns for each portfolio with respect to the benchmark valuation model that includes as factors: the excess return of the market portfolio, downloaded from the Ken French website, and the aggregate distress factor h^ν. $*, **, ***$ denote significance at $10, 5, 1$ percent, respectively.

Table 5 **5 x 5 Value-Size portfolio Betas.**

β^{h^ν}	S1	S2	S3	S4	S5
V5	0.207	-0.008	-0.124	-0.259	-0.369
se	(0.067)	(0.065)	0.084)	(0.110)	(0.151)
V4	0.163	0.020	-0.090	-0.095	-0.304
se	(0.097)	(0.078)	(0.093)	(0.097)	(0.124)
V3	0.140	-0.063	-0.129	-0.103	-0.139
se	(0.126)	(0.098)	(0.094)	(0.104)	(0.136)
V2	-0.056	-0.171	-0.067	-0.210	-0.420
se	(0.151)	(0.124)	(0.112)	(0.116)	(0.143)
V1	-0.324	-0.263	-0.270	-0.290	-0.432
se	(0.198)	(0.178)	(0.139)	(0.143)	(0.153)

For each one of the 25 value-weighted portfolios double sorted in quintiles with respect to value and size characteristics, downloaded from the Ken French website, we run a regression of the time series of returns over the period 1970-2019 with respect to the benchmark valuation model that includes as factors: the excess return of the market portfolio and the aggregate distress factor h^ν. In the Table we denote by V1-V5 the value quintiles and S1-S5 the size quintiles and report the corresponding β^{h^ν} and the corresponding standard errors.

Table 6 **Allocation of the Factor Mimicking Portfolios: Size-Value anomalies**

Value-Size 5x5	S1	S2	S3	S4	S5	Sum S1-S5	
V1	-0.11	-0.07	-0.08	-0.09	-0.18	-0.53	Short Growth
V2	0.05	-0.02	0.05	-0.04	-0.17	-0.14	
V3	0.17	0.05	0.01	0.02	0.00	0.25	
V4	0.18	0.10	0.03	0.03	-0.10	0.24	
V5	0.21	0.08	0.01	-0.07	-0.14	0.10	Long Value
Sum V1-V5	0.50	0.13	0.02	-0.15	-0.58	-0.08	
	Long Small				Short Big		

The Table shows the mean allocation required to replicate the factor mimicking portfolio for h^ν in each one of the 25 value-weighted portfolios double sorted in quintiles with respect to value and size characteristics. In the Table we denote by V1-V5 the value quintiles and S1-S5 the size quintiles. Note that effective trading of this portfolio would certainly incur in very high transaction costs, in fact it requires heavy shorting of high distress firms that might be relatively illiquid.

Table 7 Allocations in Factor Mimicking Portfolios: Idiosyncratic Volatility and Distress anomalies

	Ivol	Distress
P1	0.10	0.06
P2	0.09	0.22
P3	0.05	0.12
P4	0.05	0.10
P5	0.04	0.04
P6	-0.02	-0.02
P7	-0.04	-0.05
P8	0.03	-0.09
P9	0.03	-0.31
P10	0.04	-0.36
Sum P1-P5	0.35	0.53
Sum P6-P10	0.05	-0.83

The first (second) column in the Table shows the allocation in each one of the 10 value-weighted portfolios obtained sorting securities with respect to idiosyncratic volatility and probability of distress. Distress portfolios are selected following the rule considered in Campbell Hilsher and Szilagy (2008)

Table 8 Vulnerability Tertile portfolios.

Variable	Obs	Mean	Std. Dev.	Min	Max
ν_1^R	427	0.0149	0.0038	0.0067	0.0241
ν_2^R	427	0.0365	0.0026	0.0328	0.0450
ν_3^R	427	0.0728	0.0024	0.0687	0.0769
bm_1	427	0.675	0.231	0.435	3.7531
bm_2	427	1.082	5.016	0.464	104.34
bm_3	427	0.859	0.610	0.498	5.029
sp_1	427	1.823	0.770	0.861	5.256
sp_2	427	1.443	0.608	0.688	4.133
sp_3	427	2.688	1.126	1.409	7.900
Δsp $t > 1984m6$	427	0.865	0.562	0.025	3.421
Δsp $t < 1984m6$	174	0.463	0.310	-0.041	1.107

Summary statistics for the relevant characteristic ratios of the tertile portfolios created sorting securities according to their vulnerability. For each portfolio we report mean vulnerability, mean book-to-market ratio, mean price-to-sale ratios. Relevant financial ratios for single securities have been downloaded from the CRSP database.

Table 9 Deep Value Predictive Regression Results.

Date > June 1984	b^{h^ν}	a^{h^ν}	b^{VmR}	a^{VmR}
Holding Period				
2 years	2.66	4.42	8.12	1.87
	[0.88]	[1.05]	[1.53]	[0.52]
3 years	7.09	5.80	11.27	3.27
	[2.77]***	[1.05]	[1.86]*	[0.68]
4 years	9.73	7.40	15.29	4.18
	[3.44]***	[1.04]	[2.59]***	[0.71]
5 years	10.46	9.89	18.57	5.74
	[2.41]***	[1.04]	[3.09]***	[0.79]

The Table summarizes the results for the period 1985-2020. In fact, prior to the structural break located in June 1984, the predictive relations are not significant. Predictive variables are defined as: (i) the yearly mean level of aggregate distress h_t^ν and (ii) Vulnerability Spread VmR which is determined by the mean price-to-sales ratio for firms in high vulnerability terciles minus the mean price-to-sales ratio for firms in the low vulnerability tercile. Time variation of the two standardized regressors h_t^ν and VmR_t are those plotted in Fig.(9). Returns of the portfolio HML_t are downloaded from the Ken French website and cumulated over the relevant horizon. The specification of the regressions is $HML_{t+1:t+\tau+1} = a_\tau^{h^\nu} + b_\tau^{h^\nu} h_t^\nu + \varepsilon_{t+1:t+\tau+1}^\tau$ and $HML_{t+1,t+\tau+1} = a_\tau^{VmR} + b_\tau^{VmR} VmR_t + \eta_{t+1:t+\tau+1}^\tau$. Standard errors include a Newey-West correction (number of lags is set equal to the holding period) to take into account autocorrelation and heteroskedasticy. t-stats are reported within square brackets [.].

Table 10 **Kelley Skewness and Firm Vulnerability.**

	ν^R	KSK Mean top 5%	KSK top 2%	KSK Sub Critical
V1	0.020	+1.18%	+0.31%	-2.6%
se		(1.55%)		
V2	0.056	+0.65%	-1.24%	+4.7%
se		(2.57%)		
V3	0.069	+0.07%	-4.65%	+3.1%
se		(4.61%)		
V4	0.080	-6.83%	-4.21%	-3.3%
se		(2.53%)***		
V5	0.099	-15.01%	-19.80%	-0.48%
se		(3.27%)***		

Kelley Skewness provides a simple decomposition of the share of total dispersion that is accounted for by the left and the right tails of a distribution. It is given by: $KSK := \frac{P9050}{P9010} - \frac{5010}{P9010}$ where PQ2Q1 denote the dispersion between Quantile Q1 and Q2. A negative value of Kelley Skewness indicates that the left tail accounts for more than one-half of the total dispersion and the distribution is negatively skewed. We restrict our analysis to those periods of time characterized by high aggregate distress, in particular those that correspond to the top 5 percentiles for the distribution of h^ν reported in Fig 5. We split the full sample of firms who have quarterly data on sales in Compustat over five quintile groups of firms sorted with respect to their vulnerability ν^R and report the cross sectional skewness KSK for yearly log sales growth within each quintile group. The first column reports the mean vulnerability for each group, the second column reports the mean (standard deviation) KSK computed restricting the sample to percentiles in the range $95 - 100$. For illustration the fourth column report the point estimate for the top 2% period. The last column reports the KSK prior to the critical transition, when the economy is estimated to be in a subcritical equilibrium. $*, **, ***$ denote respectively significance at $10, 5, 1$ percent.

Table 11 **Summary Statistics Logit Regressors.**

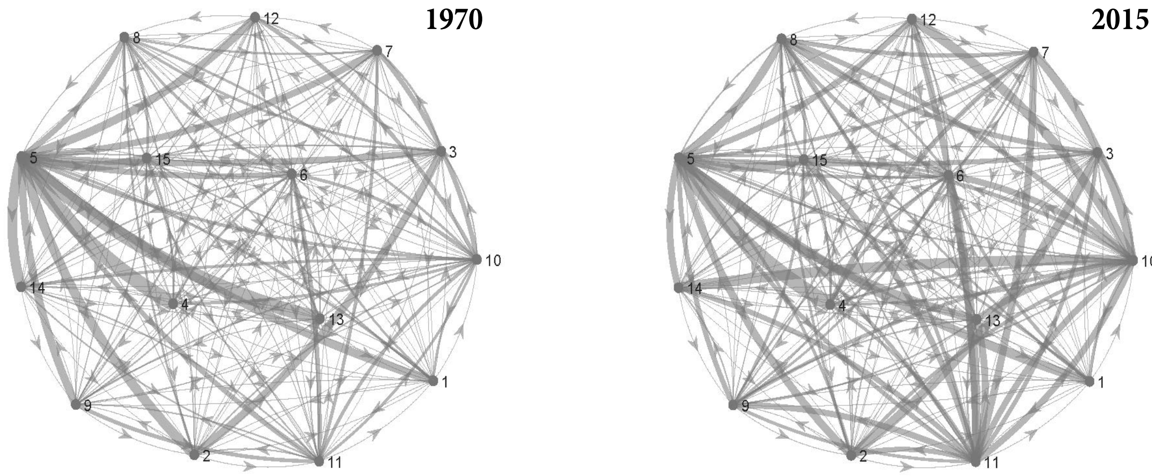
stats	NIMTA	TLMTA	ExRet	Rsize	SIGMA	CASHMTA	MB	PRICE
mean	.0004602	.4436827	-.0067406	-10.73442	.492954	.0924108	1.968377	2.213023
p50	.0023984	.4139127	-.0050476	-10.80748	.4074728	.0501727	1.531161	2.70805
sd	.0153837	.2855644	.0918598	1.90437	.3107551	.1053969	1.426622	.8844814
min	-.0428708	.0365388	-.2061729	-14.02561	.1291073	.0024027	.3722629	-4.55638
max	.0233653	.9268802	.1811677	-7.161614	1.273224	.3891692	5.780262	2.70805
N	3301426	2306478	3301426	2834563	3301063	2291299	2393267	3301426
Logit Coeff	$c_1 = -29.67$	$c_2 = 3.36$	$c_3 = -7.35$	$c_4 = 1.48$	$c_5 = 0.082$	$c_6 = -2.40$	$c_7 = 0.054$	$c_8 = -0.937$

Source: Distress Regressors from sample Jan 1970-Dec 2019 and Regression Coefficients from Campbell et al. (2008)

This table reports the descriptive statistics of the variables which have been used in the estimation of the firm specific distress probability for the sample under investigation that is extracted from CRSP-Compustat merged dataset for the period under investigation that is Jan 1970-Dec 2019. We estimate the firm distress inserting these characteristics inserting these variables in the logit estimated by Campbell Hilsher and Szilagy (2008) to forecast the short term (next month) distress probabilities. Logit parameters are reported in the last line and correspond to those originally estimated, so that the forecast is out of sample for the period following 2008. The logit expression is given by: $d_{CHS} = (c_0 + c_1 * NIMTAAVG + c_2 * TLMTA + c_3 * EXRETAVG + c_4 * SIGMA + c_5 * Rsize + c_6 * CASHMTA + c_7 * MB + c_8 * PRICE)$ and $h_i = 1/(1 + exp(-d_{CHS}))$. Following the original reference, we set $c_0 = -9.08$ while the value of the regression coefficients $c_1, .., c_8$ is reported in the table below the summary statistics of the corresponding regressor. Logit regressions of the failure indicators on predictor variables have been carried out over the sample 1963-2003. The data are constructed such that all of the predictor variables are observable at the beginning of the month over which bankruptcy or failure is measured.

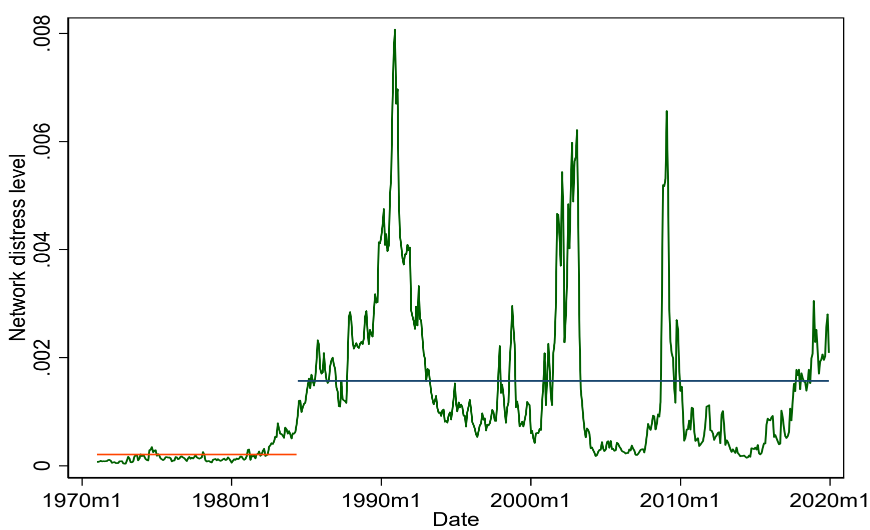
7. Figures

Figure 2 Graphical illustration of the weighted directed network.



Note. Graphical illustration of the weighted directed network for the first year 1970 IO-Table and the year 2015 IO-Table considered in the sample. Node numbering is consistent with that reported in Table 1. Size of each edge (i, j) is proportional to the entry $\Delta_{i,j}$ of the network matrix.

Figure 3 Time series evolution of the aggregate distress indicator h_t^{ν}.



Note. Time series evolution of the aggregate distress indicator h_t^{ν} computed as described in eq.(16) and its mean pre and post structural break.

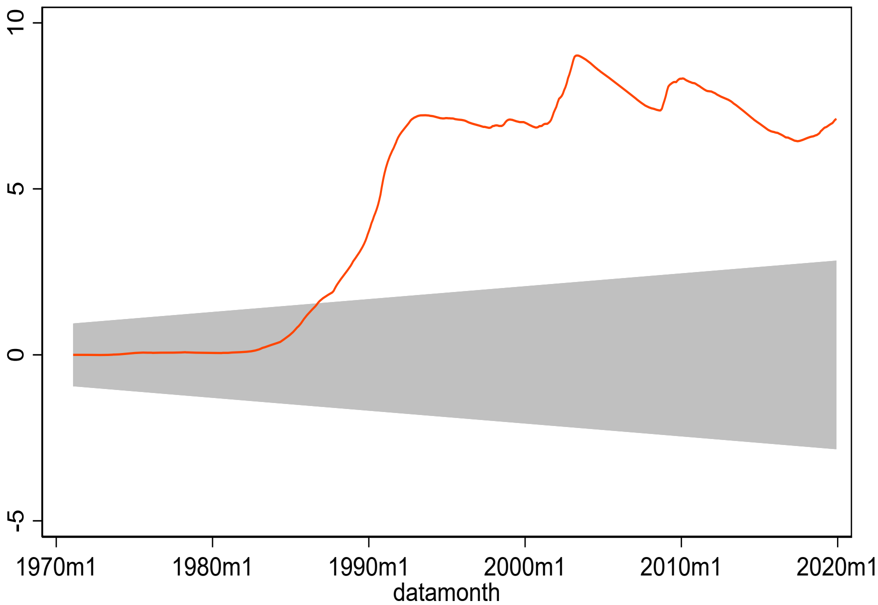
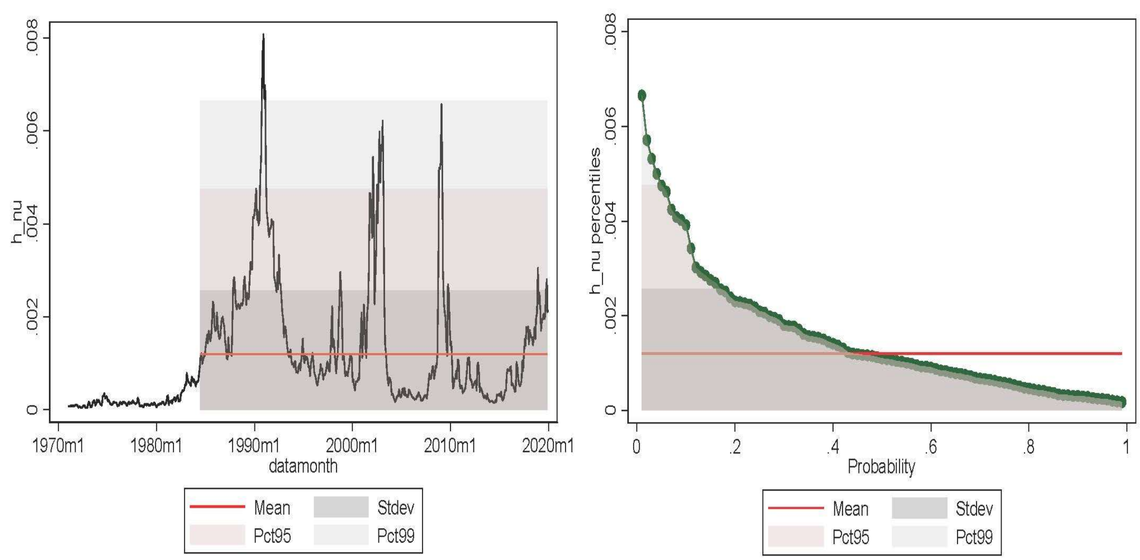


Figure 4

Parameter stability for the mean of the process h_t^ν.

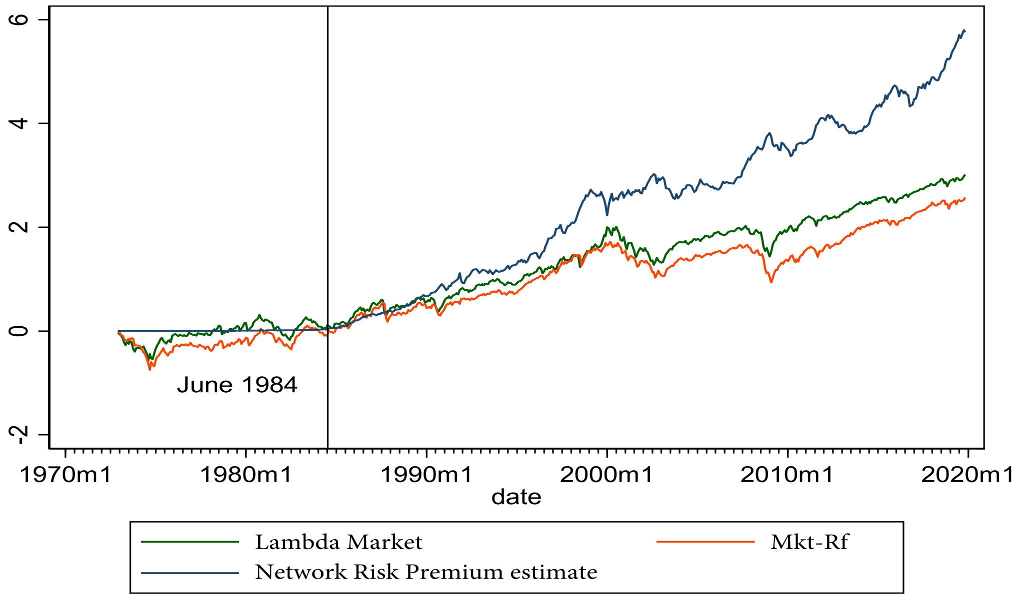
Note. Results from the recursive cumulative sum of squared residuals test with confidence bars at 95%. We run the test on the sample: $1971m1 - 2019m12$. We test the hypothesis Ho: No structural break. The null is rejected with a test statistic (3.8885) above the 1% critical threshold (1.1430). A test based on sup-Wald test statistic localizes a structural break in June 1984.

Figure 5 Distributional properties of the aggregate distress indicator.



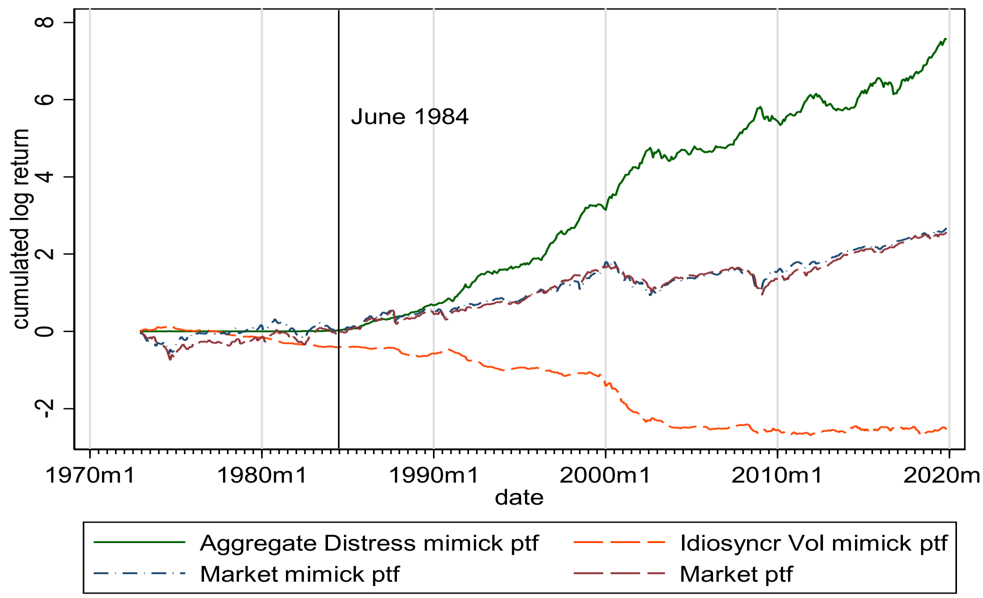
Note. Distributional properties of the aggregate distress indicator h_t^ν post structural break. Shaded areas denote respectively the level of the right standard deviation, of the 95th and of the 99th percentiles of the empirical distribution. Peaks correspond to cascades, i.e. clusters of high distress with low probability of occurrence. Panel A Time Series representation. Panel B: Cumulated empirical distribution.

Figure 6 Time series evolution of the cumulated log returns



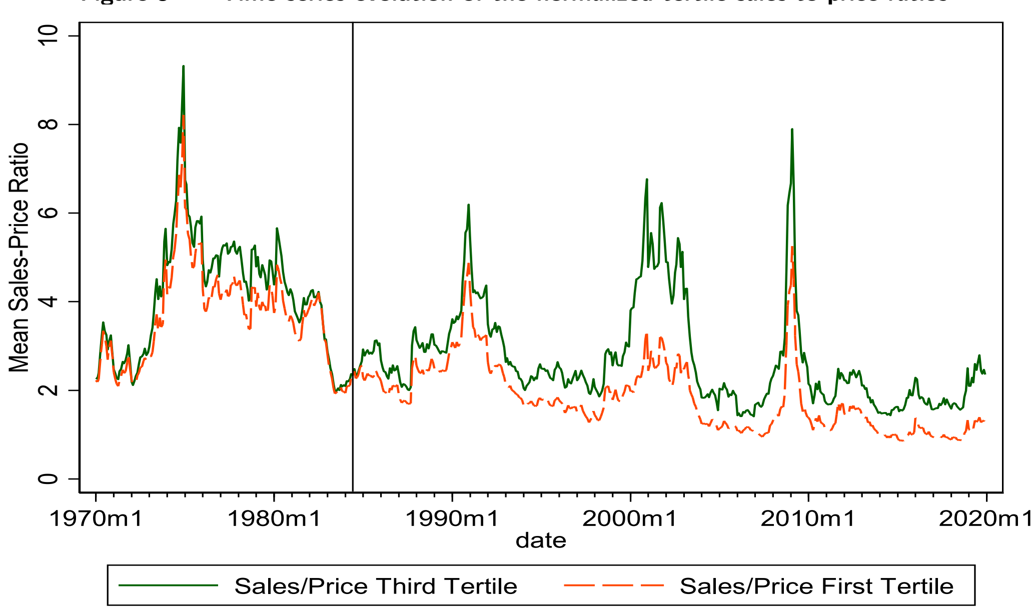
Note. Factor mimicking portfolios for the benchmark two factor model that includes the (traded) excess market return as dowloaded from the Ken French website and the aggregate distress determined from the first pass slopes of the Fama-MacBeth (1973) cross-sectional regression. Mean portfolio allocation corresponding to the factor mimicking portfolio for h^{ν} are reported in Tables 6 and 7. Since market factor is traded, we report for reference also the cumulated return of the original factor. The Sharpe Ratios of the corresponding portfolios are (in brackets those computed over the period post-1984): $SR_{MktRf} = 0.431(0.779)$, $SR_{FMP_{h^{\nu}}} = 0.853(0.983)$

Figure 7 **Time series evolution of the cumulated log returns**



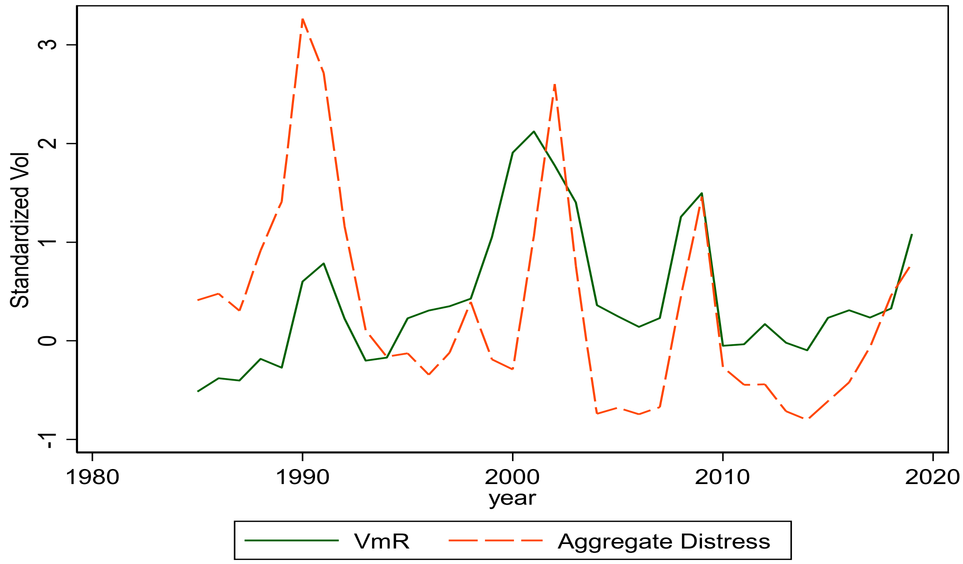
Note. Factor mimicking portfolios for the three factors model including market, aggregate distress and Common Idiosyncratic Variance as determined from the first pass slopes of the Fama-MacBeth (1973) cross-sectional regression. Since the market portfolio is traded, we report for reference also the cumulated return of the original factor. The Sharpe Ratios of the corresponding portfolios are: $SR_{MktRf} = 0.431(0.779)$, $SR_{Distress} = 0.853(0.983)$, $SR_{CIV} = -0.567(-0.556)$

Figure 8 **Time series evolution of the normalized tertile sales-to-price ratios**



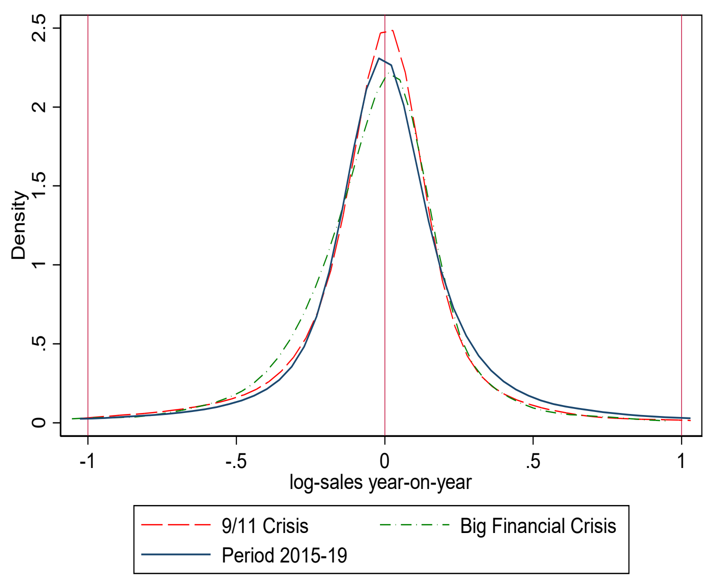
Note. In the figure we plot the monthly variation of the mean sales-to-price ratios for the third and the first tertile groups of securities sorted according to their level of vulnerability. The vertical line represents the structural break. Monthly data refer to the period 1970-2019, the individual sales-to-price ratios have been downloaded from CRSP. Vulnerability is determined from the BEA Make/USE tables provided from the BEA as described in the text.

Figure 9 Time series evolution of the predictor variables.



Note. Standardized aggregate distress indicator h_t^{ν} aggregated on a yearly basis and standardized vulnerability based spread VmR aggregated on a yearly basis.

Figure 10 Endogenous Cross-Sectional Skewness.



Note. Empirical Density of year-on-year log sales distribution. Raw data have been regularized relying on a Gaussian Kernel with Bandwidth 0.08 and 100 points.

Additional results and Proofs

EC.1. Proofs

We assume that the contagion process $\mathbf{H}_t^S$ follows a continuous time homogeneous Markov process with a finite number of states. Each configuration vector $\mathbf{H} \in \mathcal{C}$ is a N dimensional vector of dummy variables, hence the set of configurations $\mathcal{C} = \{0,1\}^N$ has 2^N elements. Relying on a well known result, see e.g. Duffie (2010), we define the continuous time contagion dynamics in a filtered probability space $\left(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, (\mathbb{P}_S^{\mathcal{G}})_{\mathbf{H}\in\mathcal{C}}, (\mathbf{H}_t^S)_{t\geq 0}\right)$ in terms of $2N$ Poisson Processes that count for each firm i the number of distress and recovery transitions occurred up to time t. The dynamics of each component H_t^i of the process $\mathbf{H}_t^S = \left(H_t^1, ..., H_t^N\right)$ are defined by:

$$dH_t^i \;:\; = (1 - H_{t-}^i) dN_t^{+,i}\left(\mathbf{H}_{t-}^S\right) - H_{t-}^i dN_t^{-,i}\left(\mathbf{H}_{t-}^S\right) \qquad i = 1,..,N \tag{EC.1}$$

$$H_0^i = \begin{cases} 1 \text{ if i} \in S, \\ 0 \text{ if i} \notin S \end{cases} \tag{EC.2}$$

and correspondingly, jump intensities of the Poisson processes $N_t^{+,i}\left(\mathbf{H}_t^S\right)$, $N_t^{-,i}\left(\mathbf{H}_t^S\right)$ are set equal to $(1-H_{t-}^i)\lambda^i(\mathbf{H}_t^S)$ and $H_{t-}^i \eta^i(\mathbf{H}_t^S)$. The corresponding continuous time dynamics is Markov in the state vector $\mathbf{H}_t^S$. In light of the above Poisson representation, only transition rates between configurations that differ at most for the state of a single firm are allowed. Correspondingly, the transition rate matrix or infinitesimal generator of the Markov process can be defined as follows:

Definition EC.1. Let $\mathbf{A}$ be the transition rate matrix for the contagion process $\mathbf{H}_t^S$. It has size $2^N \times 2^N$ and its elements $\mathbf{A}_{\mathbf{H},\mathbf{H}'}$ are:

- The unique out-of-diagonal non-zero elements are:

$$\mathbf{A}_{\mathbf{H},\mathbf{H}^{\{i\}}} := \left(1 - H^i\right)\lambda^i(\mathbf{H}) + H^i \eta^i(\mathbf{H}), \quad , i = 1,..,N. \tag{EC.3}$$

where $\mathbf{H}^{\{i\}}$ denotes the configuration that differs from configuration $\mathbf{H}$ only for the distress state of firm i, i.e. $\left(\mathbf{H}^{\{i\}}\right)_j = (\mathbf{H})_j$ for $j \neq i$ while $\left(\mathbf{H}^{\{i\}}\right)_i = 1 - \mathbf{H}^i$.

- The diagonal elements are uniquely determined by the condition that row elements sum to zero, hence:

$$\mathbf{A}_{\mathbf{H},\mathbf{H}} := -\sum_{i=1}^{N} \mathbf{A}_{\mathbf{H},\mathbf{H}^{\{i\}}}$$

Singular Value Decomposition: general formulation. We report here the basic formulation on the K factor approximation of a matrix Δ relying on the Singular Value Decomposition adapted to the present notation. For an introduction to SVD and a proof of the following proposition, see Golub and Van Loan (2012):

Proposition EC.1. *Consider a* generic directed network $\mathcal{G}$*, then the best rank-K approximation of the network matrix* Δ_{ij} $1 \leq i,j \leq N$*,* $N \geq K$*, with respect to the 2-norm,* $\|\Delta\|_2 = \max_{|v|=1} |\Delta v|$*, is given by:*

$$\Delta_{ij}^{(K)} = \alpha_0^{(N)} \nu_i^R \nu_j^L + \sum_{k=1}^{K-1} \alpha_k^{(N)} v_i^{(k)} u_j^{(k)}$$

where $\alpha_0^{(N)}$ *is the principal singular value of* Δ_{ij}*,* ν_i^R *is firm* i *vulnerability and* ν_j^L *is firm* j *systemicness,* $\alpha_k^{(N)}$ *is the* $k-th$ *order singular value while* $v_i^{(k)}$ $(u_j^{(k)})$ *is the* $k-th$ *right (left) singular component.*

The following Lemma is necessary to prove Theorem 1. Its proof follows directly from properties i) and ii) stated in Assumption 2:

LEMMA EC.1. *Assume Assumption 2 Then:*

$$\lim_{N\to+\infty}\alpha^{\mathcal{G}}\sum_{i=1}^{N}x_iy_i = L\mathbf{x}\cdot\mathbf{y} = L\sum_{k=1}^{K}p_kx_ky_k$$

and the limiting expression is finite if and only if x_k, y_k are finite for $k=1,..,K$.

Proof of Theorem 1. Consider first the process for a finite number of firms N in the economy.

• The dynamics of H_t^i has to be computed applying Ito Lemma for pure jump processes from the dynamics of the vector state variable $\mathbf{H}$ considering the projection function that maps the configuration $\mathbf{H}$ on the value of the single firm state variable:

$$H^i(\mathbf{H}):\mathbf{H}=\left(H^1,...,H^N\right)\to H^i$$

Then:

$$\begin{aligned}dH_t^i(\mathbf{H}_t) &= \sum_{j=1}^{N}\left[(+\delta_{ij})\left(1-H_t^i\right)\left(\nu_i^R\alpha\lambda H_t^{\nu}\right)+(-\delta_{ij})H_t^i\eta\right]dt\\ &\sum_{j=1}^{N}\left[(+\delta_{ij})\left(1-H_t^i\right)\left(dN_t^{+,i}(\mathbf{H}_t)-\left(\nu_i^R\alpha\lambda H_t^{\nu}\right)dt\right)+(-\delta_{ij})H_t^i\left(dN_t^{-,i}(\mathbf{H}_t)-\eta dt\right)\right]\end{aligned}$$

Hence, defining the martingale (compensated) process as:

$$dM_t^i := \left[(+1)\left(1-H_t^i\right)\left(dN_t^{+,i}(\mathbf{H}_t)-\left(\nu_i^R\alpha\lambda H_t^{\nu}\right)dt\right)+(-1)H_t^i\left(dN_t^{-,i}(\mathbf{H}_t)-\eta dt\right)\right] \tag{EC.4}$$

we can write:

$$dH_t^i = \left[(+1)\left(1-H_t^i\right)\left(\frac{\alpha^{\mathcal{G}}\lambda}{\eta}\nu_i^R\left(\sum_{i=1}^{+N}\nu_i^LH_t^i\right)\right)+(-1)H_t^i\right]\eta dt + dM_t^i$$

• Correspondingly, repeating the computation on all the firms:

$$\begin{aligned}\frac{\sum_{i=1}^{+N}\nu_i^LdH_t^i}{\sum_{i=1}^{+N}\nu_i^L} &= \frac{\sum_{i=1}^{+N}\nu_i^L\left[(+1)\left(1-H_t^i\right)\left(\nu_i^R\frac{\alpha\lambda}{\eta}H_t^{\nu}\right)+(-1)H_t^i\right]\eta dt+dM_t^i}{\sum_{i=1}^{+\infty}\nu_i^L}\\ dH_t^{\nu} &= H_t^{\nu}\left[\sum_{i=1}^{+N}\nu_i^L\nu_i^R\left(1-\widetilde{H}_t^{\nu}\right)\left(\left(\sum_{i=1}^{+N}\nu_i^L\right)\frac{\alpha^{\mathcal{G}}\lambda}{\eta}\right)+(-1)\right]\eta dt+\frac{\sum_{i=1}^{+N}\nu_i^LdM_t^i}{\sum_{i=1}^{+N}\nu_i^L}\\ \widetilde{H}_t^{\nu} &: = \frac{\sum_{i=1}^{+N}\nu_i^L\nu_i^RH_t^i}{\sum_{i=1}^{+N}\nu_i^L\nu_i^R},\end{aligned}$$

• Now we compute the limiting expression for $N\to+\infty$ of the previous expressions relying on Lemma EC.1 and on the strong law of large numbers for Poisson processes. Then:

$$\lim_{\substack{N\to+\infty\\ \frac{N_k}{N}\to p_k}}\frac{\sum_{k=1}^{+K}\frac{N_k}{N}\nu_k^L\left(\frac{1}{N_k}\sum_{i\in C_k}dM_t^i\right)}{\nu^L\cdot\mathbf{1}}=0$$

and, for $N\to+\infty$ the evolution of

$$h_t^{\nu} := \lim_{\substack{N\to+\infty\\ \frac{N_k}{N}\to p_k}}\left(\sum_{k=1}^{+K}\frac{N_k}{N}\frac{\nu_k^L}{\nu^L\cdot\mathbf{1}}\left(\frac{\sum_{i\in C_k}H_t^i}{N_k}\right)\right)$$

is given by the deterministic equation:

$$\frac{dh_t^\nu}{dt} = -\eta h_t^\nu \left[1-\left(1-\widetilde{h}_t^\nu\right) L\left(\nu^L\cdot\nu^R\right)\frac{\lambda}{\eta}\right] \tag{EC.5}$$
$$\widetilde{h}_t^\nu \;:\; = \sum_{k=1}^{K} p_k \frac{\nu_k^L \nu_k^R}{\nu^L\cdot\nu^R}\left(h_t^k\right).$$

where the dynamics for h_t^k, is deterministic and given by:

$$\begin{aligned} dh_t^k &= \lim_{N_k\to+\infty}\frac{1}{N_k}\sum_{i\in C_k} dH_t^i \\ &= \left[(+1)\left(1-h_t^k\right)\left(\frac{L\lambda}{\eta}\nu_k^R\left(\nu^L\cdot 1\right)\left(h_t^\nu\right)\right)+(-1)\,h_t^k\right]\eta dt \end{aligned}$$

- As expected $h_\infty^\nu = 0$, $h_\infty^k = 0$, $k=1,...,K$ is a solution for eq.(EC.5). A second positive solution $h_\infty^\nu > 0$ may arise only if $\left(1-\widetilde{h}_t^\nu\right)\frac{L\left(\nu^L\cdot\nu^R\right)\lambda}{\eta}-1=0$. By construction $\widetilde{h}_t^\nu \leq 1$, hence existence of the solution requires

$$\begin{cases} \dfrac{1}{\frac{L\left(\nu^L\cdot\nu^R\right)\lambda}{\eta}} = 1-\widetilde{h}_t^\nu \\ 0\leq \widetilde{h}_t^\nu < 1 \end{cases}$$

and a fortiori it is required that

$$\frac{L\left(\nu^L\cdot\nu^R\right)\lambda}{\eta} > 1$$

Hence, defining the tipping point value:

$$K(\Delta) := \frac{1}{L\left(\nu^L\cdot\nu^R\right)}$$

we can conclude that in the region $\frac{\lambda}{\eta} < K(\Delta)$ there exists a unique nonnegative solution which is given by $h_\infty^\nu = h_\infty^k = 0$.

- Now we prove that, for $\frac{\lambda}{\eta} > K(\Delta)$ there exists a strictly positive solution. Relying on the steady state condition for the dynamic evolution of the distress rates h_t^k and on the definition of h_∞^ν, it is immediate to derive by self-consistency the following fixed point equation:

$$h_\infty^\nu = \frac{\sum_{k=1}^K p_k \nu_k^L h_\infty^k}{\nu^L\cdot\mathbf{1}} = \sum_{k=1}^{K} p_k \frac{\nu_k^L}{\nu^L\cdot\mathbf{1}}\frac{\frac{L\lambda}{\eta}\nu_k^R h_\infty^\nu\left(\nu^L\cdot\mathbf{1}\right)}{\frac{L\lambda}{\eta}\nu_k^R h_\infty^\nu\left(\nu^L\cdot\mathbf{1}\right)+1}.$$

- Now we prove that in the supercritical region, i.e. for $\frac{\lambda}{\eta} > K(\Delta)$ it admits a strictly positive solution, i.e. there exists a value $h_\infty^\nu > 0$ such that:

$$1 = \sum_{k=1}^{K} p_k \frac{\frac{L\lambda}{\eta}\nu_k^L\nu_k^R}{\frac{L\lambda}{\eta}\left(\nu^L\cdot\mathbf{1}\right)\nu_k^R h_\infty^\nu + 1}.$$

Consider the function F defined as follows:

$$F\left(h_\infty^\nu\right) := 1-\sum_{k=1}^{K} p_k \frac{\nu_k^L\nu_k^R L\frac{\lambda}{\eta}}{L\left(\nu^L\cdot\mathbf{1}\right)\frac{\lambda}{\eta}\nu_k^R h_\infty^\nu+1}.$$

It is easy to verify that $F\left(h_\infty^\nu\right)$ is continuous and monotonic increasing in $[0,1]$. Now we prove also that for $\frac{L\lambda}{\eta} > \frac{1}{\nu^L\cdot\nu^R}$, $F(0) < 0$ while $F(1) > 0$:

$$\begin{aligned} h_\infty^\nu &= 0,\ L\frac{\lambda}{\eta} > \frac{1}{\nu^L\cdot\nu^R} \\ &\Rightarrow F(0) = 1 - L\frac{\lambda}{\eta}\nu^L\cdot\nu^R < 0 \end{aligned}$$

$$
\begin{aligned}
h^{\nu}_{\infty} &= 1, L\frac{\lambda}{\eta} > \frac{1}{\nu^{L}\cdot\nu^{R}} \\
&\Rightarrow F(1) = 1 - \sum_{k=1}^{K} p_k \frac{\nu^{L}_{k}\nu^{R}_{k} L\frac{\lambda}{\eta}}{L\frac{\lambda}{\eta}\nu^{R}_{k}\left(\nu^{L}\cdot\mathbf{1}\right)+1} \\
&= 1 - \sum_{k=1}^{K} p_k \frac{1}{\left(\left(\nu^{L}\cdot 1\right) + \left(L\frac{\lambda}{\eta}\nu^{R}_{k}\right)^{-1}\right)} \overset{\frac{1}{\left(\left(\nu^{L}\cdot 1\right)+\left(L\frac{\lambda}{\eta}\nu^{R}_{k}\right)^{-1}\right)} < \frac{1}{\left(\nu^{L}\cdot 1\right)}}{>} \\
&= 1 - \frac{\left(\nu^{L}\cdot 1\right)}{\left(\nu^{L}\cdot 1\right)} = 0
\end{aligned}
$$

Hence by continuity there must exist a non trivial value $0 < h^{\nu *}_{\infty} < 1$ such that $F\left(h^{\nu *}_{\infty}\right) = 0$. (Observe that the explicit expression for the steady state value $\widetilde{h^{\nu}_{\infty}}^{*}$ is determined inserting in eq.(EC.5) the values $h^{k}_{\infty}\left(h^{\nu *}_{\infty}\right)$, $k = 1, \ldots K$.

- Now we prove that the positive solution $h^{\nu *}_{\infty} > 0$ is stable in the region $\frac{\lambda}{\eta} > K(\Delta)$. Observe that an increase of h^{ν}_{t} increases also $\widetilde{h^{\nu}_{t}}$, in fact:

$$
\begin{aligned}
\frac{\partial \widetilde{h}^{\nu}_{t}}{\partial h^{k}_{t}} &= \frac{\nu^{L}_{k}\nu^{R}_{k}}{\nu^{L}\cdot\nu^{R}} > 0 \\
\frac{\partial h^{\nu}_{t}}{\partial h^{k}_{t}} &= \frac{\nu^{L}_{k}}{\nu^{L}\cdot 1} > 0
\end{aligned}
$$

hence in a neighborhood of the fixed point:

$$
\left[\frac{\partial \widetilde{h}^{\nu}_{t}}{\partial h^{k}_{t}}\right]_{\widetilde{h^{\nu}_{\infty}}^{*}} > 0 \Leftrightarrow \left[\frac{\partial h^{\nu}_{t}}{\partial h^{k}_{t}}\right]_{h^{\nu *}_{\infty}} > 0
$$

On the other hand, observe that for positive (negative) deviations $\widetilde{h^{\nu}_{\infty}}^{*} \pm \Delta$:

$$
\left[\frac{dh^{\nu}_{t}}{dt}\right]_{h^{\nu *}_{\infty}} = -\eta h^{\nu}_{t}\left[1 - \left(1 - \widetilde{h}^{\nu}_{t}\right) L\left(\nu^{L}\cdot\nu^{R}\right)\frac{\lambda}{\eta}\right]_{\widetilde{h^{\nu}_{\infty}}^{*}\pm\Delta} = \mp\eta h^{\nu}_{\infty}\Delta \lesseqgtr 0
$$

the variation is negative (positive), hence the equilibrium point corresponding to $h^{\nu *}_{\infty}$ is stable.

Proof of Corollary 1. In the large economy limit it is possible to compute the ratio between the expected consumption when dynamics is above the tipping point and the one, corresponding to the standard Lucas tree economy that is realized below the tipping point:

$$
\begin{aligned}
\frac{\sum_{i=1}^{N} x\left(H^{i}_{t}\right) Y_t}{\sum_{i=1}^{N} x\left(0\right) Y_t} &: = \frac{1}{N}\sum_{i=1}^{N}\left(\left(1 - H^{i}_{t}\right) + \frac{x\left(1\right)}{x\left(0\right)} H^{i}_{t}\right) \\
&= \sum_{k=1}^{K}\frac{1}{N}\sum_{i\in C_k}\left(1 - \left(1 - \frac{x\left(1\right)}{x\left(0\right)}\right) H^{i}_{t}\right) \\
&\overset{N\to+\infty}{\to} \sum_{k=1}^{K} p_k\left(1 - \left(1 - \frac{x\left(1\right)}{x\left(0\right)}\right) h^{k}_{t}\right)
\end{aligned}
$$

For $\frac{\lambda}{\eta} > K_c$ and small deviations from K_c the tipping point then $h^{\nu}_{\infty} << 1$ and it is possible to assume $h^{\nu}_{t} << 1$. Then to leading order in h^{ν}_{t} :

$$
\frac{\sum_{i=1}^{N} x\left(H^{i}_{t}\right) Y_t}{\sum_{i=1}^{N} x\left(0\right) Y_t} \simeq \sum_{k=1}^{K} p_k \exp\left(-\left(1 - \frac{x\left(1\right)}{x\left(0\right)}\right)\nu^{R}_{k}\alpha\frac{\lambda}{\eta}h^{\nu}_{t}\right)
$$

that will be strictly than 1 for any super-equilibrium state. Consider now the additional contribution induced by the network on the cash flow drift for a firm i:

$$
\frac{d\left[x_i\left(H_t\right)\right] Y_t}{x_i\left(0\right) Y_t} = \frac{dx_i\left(H_t\right)}{x_i\left(0\right)}
$$

Our analysis is focused on the residual component of the cash-flow conditional mean variation of the $n-th$ moment which is given by:

- Dynamics of idiosyncratic cash flow

$$
\begin{aligned}
dx_i(H_t) &= -\left(x_i(0) - x_i(1)\right)\left(1 - H_t^i\right) dN_t^{i,\uparrow} + \left(x_i(0) - x_i(1)\right) H_t^i dN_t^{i,\downarrow} \\
&= \left(x_i(0) - x_i(1)\right)\left(\left[(-)^1\left(1 - H_t^i\right)\nu_i^R \alpha\lambda H_t^\nu + \eta H_t^i\right] dt + dM_t^{(-),i}\right)
\end{aligned}
$$

- Dynamics of cash flow conditional $n-th$ order variation.

$$
\begin{aligned}
&\underbrace{\langle dx_i(H_t), \ldots . \langle dx_i(H_t), dx_i(H_t)\rangle\rangle}_{n\text{ - times}} \\
&= \left(x_i(0) - x_i(1)\right)^n \left(\left[(-1)^n\left(1 - H_t^i\right)\nu_i^R \alpha\lambda H_t^\nu + \eta H_t^i\right] dt + dM_t^{(\pm),i}\right)
\end{aligned}
$$

If I now sum over i assuming uniform levels $x(0)$, $x(1)$ in the large economy limit

$$
\begin{aligned}
&\lim_{N\to+\infty} \tfrac{1}{N}\textstyle\sum_{k=1}^K \sum_{i\in C_k} \underbrace{\frac{\langle dx_i(H_t), \ldots . \langle dx_i(H_t), dx_i(H_t)\rangle\rangle}{x(0)^n}}_{n\_th\_cumulant} \\
&= \sum_{k=1}^K p_k\left(1 - \frac{x(1)}{x(0)}\right)^n\left(\left[(-1)^n\left(1 - \nu_k^R\alpha\lambda h_t^\nu\right)\nu_k^R\alpha\lambda h_t^\nu + \eta\nu_k^R\alpha\lambda h_t^\nu\right] dt\right)
\end{aligned}
\tag{EC.6}
$$

Valuation in a finite network economy Buraschi and Porchia (2012) The computation of the equilibrium risk free rate and risk premia at finite N that we report below for completeness is the same of Buraschi and Porchia (2012) . It is derived computing the explicit dynamics of the equilibrium state price density. Since the dividend processes are $D_t^i = Y_t x_t^i$, $i = 1, \ldots, N$, aggregate consumption reads $C_t = Y_t X_t^{(N)} = Y_t \sum_{i=1}^N x_t^i$ [27]. Then, the consumption optimality condition of the representative agent implies the following equilibrium state price density, ξ_t:

$$
\xi_t \propto e^{-\delta t} Y_t^{-\gamma}\left(X_t^{(N)}\right)^{-\gamma}. \tag{EC.7}
$$

Ito computation of drift, diffusion and jump components for (EC.7) uniquely identify the expression of the state price density. The equilibrium interest rate ($r_t(\mathbf{H})$), market prices of diffusive risk (κ), and market price of distress/recovery risk ($\theta_t^i(\mathbf{H}) - 1$) are given by:

$$
\begin{aligned}
r_t(\mathbf{H}) &= \delta + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2 - \sum_{i=1}^N \left\{ H_t^i \left[1 - \left(\frac{x^i(0) + \sum x_{t-}}{x^i(1) + \sum x_{t-}}\right)^{-\gamma}\right]\eta_t^i + \right. \\
&\qquad \left. + (1 - H_t^i)\left[1 - \left(\frac{x^i(1) + \sum x_{t-}}{x^i(0) + \sum x_{t-}}\right)^{-\gamma}\right]\lambda_t^i \right\} \qquad \text{(EC.8)} \\
\kappa &= \gamma\sigma \\
\theta_t^i(\mathbf{H}) &= H_t^i\left(\frac{x^i(0) + \sum x_{t-}}{x^i(1) + \sum x_{t-}}\right)^{-\gamma} + (1 - H_t^i)\left(\frac{x^i(1) + \sum x_{t-}}{x^i(0) + \sum x_{t-}}\right)^{-\gamma} \quad i = 1, 2, \ldots N, \qquad \text{(EC.9)}
\end{aligned}
$$

where $\sum x_{t-}$ denotes the sum of trees' persistent dividend components excluding i at time t^-. $\theta_t^i(\mathbf{H}) - 1$ is the market price of firm i's risk of dividend jumps, either distress or recoveries. $\lambda^i\theta_t^i$ and $\eta^i\theta_t^i$ are the risk-neutral transition

[27] When the limiting expression as $N \to +\infty$ is required, total consumption C_t is rescaled to total consumption per firm $C_t N^{-1}$. Assumption of homothetic preferences makes the rescaling irrelevant for the pricing implications.

intensities of distress and recovery. The computation of the equilibrium conditional equity risk premium for a claim on dividends of firm i:

$$rp_t^i(\mathbf{H}) = \mathbb{E}\left[\left.\frac{dP^i(\mathbf{H})}{P^i(\mathbf{H})}\right|\mathcal{F}_t\right] + \frac{D_t^i}{P^i(\mathbf{H})} - r_t,$$

goes as follows: we apply Ito's lemma to the martingale $\mathcal{M}_t^i = \xi_t P^i(\mathbf{H}) + \int_0^t \xi_s D_s^i ds$, taking into account expression for the state-price density. We use the following notation: $\mathbf{H}^{-j}$ ($\mathbf{H}^{+j}$) coincides with the current state $\mathbf{H}$, except for firm j (not) in distress. $\theta_t^j(\mathbf{H})$ is the market price of distress/recovery risk, reported in (EC.9) while

$$R^i(\mathbf{H}^{-j}) = \frac{P^i(\mathbf{H}^{-j})}{P^i(\mathbf{H})} < 1, \quad R^i(\mathbf{H}^{+j}) = \frac{P^i(\mathbf{H}^{+j})}{P^i(\mathbf{H})} > 1$$

Then, one has:

$$\begin{aligned}
d\mathcal{M}_t^i &= \xi_t D_t^i dt + \xi_t P^i(\mathbf{H}) m_t^i dt - \xi_t P^i(\mathbf{H}) r_t dt - \xi_t P^i(\mathbf{H})\kappa\sigma dt - \xi_t P^i(\mathbf{H})(\kappa-\sigma)dZ_t + \\
&+ \xi_t P^i(\mathbf{H}) \sum_{j=1}^N \left[R^i(\mathbf{H}^{\pm j}) - 1\right] dH_t^j + \xi_t P^i(\mathbf{H}) \sum_{j=1}^N (1-2H_t^j)\left(\theta_t^j - 1\right) dM_t^j + \\
&+ \xi_t P^i(\mathbf{H}) \sum_{j=1}^N \left[R^i(\mathbf{H}^{\pm j}) - 1\right] (1-2H_t^j)\left(\theta_t^j - 1\right)\left\langle dH_t^j, dM_t^j\right\rangle \\
&= \xi_t D_t^i dt + \xi_t P^i(\mathbf{H}) m_t^i dt - \xi_t P^i(\mathbf{H}) r_t dt - \xi_t P^i(\mathbf{H})\kappa\sigma dt - \xi_t P^i(\mathbf{H})(\kappa-\sigma)dZ_t + \\
&+ \xi_t P^i(\mathbf{H}) \sum_{j=1}^N \left[R^i(\mathbf{H}^{\pm j}) - 1\right] dH_t^j + \xi_t P^i(\mathbf{H}) \sum_{j=1}^N \left(\theta_t^j - 1\right)(1-2H_t^j) dM_t^j + \\
&+ \xi_t P^i(\mathbf{H}) \sum_{j=1}^N \left[R^i(\mathbf{H}^{\pm j}) - 1\right] \underbrace{(1-2H_t^j)^2}_{1}\left(\theta_t^j - 1\right) dH_t^j \\
&= \xi_t D_t^i dt + \xi_t P^i(\mathbf{H}) m_t^i dt - \xi_t P^i(\mathbf{H}) r_t dt - \xi_t P^i(\mathbf{H})\kappa\sigma dt - \xi_t P^i(\mathbf{H})(\kappa-\sigma)dZ_t + \\
&+ \xi_t P^i(\mathbf{H}) \sum_{j=1}^N \left(\theta_t^j - 1\right)(1-2H_t^j) dM_t^j + \xi_t P^i(\mathbf{H}) \sum_{j=1}^N \left[R^i(\mathbf{H}^{\pm j}) - 1\right] \theta_t^j dM_t^j + \\
&+ \xi_t P^i(\mathbf{H}) \sum_{j=1}^N \left[R^i(\mathbf{H}^{\pm j}) - 1\right] \left(\theta_t^j \lambda_j \left(1 - H_t^j\right) + \eta_j \theta_t^j H_t^j\right) dt
\end{aligned}$$

m_t^i denotes equity i's instantaneous expected return $E[dP^i/P^i|\mathcal{F}_t]$. $\mathbf{H}^{-j}$ ($\mathbf{H}^{+j}$) is the realization of $\mathbf{H}$ to which the present state $\mathbf{H}$ jumps if firm j has a distress (recovery). Dividing both sides by $\xi_t P^i(\mathbf{H})$, taking expectations and recalling that the martingale property of $\mathcal{M}_t^i$ implies that the drift component must vanish and we obtain:

$$m_t^i + \frac{D_t^i}{P^i(\mathbf{H})} - r_t = \kappa\sigma - \sum_{j=1}^N H_t^j \left[R^i(\mathbf{H}^{\pm j}) - 1\right] \theta_t^j \eta_t^j - \sum_{j=1}^N (1 - H_t^j)\left[R^i(\mathbf{H}^{\pm j}) - 1\right] \theta_t^j \lambda_t^j$$

In conclusion, let rp_t^i denote the equilibrium risk premium of the $i-th$ security. We have:

$$\begin{aligned}
rp^i &= \kappa\sigma + rp_\lambda^i + rp_\eta^i \\
rp_\lambda^i(\mathbf{H}) &= \sum_{j=1}^N (1-H_t^j)\lambda_t^j(\mathbf{H})\,\theta_t^j(\mathbf{H})\left[1 - R^i(\mathbf{H}^{-j})\right]. \\
rp_\eta^i(\mathbf{H}) &= \sum_{j=1}^N H_t^j \eta_t^j(\mathbf{H})\,\theta_t^j(\mathbf{H})\left[1 - R^i(\mathbf{H}^{+j})\right].
\end{aligned}$$

The pure jump risk premium is easy to interpret. $\theta_t^j \lambda_t^j$ ($\theta_t^j \eta_t^j$) is the risk-neutral distress (recovery) intensity of firm j. It is greater (smaller) than the objective transition intensity λ^j (η^j). The market price of distress (recovery) risk can thus be interpreted as the risk adjustment per unit of (instantaneous) probability that the agent requires as compensation for the risk of distress (recovery). If the event materializes, security i responds with a gross returns $R^i(H^{-j})$ $(R^i(H^{+j}))$. Thus the distress risk premium is a weighted average of the risk adjusted returns on security i that would emerge if firm j had a distress (recovery), with the likelihoods of distress (recovery) as weights. Again, the network determinants of the cross-section of risk premia are hard to capture at visual inspection, as they are embedded in the transition matrix of state variables. The next Proposition states the general expression of the price-dividend ratio in a general network structure for finite N and is necessary to prove the Theorem **2**.

PROPOSITION EC.2. *Conditional on the current state* $\mathbf{H}$ *The equilibrium price-dividend ratio satisfies the following multidimensional stochastic Gordon growth condition:*

$$\frac{P^i(\mathbf{H})}{D^i(\mathbf{H})} = \frac{\widehat{P}^i(\mathbf{H})}{\mathbf{C}^i(\mathbf{H})},$$

with

$$\begin{aligned} \widehat{P}^i(\mathbf{H}) &:= \lim_{T\to\infty} \int_t^T \underbrace{\exp(\mathbf{A}(s-t))}_{1} \underbrace{e^{-a(s-t)}\mathbf{C}^i(\mathbf{H})}_{2} ds \\ \mathbf{C}^i(\mathbf{H}) &:= x^i \left(X^{(N)}(\mathbf{H})\right)^{-\gamma}, \end{aligned} \tag{EC.10}$$

where $\mathbf{A}$ *is the transition rate matrix and*

$$a = \delta - \mu(1-\gamma) - \frac{\sigma^2\gamma(1-\gamma)}{2}$$

is the inverse of the standard (representative firm) Lucas equilibrium price-dividend ratio.

REMARK EC.1. Note that Term 1 in equation (EC.10) is the transition probability matrix of the network Markov chain, from time t to s.[28] Term 2 is the conditional gross dividend of the firm discounted by the intertemporal marginal rate of substitution, namely the equilibrium pricing kernel. Therefore, $\widehat{P}^i$ is the expectation of cumulative discounted dividends, in the infinite horizon limit, conditional on the initial distress state $\mathbf{H}$.

Proof of Proposition EC.2. Let $\mathbf{H}$ denote the initial configuration, $P^i(\mathbf{H})$ the price of the claim to the $i-$th endowment stream and $D^i(\mathbf{H}) = Y_t x^i(\mathbf{H})$ the corresponding initial dividend level. Then the Theorem states that

$$\frac{P^i(\mathbf{H})}{D^i(\mathbf{H})} = \frac{\widehat{P}^i(\mathbf{H})}{\mathbf{C}^i(\mathbf{H})},$$

where:

$$\begin{aligned} \widehat{P}^i(\mathbf{H}) &:= \mathbf{1}_{\mathbf{H}}'(a\mathbb{I}_{2^N} - \mathbf{A})^{-1}\mathbf{C}^i \\ \mathbf{C}^i(\mathbf{H}) &:= x^i(\mathbf{H}) X^{(N)}(\mathbf{H})^{-\gamma} \end{aligned}$$

$1_{\mathbf{H}}$ is a vector with 2^N entries with 1 in the entry corresponding to configuration H and zero elsewhere, C^i is the vector with 2^N entries $C^i(\mathbf{H})$ while I_{2^N} is the 2^N-dimensional identity matrix and $\mathbf{A}$ is the transition rate matrix and

$$a = \delta - \mu(1-\gamma) - \frac{\sigma^2\gamma(1-\gamma)}{2}$$

[28] $\exp(\cdot)$ in this expression in a matrix exponential.

Given the equilibrium state-price density ξ_t as in (EC.7), computing the expectation of the diffusive component one gets:

$$\frac{P^i(\mathbf{H})}{D^i(\mathbf{H})} = \frac{\mathbb{E}\left[\int_t^\infty \xi_s Y_s x_s^i ds \mid \mathcal{F}_t\right]^i}{Y_t x_t^i(\mathbf{H})\xi_t(\mathbf{H})} = \frac{\widehat{P}^i(\mathbf{H})}{C^i(\mathbf{H})};$$

where

$$\widehat{P}^i(\mathbf{H}) = \mathbb{E}\left[\int_t^\infty e^{-a(s-t)} x_s^i \left(\frac{X_s^{(N)}}{X_t^{(N)}}\right)^{-\gamma} ds \mid \mathcal{F}_t\right]$$

where by assumption $a = \delta - \mu(1-\gamma) + \frac{\sigma^2}{2}(1-\gamma)\gamma > 0$ so that the integral is converging. The explicit form for the vector $\widehat{P}^i = [\ldots, \widehat{P}^i(H), \ldots]'$ is determined as follows: absence of arbitrage opportunities implies that the process

$$\int_0^t e^{-as} x_s^i \left(\sum_{j=1}^N x_s^i\right)^{-\gamma} ds + e^{-at}\widehat{P}^i(\mathbf{H}) \tag{EC.11}$$

must be a $\mathcal{F}_t-$martingale, hence its predictable component must vanish. Therefore applying Ito's lemma to (EC.11) and taking conditional expectations, the resulting expression must vanish. Using the notation $\widehat{\lambda}_t^i = H_t^i \eta^i + (1-H_t^i)\lambda_t^i$, we obtain the following system of equations

$$\left(-a - \sum_{j=1}^N \widehat{\lambda}_t^j(\mathbf{H})\right)\widehat{P}^i(\mathbf{H}) + \sum_{j=1}^N \widehat{\lambda}_t^j(\mathbf{H})\,\widehat{P}^i(\mathbf{H}^{\pm j}) + x_t^i\left(X_t^{(N)}\right)^{-\gamma} = 0 \tag{EC.12}$$

The current distress state for the economy, $\mathbf{H}$, moves to the state $\mathbf{H}^{+j}$ if firm j recovers from a distress, or to $\mathbf{H}^{-j}$ if it experiences a distress. The system of equations (EC.12) determines the functions $\widehat{P}^i(\mathbf{H})$. We can write the resulting linear system of equations in vector form:

$$(a\mathbf{I} - \mathbf{A})\,\widehat{\mathbf{P}}^i - \mathbf{C}^i = 0 \tag{EC.13}$$

In conclusion $\widehat{\mathbf{P}}^i = [\ldots, \widehat{P}^i(\mathbf{H}), \ldots]'$ contains functions $\widehat{P}^i(\cdot)$ conditional on all 2^N possible states $\mathbf{H}$.[29] Similarly $\mathbf{C}^i = [\ldots, x^i(\mathbf{H})\left(\sum_{j=1}^N x^j(\mathbf{H})\right)^{-\gamma}, \ldots]'$ contains all conditional (persistent) dividends discounted by the marginal utility of aggregate consumption. $\mathbf{I}$ is a $2^N \times 2^N$ diagonal matrix. $\mathbf{A}$ is the Markov transition matrix. From (EC.13) one gets immediately

$$\widehat{P}^i = (\mathbf{aI} - \mathbf{A})^{-1}\mathbf{C}^i$$

a is a diagonal matrix with a on the main diagonal. $\mathbf{A}$ is the transition matrix of multidimensional Markov chain. $C^i = [\ldots, x^i(\mathbf{H})\left(X^{(N)}(\mathbf{H})\right)^{-\gamma}, \ldots]'$ is the 2^N vector of dividend components paid in each state, discounted by the marginal utility. The 2^N-vector V^i of P/D ratios for all states $\mathbf{H}$ is then:

$$\mathbf{V}^i = diag\left(\mathbf{C}^i\right)^{-1}(\mathbf{a} - \mathbf{A})^{-1}\mathbf{C}^i$$

□

Proof of Theorem 2. Since $x(0) > x(1) > 0$:

$$\begin{aligned}
\theta^j(\mathbf{H}_t) \; : \; &= H_t^j \left(\frac{x_t^j(0)\left(\sum x_{t-}\right)^{-1} + 1}{x_t^j(1)\left(\sum x_{t-}\right)^{-1} + 1}\right)^{-\gamma} + (1-H_t^j)\left(\frac{x_t^j(1)\left(\sum x_{t-}\right)^{-1} + 1}{x_t^j(0)\left(\sum x_{t-}\right)^{-1} + 1}\right)^{-\gamma} \qquad j = 1, 2, \ldots N, \\
&= H_t^j\left(1 + \left(x_t^j(1) - x_t^j(\,0)\right)\left(\sum x_{t-}\right)^{-1}\right)^{-\gamma} + (1-H_t^j)\left(1 - \left(x_t^j(1) - x_t^j(0)\right)\left(\sum x_{t-}\right)^{-1}\right)^{-\gamma} \\
&= H_t^j\left(1 - \gamma\left(x_t^j(1) - x_t^j(0)\right)\left(\sum x_{t-}\right)^{-1}\right) + (1-H_t^j)\left(1 + \gamma\left(x_t^j(1) - x_t^j(0)\right)\left(\sum x_{t-}\right)^{-1}\right) \\
&= 1 + \gamma(1 - 2H_t^j)\left(x_t^j(1) - x_t^j(0)\right)\left(\sum x_{t-}\right)^{-1}
\end{aligned}$$

[29] Of course not all of them are mutually reachable, because at most one of the trees can fall in distress or recover at some time instant.

and the second term is of order $1/N$ Hence computation of prices goes as follows:

$$\begin{bmatrix} \widehat{P}_t^i(h,0) \\ \widehat{P}_t^i(h,1) \end{bmatrix} = \frac{1}{\det}\begin{bmatrix} +\eta+a & +\alpha\nu_i^R\lambda h \\ +\eta & +\alpha\nu_i^R\lambda h+a \end{bmatrix}\begin{bmatrix} x_t^i(0)\,Y_t^{-\gamma}\left(x_t^i(0)+\sum x_{t-}\right)^{-\gamma} \\ x_t^i(1)\,Y_t^{-\gamma}\left(x_t^i(1)+\sum x_{t-}\right)^{-\gamma} \end{bmatrix}$$
$$\overset{N\to+\infty}{\simeq} \frac{\left(\sum x_{t-}\right)^{-\gamma}}{\det}\begin{bmatrix} x_t^i(0)(\eta+a)+x_t^i(1)\,\alpha\nu_i^R\lambda h \\ x_t^i(0)\,\eta+x_t^i(1)\left(\alpha\nu_i^R\lambda h+a\right) \end{bmatrix}$$

and correspondingly, returns determined by state transitions are given by:

$$R^i(\mathbf{H}^{+i}) = \frac{P^i(0)}{P^i(1)} = \frac{\alpha\nu_i^R\lambda h x_t^i(1)+\eta x_t^i(0)+a x_t^i(0)}{\alpha\nu_i^R\lambda h x_t^i(1)+\eta x_t^i(0)+a x_t^i(1)}$$
$$= 1+a\frac{x_t^i(0)-x_t^i(1)}{(\alpha\nu_i^R\lambda h+a)\,x_t^i(1)+\eta x_t^i(0)} = 1+a\frac{\frac{x_t^i(0)}{x_t^i(1)}-1}{(a+\alpha\nu_i^R\lambda h)+\eta\frac{x_t^i(0)}{x_t^i(1)}}$$

and

$$R^i(\mathbf{H}^{-i}) = \frac{P^i(1)}{P^i(0)} = \frac{\alpha\nu_i^R\lambda h x_t^i(1)+\eta x_t^i(0)+a x_t^i(0)+a\left(x_t^i(1)-x_t^i(0)\right)}{\alpha\nu_i^R\lambda h x_t^i(1)+(\eta+a)\,x_t^i(0)}$$
$$= 1-a\frac{\frac{x_t^i(0)}{x_t^i(1)}-1}{(a+\eta)\frac{x_t^i(0)}{x_t^i(1)}+\alpha\nu_i^R\lambda h}$$

hence

$$R^i(\mathbf{H}_t^{-i}) \overset{N\to+\infty}{\simeq} 1-\frac{\left(\frac{x^i(0)}{x^i(1)}-1\right)}{\left(1+\frac{\eta}{a}\right)\left(\frac{x^i(0)}{x^i(1)}-1\right)+\left(1+\frac{\alpha\nu_i\lambda h}{a}+\frac{\eta}{a}\right)},$$
$$R^i(\mathbf{H}_t^{+i}) \overset{N\to+\infty}{\simeq} 1+\frac{\left(\frac{x^i(0)}{x^i(1)}-1\right)}{\left(1+\frac{\eta}{a}+\frac{\alpha\nu_i\lambda h}{a}\right)+\frac{\eta}{a}\left(\frac{x^i(0)}{x^i(1)}-1\right)}.$$

and inserting these expressions in the general expression for rp_λ and rp_η :

$$rp_\lambda^i(\mathbf{H}) = \sum_{j=1}^{N}(1-H_t^j)\lambda_t^j(\mathbf{H})\,\theta_t^j(\mathbf{H})\left[1-R^i(\mathbf{H}^{-j})\right].$$
$$rp_\eta^i(\mathbf{H}) = \sum_{j=1}^{N}H_t^j\eta_t^j(\mathbf{H})\,\theta_t^j(\mathbf{H})\left[1-R^i(\mathbf{H}^{+j})\right].$$

and in the rank one approximation, the pair (H_t^i, h_t^ν) is a sufficient statistic to describe the impact of configuration $\mathbf{H}_t$ on the risk premium of firm i then, with a slight abuse of notation, we can write:

$$rp_\lambda(h) := rp^i\left(H_t^i=0,\ h_t^\nu=h\right) = \frac{\lambda}{\eta}\alpha\nu_i h\frac{\left(\frac{x^i(0)}{x^i(1)}-1\right)}{\left(\frac{1}{\eta}+\frac{1}{a}\right)\left(\frac{x^i(0)}{x^i(1)}\right)+\frac{\alpha\nu_i\lambda h}{\eta}\frac{1}{a}}, \tag{EC.14}$$
$$rp_\eta^i(h) := rp^i\left(H_t^i=1,\ h_t^\nu=h\right) = \left[-\frac{\left(\frac{x^i(0)}{x^i(1)}-1\right)}{\left(\frac{\alpha\nu_i\lambda h}{\eta}\frac{1}{a}+\frac{1}{\eta}\right)+\left(\frac{x^i(0)}{x^i(1)}\right)\frac{1}{a}}\right].$$

which implies the desired conditional result. Note that the distress and recovery risk premia are approximated by:

$$rp_\lambda^k(h_t^\nu) \cong \alpha\frac{\lambda}{\eta}\nu_k^R h_t^\nu\frac{a}{\left(1+\frac{a}{\eta}\right)}\left(1-\frac{x^k(1)}{x^k(0)}\right)+O\left[\left(\frac{x^k(1)}{x^k(0)}\right)^2\right],$$
$$rp_\eta^k \cong (\eta)(-a)\left(1-\frac{x^k(1)}{x^k(0)}\right)+O\left[\left(\frac{x^k(1)}{x^k(0)}\right)^2\right].$$

for $\frac{x^k(1)}{x^k(0)} << 0$

□

Proof of Corollary 2 Consider the limiting expression for $N \to +\infty$, of the mean risk premium for the group of firms with vulnerability ν_k^R thanks to Assumption 2, $\frac{\sum_{i\in C_k} H_t^i}{N_k} \to h_t^i,\ i \in C_k$. Then, in light of Theorem 1 the risk premium for firms $i \in C_k$ is given by:

$$rp_t^k = \kappa\sigma + \left(1 - h_t^k\right) rp_\lambda^k\left(h_t^\nu\right) + h_t^k rp_\eta^k$$

□

Proof of Theorem 3. The expressions of the expected risk premia imply

$$rp_\lambda^i\left(0,H^\nu\right)\left[(a+\eta)\frac{x^i(0)}{x^i(1)} + \alpha\nu_i\lambda H^\nu\right] = \left[a\left(\frac{x^i(0)}{x^i(1)} - 1\right)\nu_i\alpha\lambda H^\nu\right],$$
$$rp_\eta^i\left(1,H^\nu\right)\left[\eta\frac{x^i(0)}{x^i(1)} + a + \alpha\nu_i\lambda H^\nu\right] = \left[-a\left(\frac{x^i(0)}{x^i(1)} - 1\right)\right]\eta.$$

Inserting these expressions in the equations:

$$\begin{bmatrix} \frac{\widehat{P}^i(h,0)}{x^i(0)} \\ \frac{\widehat{P}^i(h,1)}{x^i(1)} \end{bmatrix} = \frac{1}{a}\begin{bmatrix} 1 - \frac{\left(\frac{x^i(0)}{x^i(1)}-1\right)}{\left(\frac{x^i(0)}{x^i(1)}-1\right)+1}\frac{\alpha\nu_i\lambda h}{(a+\eta+\alpha\nu_i\lambda h)} \\ 1 + \left(\frac{x^i(0)}{x^i(1)} - 1\right)\frac{\eta}{(a+\eta+\alpha\nu_i\lambda h)} \end{bmatrix}$$

one gets:

$$\begin{bmatrix} \frac{\widehat{P}^i(h,0)}{x^i(0)} \\ \frac{\widehat{P}^i(h,1)}{x^i(1)} \end{bmatrix} = \frac{1}{a}\begin{bmatrix} 1 - \frac{\left(\frac{x^i(0)}{x^i(1)}-1\right)}{\left(\frac{x^i(0)}{x^i(1)}-1\right)+1}\frac{\alpha\nu_i\lambda h}{(a+\eta+\alpha\nu_i\lambda h)} \\ 1 + \left(\frac{x^i(0)}{x^i(1)} - 1\right)\frac{\eta}{(a+\eta+\alpha\nu_i\lambda h)} \end{bmatrix} \tag{EC.15}$$

reshuffling this expression one finds that it is equivalent to:

$$\begin{bmatrix} \frac{x^i(0)}{\widehat{P}^i(H^\nu,0)} - a \\ \frac{x^i(1)}{\widehat{P}^i(H^\nu,1)} - a \end{bmatrix} = \begin{bmatrix} rp_\lambda^i\left(0,H^\nu\right)\frac{\frac{a+\eta+\alpha\nu_i\lambda H^\nu\frac{x^i(1)}{x^i(0)}}{a+\alpha\nu_i\lambda H^\nu+\eta}}{1-\frac{rp_\lambda^i(0,H^\nu)}{a}\frac{a+\eta+\alpha\nu_i\lambda H^\nu\frac{x^i(1)}{x^i(0)}}{a+\alpha\nu_i\lambda H^\nu+\eta}} \\ rp_\eta^i\left(1,H^\nu\right)\frac{\frac{a+\alpha\nu_i\lambda H^\nu+\eta\frac{x^i(0)}{x^i(1)}}{a+\alpha\nu_i\lambda H^\nu+\eta}}{1-\frac{rp_\eta^i(1,H^\nu)}{a}\frac{a+\alpha\nu_i\lambda H^\nu+\eta\frac{x^i(0)}{x^i(1)}}{a+\alpha\nu_i\lambda H^\nu+\eta}} \end{bmatrix}$$

Rearranging this expression it is immediate to verify that the closed form expression of the risk premia is the fixed point solution to:

$$\begin{bmatrix} rp_\lambda^i\left(0,H^\nu\right) \\ rp_\eta^i\left(1,H^\nu\right) \end{bmatrix} = \begin{bmatrix} \left(\frac{x^i(0)}{\widehat{P}^i(H^\nu,0)} - a\right)\left(\frac{a+\alpha\nu_i\lambda H^\nu+\eta}{a+\alpha\nu_i\lambda H^\nu\frac{x^i(1)}{x^i(0)}+\eta} - \frac{rp_\lambda^i(0,H^\nu)}{a}\right) \\ \left(\frac{x^i(1)}{\widehat{P}^i(H^\nu,1)} - a\right)\left(\frac{a+\alpha\nu_i\lambda H^\nu+\eta}{a+\alpha\nu_i\lambda H^\nu+\eta\frac{x^i(0)}{x^i(1)}} - \frac{rp_\eta^i(1,H^\nu)}{a}\right) \end{bmatrix}$$

which can be obtained considering a recursive solution to an iteration with initial condition $x_\lambda^{(0)} = 0$, $x_\eta^{(0)} = 0$:

$$\begin{bmatrix} x_\lambda^{(n)} \\ x_\eta^{(n)} \end{bmatrix} = \begin{bmatrix} \left(\frac{x^i(0)}{\widehat{P}^i(H^\nu,0)} - a\right)\left(1 + \frac{\alpha\nu_i\lambda H^\nu\left(1-\frac{x^i(1)}{x^i(0)}\right)}{a+\alpha\nu_i\lambda H^\nu\frac{x^i(1)}{x^i(0)}+\eta} - \frac{1}{a}x_\lambda^{(n-1)}\right) \\ \left(\frac{x^i(1)}{\widehat{P}^i(H^\nu,1)} - a\right)\left(1 - \frac{\eta\left(1-\frac{x^i(1)}{x^i(0)}\right)}{(a+\alpha\nu_i\lambda H^\nu)\frac{x^i(1)}{x^i(0)}+\eta} - \frac{1}{a}x_\eta^{(n-1)}\right) \end{bmatrix}$$

Truncating the iteration at the first order one gets eq.(12). Note however that the first order result is in fact exact. This can be shown searching for the fixed point $\left(x_\lambda^*, x_\eta^*\right)$ that is determined by the relationship:

$$\begin{bmatrix} \frac{x_\lambda^*}{\frac{x^i(0)}{\widehat{P}^i(H^\nu,0)}-a} + \frac{x_\lambda^*}{a} \\ \frac{x_\eta^*}{\frac{x^i(1)}{\widehat{P}^i(H^\nu,1)}-a} + \frac{x_\eta^*}{a} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\alpha\nu_i\lambda H^\nu\left(1-\frac{x^i(1)}{x^i(0)}\right)}{a+\alpha\nu_i\lambda H^\nu\frac{x^i(1)}{x^i(0)}+\eta} \\ 1 - \frac{\eta\left(1-\frac{x^i(1)}{x^i(0)}\right)}{(a+\alpha\nu_i\lambda H^\nu)\frac{x^i(1)}{x^i(0)}+\eta} \end{bmatrix}$$

hence

$$\begin{bmatrix} x_\lambda^* \\ x_\eta^* \end{bmatrix} = \begin{bmatrix} \left(1+\frac{\alpha\nu_i\lambda H^\nu\left(1-\frac{x^i(1)}{x^i(0)}\right)}{a+\alpha\nu_i\lambda H^\nu\frac{x^i(1)}{x^i(0)}+\eta}\right)\left(\frac{1}{\frac{x^i(0)}{\widehat{P}^i(H^\nu,0)}-a}+\frac{1}{a}\right)^{-1} \\ \left(1-\frac{\eta\left(1-\frac{x^i(1)}{x^i(0)}\right)}{(a+\alpha\nu_i\lambda H^\nu)\frac{x^i(1)}{x^i(0)}+\eta}\right)\left(\frac{1}{\frac{x^i(1)}{\widehat{P}^i(H^\nu,1)}-a}+\frac{1}{a}\right)^{-1} \end{bmatrix}$$

that can be rewritten as:

$$\begin{bmatrix} x_\lambda^* \\ x_\eta^* \end{bmatrix} = \begin{bmatrix} \left(1+\left(c_1(0)-c_2(0)\right)n(0)-c_1(0)\,c_2(0)\,n^2(0)\right)\left(\frac{x^i(0)}{\widehat{P}^i(h,0)}-a\right) \\ \left(1+\left(c_2(1)-c_1(1)\right)n(1)-c_1(1)\,c_2(1)\,n^2(1)\right)\left(\frac{x^i(1)}{\widehat{P}^i(h,1)}-a\right) \end{bmatrix}$$

where:

$$\begin{aligned} c_2(0) &= \frac{1}{(a+\eta+\alpha\nu_i\lambda h)}, c_1(0) = \frac{1}{a+\alpha\nu_i\lambda h\frac{x^i(1)}{x^i(0)}+\eta}, n(0) = \alpha\nu_i\lambda h\left(1-\frac{x^i(1)}{x^i(0)}\right) \\ c_2(1) &= \frac{1}{a+\eta+\alpha\nu_i\lambda h}, c_1(1) = \frac{1}{a+\alpha\nu_i\lambda h+\eta\frac{x^i(0)}{x^i(1)}}, n(1) = \eta\left(\frac{x^i(0)}{x^i(1)}-1\right) \end{aligned}$$

Observe that:

$$\begin{aligned} \left(c_1(0)-c_2(0)\right) &= \frac{\alpha\nu_i\lambda h\left(1-\frac{x^i(1)}{x^i(0)}\right)}{\left(a+\alpha\nu_i\lambda h+\eta-\alpha\nu_i\lambda h\left(1-\frac{x^i(1)}{x^i(0)}\right)\right)(a+\eta+\alpha\nu_i\lambda h)} \\ &= \left(c_1(0)\,c_2(0)\right)n(0) \\ \left(c_2(1)-c_1(1)\right) &= \frac{\eta\left(\frac{x^i(0)}{x^i(1)}-1\right)}{(a+\eta+\alpha\nu_i\lambda h)\left(a+\eta+\alpha\nu_i\lambda h+\eta\left(\frac{x^i(0)}{x^i(1)}-1\right)\right)} \\ &= \left(c_2(1)\,c_1(1)\right)n(1) \end{aligned}$$

hence:

$$\begin{bmatrix} x_\lambda^* \\ x_\eta^* \end{bmatrix} = \begin{bmatrix} \left(1+\left(c_1(0)\,c_2(0)\right)n^2(0)-c_1(0)\,c_2(0)\,n^2(0)\right)\left(\frac{x^i(0)}{\widehat{P}^i(h,0)}-a\right) \\ \left(1+\left(c_2(1)\,c_1(1)\right)n^2(1)-c_1(1)\,c_2(1)\,n^2(1)\right)\left(\frac{x^i(1)}{\widehat{P}^i(h,1)}-a\right) \end{bmatrix}$$

and, since the coefficients different from 1 simplify, the final relationship is given by eq.(12). □

We can state the following:

Theorem EC.1. *Consider Assumptions 1, 2 and set* $x_k(1)=0$, $k=1,..,K$. *Let* $\mathbf{H}_0^S \neq \mathbf{0}$ *and define* $\tau_0 := \inf_{t'>0}\left\{\mathbf{H}_{t'}^S = \mathbf{0}\right\}$. *Then:*
\- *for* $\frac{\lambda}{\eta} < K(\Delta)$, $\xi_t^{\mathcal{L}}$ *is the conventional 'Lucas' stochastic discount factor:* $\frac{d\xi_t^{\mathcal{L}}}{\xi_t^{\mathcal{L}}} = -r_f dt - \kappa dW_t$. *In fact the risk free rate converges to:*

$$r^{\mathcal{L}}(\mathbf{H}_t) \overset{N\to+\infty}{\simeq} r_f := \delta+\mu\gamma-\frac{1}{2}(1+\gamma)\gamma\sigma^2,$$

and pure jump risk premia $\theta^j(\mathbf{H}_t)-1$ *vanish in the limit* $N\to+\infty$.
\- *while in a supercritical equilibrium,* $\frac{\lambda}{\eta} > K(\Delta)$ *the stochastic discount factor is given by the product* $\xi_t^{\mathcal{L}}\cdot\xi_t^\star$, *where* $\xi_t^\star = \frac{dQ}{d\mathbb{P}}$ *determines the additional martingale component of the long-term risk neutral valuation measure. Its expression restricted to any finite group of firms G in the economy is given by:*[30]

$$\begin{aligned} \frac{d\xi_{t,G}^\star}{\xi_{t,G}^\star} &= \left(\theta(\mathbf{H}_t)-1\right)\sum_{k=1}^{K}\sum_{j\in C_k\cap G}\left(1-H_{t-}^j\right)dM_t^j,\ t<\tau_0 \\ \theta(\mathbf{H}_t) &= \frac{h_\infty^\nu}{h_t^\nu}. \end{aligned} \qquad \text{(EC.16)}$$

The expression $\ln(\theta(\mathbf{H}_t))$ *sets the long-term price of cascade risk. It drives a positive price of distress risk for* $h_t^\nu < h_\infty^\nu$ *and a negative price of recovery risk for* $h_t^\nu > h_\infty^\nu$.

[30] We consider the restriction to a finite set of firms to avoid potential divergences that could arise due to the presence of an infinite number of firms in the economy.

Proof of Theorem EC.1

• To prove the first part of the theorem, it is sufficient to compute the $N \to +\infty$ limit of the steady state expressions for finite N that in the subcritical state converge to the state corresponding to $h^{\nu}_{\infty} = 0$.

• In order to prove the second part of the Theorem we briefly recall here the Girsanov Theorem for Poisson processes: define the process

$$dL_t = L_{t^-} \sum_{l=1}^{2N} J^l_t \left(dN^l_t - \lambda^l_t dt \right)$$

and such that $E^{\mathbb{P}}[L_T] = 1$. Then define $d\widetilde{\mathbb{P}} = L_T d\mathbb{P}$. Then the intensities $\widetilde{\lambda}^l_t$ of the Poisson processes dN^l_t under $\widetilde{\mathbb{P}}$ are given by:

$$\widetilde{\lambda}^l_t = \lambda^l_t \left(1 + J^l_t \right)$$

In the current framework, set $d\mathbb{P}^{LT} := \xi^*_{\tau_0} d\mathbb{P}$, and impose $E^{\mathbb{P}}[\xi^*_{\tau_0}] = 1$ where τ_0 is the first arrival time to configuration $\mathbf{H}_{\tau_0} = 0$. For each firm i, the $\mathbb{P}-$intensity of $dN^{+,i}_t(\mathbf{H}_t)$ is given by $\left(\nu^R_i \alpha\lambda h^{\nu}_t\right)$ while the $\mathbb{P}-$intensity of $dN^{-,i}_t(\mathbf{H}_t)$ is η. Hence setting:

$$\left(1 + J^{+,i}_t\right) = \frac{\nu^R_i \alpha\lambda h^{\nu}_{\infty}}{\nu^R_i \alpha\lambda h^{\nu}_{t_-}} \left(1 - H^i_t\right) \qquad \left(1 + J^{-,i}_t\right) = \frac{\eta}{\eta} H^i_t \text{ if } i \in C_k$$

implies:

$$J^{+,i}_t = \frac{h^{\nu}_{\infty}}{h^{\nu}_{t_-}} - 1 \quad J^{-,i}_t = \frac{1}{1} - 1 \text{ if } i \in C_k$$

and

$$\widetilde{\lambda}^{-,i}_t = \nu^R_i \alpha\lambda h^{\nu}_{\infty}, \quad \widetilde{\lambda}^{+,i}_t = \eta \text{ if } i \in C_k$$

and the following expression for the SDF

$$\begin{aligned} \frac{d\xi^*_t}{\xi^*_t} &= \sum_{k=1}^{K} \sum_{j \in C_k} \left(\theta^j_k(\mathbf{H}_t) - 1 \right) \left(1 - H^j_{t-} \right) dM^j_t, \\ \theta^j_k(\mathbf{H}_t) &= \frac{h^{\nu}_{\infty}}{h^{\nu}_t}, \; j \in C_k \end{aligned}$$

Note that the process is well defined only for $t < \tau_0$, i.e. only before the time of absorption to the state $\mathbf{H}_{\tau_0} = \mathbf{0}$ that, however, in the large N limit is also diverging. □

EC.2. Empirical Estimation Details

EC.2.1. Empirical proxy of distress probability

In order to estimate distress probabilities, we consider a 1-month horizon logit whose specification is the exact replication of the one proposed in Campbell et al. (2008). We identify the instantaneous probability of a distress in the model with the next-month distress probability of distress estimated using the logit:

$$h^i_t = \lim_{\tau \to 0^+} E^{\mathbb{P}}_t \left[H^i_{t+\tau} \right] \stackrel{\tau = 1m}{\simeq} h^i_{t,1m}$$

In Table 11 we report the descriptive statistics of the regressors considered in the logit estimation. Coefficients used in the logit are those estimated in the original reference on the shortest monthly period. In Table 4 we report the α^{FF3}s computed relying on the FF3 factor model for the portfolios of securities sorted w.r.t. distress probability replicating the original table reported in Campbell et al. (2008).

EC.2.2. Factor mimicking portfolio of factor h_t^ν and the Fama-MacBeth

We focus our discussion on the benchmark model that includes, $MktRf$ and h_t^ν can build the factor mimicking portfolio defined by the condition $\beta^{h^\nu} = 1$ and $\beta^{MktRf} = 0$ constructed by allocating zero total wealth in the 45 test assets. Define

$$\underset{45\times(2+1)}{P_t} := \underset{45\times(2+1)}{(B_t)} \underset{(2+1)\times(2+1)}{\left(B_t' B_t\right)^{-1}}$$

where the B_t is a matrix with 45 lines and 3 columns. Each line i is given by $b_{i,t} = \left[1, \beta_{i,t}^{MktRf}, \beta_{i,t}^{h^\nu}\right]$ and denote by p_k, $k = 0, 1, 2$ the column vectors of P_t

$$P_t = \left[p^0, p^{MktRf}, p^{h^\nu}\right]$$

Then the relevant portfolio we analyze is p^{h^ν} which is a zero investment portfolios since $\left(P_t' B_t\right)_{3,1} = p^{h^\nu} \cdot \mathbf{1} = 0$. Notice that, by construction of the linear regression coefficients, the time series of its realized returns is equal to the time series of the slopes. λ_t^{MktRf},$\lambda_t^{h^\nu}$ are plotted in Figure 6. In fact:

$$\left[\lambda_t^0, \lambda_t^{MktRf}, \lambda_t^{h^\nu}\right] = \left(B_t' B_t\right)^{-1} \left(B_t' R_{t+1}^e\right)$$

where R_{t+1}^e is the vector of excess returns for the 45 test assets. The resulting portfolio provides relevant information on the composition of the portfolio exploiting the network component of the risk premium. It is interesting to observe that this portfolio composition depends only on the size of the risk exposures $\beta_{i,t}^{MktRf}$, $\beta_{i,t}^{h^\nu}$ but its composition exploits the profitability resulting from all the distress related α_i, i.e. the mispricing that drove the selection of the 45 test assets:

- It is short big (growth) portfolios, it is long small (value) quintile portfolios, see Table 6.
- It is short high distress portfolios and with the proceeds it takes a long position in low distress and low volatility portfolios. See Table 7.