Model Matching and Passivation of MIMO Linear Systems via Dynamic Output Feedback and Feedforward

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Abstract—A model matching and passivating control architecture for MIMO linear systems, comprising dynamic feedback and feedforward, is proposed. The approach - essentially without any restriction on the relative degree and the zeroes of the underlying system and by relying only on input/output measurements - provides a closed-loop system the transfer matrix of which matches any desired matrix of rational functions. An alternative implementation of the above design allows to achieve an arbitrary approximation accuracy of a desired transfer matrix while also preserving structural properties - in particular observability - of the overall interconnected system. Such a construction can be then specialized to provide input/output decoupling or a system that is passive from a novel control input to a modified output. The result is achieved by arbitrarily assigning the relative degree and the location of poles and zeros on the complex plane of the interconnected system in a systematic way. It is also shown that similar ideas can be employed to enforce a desired, arbitrarily small, $\mathcal{L}_2$-gain from an unknown disturbance input to a modified output, while preserving the corresponding gain from the control input to the same output. The paper is concluded with applications and further discussions on the results.

I. INTRODUCTION

Due to the rapidly increasing diffusion of large-scale systems - possibly distributed over physical or communication networks - the description of complex interconnected systems in terms of input/output energy exchange patterns has gradually gained a central role in control theory. In this respect, the ability of assessing, or even imposing, desirable energetic properties for each node of the overall network is of paramount importance. Such properties are typically provided in terms of dissipation requirements for the input/output behavior of each subsystem, thus leading to the concept of passivity or more general dissipativity notions, such as finite $\mathcal{L}_2$-gain from a specific input to the interconnecting output, see e.g. [38] for a comprehensive discussion on the topic.

Such properties, in addition to the input/output energy characterization, are rendered even more desirable by their intrinsic relation with (asymptotic) stability - see e.g. [7], [25], [28], [34] for applications to the stabilization task for composite or interconnected systems - and robustness [14]. As a result, intensive research effort has been devoted to envision control architectures that enforce passivity properties of the resulting closed-loop system, see [1], [3], [6], [10], [12], [15], [21], [23], [26], [27], [41], [44] and references therein, in those cases in which they are not originally possessed by the underlying system. Interestingly, in the case of linear time-invariant systems, these concepts are intimately related to the notion of Positive Real (PR) or of Strictly Positive Real (SPR) systems, which has been deeply explored not only in control but also in circuit theory, [2], [31], [32], [40]. In [18] a generalization of the Feedback Positive Real (FPR) design, introduced in [25], is discussed and subsequently extended to the setting of switching systems in [19]. The construction relies upon an observer-based state feedback control and on the circle criterion, thus increasing the realm of applicability of the solution proposed in [30], which yields a synthesis strategy in terms of a state feedback under the assumption that the underlying system is minimum-phase and possesses relative degree one.

Several alternative techniques, such as series, feedback or feedforward interconnections, are discussed in [21] - with the aim of extending passivity-based techniques to systems that are not passive - for which an LMI characterization is given in [22]. In particular, the control architectures discussed in [1], [5], [13], [15], [27], [37] rely only on (state or output) feedback and, as a consequence, allow to impose passivity for specially structured classes of systems, while, on the other hand, preserving the original output function as a desirable side effect. In fact, for linear time invariant systems, passivity requires the properties of minimum-phasesness and relative degree one, which are not affected by feedback alone, hence the role of feedforward action has been immediately recognized and emphasized, see [3], [7] and [4] in the context of adaptive control. Therefore, a passivating technique consisting of static feedback/feedforward actions is proposed in [41], while [10], based on the characterization of SPR systems in terms of the Kalman-Yakubovich-Popov lemma [9], [33], envisions the use of observers, thus leading to a dynamic, rather than static, construction and yielding an LMI-based approach for passivation of MIMO systems. The design introduced in [17],
instead, is based on reduced-order observers for the original linear system. A suitably designed input-output transformation matrix is proposed in [43]: this allows to guarantee positive passivity indices. Similar ideas are employed in [45] in the case of event-triggered feedback systems, whereas the main objective of [42] consists in assessing passivity properties of a certain system based only on approximate models of the plant itself.

Differently from the former constructions hitherto based on feedforward actions, herein, a dynamic, instead of static, control scheme is proposed, which permits, differently from the methods available in the literature, the closed-form and exact placement of the resulting closed-loop zeros on the complex plane. Finally, it is worth mentioning that existing techniques based on feedforward compensation, see e.g. [10], [24], typically do not allow to systematically preserve observability of the overall extended system comprising the original plant and the observer, and the designer may wonder how much information on the plant is retained from the auxiliary output or if the resulting passivity property is essentially possessed only by the dynamic (feedforward) compensator itself. This crucial aspect is discussed, and tackled, in detail in the following sections.

The main contribution of the paper consists in the definition of a control architecture - comprising dynamic feedback as well as dynamic feedforward - that allows to enforce desired dissipativity properties to the closed-loop system by relying only on output measurement. More precisely, the main aspect of interest of the methodology is that such an approach, despite its intrinsic simplicity - in terms of design procedure and use of available degrees of freedom - allows to achieve, essentially without any restriction, desired passivity (relative degree, minimum phaseness) and more general dissipativity (e.g. $L_2$-gain) properties. Moreover, such objectives are obtained in a systematic, though flexible, manner. A preliminary version of the results discussed here has appeared in [35]. With respect to [35], here we provide the extension to Multi-Input/Multi-Output (MIMO) plants, together with several additional results - concerning for instance the ability of preserving structural properties in the closed-loop system as well as the (non-generic) case of plants that are not controllable - and the proof of the main results. Moreover, a more general result concerning the adaptive control of non-minimum phase systems is considered.

The rest of the paper is organized as follows. Useful notation and preliminary results on matrix transfer functions are recalled in Section II, while the problem under investigation is introduced and discussed in Section III, together with notation and the main standing assumptions. The results are specialized to the case of SISO systems in Section IV, providing additional interesting insights on the construction. The application of the design methodology to the adaptive control problem for non-minimum phase systems is the topic of Section V. Finally, conclusions are drawn in Section VI.

**II. NOTATION AND PRELIMINARIES**

Consider a Multi-Input Multi-Output (MIMO) matrix transfer function with real coefficients described by

$$W(s) = \frac{N(s)}{d(s)},$$

with $s \in \mathbb{C}$, $W(s) \in \mathbb{C}^{p \times m}$ and where $d(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$ denotes the least common multiple of the denominators of the entries of the transfer matrix $W(s)$, hence describing the minimal polynomial, and $N(s) = N_1 s^{n-1} + N_2 s^{n-2} + \ldots + N_{n-1} s + N_n$, with $N_i \in \mathbb{R}^{p \times m}$, for $i = 1, \ldots, n$. Clearly, $n$ denotes the highest degree of the polynomial $d(s)$, whereas $p$ and $m$ represent the number of outputs and inputs, respectively. Note that, differently from the previous definition, in [8, Definition 7.1] the degree of the transfer matrix is defined in terms of the least common denominators of all minors of $W(s)$, hence associated to the characteristic polynomial.

**Remark 1:** It may be possible to encompass also non-strictly proper functions in (1), i.e. in the presence of a feedthrough term. This can be achieved with minor modifications of the following discussions, which however unnecessarily increase the notational burden, hence are omitted (see also Example 4 in Section IV).

It is well-known that MIMO systems possess in general minimal realizations, namely controllable and observable state-space descriptions, with state dimension greater than the degree $n$ of $d(s)$, see e.g. [20, Chapter 6]. Similarly to SISO systems, several alternative state-space representations of minimal dimension can be considered. Among such descriptions, one of particular interest is the so-called Gilbert’s Diagonal Realization [20], obtained for systems in which the denominator possesses distinct roots, namely such that $d(s) = \Pi_{i=1}^n (s - \lambda_i)$, with $\lambda_i \neq \lambda_j$ for all $i \neq j$. To this end, consider the expansion of $W(s)$ into partial fractions

$$W(s) = \sum_{i=1}^n \frac{R_i}{s - \lambda_i},$$

and let $q_i$ denote the rank of the residual $R_i$ of dimension $p \times m$. Then, $n_{\text{min}} = \sum_{i=1}^n q_i$ and a minimal realization, of order $q_1 + \ldots + q_n \geq n$, is obtained by considering the matrices

$$A = \begin{bmatrix}
\lambda_1 I_{q_1} & 0 & \ldots & 0 \\
0 & \lambda_2 I_{q_2} & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n I_{q_n}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_n
\end{bmatrix},$$

and $C = \begin{bmatrix}
C_1 & C_2 & \ldots & C_n
\end{bmatrix}$, where the matrices $B_1$ and $C_i$, of dimensions $q_i \times m$ and $p \times q_i$, respectively, are such that $R_i = C_i B_i$.

1Note that in general (3) may provide a state-space complex realization. Nonetheless, since the coefficients of $W(s)$ are real, if there is $\lambda_i \in \mathbb{C}$ then there must be $\lambda_i \in \mathbb{C}$ such that $\lambda_i = \lambda_i^*$, namely the complex-conjugate of $\lambda_i$. As a consequence, also the corresponding individual realizations are complex-conjugate and an equivalent real realization that preserves the parallel block-diagonal structure of (3) can be derived by combining, with standard techniques, the realizations of $\lambda_i$ and $\lambda_j$. 
Remark 2: Since the set of matrices that have $n$ distinct eigenvalues is open and dense in the set of square matrices of dimension $n$, it follows that the dimension of the minimal realization $n_{\text{min}}$ is generically - namely with probability one for randomly selected problem data - equal to $\sum_{i=1}^{n} \varrho_i$, with the $\varrho_i$’s defined above. Moreover, since also the residual matrices $R_i$ are generically full-rank, given a randomly selected transfer matrix $W(s)$ as in (1), it follows that, generically, $n_{\text{min}} = \min\{np, nm\}$. ▲

As a consequence of the discussions in Remark 2 and in the case $m \geq p$ (namely systems possessing at least the same number of control inputs as outputs), which is a scenario of specific interest for the model matching and passivation problems dealt with below, it follows that also the so-called Block Observer Form, of dimension $np$, is, generically, a minimal realization of a given transfer matrix $W(s)$. More precisely, such state-space description of the transfer matrix (1) is given by a LTI system defined by the equations

$$\Sigma_p := \begin{cases} \dot{x} &= (-a \otimes C_o + A_o) x + B_o u, \\
y &= C_o x, \end{cases} \tag{4}$$

where $a = [a_1, \ldots, a_n]^T \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^{np}$, with $\otimes$ denoting the Kronecker product between two matrices, and the matrices $A_o$, $B_o$ and $C_o$ are defined by

$$A_o = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix}, \tag{5}$$

and $C_o = \begin{bmatrix} I_p & 0 & \cdots & 0 \end{bmatrix}$, where the coefficients $a_i$ and the matrices $N_i$ have been introduced in the discussion below equation (1). In the following sections, $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n \times n}$.

III. MODEL MATCHING, INPUT/OUTPUT DECOUPLING AND PASSIVATION

In this section we discuss the main results concerning model matching and passivation of LTI MIMO systems by output feedback and feedforward on the measured output. The objective is initially achieved under generic assumptions, namely for almost all transfer matrices, whereas specialized cases are discussed in what follows. For clarity, in the case in which the plant $\Sigma_p$ possesses at least the same number of inputs as outputs such assumptions are explicitly stated, despite the fact that, by the arguments in Remark 2 and the discussions therein, they hold generically.

Assumption 1: The state-space realization $\Sigma_p$ of the transfer matrix $W(s)$ in (1) is minimal, hence $\Sigma_p$ is controllable, as well as observable by construction.

The control objectives are formalized in the following statement.

Problem 1: Consider a matrix transfer function $W(s)$ as in (1) and suppose that Assumption 1 holds. Find, if possible, an integer $\nu \geq 0$ and a dynamic compensator $\Sigma_c$ described in state-space form by the equations

$$\dot{\xi} = A_c \xi + B_c y + B v, \tag{6a}$$

$$u = K_c \xi, \tag{6b}$$

$$y_c = C_c \xi + D_v, \tag{6c}$$

with $\xi(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^m$, and with matrices of appropriate dimensions, such that the interconnected system $\Sigma_c$ obtained setting $\hat{y} = y + y_c$ (see Figure 1 for the control architecture) is asymptotically stable and possesses one of the following properties.

(G1) Given a desired asymptotically stable rational matrix $W_d(s) \in \mathbb{C}^{p \times m}$, the behavior of $\Sigma_c$ from the input $v$ to the output $\hat{y} = y + y_c$, described by the transfer matrix $W_c(s)$, matches that prescribed by $W_d(s)$, namely $W_c(s) = W_d(s)$ for all $s \in \mathbb{C}$.

(G2) Assume $m = p$, $\Sigma_c$ is such that $\hat{y}_i(t)$ is affected only by $v_i(t)$, for all $t \geq 0$ and $i = 1, \ldots, m$.

(G3) Assume $m = p$, $\Sigma_c$ is passive from the input $v$ to the output $\hat{y}$, namely there exists a constant $\beta$ such that

$$\int_0^\tau \hat{y}(t)^T v(t) dt \geq \beta, \tag{7}$$

for all functions $v \in L_2(0, \tau)$ and all $\tau \geq 0$. □

The three tasks stated above formalize the problems of Model Matching, Input/Output Decoupling and Passivation, respectively.

A. Model Matching

The following statement provides a solution to the first of such objectives, under the (generic) Assumption 1. The nongeneric case, namely the case of transfer matrices (1) for which the state-space realization (4) is not controllable, is discussed below.

Proposition 1: Consider a matrix transfer function $W(s)$ as in (1) and suppose that Assumption 1 holds. Let $W_d(s) \in L_2(0, \tau)$ denote the functional space of all square integrable functions over the interval $[0, \tau]$, namely $v \in L_2(0, \tau)$ if $\int_0^\tau \|v(t)\|^2 dt < \infty$.

$$\int_0^\tau \hat{y}(t)^T v(t) dt \geq \beta, \tag{7}$$

for all functions $v \in L_2(0, \tau)$ and all $\tau \geq 0$. □


3$L_2(0, \tau)$ denotes the functional space of all square integrable functions over the interval $[0, \tau]$, namely $v \in L_2(0, \tau)$ if $\int_0^\tau \|v(t)\|^2 dt < \infty$. □
\( CP \times m \) be any desired proper, asymptotically stable rational matrix. Let the compensator \( \Sigma_c \) be described by the equations
\[
\begin{align*}
\dot{\xi} &= (-\alpha \otimes C_o + A_o - B_o K)\xi + (a - \alpha) \otimes y + B v, \\
u &= K \xi, \\
y_{\xi} &= C_o \xi + D v,
\end{align*}
\]
with \( \xi(t) \in \mathbb{R}^{np}, \alpha = [\alpha_1, ..., \alpha_n]^T, B = [\tilde{N}_1^T, ..., \tilde{N}_n^T]^T \) and \( K \) such that \( \sigma(-\alpha \otimes C_o + A_o - B_o K) \subset \mathbb{C}^- \), and consider the feedback/feedforward interconnection of \( \Sigma_p \) and \( \Sigma_c \), namely \( \Sigma_e \) with input \( v \) and output \( y = y_{\xi} \), described by the matrix transfer function \( W_e(s) \). Then there exist \( \alpha \in \mathbb{R}^n \), \( \tilde{N}_i \in \mathbb{R}^{p \times m}, i = 1, ..., n \), and \( D \in \mathbb{R}^{p \times m} \) such that \( W_e(s) = W_d(s) \) for all \( s \in \mathbb{C} \).

**Proof:** To begin with consider the change of coordinates \( z = x + \xi \), with \( x \) denoting the state of the Block Observer realization (4) of \( W(s) \), which is controllable and observable by Assumption 1. The corresponding dynamics in the transformed coordinates are given by
\[
\begin{align*}
\dot{z} &= (-\alpha \otimes C_o + A_o) x + B_o u + \dot{\xi} \\
&= (-\alpha \otimes C_o + A_o) z - (-\alpha \otimes C_o + A_o) \xi + B_o u + \dot{\xi} \\
\dot{\xi} &= (-\alpha \otimes C_o + A_o) z + B v,
\end{align*}
\]
where the last equation is obtained by the dynamics of \( \dot{\xi} \) as in (7), recalling that \( u = K \xi \) and defining \( \dot{y} = y + C_o \xi + D v \). Then, the closed-loop system is described by the equations
\[
\begin{align*}
\dot{z} &= (-\alpha \otimes C_o + A_o) z + B v, \\
\dot{\xi} &= (-\alpha \otimes C_o + A_o - B_o K)\xi + (a - \alpha) \otimes C_o z + B v, \\
\dot{y} &= C_o z + D v,
\end{align*}
\]
that is, it possesses the closed-loop matrices
\[
\begin{align*}
A_z &= \begin{bmatrix}
-\alpha \otimes C_o + A_o & 0 \\
(a - \alpha) \otimes C_o & -\alpha \otimes C_o + A_o - B_o K
\end{bmatrix}, \\
B_z &= \begin{bmatrix}
B \\
B
\end{bmatrix}, \\
C_z &= \begin{bmatrix}
C_o & 0
\end{bmatrix}.
\end{align*}
\]
Note that \( \xi \) describes an unobservable component of the extended \((z, \xi)\)-state. It is now straightforward to note that the resulting closed-loop transfer matrix \( W_c(s) \) is described by the transfer matrix
\[
W_c(s) = \frac{\tilde{N}_1 s^{n-1} + ... + \tilde{N}_n s + \tilde{N}_n}{s^n + \alpha_1 s^{n-1} + ... + \alpha_n s + \alpha_n} + D,
\]
with the coefficients \( \alpha_i \), the matrices \( \tilde{N}_i \), for \( i = 1, ..., n \) and the feed-through term \( D \) arbitrarily assignable to match any desired proper rational matrix \( W_d(s) \).

Interestingly, the proof of the result above suggests that the role of the control input \( u \) consists essentially in stabilizing the compensator \( \Sigma_c \), rather than the plant. Moreover, the statement of Proposition 1 entails that the arbitrary shaping of the closed-loop transfer matrix is achieved at the price of observability of the extended (plant and compensator) system. More precisely, the feature that, in the transformed \((z, \xi)\)-coordinates, only the state \( z \) is observable implies that in the closed-loop system the behavior of the state of the original plant \( x \) becomes essentially indistinguishable from the time evolution of the internal state of the compensator \( \xi \). This issue is resolved in the following result, in which it is shown that the resulting closed-loop transfer matrix may be shaped to approximate, with an arbitrary accuracy, a desired matrix while preserving structural properties, specifically observability, in the interconnected plant. As a by-product, such a requirement additionally rules out the somewhat trivial solution in which a direct feed-through from \( y \) is exploited in \( \Sigma_c \) to define the output as \( y_{\xi} = -y + y_{\xi} \), thus completely canceling, in the modified output \( \dot{y} \), the contribution of the plant \( \Sigma_p \), i.e. making the plant unobservable.

**Proposition 2:** Consider a matrix transfer function \( W(s) \) as in (1) and suppose that Assumption 1 holds. Let \( W_d(s) \in CP \times m \) be any proper, asymptotically stable rational matrix. Let the compensator \( \Sigma_{e,\delta} \) be described by the equations
\[
\begin{align*}
\dot{\xi} &= (-\alpha \otimes C_o + A_o - B_o K)\xi + (a - \alpha) \otimes y + \delta \xi + B v, \\
u &= K \xi, \\
y_{\xi} &= C_o \xi + D v,
\end{align*}
\]
with \( \xi(t) \in \mathbb{R}^{np}, \alpha = [\alpha_1, ..., \alpha_n]^T, B = [\tilde{N}_1^T, ..., \tilde{N}_n^T]^T \) and \( K \) such that \( \sigma(-\alpha \otimes C_o + A_o - B_o K) \subset \mathbb{C}^- \), and consider the feedback/feedforward interconnection of \( \Sigma_p \) and \( \Sigma_{e,\delta} \), namely \( \Sigma_{e,\delta} \) with input \( v \) and output \( \dot{y} = y + y_{\xi} \), described by the matrix transfer function \( W_{e,\delta}(s) \). Then there exist \( \alpha \in \mathbb{R}^n \) and \( \tilde{N}_i \in \mathbb{R}^{p \times m}, i = 1, ..., n \), such that for any \( \varepsilon > 0 \) there exists \( \delta \neq 0 \) such that
\[
\begin{align*}
i) & \| W_{e,\delta}(s) - W_d(s) \|_\infty \leq \varepsilon; \\
ii) & \text{the interconnected system } \Sigma_{e,\delta} \text{ is observable.}
\end{align*}
\]

**Proof:** The claim is shown by employing arguments similar to those in the proof of Proposition 1. More precisely, the main difference consists in the fact that, compared to \( \Sigma_e \), the closed-loop system \( \Sigma_{e,\delta} \) with the compensator defined as in (12) is described by the matrices
\[
\begin{align*}
A_z(\delta) &= \begin{bmatrix}
-\alpha \otimes C_o + A_o & \delta I \\
(a - \alpha) \otimes C_o & -\alpha \otimes C_o + A_o - B_o K + \delta I
\end{bmatrix}, \\
B_z &= \begin{bmatrix}
B \\
B
\end{bmatrix}, \\
C_z &= \begin{bmatrix}
C_o & 0
\end{bmatrix}.
\end{align*}
\]
Recall now that the observability properties of a pair \((A, C)\) are invariant with respect to static output feedback, i.e. such properties coincide with those of the pair \((A + K C, C)\) for any matrix \( K \) of appropriate dimension. Therefore, the system \( \Sigma_{e,\delta} \) described by (13) is observable if and only if the pair \((A_z(\delta) + \Gamma C_z, C_z)\), with \( \Gamma = -[0, ((a - \alpha) \otimes I_p)]^T \), is observable, namely the pair comprising \( C_z \) and
\[
\begin{bmatrix}
-\alpha \otimes C_o + A_o & \delta I \\
0 & -\alpha \otimes C_o + A_o - B_o K + \delta I
\end{bmatrix}.
\]
Observability of the latter is then implied by observability of the pair \((-a \otimes C_o + A_o - B_o K + \delta I, \delta I)\), guaranteed for any \(\delta \neq 0\), and the fact that \(C_z = [C_o, 0]\), hence proving item ii) of the statement. Item i) follows from the continuity of the entries of \(A_\varepsilon(\delta)\) with respect to \(\delta\), the property that \(W_{e,0}(s) = W_d(s)\), for all \(s \in \mathbb{C}\) and the fact that \(W_d(s)\) does not have poles on the imaginary axis.

Example 1. Consider a plant \(\Sigma_p\) described in state-space representation by matrices as in (4) with
\[
B_o = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad C_o = \begin{bmatrix} 1 & 0 \end{bmatrix},
\]
and \(a = [-3, 2]^T\). It can be easily shown that \(\Sigma_p\) is unstable and non-minimum phase. The construction discussed in

Then, it can be easily shown that the Laplace transform of the input is \(u(s) = [I - C_{u,v}(s)W(s)]^{-1}C_{u,v}(s)v(s)\). As a consequence, the Laplace transform of the auxiliary output \(\hat{y}\) is provided by
\[
\hat{y}(s) = y(s) + y_\varepsilon(s) = [I + C_{y_\varepsilon,v}(s)]y(s) + C_{y_\varepsilon,v}(s)v(s)
\]
\[= [I + C_{y_\varepsilon,y}(s)]W(s)u(s) + C_{y_\varepsilon,v}(s)v(s),
\]
\[
= \left( [I + C_{y_\varepsilon,y}(s)]W(s)[I - C_{u,y}(s)W(s)]^{-1}C_{u,v} + C_{y_\varepsilon,v}(s) \right)v(s).
\]

Since the design of the compensator \(\Sigma_c\) as in (7) is such that the matrix transfer function between \(v(s)\) and \(\hat{y}(s)\) is given by \(W_d(s)\), it follows that the selection of the matrices in (17) ensures stability of the interconnected system together with the property that
\[
W_d(s) = [I + C_{y_\varepsilon,y}(s)]W(s)[I - C_{u,y}(s)W(s)]^{-1}C_{u,v} + C_{y_\varepsilon,v}(s).
\]

Such a frequency-domain interpretation of the design provides the following interesting insight. The existence of the trivial solution to (20) in which essentially the compensator entirely bypasses the original plant via the feedforward action, namely \(C_{y_\varepsilon,v} = W_d(s) - W(s)\), \(C_{u,v}(s) \equiv I\), \(C_{y_\varepsilon,y}(s) \equiv 0\), \(C_{u,y}(s) \equiv 0\), corroborates the importance of the results stated in Proposition 2, which suggests a systematic design that allows to retain observability of the original plant, with respect to the existing literature, see e.g. [10].

It may be of interest to assess, in the context of Proposition 1, the relative contribution of the original output \(y\) and that of the feedforward output \(y_\varepsilon\) to the output \(\hat{y}\). The following construction and subsequent formal result provide a preliminary quantitative answer to the above issue, at least in terms of the static gain of the resulting transfer matrices. To provide a concise statement, for a given choice of \(\alpha\) let \(\hat{A} = -\alpha \otimes C_o + A_o\) and \(\hat{A} = -\alpha \otimes C_o + A_o\), assume that the pair \((\hat{A}, B_o)\) is controllable and consider first the change of coordinates \(\xi = T\hat{\xi}\) such that the corresponding system matrices are described by

\[
\hat{A}_T = T \hat{A} T^{-1} = \begin{bmatrix} A_{11} & A_{12} & \cdots \\
A_{21} & A_{22} & \cdots \\
\vdots & \vdots & \ddots \end{bmatrix},
\]

with
\[
A_{ii} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\
0 & 0 & 1 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
-a_{ii} & -a_{ii} & \cdots & -a_{ii} \end{bmatrix}
\]

Proposition 1 is then carried out by selecting \(\alpha = [3, 2]^T\), \(B = [-2, -1]^T\), \(D = 1\), and \(K = [-69, -27]\), yielding the closed-loop transfer function
\[
W_e(s) = \frac{s^2 + s + 1}{s^2 + 3s + 2},
\]
which is passive from the input \(v\) to the output \(\hat{y}\). Figure 2 shows the Nyquist plots of the closed-loop transfer functions obtained by relying on the constructions of Proposition 2, for various values of the parameter \(\delta\), which remain passive and for which observability of the overall interconnected system is preserved for \(\delta \neq 0\).

Remark 3: By defining the transfer matrices \(C_{u,v}(s), C_{u,y}(s), C_{y_\varepsilon,v}(s)\) and \(C_{y_\varepsilon,y}(s)\) as
\[
C_{u,v}(s) = K(s I_v - A_\varepsilon)^{-1}B_v, \quad C_{u,y}(s) = K(s I_v - A_\varepsilon)^{-1}B_\xi, \quad C_{y_\varepsilon,v}(s) = C_o(s I_v - A_\varepsilon)^{-1}B_v + D, \quad C_{y_\varepsilon,y}(s) = C_o(s I_v - A_\varepsilon)^{-1}B_\xi,
\]
with \(A_\varepsilon = (-\alpha \otimes C_o + A_o - B_o K)\) and \(B_\xi = (a - \alpha) \otimes I_p\), one could define the input/output behavior of \(\Sigma_c\) as
\[
u(s) = C_{u,v}(s)v(s) + C_{u,y}(s)y(s), \quad y_\varepsilon(s) = C_{y_\varepsilon,v}(s)v(s) + C_{y_\varepsilon,y}(s)y(s),
\]

Fig. 2. Nyquist plots of the transfer matrix \(W(s)\) (blue line) and of the closed-loop system \(W_{e,\delta}(s)\), with \(\delta = 0\), \(\delta = 0.025\) and \(\delta = 0.01\).
and

\[
A_{ij} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_{1}^{ij} & -a_{2}^{ij} & \cdots & -a_{n}^{ij}
\end{bmatrix},
\]

for some constant coefficients \(a_{ij}^{k}\). Moreover, the input matrix becomes

\[
B_T = TB_o = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1
\end{bmatrix},
\]

with the non-zero elements corresponding to the rows \(n_1, n_1 + n_2, \ldots, n\), while the entries of \(C_oT^{-1}\) are arbitrary. Consider then, in the transformed coordinates, a feedback gain matrix defined as \(K_{e,T} = \text{blkdiag}\{[\alpha_{d,1}^{i} - \alpha_{d,1}^{s}, \ldots, \alpha_{d,n_1}^{i} - \alpha_{d,n_1}^{s}]\}\), with \(\alpha_{d,1}, \ldots, \alpha_{d,n_1}\) such that the eigenvalues of \(A_i + B_iK_{e,T}\) are assigned to the set \(\lambda_{d,j}\), such that \(\text{Re}(\lambda_{d,j}) < -\delta\), \(j = 1, \ldots, n_1\), with \(\text{Re}(\lambda)\) denoting the real part of \(\lambda \in \mathbb{C}\).

**Proposition 3:** Consider a transfer matrix \(W(s)\) as in (1) and suppose that Assumption 1 holds. Let \(W_d(s) \in \mathbb{C}^{p \times m}\) be any desired strictly proper rational matrix. Let the compensator \(\Sigma_e\) be described by the equations (7) such that

i) \(\alpha = [\alpha_1, \ldots, \alpha_n]^\top\), \(B = [N_1^\top, \ldots, N_m^\top]^\top\) are arbitrary such that \(W_e(s) = W_d(s)\) and, in addition, such that the pair \((\hat{A}, B_o)\) is controllable;

ii) for any \(\delta \in \mathbb{R}_+\) there exists \(P = P^\top > 0\) such that

\[
P(\hat{A} - B_oK_{e,T}T^{-1}) + (\hat{A} - B_oK_{e,T}T^{-1})^\top P < 0.
\]

(21)

Then for any \(\mu > 0\) there exists \(K^*\) such that \(W_e(s) = W_d(s)\) for all \(s \in \mathbb{C}\) and \(|C_{y_e,v}(0)| < \mu\).

**Proof:** The first part of the claim is clearly shown in the proof of Proposition 1. As for the second point, this is achieved by assigning arbitrarily fast eigenvalues to the matrix \(\hat{A} - B_oK\), which is always possible by assumption i) of the statement while preserving asymptotic stability of the compensator by relying on assumption ii). In particular, the static gain \(C_{y_e,v}(0) = -C_oA_{\xi}^{-1}B\), which is an input/output property, is determined in the transformed coordinates \(\xi\). It then follows that \(\alpha_{d,1}^{i} = (-1)^n \prod_{j=1}^{n} \lambda_{d,j}\). Straightforward derivations then allow computing the columns of the matrix inverse \((\hat{A}_T + B_TK_{e,T})^{-1}\) multiplied by the non-zero elements of \(B_T\), denoted by \(v_i\), namely

\[
v_i = \frac{1}{\delta(\alpha_{d,1}^i)} \begin{bmatrix}
c_1(\alpha_{d,1}^i) \\
\vdots \\
c_m(\alpha_{d,1}^i)
\end{bmatrix},
\]

(22)

where \(\delta\) denotes the determinant of the matrix \((\hat{A}_T + B_TK_{e,T})\), which is such that \(\delta(\alpha_{d,1}^i) = O(\prod_{k=1}^{m} \alpha_{d,1}^{k})\), while the functions \(\alpha_{d,k}^i\) are arbitrary small with respect to the overall behavior between \(v\) and the output \(y\), which can be assigned via the choice of the desired transfer function \(W_d(s)\). Note, however, that the objective of completely reducing the contribution of the controller \(\Sigma_e\) and preserving the role of the original output \(y\) is achieved, instead, by solving the more challenging problem

\[
\min_K \left\| \begin{bmatrix}
C_{u,v}(s) & C_{u,y}(s) \\
C_{y_e,v}(s) & C_{y_e,y}(s)
\end{bmatrix} \right\|_{\infty}
\]

(23)

s.t. \(\sigma(-a \otimes C_o + A_{\xi} - B_oK) \subset \mathbb{C}^-\),

which will be addressed in future works, namely minimizing the \(H_\infty\)-norm of the controller’s transfer matrix, while simultaneously enforcing the primary task of model matching. ▲

By inspecting the structure of the internal transfer matrices of the interconnected system given in (17), the following statement is straightforward, hence its proof is omitted.

**Proposition 4:** Suppose that the assumptions of Proposition 1 hold. Then the closed-loop system \(\Sigma_e\) is internally stable provided the gain \(K\) is such that the matrix \(A_{\xi} = \hat{A} - B_oK\) is Hurwitz.

**Corollary 1:** Consider a transfer matrix \(W(s)\) as in (1) and suppose that Assumption 1 holds. Let \(W_d(s) \in \mathbb{C}^{p \times m}\) be any desired proper rational matrix. Let the compensator \(\Sigma_e\) be described by the equations (7) with \(\xi(t) \in \mathbb{R}^nP\), \(\alpha = [\alpha_1, \ldots, \alpha_n]^\top\), \(B = [N_1^\top, \ldots, N_m^\top]^\top\) and \(K\) such that \(\sigma(-a \otimes C_o + A_{\xi} - B_oK) \subset \mathbb{C}^-\), and consider the feedback/feedforward interconnection of \(\Sigma_p\) and \(\Sigma_e\). Then the zero equilibrium of the closed-loop system \(\Sigma_e\), with \(v = 0\), is GAS, provided the coefficients \(\alpha_i, i = 1, \ldots, n\), are selected such that the polynomial \(s^n + \alpha_1s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n\) is Hurwitz.

**Proof:** The claim is a straightforward consequence of the block lower triangular structure of the system (9), with the overall eigenvalues corresponding to those of \((-a \otimes C_o + A_o)\) and of \((-a \otimes C_o + A_o - B_oK)\).

**Remark 5:** The output feedback stabilizer \(\Sigma_e\) is a non-identity observer according to the definition in the seminal paper by Luenberger [29]. In fact, letting the exogenous input \(v\) be equal to zero, the state \(\xi\) asymptotically reproduces the
quantity $-x$ or, in other words, defining the error variable $e = x + \xi$ one obtains, by following steps identical to those in the proof of Proposition 1 that $\dot{e} = (-\alpha \otimes C_o + A_o)e$. Such dynamics entail that the subspace of $\mathbb{R}^n \times \mathbb{R}^n$ defined by the equation $e = x + \xi = 0$ is invariant and externally attractive. An interesting difference with respect to the classical output feedback stabilization based on identity observers is that the observable modes of the transfer function from the input $v$ to the output $\hat{y}$ coincide with the eigenvalues of the error dynamics and not with those assigned by the control input $u = K_1 \xi$. This is essentially due to the fact that the additional input $v$ enters the closed-loop system from a different channel with respect to $u$, thus guaranteeing the property of passivity from such additional input channel (compare the block diagram in Figure 1 with that of Figure 3 displaying the classical output feedback stabilization scheme by means of an identity observer).

\[ \dot{\xi} = \begin{bmatrix} A_m - GC_m + B_m K_1 & 0 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} G \\ (a_e - \alpha) \otimes I \end{bmatrix} y + \begin{bmatrix} 0 \\ B \end{bmatrix} v, \]
\[ u = \begin{bmatrix} K_1 \\ 0 \end{bmatrix} \xi, \]
\[ y_e = \begin{bmatrix} 0 \\ C_o \end{bmatrix} \xi + Dv, \]

with $\xi(t) \in \mathbb{R}^{n_m + n_o}$, $B = [\hat{N}_1^\top, \ldots, \hat{N}_n^\top]^\top$, $G$ and $K$ such that $\sigma(A_m - GC_m) \subset \mathbb{C}^-$ and $\sigma(A_m + B_m K_1) \subset \mathbb{C}^-$, respectively. $n_e$ denotes the highest degree of the polynomial $d_e(s)$ obtained as $C_e(sI - A_e)^{-1} B_e \triangleq N_e(s)/d_e(s)$, with $C_e = [C_m \ 0]$, $B_e = [B^\top_m \ B^\top_1]$ and $A_e = \begin{bmatrix} A_m \\ GC_m \\ A_m + B_m K_1 - GC_m \end{bmatrix}$, (25) and $a_e \in \mathbb{R}^{n_e}$ is such that $d_e(s) = s^{n_e} + a_{e,1}s^{n_e-1} + \ldots + a_{e,n_e}$. Consider the feedback/feedforward interconnection of $\Sigma_p$ and $\Sigma_o$, namely $\Sigma_e$ with input $v$ and output $\hat{y} = y + y_e$, described by the transfer matrix $W_e(s)$. Then there exist $\alpha_e \in \mathbb{R}^{n_e}$ and $\hat{N}_i \in \mathbb{R}^{p \times m}$, $i = 1, \ldots, n_e$, such that $W_e(s) = W_d(s)$ for all $s \in \mathbb{C}$.

**Proof:** The construction is carried out in two sequential steps based on the combination of a standard output feedback scheme and a dynamic feedback borrowed from Proposition 1. More precisely, by partitioning the state $\xi$ of (24) as $\xi(t) = [\xi_1(t) \xi_2(t)]^\top \in \mathbb{R}^{n_m + n_o}$, it can be shown that the component $\xi_1$ behaves as a Luenberger observer for a minimal realization of $W(s)$, i.e., $\dot{\xi}_1 = A_m \xi_1 + B_m u + G(y - C_m \xi_1)$, together with the stabilizing feedback $u = K_1 \xi_1$. The second component $\xi_2$, on the other hand, provides the feedforward action required to assign the desired zeros to the closed-loop system by following arguments similar to those employed in the proof of Proposition 1, specialized to the extended system of the plant and the stabilizing observer, defined with respect to the extended state $x_e = [x_m^\top, \xi_1^\top]^\top \in \mathbb{R}^{2n_m + n_o}$, dynamic matrix $A_e$ as in (25), which is Hurwitz by the selection of $K_1$ and $G$. Once the interconnection of the plant and the observer with state $\xi_1$ has been stabilized, the design of the dynamic feedforward that assigns the desired transfer matrix must be carried out. To this end, the transfer matrix resulting from the interconnection of $\Sigma_p$ and $\xi_1$-subsystem is computed as $C_e(sI - A_e)^{-1} B_e$, with the matrices defined in the statement of the proposition. Such transfer matrix is then realized in state-space by considering the Block Observer Form (5). While the latter may not be controllable, since the eigenvalues of $-\alpha \otimes C_o + A_o$ all lie on the open left-hand side of the complex plane, the construction of Proposition 1 may be carried out by setting the matrix $K$ mentioned in the statement of Proposition 1 equal to zero, hence motivating the structure of the dynamics of $\xi_2$. \qed
Example 2. Consider the transfer matrix

$$W(s) = \begin{bmatrix} -(s - 2) & 2(s - 2) \\ s + 1 & s - 1 \\ (s + 1)(s - 2) \end{bmatrix},$$

(26)

namely as in (1) with $n = 2$, $p = 2$, $m = 2$ and $d(s) = (s^2 - s - 2)$, and suppose that the objective is to decouple the two input/output channels and impose a desired frequency response on each channel characterized by $1/(s + \tau), \tau > 0$. By relying on the technicalities of Gilbert’s diagonal realization it can be shown that $n_{\min} = 3 < 4 = np$, hence the Block Observer Form is not minimal. A minimal realization is instead obtained by considering $A_m = \text{diag}(-1, -1, 2)$, together with

$$B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C_m = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$  

(27)

Note that the system is non-minimum phase since it possesses a zero at $s = 0.5$. By letting now

$$G = \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ 0 & -10 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$

(28)

one immediately obtains that $\sigma(A_c) \subset \mathbb{C}^{-}$ and

$$C_c(sI - A_c)^{-1}B_c = \begin{bmatrix} -s + 2 \\ s + 1 \\ (s + 1)(s - 2) \end{bmatrix},$$

hence $n_c = 2$ and $a_c = [3 \ 2]^{\top}$. Then the dynamic feedforward is directly obtained by letting $\alpha_c = [\tau + 1 \ \tau]^{\top}$ and $\hat{N}_i = I_p, i = 1, 2$. Therefore the overall dynamic feedback/feedforward controller

$$\dot{\xi} = \begin{bmatrix} -1 & 2 & 2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 10 & -12 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta - \tau - 1 & 0 \\ 0 & 0 & 0 & \delta - \tau & 0 \\ 0 & 0 & 0 & 0 & -\tau & 0 \end{bmatrix} \xi$$

$$+ \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ 0 & -10 \\ 2 - \tau & 0 \\ 0 & 2 - \tau \\ 2 - \tau \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} v$$

(29)

for a sufficiently small $\delta \neq 0$ in the spirit of Proposition 2, with $u = [K_1 \ 0] \xi$ is such that the closed-loop transfer matrix between $v$ and $\hat{y} = y + \begin{bmatrix} \xi_4 & \xi_5 \end{bmatrix}^{\top}$ is

$$W_d(s) = \begin{bmatrix} 1 \\ s + \tau - \delta \\ 0 \\ 1 \end{bmatrix},$$

(30)

and the interconnected system (27), (29) is observable from $\hat{y}$ for any $\delta \neq 0$.

Before giving the formal statement concerning the second alternative approach to Proposition 1, whenever Assumption 1 fails to be satisfied, the definition of reachability matrix is recalled.

Definition 1: Consider a linear time-invariant system described by the equations $\dot{x} = Ax + Bu$, with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Then the reachability matrix, denoted as $\mathcal{R}(A, B)$, associated to the system is defined as

$$\mathcal{R}(A, B) \triangleq \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}.$$  

(31)

To streamline the following statement, recall that $\bar{A} = -a \otimes C_o + A_o$ and $\bar{A} = -a \otimes C_o + A_o$.

Proposition 6: Consider a transfer matrix $W(s)$ as in (1) and suppose that $n_{\min} < np$. Let $W_d(s) \in \mathbb{C}^{n \times m}$ be any desired proper rational matrix, with denominator given by the polynomial $s^n + a_1 s^{n-1} + \ldots + a_n - 1 + a_{n+1}$ and such that $\text{Im}(\mathcal{R}(\bar{A}, [(a - a) \otimes I_p, B])) \subseteq \text{Im}(\mathcal{R}(\bar{A}, B_o))$. Then $W_d(s) = W_d(s)$ for all $s \in \mathbb{C}$.

Proof: The claim is proved by firstly considering the construction discussed in the proof of Proposition 1, hence essentially defining the dynamics of the compensator as in (7), without however requiring $u = K\xi$ to asymptotically stabilize the pair $(-a \otimes C_o + A_o, -B_o)$. In fact, since Assumption 1 does not hold, hence $n_{\min} < np$, it follows that the Observer Block state-space realization is not controllable. To overcome this issue the inclusion (32), which poses restrictions on the feasible selections of the matrices $N_i$ to satisfy the inclusion (32), ensures that the behavior of the compensator between the inputs $v$ and $y$ and the output $u$ is not affected by the uncontrollable subsystem of the pair $(-a \otimes C_o + A_o, -B_o)$. Thus, an internally stable compensator that preserves the required input/output behavior can be obtained from (7) by reducing it to the pair $(A_r, B_r)$, which can be indeed stabilized by a selection of $K$.

C. Input/Output Decoupling

The aim of this section consists in the specialization of the Model Matching machinery derived in the previous section to the case in which the desired transfer matrix possesses a special structure, namely a diagonal $W_d(s)$.

$^5\text{Im}(M)$ denotes the image of the columns of the matrix $M$. 
Consider the LTI plant $\Sigma_p$ in (4) and suppose that $p = m$ and that Assumption 1 holds. Let the compensator $\Sigma_c$ be described by the equations (7). Then the closed-loop transfer matrix $W_c(s)$ is diagonal for any $s \in \mathbb{C}$ provided that $\hat{N}_i = \beta_i I_p$, for all $\beta_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $D$ is diagonal, hence the output $\hat{y}_i(t)$ depends only on $v_i(t)$, for all $t \in \mathbb{R}$.

The proof of the above claim is obtained by a straightforward adaptation of the arguments employed for the proof of Proposition 1, hence it is omitted. Note that, in the non-generic case discussed in Section III-B in which Assumption 1 does not hold, a statement similar to Proposition 5 can be provided to generalize Corollary 2 to the context of input/output decoupling as also illustrated by Example 2.

D. Passivation

The results discussed in this section provide the solution to the objective (G3) of Problem 1. The issue of passivating a given linear time-invariant system, which may not have vector relative degree one or even not be minimum-phase originally, has become increasingly in control theory, due to the wide diffusion of passivity-based techniques for stabilization.

Corollary 3: (Passivation of MIMO plants) Consider the LTI plant $\Sigma_p$ in (4) and suppose that $p = m$ and that Assumption 1 holds. Let the compensator $\Sigma_c$ be described by the equations (7). Then
i) $\Sigma_c$ has vector relative degree equal to $\{1, 1, \ldots, 1\}$ provided $\hat{N}_i = \beta_i I_p$ and $\beta_i \neq 0$;
ii) $\Sigma_c$ is minimum-phase provided the polynomial $\mu_\beta(s) = \beta_1 s^{n-1} + \beta_2 s^{n-2} + \ldots + \beta_n - 1 s + \beta_n$ is Hurwitz and $\hat{N}_i = \beta_i I_p$. Moreover, the zero dynamics is spectrally assign able and possesses the modes $\Xi_\beta \cup \sigma(-a \otimes C_o + A_o - B_o K)$, with $\Xi_\beta$ denoting the set of roots of $\mu_\beta(s)$;
iii) $\Sigma_c$ is asymptotically stable provided $\alpha_i$, $i = 1, \ldots, n$, are such that the polynomial $\mu_\alpha(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} + \alpha_n$ is Hurwitz.

Moreover, letting $\Xi_\alpha$ denote the set of roots of $\mu_\alpha(s)$, there exist $\Xi_\alpha^+ \subset \mathbb{C}^+$ and $\Xi_\alpha^- \subset \mathbb{C}^-$ such that conditions i)-iii) hold simultaneously and $\Sigma_c$ is passive from the input $v$ to the output $\hat{y}$.

The proof of the first three claims is a straightforward adaptation of the proof of the previous statements, hence it is omitted. To render the above proposition constructive note that objective (G3) is practically achieved by requiring that the desired transfer matrix satisfies $W_d(s) + W_d(s)^\dagger \geq 0$ for all $s \in \mathbb{C}$, where $W_d(s)^\dagger$ denotes the conjugate transpose of $W_d(s)$.

IV. The Single Input/Single Output Case

The aim of this section consists in specializing the results discussed above to the case of single input/single output systems. Firstly, the theory introduced in Section III is adapted to the SISO setting and illustrated by means of a few numerical examples, then slightly modified control objectives are stated for the SISO case by allowing the presence of unmeasured disturbance inputs.

Example 3. Consider the LTI system in state-space form described by the equations

$$
\begin{align*}
\dot{x} &= \begin{bmatrix} 13 & -12 & 0 \\ 16 & -16 & 1 \\ 12 & -12 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} u, \\
y &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x,
\end{align*}
$$

(34)

with $x(t) \in \mathbb{R}^3$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$. The system (34) is controllable and observable. Its transfer function is

$$
W(s) = \frac{(s - 1)(s - 3)}{s^3 + 3s^2 - 4s - 12},
$$

(35)

yielding the coefficient vectors $a = [3, -4, -12]^\top$ and $b_o = [1, -4, 3]^\top$, from which it can be deduced that the system (34) possesses unstable poles and is non-minimum phase. In addition, note that the transfer function (35) does not satisfy the parity interlacing property, see [39], i.e. it does not have an even number of real poles between every pair of real zeros in Re(s) $\geq 0$. Let now $\alpha = [12, 47, 60]^\top$ and $B = [1, 3, 2]^\top$, and consider the dynamic feedback/feedforward compensator

$$
\begin{align*}
\dot{\xi} &= (-\alpha C_o + A_o - B_o K)\xi + (a - \alpha) y + B v, \\
u &= K\xi, \\
y_\xi &= \xi_1,
\end{align*}
$$

(36)

with $K = [-6.375, -2.875, -1.5417]$. Then, the interconnected system $\Sigma_o$ with input $v$ and output $y + y_\xi$ described by (34)-(36) is passive.

Example 4. Consider the system described by the proper transfer function

$$
W(s) = \frac{(s - 1)(s + 2)}{(s - 2)(s + 1)},
$$

(37)

which is not passive, being unstable and non-minimum phase. The model may be described in state-space form by the dynamic equation in (4), with $a = [-1, -2]^\top$ and $B_o = [2, 0]^\top$, and with output defined by $y = x_1 + u$, namely with a feed-through term. The architecture in (7) is consequently adapted as

$$
\begin{align*}
\dot{\xi} &= (-\alpha C_o + A_o - B_o K + (a - \alpha) K)\xi + (a - \alpha) y + B v, \\
u &= K\xi, \\
y_\xi &= C_o\xi - u,
\end{align*}
$$

(38)
with $\alpha = [3, 2]^T$, $B = [1, 1]^T$ and $K = [1.125, 0.594]$ such that the matrix $(-\alpha C_0 + A_o - B_oK)$ is Hurwitz. It can be easily verified that the closed-loop system (37)-(38) is passive from the input $v$ to the output $\hat{y} = y + y_c$.

Consider now the objective of enforcing a different dissipativity property to the system, namely finite, and possibly arbitrarily small, $L_2$-gain $\gamma$ between an unknown disturbance input $w$ to a possibly modified output $\hat{y}$, see e.g. [36]. Therefore, consider the linear time-invariant system $\Sigma_p$ defined in state-space form by

$$\begin{align*}
\dot{x} &= A_p x + B_p u + P_p w, \\
y &= C_p x, 
\end{align*}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of $\Sigma_p$, while $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and $w(t) \in \mathbb{R}^q$ denote the control input, the measured output and an unknown disturbance signal, respectively, and suppose that the following assumption holds.

**Assumption 2:** The pairs $(A_p, B_p)$ and $(C_p, A_p)$ are controllable and observable, respectively.

**Problem 2:** ($L_2$-gain via Dynamic Feedforward and Feedback). Consider the system $\Sigma_p$, defined in (39). Find, if possible, a dynamic compensator $\Sigma_c$ described by the equations

$$\begin{align*}
\dot{\xi} &= A_c \xi + B_c y + B_v, \\
u &= K_c \xi, \\
y_c &= C_c \xi,
\end{align*}$$

with $\xi(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}$, such that the interconnected system $\Sigma_e$ (see Figure 4 for the control architecture) is internally stable and the following holds.

**Claim (G1)*** Given $\gamma > 0$, the $L_2$-gain of $\Sigma_e$ from the disturbance $w$ to the output $\hat{y} = y + y_c$ is smaller than $\gamma$, namely

$$\int_0^\tau \|\hat{y}(t)\|^2 dt \leq \gamma^2 \int_0^\tau \|w(t)\|^2 dt,$$

for any $w \in L_2(0, \tau)$ and any $\tau > 0$.

To streamline the statement of the following result we suppose that the roots of the polynomial $\mu_\alpha(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n$ are all selected with zero imaginary part and negative real part, and that, for a generic choice of the vector $\alpha$, they are denoted by the (ordered) set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$. Finally, to provide a concise statement of the following results, let $T$ denote the change of coordinates that transforms the system (39), with $w = 0$, into canonical observability form $(-\alpha C_o + A_o, B_o, C_o)$, i.e. such that $TA_pT^{-1} = -\alpha C_o + A_o$, $TB_p = B_o$ and $C_pT^{-1} = C_o$.

**Proposition 7:** Consider the LTI plant $\Sigma_p$ in (39) and suppose that Assumption 2 holds. Let the compensator $\Sigma_c$ be described by the equations

$$\begin{align*}
\dot{\xi} &= (-\alpha C_o + A_o - B_oK)\xi + (a-\alpha)y, \\
u &= K_c \xi,
\end{align*}$$

with $\xi(t) \in \mathbb{R}^n$, $\hat{y} = y + C_c \xi$, $\alpha = [\alpha_1, \ldots, \alpha_n]^T$, and $K$ such that $\sigma(-\alpha C_o + A_o - B_oK) \subset \mathbb{C}^-$, and consider the feedback/feedforward interconnection of $\Sigma_p$ and $\Sigma_c$, namely $\Sigma_e$ with disturbance input $w$ and output $\hat{y}$. For any $\gamma > 0$, let the smallest root of the polynomial $\mu_\gamma(s)$ be such that

$$|\lambda_n| > \left(\frac{1}{2\gamma^2} \frac{1}{\sigma(TP_pP_p^T + I)} + 1 \right).$$

Then the $L_2$-gain from the disturbance input $w$ to the output $\hat{y}$ is smaller than $\gamma$, namely

$$\int_0^\tau \|\hat{y}(t)\|^2 dt \leq \gamma^2 \int_0^\tau \|w(t)\|^2 dt,$$

for all disturbances $w \in L_2(0, \tau)$ and for any $\tau > 0$.

**Proof:** To begin with, note that the closed-loop system $\Sigma_e$ is described, in the $(z, \xi)$ coordinates, with $z = Tx + \xi$, by the equations

$$\begin{align*}
\dot{z} &= (-\alpha C_o + A_o)z + TP_p w, \\
\dot{\xi} &= (-\alpha C_o + A_o - B_oK)\xi + (a-\alpha)\dot{y}, \\
\dot{y} &= C_o z,
\end{align*}$$

in which the eigenvalues of $(-\alpha C_o + A_o)$ can be arbitrarily assigned by selecting the vector $\alpha \in \mathbb{R}^n$, while the input channel provided by $B_o$ is employed to asymptotically stabilize the $z$ component of the extended closed-loop system. By performing an additional diagonalizing change of coordinates, defined in terms of the non-singular matrix $T \in \mathbb{R}^{n \times n}$, namely $\tilde{z} = T z$, the system (44a) can be written as

$$\begin{align*}
\dot{\tilde{z}} &= \Delta_o \tilde{z} + \bar{P} w, 
\end{align*}$$

with $\Delta_o$ a diagonal matrix having the eigenvalues of the matrix $(-\alpha C_o + A_o)$ as diagonal elements and $\bar{P} = TP \pm TT_p$. Consider now the algebraic Riccati inequality

$$X_{\Delta_o} + \Delta_o X + \frac{1}{\gamma^2} X T TP^{+} T^{+} X + T^{-1} T^{-1} \leq 0,$$

in the unknown symmetric, positive semidefinite matrix $X \in \mathbb{R}^{n \times n}$. It can be easily shown that if the inequality (46) is satisfied then by letting $V(\tilde{z}) = \tilde{z}^T X \tilde{z}$ one obtains $V \leq \gamma^2 w^T w - \tilde{z}^T T^{-1} T^{-1} \tilde{z}$, for any disturbance input $w$. Hence, by integrating both sides between zero and $\tau$ and since $V$ is non-negative and $\tilde{z}(0) = 0$, it follows that

$$\gamma^2 \int_0^\tau \|w(t)\|^2 dt \geq \int_0^\tau \tilde{z}(t) T^{-1} \tilde{z}(t) dt \geq \int_0^\tau \|\hat{y}(t)\|^2 dt,$$

with $\hat{y} = C_o z$. Therefore, to conclude the proof of the claim it remains to show that the inequality (46) holds. To this end, let $X = T^{-1} T^{-1} \leq 0$ and note that the inequality is satisfied by the choice of the eigenvalues of $(-\alpha C_o + A_o)$ dictated by (42) and by recalling that $\sigma(TP_pP_p^T + I) \leq \sigma(TP_pP_p^T + \sigma(I) = \sigma(TP_pP_p^T + 1)$.

**Remark 6:** It is evident that the arbitrarily small gain $\gamma$ from the disturbance $w$ to the modified output $\hat{y}$ is enforced by means of high gain-like techniques in the selection of the $\delta(M)$ denotes the maximum singular value of the matrix $M$. 
vector α. It seems then that the proposed approach may be equivalently replaced by means of an output feedback scheme, comprising a classical observer, in which the eigenvalues of the original state and those of the error dynamics are pushed arbitrarily to the left-hand side of the complex plane. However, the key difference lies in the fact that the choice of the gain matrix K is required only to stabilize the dynamics in (44b), and it should not be taken arbitrarily large to obtain a desired $L_2$-gain. This aspect is due to the fact that the resulting transfer function from the input w to the output $\hat{y}$ is given by $W_{yw}(s) = C_o(sI + A_o - A_o)^{-1}TP_p$, while adopting the alternative observer-based scheme mentioned above, one obtains $W_{yw}(s) = C(sI - A - BK)^{-1}P_p + C(sI - A - BK)^{-2}BK(sI - A + LC)^{-1}P_p$. This is the result of the fact that the dynamics of $\xi$ are independent of $w$, while in the observer-based scheme the error dynamics are indeed affected by the disturbance.

From the construction described in Proposition 7 it can be immediately deduced that, in addition to arbitrarily reducing the $L_2$-gain between the disturbance and the modified output, also the $L_2$-gain between an additional control input $v$ and the output $\hat{y}$ is decreased. This is an undesirable property in general since the control effort, e.g. to steer $\hat{y}$ to track a reference signal, is proportionally increased. This issue is solved by the following proposition.

Proposition 8: Consider the LTI plant $\Sigma_p$ in (39) and suppose that Assumption 2 holds. Let the compensator $\Sigma_c$ be described by the equations

$$
\dot{\xi} = (-\alpha C_o + A_o - B_o K)\xi + (a - \alpha)y + B(\eta + v),
$$

$$
\dot{\eta} = -\eta + (\lambda_n - 1)v,
$$

$$
u = K\xi,
$$

with $\xi(t) \in \mathbb{R}^n$, $\dot{y} = y + C_o \xi$, $\alpha = [\alpha_1, ..., \alpha_n]^\top$, $B = [\beta_1, ..., \beta_n]^\top$ such that the roots of the polynomial $\mu_B(s)$ are given by the set $\{\lambda_1, \lambda_2, ..., \lambda_n, -\lambda_n\}$, and $K$ such that $\sigma(-\alpha C_o + A_o - B_o K) \subset \mathbb{C}^-$, and consider the feedback/feedforward interconnection of $\Sigma_p$ and $\Sigma_c$, namely $\Sigma_e$ with disturbance input $w$ and output $\hat{y}$. Let $\lambda_n$ be such that (42) holds. Then the $L_2$-gain from the disturbance input $w$ to the output $\hat{y}$ is smaller than $\gamma$, for all disturbances $w \in L_2$, and the $L_2$-gain from the control input $v$ to the output $\hat{y}$ is equal to $1$.

Proof: By taking advantage of the derivations carried out in the previous proof it can be seen that the closed-loop system is described, for the z component, by the equations

$$\dot{z} = (-\alpha C_o + A_o)z + TP_p w + B(\eta + v).$$

Let now $\hat{y} = \eta + v$, then the transfer function from the input $\hat{v}$ to the output $\hat{y} = C_oz$ is defined by

$$W_{\hat{y} \hat{v}} = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + ... + \beta_{n-1}s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + ... + \alpha_{n-1}s + \alpha_n},$$

where the second equality is obtained by considering the stable zero-pole cancellations performed by the choice of the coefficients of the matrix $B$, which assigns the zeros of the resulting transfer matrix. The proof is concluded by noting that the transfer function from the input $v$ to the output $\hat{v} = \eta + v$ is then given by

$$W_{\hat{v}v} = \frac{s + \lambda_n}{s + 1},$$

hence $W_{\hat{y}v} W_{\hat{v}v} = 1/(s + 1)$, concluding the proof.

V. APPLICATION TO ADAPTIVE CONTROL OF NON-MINIMUM PHASE SYSTEMS

Consider uncertain single-input/single-output plants described by transfer functions of the form

$$W(s) = \frac{Z(s)}{R(s)},$$

where $k_p$ is referred to as the high frequency gain and $Z(s)$ and $R(s)$ are monic polynomials in the variable s. It is well-known, see e.g. [16], that the output-feedback Model Reference Adaptive Control (MRAC) can be applied to $W(s)$ provided the following assumptions are satisfied:

(C1) the order $n$ and the relative degree $r \geq 1$ are known;
(C2) the sign of the high frequency gain $k_p$ is known;
(C3) $Z(s)$ and $R(s)$ are coprime and $Z(s)$ is Hurwitz.

The most critical assumption to satisfy is typically (C3), which requires the plant to be minimum phase with respect to the measured output. Therefore, in the following we suppose that (C1) and (C2) are indeed satisfied, while (C3) may not hold. Thus, in this section, we employ the construction introduced above to remove such an assumption or replace it with weaker requirements. Towards this end, consider an uncertain transfer function defined by

$$\hat{W}(s) = \frac{\theta_{2,1}s^{n-1} + \theta_{2,2}s^{n-2} + ... + \theta_{2,n-1}s + \theta_{2,n}}{s^n + \theta_{1,1}s^{n-1} + ... + \theta_{1,n-1}s + \theta_{1,n}},$$

which satisfies the following standing assumption.

Assumption 3: The vectors $\theta_1 = [\theta_{1,1}, ..., \theta_{1,n}]^\top$ and $\theta_2 = [\theta_{2,1}, ..., \theta_{2,n}]^\top$ are unknown and belong to compact sets $\Theta_1 \subset \mathbb{R}^n$, respectively. Moreover, the numerator and the denominator of (53) are coprime for any $\theta_i \in \Theta_i, i = 1, 2$.

Note that, apart from the inclusion $\theta_2 \in \Theta_2$, Assumption 3 does not impose any constraint on the location of the resulting zeros of the plant induced by the values of $\theta_2$. A state-space representation of the above uncertain transfer function (53) can be given in terms of a linear system in observable canonical form, namely

$$\dot{x} = (-\theta_1 C_o + A_o)x + \theta_2 u,$$

together with the output function $y = x_1 = C_o x$, and the matrices $A_o$ and $C_o$ defined as in (5).

Proposition 9: Consider the uncertain plant $\Sigma_p$ defined by (54). Let the compensator $\Sigma_c$ be described by

$$\dot{\xi} = (-\alpha C_o + A_o - \theta_2 K)\xi + (\hat{\theta}_1 - \alpha)y + Bv,$$

$$u = K\xi,$$

$$y = C_o\xi,$$

with $\xi(t) \in \mathbb{R}^n$, $\alpha = [\alpha_1, ..., \alpha_n]^\top$, $B = [\beta_1, ..., \beta_n]^\top$ such that the system $(-\alpha C_o + A_o, B, C_o)$ is strictly passive, hence for any $Q = Q^\top > 0$ there exists $P = P^\top > 0$ such that

\[ P s + Q s + P^\top > 0. \]
$P(-\alpha C_o + A_o) + (-\alpha C_o + A_o)^T P + Q = 0$ and $PB = C_o^T$. Moreover, let $\hat{\theta}_i, \theta_i$ and $K$ be such that $\sigma(-\hat{\theta}_i C_o + A_o - \hat{\theta}_2 K) \subset \mathbb{C}^-$ and define $\hat{\theta}_i = \theta_i - \hat{\theta}_i$. Suppose that there exist matrices $X, Y \in \mathbb{R}^{n \times n}$, with $Y - XP^{-1}X^T > 0$, and a positive scalar $\varepsilon$ such that:

(H1) the Linear Matrix Inequality (55) (top of the next page) holds for any $\theta_i \in \Theta$, $i = 1, 2$, with $\hat{H} = \hat{\theta}_1 C_o + \hat{\theta}_2 K$,

$A_\xi = -\alpha C_o + A_o$ and $A_\xi = (-\hat{\theta}_1 C_o + A_o - \hat{\theta}_2 K)$;

(H2) $XB = 0$;

(H3) $(Y + X^T)B = 0$.

Then, the closed-loop system $\Sigma_p$, $\Sigma_c$ as in Figure 1 from the input $v$ to the output $\hat{y} = y + y_\xi$ satisfies the assumptions (C1) – (C3) of the MRAC scheme. Moreover, letting $v = -\kappa \hat{y} = -\kappa(y + y_\xi)$ with

$$\kappa > \max_{\hat{\theta}_i \in \Theta} \varepsilon \left\| \begin{bmatrix} P X^T Y \\ -\hat{\theta}_1 \\
\hat{\theta}_1 - \alpha \end{bmatrix} \right\|^2_{\eta} ,$$

(56)

all the trajectories of the uncertain closed-loop system globally exponentially converge to the origin of $\mathbb{R}^n \times \mathbb{R}^n$.

Proof: Constructions borrowed from the proofs of the previous results lead immediately to the extended closed-loop system in the $z = x + \xi$ and $\xi$ coordinates described by

$$\dot{\eta} = \begin{bmatrix} A_\xi & \hat{H} & 0 \\
\hat{\theta}_1 - \alpha & B & B \end{bmatrix} \eta + \begin{bmatrix} -\hat{\theta}_1 \\
\hat{\theta}_1 - \alpha \end{bmatrix} \hat{y},$$

(57)

with $\eta = [z^T, \xi^T]^T$. By mimicking the constructions in the nominal case discussed above, it can be easily shown that the $z$-subsystem ($A_\alpha, B, C_o$) is passive. Then, it can be shown that, letting

$$\tilde{P} = \begin{bmatrix} P & X \\
X^T & Y \end{bmatrix},$$

(58)

the item (H1) implies that $\tilde{P} A + A^T \tilde{P} < -\varepsilon I$, whereas items (H2) and (H3) together guarantee that $\tilde{P} B = C_o^T$, with $C = [C_o, 0]$, hence showing passivity, from the input $v$ to the output $\hat{y}$, of the extended system characterized by the corresponding triple ($\tilde{A}, \tilde{B}, C$), namely $\tilde{\eta} = \tilde{A} \eta + \tilde{B} v$, $\tilde{y} = \tilde{C} \eta$. Moreover, the latter property is in turn instrumental to conclude stability properties of the uncertain interconnected system (57). To this end, consider the candidate Lyapunov function $V(\eta) = (1/2) \eta^T \tilde{P} \eta$. Then

$$\dot{V} = \eta^T \tilde{P} (\tilde{A} \eta + \tilde{B} v + \tilde{G} \hat{y})$$

$$\leq -\varepsilon \eta^T \eta + \eta^T \tilde{C} v + \eta^T \tilde{P} \tilde{G} \hat{y}$$

$$\leq -\varepsilon \eta^T \eta + \hat{y}^T v + \frac{1}{\gamma} \eta^T \| \tilde{P} \tilde{G} \|_2^2 + \gamma \hat{y}^T \hat{y} < 0,$$

(59)

where the first inequality follows by passivity of the triple ($\tilde{A}, \tilde{B}, \tilde{C}$), whereas the last inequality is obtained by (56).

Remark 7: The requirements (H1) – (H3) are rather mild. Note, to begin with, that items (H1) and (H2) alone are satisfied by the trivial choice $X = 0$ and $Y$ selected as any Lyapunov function for the matrix $A_\xi$, which is guaranteed to exist, by construction. Focusing instead on the conditions (H2) and (H3) it can be immediately seen that they consist in $2n$ linear equations in $n^2 + n(n + 1)/2$ unknowns, hence generically solvable for $n > 1$.

To partially specialize the above results to specific cases in which closed-form selections can be proposed, consider an uncertain double integrator described by the equations

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 \\
0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} \theta_1 \\
\theta_2 \end{bmatrix} v,$$

(60)

$$\bar{y} = \bar{x}_1,$$

with the parameters $\theta_i$ unknown and such that $\theta_1 \in \theta_{1,min}, \theta_{1,max}$ and $\theta_2 \in \theta_{2,min}, \theta_{2,max}$. We suppose that the plant (60) is not minimum-phase, hence we assume that $\theta_{1,min} > 0$ and $\theta_{2,max} < 0$.

Proposition 10: Consider the LTI plant $\Sigma_p$ defined by (60). Let the compensator $\Sigma_c$ be described by (55) with $\xi(t) \in \mathbb{R}^2$, $\alpha = [\alpha_1, \alpha_2]^T$, $B = [0, 1]^T$, $\hat{\theta}_1 = [0, 0]^T$, $\hat{\theta}_2 = [0, 1]^T$. Let $K = [\kappa_1, \kappa_2]$ be such that

$$\frac{\theta_{2,max}}{\theta_{1,max}} < \kappa_1 < 0,$$

(61a)

$$-\frac{1}{\theta_{1,max}} < \kappa_2 < \frac{\theta_{2,max}}{\theta_{1,max}} K_1.$$

(61b)

Then, the interconnected system (60)-(55) from the input $v$ to the output $\hat{y} = x_1 + \xi_1$ satisfies the assumptions (C1) – (C3) of the MRAC scheme.

Proof: The characteristic polynomial associated to the extended plant becomes

$$s^2(1 + \kappa_2 \kappa_1) + s(\kappa_2 \theta_2 + \kappa_1 \theta_1) + \kappa_1 \theta_2,$$

(62)

which is then Hurwitz, hence the interconnected plant is minimum-phase, if and only if $1 + \theta_1 \kappa_2 > 0$, $\kappa_2 \theta_1 + \kappa_1 \theta_1 > 0$ and $\kappa_1 \theta_2 > 0$ for any $\theta \in \Theta$. The last inequality implies that $\kappa_1 < 0$, hence the need for the right inequality in (61a). This, in turn, implies that also $\kappa_2$ must be negative to satisfy the second inequality. Finally, rewriting the first two inequalities above in terms of the worst case value for $\theta_1$ and $\theta_2$ one obtains the constraints in (61). □

Note that the choice of the gain $K$ as suggested in the statement of Proposition 10 is such that indeed the interconnected plant is minimum-phase, but nothing can be a priori concluded about the location of the closed-loop poles.

As a numerical example consider the case in which $\theta_1 \in [1, 10]$ and $\theta_2 \in [-10, -1]$. Since the numerator of the transfer function from $u$ to $y$ is defined by $\theta_1 s + \theta_2$, the ranges of variation of the two unknown parameters imply that the system is always non-minimum phase, with the unstable zero ranging from 0.1 to 10. The compensator $\Sigma_c$ described by the equations

$$\dot{x} = \begin{bmatrix} 0 & 1 \\
-\kappa_1 & -\kappa_2 \end{bmatrix} x + \begin{bmatrix} \alpha_1 \\
\alpha_2 \end{bmatrix} y + \begin{bmatrix} 0 \\
1 \end{bmatrix} v,$$

(63)

is such that the interconnected system (60)-(63) is minimum-phase, from the input $v$ to the modified output $\hat{y}$, for any admissible uncertain parameter $\theta$, provided $\kappa_1$ and $\kappa_2$ are selected according to $-0.01 < \kappa_1 < 0$ and $-0.1 < \kappa_2 < 10 \kappa_1$. In the following simulations, the parameters are selected as $\kappa_1 = -0.005$ and $\kappa_2 = -0.075$. The control objective consists in steering the output of the original non-minimum phase plant (60) to track a sinusoidal function of time. To
this end, by following steps of the classical MRAC design [16], we introduce the standard desired model reference and, in addition, the dynamics required to compensate at steady-state the fact that the output $\hat{y} = y + C_\alpha \xi$, in place of $y$ alone, is employed for feedback, namely

$$
\begin{align*}
\dot{x}_m &= A_m x_m + B_m r, \\
\dot{\xi}_m &= A_\xi \xi_m - \alpha C_m x_m + B_v, \\
\hat{y}_m &= C_m x_m + C_\alpha \xi_m \\
y_m &= y + C_\xi \xi_m,
\end{align*}
$$

(64)

with $r(t) = \cos(0.5t)$, $C_m = [1, 0]$ and

$$
A_m = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
$$

(65)

and with the error defined as $e = \hat{y} - \hat{y}_m = y - C_m x_m + C_\alpha (\xi - \xi_m)$. The numerical simulations reported in Figure 5 show that, after the transient response, $C_\alpha \xi(t)$ and $C_\xi \xi_m(t)$ coincide and the adaptive control scheme is then such that the output of the original non-minimum phase plant (60) converges to the desired reference signal $C_m x_m$ for any value of the uncertain parameter vector $\theta \in \Theta$.

### VI. Conclusions

A model matching and passivating control architecture for MIMO linear systems, comprising dynamic feedback and feedforward, has been proposed. The design methodology does not rely on assumptions concerning the relative degree or minimum phaseness of the plant and employs only input/output measurements. The construction provides a closed-loop system that from suitably modified inputs and outputs matches any desired transfer matrix. This is achieved, in its basic formulation, at the price of observability of the overall interconnected system. Therefore an alternative implementation of the above design has been proposed that allows to achieve an arbitrary approximation accuracy of the desired matrix while also preserving structural properties - in particular observability - of the overall interconnected system. Such a construction can then be specialized to provide input/output decoupling or a system that is passive from a novel control input to a modified output. The result is achieved by arbitrarily assigning the relative degree and the location of poles and zeros on the complex plane of the interconnected system in a systematic way. It is also shown that similar ideas can be employed to enforce a desired, arbitrarily small, $L_2$-gain from an unknown disturbance input to a modified output, while preserving the corresponding gain from the control input to the same output. Finally, similar constructions are employed to study the adaptive control problem for uncertain linear systems that are potentially not minimum-phase.

### References


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