Contingent Stimulus in Crowdfunding

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Reward-based crowdfunding is a form of innovative financing that allows project creators to raise funds from potential backers to start their ventures. A crowdfunding project is successfully funded if and only if the predetermined funding goal is achieved within a given time. We study the optimal timing of contingently placing a "fulcrum" in the random pledging process, with the potential of tilting it towards success, which would be a win-win-win for the creator, backers, and platform. Specifically, we consider a model where backers arrive sequentially at a crowdfunding project. Upon arrival, a backer makes her pledging decision by taking into account the expected success of the project. We characterize the dynamics of the project's pledging process. We show that there exists a *cascade effect* on backers' pledging, which is mainly driven by the *all-or-nothing* nature of crowdfunding projects. According to our data collected from the most popular online crowdfunding platform, Kickstarter, the majority of projects fail to achieve their goals. To address this issue, we propose three *contingent* stimulus policies, namely, seeding, feature upgrade, and limited-time offer. As a result of the cascade effect on backers' pledging, the optimal timing to apply stimulus policies has a cutoff-time structure. Lastly, we show that the benefit of contingent policies is greatest in the *middle* of crowdfunding campaigns. Testing with the dataset of Kickstarter, we obtain empirical evidence that the projects' success rates improve by 14.6% on average with updates in the middle of the campaign and when the pledging progress is lagging.

Key words: crowdfunding; dynamic/contingent policy; dynamic programming; empirical

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1. Introduction

Reward-based crowdfunding is a form of innovative financing that has grown enormously in recent years. It is reported that the crowdfunding industry will soon account for more funding than venture capital (Barnett 2015). One of the leading crowdfunding platforms is Kickstarter, on which creators can raise funds from potential backers to start their ventures, and backers are rewarded with variations of the products being produced. As of February 4, 2021, 195,671 projects have been successfully funded on Kickstarter, raising around $5.02 billion from 19 million people from nearly every country on the planet.[1]

A typical reward-based crowdfunding project has a predetermined monetary goal. The project will be successfully funded only if the goal is reached within a specified time period. Improving chances of successfully raising the required funds lies at the core of the design of crowdfunding projects for project creators as well as for the platforms. Higher success rates benefit all parties: creators receive much-needed funds to initiate their ventures; backers get a chance to support their favorite projects and are rewarded with products being produced; and platforms receive a commission from every successfully funded projects. However, owing to the unpredictability of how many backers will arrive and what their preferences and valuations will be, there is much uncertainty about the outcome of a project, especially since every project has a limited time to meet its target. Using a dataset that we collected from Kickstarter from January 30 to June 27, 2015, we found that 63.4% $(13,745)$ of the projects failed to collect more than 20% of their goals before the deadline. An additional 8.45% $(1,831)$ of projects collected at least 20% of their goals but eventually failed to meet their target.

Traditionally, the effort to improve the success rates of projects concentrates on optimizing the upfront design of project characteristics, such as the targeted amount, reward levels and corresponding prices, which are fixed during the campaign horizon (e.g., Hu et al. 2015, Alaei et al. 2016 and Zhang et al. 2018). However, because of the inherent *uncertainty* and *all-or-nothing* mechanism of crowdfunding projects, we advocate that *contingently* providing incentives or adjusting project characteristics over the course of a crowdfunding campaign is as important as, if not more important than, the ex ante optimal design.

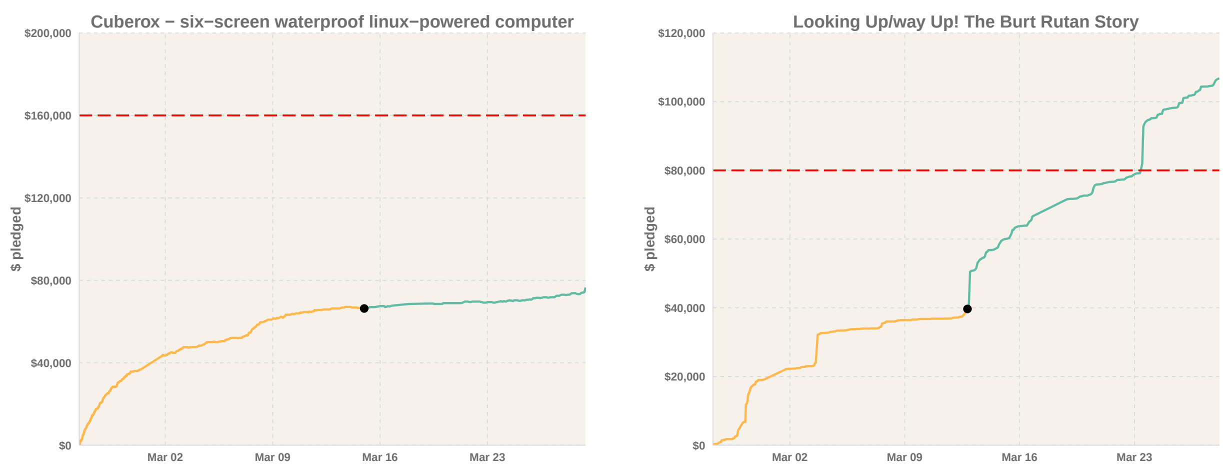
Most crowdfunding platforms do allow project creators to update their projects and post related information on projects' web pages. Updates can range from simple reminders and expressions of appreciation to tangible modifications to the project, such as new designs or extra features. As a matter of fact, both Kickstarter and Indiegogo describe updates as a good way to raise awareness and boost success rates.[2] Our data suggests that, on average, successful projects make 1.1 updates per week, whereas the failed ones make only 0.2.

[1] Source: https://www.kickstarter.com/help/stats.

[2] See https://go.indiegogo.com/blog/2015/10/crowdfunding-statistics-trends-infographic.html.

We use two projects posted on Kickstarter to illustrate the effect of contingent updates on projects' success. The creators of project "Cuberox" seek to develop a waterproof six-screen computer powered by the Linux operating system. The project was launched on February 24, 2015, aiming to gather $150,000 by March 30, 2015. Figure 1(a) displays the cumulative amount pledged to the project during its crowdfunding campaign. As the figure suggests, at first the amount pledged grew steadily; however the increase slowed down significantly in the middle of the campaign. A few backers also expressed a concern that the project might not reach its goal. But the creators did not take any action. The pledging almost halted, and the project eventually failed as shown in Figure 1(a).

Figure 1 Pledging Trajectories of Two Projects on Kickstarter



(a) Cuberox (b) Looking Up, Way Up

Note. The horizontal axis indicates the time, and the vertical axis indicates the cumulative amount pledged. The dashed horizontal lines represent the targeted amount.

Another project launched around the same time is "Looking Up, Way Up!", which is a proposed documentary about Burt Rutan, a celebrated aerospace engineer. The project was launched on February 25, 2015, with a deadline of March 28, 2015, and a goal of $80,000. The cumulative amount pledged to the project over time is displayed in Figure 1(b). We can see that the first half of the pledging trajectory resembles that of "Cuberox." However, the number picked up again in the middle of the campaign and eventually reached its target. A closer look at the project timeline shows that project creators announced two raffles for a few free limited-edition items on March 13 and March 17, 2015, which contributed to a significant increase in the pledging number. Whereas the high funding goal certainly contributed to the failure of the "Cuberox" project, updates of

"Looking Up, Way Up!" that stimulated pledges in the middle of the campaign is arguably one of the main reasons why the project eventually reached its funding target.

Motivated by the preceding examples, we study *contingent* stimulus policies commonly used by creators during their project campaigns to improve their chances of raising the required funds. Specifically, we consider a situation where backers with a heterogeneous, privately known willingness to pledge (or valuation) arrive sequentially at a crowdfunding project. Upon arriving, a backer makes her pledging decision according to her valuation which depends on project characteristics, as well as according to the expected success of the project which depends on the time of arrival and the amount pledged at that time. We first study, as a benchmark, the random pledging process without any creators' contingent stimulus. Specifically, we characterize the dynamics of a project's pledging process and the structural properties of the project's success rate, using the concept of rational expectations equilibrium. In particular, due to the all-or-nothing nature of crowdfunding projects, we show that there exists a *cascade effect* on backers' pledging. That is, a backer's pledge not only reduces the required number of pledgers by one, but also boosts the confidence of backers who arrive later, leading to a greater likelihood of pledging by future arrivals. Overall, a backer's pledge results in a relatively much higher success rate compared to without the pledge. The boost in the success rate due to a pledge (in the form of a ratio of success rates with and without the pledge) is more salient when the pledge is made closer to the deadline (hence there is less time in the horizon to attract pledges) or when the number of additional pledgers needed in order to reach the target is larger for a given time. In other words, the *relative* benefit of adding one more pledger improves as the chance of success grows dimmer.

Next, we consider three different types of contingent stimulus policies that are costly to implement and the optimal timing of using them. For simplicity, we focus on the decision on whether and when to use those costly stimulus policies for once.[3]

First, we consider a seeding policy, where the project creator has the option to acquire backers at a cost. Owing to the cascade effect, the addition of pledges increases the pledging likelihood of future arrivals and thus leads to a higher success rate. Second, inspired by a common practice,

[3] We extend the model to consider multiple rounds of stimulus offerings for the two reactive stimulus policies: seeding and feature upgrade, in the appendix. With multiple rounds of stimuli, the problem becomes much more complicated because, for a given remaining time, the expected profit of activating the stimulus policy is difficult to pinpoint, due to the fact that the creator is able to apply multiple stimuli at the same time. Nonetheless, through careful analysis, we prove analytically that the optimal strategy is a threshold policy, and the threshold increases in the number of additional pledgers required. For limited-time offers, when there are multiple LTOs in effect, the decision to end one of them would depend on the total funds collected at the time, which makes the problem significantly more complicated. While we hypothesize that the optimal strategy is a threshold policy, the proof is beyond the scope of this paper, which we leave for future research.

we consider a feature upgrade policy, where project creators are able to upgrade project features once over the course of the crowdfunding campaign. These two policies are similar in the sense that they are both *reactive*; i.e., both of them seek to increase the likelihood of future pledging if there are fewer early pledgers than expected. As a result, the optimal policies for these two policies follow a similar structure. That is, for any number of additional pledgers required to reach the target, there exists a cutoff time such that the creator should implement the stimulus if and only if the remaining time is less than or equal to the cutoff. As a direct consequence of this cutoff structure, both seeding and feature upgrade policies can be implemented easily, where the cutoff time only needs to be updated when the number of pledgers changes. The main driving force behind this cutoff structure is the cascade effect on backers' pledging due to the all-or-nothing nature of crowdfunding projects. When it comes to implementing stimulus policies, there is an essential tradeoff between the cost of stimulus policies and the potential benefit measured by the improvement in the success likelihood. Because the cascade effect becomes stronger as it comes closer to the end of a crowdfunding campaign, the stimulus policy offers a greater boost in the success rate, and thus the expected gain always outweighs the cost of the stimulus policies when remaining time is shorter than a certain value. We also show that the cutoff time increases in the number of additional pledgers required, which indicates that the further the total amount pledged is from the goal, the earlier the stimulus policies should be applied.

The third policy is a limited-time offer, where project creators are able to offer extra bonuses to early adopters. Compared with the other two policies, a limited-time offer is more *proactive* in the sense that it encourages backers to pledge early with the hope of attracting more backers later on owing to the cascade effect. Because of this difference, the optimal use of the limited-time offer contrasts with that of the other two policies. There is still a cutoff time for any number of additional pledgers required to reach the target; however, the creator should end limited-time offers if and only if the remaining time is greater than or equal to this cutoff.

Though all three policies indirectly benefit all backers through the boost in the success rate, seeding and limited-time offer only directly benefit a few of those who get the promotions, whereas feature upgrade directly benefits all, once the project becomes successful.

The cutoff-time structure in the optimal policies suggests that the project creators should wait and apply (or end) the stimulus only when the early pledging trajectory is unsatisfactory (or satisfactory). In addition, what all three policies share in common is that their benefit in *absolute* terms vanishes when the remaining time is either too long or too short. On the one hand, when there is ample time left, a project is likely to be successful without any stimulus. On the other hand, when time is very limited, the chance of reaching the funding goal can still be low even with stimulus policies. Hence, it tends to be more effective to apply stimulus policies in the *middle* of the pledging process. This is validated by our empirical analysis of a dataset collected from Kickstarter. We show that, although making updates during the funding campaign always improves a project's chance of success, updates are most effective in the middle of a campaign, especially when the pledging is lagging. On average, updating under this scenario improves success rates by 14.6%.

We summarize the contributions of our paper as follows. First, we characterize the cascade effect on backers' pledging which is driven by the all-or-nothing nature of crowdfunding projects. Second, as a result of the cascade effect, we show that the optimal timing to apply stimulus policies has a cut-off structure that is *contingent* upon the progress of the pledging. A project where the amount pledged grows at a healthy pace does not need interference, whereas one whose pledging progress turns out unsatisfactory would benefit from applying stimuli. Last, we corroborate this finding with the data we collected from Kickstarter. Project updates are shown to offer the greatest boost to success rates when the middle of the campaign is reached and the total amount pledged falls behind.

2. Literature Review

This paper contributes to the growing literature on the crowdfunding scheme (see Chen et al. 2020, Section 4.5 and Allon and Babich 2020 for surveys on crowdfunding in the operations management literature). The origin of crowdfunding can be traced back to the provision point mechanism that is traditionally used in the provision of public goods from private contributions (see, e.g., Bagnoli and Lipman 1989 and Varian 1994). Crowdfunding differs from this stream of literature in that a backer cannot benefit from a crowdfunding project without actually pledging, and thus the free-riding problem that commonly arises in the provision of public goods is not a salient concern in the context of crowdfunding.

The recent emergence of online crowdfunding platforms, such as Kickstarter and Indiegogo, has attracted a wide range of researchers who have studied the phenomenon both empirically and analytically. On the empirical side, researchers have studied many different aspects of the crowdfunding mechanism, including geographic dispersion of investors (Agrawal et al. 2011), backer dynamics over the project funding cycle (Kuppuswamy and Bayus 2013), positive network externalities (Li and Duan 2016), factors that lead to successful projects (Mollick 2014), the long-term benefit from launching crowdfunding campaigns (Mollick and Kuppuswamy 2014), and backers' prosocial behavior due to the existence of funding goals (Dai and Zhang 2019).

On the analytical side, Belleflamme et al. (2014) discuss the optimal choices between reward-based and equity-based crowdfunding under various conditions. Hu et al. (2015) study pricing and product design decisions and demonstrate unique benefits of menu pricing in the context of crowdfunding. Chakraborty and Swinney (2021) study how the creators may signal the quality of their projects through funding targets and how the creators' behavior can be different under the objective of profit-maximization versus success-maximization. Roma et al. (2018) study an entrepreneur who essentially needs venture capital but could use a crowdfunding campaign to learn what the market is. The authors study whether the entrepreneur should launch a crowdfunding campaign and, if so, how to choose the campaign instruments. Chakraborty and Swinney (2018) suggest a multi-reward strategy with limited quantities for the more attractive options to mitigate the strategic behavior of backers (who delay pledging until the campaign is more likely to succeed). Zhang et al. (2018) model the pledging dynamics with a diffusion process that aligns with the U-shape and L-shape patterns commonly observed in practice. With this model, they investigate the optimal design of crowdfunding projects in terms of the pledge levels and campaign duration. Alaei et al. (2016) seek to unravel the commonly observed phenomenon that crowdfunding projects either succeed or fail by large margins, by modeling the detailed pledging process (see more discussion below). The authors then study the creator's ex ante decisions of reward pricing and funding target. Unlike the analytical works that mainly address the upfront design of crowdfunding projects in terms of price, target, and mechanism, our work focuses on the *contingent* policies that creators can apply to the dynamic pledging progress after the project design has been determined. We demonstrate the importance of contingent policies, analyze three implementable policies, and show their benefits analytically and empirically.

The closest theoretical work to ours is Alaei et al. (2016), because both papers model the dynamic pledging process in which backers anticipate the pledging behavior of later arrivals and take the project's success rate into account when making pledging decisions. They model the stochastic process as an anticipating random walk. As a base, we model the pledging process with backers' anticipation, using a different approach, namely, the differential and difference equations, which are a tool commonly used in revenue management. Moreover, our model works under a more general set of assumptions, namely, that the distribution of backers' valuations takes a general form and their arrivals follow a non-homogeneous Poisson process, as opposed to a two-point distribution of backer valuations and the assumption of one backer per time period in Alaei et al. (2016). Lastly, as mentioned above, the primary difference is that they consider upfront pricing and target decisions, taking into account the resulting pledging process, whereas we study contingent policies as the pledging process evolves. Moreover, another closely related paper is Burtch et al. (2021). This work is a combination of analytical and empirical studies with a dynamic program model on how the creators should dynamically send out referral links. Their model has no microfoundation on how individual potential backers decide on their pledging decisions, and neither is the random success rate taken into account. A feature of their model is that the referral links can be sent out at different times. But due to this complexity, the obtained theoretical structural results are somewhat limited. The authors further estimate their models using proprietary data from a crowdfunding platform. In contrast, by focusing on a set of one-time stimulus policies, we are able to fully characterize the structure of the optimal stimulus policies under endogenized backing decisions that depend on the randomly evolving state of the pledging process.

The closest empirical work to the theme of our paper is by Li and Duan (2016). They study the pledging process empirically and demonstrate that the portion of funds already raised has a positive effect on investors' pledging decisions (i.e., positive network externality), and that the time elapsed has a negative effect (i.e., negative time effect). Those empirical findings are consistent with the structural properties of the pledging process (without stimulus) derived analytically from our model. The authors also briefly study the dynamic promotions based on simulations. For a promotion policy that informs a larger number of investors (similar to our seeding policy), they suggest a heuristic, which is to carry out the promotion when the simulated success rate falls under a predetermined threshold. We show analytically that the optimal timing of one-shot promotions has a cutoff-time structure, which is simpler to implement than a policy depending on the simulated likelihood of success. Moreover, we demonstrate theoretically the effectiveness of contingent policies, whereas their support for dynamic promotions is based on simulated counterfactual analysis.

Crowdfunding shares some similarities with group buying, which also uses the all-or-nothing mechanism with a threshold. Anand and Aron (2003) compare the group-buying mechanism against the listed price mechanism, and illustrate its superiority when the market size is uncertain. Chen et al. (2010) study the optimal design of group-buying mechanisms under quantity discounts. Jing and Xie (2011) explore the role of group buying in facilitating consumer social interactions. Hu et al. (2013) show analytically the impact of sign-up information disclosure on the success rates of group-buying deals. Using data from Groupon, Wu et al. (2014) find two types of threshold-induced effects. Marinesi et al. (2018) study the benefit of group buying as a means of moderating demand between peaks and troughs. Ming and Tunca (2016) characterize the dynamic sign-up process in group buying by capturing consumer purchase equilibrium with rational expectations of future.

Then based on the model, they perform structural estimation and find that consumers do not exhibit large-scale systematic waiting behavior.

On the methodological side, the contingent policies we study in this paper are similar to the dynamic/contingent policies in revenue management (for comprehensive surveys, see, e.g., McGill and van Ryzin 1999, Bitran and Caldentey 2003, Elmaghraby and Keskinocak 2003). In traditional revenue management, firms seek to maximize the revenue from selling limited inventory over a fixed time horizon by changing prices dynamically depending on the progress of sales. In our work, we adopt the rational expectations equilibrium (REE) framework that has been used in the revenue management literature to analyze forward-looking behavior of customers (see, e.g., Su 2007, Liu and van Ryzin 2008, Zhang and Cooper 2008, Levin et al. 2009 and Liu and Zhang 2013). Our work differs from studies of traditional revenue management in that, because of the all-or-nothing nature of crowdfunding projects, backers' pledging decisions are *temporally linked* in a direct way as captured in the cascade effect, whereas in revenue management they are typically moderated by prices alone (though earlier prices may be indirectly linked with later ones through the inventory depleting process).

In the revenue management settings, Levin et al. (2008) consider a risk-averse objective that takes into account the probability of meeting a revenue target through a chance constraint. Besbes and Maglaras (2012) study financial milestone constraints on the revenues and sales that are imposed at different time points along the sales horizon. Those constraints are soft in the sense that the constraints can be violated with a penalty. In contrast, in all-or-nothing crowdfunding, a successfully funded project requires the predetermined funding goal to be achieved within a given time, as a hard constraint. This situation is similar to the setting of Besbes et al. (2018) in which the firm under debt would earn nothing if the generated revenues are not more than the debt at the end of the sales horizon. The slight difference is that there the firm would only collect the residual revenues after paying the debt, whereas crowdfunding creators collect all revenues if the project is successful. Du et al. (2020) study a setting similar to Besbes et al. (2018) in the sense that the firm can continuously update its decisions (prices in Besbes et al. 2018 and sales rates in Du et al. 2020) under an all-or-nothing constraint. In contrast, our focus is on whether and when to apply a one-shot stimulus policy in a setting where the pledges exhibit a cascade effect that is absent in Besbes et al. (2018) and Du et al. (2020). Moreover, in our crowdfunding setting, the cost of a stimulus is paid out only if the hard constraint is met; however, the cost associated with an action in those two papers is sunk regardless of whether the hard constraint is satisfied. Lastly, Swinney et al. (2011) consider a start-up which maximizes the survival probability in an investment timing game. In our setting, the creators need to not only consider the project's success probability, but also take into account the cost of stimuli. Given a healthy growth of the pledging process, the creators may not want to offer the stimulus even though doing so can increase the success probability.

3. The Model

We consider a crowdfunding platform where creators (such as entrepreneurs or artists) are able to raise funds from potential backers to start their ventures. Initially, the creator posts its crowdfunding project, which is characterized by a targeted goal G, a fixed time horizon T, and prices for rewards. A project is deemed to be successful only when the total pledged amount reaches or exceeds the target G by the end of the time horizon.

Although creators are allowed or even advised to choose a price menu for rewards on most crowdfunding platforms (Hu et al. 2015), we make a simplification assumption that there is only one price tier p in our model. Each backer who contributes the amount of p will be rewarded with a copy of the final product at the end of the crowdfunding campaign. This assumption allows us to characterize precisely the pledging dynamics. Indeed, most analytical works in the crowdfunding literature adopt this single-tier-pricing assumption (see, e.g., Alaei et al. 2016 and Zhang et al. 2018), and our key insights on contingent stimuli are not expected to change even with the presence of a price menu. As we focus on the contingent policies during the campaign, the upfront design of the project, including the target G, the duration T, and the price p are assumed to be exogenously given.

3.1. Individuals' Pledging Decisions

We start by analyzing individual backers' optimal pledging decisions. To facilitate our discussion, we denote by t the time remaining until the end of the crowdfunding project, i.e., the time-to-go. Potential backers patronize the project's webpage sequentially according to a non-homogeneous Poisson process with a time-varying rate λ_t. Upon arrival, they are able to observe the cumulative amount pledged. This information assumption is consistent with the common practice by most crowdfunding platforms such as Kickstarter and Indiegogo. In making her pledging decision, a potential backer takes into account her valuation of the project, which is dependent on the project's characteristics, and her expectation of the success of the crowdfunding project, which is dependent on the elapsed time and the cumulative amount pledged when she arrives. We assume that potential backers form a rational expectations equilibrium. That is, potential backers act on their rational expectations of the project's success when making pledging decisions and the final outcome is consistent with their expectations. A potential backer decides to contribute to the project if and only if she expects her utility from contributing to the project higher than that of not contributing. We do not allow potential backers to wait strategically. That is, upon arrival, potential backers either make a pledge or leave the system. In a similar context of online group buying, which also adopts an all-or-nothing mechanism, Ming and Tunca (2016) empirically show that customers' strategic waiting behavior is not significant.

The willingness to pledge of backers is private information. In the eye of creators, the pledging behavior can be characterized through pledging likelihood functions defined as follows.

DEFINITION 1. (INDIVIDUAL'S PLEDGING LIKELIHOOD) $H(q)$ denotes the probability that a backer pledges to the project upon arrival, given her expectation of the success rate of the crowdfunding project being q.

By using this notation, we emphasize the dependence of a backer's pledging likelihood on the success probability of the project. But we keep in mind that a backer's pledging likelihood depends on the project's characteristics as well. We will discuss policies that involve contingent control of those characteristics later in the paper. We further assume that $H(q)$ satisfies the following properties throughout the rest of the paper.

ASSUMPTION 1. (PROPERTIES OF INDIVIDUAL'S PLEDGING LIKELIHOOD)

(i) $H(q)$ *increases in* q.[4]

(ii) *For any* $q > 0$, $H(q) > 0$.

(iii) $\frac{H(\alpha q)}{H(q)}$ *increases in* q *for any* $0 < \alpha < 1$.

Assumption 1(i) is consistent with the intuition that a backer is more likely to pledge when the project is more likely to succeed eventually. Assumption 1(ii) says that, as long as the success rate of the crowdfunding project is not zero, there will be some backers who are willing to pledge. Assumption 1(iii) implies that the influence of the project's success rate on backer's pledging decisions becomes less salient when the likelihood of success is higher. In other words, a backer's pledging decision becomes less sensitive to success-rate perturbations when the success likelihood is higher. The first two conditions are innocuous. The last condition is more involved but still seems not unreasonable. We use the following example to illustrate the generality of Assumption 1.

EXAMPLE 1. To gain granularity on how exactly backers' pledging decisions may depend on the success likelihood, we consider an example where the creator chooses the quality of the project as θ. For a given quality level θ, a type-v backer has a willingness-to-pledge $v \cdot \theta$ for the project, where

[4] In this paper, the monotonicity is in its weaker sense unless otherwise stated.

v is assumed to be the realization of a continuous random variable, drawn from an unbounded distribution with cumulative distribution function $F(\cdot)$ and probability density function $f(\cdot)$. If the backer chooses to pledge but the project fails eventually, an "inconvenience penalty" c will be incurred, where $0 \le c < p$.[5] Therefore, the expected surplus from pledging for the crowdfunding project includes two components: if the project turns out to be successful, at the end of the campaign the backer enjoys a payoff of $v\theta - p$; otherwise, a cost of c is incurred. Any backer whose belief in the project's success likelihood is q will pledge if and only if

$$(v\theta - p) \cdot q - c \cdot (1-q) > 0 \qquad \Rightarrow \qquad H(q) = \bar{F}\left(\frac{1}{\theta}\left[p + c \cdot \left(\frac{1}{q} - 1\right)\right]\right). \tag{1}$$

Lemma 1. *$H(q)$ in (1) satisfies Assumption 1 if the distribution of backers' types has an increasing generalized failure rate (IGFR), i.e., $v \cdot \frac{f(v)}{\bar{F}(v)}$ is an increasing function in v.*

Lemma 1 gives a sufficient condition for Assumption 1 for the specific form of $H(q)$ in (1). The IGFR is a very general assumption as it captures many commonly used distributions, such as normal and uniform distributions. □

Example 1 specifies an individual discrete choice model where the pledger has full information about the project's success rate. The general form of the pledging likelihood function can also accommodate observational learning behavior in which a pledger may not have complete information about the project but can rationally anticipate the future arrivals' pledging behaviors.

3.2. Pledging Dynamics

The previous discussion of individuals' pledging decisions sets the stage for our characterization of the dynamics of the pledging process. Since backers' pledging decisions are determined by the expected success through the individual's pledging likelihood function, the pledging dynamics can be captured by the evolution of the project's likelihood of success over time. Recall that the crowdfunding project needs to gather G dollars before the end of a fixed time horizon. Given the price p charged to each backer, the project requires at least

$$N \equiv \lceil \frac{G}{p} \rceil$$

pledgers before time expires. From now on, we may refer to N as the target of the crowdfunding campaign. We denote by n, where $0 \le n \le N$, the additional number of pledgers required to reach the project's target, i.e., the pledges needed. The funding progress of the project towards reaching the goal is uniquely captured by the state space $\{(t,n) : 0 \le t \le T, 0 \le n \le N\}$.

[5] The cost may consist of psychological frustration in backers who failed to get the product or service they desired. It may also stem from economic losses. When a crowdfunding project fails to reach its goal, backers will not be charged. However, since they will not know that and be able to use the money for other purposes until the time expires, they will have experienced a loss because of the time value of money.

3.2.1. Success Rate. For a backer who arrives at the state of time-to-go t and pledges needed n, her expected project's success rate, conditional on her pledging, is denoted by $Q_t(n-1)$. Under the rational expectations equilibrium, her expectation will be fulfilled by backers who arrive later and act on their rational expectations. Then the dynamics of the project's success likelihood in equilibrium can be summarized as follows.

Proposition 1. (Rational Expectations Equilibrium (REE)) *There exists a unique REE, such that the probability $Q_t(n)$ of the project being successfully funded at state (t,n), is given by*

$$\frac{\partial Q_t(n)}{\partial t} = \lambda_t \cdot H(Q_t(n-1)) \cdot (Q_t(n-1) - Q_t(n)), \tag{2}$$

with boundary conditions $Q_t(0) = 1$ for all t and $Q_0(n) = 0$ for all $n > 0$.

The success likelihood at any state (t,n) can be solved by backward induction. However, in general, obtaining the closed form of $Q_t(n)$ is extremely difficult, if not impossible, even for special forms of $H(\cdot)$. Nevertheless, we are able to show a set of structural properties of $Q_t(n)$.

Theorem 1. (Structural Properties of Equilibrium Success Likelihood)

(i) $Q_t(n)$ strictly increases in t for any $n \geq 1$ and strictly decreases in n for any $t > 0$.

(ii) $\frac{Q_t(n-1)-Q_t(n)}{Q_t(n)} \geq \frac{1}{e^{\bar{\lambda} t}-1}$, where $\bar{\lambda} \equiv \sup\{\lambda_t : 0 \leq t \leq T\}$.

(iii) For any $n \geq 1$ and $t > 0$, both $\frac{Q_t(n-1)}{Q_t(n)}$ and $\frac{H(Q_t(n-1))}{H(Q_t(n))}$ decrease in t and increase in n. Moreover, $\lim_{t \to 0} \frac{Q_t(n-1)}{Q_t(n)} = \infty$.

(iv) For any $h > 0$, $\frac{Q_{t+h}(n)}{Q_t(n)}$ strictly increases in n and decreases in t.

Theorem 1(i) shows that the chance of the project being successful increases with more time remaining and fewer pledgers required. Theorem 1(ii) gives a lower bound on the relative change in the success likelihood by adding one more pledger. The guaranteed relative improvement in the likelihood of success with one more pledger is larger if the arrival rates are smaller.

The most interesting property of $Q_t(n)$ is shown in Theorem 1(iii). The effect of backers' pledging decisions on a project's success likelihood is twofold: (1) On one hand, a backer's pledging reduces the required number of pledgers by one and thus leads to a higher likelihood of success; (2) On the other hand, the backer's pledging also boosts the confidence of backers who arrive later, leading to a higher likelihood that future arrivals will pledge. These two factors add up to what we referred to as the *cascade effect* of an individual's pledging on future backers' pledging decisions. Theorem 1(iii) shows that this compounding cascade effect is more salient when the time is closer to the deadline and/or the number of additional pledgers required is larger. It would also be interesting to contrast this property with results from a typical revenue management setting, where the firm has to sell a limited amount of inventory within a fixed period of time. There a customer's valuation of the product is not directly affected by the purchase decisions of other customers. However, in our crowdfunding situation, any individual backer's pledging decision would directly and positively affect subsequent backers' decisions. Because of this cascade effect, all optimal stimulus policies that we will discuss in the next section follow a cutoff-time structure.

Theorem 1(iv) shows the impact of time-to-go on the project's success likelihood for a fixed pledges needed. A longer remaining time results in a higher likelihood of success for the project as shown in Theorem 1(i). Theorem 1(iv) further shows that this effect is more significant when the number of additional pledgers required is larger, or when the remaining time is shorter.

3.2.2. Upfront Design. Given the cascade effect on backers' pledging decisions, it is important to carefully consider the project's characteristics before launching the crowdfunding campaign. Consider two designs of a project, namely, design a and design b, which can differ in various project characteristics, such as price and quality. Suppose that design b is more attractive in the sense that $H^a(q) < H^b(q)$ for any $q > 0$. We have the following structural results from the comparisons of the project's success likelihood and backers' pledging likelihood between the two projects.

Proposition 2. (Upfront Design of Crowdfunding Projects) *Consider two pledging likelihood functions $H^a(q)$ and $H^b(q)$. If $H^a(q) < H^b(q)$ for any $q > 0$, and $\frac{H^a(q)}{H^b(q)}$ increases in q, then both the ratios of success likelihoods, $\frac{Q_t^a(n)}{Q_t^b(n)}$, and pledging likelihoods, $\frac{H^a(Q_t^a(n))}{H^b(Q_t^b(n))}$, increase in t and decrease in n.*

Proposition 2 underscores the importance of the design of project characteristics. A small difference in backers' pledging likelihoods may lead to a huge gap in the project's success likelihoods because of the cascade effect. Proposition 2 states that, given two different project designs, the relative difference in the project's success likelihoods is more significant when the time is closer to the deadline and/or the number of additional pledgers required is larger. The same applies to backers' pledging likelihood as well.

Recall that design a is less attractive. The assumption that $\frac{H^a(q)}{H^b(q)}(< 1)$ is an increasing function of q requires that the relative difference in the pledging likelihoods under two designs increases when the project's likelihood of success decreases. That is, the inferior design hurts backers' pledging likelihood more significantly when the success likelihood of the project is lower. We revisit the typical case introduced in Example 1 and investigate when this assumption is satisfied. Two sufficient conditions are summarized below. It turns out that the assumption can be easily satisfied when the project can be configured with different prices or qualities.

Lemma 2. (Properties of Pledging Likelihood) *Consider the pledging likelihood function derived in Example 1.*

(i) For two quality levels $\theta_a < \theta_b$*, the ratio of pledging likelihoods,* $\frac{H^{\theta_a}(q)}{H^{\theta_b}(q)}$*, is an increasing function of* q*.*

(ii) If the distribution of backers' valuations in Assumption 1 has an increasing failure rate (IFR), then for two prices $p_a > p_b$*, the ratio of pledging likelihoods,* $\frac{H^{p_a}(q)}{H^{p_b}(q)}$*, is an increasing function of* q*.*

3.2.3. Expected Profit. All of the above structural properties are about the success rates and pledging likelihood. Next we derive those for the expected profit of a crowdfunding project. Conditional on reaching the funding target G, the creator would have collected enough capital to potentially launch the new product in a mass market. As a result, in addition to the immediate profit gained during the rest of the campaign, the creator is able to continue selling the products beyond the campaign deadline. For analytical tractability, we do not differentiate between the profit gained during the campaign after the funding goal is reached and the potential profit from selling products after the campaign, and denote the two of them combined by a long-term profit $B \geq 0$. B can be interpreted as the total life-time discounted profit after reaching the funding goal, e.g., $B = \int_0^\infty \Lambda p \delta^t dt$, where Λ is the sales rate and $0 < \delta < 1$ is the discount factor.[6]

Without loss of generality, we normalize the marginal cost of production to 0. The total expected profit at state (t, n) is therefore given by $J_t^b(n) = (G + B) \cdot Q_t(n)$. It is obvious that $J_t^b(n)$ increases in t and decreases in n. The impact of an additional pledger on the expected profit is summarized in the proposition below, which is derived from Theorem 1(iii).

Proposition 3. (Marginal Value of a Pledger) *The marginal increase in the expected profit with one more pledger at state* (t, n)*,* $\frac{J_t^b(n-1) - J_t^b(n)}{J_t^b(n)}$*, decreases in* t *and increases in* n*.*

Like Theorem 1(iii), Proposition 3 shows that an additional pledger is more valuable when the time is closer to the deadline and/or the number of additional pledgers required is larger. In the traditional revenue management literature, monotonicity properties are derived for the *absolute* difference between the expected profits. However, because of the cascade effect demonstrated in

[6] The main source of uncertainty considered in this paper is w.r.t. whether the funding target can be successfully reached by the end of the crowdfunding campaign, which will affect the profit gained during the campaign after the funding goal is reached and the potential profit from selling products after the campaign in the same way. As a result, we do not differentiate between the two sources of profit. But in practice, there can be other sources of uncertainties, especially regarding whether the product can be successfully developed, as well as the quality of the product. A higher funding ratio, i.e., the total pledged amount during the campaign to the funding target, helps reduce these uncertainties, and thus it becomes necessary to differentiate between the two sources of profit when they are accounted for.

Theorem 1 in the context of crowdfunding, analogous properties exist but they are for the *relative* difference.

4. Contingent Stimulus Policies

We have illustrated the importance of the upfront design of projects' characteristics in Proposition 2. Given the stochastic nature of arriving backers and their willingness to pledge to the project, the pledging process may still fail to meet the creator's expectations even if the project's characteristics are optimized ex ante. In such cases, the creator can be better off taking ex post actions to influence backers during the campaign. In this section, we consider three types of contingent stimulus policies from the perspective of project creators, namely, seeding, feature upgrade, and limited-time offer. They are different in their effect on the cost structure and pledging, but they share the common feature that the associated costs to the creators do not materialize unless the project is successful. We discuss the optimal ways of applying these three policies, and quantify their potential benefit.

4.1. Seeding Policy

We first study the seeding policy where the creator has the option to acquire n_0 number of pledges ($1 \leq n_0 < N$) at a cost of R exactly *once* during the campaign. Seeding strategies have been widely used in marketing campaigns where firms recruit customers to speed up the diffusion of the new products. The difference in crowdfunding is that the acquiring cost R will be incurred only if the project reaches its funding target. In practice, a broad class of strategies may be classified under the umbrella of the seeding policy, with which the creator is able to obtain a number of pledges under some contingent cost. For instance, the creator of "Looking Up, Way Up!" offered free samples to backers, which is a straightforward approach but may pose fairness concerns for early backers. Some less intrusive alternatives include the commonly adopted referral incentives where the creators offer bonuses to existing backers if they are able to bring in additional backers. It is also common for the creators to seek backing from friends and family. Those pledges are not without cost, as the creators may ask for favors.

The adoption of the stimulus effectively decreases the target level from N to $N - n_0$. The superiority of this seeding policy over the manipulation of the target level is obvious. The creator would choose to seed only along certain sample paths in which the early pledging progress is not satisfactory. When the pledging process materializes in a way that favors the creator, the incentives could be saved, allowing the creator to obtain a higher profit. We limit our discussion to the case where the incentives are offered once at most. We focus on the change in the number of pledgers required because it is assumed there is only one price tier p in this paper. In practice, when there are more than one price tier, the amount of fund required to reach the funding target may be more relevant for estimation of the likelihood of project success. In terms of the funding amount, the adoption of the stimulus decreases the funding level from G to $G - n_0 p$.

We assume that the n_0 pledges will be added immediately and that backers do not expect future seeding when they make their pledging decisions. If they do, under our assumption of no strategic waiting, the incentive for backers to pledge now will be even higher, thus leading to a higher value of contingent seeding. This is because backers will be more confident in the project's success since they expect an intervention by the creator when the pledging progress stalls.

Theorem 2. (Optimal Cutoff for Seeding) *For each $n \geq 1$, there exists a cutoff time $\tau^s(n)$, such that the creator will activate the seeding stimulus if and only if $t \leq \tau^s(n)$.*

Theorem 2 sheds light on the conditions under which the creator is better off activating the seeding stimulus. For any current pledges needed n, there exists a cutoff $\tau^s(n)$ such that the creator should implement seeding if and only if the time-to-go is no more than this cutoff. Although details of the proof are more involved and can be found in the appendix, we describe the intuition as follows. The creator makes the optimal stopping decision by comparing the optimal expected profits with and without using the seeding stimulus. In particular, from Theorem 1, we show that the relative improvement in the success likelihood by seeding decreases in t. Thus, when there is ample time left, the cost of seeding outweighs the improvement in the likelihood of success, and the project creator will choose to hold out as a result. On the other hand, when the time-to-go is short enough, it is optimal to use the seeding option immediately to boost the chances of success.

We present the monotonicity properties of the cutoffs as follows.

Corollary 1. *(i) $\tau^s(n)$ increases in n, i.e.,*

$$\tau^s(N) \geq \tau^s(N-1) \geq \cdots \geq \tau^s(n_0) = \cdots = \tau^s(1) = 0.$$

(ii) $\tau^s(n)$ increases in B and decreases in R.

In Corollary 1(i), we show that the cutoff $\tau^s(n)$ is increasing in the pledges needed n. This implies that the seeding policy is more likely to be used at a time when the pledging number is further away from the target. Again, the monotonicity of $\tau^s(n)$ with respect to (w.r.t.) n can be derived from Theorem 1, where we show that the cascade effect is stronger when the number of additional pledgers required is larger. Corollary 1(ii) implies that the seeding policy is more likely to be used earlier when the long-term profit B is larger and/or the cost of stimulus R is lower.

While the latter is intuitive, the former is sensible because the potential loss from failing to reach the funding target becomes greater with a higher long-term profit B.

In general, it is very hard to derive the closed form solution of $\tau^s(n)$. To see how $\tau^s(n)$ may look like, we consider two special cases: Case (i) $H(q)\equiv H$, i.e., a backer's pledging decision is based solely on the project's characteristics, rather than the likelihood of success. From the threshold characterization (see the proof in the appendix), for this case, we have $\tau^s(n)=0$ for all $n\geq 1$. That is, the creator will never activate the seeding strategy before time expires. This is sensible considering that the benefit of the seeding policy is driven by the cascade effect of backers' pledging decisions. The seeding policy has no influence when backers are not affected by the decisions of others. Case (ii) $H(q)=\begin{cases}1 & \text{if } q>\bar{q}\\ 0 & \text{if } q\leq\bar{q}\end{cases}$, as a result of that backers have homogeneous willingness to pledge. Then the creator will seed if and only if backers' perceived project success likelihood drops to $\bar{q}$ for the first time; Otherwise, backers are expected to pledge upon arrival, rendering seeding unnecessary.

Denote by $J^s_{T,N}$ the optimal expected profit with the option of seeding when the deadline is T and the goal is N. We compare $J^s_{T,N}$ with the expected profit under no stimulus $J^b_{T,N}$, and obtain the following structural properties:

Theorem 3. *(i) For any $N\geq 1$, $\frac{J^s_{T,N}}{J^b_{T,N}}$ decreases in T.*
(ii) For any $N>n_0$, $\lim_{T\to\infty} J^s_{T,N}-J^b_{T,N}=\lim_{T\to 0} J^s_{T,N}-J^b_{T,N}=0$.

The seeding policy always benefits the project because it gives extra flexibility to the project creator, allowing him to keep the pledging process at a healthy pace by using the stimulus if necessary. From Theorem 3, we can see that the *relative* benefit of seeding becomes more significant as the time remaining gets shorter. However, its *absolute* benefit vanishes as T approaches either infinity or zero. When the time is long enough, having few pledgers at the beginning of the process will not have a huge negative impact because future arrivals may still reverse the trend, resulting in a low value of seeding. On the other end of the spectrum, when the time is very short, few backers are expected to come to the project, leading to the ineffectiveness of the cascade effect, as well as the seeding policy. Consequently, the benefit of seeding is significant when time is limited but not impossibly short. We further confirm this finding numerically in Section 4.4.

4.2. Feature Upgrade

In the second policy, we allow the creator to upgrade project features for once during the campaign. This policy is motivated by the common practice of popular crowdfunding platforms, such as Kickstarter and Indiegogo, on which project creators can update project features over the course of the pledging process. The new feature could be, for example, a new color for a fashion product or a bonus soundtrack for an album. In the context of the "Looking Up, Way Up!" project, the creator could have offered to release behind-the-scenes shots to complement the original film as an upgraded feature. Another alternative for upgrade is to release the film of a higher video quality, such as in 4K resolution. With the upgrades the project creator hopes that backers will be more willing to pledge. However, upgrading project features could be costly. Consequently, the key question here is whether and when the project creators should offer an upgraded version of their project.

To answer this question, we enrich the base model as follows. Assume that the cost of an upgrade is K. In the context of Example 1, we can interpret the feature upgrade as that the quality level of the project increases from θ to $\tilde{\theta}$. As a result of the upgraded project, backers' pledging likelihood increases to $\tilde{H}(q)$, where $\tilde{H}(q) \geq H(q)$ for any q. We assume that $\frac{\tilde{H}(q)}{H(q)}$ increases in q. This assumption is consistent with Assumption 1(iii), and can be satisfied when the distribution of backers' types has the IGFR property in the context of Example 1. The corresponding likelihood of success is denoted by $\tilde{Q}_t(n)$.

Theorem 4. (Optimal Cutoff for Feature Upgrade) *For each n, there exists a cutoff time $\tau^u(n)$, such that the creator will upgrade if and only if $t \leq \tau^u(n)$.*

The policy of feature upgrade differs from the seeding policy in that it does not directly interfere with the pledging number. However, both of them rely on the cascade effect of backers' pledging decisions to be effective. As a result, the optimal policy of feature upgrade is similar to that of the seeding policy. That is, for any pledges needed n, there exists a cutoff in time $\tau^u(n)$ such that the creator should upgrade the project features if and only if the remaining time towards the end of the campaign is less than or equal to this cutoff.

Corollary 2. *(i) $\tau^u(n)$ increases in n, i.e., $\tau^u(N) \geq \tau^u(N-1) \geq \cdots \geq \tau^u(1)$.*
(ii) $\tau^u(n)$ increases in B and decreases in K.

Corollary 2(i) implies that the feature upgrade policy is more likely to be used at a time when the pledging number is further away from the target. Similar to Corollary 1(ii), Corollary 2(ii) shows that the feature upgrade stimulus tends to be implemented earlier if the long-term benefit B is higher and/or the cost of upgrading features K is lower.

Lastly, denote by $J^u_{T,N}$ the optimal expected profit with the option of feature upgrade when the duration is T and the goal is N. Following a similar proof as that of Theorem 3, we show that the relative difference in expected profits with and without feature upgrade decreases in the campaign duration T, but the absolute benefit vanishes as T approaches infinity or zero.

Theorem 5. *(i) For any* $N \geq 1$, $\frac{J^u_{T,N}}{J^b_{T,N}}$ *decreases in* T.

(ii) For any $N \geq 1$, $\lim_{T\to\infty} J^u_{T,N} - J^b_{T,N} = \lim_{T\to 0} J^u_{T,N} - J^b_{T,N} = 0$.

When the duration is sufficiently long, the chance that the project will be successfully funded is high, and that eliminates any incentive for the project creator to upgrade the project features. When the duration is very short, a project upgrade will affect decisions by only a negligible fraction of backers. Consequently, the stimulus will bring only a limited benefit. The implication of Theorem 5(ii) is that the benefit of a feature upgrade is greatest when the project duration is moderate. We further confirm this finding numerically in Section 4.4 and empirically in Section 5.

4.3. Limited-Time Offer (LTO)

Because of the cascade effect on backers' pledging decisions, it is important to encourage backers to pledge early in the process. One way to achieve this is to introduce a limited-time offer (LTO) to those who pledge early. For instance, the creator of the project "Looking up, Way Up!" could have offered physical copies of the film signed by Burt Rutan as a bonus to early adopters. Conceptually, in the context of Example 1, it means that the creator may offer products of higher quality $\hat{\theta}$ for the same price p to early arrivals. The creator may choose to end the LTO and switch back to normal quality θ whenever the momentum is established. The use of limited-time offers is prevalent in a wide range of industries, especially when new products are being introduced to the market. LTO is also related to nudging (Thaler and Sunstain 2009), which is commonly used by governments and firms to influence the decision-making of individuals. However, the difference is also obvious as, in our context, customers are assumed to be rational utility-maximizers whereas nudging takes advantage individuals' bounded rationality so that by altering the environment, it makes an individual more likely to make a particular choice. In this subsection, we seek to quantify the value of LTOs in the context of crowdfunding, and discuss related issues.

LTO differs from the preceding two policies, namely seeding and feature upgrade, in one important aspect: LTO is a *proactive* policy in which the creator induces early pledging by making the project more attractive at the beginning, whereas seeding and feature upgrade policies are *reactive* in the sense that the creator responds to the progress of the pledging, and chooses to apply the policies only if the number of early pledgers is low. As a result, the optimal use of an LTO differs inherently from that of those two policies.

For the creator, there is an increase in the marginal cost for each unit purchased by backers during an LTO, which we denote by k. Compared with feature upgrade, the promotional product being offered during an LTO is typically a standard version of the product plus some extras. Thus, the creator can conveniently stop the LTO and switch back to the standard product. In contrast, feature upgrade typically involves a permanent upgrade of certain characteristics of the product, e.g., making a proposed smart watch waterproof. Thus a fixed cost is incurred for producing the superior product. During an LTO, backers' pledging likelihood increases to $\hat{H}(q)$, whereas that corresponding to the normal quality level is $H(q)(\leq \hat{H}(q))$ for any likelihood of success q.

THEOREM 6. (OPTIMAL CUTOFF FOR LTO) *For any n, there exists a cutoff time $\tau^l(n)$, such that the creator will end the limited-time offer if and only if $t \geq \tau^l(n)$.*

Theorem 6 shows that, for any pledges needed n, there exists a cutoff in time $\tau^l(n)$ such that the creator should end the LTO if and only if the time remaining before the end of the project is *greater* than or equal to this cutoff. In other words, if the project has already attracted a large number of pledgers while the remaining time is long, the creator can end the LTO immediately to enjoy a lower unit cost without jeopardizing the project's success. However, if the remaining time is short, in particular if it is less than the cutoff time $\tau^l(n)$, the creator is better off continuing the LTO. The profit margin from each backer is lower in such circumstances; however, it is compensated for by a greater chance of reaching the target.

COROLLARY 3. *$\tau^l(n)$ increases in B and decreases in k.*

Corollary 3 implies that the creator is more likely to run LTO for a longer period of time when the long-term profit B is higher and/or the per unit cost of LTO k is lower. This result is consistent with Corollaries 1(ii) and 2(ii). However, unlike the seeding and feature upgrade policies, the cutoff $\tau^l(n)$ is not monotonic in the pledge-to-go n in general. This is because the overall cost of LTO is a function of pledge-to-go n, rather than a fixed cost as in the preceding two stimulus policies. If the creator chooses to run LTO longer, the campaign is indeed more likely to succeed, however the profit is also lower should it succeed due to the higher per-unit cost associated with running LTO. As a result, the expected profit with the option of LTO is not necessarily monotonic w.r.t. n, leading to possible non-monotonicity of $\tau^l(n)$ in n.

It is not surprising that the benefit of LTOs also vanishes as T approaches either infinity or zero, as does the benefit of the other two policies. The result is summarized as follows, where $J^l_{T,N}$ is the optimal expected profit with the option of an LTO when the duration is T and the goal is N.

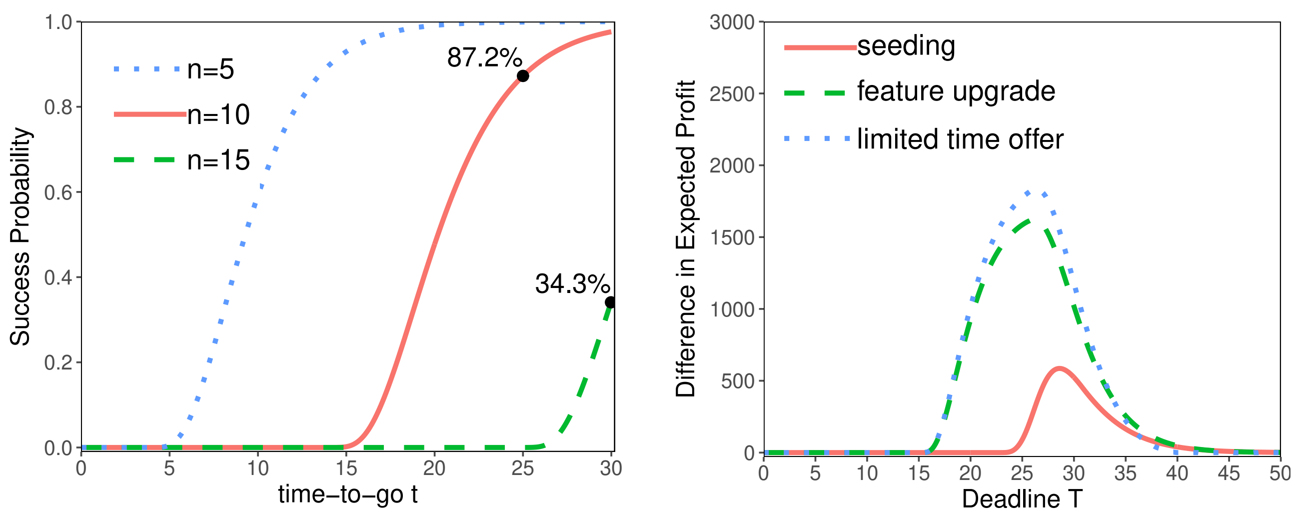
THEOREM 7. *(i) For any $N \geq 1$, $\frac{J^l_{T,N}}{J^b_{T,N}}$ decreases in T.*
(ii) For any $n \geq 1$, $\lim_{T\to\infty} J^l_{T,N} - J^b_{T,N} = \lim_{T\to 0} J^l_{T,N} - J^b_{T,N} = 0$.

4.4. Numerical Examples

We now demonstrate the effectiveness of stimulus policies with numerical experiments. We consider the setup as described in Example 1, where the creator can make the project more attractive by improving the quality of the project. The parameters in the numerical experiments are specified as follows. A backer's valuation v is drawn from an exponential distribution with mean of \$100. The contribution p required from each backer is \$120, the quality level θ of the project is 1, and the penalty cost c for each consumer if the project fails to reach its target is \$30. The goal G of the project is set to be \$1,800, which is equivalent to requiring at least $N = 15$ pledgers. The duration of the campaign is 30 days, and the arrival rate λ_t at which potential backers land on the project's webpage is assumed to be time-invariant and equals to 2 per day. The long-term benefit B is assumed to be \$500.

Using Proposition 1 and backward induction, we can compute the success likelihood $Q_t(n)$ without any contingent stimulus policy. The result is displayed in Figure 2(a). The expected success rate right after the project launch is 34.3%. Of course, whether this project indeed succeeds by the end of the campaign depends on the realized sample path, especially the number of pledgers appearing in the early stage of the crowdfunding campaign, due to the cascade effect. For instance, if 5 backers pledge during the first five days, then the project's likelihood of success increases to over 87%. On the other hand, that drops to nearly zero if nobody pledges during the first five days. The latter case is when the project creator may be able to save the project with stimulus policies.

Figure 2 A Numerical Result illustrating Benefits of Stimulus Policies



(a) Success Probability $Q_t(n)$ (b) Improvement in Expected Profit (vs. Base Model)

Note: $V \sim \exp(\frac{1}{100})$, $p = \$120$, $\theta = 1$, $c = \$30$, $G = \$1,800$ (i.e., $N = 15$), $B = \$500$, $T = 30$ and $\lambda_t = 2$.

Next, we evaluate the optimal expected profit under each of the three policies referred to in the preceding subsections. The results are shown in Figure 2(b). Here, we assume that the creator has the option to acquire 1 pledge at a cost of \$120 for the seeding strategy. For feature upgrade, she can also improve the project's quality level to $\tilde{\theta} = 1.5$ with a cost of $K = \$600$ under the feature upgrade policy. Alternatively, the creator is able to offer products at the higher quality level $\tilde{\theta} = 1.5$ to early arrivals with an LTO at an additional cost of $k = \$30$ per unit.

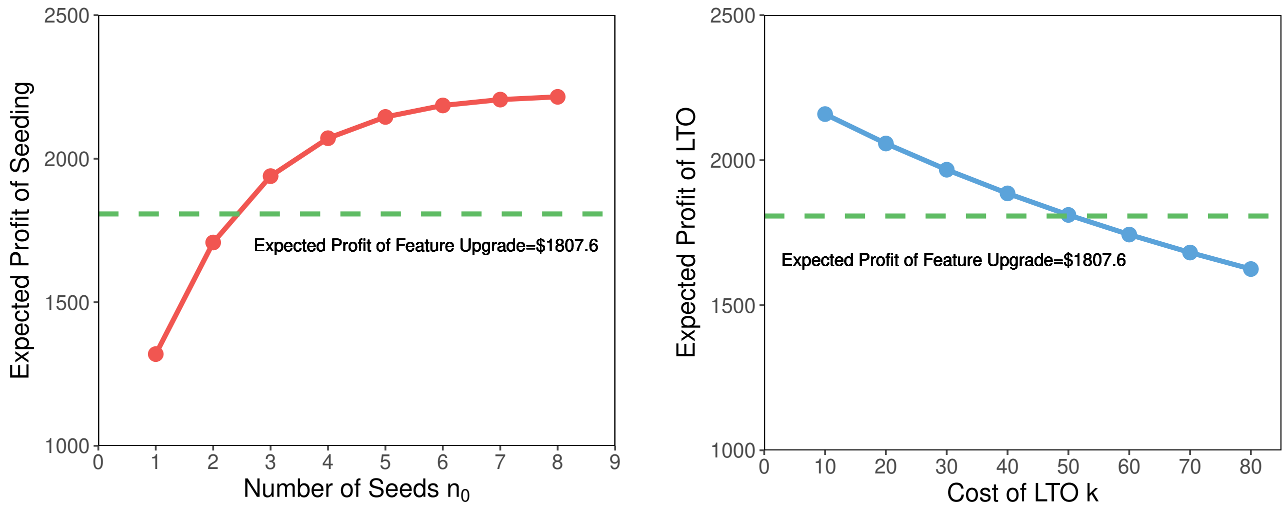
From Figure 2(b), we first observe that benefits of stimulus policies are not monotonic in the duration of projects, given the same target $N = 15$. When the project duration is short (i.e., $T < 15$), the benefit of stimulus policies is marginal because projects are likely to fail no matter what policies the project creator uses to attract backers. On the other end of the spectrum, when there is ample time (i.e., $T > 35$), project are highly likely to succeed even without stimulus. The benefit of stimulus policies is most salient with a moderate project duration (i.e., $15 \leq T \leq 35$ for this particular example). In other words, for those projects that have potential but are not overwhelmingly popular, offering stimulus at the right time could help tremendously. For instance, let us compare the results with and without stimulus when $T = 30$, which is the duration of the crowdfunding campaign in our baseline setup. The expected profit without any stimulus policy is \$790. With the optimal seeding policy, the expected profit increases to \$1,313, i.e., a 66.2% increase benchmarked with the expected profit without stimulus. Similarly, the expected profits increase by 128.8% and 149.2% with the optimal feature upgrade and LTOs, respectively.

Next we compare the efficacy of three policies under different parameters. We use the expected profit of the feature upgrade as a benchmark, and compare it against the expected profits of seeding and LTO under various costs. The results are summarized in Figure 3. Figure 3(a) shows the expected profit of the seeding policy with the same fixed cost $R = \$120$ but a different number of seeds n_0. As n_0 increases, the creator is also able to recruit customers more cost-effectively, leading to a higher profit. In our numerical analysis, the seeding strategy would yield a higher profit than feature upgrade as long as n_0 is larger than 2. Similarly, the expected profit of LTO with different cost k is shown in Figure 3(b). The expected profit of LTO decreases in k, and is lower than that of feature upgrade as long as k is greater than \$50. The numerical results show that no strategy strictly dominates, and a creator needs to carefully evaluate the costs of implementing different stimuli when it comes to the choice of the optimal stimulus policies.

5. Empirical Evidence

We built a data crawler on the Google App Engine platform to collect data from Kickstarter between January 30 and June 27, 2015. Whenever a new project was posted, the data crawler

Figure 3 Comparison of Stimulus Policies



(a) Expected Profit of Seeding (b) Expected Profit of LTO

Note: $V \sim \exp(\frac{1}{100})$, $p = \$120$, $\theta = 1$, $c = \$30$, $G = \$1,800$ (i.e., $N = 15$), $B = \$500$, $T = 30$ and $\lambda_t = 2$. Seeding: $R = \$120$; Feature Upgrade: $\tilde{\theta} = 1.5$ and $K = \$150$; LTO: $\hat{\theta} = 1.5$.

extracted static project information, such as the project name, goal, and campaign duration. It also kept track of the pledging in terms of the intertemporal number of pledgers, cumulative pledged amount, project creators' updates and backers' comments whenever there was any change to the project. This real-time dataset allows us to uncover the pledging patterns, as well as the impact of the creators' updates.

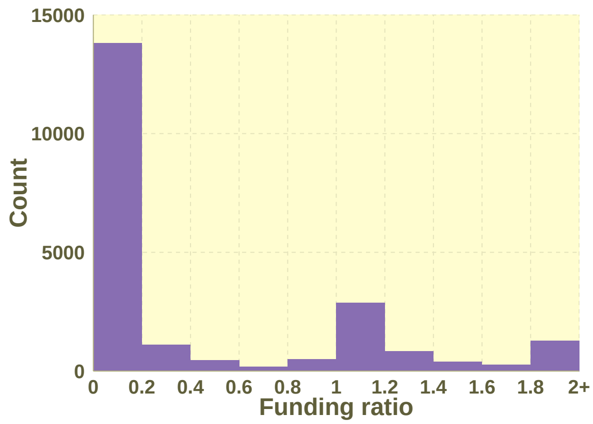
Table 1 Summary Statistics of Kickstarter Data

Project Attributes	Mean	St. Dev.	Min	Max
Goal ($)	67,009	1,401,462	1	100,000,000
Funding ratio	1.90	99.93	0	12,984
Duration (days)	33.63	11.66	1	60
# of updates per week	0.69	1.40	0	27.30

In total, our data includes 21,657 Kickstarter projects. Table 1 shows the summary statistics for all of those projects. The average project target in the sample was $67,009.[7] The average crowdfunding campaign duration was 33.63 days. We compute the funding ratio as the total pledged amount to the target. As shown in Figure 4, although 1,110 projects managed to collect over 200% of the goals, the majority of successful projects collected no more than 120% of their goals.

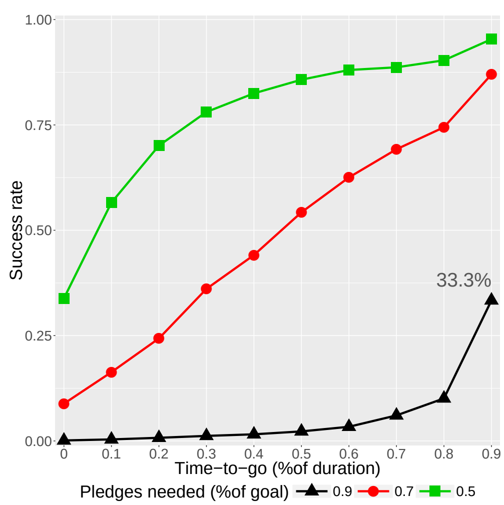
[7] Project targets may be in different currencies depending on where the project creators were located. We ignore the differences and assume that they were all measured in dollars.

Figure 4 Funding Ratio Distribution

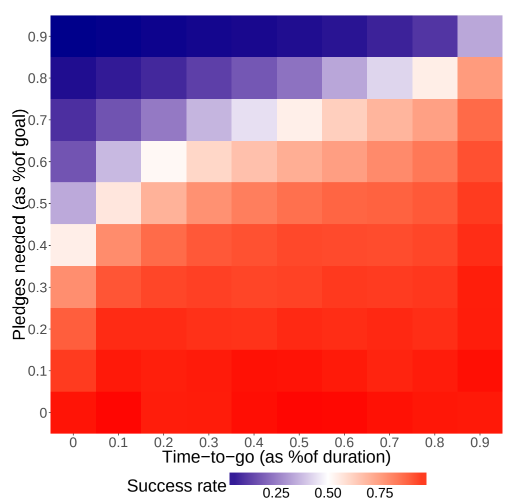


Project creators are allowed to make changes to their project over the course of their crowdfunding campaign. On average, project creators updated their project 0.69 times per week. We also observe significant variations in project update frequencies in our sample, ranging from 0 to 27.30 times per week. This variation allows us to study the effect of project updates on the project's likelihood of success.

Figure 5 Average Project Success Rate as a Function of Time-to-Go and Pledges Needed



(a) Success Rate by Time-to-Go



(b) Success Rate by Time-to-Go and Pledges Needed

We first display the project's success rate as a function of time-to-go and pledges needed by investigating the trajectories of all projects in the sample. Specifically, we break down time-to-go and pledges needed of each project into 10 stages, i.e., $0-10\%$, $10\%-20\%$, …, $90\%-100\%$, and compute the average success rate for those projects that fall into the same time stage and pledge stage. The results are summarized in Figure 5. The first observation is that, on average, a project is less likely to succeed with either a shorter remaining time given the same pledges needed, or a higher amount required to reach the target given the same time-to-go. This is consistent with our theoretical results on the pledging likelihood function $Q_t(n)$, as shown in Theorem 1. The empirical evidence also shows the importance of maintaining the momentum of the pledging, especially at the beginning of a campaign. For instance, Figure 5(b) shows that over 94% of projects will fail if they do not secure at least 10% of their goal after one-fifth of the time has passed. Secondly, we see from Figure 5(a) that the probability that a newly launched project will eventually reach its goal is around 33%, which is about the same as the expected success of the project shown in the numerical example in Section 4.4. In other words, in terms of the success rate, our numerical example is a "typical" project, and the effectiveness demonstrated in the numerical experiments further lends some credibility to the importance of stimulus policies in practical settings.

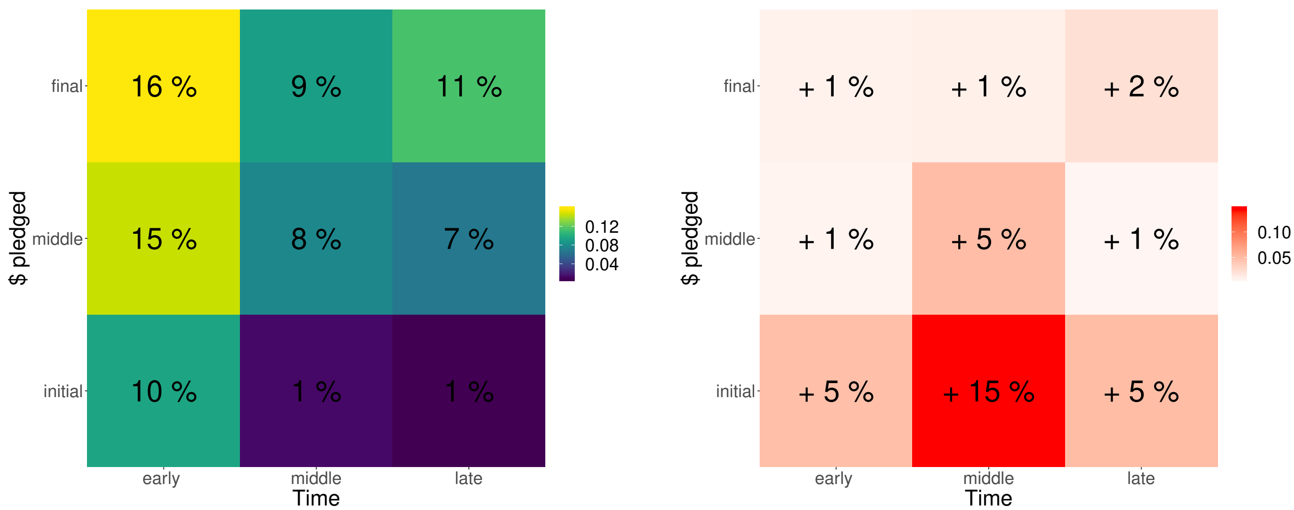
Table 2 Number of Updates in Successful and Failed Projects Per Week

	Project Count	Mean	Std. Error
Successful projects	6089	1.136	0.0179
Failed projects	15568	0.186	0.0039

Next, we study the effect of the creator's updates on the project's likelihood of success. The effectiveness of creators' updates is supported by our data as well. We find that, on average, successful projects made 1.136 updates per week, while failed ones made only 0.186 weekly updates and the difference is statistically significant (see Table 2).

It is complicated to quantify the exact benefit of updates because of data and identification issues. On the data side, the nature of updates, whether it is seeding, feature upgrade or LTO, may be hard to classify accurately using natural language algorithms. The identification could also be challenging because the difference in the number of updates may be a reflection of the creators' intrinsic motivation, which also affects campaign outcomes. A rigorous full-scale econometric model is beyond the scope of this paper. However, we provide some model-free evidence which demonstrates the importance of update timings. We divide campaigns along the time dimension into three stages of equal length: early, middle, and late. Similarly, using the ratio of the pledged amount to the project's target, we divide campaigns along the pledging-ratio dimension into three different stages, namely, initial, middle, and final. We calculate the average pledged amount (as % of the project goal) in each of nine categories. As seen from Figure 6(a), at the start of crowdfunding campaigns (i.e., early stage), the pledged amount of most projects increases at a steady pace. However,

Figure 6 Success Rate and Pledge Rate under Different Stages



(a) $ Pledged as % of Goal (b) Difference in Success Rates with and without Updates

the same cannot be said for the middle and late stages of the campaigns. For those projects where the cumulative pledged amount is greater than 33% of the funding target (i.e., middle and final stages), pledging rates remain relatively stable and healthy at around 8-9% on average. However, if a project is not progressing well (i.e., in the initial stage where the cumulative pledged amount is less than 33% of the funding target), the pledging comes to a nearly complete stop with the pledging rate stays at 1% during the middle or late stage of the campaigns. This is consistent with the examples in Figure 1 and underscores the importance of using stimulus strategies to keep the momentum.

We then investigate the effect of updates by comparing the outcomes of the projects for which creators made updates and the projects without updates in each of the nine categories. The results are summarized in Figure 6(b). In general, projects with updates have, on average, a higher success likelihood across all nine categories. The difference is greatest in the *middle* of a crowdfunding campaign and in the *initial* stage when the pledging amount is falling behind. In this scenario, the average success rate increases from 11.9% to 26.5% with updates. This scenario is consistent with our theoretical results in Theorems 2 and 4, where we show that when applying stimuli, it is optimal to do so only if the pledging slows down but not when the pledging is going smoothly. Moreover, the benefit in this particular scenario as the greatest is consistent with our results in Theorems 3(ii) and 5(ii), where we show that the benefit of stimuli is the most significant when the time-to-go is in an intermediate range.

6. Conclusion

Archimedes once said “give me a fulcrum, and I shall move the world.” In this paper, we study the optimal timing of contingently placing a “fulcrum” in the context of crowdfunding, with the potential of tilting the random pledging process from failure to success. In particular, we evaluate three different policies in detail, namely, seeding, feature upgrade, and limited-time offer. The three policies seek to encourage backers’ pledging in different ways. Seeding directly interacts with the pledging process by reducing the number of pledgers needed to reach the target and making the project more promising for future arrivals. With feature upgrade, project creators offer a superior version of the product with the hope of attracting more backers. This upgraded product is offered to future arrivals, as well as those who have already pledged. On the other hand, limited-time offer seeks to exploit the cascade effect in the pledging process by using promotional products to encourage potential backers to pledge early. However, unlike feature upgrade, promotional products are offered only during the LTO period.

Our analysis provides useful guidance on whether, when, and how project creators should apply these policies. We show that the potential benefits of the three policies vanish when the remaining time approaches either infinity or zero. It implies that these policies would be most effective in the *middle* of the pledging process. This is also consistent with the contingent nature of these policies. That is, project creators may want to “wait and see” and implement them only when the pledging trajectory is unsatisfactory in the early stage of the campaign.

On a related note, in practice, project creators may benefit from using a combination of the three policies. LTO is a proactive policy which induces customers to pledge early on. As shown in our analysis, the creator should end LTO if the project has already attracted a large number of pledgers while the remaining time is long. However, this does not guarantee that the project will succeed 100%. There is still a chance that the pledging process slows down after the end of LTO, and this is where the two reactive policies, i.e., seeding and feature upgrade, come in handy. Using a combination of proactive and reactive stimulus policies may lead to a higher profit that is unachievable with any policy alone.

Our study serves as the first step towards an understanding of the dynamics of crowdfunding projects. Future research may consider other types of information uncertainty beyond the project’s likelihood of success and may investigate their influence on the pledging dynamics. For instance, one salient concern from backers is whether and when project creators will successfully deliver the products (Mollick and Kuppuswamy 2014). This type of information asymmetry and uncertainty may affect backers’ pledging decisions even after the target is reached when the success uncertainty is resolved. To assure backers, it might be beneficial for the creators to deposit part of the funding beforehand to a trustworthy third-party, as a way to signal the quality of their products. On the empirical side, whether and to what extent backers take into account the probability of the final product's delivery needs to be verified with real data. In fact, the significance of various effects may well depend on project characteristics, and thus empirical analysis can offer useful guidance on the choice of policies for project creators.

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Online Appendix to "Contingent Stimulus in Crowdfunding"

OA.1. Proofs

Proof of Lemma 1. (i) Taking derivative of $H(q)$ w.r.t. q, we have

$$\frac{dH(q)}{dq}=\frac{c}{\theta q^2}f\left(\frac{1}{\theta}\left(p+c\cdot\left(\frac{1}{q}-1\right)\right)\right)>0.$$

(ii) Assumption 1(ii) is guaranteed by the fact that the support of the distribution $F(\cdot)$ is unbounded.

(iii) We prove Assumption 1(iii) by contradiction. Taking derivative of $H(\alpha q)/H(q)$ w.r.t. q, we have

$$\frac{d}{dq}\left(\frac{H(\alpha q)}{H(q)}\right)=\frac{H(\alpha q)}{H(q)}\left[\frac{c}{\theta\alpha q^2}\frac{f(\frac{1}{\theta}(p+c(\frac{1}{\alpha q}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{\alpha q}-1)))}-\frac{c}{\theta q^2}\frac{f(\frac{1}{\theta}(p+c(\frac{1}{q}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{q}-1)))}\right].$$

Suppose there exists a q' such that $\frac{d}{dq}\left(\frac{H(\alpha q')}{H(q')}\right)\leq 0$, which implies that $\frac{f(\frac{1}{\theta}(p+c(\frac{1}{\alpha q'}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{\alpha q'}-1)))}\leq \alpha\cdot\frac{f(\frac{1}{\theta}(p+c(\frac{1}{q'}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{q'}-1)))}$. Coupling with the IGFR property that $\frac{1}{\theta}\left[p+c(\frac{1}{\alpha q'}-1)\right]\frac{f(\frac{1}{\theta}(p+c(\frac{1}{\alpha q'}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{\alpha q'}-1)))}\geq \frac{1}{\theta}\left[p+c(\frac{1}{q'}-1)\right]\frac{f(\frac{1}{\theta}(p+c(\frac{1}{q'}-1)))}{\bar{F}(\frac{1}{\theta}(p+c(\frac{1}{q'}-1)))}$, we have $p+c(\frac{1}{\alpha q'}-1)\geq\frac{1}{\alpha}\left[p+c(\frac{1}{q'}-1)\right]$. A direct consequence of the preceding inequality is that $(p-c)\geq\frac{p-c}{\alpha}$, which contradicts with $0<\alpha<1$ and $p>c$. Thus, we obtain the desired result. □

Proof of Proposition 1. Suppose that a backer arrives with time-to-go $t>0$ and pledges needed $n\geq 1$. This focal backer would decide whether or not to pledge based on her expected project's success rate conditional on her pledging, i.e., $Q_t(n-1)$. Consider what happens in a small time interval δ, and we have

$$Q_t(n)=(1-\delta\lambda_t H(Q_t(n-1)))\cdot Q_{t-\delta}(n)+\delta\lambda_t H(Q_t(n-1))\cdot Q_{t-\delta}(n-1)+o(\delta).$$

Rearranging and taking the limit as $\delta\to 0$, we obtain Equation (2). With the boundary conditions, the solution to Equation (2), which is an ordinary differential equation solved by induction, is unique. □

Proof of Theorem 1. (i) We prove this by induction. First when $n=1$, because $Q_t(0)=1$, it is easy to verify that $Q_t(1)=1-\exp\left(-\int_0^t\lambda_s H(1)ds\right)$ is the unique solution of Equation (2). Hence $Q_t(1)$ increases in t, and $Q_t(1)<Q_t(0)$.

Now assume the statement is true for $n-1$ ($n\geq 2$), then for n:

$$\frac{\partial}{\partial t}\Big[Q_t(n-1)-Q_t(n)\Big]=\lambda_t\Big[H(Q_t(n-2))\left(Q_t(n-2)-Q_t(n-1)\right)-H(Q_t(n-1))\left(Q_t(n-1)-Q_t(n)\right)\Big].$$

Since $Q_t(n-2) - Q_t(n-1) > 0$, $\frac{\partial}{\partial t}\Big[Q_t(n-1) - Q_t(n)\Big] > \lambda_t H(Q_t(n-1))\big[Q_t(n-1) - Q_t(n)\big]$. Based on Grönwall's Inequality and the fact that $Q_t(n-1) - Q_t(n)\Big|_{t=0} = 0$, we have $Q_t(n-1) - Q_t(n) > 0$ for any $t > 0$. This also implies that $\frac{\partial Q_t(n)}{\partial t} > 0$. Therefore the statement is also true for n.

(ii) The inequality is equivalent to $\frac{Q_t(n)}{Q_t(n-1)} \le 1 - e^{-\bar{\lambda}t}$. Consider the function $e^{\bar{\lambda}t}Q_t(n)$. Taking the derivative w.r.t. t, we have

$$\frac{\partial(e^{\bar{\lambda}t}Q_t(n))}{\partial t} = \bar{\lambda}e^{\bar{\lambda}t}Q_t(n) + e^{\bar{\lambda}t}\frac{\partial Q_t(n)}{\partial t} \le \bar{\lambda}e^{\bar{\lambda}t}Q_t(n) + \bar{\lambda}e^{\bar{\lambda}t}[Q_t(n-1) - Q_t(n)] = \bar{\lambda}e^{\bar{\lambda}t}Q_t(n-1),$$

where the inequality is due to $\frac{\partial Q_t(n)}{\partial t} > 0$ and $\frac{\partial Q_t(n)}{\partial t} \le \bar{\lambda}[Q_t(n-1) - Q_t(n)]$, as implied by Equation (2). Integrating from 0 to t on both sides, we have

$$Q_t(n) \le \int_0^t \bar{\lambda}e^{-\bar{\lambda}(t-s)}Q_s(n-1)ds \le \bar{\lambda}Q_t(n-1)\int_0^t e^{-\bar{\lambda}(t-s)}ds = (1 - e^{-\bar{\lambda}t})Q_t(n-1).$$

where the second inequality is due to the increasing monotonicity of $Q_t(n-1)$ in t as shown in Theorem 1(i). Therefore, we conclude that $\frac{Q_t(n)}{Q_t(n-1)} \le 1 - e^{-\bar{\lambda}t}$.

(iii) We will prove that $\frac{Q_t(n)}{Q_t(n-1)}$ strictly increases in t and $\frac{H(Q_t(n))}{H(Q_t(n-1))}$ increases in t by induction. Consider first when $n = 1$. Because $\frac{Q_t(1)}{Q_t(0)} = Q_t(1)$ and $\frac{H(Q_t(1))}{H(Q_t(0))} = \frac{H(Q_t(1))}{H(1)}$, the monotonicity is guaranteed by part (i) and Assumption 1(i). Now assume that the monotonicity in t holds for $n-1$. We next show that $r_t(n) \equiv \frac{Q_t(n)}{Q_t(n-1)}$ strictly increases in t and $\varphi_t(n) \equiv \frac{H(Q_t(n))}{H(Q_t(n-1))}$ increases in t. First from part (i), we observe that $0 < r_t(n) < 1$ for $t > 0$. Taking the derivative of $r_t(n)$ w.r.t. t, we have

$$\begin{aligned}
\frac{\partial r_t(n)}{\partial t} &= \frac{\lambda_t H(Q_t(n-1))\,[Q_t(n-1) - Q_t(n)]}{Q_t(n-1)} - \frac{Q_t(n)\lambda_t H(Q_t(n-2))\,[Q_t(n-2) - Q_t(n-1)]}{Q_t^2(n-1)} \\
&= \lambda_t\left[H(Q_t(n-1))\left(1 - \frac{Q_t(n)}{Q_t(n-1)}\right) - \frac{Q_t(n)}{Q_t(n-1)}H(Q_t(n-2))\left(\frac{Q_t(n-2)}{Q_t(n-1)} - 1\right)\right] \\
&= \lambda_t\frac{Q_t(n)}{Q_t(n-1)}H_t(Q_t(n-2))\left[\frac{H(Q_t(n-1))}{H(Q_t(n-2))}\left(\frac{Q_t(n-1)}{Q_t(n)} - 1\right) - \left(\frac{Q_t(n-2)}{Q_t(n-1)} - 1\right)\right] \\
&= \lambda_t r_t(n)H(Q_t(n-2))\left[\varphi_t(n-1)\left(\frac{1}{r_t(n)} - 1\right) - \left(\frac{1}{r_t(n-1)} - 1\right)\right].
\end{aligned}$$

Suppose that there exists some t_1 such that $\frac{\partial r_t(n)}{\partial t}\Big|_{t=t_1} \le 0$. Then, there must exist some $t_2 \in (0, t_1)$ such that $\frac{\partial r_t(n)}{\partial t}\Big|_{t=t_2} > 0$. Otherwise, if $\frac{\partial r_t(n)}{\partial t} \le 0$ for all $t < t_1$, then $\lim_{t\to 0} r_t(n) = 0 \ge r_{t_1}(n)$, which contradicts with the fact that $Q_t(n) > 0$. Due to the continuity of $\frac{\partial r_t(n)}{\partial t}$, there exists some $t_3 \in [t_2, t_1)$, such that $\frac{\partial r_t(n)}{\partial t}\Big|_{t=t_3} = 0$. That is,

$$\varphi_{t_3}(n-1)\left(\frac{1}{r_{t_3}(n)} - 1\right) - \left(\frac{1}{r_{t_3}(n-1)} - 1\right) = 0.$$

Because $\varphi_t(n-1)$ strictly increases in t and $r_t(n-1)$ increases in t, and $r_t(n)$ decreases in t between $[t_3, t_1]$, we have

$$\varphi_{t_1}(n-1)\left(\frac{1}{r_{t_1}(n)}-1\right)-\left(\frac{1}{r_{t_1}(n-1)}-1\right)>\varphi_{t_3}(n-1)\left(\frac{1}{r_{t_3}(n)}-1\right)-\left(\frac{1}{r_{t_3}(n-1)}-1\right)=0,$$

which implies that $\frac{\partial r_t(n)}{\partial t}|_{t=t_1}>0$. However, this contradicts with the preceding statement that $\frac{\partial r_t(n)}{\partial t}|_{t=t_1}\le 0$. Therefore, we conclude that $\frac{\partial r_t(n)}{\partial t}>0$ for any $t>0$.

Next we show that $\frac{H(Q_t(n))}{H(Q_t(n-1))}$ increases in t. For any $t'>t$, we have

$$H(Q_{t'}(n))=H\Big(\frac{Q_{t'}(n)}{Q_{t'}(n-1)}Q_{t'}(n-1)\Big)\ge H\Big(\frac{Q_t(n)}{Q_t(n-1)}Q_{t'}(n-1)\Big),$$

where the inequality is due to the increasing monotonicity of $\frac{Q_t(n)}{Q_t(n-1)}$ in t and Assumption 1(i). Due to Assumption 1(iii) and Theorem 1(i), we have

$$\frac{H(Q_{t'}(n))}{H(Q_{t'}(n-1))}\ge\frac{H\left(\frac{Q_t(n)}{Q_t(n-1)}Q_{t'}(n-1)\right)}{H(Q_{t'}(n-1))}\ge\frac{H\left(\frac{Q_t(n)}{Q_t(n-1)}Q_t(n-1)\right)}{H(Q_t(n-1))}=\frac{H(Q_t(n))}{H(Q_t(n-1))}.$$

We hence prove the increasing monotonicity of $\frac{H(Q_t(n))}{H(Q_t(n-1))}$ in t.

For the monotonicity in n, because we have shown that $\frac{\partial r_t(n)}{\partial t}>0$ for any $t>0$, $\varphi_t(n-1)\big(\frac{1}{r_t(n)}-1\big)-\big(\frac{1}{r_t(n-1)}-1\big)>0$. Since $\varphi_t(n-1)\le 1$, we have $r_t(n)<r_t(n-1)$, i.e., $\frac{Q_t(n)}{Q_t(n-1)}<\frac{Q_t(n-1)}{Q_t(n-2)}$. A direct consequence is that $\frac{H(Q_t(n))}{H(Q_t(n-1))}=\frac{H\left(\frac{Q_t(n)}{Q_t(n-1)}Q_t(n-1)\right)}{H(Q_t(n-1))}<\frac{H\left(\frac{Q_t(n-1)}{Q_t(n-2)}Q_t(n-1)\right)}{H(Q_t(n-1))}$. Due to Assumption 1(iii) and part (i), we have

$$\frac{H\left(\frac{Q_t(n-1)}{Q_t(n-2)}Q_t(n-1)\right)}{H(Q_t(n-1))}\le\frac{H\left(\frac{Q_t(n-1)}{Q_t(n-2)}Q_t(n-2)\right)}{H(Q_t(n-2))}=\frac{H(Q_t(n-1))}{H(Q_t(n-2))}.$$

Therefore, we conclude that $\frac{H(Q_t(n))}{H(Q_t(n-1))}\le\frac{H(Q_t(n-1))}{H(Q_t(n-2))}$ for any $t>0$.

(iv) For any $n\ge 1$, we have

$$\frac{Q_{t+h}(n)}{Q_t(n)}=\frac{Q_{t+h}(n)}{Q_{t+h}(n-1)}\cdot\frac{Q_{t+h}(n-1)}{Q_t(n-1)}\cdot\frac{Q_t(n-1)}{Q_t(n)}>\frac{Q_{t+h}(n-1)}{Q_t(n-1)},$$

where the inequality is due to $\frac{Q_{t+h}(n)}{Q_{t+h}(n-1)}>\frac{Q_t(n)}{Q_t(n-1)}$ as shown in Theorem 1(iii).

Last, we prove the monotonicty in t by induction. When $n=0$ the statement is obvious. Suppose that the statement is true for $n-1$, where $n\ge 1$. Then for any $t_2>t_1\ge 0$,

$$\frac{H(Q_{t_2}(n-1))}{H(Q_{t_2+h}(n-1))}=\frac{H\left(\frac{Q_{t_2}(n-1)}{Q_{t_2+h}(n-1)}Q_{t_2+h}(n-1)\right)}{H(Q_{t_2+h}(n-1))}\ge\frac{H\left(\frac{Q_{t_1}(n-1)}{Q_{t_1+h}(n-1)}Q_{t_2+h}(n-1)\right)}{H(Q_{t_2+h}(n-1))}.$$

Based on Assumption 1(iii), we have

$$\frac{H(Q_{t_2}(n-1))}{H(Q_{t_2+h}(n-1))}\ge\frac{H\left(\frac{Q_{t_1}(n-1)}{Q_{t_1+h}(n-1)}Q_{t_1+h}(n-1)\right)}{H(Q_{t_1+h}(n-1))}=\frac{H(Q_{t_1}(n-1))}{H(Q_{t_1+h}(n-1))},$$

due to $Q_{t_2+h}(n-1) \geq Q_{t_1+h}(n-1)$ and $\frac{Q_{t_1}(n-1)}{Q_{t_1+h}(n-1)} \leq 1$. Thus $\frac{H(Q_t(n-1))}{H(Q_{t+h}(n-1))}$ increases in t. Next we take derivative of $\frac{Q_{t+h}(n)}{Q_t(n)}$ w.r.t. t:

$$\begin{aligned}\frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)} &= \frac{H_{t+h}(n)\left[Q_{t+h}(n-1)-Q_{t+h}(n)\right]}{Q_t(n)} - \frac{Q_{t+h}(n)}{Q_t(n)}\frac{H_t(n)\left[Q_t(n-1)-Q_t(n)\right]}{Q_t(n)} \\ &= H_{t+h}(n)\frac{Q_t(n-1)}{Q_t(n)}\left[\frac{Q_{t+h}(n-1)}{Q_t(n-1)} - \frac{Q_{t+h}(n)}{Q_t(n)}\left(\frac{Q_t(n)}{Q_t(n-1)} + \frac{H_t(n)}{H_{t+h}(n)}\left(1-\frac{Q_t(n)}{Q_t(n-1)}\right)\right)\right] \\ &= H_{t+h}(n)\frac{Q_t(n-1)}{Q_t(n)}\left[\frac{Q_{t+h}(n-1)}{Q_t(n-1)} - \frac{Q_{t+h}(n)}{Q_t(n)}\left[1-\left(1-\frac{H(Q_t(n-1))}{H(Q_{t+h}(n-1))}\right)\left(1-\frac{Q_t(n)}{Q_t(n-1)}\right)\right]\right].\end{aligned}$$

Note that $\frac{Q_{t+h}(n)}{Q_t(n)} \to 1$ when $t \to \infty$, and $\frac{Q_{t+h}(n)}{Q_t(n)} > 1$ for any finite t. Thus $\frac{Q_{t+h}(n)}{Q_t(n)}$ decreases in t when t is sufficiently large. Suppose that $\frac{Q_{t+h}(n)}{Q_t(n)}$ is not monotonically decreasing in t. Then there must exist a $t_3 > t_2 > t_1$ such that $\frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)}\Big|_{t=t_1} = 0$ and $\frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)} > 0$ for any $t \in (t_2, t_3)$. However, $\frac{Q_{t+h}(n-1)}{Q_t(n-1)}$ decreases in t by the induction assumption. We also know that $\frac{H(Q_t(n-1))}{H(Q_{t+h}(n-1))}$ increases in t, which would imply that $1-\left(1-\frac{H(Q_t(n-1))}{H(Q_{t+h}(n-1))}\right)\left(1-\frac{Q_t(n)}{Q_t(n-1)}\right)$ increases in t over (t_2, t_3). Consequently $\frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)} \leq \frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)}\Big|_{t=t_2} = 0$ for $t \in (t_2, t_3)$, which contradicts with $\frac{\partial}{\partial t}\frac{Q_{t+h}(n)}{Q_t(n)} > 0$ for any $t \in (t_2, t_3)$. We thus obtain the announced results. □

Proof of Lemma 2. (i) Taking derivative of $\frac{H^{\theta_a}(q)}{H^{\theta_b}(q)}$ w.r.t. q, we have

$$\frac{\partial}{\partial q}\left(\frac{H^{\theta_a}(q)}{H^{\theta_b}(q)}\right) = \frac{H^{\theta_a}(q)}{H^{\theta_b}(q)}\frac{c}{q^2}\Big[\frac{1}{\theta_a}\frac{f\big(\frac{1}{\theta_a}(p+c(\frac{1}{q}-1))\big)}{\bar{F}\big(\frac{1}{\theta_a}(p+c(\frac{1}{q}-1))\big)} - \frac{1}{\theta_b}\frac{f\big(\frac{1}{\theta_b}(p+c(\frac{1}{q}-1))\big)}{\bar{F}\big(\frac{1}{\theta_b}(p+c(\frac{1}{q}-1))\big)}\Big].$$

Because $\theta_a < \theta_b$ and Assumption 1, we conclude that $\frac{\partial}{\partial q}\left(\frac{H^{\theta_a}(q)}{H^{\theta_b}(q)}\right) > 0$. Thus, we obtain the announced results.

(ii) Taking derivative of $\frac{H^{p_a}(q)}{H^{p_b}(q)}$ w.r.t. q, we have

$$\begin{aligned}\frac{\partial}{\partial q}\left(\frac{H^{p_a}(q)}{H^{p_b}(q)}\right) &= \frac{1}{[H^{p_b}(q)]^2}\Bigg[\frac{c}{\theta q^2}f\left(\frac{1}{\theta}\left[p_a + c\cdot\left(\frac{1}{q}-1\right)\right]\right)\bar{F}\left(\frac{1}{\theta}\left[p_b + c\cdot\left(\frac{1}{q}-1\right)\right]\right) \\ &\quad -\frac{c}{\theta q^2}f\left(\frac{1}{\theta}\left[p_b + c\cdot\left(\frac{1}{q}-1\right)\right]\right)\bar{F}\left(\frac{1}{\theta}\left[p_a + c\cdot\left(\frac{1}{q}-1\right)\right]\right)\Bigg] \\ &= \frac{H^{p_a}(q)}{H^{p_b}(q)}\frac{c}{\theta q^2}\left[\frac{f\left(\frac{1}{\theta}\left[p_a + c\cdot\left(\frac{1}{q}-1\right)\right]\right)}{\bar{F}\left(\frac{1}{\theta}\left[p_a + c\cdot\left(\frac{1}{q}-1\right)\right]\right)} - \frac{f\left(\frac{1}{\theta}\left[p_b + c\cdot\left(\frac{1}{q}-1\right)\right]\right)}{\bar{F}\left(\frac{1}{\theta}\left[p_b + c\cdot\left(\frac{1}{q}-1\right)\right]\right)}\right].\end{aligned}$$

Due to $p_a > p_b$ and that $\frac{f(v)}{\bar{F}(v)}$ increases in v, we conclude that $\frac{\partial}{\partial q}\left(\frac{H^{p_a}(q)}{H^{p_b}(q)}\right) > 0$. □

Proof of Proposition 2. Denote $x_t(n) = \frac{Q_t^a(n)}{Q_t^b(n)}$ and $\gamma_t(n) = \frac{H^a(Q_t^a(n))}{H^b(Q_t^b(n))}$. We first prove that $x_t(n)$ and $\gamma_t(n)$ increase in t by induction. When $n=0$, $x_t(0) = 1$ and $\gamma_t(0) = \frac{H^a(1)}{H^b(1)}$, and thus the monotonicity holds trivially. Now suppose that the statement is true for $n-1$. Taking the derivative of $x_t(n)$ w.r.t. t, we have

$$\frac{dx_t(n)}{dt} = \frac{\lambda_t H^a(Q_t^a(n-1))\left[Q_t^a(n-1)-Q_t^a(n)\right]}{Q_t^b(n)} - \frac{Q_t^a(n)\lambda_t H^b(Q_t^b(n-1))\left[Q_t^b(n-1)-Q_t^b(n)\right]}{[Q_t^b(n)]^2}$$

$$
\begin{aligned}
&= \lambda_t \frac{Q_t^a(n)}{Q_t^b(n)} \left[H^a(Q_t^a(n-1)) \left(\frac{Q_t^a(n-1)}{Q_t^a(n)} - 1 \right) - H^b(Q_t^b(n-1)) \left(\frac{Q_t^b(n-1)}{Q_t^b(n)} - 1 \right) \right] \\
&= \lambda_t \left[H^a(Q_t^a(n-1)) \left(\frac{Q_t^a(n-1)}{Q_t^b(n-1)} \frac{Q_t^b(n-1)}{Q_t^b(n)} - \frac{Q_t^a(n)}{Q_t^b(n)} \right) - H^b(Q_t^b(n-1)) \frac{Q_t^a(n)}{Q_t^b(n)} \left(\frac{Q_t^b(n-1)}{Q_t^b(n)} - 1 \right) \right] \\
&= \lambda_t H^a(Q_t^a(n-1)) \frac{Q_t^b(n-1)}{Q_t^b(n)} \cdot \\
&\quad \left[\frac{Q_t^a(n-1)}{Q_t^b(n-1)} - \frac{Q_t^b(n)}{Q_t^b(n-1)} \frac{Q_t^a(n)}{Q_t^b(n)} - \frac{H^b(Q_t^b(n-1))}{H^a(Q_t^a(n-1))} \frac{Q_t^a(n)}{Q_t^b(n)} \left(1 - \frac{Q_t^b(n)}{Q_t^b(n-1)} \right) \right] \\
&= \lambda_t H^a(Q_t^a(n-1)) \frac{Q_t^b(n-1)}{Q_t^b(n)} \left[x_t(n-1) - \frac{Q_t^b(n)}{Q_t^b(n-1)} x_t(n) - \frac{x_t(n)}{\gamma_t(n-1)} \left(1 - \frac{Q_t^b(n)}{Q_t^b(n-1)} \right) \right] \\
&= \lambda_t H^a(Q_t^a(n-1)) \frac{Q_t^b(n-1)}{Q_t^b(n)} \left[x_t(n-1) - x_t(n) - \left(\frac{1}{\gamma_t(n-1)} - 1 \right) \left(1 - \frac{Q_t^b(n)}{Q_t^b(n-1)} \right) x_t(n) \right].
\end{aligned}
$$

Denote $L(t) = x_t(n-1) - \left[1 + \left(\frac{1}{\gamma_t(n-1)} - 1\right)\left(1 - \frac{Q_t^b(n)}{Q_t^b(n-1)}\right)\right] x_t(n)$. Next we show that if there exists some t_1 such that $L(t_1) < 0$, there must exist some $t_2 \in (0, t_1)$ such that $L(t_2) \geq 0$. Consider the following two cases.

(1) $\lim_{t\to 0} \gamma_t(n-1) = 0$. Using L' Hopital's rule, we have

$$
\begin{aligned}
\lim_{t\to 0} x_t(n) &= \lim_{t\to 0} \frac{\frac{\partial Q_t^a(n)}{\partial t}}{\frac{\partial Q_t^b(n)}{\partial t}} = \lim_{t\to 0} \frac{\lambda_t H^a(Q_t^a(n-1))\,(Q_t^a(n-1) - Q_t^a(n))}{\lambda_t H^b(Q_t^b(n-1))\,(Q_t^b(n-1) - Q_t^b(n))} \\
&= \lim_{t\to 0} \gamma_t(n-1) \cdot \frac{Q_t^a(n-1)\left[1 - \frac{Q_t^a(n)}{Q_t^a(n-1)}\right]}{Q_t^b(n-1)\left[1 - \frac{Q_t^b(n)}{Q_t^b(n-1)}\right]} = 0.
\end{aligned}
$$

Suppose there exists some $t_1 > 0$ such that $\frac{\partial x_t(n)}{\partial t}|_{t=t_1} < 0$. Then, there must exist some $t_2 \in (0, t_1)$ such that $\frac{\partial x_t(n)}{\partial t}|_{t=t_2} \geq 0$; otherwise $x_t(n)$ decreases within $(0, t_1]$, which implies that $x_{t_1} \leq \lim_{t\to 0} x_t(n) = 0$. This contradicts with the fact that $x_t(n) > 0$ for $t > 0$.

(2) $\lim_{t\to 0} \gamma_t(n-1) > 0$. Because of $\lim_{t\to 0} \frac{Q_t^b(n)}{Q_t^b(n-1)} = 0$ as shown in Theorem 1(ii), $\lim_{t\to 0} L(t) = \lim_{t\to 0} x_t(n-1) - \frac{x_t(n)}{\gamma_t(n-1)}$. Again using L' Hopital's rule, we have

$$
\lim_{t\to 0} L(t) = \lim_{t\to 0} x_t(n-1) - \lim_{t\to 0} \frac{1}{\gamma_t(n-1)} \frac{H^a(Q_t^a(n-1)) \cdot Q_t^a(n-1)\left[1 - \frac{Q_t^a(n)}{Q_t^a(n-1)}\right]}{H^b(Q_t^b(n-1)) \cdot Q_t^b(n-1)\left[1 - \frac{Q_t^b(n)}{Q_t^b(n-1)}\right]} = 0.
$$

Suppose there exists some $t_1 > 0$ such that $\left.\frac{\partial x_t(n)}{\partial t}\right|_{t=t_1} < 0$, i.e., $L(t_1) < 0$. Then, there must exist some $t_2 \in (0, t_1)$ such that $\frac{\partial x_t(n)}{\partial t}|_{t=t_2} \geq 0$; otherwise, $x_t(n)$ decreases within $(0, t_1]$. Combined with the results that $x_t(n-1)$, $\gamma_t(n-1)$ and $\frac{Q_t^b(n)}{Q_t^b(n-1)}$ all increase in t, we have that $L(t)$ increases in $(0, t_1]$, which suggests that $L(t_1) \geq \lim_{t\to 0} L(t) = 0$. This contradicts with the preceding argument that $L(t_1) < 0$.

Therefore, if there exists some t_1 such that $L(t_1) < 0$, there must exist a $t_2 \in (0, t_1)$ such that $L(t_2) \geq 0$. Coupling with the continuity of $L(t)$, there exists a $t_3 \in [t_2, t_1)$ such that $L(t_3) = 0$.

This implies that $x_t(n)$ strictly decreases within $(t_3, t_1]$. Combined with the results that $x_t(n-1)$, $\gamma_t(n-1)$ and $\frac{Q_t^b(n)}{Q_t^b(n-1)}$ all increase in t, we have that $L(t)$ increases in $(t_3, t_1]$, which suggests that $L(t_1) \geq L(t_3) = 0$. This contradicts with the preceding argument that $L(t_1) < 0$. Therefore, we conclude that $\frac{\partial x_t(n)}{\partial t} \geq 0$ for all $t > 0$.

Given that $x_t(n)$ increases in t, for any $\delta > 0$, we have

$$\frac{H^b(Q_{t+\delta}^a(n))}{H^b(Q_{t+\delta}^b(n))} = \frac{H^b\left(x_{t+\delta}(n)Q_{t+\delta}^b(n)\right)}{H^b(Q_{t+\delta}^b(n))} \geq \frac{H^b\left(x_t(n)Q_{t+\delta}^b(n)\right)}{H^b(Q_{t+\delta}^b(n))} \geq \frac{H^b\left(x_t(n)Q_t^b(n)\right)}{H^b(Q_t^b(n))} = \frac{H^b(Q_t^a(n))}{H^b(Q_t^b(n))},$$

where the second inequality is a result of Assumption 1(iii). Hence $\frac{H^b(Q_t^a(n))}{H^b(Q_t^b(n))}$ increases in t. Combining with the assumption that $\frac{H^a(q)}{H^b(q)}$ increases in q, we conclude that $\frac{H^a(Q_t^a(n))}{H^b(Q_t^b(n))}$ increases in t.

Next we prove that $x_t(n)$ and $\gamma_t(n)$ decrease in n. Because $x_t(n)$ increases in t, we have $L(t) = x_t(n-1) - x_t(n) - \left(\frac{1}{\gamma_t(n-1)} - 1\right)\left(1 - \frac{Q_t^b(n)}{Q_t^b(n-1)}\right)x_t(n) > 0$ for any $t > 0$. Coupling with the results that $\gamma_t(n-1) \leq 1$, $x_t(n) \geq 0$ and Theorem 1(i), we thus have that $x_t(n-1) > x_t(n)$.

Given that $\frac{Q_t^a(n)}{Q_t^b(n)}$ decreases in n, we have

$$\frac{H^a(Q_t^a(n))}{H^a(Q_t^b(n))} = \frac{H^a\left(\frac{Q_t^a(n)}{Q_t^b(n)}Q_t^b(n)\right)}{H^a(Q_t^b(n))} \leq \frac{H^a\left(\frac{Q_t^a(n-1)}{Q_t^b(n-1)}Q_t^b(n)\right)}{H^a(Q_t^b(n))} \leq \frac{H^a\left(\frac{Q_t^a(n-1)}{Q_t^b(n-1)}Q_t^b(n-1)\right)}{H^a(Q_t^b(n-1))} = \frac{H^a(Q_t^a(n-1))}{H^a(Q_t^b(n-1))},$$

where the second inequality is a result of Assumption 1(iii). Moreover, $\frac{H^a(Q_t^b(n))}{H^b(Q_t^b(n))} \leq \frac{H^a(Q_t^b(n-1))}{H^b(Q_t^b(n-1))}$ because of the assumption that $\frac{H^a(q)}{H^b(q)}$ increases in q. Therefore, we have

$$\frac{H^a(Q_t^a(n))}{H^b(Q_t^b(n))} = \frac{H^a(Q_t^a(n))}{H^a(Q_t^b(n))}\frac{H^a(Q_t^b(n))}{H^b(Q_t^b(n))} \leq \frac{H^a(Q_t^a(n-1))}{H^a(Q_t^b(n-1))}\frac{H^a(Q_t^b(n-1))}{H^b(Q_t^b(n-1))} = \frac{H^a(Q_t^a(n-1))}{H^b(Q_t^b(n-1))}.$$

We thus complete the proof. □

For notational convenience, We denote $H_t(n) \equiv H(Q_t(n-1))$ in the following proofs.

Proof of Theorem 2. Denote $J_t^s(n)$ as the optimal expected profit at state (t, n) assuming that the seeding stimulus has not been activated. We prove that $\tau^s(n)$ is given by

$$\tau^s(n) = \sup\Big\{t : H_t((n-n_0)^+) \cdot Q_t((n-n_0-1)^+) - \left[H_t((n-n_0)^+) - H_t(n)\right] \cdot Q_t((n-n_0)^+) \geq H_t(n)\frac{J_t^s(n-1)}{G+B-R}\Big\}. \tag{OA.1}$$

Expected profit $J_t^s(n)$ at state (t, n) is given by

- when $n \geq 1$ and $t \leq \tau^s(n)$, $J_t^s(n) = (G+B-R) \cdot Q_t((n-n_0)^+)$;
- when $t > \tau^s(n)$, $J_t^s(n)$ is given by

$$\frac{\partial J_t^s(n)}{\partial t} = \lambda_t H_t(n)\left[J_t^s(n-1) - J_t^s(n)\right], \tag{OA.2}$$

with boundary conditions $J_{\tau^s(n)}^s(n) = (G+B-R) \cdot Q_{\tau^s(n)}((n-n_0)^+)$ and $J_t^s(0) = G+B$.

Denote $l_t(n) \equiv \frac{J_t^s(n)}{Q_t((n-n_0)^+)}$. We add to the statement that $l_t(n)$ increases in t, and prove by induction. When $n \le n_0$, the optimal expected profit is given by $J_t^s(n) = (G+B)\cdot Q_t(n) + (G+B-R)\cdot(1-Q_t(n))$. That is, the creator's optimal policy is to hold off until right before the deadline, and to activate "seeding" if no backer pledges by then. It is not hard to verify that it is the unique solution to the differential equation characterized by Equation (OA.2). We thus conclude that $l_t(n) = J_t^s(n)$ increases in t for $n \le n_0$.

Assume that the statement is true for $n-1$, where $n \ge n_0+1$. Next, we seek to derive $J_t^s(n)$ by showing that the creator's optimal policy is to "seed" immediately when $t \le \tau_t^s(n)$ and to hold off when $t > \tau_t^s(n)$. We can rewrite the inequality within the curly brackets in Equation (OA.1) as follows.

$$1 + \left[\frac{H_t(n-n_0)}{H_t(n)} - 1\right]\left[1 - \frac{Q_t(n-n_0)}{Q_t(n-n_0-1)}\right] \ge \frac{J_t^s(n-1)}{(G+B-R)\cdot Q_t(n-n_0-1)}.$$

RHS of the inequality increases in t because $l_t(n-1)$ increases in t, while LHS decreases in t due to Theorem 1(iii). Therefore, for any $t \le \tau^s(n)$, the inequality within the curly brackets in Equation (OA.1) holds; whereas the direction of the inequality is flipped for any $t > \tau^s(n)$.

Suppose there exists some $t_1 > \tau^s(n)$ such that the creator's optimal policy is to activate the seeding stimulus immediately, i.e., $J_{t_1}^s(n) = (G+B-R)\cdot Q_{t_1}(n-n_0)$. Comparing the case without activating the stimulus at time t_1, we have

$$\begin{aligned} J_{t_1}^s(n) &\ge \lambda_{t_1} H_{t_1}(n)\delta \cdot J_{t_1-\delta}^s(n-1) + (1-\lambda_{t_1} H_{t_1}(n)\delta)\cdot J_{t_1-\delta}^s(n) + o(\delta) \\ &\ge \lambda_{t_1} H_{t_1}(n)\delta \cdot J_{t_1-\delta}^s(n-1) + (1-\lambda_{t_1} H_{t_1}(n)\delta)\cdot (G+B-R)\cdot Q_{t_1-\delta}(n-n_0) + o(\delta). \end{aligned}$$

Plugging $Q_{t_1}(n-n_0) = (1-\lambda_{t_1} H_{t_1}(n-n_0)\delta)\cdot Q_{t_1-\delta}(n-n_0) + \lambda_{t_1} H_{t_1}(n-n_0)\delta \cdot Q_{t_1-\delta}(n-n_0-1) + o(\delta)$ into $J_{t_1}^s(n)$ in the inequality above, rearranging and taking the limit as $\delta \to 0$, we have

$$(G+B-R)\Big[H_{t_1}(n-n_0)Q_{t_1}(n-n_0-1) - (H_{t_1}(n-n_0) - H_{t_1}(n))Q_{t_1}(n-n_0)\Big] \ge H_{t_1}(n)J_{t_1}^s(n-1).$$

This contradicts with the fact that $t_1 > \tau^s(n)$. Therefore, the creator's optimal policy is to hold off when $t > \tau^s(n)$, i.e., $J_t^s(n) > (G+B-R)\cdot Q_t(n-n_0)$. Consider what happens in a small time interval δ, we have

$$J_t^s(n) = (1-\delta\lambda_t H_t(n))\cdot J_{t-\delta}^s(n) + \delta\lambda_t H_t(n)\cdot J_{t-\delta}^s(n-1) + o(\delta).$$

Rearranging and taking the limit as $\delta \to 0$, we obtain Equation (OA.2).

We next show that the creator's optimal policy is to "seed" immediately when $t < \tau^s(n)$. Suppose that there exists some $t_2 < \tau^s(n)$, such that $J_t^s(n) = (G+B-R)\cdot Q_t(n-n_0)$ for any $t \le t_2$, and

$J_t^s(n) > (G+B-R)\cdot Q_t(n-n_0)$ when $t\in(t_2,t_2+h]$. (Because $J_0^s(n)=0$ for any $n>n_0$, we can always find some t_2 such that $J_t^s(n)=(G+B-R)\cdot Q_t(n-n_0)$ for any $t\le t_2$.) Then, for any $t\in(t_2,t_2+h]$

$$J_{t+\delta}^s(n) = (1-\lambda_{t+\delta}H_{t_2+\delta}(n)\delta)\cdot J_t^s(n) + \lambda_{t+\delta}H_{t+\delta}(n)\delta\cdot J_t^s(n-1) + o(\delta).$$

Let $\delta\to 0$, we obtain $\frac{\partial J_t^s(n)}{\partial t} = \lambda_t H_t(n)\left[J_t^s(n-1) - J_t^s(n)\right]$ over interval $(t_2,t_2+h]$. According to Equation (OA.1), $J_t^s(n-1)\le \frac{G+B-R}{H_t(n)}\left[H_t(n-n_0)Q_t(n-n_0-1) - \left(H_t(n-n_0)-H_t(n)\right)Q_t(n-n_0)\right]$. Also because $J_t^s(n) > (G+B-R)\cdot Q_t(n-n_0)$ when $t\in(t_2,t_2+h]$, we have

$$\begin{aligned}
&\frac{\partial J_t^s(n)}{\partial t}\\
&< \lambda_t(G+B-R)\left[H_t(n-n_0)Q_t(n-n_0-1) - \left(H_t(n-n_0)-H_t(n)\right)Q_t(n-n_0) - H_t(n)Q_t(n-n_0)\right]\\
&= \lambda_t(G+B-R)\cdot H_t(n-n_0)\left[Q_t(n-n_0-1) - Q_t(n-n_0)\right].
\end{aligned}$$

However, we know from Equation (2) that $\frac{\partial}{\partial t}\left[(G+B-R)\cdot Q_t(n-n_0)\right] = \lambda_t(G+B-R)\cdot H_t(n-n_0)\left[Q_t(n-n_0-1)-Q_t(n-n_0)\right]$. Therefore, $\frac{\partial}{\partial t}\left[J_t^s(n) - (G+B-R)\cdot Q_t(n-n_0)\right] < 0$ for any $t\in(t_2,t_2+h]$. Since $\left[J_t^s(n)-(G+B-R)\cdot Q_t(n-n_0)\right]\Big|_{t=t_2} = 0$, we obtain that $J_t^s(n) < (G+B-R)\cdot Q_t(n-n_0)$ when $t\in(t_2,t_2+h]$. This contradicts with the assumption we made earlier. Hence, the creator's optimal policy is to "seed" immediately for any $t<\tau_t^s(n)$, i.e., $J_t^s(n)=(G+B-R)\cdot Q_t(n-n_0)$ for any $t<\tau^s(n)$.

Lastly, we show that $l_t(n)$ is an increasing function of t. This is obvious when $t\le\tau^s(n)$, as $\frac{J_t^s(n)}{Q_t(n-n_0)} = G+B-R$. When $t>\tau^s(n)$, taking the derivative of $l_t(n)$ w.r.t. t, we have

$$\begin{aligned}
\frac{\partial l_t(n)}{\partial t} &= \frac{\lambda_t H_t(n)\left[J_t^s(n-1)-J_t^s(n)\right]}{Q_t(n-n_0)} - \frac{\lambda_t H_t(n-n_0)J_t^s(n)\left[Q_t(n-n_0-1)-Q_t(n-n_0)\right]}{[Q_t(n-n_0)]^2}\\
&= \lambda_t\left\{H_t(n)\left[\frac{J_t^s(n-1)}{Q_t(n-n_0-1)}\frac{Q_t(n-n_0-1)}{Q_t(n-n_0)} - \frac{J_t^s(n)}{Q_t(n-n_0)}\right] - H_t(n-n_0)\frac{J_t^s(n)}{Q_t(n-n_0)}\left[\frac{Q_t(n-n_0-1)}{Q_t(n-n_0)}-1\right]\right\}\\
&= \lambda_t H_t(n)\frac{Q_t(n-n_0-1)}{Q_t(n-n_0)}\left\{l_t(n-1) - l_t(n)\frac{Q_t(n-n_0)}{Q_t(n-n_0-1)} - l_t(n)\frac{H_t(n-n_0)}{H_t(n)}\left[1-\frac{Q_t(n-n_0)}{Q_t(n-n_0-1)}\right]\right\}\\
&= \lambda_t H_t(n)\frac{Q_t(n-n_0-1)}{Q_t(n-n_0)}\left[l_t(n-1) - l_t(n) - \left(\frac{H_t(n-n_0)}{H_t(n)}-1\right)\left(1-\frac{Q_t(n-n_0)}{Q_t(n-n_0-1)}\right)l_t(n)\right].
\end{aligned}$$

Notice that $J_t^s(n) > (G+B-R)\cdot Q_t(n-n_0)$ when $t>\tau^s(n)$, and thus $l_t(n) > G+B-R$ when $t>\tau^s(n)$. Suppose that there exists some $t_3>\tau^s(n)$ such that $\frac{\partial l_t(n)}{\partial t}\Big|_{t=t_3} < 0$. Then, there must be some $t_4\in(\tau^s(n),t_3)$, such that $\frac{\partial l_t(n)}{\partial t}|_{t=t_4}\ge 0$; otherwise, $\frac{\partial l_t(n)}{\partial t}<0$ for any $\tau_s(n)<t\le t_3$, leading to $l_{t_3} < l_{\tau^s(n)}(n) = G+B-R$, which contradicts with the result that $l_t(n) > (G+B-R)$ when $t>\tau^s(n)$.

Due to the continuity of $\frac{\partial l_t(n)}{\partial t}$, there exists some $t_5 \in [t_4, t_3)$ such that $\frac{\partial l_t(n)}{\partial t}\big|_{t=t_5} = 0$, and $\frac{\partial l_t(n)}{\partial t} < 0$ on $(t_5, t_3]$. That is,

$$l_{t_5}(n-1) - l_{t_5}(n) - \left(\frac{H_{t_5}(n-n_0)}{H_{t_5}(n)} - 1\right)\left(1 - \frac{Q_{t_5}(n-n_0)}{Q_{t_5}(n-n_0-1)}\right) l_{t_5}(n) = 0.$$

According to Theorem 1(iii), $\frac{H_t(n-1)}{H_t(n)}$ decreases in t and $\frac{Q_t(n-n_0)}{Q_t(n-n_0-1)}$ increases in t. Coupling with the result that $l_t(n)$ strictly decreases within $(t_5, t_3]$, we have

$$\begin{aligned}
&l_{t_3}(n-1) - l_{t_3}(n) - \Big(\frac{H_{t_3}(n-n_0)}{H_{t_3}(n)} - 1\Big)\Big(1 - \frac{Q_{t_3}(n-n_0)}{Q_{t_3}(n-n_0-1)}\Big) l_{t_3}(n) \\
&> l_{t_5}(n-1) - l_{t_5}(n) - \Big(\frac{H_{t_5}(n-1)}{H_{t_5}(n)} - 1\Big)\Big(1 - \frac{Q_{t_5}(n-n_0)}{Q_{t_5}(n-n_0-1)}\Big) l_{t_5}(n) = 0.
\end{aligned}$$

This implies that $\frac{\partial l_t(n)}{\partial t}\big|_{t=t_3} > 0$ and contradicts with our assumption that $\frac{\partial l_t(n)}{\partial t}\big|_{t=t_3} < 0$. We thus complete the proof. □

Proof of Corollary 1. (i) We prove by induction. When $n = n_0 + 1$, it is straightforward that $\tau^s(n_0+1) \geq \tau^s(n_0) = \cdots = \tau^s(1) = 0$. Now assume the statement is true for $n-1$, i.e., $\tau^s(1) \leq \cdots \leq \tau^s(n-1)$ for some $n > n_0$. We prove $\tau^s(n-1) \leq \tau^s(n)$ by showing that for any $t < \tau^s(n-1)$, the creator's optimal action is not to activate the seeding stimulus at state (t, n). Suppose this is not true, then $t > \tau^s(n)$. From Equation (OA.1), we have

$$H_t(n-n_0)Q_t(n-n_0-1) - (H_t(n-n_0) - H_t(n))\, Q_t(n-n_0) < H_t(n)\frac{J_t^s(n-1)}{G+B-R}.$$

Because $t < \tau^s(n-1)$, $J_t^s(n-1) = (G+B-R) \cdot Q_t(n-n_0-1)$. Plugging $J_t^s(n-1)$ into the inequality above, we have

$$\begin{aligned}
&H_t(n-n_0)Q_t(n-n_0-1) - (H_t(n-n_0) - H_t(n))\, Q_t(n-n_0) < H_t(n)Q_t(n-n_0-1) \\
\Rightarrow\ &(H_t(n-n_0) - H_t(n))\, (Q_t(n-n_0-1) - Q_t(n-n_0)) < 0.
\end{aligned}$$

However, it contradicts with Theorem 1(i) and Assumption 1(i). We thus obtain the announced results.

(ii) Denote $Y_t(n; B, R) = \frac{J_t^s(n;B,R)}{G+B-R}$. Here, we use the notation $J_t^s(n; B, R) \equiv J_t^s(n)$ to emphasize the dependence of $J_t^s(n)$ on B and R. Similarly, we denote $\tau^s(n; B, R) \equiv \tau^s(n)$. We add to the statement that $Y_t(n; B, R)$ decreases in B and increases in R, and prove by induction. For any $n \leq n_0$, the statement is obviously true since $\tau^s(n; B, R) = 0$ and $Y_t(n; B, R) = \frac{(G+B)\cdot Q_t(n) + (G+B-R)\cdot(1-Q_t(n))}{G+B-R} = 1 + \frac{R \cdot Q_t(n)}{G+B-R}$ decreases in B and increases in R. Now suppose $\tau^s(n; B_1, R) \geq \tau^s(n; B_2, R)$ and $Y_t(n; B_1, R) \leq Y_t(n; B_2, R)$, for any $n \leq n_0$ and $B_1 > B_2 \geq 0$. From Equation (OA.1), for any $t > \tau^s(n+1; B_1, R)$,

$$1 + \left[\frac{H_t(n+1-n_0)}{H_t(n+1)} - 1\right]\left[1 - \frac{Q_t(n+1-n_0)}{Q_t(n+1-n_0-1)}\right] < \frac{Y_t(n; B_1, R)}{Q_t(n+1-n_0)} \leq \frac{Y_t(n; B_2, R)}{Q_t(n+1-n_0)}.$$

Therefore, $\tau^s(n+1;B,R)$ increases in B. Similarly we can show that $\tau^s(n+1;B,R)$ decreases in R.

Now we show the monotonicity of $Y_t(n+1;B,R)$ w.r.t B and R. For any $t \le \tau^s(n+1;B_2,R)$, $Y_t(n+1;B_1,R) = Y_t(n+1;B_2,R) = Q_t(n+1-n_0)$.

When $t \in (\tau^s(n+1;B_2,R), \tau^s(n+1;B_1,R)]$, $Y_t(n+1;B_1,R) = Q_t(n+1-n_0)$ whereas the $Y_t(n+1;B_2,R) \ge Q_t(n+1-n_0)$ because of the definition of $J_t^s(n)$.

When $t > \tau^s(n+1;B_1,R)$, $Y_t(n+1;B_i,R)$ is the solution of

$$\frac{\partial y}{\partial t} = \lambda_t H_t(n+1)[Y_t(n;B_i,R) - y],$$

with the boundary condition $y_{\tau^s(n+1;B_1,R)} = Y_{\tau^s(n+1;B_1,R)}(n+1;B_i,R)$ where $i=1,2$. Note that RHS of the equation decreases in B based on the induction hypothesis of n. Coupling with the fact that $Y_{\tau^s(n+1;B_1,R)}(n+1;B_1,R) \le Y_{\tau^s(n+1;B_1,R)}(n+1;B_2,R)$, $Y_t(n+1;B,R)$ decreases in B when $t > \tau^s(n+1;B_1,R)$. In a similar fashion, we can show that $Y_t(n+1;B,R)$ increases in R. We thus obtain the announced results. $\square$

Proof of Theorem 3. (i) Since $J_{T,N}^b = (G+B)\cdot Q_T(N)$ and $J_{T,N}^s = J_T^s(N)$, it is sufficient to show that $\frac{Q_t(n)}{J_t^s(n)}$ increases in t.

When $n=0$, the statement is obvious as $Q_t(0)=1$ and $J_t^s(0) = G+B$. Now assume that $\frac{Q_t(n-1)}{J_t^s(n-1)}$ weakly increases in t. In that case:

When $t < \tau^s(n)$, $J_t^s(n) = (G+B-R)\cdot Q_t((n-n_0)^+)$. Therefore $\frac{Q_t(n)}{J_t^s(n)} = \frac{Q_t(n)}{(G+B-R)\cdot Q_t((n-n_0)^+)}$. According to Theorem 1, it increases in t.

When $t \ge \tau^s(n)$,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^s(n)} &= \frac{\lambda_t H_t(n)\,[Q_t(n-1)-Q_t(n)]}{J_t^s(n)} - \frac{Q_t(n)}{J_t^s(n)}\frac{\lambda_t H_t(n)\,[J_t^s(n-1)-J_t^s(n)]}{J_t^s(n)} \\ &= \lambda_t H_t(n)\frac{Q_t(n)}{J_t^s(n)}\left[\frac{Q_t(n-1)}{Q_t(n)} - \frac{J_t^s(n-1)}{J_t^s(n)}\right] = \lambda_t H_t(n)\frac{J_t^s(n-1)}{J_t^s(n)}\left[\frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)}\right].\end{aligned}$$

When $t = \tau^s(n)$, because $J_t^s(n) = (G+B-R)\cdot Q_t(n-1)$,

$$\frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} = \frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{(G+B-R)\cdot Q_t(n-1)}.$$

Also, according to Equation (OA.1), at $t = \tau^s(n)$,

$$J_t^s(n-1) = (G+B-R)\cdot\left[\frac{H_t(n-1)}{H_t(n)}Q_t(n-2) - \left(\frac{H_t(n-1)}{H_t(n)} - 1\right)Q_t(n-1)\right].$$

Hence,

$$\frac{Q_t(n-1)}{J_t^s(n-1)} - \frac{Q_t(n)}{J_t^s(n)} = \frac{1}{G+B-R}\frac{Q_t(n-1)}{\frac{H_t(n-1)}{H_t(n)}Q_t(n-2) - \left(\frac{H_t(n-1)}{H_t(n)} - 1\right)Q_t(n-1)} - \frac{Q_t(n)}{(G+B-R)\cdot Q_t(n-1)}$$

$$= \frac{1}{G+B-R} \frac{Q_t(n)}{\frac{H_t(n-1)}{H_t(n)}Q_t(n-2) - \left(\frac{H_t(n-1)}{H_t(n)} - 1\right)Q_t(n-1)} \cdot \left[\frac{G+B-R}{Q_t(n)} - \frac{\frac{H_t(n-1)}{H_t(n)}Q_t(n-2) - \left(\frac{H_t(n-1)}{H_t(n)} - 1\right)Q_t(n-1)}{Q_t(n-1)}\right]$$

$$= \frac{1}{G+B-R} \frac{Q_t(n)}{\frac{H_t(n-1)}{H_t(n)}Q_t(n-2) - \left(\frac{H_t(n-1)}{H_t(n)} - 1\right)Q_t(n-1)} \left[\left(\frac{Q_t(n-1)}{Q_t(n)} - 1\right) - \frac{H_t(n-1)}{H_t(n)}\left(\frac{Q_t(n-2)}{Q_t(n-1)} - 1\right)\right].$$

Recall that in the proof of Theorem 1, we have shown that for any $t > 0$, $\frac{H_t(n)}{H_t(n-1)}\left(\frac{Q_t(n-1)}{Q_t(n)} - 1\right) - \left(\frac{Q_t(n-2)}{Q_t(n-1)} - 1\right) > 0$. Therefore $\frac{Q_t(n-1)}{J^s_t(n-1)} - \frac{Q_t(n)}{J^s_t(n)}\Big|_{t=\tau^s(n)} > 0$.

Suppose that there exists a $t' > \tau^s(n)$ such that $\frac{Q_t(n-1)}{J^s_t(n-1)} - \frac{Q_t(n)}{J^s_t(n)} < 0$, then because of continuity, there must exists a $\tau^s(n) < t_1 < t'$ such that $\frac{Q_t(n-1)}{J^s_t(n-1)} - \frac{Q_t(n)}{J^s_t(n)}\Big|_{t=t_1} = 0$ and $\frac{Q_t(n-1)}{J^s_t(n-1)} - \frac{Q_t(n)}{J^s_t(n)} < 0$ when $t \in (t_1, t')$. This also means that $\frac{Q_t(n)}{J^s_t(n)}$ decreases in t over the interval. However, since $\frac{Q_t(n-1)}{J^s_t(n-1)}$ increases in t, $\frac{Q_t(n-1)}{J^s_t(n-1)} - \frac{Q_t(n)}{J^s_t(n)}$ must be increasing in t within (t_1, t'). This indicates $\frac{Q_t(n-1)}{J^s_t(n-1)} - \frac{Q_t(n)}{J^s_t(n)}\Big|_{t=t'} \geq 0$, which leads to contradiction. Therefore $\frac{Q_t(n)}{J^s_t(n)}$ increases in t for any $t > 0$.

(ii) Note that $J^b_{T,N} \leq J^s_{T,N} \leq (G+B) \cdot Q_T((N-n_0)^+)$. Consequently, we have

$$0 \leq J^s_{T,N} - J^b_{T,N} \leq (G+B) \cdot \left[Q_T((N-n_0)^+) - Q_T(N)\right].$$

Letting $T \to \infty$ or $T \to 0$, we thus obtain the announced results. $\square$

Proof of Theorem 4. We show that $\tau^u(n)$ is given by

$$\tau^u(n) = \sup\left\{t : \tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n) \geq H_t(n)\frac{J^u_t(n-1)}{G+B-K}\right\}, \tag{OA.3}$$

where $J^u_t(n)$ is the expected profit at state (t, n). It is given by

- when $t \leq \tau^u(n)$, $J^u_t(n) = (G+B-K)\tilde{Q}_t(n)$;
- when $t > \tau^u(n)$,

$$\frac{\partial J^u_t(n)}{\partial t} = \lambda_t H_t(n)\left[J^u_t(n-1) - J^u_t(n)\right], \tag{OA.4}$$

with boundary conditions $J^u_{\tau^u(n)}(n) = (G+B-K)\tilde{Q}_t(n)$, and $J^u_t(0) = G+B$.

First we show that if $\frac{J^u_t(n-1)}{\tilde{Q}_t(n-1)}$ increases in t, then the creator would activate the stimulus if and only if $t \leq \tau^u(n)$. To see that, we can rewrite the inequality in the bracket in Equation (OA.3) as follows.

$$1 + \left(\frac{\tilde{H}_t(n)}{H_t(n)} - 1\right)\left(1 - \frac{\tilde{Q}_t(n)}{\tilde{Q}_t(n-1)}\right) \geq \frac{J^u_t(n-1)}{(G+B-K)\tilde{Q}_t(n-1)}.$$

According to Theorem 1(iii), LHS of the above inequality strictly decreases in t; while RHS increases in t due to our induction hypothesis. Therefore, for any $t < \tau^u(n)$, the inequality holds; whereas the direction of the inequality is flipped for any $t > \tau^u(n)$.

Suppose that there exists some $t_1 > \tau^u(n)$, such that the creator's optimal policy is to upgrade immediately, i.e., $J^u_{t_1}(n) = (G+B-K)\tilde{Q}_{t_1}(n)$. Then, we have

$$\begin{aligned}(G+B-K)\tilde{Q}_{t_1}(n) &> (1-\delta\lambda_{t_1}H_{t_1}(n))J^u_{t_1-\delta}(n) + \delta\lambda_{t_1}H_{t_1}(n)J^u_{t_1-\delta}(n-1) + o(\delta)\\ &\geq (1-\delta\lambda_{t_1}H_{t_1}(n))(G+B-K)\tilde{Q}_{t_1-\delta}(n) + \delta\lambda_{t_1}H_{t_1}(n)J^u_{t_1-\delta}(n-1) + o(\delta).\end{aligned}$$

Plugging $\tilde{Q}_{t_1}(n) = (1-\delta\lambda_{t_1}\tilde{H}_{t_1}(n))\tilde{Q}_{t_1-\delta}(n) + \delta\lambda_{t_1}\tilde{H}_{t_1}(n)\tilde{Q}_{t_1-\delta}(n-1) + o(\delta)$ into the inequality above, rearranging and taking the limit as $\delta \to 0$, we have

$$\tilde{H}_{t_1}(n)\tilde{Q}_{t_1}(n-1) - \left(\tilde{H}_{t_1}(n) - H_{t_1}(n)\right)\tilde{Q}_{t_1}(n) \geq \frac{H_{t_1}(n)J^u_{t_1}(n-1)}{G+B-K}.$$

This contradicts with our assumption that $t_1 > \tau^u(n)$. Therefore, the creator would not upgrade when $t > \tau^u(n)$, i.e., $J^u_t(n) > (G+B-K)\tilde{Q}_t(n)$ for any $t > \tau^u(n)$. Consider what happens in a small time interval δ, we have

$$J^u_t(n) = (1-\delta\lambda_t H_t(n))J^u_{t-\delta}(n) + \delta\lambda_t H_t(n)J^u_{t-\delta}(n-1) + o(\delta).$$

Rearranging and taking the limit as $\delta \to 0$, we thus obtain Equation (OA.4).

We next show that the creator's optimal policy is to upgrade immediately when $t < \tau^u(n)$. Suppose that there exists some $t_2 < \tau^u(n)$, such that $J^u_t(n) = (G+B-K)\tilde{Q}_t(n)$ for all $t \leq t_2$, and $J^u_t(n) > (G+B-K)\tilde{Q}_t(n)$ for $t \in (t_2, t_2+\delta]$. Then, we have

$$\begin{aligned}(G+B-K)\tilde{Q}_{t_2+\delta}(n) &< J^u_{t_2+\delta}(n) = (1-\delta\lambda_{t_2+\delta}H_{t_2+\delta}(n))J^u_{t_2}(n) + \delta\lambda_{t_2+\delta}H_{t_2+\delta}(n)J^u_{t_2}(n-1) + o(\delta)\\ &= (1-\delta\lambda_{t_2+\delta}H_{t_2+\delta}(n))(G+B-K)\tilde{Q}_{t_2}(n) + \delta\lambda_{t_1+\delta}H_{t_2+\delta}(n)J^u_{t_2}(n-1) + o(\delta).\end{aligned}$$

Plugging $\tilde{Q}_{t_2+\delta}(n) = (1-\delta\lambda_{t_2+\delta}\tilde{H}_{t_2+\delta}(n))\tilde{Q}_{t_2}(n) + \delta\lambda_{t_2+\delta}\tilde{H}_{t_2+\delta}(n)\tilde{Q}_{t_2}(n-1) + o(\delta)$ into the inequality above, rearranging and taking the limit as $\delta \to 0$, we have

$$(G+B-K)\left[\tilde{H}_{t_2}(n)\tilde{Q}_{t_2}(n-1) - \left(\tilde{H}_{t_2}(n) - H_{t_2}(n)\right)\tilde{Q}_{t_2}(n)\right] \leq H_{t_2}(n)J^u_{t_2}(n-1).$$

This contradicts with the assumption that $t_2 < \tau^u(n)$. Therefore, the creator would upgrade immediately when $t < \tau^u(n)$, i.e., $J^u_t(n) = (G+B-K)\tilde{Q}_t(n)$ for any $t < \tau^u(n)$.

Therefore, to prove Theorem 4, it is sufficient to show that $\frac{J^u_t(n)}{\tilde{Q}_t(n)}$ increases in t. We do this by induction. For $n = 0$, the statement is obvious since $\frac{J^u_t(0)}{\tilde{Q}_t(0)} = G+B$.

Now assume that the statement is true for $n-1$, and consider the case n. It is trivial when $t \leq \tau^u(n)$ because $\frac{J^u_t(n)}{\tilde{Q}_t(n)} = G+B-K$. Consider next when $t > \tau^u(n)$. Suppose that there exists some $t_3 > \tau^u(n)$ such that $\frac{\partial}{\partial t}\frac{J^u_t(n)}{\tilde{Q}_t(n)}\Big|_{t=t_3} < 0$. Then, there must exist some $t_4 \in (\tau^u(n), t_3)$ such that

$\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}\Big|_{t=t_4} \geq 0$; otherwise, $\frac{J_{t_3}^u(n)}{\tilde{Q}_{t_3}(n)} < \frac{J_{\tau^u(n)}^u(n)}{\tilde{Q}_{\tau^u(n)}(n)} = G+B-K$, which contradicts with the result that $J_t^u(n) > (G+B-K)\cdot\tilde{Q}_t(n)$ for any $t > \tau^u(n)$. Due to the continuity of $\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}$, there exists some $t_5 \in [t_4, t_3)$, such that $\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}\Big|_{t=t_5} = 0$. That is,

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}\Big|_{t=t_5} &= \frac{\lambda_{t_5} H_{t_5}(n)\left[J_{t_5}^u(n-1) - J_{t_5}^u(n)\right]}{\tilde{Q}_{t_5}(n)} - \frac{\lambda_{t_5}\tilde{H}_{t_5}(n) J_{t_5}^u(n)\left[\tilde{Q}_{t_5}(n-1) - \tilde{Q}_{t_5}(n)\right]}{[\tilde{Q}_{t_5}(n)]^2} \\
&= \lambda_{t_5} H_{t_5}(n)\frac{\tilde{Q}_{t_5}(n-1)}{\tilde{Q}_{t_5}(n)}\left[\frac{J_{t_5}^u(n-1)}{\tilde{Q}_{t_5}(n-1)} - \frac{J_{t_5}^u(n)}{\tilde{Q}_{t_5}(n)} - \left(\frac{\tilde{H}_{t_5}(n)}{H_{t_5}(n)} - 1\right)\left(1 - \frac{\tilde{Q}_{t_5}(n)}{\tilde{Q}_{t_5}(n-1)}\right)\frac{J_{t_5}^u(n)}{\tilde{Q}_{t_5}(n)}\right] = 0.
\end{aligned}$$

Because $\frac{\tilde{Q}_t(n)}{\tilde{Q}_t(n-1)}$ increases in t, $\frac{\tilde{H}_t(n)}{H_t(n)}$ decreases in t as shown in Theorem 1(iii), and the induction hypothesis that $\frac{J_t^u(n-1)}{\tilde{Q}_t(n-1)}$ increases in t, we have $\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}\Big|_{t=t_3} \geq 0$, which contradicts with the assumption that $\frac{\partial}{\partial t}\frac{J_t^u(n)}{\tilde{Q}_t(n)}\Big|_{t=t_3} < 0$. Therefore, $\frac{J_t^u(n)}{\tilde{Q}_t(n)}$ increases in t for any $t > \tau^u(n)$, and we thus complete the proof. □

Proof of Corollary 2. (i) Suppose that there exists an n, such that $\tau^u(n) < \tau^u(n-1)$. For any $t \in (\tau^u(n), \tau^u(n-1))$, $J_t^u(n-1) = (G+B-K)\tilde{Q}_t(n-1)$. Using the definition of $\tau^u(n)$, we have

$$\begin{aligned}
&(G+B-K)\left[\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)\right] < H_t(n) J_t^u(n-1) \\
\Rightarrow\ &(\tilde{H}_t(n) - H_t(n))(\tilde{Q}_t(n-1) - \tilde{Q}_t(n)) < 0.
\end{aligned}$$

This contradicts with Theorem 1(i) and Assumption 1(i). We thus obtain the announced results.

(ii) Denote $Z_t(n; B, K) = \frac{J_t^u(n;B,R)}{G+B-K}$. Here, we use the notation $J_t^u(n; B, R) \equiv J_t^u(n)$ to emphasize the dependence of $J_t^u(n)$ on B and K. Similarly, we denote $\tau^u(n; B, K) \equiv \tau^u(n)$. Note that Equation (OA.3) can be rewritten as:

$$\tau^u(n; B, K) = \sup\left\{t:\ \tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n) \geq H_t(n) Z_t(n-1; B, K)\right\}.$$

Only the RHS of the above inequality depends on B and K. Thus, $\tau^u(n; B, K)$ increases in B and decreases in K if and only if $Z_t(n-1; B, K)$ decreases in B and increases in K. We prove the monotonicity of $Z_t(n; B, K)$ w.r.t. B and K for any $n \geq 0$ by induction.

First when $n = 0$, $Z_t(0; B, K) = \frac{G+B}{G+B-K}$. The statement is obvious. Now suppose it is true for any $m \leq n-1$. This implies that $\tau^u(m; B, K)$ increases in B and decreases in K for any $m \leq n$. Thus, we have $\tau^u(n; B_1, K) > \tau^u(n; B_2, K)$ for any $B_1 > B_2 \geq 0$. Consider the following cases w.r.t. t:

When $t \leq \tau^u(n; B_2, K)$, creators of both projects would upgrade project features immediately. Hence $Z_t(n; B_1, K) = Z_t(n; B_2, K) = \tilde{Q}_t(n)$.

When $\tau^u(n;B_2,K) < t \le \tau^u(n;B_1,K)$, only the project with a long-term profit of B_1 would use the stimulus. Therefore, $Z_t(n;B_1,K) = \tilde{Q}_t(n)$ whereas $Z_t(n;B_2,K) \ge \tilde{Q}_t(n) = Z_t(n;B_1,K)$.

When $t > \tau^u(n;B_1,K)$, neither projects activates the stimulus policy. For $i=1,2$, $Z_t(n;B_i,K)$ is the solution of

$$\frac{dz}{dt} = \lambda_t H_t(n)\left(Z_t(n-1;B_i,K) - z\right),$$

with boundary condition $z(\tau^u(n;B_1,K)) = Z_{\tau^u(n;B_1,K)}(n,B_i,K)$. RHS of the above equation decreases in B_1. Coupling with the inequality $Z_{\tau^u(n;B_1,K)}(n,B_1,K) \le Z_{\tau^u(n;B_1,K)}(n,B_2,K)$, we have $Z_t(n;B_1,K) \le Z_t(n;B_2,K)$.

In a similar fashion, we can show that $Z_t(n;B,K)$ increases in K. This completes the proof. □

Proof of Theorem 5. (i) Since $J_t^b(n) = (G+B)\cdot Q_t(n)$. It is equivalent to show that $\frac{Q_t(n)}{J_t^u(n)}$ increases in t.

When $n=0$, the statement is obvious as $Q_t(n)=1$ and $J_t^u(n) = G+B$. Now assume that it's true for $n-1$. Then for n:

When $t < \tau^u(n)$, $J_t^u(n) = (G+B-K)\tilde{Q}_t(n)$. Hence $\frac{Q_t(n)}{J_t^u(n)} = \frac{1}{G+B-K}\frac{Q_t(n)}{\tilde{Q}_t(n)}$. According to Proposition 2, $\frac{Q_t(n)}{J_t^u(n)}$ increases in t.

When $t \ge \tau^u(n)$,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)} &= \frac{\lambda_t H_t(n)\left[Q_t(n-1) - Q_t(n)\right]}{J_t^u(n)} - \frac{Q_t(n)}{J_t^u(n)}\frac{\lambda_t H_t(n)\left[J_t^u(n-1) - J_t^u(n)\right]}{J_t^u(n)} \\ &= \lambda_t H_t(n)\frac{Q_t(n)}{J_t^u(n)}\left[\frac{Q_t(n-1)}{Q_t(n)} - \frac{J_t^u(n-1)}{J_t^u(n)}\right] = \lambda_t H_t(n)\frac{J_t^u(n-1)}{J_t^u(n)}\left[\frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{J_t^u(n)}\right].\end{aligned}$$

At $t = \tau^u(n)$, because $J_t^u(n) = (G+B-K)\tilde{Q}_t(n)$,

$$\frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{J_t^u(n)} = \frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{(G+B-K)\tilde{Q}_t(n)}.$$

Also according to Equation (OA.3), $J_t^u(n-1) = \frac{G+B-K}{H_t(n)}[\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)]$ at $t = \tau^u(n)$. Hence,

$$\begin{aligned}&\frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{J_t^u(n)} = \frac{H_t(n)\cdot Q_t(n-1)}{(G+B-K)\left[\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)\right]} - \frac{Q_t(n)}{(G+B-K)\tilde{Q}_t(n)} \\ &= \frac{1}{G+B-K}\left[\frac{H_t(n)Q_t(n-1)}{\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)} - \frac{Q_t(n)}{\tilde{Q}_t(n)}\right] \\ &= \frac{1}{G+B-K}\frac{Q_t(n)}{\tilde{H}_t(n)\tilde{Q}_t(n-1) - (\tilde{H}_t(n) - H_t(n))\tilde{Q}_t(n)}\left[H_t(n)\left(\frac{Q_t(n-1)}{Q_t(n)} - 1\right) - \tilde{H}_t(n)\left(\frac{\tilde{Q}_t(n-1)}{\tilde{Q}_t(n)} - 1\right)\right].\end{aligned}$$

In the proof of Proposition 2, we have shown that for any $t > 0$, $H_t(n)\left(\frac{Q_t(n-1)}{Q_t(n)} - 1\right) - \tilde{H}_t(n)\left(\frac{\tilde{Q}_t(n-1)}{\tilde{Q}_t(n)} - 1\right) > 0$. Therefore, $\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)}\Big|_{t=\tau^u(n)} > 0$. Suppose there exists some $t' > \tau^u(n)$ such that $\frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{J_t^u(n)} < 0$, then according to the continuity of the functions, there must exist some $\tau^u(n) < t_0 < t'$, such that $\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)}\Big|_{t=t_0} = 0$ and $\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)} < 0$ in the interval $(t_0, t']$. However, since $\frac{Q_t(n-1)}{J_t^u(n-1)}$ increases in t, $\frac{Q_t(n-1)}{J_t^u(n-1)} - \frac{Q_t(n)}{J_t^u(n)}$ must strictly increase in the interval $(t_0, t']$, implying $\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)}\Big|_{t=t'} > 0$. This leads to contradiction. Therefore, $\frac{\partial}{\partial t}\frac{Q_t(n)}{J_t^u(n)} \geq 0$ for any $t > 0$.

(ii) Following a similar approach as the proof for Theorem 3(ii), we can show that $\lim_{T\to\infty} J_{T,N}^u - J_{T,N}^b = \lim_{T\to 0} J_{T,N}^u - J_{T,N}^b = 0$ for any $N \geq 1$. □

Proof of Theorem 6. Denote $A_t(n)$ the optimal expected profit at state (t,n) assuming that the creator has not ended LTO yet. The optimal expected profit over the course of the entire pledging process is denoted by $J_t^l(n)$. We show that $\tau^l(n)$ is given by

$$\tau^l(n) = \sup\Big\{t:\ A_t(n) \geq [G + B - (N-n)k]\cdot Q_t(n)\Big\}, \tag{OA.5}$$

where $A_t(n)$ is the solution of

$$\frac{\partial A_t(n)}{\partial t} = \lambda_t \hat{H}_t(n)\left[J_t^l(n-1) - A_t(n)\right], \tag{OA.6}$$

with boundary conditions $A_0(n) = 0$ for any $n \geq 1$, and $A_t(0) = G + B - Nk$.

Expected profit $J_t^l(n)$ at state (t,n) is given by

$$J_t^l(n) = \begin{cases} A_t(n), & \text{if } t < \tau^l(n) \\ [G+B-(N-n)k]\cdot Q_t(n), & \text{if } t \geq \tau^l(n) \end{cases}.$$

Denote $d_t(n) = \frac{A_t(n)}{Q_t(n)}$. We add to the statement that $d_t(n)$ decreases in t and prove by induction. It's trivial when $n = 0$ because $d_t(0) = \frac{A_t(0)}{Q_t(0)} = G + B - Nk$. Suppose that the statement is true for $n-1$. Taking the derivative of $d_t(n)$ w.r.t. t, we have

$$\begin{aligned}
\frac{\partial d_t(n)}{\partial t} &= \frac{\lambda_t \hat{H}_t(n)\,[J_t^l(n-1) - A_t(n)]}{Q_t(n)} - \frac{\lambda_t H_t(n) A_t(n)\,[Q_t(n-1) - Q_t(n)]}{[Q_t(n)]^2} \\
&= \lambda_t\left(\hat{H}_t(n)\left[\frac{J_t^l(n-1)}{Q_t(n-1)}\frac{Q_t(n-1)}{Q_t(n)} - \frac{A_t(n)}{Q_t(n)}\right] - H_t(n)\frac{A_t(n)}{Q_t(n)}\left[\frac{Q_t(n-1)}{Q_t(n)} - 1\right]\right) \\
&= \lambda_t \hat{H}_t(n)\frac{Q_t(n-1)}{Q_t(n)}\left[\frac{J_t^l(n-1)}{Q_t(n-1)} - \left[1 - \left(1 - \frac{H_t(n)}{\hat{H}_t(n)}\right)\left(1 - \frac{Q_t(n)}{Q_t(n-1)}\right)\right]d_t(n)\right].
\end{aligned}$$

Taking the limit as $t \to 0$ and using L'Hopital's rule, we have

$$\lim_{t\to 0} d_t(n) = \lim_{t\to 0}\frac{\lambda_t \hat{H}_t(n)\,[J_t^l(n-1) - A_t(n)]}{\lambda_t H_t(n)\,[Q_t(n-1) - Q_t(n)]} = \lim_{t\to 0}\frac{\hat{H}_t(n)}{H_t(n)}\frac{J_t^l(n-1)}{Q_t(n-1)}.$$

By the induction hypothesis, we know that $\frac{J_t^l(n-1)}{Q_t(n-1)}$ decreases in t since $J_t^l(n-1)$ is equal to either $A_t(n-1)$ or $[G+B-(N-n)k]\cdot Q_t(n-1)$. From Theorem 1(iii), $\frac{\hat{H}_t(n)}{H_t(n)}$ decreases in t. Therefore

$\lim_{t\to 0} d_t(n) = \lim_{t\to 0} \frac{\hat{H}_t(n)}{H_t(n)} \frac{J_t^l(n-1)}{Q_t(n-1)}$ exists. Next we show that, if there exists some t_1 such that $\frac{\partial d_t(n)}{\partial t}\Big|_{t=t_1} > 0$, there must be some $t_2 \in (0, t_1)$ such that $\frac{\partial d_t(n)}{\partial t}\Big|_{t=t_2} < 0$. Consider the following two cases.

(1) $\lim_{t\to 0} d_t(n) = \infty$. If there exists a t_1 such that $\frac{\partial d_t(n)}{\partial t}|_{t=t_1} > 0$, then there must exist a $t_2 \in (0, t_1)$ such that $\frac{\partial d_t(n)}{\partial t}|_{t=t_2} \leq 0$; Otherwise $d_{t_1} \geq \lim_{t\to 0} d_t(n) = \infty$, which is impossible.

(2) $\lim_{t\to 0} d_t(n) < \infty$. This implies that $\lim_{t\to 0} \frac{J_t^l(n-1)}{Q_t(n-1)} < \infty$ and $\lim_{t\to 0} \frac{H_t(n)}{\hat{H}_t(n)} > 0$. Let $S(t) = \frac{J_t^l(n-1)}{Q_t(n-1)} - \left[1 - \left(1 - \frac{H_t(n)}{\hat{H}_t(n)}\right)\left(1 - \frac{Q_t(n)}{Q_t(n-1)}\right)\right] d_t(n)$. Because $\lim_{t\to 0} 1 - \left(1 - \frac{H_t(n)}{\hat{H}_t(n)}\right)\left(1 - \frac{Q_t(n)}{Q_t(n-1)}\right) = \lim_{t\to 0} \frac{H_t(n)}{\hat{H}_t(n)} > 0$ and $\lim_{t\to 0} d_t(n) = \lim_{t\to 0} \frac{\hat{H}_t(n)}{H_t(n)} \frac{J_t^l(n-1)}{Q_t(n-1)} < \infty$, we have $\lim_{t\to 0} S(t) = 0$.

Recall that $\frac{\partial d_t(n)}{\partial t} = \lambda_t \hat{H}_t(n) \frac{Q_t(n-1)}{Q_t(n)} \cdot S(t)$. Suppose there exists a t_1 such that $\frac{\partial d_t(n)}{\partial t}\Big|_{t=t_1} > 0$, then there must exist a $t_2 \in (0, t_1)$ such that $\frac{\partial d_t(n)}{\partial t}\Big|_{t=t_2} < 0$; Otherwise, $d_t(n)$ increases within $[0, t_1]$. Coupling with the results that $\frac{H_t(n)}{\hat{H}_t(n)}$ and $\frac{Q_t(n)}{Q_t(n-1)}$ both increase in t, we conclude that $S(t)$ decreases in t. A direct consequence is that $S(t_1) \leq S(0) = 0$, which contradicts with the assumption that $\frac{\partial d_t(n)}{\partial t} > 0$.

Consequently, if there exists a t_1 such that $\frac{\partial d_t(n)}{\partial t}|_{t=t_1} > 0$, there must exist a $t_2 \in [0, t_1)$ such that $\frac{\partial d_t(n)}{\partial t}|_{t=t_2} \leq 0$. Due to the continuity of $\frac{\partial d_t(n)}{\partial t}$, there exists some $t_3 \in [t_2, t_1)$ such that $S(t_3) = 0$, and $S(t) > 0$ for any $t \in (t_3, t_1]$. However, because $d_t(n)$, $\frac{H_t(n)}{\hat{H}_t(n)}$ and $\frac{Q_t(n)}{Q_t(n-1)}$ increase in t, and $\frac{J_t^l(n-1)}{Q_t(n-1)}$ decreases in t for any $t \in (t_3, t_1)$, $S(t)$ should decrease in t, which contradicts with the preceding result. Therefore, $d_t(n)$ must decrease in t for any $t > 0$. Moreover, because $\frac{Q_t(n)}{Q_t(n-1)}$ strictly increases in t, $S(t) \neq 0$ for any t. Therefore, for any $t > 0$, $S(t) < 0$ and $d_t(n)$ strictly decreases in t. As a result, $A_t(n) > [G + B - (N-n)k] \cdot Q_t(n)$ for any $t < \tau^l(n)$, and the direction of the inequality is flipped for any $t > \tau^l(n)$.

Next we show that the creator's optimal policy is to end the limited-time offer if and only if $t > \tau^l(n)$. Suppose that there exists some $t_4 < \tau^l(n)$, such that the creator's optimal decision is to end the limited-time offer immediately, i.e., $J_{t_4}^l(n) = [G + B - (N-n)k] \cdot Q_{t_4}(n)$. Then, we have

$$
\begin{aligned}
&[G + B - (N-n)k] \cdot Q_{t_4}(n) \\
&> \lambda_{t_4} \hat{H}_{t_4}(n) \delta J_{t_4-\delta}^l(n-1) + (1 - \lambda_{t_4} \hat{H}_{t_4}(n)\delta) J_{t_4-\delta}^l(n) + o(\delta) \\
&\geq \lambda_{t_4} \hat{H}_{t_4}(n) \delta J_{t_4-\delta}^l(n-1) + \left(1 - \lambda_{t_4} \hat{H}_{t_4}(n)\delta\right) A_{t_4-\delta}(n) + o(\delta) = A_{t_4}(n) + o(\delta),
\end{aligned}
$$

which contradicts with $t_4 < \tau^l(n)$. Therefore, the creator would not end the limited-time offer for any $t \leq \tau^l(n)$. Consider what happens in a small time interval δ, we have

$$J_t^l(n) = (1 - \delta\lambda_t \hat{H}_t(n)) J_{t-\delta}^l(n) + \delta\lambda_t \hat{H}_t(n) J_{t-\delta}^l(n-1) + o(\delta).$$

Rearranging, taking the limit as $\delta \to 0$, and replacing $J_t^l(n)$ with $A_t(n)$, we thus have

$$\frac{\partial A_t(n)}{\partial t} = \lambda_t \hat{H}_t(n) \left[J_t^l(n-1) - A_t(n)\right].$$

Suppose that there exists some $t_5 \geq \tau^l(n)$ such that $J_t^l(n) = [G+B-(N-n)k] \cdot Q_t(n)$ for any $t \leq t_5$ but $J_t^l(n) > [G+B-(N-n)k] \cdot Q_t(n)$ for any $t \in (t_5, t_5+\delta]$. (Because $J_{\tau^l(n)}^l = A_{\tau^l(n)}(n) = [G+B-(N-n)k] \cdot Q_{\tau^l(n)}(n)$, we can always find such $t_5 \geq \tau^l(n)$.) Thus, we have

$$\begin{aligned}
&[G+B-(N-n)k] \cdot Q_{t_5+\delta}(n) \\
< &\lambda_{t_5+\delta}\hat{H}_{t_5+\delta}(n)\delta J_{t_5}^l(n-1) + \left(1-\lambda_{t_5+\delta}\hat{H}_{t_5+\delta}(n)\delta\right) J_{t_5}^l(n) + o(\delta) \\
= &\lambda_{t_5+\delta}\hat{H}_{t_5+\delta}(n)\delta J_{t_5}^l(n-1) + \left(1-\lambda_{t_5+\delta}\hat{H}_{t_5+\delta}(n)\delta\right) [G+B-(N-n)k] \cdot Q_{t_5}(n) + o(\delta).
\end{aligned}$$

Plugging $Q_{t_5+\delta}(n) = \lambda_{t_5+\delta}H_{t_5+\delta}(n)\delta Q_{t_5}(n-1) + (1-\lambda_{t_5+\delta}H_{t_5+\delta}(n)\delta)Q_{t_5}(n) + o(\delta)$ into the inequality above, rearranging, and taking the limit as $\delta \to 0$, we have

$$[G+B-(N-n)k] \cdot \left(H_{t_5}(n)Q_{t_5}(n-1) + (\hat{H}_{t_5}(n) - H_{t_5}(n))Q_{t_5}(n)\right) \leq \hat{H}_{t_5}(n)J_{t_5}^l(n-1).$$

Because that $S(t) = \frac{J_t^l(n-1)}{Q_t(n-1)} - \left[1 - \left(1 - \frac{H_t(n)}{\hat{H}_t(n)}\right)\left(1 - \frac{Q_t(n)}{Q_t(n-1)}\right)\right] d_t(n) < 0$ for any t, we have

$$\hat{H}_{t_5}(n)J_{t_5}^l(n-1) < \left[H_{t_5}(n)Q_{t_5}(n-1) + (\hat{H}_{t_5}(n) - H_{t_5}(n))Q_{t_5}(n)\right] d_{t_5}(n).$$

Combining the preceding two inequalities, we have that $d_{t_5}(n) = \frac{A_{t_5}(n)}{Q_{t_5}(n)} > G+B-(N-n)k$, which contradicts with $t_5 > \tau^l(n)$. Therefore, the creator's optimal policy is to end the limited-time offer for any $t > \tau^l(n)$, i.e., $J_t^l(n) = [G+B-(N-n)k] \cdot Q_t(n)$ for any $t > \tau^l(n)$. We thus obtain the announced results. □

Proof of Corollary 3. Denote $W_t(n;B,k) = \frac{A_t(n)}{G+B-(N-n)k}$. We prove by induction that $W_t(n;B,k)$ increases in B and decreases in k. When $n=0$, this statement is trivial as $W_t(0;B,k) = 1$. Now suppose it is true for $n-1$ and consider the case n:

$$\frac{\partial W_t(n;B,k)}{\partial t} = \lambda_t\hat{H}_t(n)\left[\frac{J_t^l(n-1)}{G+B-(N-n+1)k}\frac{G+B-(N-n+1)k}{G+B-(N-n)k} - W_t(n;B,k)\right],$$

with boundary condition $W_0(n;B,k) = 0$ for any $n > 0$. Since $J_t^l(n-1) = \max\{A_t(n-1), [G+B-(N-n+1)k]Q_t(n-1)\}$, $\frac{J_t^l(n-1)}{G+B-(N-n+1)k}$ increases in B and decreases in k based on the induction hypothesis for $n-1$. It is also obvious that $\frac{G+B-(N-n+1)k}{G+B-(N-n)k}$ increases in B and decreases in k. Therefore, the RHS of the equation above increases in B and decreases in k, which implies that $W_t(n;B,k)$ increases in B and decreases in k. We thus proved the statement for n.

Denote $\tau^l(n;B,k) \equiv \tau^l(n)$ to emphasize the dependence of $\tau^l(n)$ on B and k. From Equation (OA.5), we have $\tau^l(n;B,k) = \sup\{t : W_t(n;B,k) \geq Q_t(n)\}$. Consider any $B_1 > B_2$. For any $t \leq \tau^l(n;B_2,k)$, we have $W_t(n;B_1,k) \geq W_t(n;B_2,k) \geq Q_t(n)$. Therefore, $\tau^l(n;B_1,k) \geq \tau^l(n;B_2,k)$. Similarly we can show that $\tau^l(n;B,k)$ decreases in k. We thus obtain the announced result. □

Proof of Theorem 7. (i) follows directly from the proof of Theorem 6, where we show that $\frac{J_t^l(n)}{Q_t(n)}$ decreases in t.

Next we prove (ii). When $T \to 0$, both $J_{T,N}^l$ and $J_{T,N}^b \to 0$. On the other hand, when $T \geq \tau^l(N)$, the creator ends the LTO immediately, so we have $J_{T,N}^l = J_{T,N}^b$. Thus, $\lim_{T\to\infty} J_{T,N}^l - J_{T,N}^b = \lim_{T\to 0} J_{T,N}^l - J_{T,N}^b = 0$, and thus we obtain the announced results. □

OA.2. Extension: Multiple Rounds of Stimulus

For analytical tractability, we restrict our attention to the circumstance where a creator can apply the stimulus only once in the main text. However, as we can see from Table 2, creators typically update their projects rather frequently in practice, especially for those successful projects. In this section, we extend the model in Section 4 to consider multiple rounds of stimulus offerings for the two reactive stimulus policies: seeding and feature upgrade. We show that the optimal strategies still follow the threshold structure in the sense that the creator should adopt the stimuli if and only the time-to-go is shorter than a cutoff, and that the cutoff increases in pledge-to-go n. For limited-time offers, when there are multiple LTOs in effect, the decision to end one of them would depend on the total funds collected at a given time, which makes the problem significantly more complicated. While we hypothesize that the optimal strategy is a threshold policy, the proof is beyond the scope of this paper, which we leave for future research.

OA.2.1. Seeding

Suppose that the creator is able to offer up to $n_0 \geq 1$ seeds, potentially in multiple rounds. Denote $0 \leq m \leq n_0$ as the number of seeds left at a given point during the crowdfunding campaign. At the state of time-to-go t, pledges needed n and seeds left m, the expected profit is denoted as $J_t^s(n,m)$. The cost of the ith seed is assumed to be $R_i \geq 0$ for any $1 \leq i \leq n_0$.

Proposition OA.1. *For any (n,m), there exists a $0 \leq \tau^s(n,m) \leq \infty$, such that:*

- *When $t \leq \tau^s(n,m)$, the creator will activate seeding stimulus right away. That is, $J_t^s(n,m) = J_t^s(n-1,m-1)$ for any $t \leq \tau^s(n,m)$;*
- *When $t > \tau^s(n,m)$, the creator is better-off withholding seeding stimulus. The expected profit $J_t^s(n,m)$ in this case is given by:*

$$\frac{\partial J_t^s(n,m)}{\partial t} = \lambda_t H_t(n) \left(J_t^s(n-1,m) - J_t^s(n,m)\right),$$

with boundary condition $J_{\tau^s(n,m)}^s(n,m) = J_{\tau^s(n,m)}^s(n-1,m-1)$, $J_t^s(n,0) = (G+B-\sum_{i=1}^{n_0} R_i)Q_t(n)$, $J_0^s(0,m) = G+B-\sum_{i=m+1}^{n_0} R_i$, and $J_0^s(n,m) = 0$ for all $n > 0$.

Moreover, $\tau^s(n,m)$ increases in n.

Proof of Proposition OA.1. First for any (n,m) and $t>0$,

$$J^s_{t+\delta}(n,m) \geq (1-\lambda_t H_t(n)\delta) J^s_t(n,m) + \lambda_t H_t(n)\delta J^s_t(n-1,m) + o(\delta).$$

Let $\delta \to 0$, we get the following inequality:

$$\frac{\partial J^s_t(n,m)}{\partial t} \geq \lambda_t H_t(n) \left[J^s_t(n-1,m) - J^s_t(n,m) \right]. \tag{OA.7}$$

We add the following statements to the proposition and prove by induction.

(1) For any $1 \leq j \leq m$, $\frac{J^s_t(n,m)}{J^s_t(n-j,m-j)}$ increases in t.

(2) $\frac{J^s_t(n,m)}{J^s_t(n-1,m)}$ increases in t when $t \geq \tau^s(n,m)$.

First when $m=1$, we already prove the threshold structure and the monotonicity of the thresholds in Theorem 2. We also show that statement (1) holds for $m=1$ in the proof of Theorem 2. In addition, in the proof of Theorem 3, we show that $\frac{J^s_t(n,0)}{J^s_t(n,1)}$ increases in t for any n. Therefore $\frac{J^s_t(n,1)}{J^s_t(n-1,1)} = \frac{J^s_t(n,1)}{J^s_t(n-1,0)} \frac{J^s_t(n-1,0)}{J^s_t(n-1,1)}$ increases in t. Thus the statements are true for $m=1$.

Now consider $m>1$. Suppose the statements are true for any n and $m-i$ where $i \geq 1$. When $n=1$, it is obvious that the optimal strategy is to wait to use the seeding stimulus right before time expires, i.e., $\tau^s(1,m)=0$. So the statements are true for $n=1$.

Assume that the statements are true for some $n-1$, where $n>1$. We prove the threshold structure for n by contradiction. Suppose it is not true, then there exists an time interval $(\underline{t}, \underline{t}+h)$ over which the stimulus will not be used. Because of the continuity of $J^s_t(n,m)$, $J^s_{\underline{t}}(n,m) = J^s_{\underline{t}}(n-1,m-1)$ and $J^s_{\underline{t}+h}(n,m) = J^s_{\underline{t}+h}(n-1,m-1)$. For any $t \in (\underline{t}, \underline{t}+h)$, $J^s_t(n,m) > J^s_t(n-1,m-1)$. Now for every $t \in [\underline{t}, \underline{t}+h]$, we find $j = \min\{i \geq 1 : \tau^s(n-i,m-i) \leq t\}$. We collect all those unique j's, and denote them as $j_0 > j_1 > \cdots > j_\kappa$, where $j_0 = \min\{i \geq 1 : \tau^s(n-i,m-i) \leq \underline{t}\}$ and $j_\kappa = \min\{i \geq 1 : \tau^s(n-i,m-i) \leq \underline{t}+h\}$. For any $\tau^s(n-j_i,m-j_i) \leq t \leq \tau^s(n-j_{i+1},m-j_{i+1})$, $J^s_t(n,m) = J^s_t(n-1,m-1) = \cdots = J^s_t(n-j_i,m-j_i)$.

Since the optimal strategy is not to use the stimulus at $\underline{t}+\delta$ and $J^s_{\underline{t}}(n,m) = J^s_{\underline{t}}(n-j_0,m-j_0)$,

$$\begin{aligned} J^s_{\underline{t}+\delta}(n,m) &= (1-\lambda_{\underline{t}} H_{\underline{t}}(n)\delta) J^s_{\underline{t}}(n,m) + \lambda_{\underline{t}} H_{\underline{t}}(n)\delta J^s_{\underline{t}}(n-1,m) + o(\delta) \\ &= (1-\lambda_{\underline{t}} H_{\underline{t}}(n)\delta) J^s_{\underline{t}}(n-j_0,m-j_0) + \lambda_{\underline{t}} H_{\underline{t}}(n)\delta J^s_{\underline{t}}(n-1,m) + o(\delta) \\ &> (1-\lambda_{\underline{t}} H_{\underline{t}}(n-j_0)\delta) J^s_{\underline{t}}(n-j_0,m-j_0) + \lambda_{\underline{t}} H_{\underline{t}}(n-j_0)\delta J^s_{\underline{t}}(n-j_0-1,m-j_0) + o(\delta) \end{aligned}$$

Let $\delta \to 0$, we have

$$H_t(n-j_0) \left[J^s_t(n-j_0-1,m-j_0) - J^s_t(n-j_0,m-j_0) \right] < H_t(n) \left[J^s_t(n-1,m) - J^s_t(n-j_0,m-j_0) \right],$$

at $t = \underline{t}$. Rearrange the terms:

$$1 + \left[\frac{H_t(n-j_0)}{H_t(n)} - 1\right]\left[1 - \frac{J_t^s(n-j_0, m-j_0)}{J_t^s(n-j_0-1, m-j_0)}\right] < \frac{J_t^s(n-1,m)}{J_t^s(n-j_0-1, m-j_0)},$$

at $t = \underline{t}$. According to our induction assumptions, the LHS decreases in t for any $t \geq \tau^s(n-j_0, m-j_0)$ and RHS increases in t. Thus the inequality holds for any $t > \underline{t}$. Also for any $\underline{t} \leq t < \tau^s(n-j_1, m-j_1)$, $J_t^s(n-j_1, m-j_1) = J_t^s(n-j_0, m-j_0)$. According to Inequality (OA.7), we have

$$\begin{aligned}
&\frac{\partial J_t^s(n, m-j_1)}{\partial t} = \frac{\partial J_t^s(n, m-j_0)}{\partial t} \\
&= H_t(n-j_0)\left[J_t^s(n-j_0-1, m-j_0) - J_t^s(n-j_0, m-j_0)\right] \\
&\geq H_t(n-j_1)\left[J_t^s(n-j_1-1, m-j_1) - J_t^s(n-j_1, m-j_1)\right]
\end{aligned}$$

Thus, the following inequality holds for any $\underline{t} \leq t \leq \tau^s(n-j_1, m-j_1)$.

$$H_t(n-j_1)\left[J_t^s(n-j_1-1, m-j_1) - J_t^s(n-j_1, m-j_1)\right] < H_t(n)\left[J_t^s(n-1, m) - J_t^s(n-j_0, m-j_0)\right].$$

Note that $J_t^s(n-j_0, m-j_0) = J_t^s(n-j_1, m-j_1)$ at $t = \tau^s(n-j_1, m-j_1)$. Thus,

$$H_t(n-j_1)\left[J_t^s(n-j_1-1, m-j_1) - J_t^s(n-j_1, m-j_1)\right] < H_t(n)\left[J_t^s(n-1, m) - J_t^s(n-j_1, m-j_1)\right],$$

at $t = \tau^s(n-j_1, m-j_1)$. In a similar manner, we can show that for any $t > \tau^s(n-j_\kappa, m-j_\kappa)$,

$$H_t(n-j_\kappa)\left[J_t^s(n-j_\kappa-1, m-j_\kappa) - J_t^s(n-j_\kappa, m-j_\kappa)\right] < H_t(n)\left[J_t^s(n-1, m) - J_t^s(n-j_\kappa, m-j_\kappa)\right].$$

On the other hand, at $t = \underline{t} + h$, the optimal strategy is to activate the stimulus, which means that

$$\begin{aligned}
&J_{\underline{t}+h}^s(n, m) = J_{\underline{t}+h}^s(n-j_\kappa, m-j_\kappa) \\
&= \left(1 - \lambda_{\underline{t}+h} H_{\underline{t}+h}(n-j_\kappa)\delta\right) J_{\underline{t}+h-\delta}^s(n-j_\kappa, m-j_\kappa) + \lambda_{\underline{t}+h} H_{\underline{t}+h}(n-j_\kappa)\delta J_{\underline{t}+h-\delta}^s(n-j_\kappa-1, m-j_\kappa) + o(\delta) \\
&\geq \left(1 - \lambda_{\underline{t}+h} H_{\underline{t}+h}(n)\delta\right) J_{\underline{t}+h-\delta}^s(n, m) + \lambda_{\underline{t}+h} H_{\underline{t}+h}(n)\delta J_{\underline{t}+h-\delta}^s(n-1, m) + o(\delta) \\
&\geq \left(1 - \lambda_{\underline{t}+h} H_{\underline{t}+h}(n)\delta\right) J_{\underline{t}+h-\delta}^s(n-j_\kappa, m-j_\kappa) + \lambda_{\underline{t}+h} H_{\underline{t}+h}(n)\delta J_{\underline{t}+h-\delta}^s(n-1, m) + o(\delta).
\end{aligned}$$

This would imply that

$$H_t(n-j_\kappa)\left[J_t^s(n-j_\kappa-1, m-j_\kappa) - J_t^s(n-j_\kappa, m-j_\kappa)\right] \geq H_t(n)\left[J_t^s(n-1, m) - J_t^s(n-j_\kappa, m-j_\kappa)\right],$$

and therefore leads to contradiction.

Next we show that $\tau^s(n,m) < \tau^s(n-1,m)$. Suppose this is not true. Then for any $\tau^s(n,m) \le t \le \tau^s(n-1,m)$, $J_t^s(n,m) > J_t^s(n-1,m-1)$ and $J_t^s(n-1,m) = J_t^s(n-2,m-1)$. Thus,

$$\frac{\partial J_t^s(n,m)}{\partial t} = \lambda_t H_t(n)\left[J_t^s(n-1,m) - J_t^s(n,m)\right] \le \lambda_t H_t(n-1)\left[J_t^s(n-2,m-1) - J_t^s(n,m)\right].$$

On the other hand, $\frac{\partial J_t^s(n-1,m-1)}{\partial t} \ge \lambda_t H_t(n-1)[J_t^s(n-2,m-1) - J_t^s(n-1,m-1)]$ according to Inequality (OA.7). Since $J^s_{\tau^s(n,m)}(n,m) = J^s_{\tau^s(n,m)}(n-1,m-1)$, $J_t^s(n,m) \le J_t^s(n-1,m-1)$ for any $t \in (\tau^s(n,m), \tau^s(n-1,m))$, which leads to contradiction.

Finally we prove statements (1) and (2) for n and m. For (1), if $t \le \tau^s(n,m)$, $\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)} = \frac{J_t^s(n-1,m-1)}{J_t^s(n-j,m-j)}$ increases in t from the induction assumption. Thus all we need to show is that for a given $t > \tau^s(n,m)$, $\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)} \ge 0$ for any $j \le \min\{j \ge 1 : \tau^s(n-j,m-j) < t\}$. This derivative is given by

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\\
=&\frac{\lambda_t H_t(n)[J_t^s(n-1,m) - J_t^s(n,m)]}{J_t^s(n-j,m-j)} - \frac{\lambda_t H_t(n-j) J_t^s(n,m)\left[J_t^s(n-j-1,m-j) - J_t^s(n-j,m-j)\right]}{[J_t^s(n-j,m-j)]^2}\\
=&\lambda_t\Bigg\{H_t(n)\left[\frac{J_t^s(n-1,m)}{J_t^s(n-j-1,m-j)}\frac{J_t^s(n-j-1,m-j)}{J_t^s(n-j,m-j)} - \frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\right] -\\
&H_t(n-j)\frac{J_t^s(n,m)}{J_t(n-j,m-j)}\left[\frac{J_t^s(n-j-1,m-j)}{J_t^s(n-j,m-j)} - 1\right]\Bigg\}\\
=&\lambda_t H_t(n)\frac{J_t^s(n-j-1,m-j)}{J_t^s(n-j,m-j)}\Bigg\{\frac{J_t^s(n-1,m)}{J_t^s(n-j-1,m-j)} - \frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\frac{J_t^s(n-j,m-j)}{J_t^s(n-j-1,m-j)} -\\
&\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\frac{H_t(n-j)}{H_t(n)}\left[1 - \frac{J_t^s(n-j,m-j)}{J_t^s(n-j-1,m-j)}\right]\Bigg\}\\
=&\lambda_t H_t(n)\frac{J_t^s(n-2,m-1)}{J_t^s(n-1,m-1)}\Bigg\{\frac{J_t^s(n-1,m)}{J_t^s(n-j-1,m-j)} -\\
&\left[1 + \left(\frac{H_t(n-j)}{H_t(n)} - 1\right)\left(1 - \frac{J_t^s(n-j,m-j)}{J_t^s(n-j-1,m-j)}\right)\right]\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\Bigg\}.
\end{aligned}$$

Note that $\left.\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\right|_{t=\tau^s(n,m)} = 1$ and $\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)} > 1$ for any $t > \tau^s(n,m)$. Thus $\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}$ increases in t initially. Now suppose there exists a t_1 such that $\left.\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\right|_{t=t_1} < 0$. We can then find a $t_2 < t_1$ such that $\left.\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}\right|_{t=t_2} = 0$ and $\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)} < 0$ for any $t \in (t_2, t_1]$. However, $\frac{J_t^s(n-1,m)}{J_t^s(n-j-1,m-j)}$ increases in t for any $t \ge \tau^s(n-1,m)$, $\frac{J_t^s(n-j,m-j)}{J_t^s(n-j-1,m-j)}$ increases in t for any $t \ge \tau^s(n-j,m-j)$, and $\frac{H_t(n-j)}{H_t(n)}$ decreases in t. This means that $\frac{\partial}{\partial t}\frac{J^s(n,m)}{J^s(n-j,m-j)} > 0$ over $(t_1,t_2]$, which leads to contradiction. Thus $\frac{J_t^s(n,m)}{J_t^s(n-j,m-j)}$ increases in t when $t \ge \tau^s(n,m)$.

Finally we prove statement (ii). Since $\tau^s(n,m) \geq \tau^s(n-1,m)$, for any $t \geq \tau^s(n,m)$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-1,m)} =& \lambda_t H_t(n)\left(1-\frac{J_t^s(n,m)}{J_t^s(n-1,m)}\right) - \lambda_t H_t(n-1)\frac{J_t^s(n,m)}{J_t^s(n-1,m)}\left(\frac{J_t^s(n-2,m)}{J_t^s(n-1,m)}-1\right)\\ =& \lambda_t H_t(n)\frac{J_t^s(n,m)}{J_t^s(n-1,m)}\left[\frac{J_t^s(n-1,m)}{J_t^s(n,m)}-1-\frac{H_t(n-1)}{H_t(n)}\left(\frac{J_t^s(n-2,m)}{J_t^s(n-1,m)}-1\right)\right].\end{aligned}$$

Note that $\frac{J_t^s(n,m)}{J_t^s(n-1,m)} < 1$ for any finite t and $\lim_{t\to\infty}\frac{J_t^s(n,m)}{J_t^s(n-1,m)} = 1$. This means that $\frac{J_t^s(n,m)}{J_t^s(n-1,m)}$ approaches 1 from below. Thus, if it does not increase in t for any $t \geq \tau^s(n+1,m)$, there must exist $t_2 > t_1 \geq \tau^s(n,m)$ such that $\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-1,m)}\Big|_{t=t_1} = 0$ and $\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-1,m)} \leq 0$ for any $t \in (t_1, t_2]$. However $\frac{H_t(n-1)}{H_t(n)}\left(\frac{J_t^s(n-2,m)}{J_t^s(n-1,m)}-1\right)$ strictly decreases in t from our induction assumption about $n-1$. This means that $\frac{\partial}{\partial t}\frac{J_t^s(n,m)}{J_t^s(n-1,m)} > 0$ over $(t_1,t_2]$, which leads to contradiction. We thus prove the announced results. □

OA.2.2. Feature Upgrade

Suppose the creator can make at most n_u updates according to a predefined sequence (, which is possibly determined by the potential benefit of each update). We use $m(\leq n_u)$ to denote the number of feature upgrades that remains to be implemented. With m upgrades left, the corresponding pledge likelihood is denoted as $\tilde{H}^{(m)}(q)$. More upgrades make the project more attractive in the sense that $\tilde{H}^{(0)}(q) \geq \tilde{H}^{(1)}(q) \geq \cdots \geq \tilde{H}^{(n_u)}(q) = H(q)$. Similarly, the corresponding success probability $\tilde{Q}_t^m(n)$ satisfies the condition $\tilde{Q}_t^{(0)}(n) > \tilde{Q}_t^{(1)}(n) > \cdots > \tilde{Q}_t^{(n_u)}(n) = Q_t(n)$. As a direct extension of Assumption 1(iii), we assume that $\frac{\tilde{H}^{(m-1)}(q)}{\tilde{H}^{(m)}(q)}$ decreases in q for any m.

Cost of ith update is assumed to be K_i, $i = 1, \ldots, n_u$. At the state of time-to-go t, pledges needed n and upgrades remaining m, the expected profit is denoted as $J_t^u(n,m)$.

Proposition OA.2. *For any (n,m), there exists a $0 \leq \tau^u(n,m) \leq \infty$, such that:*

- *When $t \leq \tau^u(n,m)$, the optimal strategy is to upgrade project features right away. That is, $J_t^u(n,m) = J_t^u(n,m-1)$ for any $t \leq \tau^u(n,m)$;*
- *When $t > \tau^u(n,m)$, the creator is better-off withholding feature upgrades. The expected profit $J_t^u(n,m)$ in this case is given by:*

$$\frac{\partial J_t^u(n,m)}{\partial t} = \lambda_t \tilde{H}_t(n,m)\left(J_t^u(n-1,m) - J_t^u(n,m)\right),$$

with boundary conditions $J^u_{\tau^u(n,m)}(n,m) = J^u_{\tau^u(n,m)}(n,m-1)$, $J_t^u(n,0) = (G+B-\sum_{i=1}^{n_u} K_i)\tilde{Q}_t^{(0)}(n)$, $J_0^u(0,m) = G+B-\sum_{i=m+1}^{n_u} K_i$, and $J_0^u(n,m) = 0$ for all $n > 0$.

Moreover, $\tau^u(n,m)$ increases in n.

Proof of Proposition OA.2. First, for any (n,m) and $t>0$, we have

$$J^u_{t+\delta}(n,m) \geq \left[1-\lambda_t \tilde{H}_t(n,m)\delta\right] J^u_t(n,m) + \lambda_t \tilde{H}_t(n,m)\delta J^u_t(n-1,m) + o(\delta).$$

Let $\delta \to 0$, we obtain the following inequality:

$$\frac{\partial J^u_t(n,m)}{\partial t} \geq \lambda_t \tilde{H}_t(n,m)\left[J^u_t(n-1,m) - J^u_t(n,m)\right]. \tag{OA.8}$$

For proof convenience, let $J^u_t(-1,m) = J^u_t(0,m)$ and $\tau^u_t(-1,m)=0$. It's easy to see that $J^u_t(0,m)$ is indeed the unique solution of the differential equation where $n=0$.

We add the following statements to the Proposition and prove by induction

(i) For any $0 \leq j \leq m$, $\frac{J^u_t(n,m)}{J^u_t(n,m-j)}$ increases in t.

(ii) $\frac{J^u_t(n,m)}{J^u_t(n-1,m)}$ increases in t for $t \geq \tau^u(n,m)$.

First when $m=0$, the threshold structure and the monotonicity of the thresholds follow immediately from our earlier denotation. Statement (i) is trivial as j can only be 0, and Statement (ii) holds according to Proposition 2.

Now suppose that the statements are true for any $n \geq 0$ and $m-1$ where $m \geq 1$. We next show that they must also hold for n and m. First it is obvious that the statements hold for $n=0$. Now assume that they hold for $n-1$ where $n \geq 1$. We first show that the optimal stimulus strategy is a threshold policy. If this is not true, then we can find a time interval $(\underline{t}, \underline{t}+h)$, over which the optimal strategy is not to upgrade the project features, i.e., $J^u_t(n,m) > J^u_t(n,m-1)$ over $(\underline{t}, \underline{t}+h)$. Because of the continuity of $J^u_t(n,m)$, we have $J^u_{\underline{t}}(n,m) = J^u_{\underline{t}}(n,m-1)$ and $J^u_{\underline{t}+h}(n,m) = J^u_{\underline{t}+h}(n,m-1)$. For each $t \in [\underline{t}, \underline{t}+h]$, there exists a $j = \min\{i \geq 1 : \tau^u(n,m-i) \leq t\}$. We collect all those unique j's and denote them as $j_0 > j_1 > \cdots > j_\kappa$, where $j_0 = \min\{i \geq 1 : \tau^u(n,m-i) \leq \underline{t}\}$ and $j_\kappa = \min\{i \geq 1 : \tau^u(n,m-i) \leq \underline{t}+h\}$. When $t = \underline{t}+\delta$, the optimal strategy is not to upgrade project features, but $J^u_{\underline{t}}(n,m) = J^u_{\underline{t}}(n,m-j_0)$. Thus,

$$\begin{aligned} J^u_{\underline{t}+\delta}(n,m) &= \left(1-\lambda_{\underline{t}}\tilde{H}_{\underline{t}}(n,m)\delta\right) J^u_{\underline{t}}(n,m) + \lambda_{\underline{t}}\tilde{H}_{\underline{t}}(n,m)\delta J^u_{\underline{t}}(n-1,m) + o(\delta) \\ &> \left(1-\lambda_{\underline{t}}\tilde{H}_{\underline{t}}(n,m-j_0)\delta\right) J^u_{\underline{t}}(n,m-j_0) + \lambda_{\underline{t}}\tilde{H}_{\underline{t}}(n,m-j_0)\delta J^u_{\underline{t}}(n-1,m-j_0) + o(\delta). \end{aligned}$$

Let $\delta \to 0$, we have

$$\tilde{H}_t(n,m)\left[J^u_t(n-1,m) - J^u_t(n,m-j_0)\right] > \tilde{H}_t(n,m-j_0)\left[J^u_t(n-1,m-j_0) - J^u_t(n,m-j_0)\right],$$

at $t=\underline{t}$. Rearrange the terms, at $t=\underline{t}$, we have

$$\left(\frac{\tilde{H}_t(n,m-j_0)}{\tilde{H}_t(n,m)} - 1\right)\left[1 - \frac{J^u_t(n,m-j_0)}{J^u_t(n-1,m-j_0)}\right] \leq \frac{J^u_t(n-1,m)}{J^u_t(n-1,m-j_0)}.$$

According to our induction assumptions, LHS decreases in t for $t \geq \underline{t} \geq \tau^u(n,m-j_0)$ and RHS increases in t. Thus the above inequality holds for any $t > \underline{t}$. In addition, $J_t^u(n,m-j_1) = J_t^u(n,m-j_0)$ for any $\tau^u(n,m-j_0) < t \leq \tau^u(n,m-j_1)$, which leads to

$$\frac{\partial J_t^u(n,m-j_1)}{\partial t} = \frac{\partial J_t^u(n,m-j_0)}{\partial t} = \tilde{H}_t(n,m-j_0)\left[J_t^u(n-1,m-j_0) - J_t^u(n,m-j_0)\right].$$

According to Inequality (OA.8),

$$\tilde{H}_t(n,m-j_0)\left[J_t^u(n-1,m-j_0) - J_t^u(n,m-j_0)\right] \geq \tilde{H}_t(n,m-j_1)\left[J_t^u(n-1,m-j_1) - J_t^u(n,m-j_1)\right].$$

Coupling with the fact that $J_t^u(n,m-j_0) = J_t^u(n,m-j_1)$ for any $t \in [\tau^u(n,m-j_0), \tau^u(n,m-j_1)]$, we thus have

$$\tilde{H}_t(n,m)\left[J_t^u(n-1,m) - J_t^u(n,m-j_1)\right] > \tilde{H}_t(n,m-j_1)\left[J_t^u(n-1,m-j_1) - J_t^u(n,m-j_1)\right].$$

Similarly, we can show that for any $t > \tau^u(n,m-j_\kappa)$,

$$\tilde{H}_t(n,m)\left[J_t^u(n-1,m) - J_t^u(n,m-j_\kappa)\right] > \tilde{H}_t(n,m-j_\kappa)\left[J_t^u(n-1,m-j_\kappa) - J_t^u(n,m-j_\kappa)\right].$$

However for any $t > \underline{t} + h$,

$$\begin{aligned}
&J_t^u(n,m) = J_t^u(n,m-j_\kappa)\\
&= \left(1 - \lambda_{t-\delta}\tilde{H}_{t-\delta}(n,m-j_\kappa)\delta\right) J_{t-\delta}^u(n,m-j_\kappa) + \lambda_{t-\delta}\tilde{H}_{t-\delta}(n,m-j_\kappa)\delta J_{t-\delta}^u(n-1,m-j_\kappa)\\
&\geq \left(1 - \lambda_{t-\delta}\tilde{H}_{t-\delta}(n,m)\delta\right) J_{t-\delta}^u(n,m-j_\kappa) + \lambda_{t-\delta}\tilde{H}_{t-\delta}(n,m)\delta J_{t-\delta}^u(n-1,m) + o(\delta).
\end{aligned}$$

Let $\delta \to 0$,

$$\tilde{H}_t(n,m)\left[J_t^u(n-1,m) - J_t^u(n,m-j_\kappa)\right] \leq \tilde{H}_t(n,m-j_\kappa)\left[J_t^u(n-1,m-j_\kappa) - J_t^u(n,m-j_\kappa)\right],$$

which leads to contradiction. Thus a unique threshold $\tau^u(n,m)$ exists, such that the optimal policy is to upgrade the features when $t \leq \tau^u(n,m)$, and not to upgrade when $t > \tau^u(n,m)$.

Next we show $\tau^u(n,m) \geq \tau^u(n-1,m)$ by contradiction. Suppose this is not true. Then for any $\tau^u(n,m) < t \leq \tau^u(n-1,m)$, $J^u(n,m) > J_t^u(n,m-1)$ and $J_t^u(n-1,m) = J_t^u(n-1,m-1)$. Thus,

$$\begin{aligned}
\frac{\partial J_t^u(n,m)}{\partial t} &= \lambda_t\tilde{H}_t(n,m)\left[J_t^u(n-1,m) - J_t^u(n,m)\right]\\
&= \lambda_t\tilde{H}_t(n,m)\left[J_t^u(n-1,m-1) - J_t^u(n,m)\right]\\
&< \lambda_t\tilde{H}_t(n,m-1)\left[J_t^u(n-1,m-1) - J_t^u(n,m)\right].
\end{aligned}$$

On the other hand, $\frac{\partial J_t^u(n,m-1)}{\partial t} \geq \lambda_t \tilde{H}_t(n,m-1)[J_t^u(n-1,m-1) - J_t^u(n,m-1)]$ according to Inequality (OA.8). Since $J^u_{\tau^u(n,m)}(n,m) = J^u_{\tau^u(n,m)}(n,m-1)$, $J_t^u(n,m) < J_t^u(n,m-1)$ for any $\tau^u(n,m) < t \leq \tau^u(n-1,m)$, which leads to contradiction.

Now we show that statements (i) and (ii) are true for n. For any $t \leq \tau^u(n,m)$, $\frac{J_t^u(n,m)}{J_t^u(n,m-j)} = \frac{J_t^u(n,m-1)}{J_t^u(n,m-j)}$ increases in t based on the induction assumptions. Thus in order to prove statement (i), we only need to focus on $t > \tau^u(n,m)$. Without loss of generality, we assume that $\tau^u(n,m-j) < \tau^u(n,m)$ (Otherwise $J_t^u(n,m-j) = J_t^u(n,m-j')$, where $j' = \min\{i < j : \tau^u(n,m-i) < \tau^u(n,m)\}$ for any $t > \tau^u(n,m)$).

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n,m-j)} =& \frac{\lambda_t \tilde{H}_t(n,m)[J_t^u(n-1,m) - J_t^u(n,m)]}{J_t^u(n,m-j)} - \\
& \frac{\lambda_t \tilde{H}_t(n,m-j) J_t^u(n,m)\left[J_t^u(n-1,m-j) - J_t^u(n,m-j)\right]}{[J_t^u(n,m-j)]^2} \\
=& \lambda_t \Bigg\{ \tilde{H}_t(n,m)\left[\frac{J_t^u(n-1,m)}{J_t^u(n-1,m-j)}\frac{J_t^u(n-1,m-j)}{J_t^u(n,m-j)} - \frac{J_t^u(n,m)}{J_t^u(n,m-j)}\right] - \\
& \tilde{H}_t(n,m-j)\frac{J_t^u(n,m)}{J_t^u(n,m-j)}\left[\frac{J_t^u(n-1,m-j)}{J_t^u(n,m-j)} - 1\right]\Bigg\} \\
=& \lambda_t \tilde{H}_t(n,m)\frac{J_t^u(n-1,m-j)}{J_t^u(n,m-j)}\Bigg\{\frac{J_t^u(n-1,m)}{J_t^u(n-1,m-j)} - \frac{J_t^u(n,m)}{J_t^u(n,m-j)}\frac{J_t^u(n,m-j)}{J_t^u(n-1,m-j)} - \\
& \frac{J_t^u(n,m)}{J_t^u(n,m-j)}\frac{\tilde{H}_t(n,m-j)}{\tilde{H}_t(n,m)}\left[1 - \frac{J_t^u(n,m-j)}{J_t^u(n-1,m-j)}\right]\Bigg\} \\
=& \lambda_t \tilde{H}_t(n,m)\frac{J_t^u(n-1,m-j)}{J_t^u(n,m-j)}\Bigg\{\frac{J_t^u(n-1,m)}{J_t^u(n-1,m-j)} - \\
& \left[1 + \left(\frac{\tilde{H}_t(n,m-j)}{\tilde{H}_t(n,m)} - 1\right)\left(1 - \frac{J_t^u(n,m-j)}{J_t^u(n-1,m-j)}\right)\right]\frac{J_t^u(n,m)}{J_t^u(n,m-j)}\Bigg\}.
\end{aligned}
$$

When $t = \tau^u(n,m)$, $\frac{J_t^u(n,m)}{J_t^u(n,m-j)} = 1$, whereas $\frac{J_t^u(n,m)}{J_t^u(n,m-j)} > 1$ for any $t > \tau^u(n,m)$. Thus, $\frac{J_t^u(n,m)}{J_t^u(n,m-j)}$ increases in t at $t = \tau^u(n,m)$. Suppose there exists a $t_1 > \tau^u(n,m)$ such that $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n,m-j)} < 0$. Because of the continuity of $\frac{J_t^u(n,m)}{J_t^u(n,m-j)}$, there must exist a $t_2 < t_1$ such that $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n,m-j)}\Big|_{t=t_2} = 0$ and $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n,m-1)} < 0$ for any $t \in (t_2, t_1]$. However, $\frac{J_t^u(n-1,m)}{J_t^u(n-1,m-j)}$ strictly increases in t according to our induction assumption, and $\frac{\tilde{H}_t(n,m-j)}{\tilde{H}_t(n,m)}$ decreases in t according to our assumption. This means that $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n,m-j)} \geq 0$ over $(t_2, t_1]$, which leads to contradiction.

Finally we prove statement (ii) by contradiction. Suppose it is not true. Then there must exist $t_2 > t_1 \geq \tau^u(n,m)$, such that $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n-1,m)}\big|_{t=t_1} = 0$ and $\frac{\partial}{\partial t}\frac{J_t^u(n,m)}{J_t^u(n-1,m)} < 0$ for any $t \in (t_1, t_2]$. Because $\tau^u(n,m) \geq \tau^u(n-1,m)$, we have

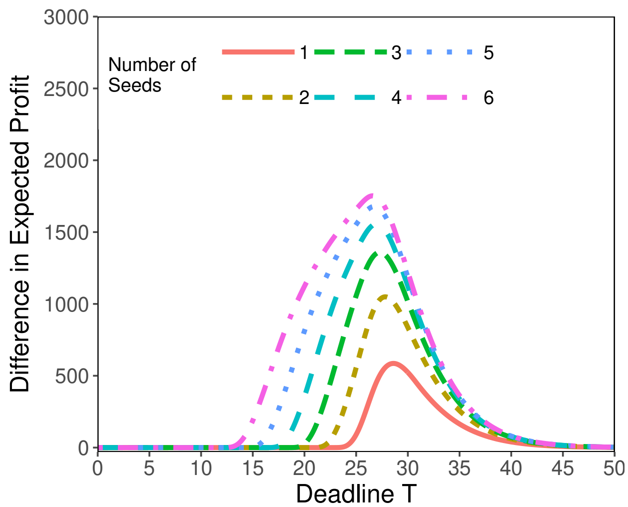
$$\frac{\partial}{\partial t}\frac{J^u_t(n,m)}{J^u_t(n-1,m)} = \lambda_t\tilde{H}_t(n,m)\left(1-\frac{J^u_t(n,m)}{J^u_t(n-1,m)}\right) - \lambda_t\tilde{H}_t(n-1,m)\frac{J^u_t(n,m)}{J^u_t(n-1,m)}\left(\frac{J^u_t(n-2,m)}{J^u_t(n-1,m)}-1\right)$$
$$= \lambda_t\tilde{H}_t(n,m)\frac{J^u_t(n,m)}{J^u_t(n-1,m)}\left[\frac{J^u_t(n-1,m)}{J^u_t(n,m)} - 1 - \frac{\tilde{H}_t(n-1,m)}{\tilde{H}_t(n,m)}\left(\frac{J^u_t(n-2,m)}{J^u_t(n-1,m)}-1\right)\right].$$

However, $\frac{J^u_t(n-2,m)}{J^u_t(n-1,m)}$ decreases in t as $t \geq \tau^u(n,m) \geq \tau^u(n-1,m)$, and $\frac{\tilde{H}_t(n-1,m)}{\tilde{H}_t(n,m)}$ decreases in t according to Theorem 1(iii). Consequently, if $\frac{J^s_t(n-1,m)}{J^s_t(n,m)}$ decreases in t, then $\frac{\partial}{\partial t}\frac{J^u_t(n,m)}{J^u_t(n-1,m)} \geq 0$ over $(t_1, t_2]$, which leads to contradiction. We thus obtained the announced results. □

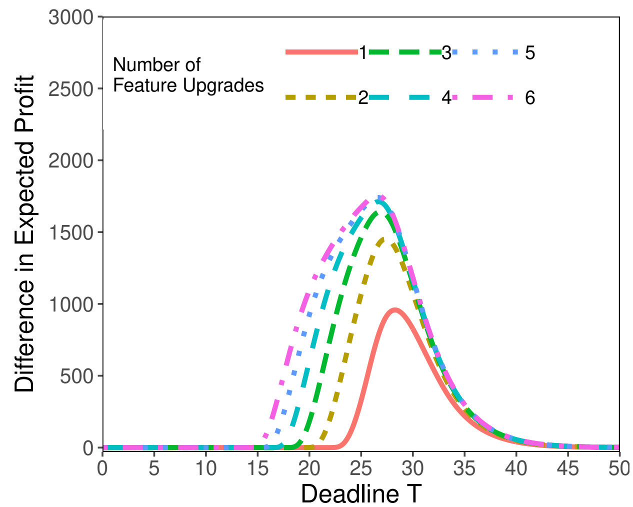
OA.2.3. Numerical Experiments

In this section, we complement our analytical results with a numerical analysis illustrating the benefit of multiple rounds of stimuli. Parameters of the numerical experiments are specified as follows. For seeding, we consider the case where each seeding stimulus allows the creator to acquire 1 pledge at a cost of \$120. For feature upgrade, we consider the case where each upgrade in project features costs the creator $K = \$120$, and improves the project quality by 0.1.

Figure OA.1 Benefits of Mutiple Rounds of Stimulus Policies



(a) Improvement in Expected Profit: *Seeding*



(b) Improvement in Expected Profit: *Feature Upgrade*

Note: $V \sim \exp(\frac{1}{100})$, $p = \$120$, $\theta = 1$, $c = \$30$, $G = \$1{,}800$ (i.e., $N = 15$), $B = \$500$, $T = 30$ and $\lambda_t = 2$. The benchmark is the base model with no stimulus.

Figure OA.1 illustrates the change in the expected profit w.r.t. the number of stimuli and the deadline. While access to additional rounds of stimuli always improve the expected profit, the absolute benefit is non-monotonic w.r.t. the deadline T. When the deadline T is small, having more rounds of the stimuli helps little because the project has little chance to succeed even if multiple stimuli are applied. At the other end of the spectrum, when the deadline T is sufficiently large, again multiple rounds of stimuli render little benefit as the project is likely to reach the target without help of any stimulus policies. Similar to our observation from the numerical analysis in Section 4.4, we also observe that the stimulus policy with multiple rounds of updates is the most effective when the remaining time is neither too long or too short.