LOCALIC COMPLETION OF QUASIMETRIC SPACES

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Abstract
We give a constructive localic account of the completion of quasimetric spaces. In the context of Lawvere's approach, using enriched categories, the points of the completion are flat left modules over the quasimetric space. The completion is a triquotient surjective image of a space of Cauchy sequences and can also be embedded in a continuous dcpo, the "ball domain". Various examples and constructions are given, including the lower, upper and Vietoris powerlocales, which are completions of finite powerspaces. The exposition uses the language of locales as "topology-free spaces".

Keywords
topology, locale, topos, geometric theory, metric, quasimetric, completion, enriched category, triquotient, powerlocale.

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1. Introduction

The aim of this paper is to give a constructive localic account of metric completion (and it turns out that the techniques work well also in the quasimetric case, i.e. dropping the symmetry axiom). It brings together a number of different ideas, and to give some immediate overview I ought to point out that there are at least three distinct novelties.

Conceptually, we take a somewhat unorthodox view of the nature of completion. It is usual, amongst the general spaces, all topologized, to distinguish the complete ones, and to give a general "completion" construction that always yields a complete space. By contrast we regard the uncompleted space as being not itself topologized, but a set-theoretic structure that is used simply as a presentation of the completion, which *is* topologized. There is a localic reason for doing it this way. Normally, one thinks of a metric as defining a new topology on a set of points, which of course already has its discrete topology. However, in the constructive treatment of locales it is not possible to change the topology without also changing the points – at least, when points are regarded in the "generalized" sense, which allows set theory to vary.

If we start from a metric space (X, d) then we can construct the metric topology on X – as a subframe of its powerset – and hence define a locale X'. In our base set theory, the points of X' will be the elements of X, but this fails elsewhere. Moreover, the very construction of this metric topology is not preserved under change of base, so in effect we have fudged the construction of the topology in attempting to get a result (points are elements of X) that still falls apart. Our view of completion is that it is the good construction of the metric topology, but that even in the base category of sets it creates new points.

Hence completion is viewed as a construction that changes the nature of the object that it is applied to – from set with structure to a locale.

Expositionally, though we are working with locales, we experiment with a new approach to them as "topology-free spaces", as outlined in [21] and [23]. In execution, this has the appearance of working with topological spaces but ignorantly neglecting to deal with topologies: they are not explicitly defined, nor is continuity explicity proved. However, this is justified by adherence to a constructivist discipline that makes topology implicit.

Technically, we use an approach to completion that derives from enriched category theory – this was first proposed in the study of quasimetric spaces in [11]. According to our definition, the points of the completion are the "flat left modules" over the quasimetric space, in a sense that is already well-known for categories enriched over Abelian groups or sets. However, the clearest topological way of understanding this is that instead of constructing the completion using Cauchy sequences, we use a definition that more directly reflects filter definitions of completeness.

Cauchy completion by distance functions

Traditionally, one completes a metric space X by taking Cauchy sequences modulo an equivalence relation. This trick is not available localically, so we shall need *canonical* representations of the points of the completion $\overline{X}$. We give here a classically valid result that provides these in the form of certain maps from X to the real line. Although the result itself is not constructively valid, we shall take it as justification for using the maps rather than the Cauchy sequences as the constructive basis for metric completion.

If $\xi = (x_n) \in \overline{X}$ is a Cauchy sequence, then we can define a map $M: X \to [0, \infty)$ by $M(x) = \lim_{n\to\infty} d(x,x_n) = d(x,\xi)$ in $\overline{X}$.

Proposition 1.1 *(Classically)* M satisfies the following conditions:

(i) $M(y) \leq M(x)+d(x,y)$

(ii) $d(x,y) \le M(x)+M(y)$
(iii) $\inf_x M(x) = 0$

Proof (i) and (ii) are instances of the triangle inequality in $\overline{X}$, (iii) is obvious.]

Proposition 1.2 *(Classically)* Let $\xi = (x_n)$ and $\xi' = (x'_n)$ be two Cauchy sequences, giving rise to functions M and M' as above. Then the sequences are equivalent iff M = M'.
Proof $\Rightarrow$: ξ and ξ' are equal in $\overline{X}$, so for all x, $M(x) = d(x,\xi) = d(x,\xi') = M'(x)$.
$\Leftarrow$: $d(\xi,\xi') = \lim_{n\to\infty} d(x_n,\xi') = \lim_{n\to\infty} M'(x_n) = \lim_{n\to\infty} M(x_n) = \lim_{n\to\infty} d(x_n,\xi) = 0$, so the sequences are equivalent.]

Proposition 1.3 *(Classically)* If M satisfies the conditions of Proposition 1.1, then there is a Cauchy sequence (x_n) from which M is derived by the definition above.
Proof
By (iii) we can find a sequence (x_n) such that $M(x_n) < 2^{-n}$. Then by (ii),

$$d(x_n, x_{n+k}) \le M(x_n)+M(x_{n+k}) < 2^{-n} + 2^{-n-k} \le 2^{-n+1}$$

and it follows that (x_n) is Cauchy. For any x, we have

$$M(x) \le M(x_n) + d(x,x_n) < 2^{-n} + d(x,x_n) \qquad \text{(using (i))}$$
$$d(x,x_n) \le M(x) + M(x_n) < 2^{-n} + M(x) \qquad \text{(using (ii))}$$

and it follows that $M(x) = \lim_{n\to\infty} d(x,x_n)$.]

In summary, we have

Theorem 1.4 *(Classically)* If X is a metric space, then the points of its Cauchy completion are in bijective correspondence with the maps $M: X \to [0,\infty)$ satisfying the condition of Proposition 1.1.]

It turns out that these functions M fit perfectly in Lawvere's account of metric spaces using enriched category theory: the metric space X itself is the enriched category (enriched over $[0,\infty]$), condition 1.1 (i) makes M a "presheaf" over X, and conditions (ii) and (iii) say that it is "flat". However, an equivalent topological view is that M describes the open balls $B_\varepsilon(x)$ that contain the point (i.e. for which $M(x) < \varepsilon$), and our definition will fairly naturally turn out to be equivalent to filters of such open balls (Proposition 4.8).

Quasimetric spaces via enriched categories

In [11] metric spaces are discussed as $\mathcal{V}$-enriched categories ($\mathcal{V}$ here being the extended non-negative real line $[0,\infty]$) and despite the abstractness of this account it has a solid conceptual basis: it views a metric space as a "set" in which equality formulae receive their truth values as real numbers. The distance d(x,y) is the "numerical falsity" of the formula x=y, so that the bigger d(x,y) is, the "less equal" x and y are. A zero value represents utter truth: if $d(x,y) = 0$ then x and y are considered equal. The conditions on metrics correspond to properties of equality, and in particular the triangle law corresponds to transitivity. (Actually, Lawvere deals with quasimetrics, i.e. without symmetry, which correspond to partial orders or preorders.)

The use of $\mathcal{V}$-enriched categories invites comparison with other possibilities for $\mathcal{V}$, and we shall refer particularly to the case of Abelian groups. Of particular importance there is tensor product of modules, and the notion of *flatness:* a module is flat iff tensoring by it preserves finite limits. (General abstract nonsense ([1], p.28) shows that tensoring always preserves arbitrary colimits, for it has a right adjoint given by modules of homomorpisms.) We show that over a metric space, the flat modules are the points of the Cauchy completion, and develop the idea with quasimetric spaces.

Locales as topology-free spaces

The results here are localic, but the casual reader might be excused for not realising this. Locale theory is often described as "point-free topology": the frame is an abstract topology that does not rely for its description on a set of points of whose powerset it is a subframe. In the standard introductions such as [6] or [16], the frames appear very explicitly. By contrast, what you see here is almost entirely in terms of points, with hardly any mention of topology at all – so little indeed, that even as conventional topology it is somewhat negligent. Turning the usual description on its head, we treat locales as "topology-free spaces".

The trick lies in the nature of the mathematical discussion, for it is of a restricted "geometric" form. It turns out that this is sufficient to give topologies and continuity automatically.

A fuller technical account will be given in Section 2, but let us here set out the ground rules.

(1) "Geometric" mathematics comprises those constructions and properties that can be interpreted in any Grothendieck topos and are preserved by the inverse image functors of geometric morphisms.

(We shall actually use a different language, analogous to that of [18, 20], that makes a systematic distinction between "toposes as generalized topological spaces" – for which we reserve the word *topos* – and "toposes as generalized categories of sets" – which we call *geometric universes,* or *GUs.* For a topos D, the corresponding geometric universe will be written $\mathcal{S}$D and its objects will be called sheaves over D. Inverse image functors of geometric morphisms will be called "GU-homomorphisms".)

(2) If certain structures are described as being the models of a geometric theory, that is to say, they are specified by structure and properties within geometric mathematics, then there is a corresponding classifying topos of which those structures are the points.

(3) If such a theory in (2) is "essentially propositional", that is to say, it has no sorts (other than what can be constructed geometrically out of thin air), then its classifying topos is actually a locale.

(4) Suppose D and E are two toposes. Then construction of points of E out of points of D, if it is geometric, describes a geometric morphism from D to E.

(5) Geometric morphisms between locales are the same as continuous maps.

Consequently, we describe a locale by giving a geometric description of its points; and we describe a continuous map by giving a geometric description of how it transforms points to points. No discussion of topology is needed – the geometricity already covers that –, and so the locales appear as "topology-free spaces".

We shall examine what is allowed in this geometric mathematics, but first let us mention some things that are *not* allowed.

- *The logic is non-classical.* Intuitionistic logic is valid in geometric universes, but in general excluded middle and choice are not valid. More subtly, intuitionistic negation is not preserved by GU-homomorphisms, and nor are implication and universal quantification – so we can't use them in general. The geometric logic is therefore more restricted than intuitionistic logic.

However, if we can prove or postulate that two propositions P and Q are logical complements ($P \wedge Q \vdash$ **false**, **true** $\vdash P \vee Q$), then that fact is preserved by GU-homomorphisms and so gives an instance of a geometric negation.

- We can't use exponentials X^Y, powersets $\mathcal{P}X$, or the subobject classifier Ω – none of these are preserved by GU-homomorphisms.

I shall not attempt to formalize the geometric constructions, but they include finite limits, set-indexed colimits, image factorization, monicness, epiness, inclusion between subobjects, finite intersections and arbitrary set-indexed unions of subobjects, existential quantification, free algebra constructions, $\mathbb{N}$ (natural numbers), $\mathbb{Q}$ (rationals), Kuratowski finiteness, finite powersets $\mathcal{F}X$(free semilattices) and universal quantification bounded over finite objects.

A couple of specific issues worth mentioning are decidability and finiteness. Equality is part of the geometric logic, but inequality is not (because there is no negation). Nonetheless, certain "decidable" sets come equipped with inequality, a relation complementary to equality – two good examples are $\mathbb{N}$ and $\mathbb{Q}$. Finiteness is – as remarked above – *Kuratowski finiteness* [5]: X is Kuratowski finite iff the free semilattice $\mathcal{F}X$ has an element T such that $\{x\} \subseteq T$ for every x. This notion can sometimes behave surprisingly from the point of view of classical mathematics: for instance, subsets of finite sets, or intersections of finite subsets, need not themselves be finite. A summary of the mathematics of this finiteness is provided in [24], which also include "observational" intuitions that explain the surprises.

2. Technicalities on locales and toposes

The ideas of "locales as topology-free spaces", as outlined above, have already been described informally in [21, 23]. However, a more detailed technical justification has been lacking and we take the opportunity to present one here. The reader who is more interested in the localic account of completion is invited to skip this section and admire the audacity of the subsequent treatment.

The notion of geometric theory can be found in standard texts such as [5] and [12]: it is a first-order, many-sorted theory in which the axioms are of the form $\phi \vdash_S \psi$, where ϕ and ψ are geometric formulae (the connectives allowed are finite conjunction, arbitrary disjunction, equality and existential quantification) all of whose free variables are in the finite set S. If T is a geometric theory, then we shall write [T] for its classifying topos, and $\mathcal{S}$[T] for the corresponding geometric universe. (Note that by happy coincidence our notation "$\mathcal{S}$[T]" – "$\mathcal{S}$heaves over [T]" – agrees with one that is already commonly used, and denotes the same category – it is $\mathcal{S}$ (**Sets**) with a model of T freely adjoined. Where we depart from convention is in refusing to call this geometric universe the classifying topos.)

However, we shall extend this notion slightly to allow geometric constructions to be used as notational definitions. It is clear what a model would be for such a theory. It is not immediate that such theories have classifying toposes, but we shall prove that they do.

As mentioned in the Introduction, we use the phrase *geometric universe* (or *GU*) for (Grothendieck) topos as generalized category of sets. Hence a category is a GU iff it has the structure and properties given in Giraud's Theorem (see [5]).

2.1 Propositional geometric theories

A geometric theory is *propositional* if it has no sorts at all. Since terms must have sorts, we see that there can be no functions at all (no sorts for their results) and predicates can have no variables: so they are propositions. A formula then is equivalent to a disjunction of finite conjunctions of propositional symbols, and an axiom is $\phi \vdash \psi$. It follows that a propositional geometric theory T is formally identical to a presentation of a frame by generators and relations [16], and so presents a frame Ω[T] (say), corresponding to a locale [T].

Let us now write $\mathcal{S}$[T] for the category of sheaves over [T], according to the normal definition of sheaf. $\mathcal{S}$[T] is a geometric universe, so there is a corresponding topos which, according to our earlier remarks, should be written [T]. But then is [T] the locale or the topos? Actually, it doesn't matter, for we have ample notation to discriminate between the frame Ω[T] and the geometric universe $\mathcal{S}$[T], and this point of view allows us to say that a topos truly is a generalized locale.

So far, this is really no more than a recasting of standard results, part of which is the fact that continuous maps between the locales are equivalent to geometric morphisms between the toposes:

Theorem 2.1.1 Let f: E $\rightarrow$ B be a localic geometric morphism. Then for any other geometric morphism f': E' $\rightarrow$ B, there is an equivalence between geometric morphisms from E' to E over B, and frame homomorphisms from $f_*\Omega_E$ to $f'_*\Omega_{E'}$ in $\mathcal{S}$B. (Ω_E and $\Omega_{E'}$ are the subobject classifiers in $\mathcal{S}$E and $\mathcal{S}$E'.)
Proof [10].]

If we apply this with B = 1 (the topos classifying the theory with empty presentation; $\mathcal{S}1 = \mathcal{S}$) and E = [U], U a propositional geometric theory, then we find that for any topos E', with f': E' $\rightarrow$ 1 the essentially unique geometric morphism, geometric morphisms from E' to [U] are equivalent to frame homomorphisms from Ω[U] to $f'_*\Omega_{E'}$. If we then take E' to be [T] for another propositional geometric theory, then we find that geometric morphisms from [T] to [U] (qua toposes) are equivalent to frame homomorphisms from Ω[U] to Ω[T], i.e. continuous maps from [T] to [U] (qua locales).

The topos [T] was constructed by sheaf theory, as the geometric universe of sheaves $\mathcal{S}$[T] constructed from the frame Ω[T]. Our notation was devised to suggest that [T] classifies the theory T, but that remains to be proved. Another way of viewing this is that the standard theory works from the frame Ω[T], but we want a firmer grasp of how it relates to the presentations T. We shall therefore be more precise about the structure of a propositional theory.

A frame presentation will include sets G and R of generators and relations, and the relations can be written in the form $e_1 \leq e_2$, where e_1 and e_2 are frame expressions in the generators. Using frame distributivity, each e_i can be written as a join of finite meets of generators; and then the relation can be replaced by a set of relations, one for each disjunct in e_1, saying that the disjunct is $\leq e_2$. After all this rewriting we have that each relation r is of the form –

finite meet of generators $\leq$ join of finite meets of generators

Let us write $\lambda(r) \in \mathcal{F}$G for the finite set of conjuncts on the left. For the right-hand side, we have an arbitrary set of disjuncts: so what we should do is take the set D of all disjuncts in all relations, fibred over R by some π: D $\rightarrow$ R. Each disjunct d is a conjunction of a finite set ρ(d) of generators, so the relation r has been formalized as –

$\bigwedge\lambda(r) \leq \bigvee_{\pi(d)=r} \bigwedge\rho(d)$

Definition 2.1.2 A *frame presentation* is a structure comprising three sets G, R and D with functions λ: R $\to \mathcal{F}$G, ρ: D $\to \mathcal{F}$G, and π: D $\to$ R.

We write **FrPr** for the (geometric) theory of frame presentations.

Given such a frame presentation, we shall as usual write Fr $\langle$ G | R $\rangle$ for the frame presented by it. Then a frame homomorphism from Fr $\langle$ G | R $\rangle$ to a frame A is given by a function γ: G $\to$ A that respects the relations in the following way. Since A is a semilattice under $\wedge$, and $\mathcal{F}$G is the free semilattice over G, γ extends uniquely to a semilattice homomorphism γ': ($\mathcal{F}$G, $\cup$) $\to$ (A, $\wedge$) such that $\gamma'(\{g\}) = \gamma(g)$. We want for each relation r that $\gamma' \circ \lambda(r) \leq \bigvee\{\gamma' \circ \rho(d): \pi(d) = r\}$.

Theorem 2.1.3 Let T be a geometric theory whose ingredients include a frame presentation as above, and let A be the frame in $\mathcal{S}$[T] presented by it. Then the corresponding locale over [T] classifies the theory T' that is T extended by –

- a predicate symbol I(g) (g: G)
- an axiom

$\forall g \in \lambda(r).\ I(g) \vdash_{r:R} \exists d{:}D.\ (\pi(d) = r \wedge \forall g \in \rho(d).\ I(g))$

Proof

Let f: E $\to$ [T] be any topos over [T]. We know (by Theorem 2.1.1) that geometric morphisms over [T] from E to the locale are equivalent to frame homomorphisms in $\mathcal{S}$[T] from A to $f_*\Omega_E$, and these are equivalent to functions from G to $f_*\Omega_E$ that respect the relations. On the other hand, geometric morphisms over [T] from E to [T'] are equivalent to subsets of f*(G) that satisfy the axiom. We show that these are equivalent, and that suffices to show that the locale and [T'] are equivalent.

Functions γ: G $\to f_*\Omega_E$ are equivalent to functions from f*G to Ω_E, which in turn are equivalent to subsets I of f*G. The difficult part is to show that the one respects the relations iff the other satisfies the axiom.

Recall some general properties about how f_* and f* relate to algebras for any finitary algebraic theory. First, because both f_* and f* preserve finite products, they transform algebras into algebras. (In particular, this gives the distributive lattice structure on $f_*\Omega_E$, though not the frame structure – for Mikkelson's description of joins in $f_*\Omega_E$ see [5], Proposition 5.36.) Moreover, if X and Y are algebras in $\mathcal{S}$[T] and $\mathcal{S}$E respectively, and θ: X $\to f_*$Y and ϕ: f*X $\to$ Y are adjoint transposes of each other, then θ is a homomorphism iff ϕ is. Finally, if $\mathbb{F}$ denotes the free algebra construction, then f* preserves it: $f^*\mathbb{F}X \cong \mathbb{F}(f^*X)$ by an algebra isomorphism in $\mathcal{S}$E (we shall apply this for the theory of semilattices, so that $\mathbb{F}$ is $\mathcal{F}$).

Lemma 2.1.4 Let θ: X $\to f_*\Omega_E$ in $\mathcal{S}$[T] give semilattice homomorphism θ': $\mathcal{F}$X $\to$ ($f_*\Omega_E$, $\wedge$), and let their adjoint transposes be ϕ: f*X $\to \Omega_E$ and ϕ': f* $\mathcal{F}$X $\to \Omega_E$. Then the function

$\cong;\phi'$: $\mathcal{F}(f^*X) \cong f^*\mathcal{F}X \to \Omega_E$

maps S to the truth value $[\forall x \in S.\ \phi(x)]$.

Proof

$$\begin{tikzcd}
& f^*X \arrow[r, "\phi"] \arrow[dl, "\{-\}"'] \arrow[d, "f^*\{-\}"] & \Omega_E \\
\mathcal{F}f^*X \arrow[r, "\cong"'] & f^*\mathcal{F}X \arrow[ur, "\phi'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\theta"] \arrow[d, "\{-\}"] & f_*\Omega_E \\
\mathcal{F}X \arrow[ur, "\theta'"'] &
\end{tikzcd}$$

In the left-hand diagram, both triangles commute: the left-hand one by definition of the isomorphism, and the right-hand one by naturality of the adjoint transpose. Also, ϕ' is a semilattice homomorphism (Ω_E as $\wedge$-semilattice) because θ' is. It follows that $\cong;\phi'$ is the unique semilattice homomorphism mapping each $\{x\}$ to $\phi(x)$, but $S \mapsto [\forall x \in S.\ \phi(x)]$ is such a one.]

From the lemma (applied to γ: $G \to f_*\Omega_E$), and using naturality, it follows that the adjoint transposes of $\lambda;\gamma'$: $R \to f_*\Omega_E$ and $\rho;\gamma'$: $D \to f_*\Omega_E$ correspond to the subsets $\{r\colon \forall g \in f^*\lambda(r).\ I(g)\}$ and $\{d\colon \forall g \in f^*\rho(d).\ I(g)\}$of f^*R and f^*D. It remains to show that the adjoint transpose of the function $r \mapsto \bigvee\{\gamma' \circ \rho(d)\colon \pi(d) = r\}$ from R to $f_*\Omega_E$ corresponds to the subset

$$\{r\colon \exists d.(f^*\pi(d) = r \wedge \forall g \in f^*\rho(d).\ I(g)\}$$

of f^*R. Our function from R to $f_*\Omega_E$ calculates the join of a generalized element of $\Omega_{[T]}{}^{f_*\Omega_E}$, namely $r \mapsto \{\gamma' \circ \rho(d)\colon \pi(d) = r\}$. The generalized element comes from a function from $R \times f_*\Omega_E$ to $\Omega_{[T]}$, and this corresponds to a subset of $R \times f_*\Omega_E$, namely the image of $\langle \pi, \gamma' \circ \rho\rangle\colon D \to R \times f_*\Omega_E$. We obtain a function $\langle f^*\pi, \alpha\rangle\colon f^*D \to f^*R \times \Omega_E$, where $\alpha\colon f^*D \to \Omega_E$ is the adjoint transpose of $\gamma' \circ \rho$, which by the lemma corresponds to the subset $\{d\colon \forall g \in f^*\rho(d).\ I(g)\}$of f^*D. The image of $\langle f^*\pi, \alpha\rangle$ gives a function from f^*R to $\Omega_E{}^{\Omega_E}$, $r \mapsto \{a \in \Omega_E \mid \exists d{:}f^*D.(f^*\pi(d) = r \wedge a = [\forall g \in f^*\rho(d).\ I(g)])\}$ and the join of this set is the truth value $[\exists d{:}f^*D.(f^*\pi(d) = r \wedge \forall g \in f^*\rho(d).\ I(g)]$, and Mikkelson's description says that this is the adjoint transpose of $r \mapsto \bigvee\{\gamma' \circ \rho(d)\colon \pi(d) = r\}$.]

2.2 Essentially propositional geometric theories

Though the propositional theories are sufficient to describe locales, it turns out to be very convenient to use equivalent non-propositional theories. We shall call a geometric theory *essentially propositional* if it has no base sorts: so all the sorts used must be constructed geometrically out of nothing.

Proposition 2.2.1 Essentially propositional geometric theories are equivalent to propositional ones.

Proof

Let T be an essentially propositional geometric theory. Without loss of generality, we can assume that it has no function symbols: for the functions can be replaced by their graphs (as predicates, axiomatized to require single-valuedness and totality).

Each type is constructed out of nothing, and so is interpreted uniquely (up to isomorphism) as an object A in $\mathcal{S}$, the base category of sets; moreover, the construction is preserved by GU-homomorphisms, giving the type an essentially unique interpretation in any geometric universe – in fact, as an A-indexed coproduct of copies of 1.

Now consider an n-ary predicate $P(x_1, \ldots, x_n)$ in T. Each x_i is typed; let X be the product of the corresponding interpretations of the types in $\mathcal{S}$. Then in any model of T, the interpretation of P is a subobject of the interpretation of X, which is equivalent to an X-indexed family of truth values. It follows that P in the theory can be replaced by an X-indexed family of propositional symbols.

Finally, we must look at the axioms. In [18] it is shown that any geometric formula $\phi(\mathbf{x})$ is equivalent to one of the form $\bigvee_i (E_i \wedge \exists \mathbf{y}^{(i)}. \bigwedge_{j=1}^{n_i} P_{ij})$ where (1) for each i the vectors $\mathbf{x}$ and $\mathbf{y}^{(i)}$ are disjoint, (2) each E_i is a conjunction of equations among the free variables $\mathbf{x}$, and (3) each P_{ij} is a predicate applied to variables from $\mathbf{x}$ and $\mathbf{y}^{(i)}$. We can represent ϕ by an X-indexed family of propositional formulae, where X is the interpretation of the product type for $\mathbf{x}$. Suppose $\xi \in X$. Then we get a truth value $\phi(\xi)$ constructed in propositional geometric logic: $\bigvee_i$ is – obviously – just disjunction, "$E_i \wedge \ldots$" is $\bigvee\{\ldots$: equations E_i hold for $\xi\}$ (a disjunction of a subsingleton), "$\exists \mathbf{y}^{(i)}. \ldots$" is $\bigvee\{\ldots(\xi, \eta^{(i)})$: $\eta^{(i)}$ in interpretation of product type for $\mathbf{y}^{(i)}\}$, $\bigwedge$ is conjunction, and $P_{ij}(\xi,\eta^{(i)})$ is the corresponding proposition. Hence any geometric axiom $\phi(\mathbf{x}) \vdash_{\mathbf{x}} \psi(\mathbf{x})$ gives an X-indexed family of propositional relations.]

It follows that by writing down an essentially propositional geometric theory T, we can present a locale [T] whose points (anywhere) are equivalent to the models of T. Its frame Ω[T] is presented by generators and relations got from an equivalent propositional theory.

2.3 Continuous maps

We have already seen that continuous maps between locales are equivalent to geometric morphisms between the corresponding toposes; and the theory of classifying toposes makes it easy to define these as geometric constructions.

In passing, let us note that any geometric theory in our extended sense must have a classifying topos. We have already covered the essentially propositional case in the previous section. More generally, if a theory T has base sorts, then we can form a classifying topos [T'] for those base sorts on their own. If we then relativize ourselves to $\mathcal{S}$[T'] then T is essentially propositional and so has a classifying topos over [T'].

Returning to our discussion of geometric morphisms, suppose T and U are two geometric theories (not necessarily propositional), and let F be a geometric construction that transforms models M of T to models F(M) of U. Since F is geometric, it can be applied in any geometric universe, so in particular we can apply it to the generic T-model M_0 (say) in $\mathcal{S}$[T] to give a U-model $F(M_0)$, also in $\mathcal{S}$[T]. By the theory of classifying toposes, this then gives us a geometric morphism f: [T] $\rightarrow$ [U] such that f* applied to the generic model N_0 of U gives (something isomorphic to) $F(M_0)$. Now suppose M is an arbitrary model of T, in $\mathcal{S}$E, say; it will correspond to a geometric morphism g: E $\rightarrow$ [T]. Since F is geometric, it is preserved by GU-homomorphisms, so $F(M) \cong F(g^*(M_0)) \cong g^*(F(M_0)) \cong g^*(f^*(N_0)) \cong (g;f)^*(N_0)$. Hence, applying F is equivalent to postcomposing with f.

We thus see that geometric transformations give us geometric morphisms. In the circumstances, it seems fussy to distinguish notationally between F and f – we might as well call them both by the same name and think of a geometric morphism as a geometric transformation.

As a corollary, note that geometric transformations are therefore automatically functorial with respect to homomorphisms between models, and moreover preserve filtered colimits: the homomorphisms are the natural transformations between geometric morphisms (in the 2-category of toposes), and postcomposition by geometric morphisms has these properties. In particular, in the localic case, geometric transformations (or continuous maps) are automatically monotone with respect to the specialization order, and preserve directed joins (Scott continuity).

3. Quasimetric spaces

In this section, we give a localic discussion of quasimetric spaces (*quasi*metric means dropping the symmetry law). As already mentioned, we do not try to consider the space itself to be a locale "under its metric topology" – it is a set (discrete locale) with some additional structure given by the quasimetric. However, we do need to take care over our localic account of the real line in which the quasimetric takes its values, for we don't assume the usual topology (which, constructively, means the points are slightly different too).

Definition 3.1 $\overleftarrow{[0,\infty]}$ is the locale whose points are rounded upper sets of positive rationals (we write Q_+ for the set of positive rationals) – we shall refer to its points as *upper reals.* We shall frequently use conventional notation such as "x" for an upper real considered abstractly, and R_x for the corresponding subset of Q_+.

To show how this already defines the topology, we have

$$\Omega\overleftarrow{[0,\infty]} = \mathrm{Fr}\,\langle\, [0,q)\ (q\in Q_+) \mid [0,q) = \bigvee_{q'<q} [0,q') \,\rangle$$

The symbolic generator $[0,q)$ corresponds to "$q \in R_x$", and the relations to the property of being rounded (the $\leq$ direction of the relation) and upper ($\geq$).

This is the rounded ideal completion of the continuous information system ([17], where it is called an "infosys") whose tokens are the extended positive rationals ordered by $>$ (with $\infty > \infty > q$) – in other words, "<" in the infosys sense is numerical ">". Hence it is a continuous dcpo (i.e. its frame is constructively completely distributive).

Classically, it is the space whose points are the non-negative extended reals, $[0,\infty]$, under the Scott topology for $\geq$. But constructively we cannot change the topology without changing the points, and the points of $\overleftarrow{[0,\infty]}$ are not real numbers – i.e. Dedekind sections – because they lack half the data and it can't be reconstructed geometrically.

We now give localic expression to the symmetric monoidal category (or, rather, poset) structure on $\overleftarrow{[0,\infty]}$. The poset structure is the specialization order, the reverse of the usual numerical order (a big set of rationals is numerically small).

The monoidal product is sum:

$$R_{x+y} = \{q+r \colon q \in R_x,\ r \in R_y\}$$

giving a map $+\colon \overleftarrow{[0,\infty]}^{\,2} \to \overleftarrow{[0,\infty]}$. This is associative and commutative, and its unit is 0 (R_0 contains all the positive rationals).

Finite limits – meets – are given by right adjoints to the finite diagonals $\Delta\colon \overleftarrow{[0,\infty]} \to \overleftarrow{[0,\infty]}^{\,n}$. Specifically, for n = 0 we have a top point 0, and for n = 2 we have max: $\overleftarrow{[0,\infty]}^{\,2} \to \overleftarrow{[0,\infty]}$, $R_{max(x,y)} = R_x \cap R_y$.

Similarly, finite colimits are given by left adjoints to the finite diagonals. The nullary join is ∞ ($R_\infty = \emptyset$), and binary join is min, $R_{min(x,y)} = R_x \cup R_y$. However, since a locale always has directed joins, we actually have arbitrary joins (numerical inf) in $\overleftarrow{[0,\infty]}$: $R_{\inf X} = \bigcup_{x\in X} R_x$.

The need to work with extended reals, i.e. including ∞, is a minor nuisance, but seems constructively unavoidable if we are to give meaning to the empty infs that arise in connection with empty metric spaces. Of course, for the usual constructivist reasons, it would be quite wrong to try to exclude these empty metric spaces.

Morally, of course, $\overleftarrow{[0,\infty]}$ should be monoidal closed. However, the monoidal hom can't be defined as a continuous map, for it would have to be contravariant in one argument. It would be interesting to know whether the closedness can still be expressed somehow in this localic setting, but fortunately it seems that we can manage without having it as an explicit piece of structure.

Definition 3.2 A *metric space* is a set X equipped with a map $d\colon X^2 \to \overleftarrow{[0,\infty]}$ satisfying

- $d(x,x) = 0$
- $d(x,z) \le d(x,y)+d(y,z)$
- $d(x,y) = d(y,x)$

We do not assume that $d(x,y) = 0$ implies $x = y$: in other words, we are really defining a *pseudo*metric space. However, the omitted property does not arise naturally in the enriched category setting, and makes no difference to our theory of completion. Therefore we drop it without bothering to write "pseudo" everywhere.

If the symmetry axiom is dropped, then we have a *quasimetric* space. Much of the general theory works in the quasimetric case (as Lawvere pointed out).

We shall call the metric space, or quasimetric space, –

- *finitary* iff $d(x,y)$ is finite for every x, y;
- *Dedekind* iff d factors via $[0,\infty] \to \overleftarrow{[0,\infty]}$.

There are geometric theories of quasimetric spaces. The trick is to replace d by a ternary relation, a subset of $X\times X\times Q_+$. We shall write this relation using the suggestive notation "$d(x,y) < \varepsilon$". Then the axioms are –

$$d(x,y) < \varepsilon \wedge \varepsilon < \varepsilon' \vdash d(x,y) < \varepsilon'$$
$$d(x,y) < \varepsilon' \vdash \exists\varepsilon.\ (d(x,y) < \varepsilon \wedge \varepsilon < \varepsilon')$$
$$\vdash d(x,x) < \varepsilon$$
$$d(x,y) < \delta \wedge d(y,z) < \varepsilon \vdash d(x,z) < \delta+\varepsilon$$

and, for metrics (symmetric), –

$$d(x,y) < \varepsilon \vdash d(y,x) < \varepsilon$$

(The first two of these express the fact the d(x,y) is a point of $\overleftarrow{[0,\infty]}$, and the last three express the conditions given in Definition 3.2.) In the case of Dedekind spaces, we should also need a second ternary relation "$d(x,y) > \varepsilon$". From the geometricity, it follows that there are toposes classifying the various kinds of spaces, in particular [MS] and [QMS] for metric or quasimetric spaces.

Given a metric space, the functions M as in Proposition 1.1 are the models of a geometric theory, using the trick of expressing M by through the relation "$M(x) < \varepsilon$" (ε rational). But since X is given and Q_+ is geometrically constructible, the theory is essentially propositional and hence presents a locale. These propositions correspond to the rational open balls $B_\varepsilon(x)$, and hence the topology induced by this locale is the usual one. It follows that we have a localic completion; explicitly, it is given by

$$\begin{aligned}\Omega\overline{X} = \mathrm{Fr}\,\langle\, B_\varepsilon(x)\ (x \in X, \varepsilon \in Q_+) \mid\ & B_\varepsilon(x) \leq B_{\varepsilon'}(x)\ (\varepsilon < \varepsilon')\\ & B_{\varepsilon'}(x) \leq \textstyle\bigvee_{\varepsilon<\varepsilon'} B_\varepsilon(x)\\ & B_\delta(x) \leq B_{\delta+\varepsilon}(y) \qquad (d(x,y) < \varepsilon)\\ & B_\delta(x) \wedge B_\varepsilon(y) \leq \textstyle\bigvee\{\mathbf{true}\colon d(x,y) < \delta+\varepsilon\}\\ & \mathbf{true} \leq \textstyle\bigvee_x B_\varepsilon(x) \qquad \rangle\end{aligned}$$

Classically, by Theorem 1.4, its global points are in bijection with those of the completion by Cauchy sequences. Moreover the frame, generated by rational open balls, puts the right topology on those points. This leaves open the issue of whether the locale is spatial, but, leaving that aside, we now have a localic completion of *metric* spaces X.

We shall not exploit this possibility here, but by using the generic metric space in $\mathcal{S}$[MS] we get a locale over [MS] (i.e. a localic geometric morpism to [MS]) that is in effect the generic localic completion.

In the next section, we shall see how to extend the construction to quasimetrics.

4. Enriched categories and flat modules

We follow with a quick exposition of how the module language (of ringoids, i.e. categories enriched over Abelian groups) applies to metric spaces (treated as enriched categories following [11]), and in particular leads to a notion of flat module. The material here applies easily to quasimetric spaces, and we shall therefore express it in that generality.

For the rest of this section, let X be a quasimetric space. For reasons of notational slickness, we shall frequently write X_{xy} for the distance d(x,y) in X.

Definition 4.1

(i) A *right module* over X is a map $M\colon X \to \overleftarrow{[0,\infty]}$ (we shall often write M_x rather than M(x)) such that $M_x+X_{xy} \geq M_y$. These form a locale, Mod-X, in which the specialization order is the pointwise reverse numerical order.

(ii) A *left module* over X is a right module over X^{op}, in other words a map $M\colon X \to \overleftarrow{[0,\infty]}$ such that $X_{xy}+M_y \geq M_x$. We write X-Mod for the locale of left X-modules.

Proposition 4.2 Mod-X is a localic distributive lattice with meet and join given by pointwise numerical max and min.]

Some important examples are the *representable modules:* if $x \in X$, then X_{x-} is the right module given by $(X_{x-})_y = X_{xy}$, and X_{-x} is the left module given by $(X_{-x})_y = X_{yx}$. We thus get two maps, one from X to X-Mod and one from X to Mod-X. One of them (it's a matter of opinion which, but we shall choose X-Mod) should be thought of as the Yoneda embedding.

If M is right module, then so is $\lambda \otimes M$ for any point λ of $\overleftarrow{[0,\infty]}$, defined by $(\lambda \otimes M)_x = \lambda + M_x$. This gives a continuous map

$$\otimes: \overleftarrow{[0,\infty]} \times \text{Mod-X} \to \text{Mod-X}$$

Similarly if M is a left module, then we write $M \otimes \lambda$, giving a map from X-Mod$\times\overleftarrow{[0,\infty]}$ to X-Mod.

Definition 4.3

(i) Let M and N be right and left modules respectively. Then their *tensor product* $M \otimes_X N$ is $\inf_x(M_x+N_x)$, giving a map

$$\otimes_X: \text{Mod-X} \times \text{X-Mod} \to \overleftarrow{[0,\infty]} \ .$$

(ii) A left X-module M is *flat* iff the map $-\otimes_X M$: Mod-X $\to \overleftarrow{[0,\infty]}$ preserves finite meets.

Note that $-\otimes_X M$ preserves the nullary meet iff $0 \otimes_X M = 0$, i.e. $\inf_z M_z = 0$. If X is finitary (no infinite distances), then this condition in itself is enough to show tht M too is finitary: for if we choose z so that $M_z < 1$, then for any x we have $M_x \le X_{xz}+M_z \le X_{xz}+1$, which is finite.

Proposition 4.4 $M \otimes_X X_{-y} = M_y$, and $X_{x-} \otimes_X N = N_x$.

Proof $M \otimes_X X_{-y} = \inf_x (M_x+X_{xy})$. By the module law this is $\ge M_y$, but by choosing $x = y$ we can attain that lower bound.]

From this it is plain that representable modules are flat.

Proposition 4.5 Let M be a left module. Then the following conditions are equivalent.

(i) $-\otimes_X M$ preserves binary meets.

(ii) For all upper reals λ and μ, $\inf_z (\max(\lambda+X_{xz}, \mu+X_{yz}) + M_z) \le \max(\lambda+M_x, \mu+M_y)$.

(iii) For all *rationals* λ and μ, $\inf_z (\max(\lambda+X_{xz}, \mu+X_{yz}) + M_z) \le \max(\lambda+M_x, \mu+M_y)$.

(iv) For all $x, y \in X$ and rationals α and β, if $M_x < \alpha$ and $M_y < \beta$ then there is some z for which $d(x,z) + M_z < \alpha$ and $d(y,z) + M_z < \beta$.

(v) If $m \ge 1$ and for each i, $1 \le i \le m$, we have $x_i \in X$ and $M(x_i) < \alpha_i \in Q_+$, then there is some z for which for all i, $d(x_i,z) + M_z < \alpha_i$.

Proof

(i) $\Rightarrow$ (ii): For any $x, y \in X$, we must have

$$\begin{aligned}(\lambda \otimes X_{x-} \wedge \mu \otimes X_{y-}) \otimes_X M &= \lambda \otimes X_{x-} \otimes_X M \wedge \mu \otimes X_{y-} \otimes_X M \\ &= \max(\lambda+M_x, \mu+M_y).\end{aligned}$$

But the left hand side of this is just $\inf_z (\max(\lambda+X_{xz}, \mu+X_{yz}) + M_z)$.

(ii) $\Rightarrow$ (i): Let N and N' be right X-modules, so we want to show that $(N \wedge N') \otimes_X M = (N \otimes_X M) \wedge (N' \otimes_X M)$, i.e.

$$\inf_z (\max(N_z, N'_z) + M_z) = \max(\inf_z(N_z+M_z), \inf_z(N'_z+M_z))$$

The $\geq$ direction is obvious. For $\leq$, we see that the right hand side is $\inf_{xy}(\max(N_x+M_x, N'_y+M_y))$, so we must show that for every x and y we have

$$\inf_z (\max(N_z, N'_z) + M_z) \leq \max(N_x+M_x, N'_y+M_y)$$

But $\max(N_z, N'_z) + M_z \leq \max(N_x+X_{xz}, N'_y+X_{yz}) + M_z$, so we can apply condition (ii) with $\lambda = N_x$, $\mu = N'_y$.

(ii) $\Leftrightarrow$ (iii): $\Rightarrow$ is a fortiori. For $\Leftarrow$, use the fact that any upper real is the inf of the rationals greater than it.

(iii) $\Rightarrow$ (iv): If $M_x < \alpha$, $M_y < \beta$ then $\max(\beta+M_x, \alpha+M_y) < \alpha+\beta$, so there is some z with $\beta+X_{xz}+M_z < \alpha+\beta$, $\alpha+X_{yz}+M_z < \alpha+\beta$.

(iv) $\Rightarrow$ (iii): If $\max(\lambda+M_x, \mu+M_y) < q$ then for some α, β we have $M_x < \alpha$, $M_y < \beta$, $\lambda+\alpha \leq q$, $\mu+\beta \leq q$. Find z with $X_{xz}+M_z < \alpha$, $X_{yz}+M_z < \beta$; then LHS $\leq \max(\lambda+X_{xz}, \mu+X_{yz})+M_z < \max(\lambda+\alpha,\mu+\beta) \leq q$.

(iv) $\Leftrightarrow$ (v): $\Leftarrow$ is a fortiori. We prove $\Rightarrow$ by induction on m. If m = 1, then we can take $z = x_1$. Suppose $m \geq 2$. By induction we can find z' such that $d(x_i, z') + M_{z'} < \alpha_i$ $(1 \leq i \leq m-1)$. It follows that for each i we can find γ_i and ε_i' such that $d(x_i, z') < \gamma_i$, $M_{z'} < \varepsilon_i'$ and $\gamma_i+\varepsilon_i' \leq \alpha_i$. Let $\varepsilon' = \min_i \varepsilon_i'$. Now by (iv) we can find z such that $d(x_m, z) + M_z < \alpha_m$ and $d(z', z) + M_z < \varepsilon'$. Then for $1 \leq i \leq m-1$ we have $d(x_i, z) + M_z \leq d(x_i, z') + d(z', z) + M_z < d(x_i, z') + \varepsilon' < \gamma_i+\varepsilon_i' \leq \alpha_i$.]

Theorem 4.6 A left module M is flat iff it has $\inf_z M_z = 0$ and satisfies any one of the equivalent conditions (i)-(v) in Proposition 4.5.]

We write X-$_f$Mod and Mod$_f$-X for (respectively) the locales of flat left and right modules of X. However, we shall generally abbreviate X-$_f$Mod to $\overline{X}$.

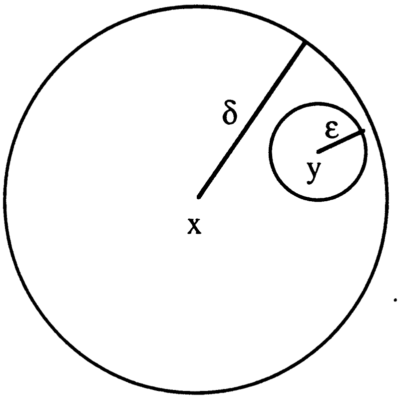
From 4.5 (v) we can deduce that the propositions $M_x < \delta$ form a basis for the topology on $\overline{X}$ (in effect they're the open balls $B_\delta(x)$): for a conjunction $\bigwedge_{1\leq i\leq m} M(x_i) < \alpha_i$ is a disjunction of propositions $M_z < \varepsilon$ such that for each i there is ε_i' with $d(x_i, z) < \varepsilon_i'$ and $\varepsilon_i'+\varepsilon \leq \alpha_i$.

We shall often treat points of $\overline{X}$ straightforwardly as the maps M, but it is also useful to treat them via the relations "$M_x < \delta$". In doing so, we shall use the following language.

Definition 4.7 We introduce the symbol "$B_\delta(x)$", a "formal open ball", as alternative notation for the pair (x,δ) $(x \in X, \delta \in Q_+)$, and write

$$B_\delta(x) < B_\varepsilon(y) \quad \text{iff} \quad d(x,y) + \varepsilon < \delta$$

This formal relation is intended to represent the notion that $\{z: d(y,z) < \varepsilon\}$ is contained in $\{z: d(x,z) < \delta\}$, with a bit to spare:



Note that if $B_\delta(x) < B_\varepsilon(y)$ then $B_\varepsilon(y)$ is the *smaller* ball: this is because $<$ is being used as an information ordering – high in the order means more precise. We shall say that $B_\varepsilon(y)$ *refines* $B_\delta(x)$.

In these terms, 4.5 (v) can be understood as saying that in $\Omega\overline{X}$,

$$\bigwedge_i B_{\alpha_i}(x_i) = \bigvee\{B_\varepsilon(z) \colon \forall i.\ B_{\alpha_i}(x_i) < B_\varepsilon(z)\}$$

Proposition 4.8 A point of $\overline{X}$ can be represented as a subset M of $X \times Q_+$ such that –

(i) $B_\delta(x) \in M \Leftrightarrow B_\varepsilon(y) \in M$ for some $B_\varepsilon(y) > B_\delta(x)$

(ii) If $B_\alpha(x)$ and $B_\beta(y)$ are both in M, then so is some $B_\varepsilon(z)$ with $B_\alpha(x) < B_\varepsilon(z)$ and $B_\beta(y) < B_\varepsilon(z)$.

(iii) For every δ in Q_+ there is some x with $B_\delta(x) \in M$.

Proof

A map from X to $\overleftarrow{[0,\infty]}$ is equivalent to a subset M of $X \times Q_+$ such that

$$B_\delta(x) \in M \Leftrightarrow \exists \varepsilon < \delta.\ B_\varepsilon(x) \in M.$$

This condition is implied by (i), using the fact that

$$B_\varepsilon(y) > B_\delta(x) \Rightarrow \varepsilon < \delta \Rightarrow B_\varepsilon(x) > B_\delta(x),$$

and it also implies the $\Rightarrow$ direction of (i). However, the $\Leftarrow$ direction of (i) is a direct translation of the module law: for $X_{xy}+M_y < \delta$ iff there is some ε for which $B_\delta(x) < B_\varepsilon(y) \in M$. We thus see that left X-modules are equivalent to subsets of $X \times Q_+$ satisfying (i). Then (iii) holds iff $\inf_x M_x = 0$, and (ii) is equivalent to 4.5 (iv).]

This presentation leads to a useful embedding. Recall [17] that a *continuous information system* is a set D (of *tokens*) with a transitive, interpolative order $<$ (so $<\circ< \, = \, <$). Then its rounded ideal completion RIdl(D) is a continuous dcpo. Now the order $<$ just defined on $X \times Q_+$ is transitive and interpolative.

Definition 4.9 The *ball domain* Ball(X) is the rounded ideal completion of $(X \times Q_+, <)$.

A point of this is a subset of $X \times Q_+$ satisfying the conditions of 4.8, except that (iii) is weakened to inhabitedness of M (for *some* δ there is x with $B_\delta(x) \in M$).

Hence $\overline{X}$ is (homeomorphic to) a sublocale of a continuous dcpo Ball(X). This is obviously related to the continuous dcpo of "formal balls" used in [2], though the spatial construction there is constructively inequivalent to ours.

Given the unspatial direction of the ordering on $X \times Q_+$ (big balls are small for <), it is conventionally more natural to think of the rounded ideals as filters of balls, and then the points of $\overline{X}$ are the Cauchy filters, those that contain balls of arbitrarily small radius.

We conclude this section with a result that expresses the usual dense embedding of a metric space in its completion. For constructivist reasons we use the notion of "strong density" that is defined in [8]. From the definition there it is not hard to see that a map f: D → E is strongly dense iff for any proposition $p \in \Omega$, and $a \in \Omega E$, if $\Omega f(a) \leq \Omega!(p)$ then $a \leq \Omega!(p)$. (! denotes a unique locale map to 1. Classically, p is either **true** or **false. true** contributes only trivially to the definition, and **false** gives the definition of density.)

Proposition 4.10 The Yoneda embedding $\mathcal{Y}: X \to \overline{X}$, $\mathcal{Y}(x) = X_{-x}$, is strongly dense.

Proof

Suppose $a \in \Omega\overline{X}$. Without loss of generality we can take a to be a basic open $B_\varepsilon(x)$. $\Omega\mathcal{Y}(B_\varepsilon(x))$ is the set $\{y \in X: X_{xy} < \varepsilon\}$. This contains x, so if $\Omega\mathcal{Y}(B_\varepsilon(x)) \leq \Omega!(p)$ we deduce that $x \in \Omega!(p) = \bigcup\{X: p\}$ and so p (is true). $a \leq \Omega!(p)$ is then immediate.]

5. Examples

5.1 Some dcpos

Example 5.1.1 *Flat quasimetric completion subsumes ideal completion*
Let (P, ≤) be a preorder, and define a quasimetric d on it by

$$d(x, y) = \inf \{0: x \leq y\}$$

Then $\overline{P}$ is homeomorphic to Idl P.
Proof Suppose N is a flat left module over P. The module property tells us that if $x \leq y$ then $N_x \leq N_y$.

We claim that if N_x is finite, then it is 0. For suppose $N_x < \alpha \in Q_+$. If $\varepsilon > 0$, then we can find y such that $N_y < \varepsilon$, and then z such that $d(x,z)+N_z < \alpha$, $d(y,z)+N_z < \varepsilon$. Since $d(x,z) < \alpha$, it follows that $\alpha \in \bigcup\{Q_+: x \leq z\}$, so $x \leq z$, so $N_x \leq N_z < \varepsilon$. Hence $N_x = 0$.

Now let $I = \{x \in P: N_x < 1\}$. It is easy to see that this is an ideal, and what we have just shown proves that $N_x = \inf\{0: x \in I\}$. Conversely, if I is any ideal, then this definition of N gives a flat left module for which $x \in I$ iff $N_x < 1$, and putting this together gives the bijection between ideals and flat left modules.]

Example 5.1.2 Let the rationals Q be equipped with a quasimetric $d(x,y) = x \dot{-} y = \max(0, x-y)$ (truncated minus). Then its left flat completion is homeomorphic to the rounded ideal completion of Q, which we may write as $\overrightarrow{(-\infty,+\infty]}$.
Proof Let N be a flat left module over (Q, d). From the module property, we get that if $x \leq y$ then

$$N_x \leq N_y \leq N_x + (y \dot{-} x)$$

We claim that if $N_x < \alpha$, then $N_{x-\alpha} = 0$. For if $\varepsilon > 0$, then we can find y with $N_y < \varepsilon$, and then z with $(x \dot{-} z) + N_z < \alpha$ and $(y \dot{-} z) + N_z < \varepsilon$. Hence $x-z < \alpha$, so $x-\alpha < z$ and $N_{x-\alpha} \leq N_z < \varepsilon$.

Now define $I = \{x \in \mathbb{Q}: \exists x' \in \mathbb{Q}.\ x < x' \text{ and } N_{x'} = 0\}$. Clearly I is rounded lower; it is also inhabited, for if we choose x with $N_x < 1$, then $N_{x-1} = 0$ and $x-2 \in I$. Hence I is a rounded ideal.

Furthermore, for all x we have $N_x = \inf \{x \dot{-} y: y \in I\}$. For $\leq$, if $y \in I$ then $N_x \leq (x \dot{-} y) + N_y = x \dot{-} y$. For $\geq$, suppose $N_x < \alpha$. Then $N_x < \alpha' < \alpha$ for some α', $N_{x-\alpha'} = 0$, and $x-\alpha \in I$; $x \dot{-} (x-\alpha) = \alpha$.

Conversely, if I is any rounded ideal, then defining $N_x = \inf \{x \dot{-} y: y \in I\}$ gives a flat left module. The only part of any intricacy here is when $N_x < \alpha$ and $N_y < \beta$. We can then find z_1 and z_2 in I such that $x \dot{-} z_1 < \alpha$ and $y \dot{-} z_2 < \beta$. Taking z to be $\max(z_1, z_2)$, we find $(x \dot{-} z) + N_z < \alpha$ and $(y \dot{-} z) + N_z < \beta$. Furthermore, we have $x \in I$ iff there is some $x' > x$ with $N_{x'} = 0$. For $\Rightarrow$, if $x \in I$ then $x < x' \in I$ for some x', and $N_{x'} = 0$. For $\Leftarrow$, since $x'-x > 0$, we can find $z \in I$ with $x' \dot{-} z \leq x'-x$, so $x \leq z$ and $x \in I$.

Putting all these together we get that flat left modules are equivalent to rounded ideals.]

5.2 Completion of metric spaces

For this section, we take X to be a metric space. In this symmetric case, we can simplify the characterization of flatness somewhat. This will complete the connection with Theorem 1.4, for condition (ii) in 5.2.1 corresponds to condition (ii) in Proposition 1.1. In other words, in the metric case, the flat completion of Section 4 is the same as the completion mentioned at the end of Section 3.

Theorem 5.2.1 Let M be a module over X for which $\inf_x M_x = 0$. Then the following are equivalent:

(i) M is flat.

(ii) $\forall x,y{:}X.\ X_{xy} \leq M_x+M_y$

(iii) $\forall x,y{:}X.\ \inf_z (\max(X_{xz}, X_{yz}) + M_z) \leq \max(M_x, M_y)$

Proof

(i)$\Rightarrow$(iii): (iii) states that $-\otimes_X M$ preserves binary meets of *representable* modules. Note that the Theorem is showing that this is sufficient for flatness in the metric case; in the quasimetric generality of 4.5 (ii) we needed modules of the form $\lambda \otimes X_{x-}$.

(iii)$\Rightarrow$(ii): This is the hard part. We shall use a lemma; for it we shall presume (iii) as hypothesis.

Lemma 5.2.2 Suppose $M_y < \delta$. Then for all x we have $X_{xy} \leq M_x+M_y+2\delta$.

Proof

We first prove by induction on n the following:

$$\forall n.\ \forall x.\ \forall q.\ (M_x < q \wedge (3/4)^n q \leq 2\delta \rightarrow X_{xy} \leq M_x+M_y+2\delta)$$

Suppose we have such n, x and q. Suppose in addition that $M_x < q'$ and $M_y < \delta'$; we must show that $X_{xy} < q'+\delta'+2\delta$. Without loss of generality we can replace q' by min(q',q) (so that $(3/4)^n q' \leq 2\delta$) and δ' by $\min(\delta', \delta)$, and thereby take it that $q' \leq q$ and $\delta' \leq \delta$. Since $\max(M_x, M_y) < \max(q', \delta')$, we can find z such that $X_{xz}+M_z$ and $X_{yz}+M_z$ are both $< \max(q', \delta')$.

If $q' \leq 2\delta$, then $X_{xy} \leq X_{zx}+X_{zy} < 2 \max(q', \delta') \leq q'+\delta'+2\delta$ as required.

Otherwise, $q' > 2\delta$. Then $\max(q', \delta') = q'$. $M_z \leq X_{zy}+M_y$, so

$$2 \cdot M_z \leq M_y+M_z+X_{yz} < M_y+q' < \delta+q' < 3q'/2$$

and so $M_z < 3q'/4$. Hence by induction, $X_{zy} \leq M_z+M_y+2\delta$, so

$$X_{xy} \leq X_{zx}+X_{zy} \leq X_{zx}+M_z+M_y+2\delta < q'+\delta'+2\delta$$

Returning now to the main statement of the Lemma, suppose without n and q that we have $M_x <$ q' and $M_y < \delta'$. For some n we have $(3/4)^n q' \leq 2\delta$, so by our inductive result we have $X_{xy} \leq M_x+M_y+2\delta$. This completes the proof of Lemma 5.2.2.]

Returning now to 5.2.1, let x and y be arbitrary and suppose $M_x < q$ and $M_y < r$. If $\delta > 0$, choose z such that $M_z < \min(\delta,q,r)$. By the lemma we have $X_{xz} \leq M_x+M_z+2\delta < q+3\delta$, $X_{yz} < r+3\delta$, so it follows that $X_{xy} < q+r+6\delta$. Since δ was arbitrary, $X_{xy} \leq q+r$, so $X_{xy} \leq M_x+M_y$.
(ii)⇒(i): $\lambda+X_{xz}+M_z \leq \lambda+M_x+2M_z$, and $\mu+X_{yz}+M_z \leq \mu+M_y+2M_z$. Hence

$$\inf_z(\max(\lambda+X_{xz}, \mu+X_{yz})+M_z) \leq \inf_z (\max(\lambda+M_x, \mu+M_y) + 2M_z) = \max(\lambda+M_x, \mu+M_y)$$

]

A notable fact about completion in the metric case is that we can put a metric on $\overline{X}$ in a simple way.

Proposition 5.2.3 Let X be a metric space. By symmetry Mod-X and X-Mod are homeomorphic, so the tensor product ⊗: Mod-X × X-Mod → $\overleftarrow{[0,\infty]}$ restricted to the flat modules gives a map d = ⊗: $\overline{X} \times \overline{X} \to \overleftarrow{[0,\infty]}$, $d(M,N) = \inf_x (M(x)+N(x))$.

(i) d satisfies the axioms for a metric.
(ii) The Yoneda map $\mathcal{Y}: X \to \overline{X}$ is an isometry.

Proof
(i) Symmetry is obvious, and $d(M,M) = \inf_x (M(x)+M(x)) = 2 \inf_x M(x) = 0$. For the triangle inequality, suppose $d(L,M) < \alpha$ and $d(M,N) < \beta$, so we want to prove $d(L,N) < \alpha+\beta$. We can find x, α_1 and α_2 such that $L(x) < \alpha_1$, $M(x) < \alpha_2$ and $\alpha_1+\alpha_2 < \alpha$, and y, β_1 and β_2 such that $M(y) < \beta_1$, $N(y) < \beta_2$ and $\beta_1+\beta_2 < \beta$. Choose z such that $d(x,z)+M(z) < \alpha_2$ and $d(y,z)+M(z) < \beta_1$. Then $L(z)+N(z) \leq L(x)+d(x,z)+d(y,z)+N(y) < \alpha_1+\alpha_2+\beta_1+\beta_2 < \alpha+\beta$.
(ii) $d(X_{-x},X_{-y}) = X_{xy}$ (by Proposition 4.4) $= d(x,y)$.]

Next, we show that, at least in the metric case, there's a sense in which our completion really does complete.

Proposition 5.2.4 Let X be a metric space, and let X' be the set of points of $\overline{X}$ (the construction of X' is not geometric, but it is intuitionistically constructive), equipped with the metric arising from 5.2.3. Then $\overline{X'}$ is homeomorphic to $\overline{X}$.

Proof
Let K be a flat module over X'. We show that for every M in X', $K(M) = \inf_x (K(X_{-x})+M(x))$. ≤ comes from the module property of K, for $M(x) = d(M, X_{-x})$. For ≥, suppose $K(M) < \varepsilon$ We can find $\varepsilon' < \varepsilon$ such that $K(M) < \varepsilon'$, and x such that $M(x) < (\varepsilon-\varepsilon')/2$; then $K(X_{-x})+M(x) \leq K(M)+2M(x) < \varepsilon$. It follows that a flat module over X' is determined by its restriction to X.]

5.3. Dedekind sections

In this section we present the relationship between two different completions of the rationals: by Dedekind sections, and by flat modules as in Section 4. They are in fact equivalent, though the proof is surprisingly intricate. This arises from the fact that the two constructions are quite different. A Dedekind section relies heavily on the ordering of $\mathbb{Q}$, describing which rationals are bigger than a given real, and which are smaller. The modules, on the other hand rely entirely on undirected distances.

Definition 5.3.1 A *Dedekind section* is a disjoint pair (L,R) of inhabited subsets of the rationals $\mathbb{Q}$ such that L is rounded lower, R is rounded upper, and if $q < r$ then either $q \in L$ or $r \in R$.

This is the usual definition of the reals, as in [6]. However, there the following lemma is incorporated into the definition.

Lemma 5.3.2 If (L, R) is a Dedekind section and $0 < \varepsilon \in \mathbb{Q}$, then we can find $q \in L$, $r \in R$ such that $r-q < \varepsilon$.

Proof

We can take $q_0 \in L$ and $r_0 \in R$ (for both are inhabited), and then use induction on natural numbers n for which $r_0-q_0 < \varepsilon.2^n$. For the base case, we can take $q = q_0$, $r = r_0$. For the induction step, let $s_i = q_0 + \frac{(r_0-q_0)i}{4}$ $(0 \le i \le 4)$. We have $s_1 \in L$ or $s_2 \in R$, and $s_2 \in L$ or $s_3 \in R$. Then if $s_2 \in R$ we can define $q_1 = q_0$, $r_1 = s_2$; if $s_2 \in L$, define $q_1 = s_2$, $r_1 = r_0$; and if $s_1 \in L$, $s_3 \in R$, define $q_1 = s_1$, $r_1 = s_3$. Then $q_1 \in L$, $r_1 \in R$, $r_1-q_1 = \frac{r_0-q_0}{2} < \varepsilon.2^{n-1}$ and the result follows by induction.]

Theorem 5.3.3 $\mathbb{R}$, the locale of Dedekind sections of $\mathbb{Q}$, is homeomorphic to the completion of $\mathbb{Q}$ as (finitary) metric space.

Proof

If (L, R) is a Dedekind section, then we can define a map $M: \mathbb{Q} \to \overleftarrow{[0,\infty]}$ (which in fact is always finite) by

$$M(x) = \inf\{\varepsilon\colon x+\varepsilon \in R \text{ and } x-\varepsilon \in L\}$$

To show $\inf_x M(x) = 0$, suppose $\varepsilon > 0$ and choose $q \in L$, $r \in R$ such that $r-q < 2\varepsilon$. Then $M((q+r)/2) < \varepsilon$. To show $M(x) \le M(y)+d(x, y)$, suppose $M(y) < \varepsilon$. Without loss of generality (since the order on $\mathbb{Q}$ is decidable), we can take $x \ge y$. Then

$$x-\varepsilon-|x-y| = x-\varepsilon-x+y = y-\varepsilon \in L$$
$$x+\varepsilon+|x-y| \ge y+\varepsilon \in R$$

so $M(x) \le \varepsilon+d(x,y)$. To show $d(x,y) \le M(x)+M(y)$ (as in Theorem 5.2.1), suppose $M(x) < \delta$ and $M(y) < \varepsilon$. Again, without loss of generality $x \ge y$. $x-\delta \in L$ and $y+\varepsilon \in R$, so $x-\delta < y+\varepsilon$, and $d(x,y) = x-y < \delta+\varepsilon$.

Now suppose M is an arbitrary map satisfying the properties. We define L and R by

$$r \in R \quad \text{iff} \quad \exists r' < r.\ M(r') < r-r'$$
$$q \in L \quad \text{iff} \quad \exists q' > q.\ M(q') < q'-q$$

The idea here is that r is bigger than a real ξ iff some rational r' smaller than r is closer to ξ than to r. (Obviously this is impossible if $r \le \xi$.)

We must show that these give a Dedekind section. Choose s such that $M(s) < 1$; then $s+1 \in R$. Hence R is inhabited. It is clearly upper. To show that it is rounded, suppose $r \in R$ with corresponding r'. Then $M(r') < r-r'-\varepsilon$ for some $\varepsilon > 0$. $0 < r-r'-\varepsilon$, so $r' < r-\varepsilon$ and it follows that $r-\varepsilon \in R$. Similarly, L is inhabited rounded lower.

To show L and R are disjoint, suppose $q \in L \cap R$ with $r' < q < q'$, $M(r') < q-r'$ and $M(q') < q'-q$. Then $q'-r' < q-r'+q'-q = q'-r'$, a contradiction.

Now suppose q and r are any rationals with $q < r$. Let $\varepsilon = (r-q)/3$, and find s such that $M(s) < \varepsilon$. We have $q+\varepsilon < r-\varepsilon$, so either $s < r-\varepsilon$ or $s > q+\varepsilon$. Suppose the former. $M(s) < \varepsilon$, so $r > s+\varepsilon \in R$. Similarly, if $s > q+\varepsilon$ then $q \in L$.

We have now shown that (L, R) is a Dedekind section.

Lemma 5.3.4 $M(x) < \varepsilon \Leftrightarrow x+\varepsilon \in R$ and $x-\varepsilon \in L$.

Proof

$\Rightarrow$: We can find $\varepsilon' < \varepsilon$ such that $M(x) < \varepsilon'$. $x-\varepsilon < x-\varepsilon'$, so either $x-\varepsilon \in L$ or $x-\varepsilon' \in R$. In the latter case, we have $r' < x-\varepsilon'$ with $M(r') < x-\varepsilon'-r'$. Then $x-r' \le M(x)+M(r') < \varepsilon'+x-\varepsilon'-r'$, a contradiction. It follows that in either case $x-\varepsilon \in L$, and similarly $x+\varepsilon \in R$.

$\Leftarrow$: We first prove a sublemma:

Lemma 5.3.5 If $q \in L$, $r \in R$, $r-q < \delta$ and $q \le x \le r$, then $M(x) < \delta$.

Proof We can find $q < q' \in L$ and $r > r' \in R$. Let $\varepsilon = \min(q'-q, r-r')/2$, and choose s such that $M(s) < \varepsilon$. By the first part of 5.3.4 we have $s-\varepsilon \in L$ and $s+\varepsilon \in R$, so $q' < s+\varepsilon$ and $s-\varepsilon < r'$. Hence

$$q \le q'-2\varepsilon < s-\varepsilon < s < s+\varepsilon < r'+2\varepsilon \le r$$

If $s \ge x$ then $s-x \le s-q = (r-q)-(r-s) < \delta-\varepsilon$, and so $M(x) < \varepsilon+(\delta-\varepsilon) = \delta$, and similarly if $s \le x$. This completes the proof of Lemma 5.3.5.]

Returning to 5.3.4, we can find $\varepsilon' < \varepsilon$ such that $x-\varepsilon' \in L$ and $x+\varepsilon' \in R$. Consider the elements $x-\varepsilon'/2 < x < x+\varepsilon'/2$. We have $x-\varepsilon'/2 \in L$ or $x \in R$, and $x \in L$ or $x+\varepsilon'/2 \in R$. In every case we can find $q \in L$ and $r \in R$ such that $q \le x \le r$ and $r-q = \varepsilon' < \varepsilon$, so by 5.3.5 we have $M(x) < \varepsilon$. This completes the proof of Lemma 5.3.4.]

We now return to the main theorem. Lemma 5.3.4 has shown us that starting from M, defining a Dedekind section, and then defining a new M from that, we have actually recovered the original one. It remains to show that if we start from a Dedekind section (L, R), define M, and then define the corresponding Dedekind section, we again have the original one. In other words,

$$r \in R \Leftrightarrow \exists r'<r.\ (r'+(r-r') \in R \wedge r'-(r-r') \in L)$$
$$q \in L \Leftrightarrow \exists q'>q.\ (q'+(q'-q) \in R \wedge q'-(q'-q) \in L)$$

In both parts, $\Leftarrow$ follows a fortiori. For $\Rightarrow$, let us consider the first part, concerning R (the second part is similar). We can find an element r' of L with $r' < r$. Then $r'+(r-r') = r \in R$ and $r'-(r-r') < r' \in L$.]

6. Powerlocales

We turn now to the powerlocale constructions, lower (P_L), upper (P_U) and Vietoris (V), and show that the class of quasimetric completions is closed under all three. A summary of the constructive theory can be found in [22]. Quite apart from any intrinsic interest, they are also crucial to the "topology-free space" approach to locale theory, for they can be used in characterizing certain important properties. For instance [19] a locale D is open iff P_LD has a top point and compact iff P_UD has a bottom point, and we exploit this in 6.1.4 and 6.2.4 (which leads to another proof of the localic Heine-Borel theorem).

Recall that ΩP_LD and ΩP_UD are the frames generated freely over ΩD qua suplattice and qua preframe respectively, with generators written as $\Diamond a$ and $\Box a$ ($a \in \Omega D$). VD is the sublocale of $P_LD \times P_UD$ presented by relations $\Diamond a \wedge \Box b \le \Diamond(a \wedge b)$ and $\Box(a \vee b) \le \Box a \vee \Diamond b$.

In the case of algebraic (or even continuous) dcpos, these three powerlocale constructions on the ideal (or rounded ideal) completions can be constructed by defining preorders on the finite power sets of the informations systems [17]. If D is an information system, as defined just before Definition 4.9, and RIdl(D) its localic rounded ideal completion, then the powerlocales P_LRIdl(D), P_URIdl(D) and V RIdl(D) are also continuous dcpos, with tokens all taken from the finite powerset $\mathcal{F}$D, but with three different orders: respectively,

- the *lower* order $S <_L T$ iff $\forall s \in S.\ \exists t \in T.\ s < t$
- the *upper* order $S <_U T$ iff $\forall t \in T.\ \exists s \in S.\ s < t$
- the *convex* order $S <_C T$ iff $S <_L T$ and $S <_U T$

We show that a parallel idea works for quasimetric completions, defining quasimetrics on the finite powerset of a quasimetric space. It is clearly reminiscent of the Vietoris metric and its quasimetric parts, but the information system flavour shows up in the fact that we define it only on finite subsets.

The ball domain (Definition 4.9) turns out to be technically useful in relating the quasimetric powerlocale constructions to continuous dcpos.

6.1 The lower powerlocale, P_L

Definition 6.1.1 Let X be a quasimetric space. We define its *lower powerspace,* $\mathcal{F}_L$X, by taking the elements to be the finite subsets of X, with distance $d_L(S,T) = \max_{x \in S} \min_{y \in T} d(x,y)$.

Theorem 6.1.2 $\overline{\mathcal{F}_L X}$ is homeomorphic to the lower powerlocale $P_L\overline{X}$.

Proof

We work by embedding $\overline{\mathcal{F}_L X}$ and $P_L\overline{X}$ in two continuous dcpos, namely Ball($\mathcal{F}_L$X) and P_LBall(X) respectively.

A point of Ball($\mathcal{F}_L$X) is a rounded ideal of $\mathcal{F}X \times Q_+$, while a point of P_LBall(X) is a rounded ideal of $\mathcal{F}(X \times Q_+)$. Let us define ϕ: $\mathcal{F}X \times Q_+ \to \mathcal{F}(X \times Q_+)$ by $\phi(B_\delta(S)) = \{B_\delta(s) : s \in S\}$. We have

$$\begin{aligned} B_\delta(S) < B_\varepsilon(T) &\Leftrightarrow d_L(S,T)+\varepsilon < \delta \\ &\Leftrightarrow \forall s \in S.\ \exists t \in T.\ d(s,t)+\varepsilon < \delta \\ &\Leftrightarrow \forall s \in S.\ \exists t \in T.\ B_\delta(s) < B_\varepsilon(t) \\ &\Leftrightarrow \phi(B_\delta(S)) <_L \phi(B_\varepsilon(T)). \end{aligned}$$

It follows that we get a continuous map ϕ': $\mathrm{Ball}(\mathcal{F}_L X) \to P_L\mathrm{Ball}(X)$, mapping I to $\downarrow\{\phi(B_\delta(S))$: $B_\delta(S) \in I\}$. ϕ' is not itself a homeomorphism, but we show it restricts to a homeomorphism between $\overline{\mathcal{F}_L X}$ and $P_L\overline{X}$.

We must identify the points of $P_L\mathrm{Ball}(X)$ that lie in $P_L\overline{X}$.

Lemma 6.1.3 Let J be a point of $P_L\mathrm{Ball}(X)$. Then the following are equivalent:

(i) J is in $P_L\overline{X}$.

(ii) Singletons in J have arbitrarily small singleton refinements: in other words, if $\alpha > 0$ and $\{B_\delta(x)\} \in J$, then there is some $\{B_\varepsilon(y)\} \in J$ with $\varepsilon < \alpha$ and $B_\delta(x) < B_\varepsilon(y)$.

(iii) if $\alpha > 0$ and $U \in J$, then there is some $B_\varepsilon(T)$ with $\varepsilon < \alpha$ and $U <_L \phi(B_\varepsilon(T)) \in J$.

Proof

Let $D \to E$ be an arbitrary locale embedding, with ΩD presented over ΩE by relations $a \le b$ for $(a,b) \in R \subseteq \Omega E \times \Omega E$. By a routine application of the coverage theorem (see [22]), we have

$$\Omega D \cong \mathrm{SupLat}\,\langle \Omega E \text{ (qua SupLat)} \mid a \wedge c \le b \wedge c \ ((a,b) \in R,\ c \in \Omega E)\rangle$$

and it follows that

$$\begin{aligned}\Omega P_L D &= \mathrm{Fr}\,\langle \Omega D \text{ (qua SupLat)}\rangle\\ &\cong \mathrm{Fr}\,\langle \Omega E \text{ (qua SupLat)} \mid a\wedge c \le b \wedge c\ ((a,b) \in R,\ c \in \Omega E)\rangle\\ &\cong \mathrm{Fr}\,\langle \Omega P_L E \text{ (qua Fr)} \mid \Diamond(a \wedge c) \le \Diamond(b\wedge c)\ ((a,b) \in R,\ c \in \Omega E)\rangle\end{aligned}$$

In our present case we have $D = \overline{X}$, $E = \mathrm{Ball}(X)$, with $\Omega\overline{X}$ presented over $\Omega\mathrm{Ball}(X)$ by relations $\mathbf{true} \le \bigvee_y B_\alpha(y)$ $(\alpha > 0)$. Hence, using the fact that the $B_\delta(x)$'s are a base for Ball(X), we find that $\Omega P_L\overline{X}$ is presented over $\Omega P_L\mathrm{Ball}(X)$ by relations

$$\begin{aligned}\Diamond B_\delta(x) &\le \bigvee_y \Diamond(B_\delta(x) \wedge B_\alpha(y))\\ &= \bigvee\{\Diamond B_{\varepsilon'}(y'):\ B_\delta(x) < B_{\varepsilon'}(y') \text{ and } \varepsilon' < \alpha\}\end{aligned}$$

Equivalence of (i) with (ii) now follows, because [17] J satisfies $\Diamond B_\delta(x)$ iff $\{B_\delta(x)\} \in J$.

(iii) $\Rightarrow$ (ii) follows a fortiori. For the converse, let J satisfy (ii), and let $U \in J$, $\alpha > 0$. We can find $U <_L U' \in J$, and by pressing the finitely many strict inequalities involved we can find $\eta > 0$ such that

$$\forall B_\delta(x) \in U.\ \exists B_\varepsilon(y) \in U'.\ B_\delta(x) < B_{\varepsilon+\eta}(y)$$

In addition, we can require $\eta < \alpha$. For each $B_\varepsilon(y) \in U'$ we have $\{B_\varepsilon(y)\} \in J$, and so we can find $\{B_{\eta'}(z')\} \in J$ with $\eta' < \eta$ and $B_\varepsilon(y) < B_{\eta'}(z')$. We can therefore find a finite set $U'' \in J$ with $U' <_L U''$ and for every $B_{\eta'}(z')$ in U'', $\eta' < \eta$. Let T be $\{z': \exists\eta'.\ B_{\eta'}(z') \in U''\}$. $\phi(B_\eta(T)) <_L U''$, so $\phi(B_\eta(T))$ is in J. Given $B_\delta(x)$ in U, find $B_\varepsilon(y) \in U'$ with $B_\delta(x) < B_{\varepsilon+\eta}(y)$ and $B_{\eta'}(z') \in U''$ with $B_\varepsilon(y) < B_{\eta'}(z')$. Then $B_{\varepsilon+\eta}(y) < B_{\eta'+\eta}(z') < B_\eta(z')$ and so $U <_L \phi(B_\eta(T))$.]

Returning to the proof of 6.1.2, suppose we are given I in $\mathrm{Ball}(\mathcal{F}_L X)$. We see that if I is in $\overline{\mathcal{F}_L X}$ (i.e. I has balls of arbitrarily small radius) then $\phi'(I)$ satisfies (iii) in the lemma, so ϕ' restricts to a map from $\overline{\mathcal{F}_L X}$ to $P_L\overline{X}$. Its inverse is given by $J \mapsto \phi^{-1}(J)$. Note that $J = \phi'(\phi^{-1}(J))$ follows from 6.1.3 (iii).]

Recall [10] that a locale D is *open* iff the unique map $!: D \to 1$ is an open map of locales. By [19] this holds iff $P_L D$ has a top point.

Corollary 6.1.4 If X is a quasimetric space then $\overline{X}$ is open (i.e. as a locale).
Proof $\mathcal{F}X \times Q_+$ is a point of $\overline{\mathcal{F}_L X}$, and hence must be the top point. It follows that $P_L\overline{X}$ has a top point and so $\overline{X}$ is open.]

The lower powerlocale $P_L D$ always has a bottom point, corresponding to the empty sublocale of D. For many purposes it is desirable to exclude this and work with the open sublocale $P_L{}^+D$ (or $\Diamond$**true**). We show that this corresponds to excluding the empty set from our finite powerspace (recall that for finite sets, emptiness is decidable). Notice that including the empty set had inevitably taken us beyond finitary metrics, for $d_L(T,\emptyset) = \infty$ if $T \neq \emptyset$. ($d_L(\emptyset,T) = 0$ always.)

Proposition 6.1.5 The homeomorphism of Theorem 6.1.2 restricts to a homeomorphism between $P_L{}^+\overline{X}$ and $\overline{\mathcal{F}_L{}^+X}$, where the elements of the space $\mathcal{F}_L{}^+X$ are the finite non-empty subsets of X, and its metric d_L is as before.
Proof
Suppose $I \subseteq \mathcal{F}X \times Q_+$ is a point of $\overline{\mathcal{F}_L X}$. For every ε we have $B_\varepsilon(\emptyset) < B_{\varepsilon/2}(T) \in I$ for some T. But $\{B_\varepsilon(\emptyset): \varepsilon \in Q_+\}$ is already a point of $\overline{\mathcal{F}_L X}$ and hence must be the bottom point, so $P_L{}^+\overline{X}$ corresponds to those I that contain some $B_\varepsilon(T)$ with $T \neq \emptyset$. But once we have some such $B_\varepsilon(T)$ then we have arbitrarily small ones, for an upper bound $B_\alpha(S)$ of $B_\varepsilon(T)$ and $B_\delta(\emptyset)$ must have $S \neq \emptyset$ and $\alpha < \delta$. The result now follows.]

6.2 The upper powerlocale, P_U

Definition 6.2.1 Let X be a quasimetric space. We define its *upper powerspace,* $\mathcal{F}_U X$, by taking the elements to be the finite subsets of X, with distance $d_U(S,T) = \max_{y \in T} \min_{x \in S} d(x,y)$.

Theorem 6.2.2 $\overline{\mathcal{F}_U X}$ is homeomorphic to the upper powerlocale $P_U\overline{X}$.
Proof
The proof, somewhat similar to that of 6.1.2, works by embedding $\overline{\mathcal{F}_U X}$ in $\mathrm{Ball}(\mathcal{F}_U X)$ and $P_U\overline{X}$ in $P_U\mathrm{Ball}(X)$. The same function $\phi: \mathcal{F}X \times Q_+ \to \mathcal{F}(X \times Q_+)$ preserves and reflects order: $B_\delta(S) < B_\varepsilon(T)$ (but this time with respect to the upper quasimetric – $d_U(S,T)+\varepsilon < \delta$) iff $\phi(B_\delta(S)) <_U \phi(B_\varepsilon(T))$. It follows that we get a continuous map ϕ': $\mathrm{Ball}(\mathcal{F}_U X) \to P_U\mathrm{Ball}(X)$, which we show restricts to a homeomorphism between $\overline{\mathcal{F}_U X}$ and $P_U\overline{X}$. Again, the bulk of the work lies in identifying the points of $P_U\mathrm{Ball}(X)$ that lie in $P_U\overline{X}$.

Lemma 6.2.3 Let J be a point of $P_U\mathrm{Ball}(X)$. Then the following are equivalent:

(i) J is in $P_U\overline{X}$.
(ii) J contains elements $\phi(B_\varepsilon(T))$ for arbitrarily small ε.
(iii) if $\alpha > 0$ and $U \in J$, then there is some $B_\varepsilon(T)$ with $\varepsilon < \alpha$ and $U <_U \phi(B_\varepsilon(T)) \in J$.

Proof
Let $D \to E$ be an arbitrary locale embedding, with ΩD presented over ΩE by relations $a \leq b$ for $(a,b) \in R \subseteq \Omega E \times \Omega E$. By a routine application of the preframe coverage theorem [9], we have

$$\Omega D \cong \mathrm{PreFr}\,\langle \Omega E \text{ (qua PreFr)} \mid a \vee c \leq b \vee c \ ((a,b) \in R,\ c \in \Omega E) \rangle$$

and it follows that

$$\begin{aligned}\Omega P_U D &= \mathrm{Fr}\,\langle \Omega D \text{ (qua PreFr)} \rangle\\ &\cong \mathrm{Fr}\,\langle \Omega E \text{ (qua PreFr)} \mid a \vee c \le b \vee c\ ((a,b) \in R,\ c \in \Omega E) \rangle\\ &\cong \mathrm{Fr}\,\langle \Omega P_U E \text{ (qua Fr)} \mid \Box(a \vee c) \le \Box(b \vee c)\ ((a,b) \in R,\ c \in \Omega E) \rangle\end{aligned}$$

In our present case we have $D = \overline{X}$, $E = \mathrm{Ball}(X)$, with $\Omega\overline{X}$ presented over $\Omega\mathrm{Ball}(X)$ by relations $\mathbf{true} \le \bigvee_y B_\varepsilon(y)$ $(\varepsilon > 0)$. The c's appearing above make no difference ($\mathbf{true} \vee c = \mathbf{true}$), so we find that $\Omega P_U\overline{X}$ is presented over $\Omega P_U\mathrm{Ball}(X)$ by relations

$$\mathbf{true} \le \Box\bigvee_y B_\varepsilon(y) = \bigvee^{\uparrow}\{\Box\bigvee_{y\in T} B_\varepsilon(y) : T \subseteq_{\mathrm{fin}} X\}$$

Equivalence of (i) with (ii) now follows, because J satisfies $\Box\bigvee_{y\in T} B_\varepsilon(y)$ iff $\phi(B_\varepsilon(T)) \in J$.

(iii) $\Rightarrow$ (ii) follows easily because J is inhabited (so we can find a U in it). For the converse, let J satisfy (ii), and let $U \in J$, $\alpha > 0$. We can find $U <_U U' \in J$, and by pressing the finitely many strict inequalities involved we can find $\varepsilon > 0$ such that

$$\forall B_\gamma(y) \in U'.\ \exists B_\beta(x) \in U.\ B_\beta(x) < B_{\gamma+\varepsilon}(y)$$

In addition, we can require $\varepsilon < \alpha$. Choose S such that $\phi(B_\varepsilon(S)) \in J$, let V' be a common refinement of U' and $\phi(B_\varepsilon(S))$ in J, and let $T = \{z : \exists\delta.\ B_\delta(z) \in V'\}$. If $B_\delta(z) \in V'$ then $\delta < \varepsilon$, and it follows that $\phi(B_\varepsilon(T)) \in J$. If $B_\delta(z) \in V'$ then we can find $B_\gamma(y) \in U'$ and $B_\beta(x) \in U$ with $B_\gamma(y) < B_\delta(z)$ and $B_\beta(x) < B_{\gamma+\varepsilon}(y)$, and so $B_\beta(x) < B_{\delta+\varepsilon}(z) < B_\varepsilon(z)$. It follows that $U <_U \phi(B_\varepsilon(T))$.]

Returning to 6.2.2, suppose we are given I in $\mathrm{Ball}(\mathcal{F}_U X)$. If I is in $\overline{\mathcal{F}_U X}$ (i.e. I has balls of arbitrarily small radius) then $\phi'(I)$ satisfies (ii) in the lemma, so ϕ' restricts to a map from $\overline{\mathcal{F}_U X}$ to $P_U\overline{X}$. Its inverse is given by $J \mapsto \phi^{-1}(J)$.]

Corollary 6.2.4 Let X be a quasimetric space. Then $\overline{X}$ is compact iff X is totally bounded, i.e. for every $\varepsilon > 0$ there is some finite ε-cover, i.e. some $S \subseteq_{\mathrm{fin}} X$ such that for every x in X there is an s in S with $d(s,x) < \varepsilon$.

Proof By [19], a locale is compact iff its upper powerlocale has a least point, so by 6.2.2 we see that $\overline{X}$ is compact iff $\overline{\mathcal{F}_U X}$ has a least point.

$\Rightarrow$: Let K be the least point of $\overline{\mathcal{F}_U X}$. If $\varepsilon > 0$, we can find S such that $B_\varepsilon(S) \in K$. If $x \in X$, then we have a point of $\overline{\mathcal{F}_U X}$ comprising those $B_\delta(U)$ for which there is some u in U with $d(u,x) < \delta$. Since K is the least point, it follows that $B_\varepsilon(S)$ is in this other point and so for some s in S, $d(s,x) < \varepsilon$. Hence X is totally bounded.

$\Leftarrow$: We define K to contain $B_\varepsilon(S)$ iff for some $\varepsilon' < \varepsilon$, S is an ε'-cover. To show that this is a point of $\overline{\mathcal{F}_U X}$, most of the parts are easy. If S_i is an α_i'-cover, $\alpha_i' < \alpha_i$ $(i = 1,2)$, let T be an $\varepsilon/2$-cover where $\alpha_i'+\varepsilon < \alpha_i$. $d(S_i,T)+\varepsilon < \alpha_i$, and $B_\varepsilon(T) \in K$.

Now let M be another point of $\overline{\mathcal{F}_U X}$ – we must show $K \sqsubseteq M$, so if S is an ε'-cover (so for all T, $d_U(S,T) < \varepsilon'$) with $\varepsilon' < \varepsilon$, we want $B_\varepsilon(S) \in M$. Let $\delta = \varepsilon-\varepsilon'$, and choose T such that $B_\delta(T) \in M$. Then $B_\varepsilon(S) \in M$. It follows that K is the least point.]

As a corollary, we get a new constructive proof of the localic Heine-Borel theorem [3]. Incidentally, this is an illustration of the disadvantages of trying to use point-set topology constructively. In any geometric universe we can construct (non-geometrically) a real number object R, the set of points of the locale $\mathbb{R}$; then we can construct (again non-geometrically) a spatial locale of "spatial reals" whose frame is the subframe of $\mathcal{P}\mathrm{R}$ given by the usual topology. But then the subspace [0,1] is not in general compact [4].

Theorem 6.2.5 *(Heine-Borel Theorem)*
If x and y are reals, then the closed interval [x,y] in $\mathbb{R}$ is compact.

Proof

It is not hard to show that [0,1] is homeomorphic to $\overline{(0,1)\cap\mathbb{Q}}$, after which its compactness follows from 6.2.4, and a similar technique works for other closed intervals. However, we shall use a different method that also shows that [x,y] depends continuously on x and y. We shall first define a map H-B: $\mathbb{R}\times\mathbb{R} \to P_U\mathbb{R}$ which, by 5.3.3 and 6.2.2, is homeomorphic to $\overline{\mathcal{F}_U\mathbb{Q}}$. The points of $P_U\mathbb{R}$ are equivalent to compact fitted sublocales of $\mathbb{R}$ [22], and we then show that a point z of $\mathbb{R}$ is in H-B(x,y) (i.e. $\uparrow z \sqsupseteq$ H-B(x,y) where $\uparrow$ is the unit of the P_U monad – see [19]) iff $x \leq z \leq y$. This will complete the proof that [x,y] is compact.

Suppose x and y are reals. We define H-B(x,y) as a point M of $\overline{\mathcal{F}_U\mathbb{Q}}$: if $S \subseteq_{fin} \mathbb{Q}$, then M(S) is $\inf\{\varepsilon: \{B_\varepsilon(s): s \in S\}$ covers $[x,y]\}$, where we say that a finite set of balls (i.e., here, rational open intervals) covers [x,y] iff either $y < x$ or if we can find a finite list of balls $(B_{\varepsilon_i}(s_i): 1 \leq i \leq n)$ from the set, with $s_1 \leq s_2 \leq \ldots \leq s_n$, $s_1-\varepsilon_1 \in L_x$, $s_i+\varepsilon_i > s_{i+1}-\varepsilon_{i+1}$ $(1 \leq i \leq n-1)$ and $s_n+\varepsilon_n \in R_y$. This is a geometric construction, and a little calculation verifies that it is indeed a point of $\overline{\mathcal{F}_U\mathbb{Q}}$.

Now we have that $\uparrow z \sqsupseteq$ H-B(x,y) iff whenever $\{B_\varepsilon(s): s \in S\}$ covers [x,y] then z is in $B_\varepsilon(s)$ for some s in S (i.e. $s-\varepsilon \in L_z$, $s+\varepsilon \in R_z$, by the proof of 5.3.3). Suppose this holds. We wish to show $x \leq z$ (i.e. that (x,z) is in the closed complement of the open sublocale > of $\mathbb{R}^2$), so suppose $z < x$. We can find a rational q in $R_z \cap L_x$, after which it is not hard to find a cover of [x,y] that lies wholly in R_z – a contradiction. Similarly, $z \leq y$. For the converse, suppose that $x \leq z \leq y$ and $\{B_\varepsilon(s): s \in S\}$ covers [x,y]. If $y < x$ we get instant contradiction. Otherwise, suppose we have a list $(B_\varepsilon(s_i): 1 \leq i \leq n)$ as above. We have $s_1-\varepsilon \in L_z$ and $s_n+\varepsilon \in R_z$, and for $1 \leq i \leq n-1$ either $s_{i+1}-\varepsilon \in L_z$ or $s_i+\varepsilon \in R_z$. It can be deduced that z is in some $B_\varepsilon(s_i)$.]

(I conjecture that, at least in the case when $x \leq y$, it is possible to strengthen the construction H-B by defining H-B(x,y) as a point in the Vietoris powerlocale $V^+\mathbb{R}$ described in Section 6.3.)

Just as for the lower powerlocale, we can "exclude the empty set", which in the upper powerlocale corresponds to a *top* point. Excluding this gives us a closed sublocale $P_U{}^+D$ (the complement of $\Box$**false**). Again, including the empty set had taken us beyond finitary metrics: $d_U(\emptyset,T) = \infty$ if $T \neq \emptyset$.

Proposition 6.2.6 The homeomorphism of Theorem 6.2.2 restricts to a homeomorphism between $\overline{P_U{}^+X}$ and $\overline{\mathcal{F}_U{}^+X}$, where the elements of the space $\mathcal{F}_U{}^+X$ are the finite non-empty subsets of X, and its metric d_U is as before.

Proof

Identifying points of $\overline{\mathcal{F}_U X}$ with certain subsets I of $\mathcal{F}X\times Q_+$, it is clear that the top point is the whole of $\mathcal{F}X\times Q_+$. Now suppose a point I contains $B_\varepsilon(\emptyset)$ for some ε. For any δ it will contain some $B_\delta(S)$, and hence an upper bound $B_\alpha(T)$ for $B_\delta(S)$ and $B_\varepsilon(\emptyset)$. But $B_\varepsilon(\emptyset) < B_\alpha(T)$ implies $d_U(\emptyset,T)$ is finite, and hence that $T = \emptyset$; and since also $\alpha < \delta$ we deduce that $B_\delta(\emptyset) \in$ I for every δ and hence

$I = \mathcal{F}X\times Q_+$. We deduce that the points of $P_U{}^+\overline{X}$ are those I containing no $B_\varepsilon(\emptyset)$, and the result follows.]

6.3 The Vietoris powerlocale, V

Definition 6.3.1 Let X be a quasimetric space. We define its *convex powerspace,* $\mathcal{F}_C X$, by taking the elements to be the finite subsets of X, with distance $d_C(S,T) = \max(d_L(S,T), d_U(S,T))$.

Note that if X is actually a metric space, then so is $\mathcal{F}_C X$: for $d_L(S,T) = d_U(T,S)$.

Theorem 6.3.2 $\overline{\mathcal{F}_C X}$ is homeomorphic to the Vietoris powerlocale V $\overline{X}$.

Proof

Again, we embed $\overline{\mathcal{F}_C X}$ in Ball($\mathcal{F}_C X$) and V $\overline{X}$ in V Ball(X). The same function ϕ: $\mathcal{F}X\times Q_+ \to \mathcal{F}(X\times Q_+)$ preserves and reflects order: we have $B_\delta(S) < B_\varepsilon(T)$ (but this time with respect to the convex quasimetric) iff $\phi(B_\delta(S)) <_C \phi(B_\varepsilon(T))$. It follows that we get a continuous map ϕ': Ball($\mathcal{F}_C X$) $\to$ V Ball(X), which we show restricts to a homeomorphism between $\overline{\mathcal{F}_C X}$ and V $\overline{X}$. Again, the bulk of the work lies in identifying the points of V Ball(X) that lie in V $\overline{X}$.

Lemma 6.3.3 Let J be a point of V Ball(X). Then the following are equivalent:

(i) J is in V $\overline{X}$.
(ii) J contains elements $\phi(B_\varepsilon(T))$ for arbitrarily small ε.
(iii) if $\alpha > 0$ and $U \in J$, then there is some $B_\varepsilon(T)$ with $\varepsilon < \alpha$ and $U <_C \phi(B_\varepsilon(T)) \in J$.

Proof

Again, let $D \to E$ be an arbitrary locale embedding, with ΩD presented over ΩE by relations $a \le b$ for $(a,b) \in R \subseteq \Omega E\times\Omega E$. Combining the calculations of 6.1.2 and 6.2.2, we get

$$\begin{aligned}\Omega VD &= \mathrm{Fr}\,\langle \Omega D \text{ (qua SupLat)}, \Omega D \text{ (qua PreFr)} \mid \\ &\qquad\qquad \Box d \wedge \Diamond e \le \Diamond(d\wedge e),\ \Box(d\vee e) \le \Box d \vee \Diamond e\,\rangle \\ &\cong \mathrm{Fr}\,\langle \Omega VE \text{ (qua Fr)} \mid \Diamond(a\wedge c) \le \Diamond(b\wedge c),\ \Box(a\vee c) \le \Box(b\vee c) \\ &\qquad\qquad\qquad\qquad ((a,b) \in R,\ c \in \Omega E)\,\rangle\end{aligned}$$

In our present case, these relations reduce to

$$\Diamond B_\delta(x) \le \bigvee\{\Diamond B_{\varepsilon'}(y'): B_\delta(x) < B_{\varepsilon'}(y') \text{ and } \varepsilon' < \alpha\}$$
$$\mathbf{true} \le \bigvee^{\uparrow}\{\Box \textstyle\bigvee_{y\in T} B_\varepsilon(y): T \subseteq_{fin} X\}$$

However, given the second, we have the first (in fact they are equivalent): for

$$\begin{aligned}\Diamond B_\delta(x) &\le \bigvee^{\uparrow}\{\Diamond B_\delta(x)\wedge\Box \textstyle\bigvee_{y\in T} B_\varepsilon(y): T \subseteq_{fin} X\} \\ &\le \bigvee^{\uparrow}\{\Diamond(B_\delta(x)\wedge\textstyle\bigvee_{y\in T} B_\varepsilon(y)): T \subseteq_{fin} X\} \\ &= \textstyle\bigvee_{y\in X} \Diamond(B_\delta(x)\wedge B_\varepsilon(y)) \le \text{RHS of first}\end{aligned}$$

From [17] we see that J satisfies $\Box\bigvee_{y\in T} B_\varepsilon(y)$ iff it contains $\phi(B_\varepsilon(T'))$ for some $T' \subseteq_{fin} T$. The conditions therefore reduce to (ii).

(iii) $\Rightarrow$ (ii) is easy. For (ii) $\Rightarrow$ (iii), suppose $U \in J$; choose U' with $U <_C U' \in J$, and $\eta > 0$ such that $\eta < \alpha$, $\forall B_\delta(x)\in U.\ \exists B_\varepsilon(y)\in U'.\ B_\delta(x) < B_{\varepsilon+\eta}(y)$ and $\forall B_\varepsilon(y)\in U'.\ \exists B_\delta(x)\in U.\ B_\delta(x) < B_{\varepsilon+\eta}(y)$. Find S such that $\phi(B_\eta(S)) \in J$, and let V be a common refinement of U' and $\phi(B_\eta(S))$ in J. Then, much as before, we see that if $T \doteq \{z: \exists\beta.\ B_\beta(z) \in V\}$ then $U <_C \phi(B_\eta(T)) <_C V$.]

Returning to 6.3.2, suppose we are given I in Ball($\mathcal{F}_C$X). If I is in $\overline{\mathcal{F}_C X}$ (i.e. I has balls of arbitrarily small radius) then $\phi'(I)$ satisfies (iii) in the lemma, so ϕ' restricts to a map from $\overline{\mathcal{F}_C X}$ to V $\overline{X}$. Its inverse is given by J $\mapsto \phi^{-1}(J)$.]

Again we can "exclude the empty set", which in the Vietoris powerlocale VD is neither bottom nor top, but is isolated. Excluding it gives us a clopen sublocale V^+D ($\Diamond$**true** is now the complement of $\Box$**false**).

Proposition 6.3.4 The homeomorphism of Theorem 6.3.2 restricts to a homeomorphism between $V^+\overline{X}$ and $\overline{\mathcal{F}_C{}^+X}$, where the elements of the space $\mathcal{F}_C{}^+X$ are the finite non-empty subsets of X, and its metric d_C is as before.

Proof

If $B_\delta(S) < B_\varepsilon(T)$ then $d_C(S,T)$ is finite, and it follows that S and T are either both empty or both non-empty. Identifying points of $\overline{\mathcal{F}_C X}$ with certain subsets I of $\mathcal{F}X \times Q_+$, we therefore see that either $I = \{B_\varepsilon(\emptyset): \varepsilon \in Q_+\}$ (which corresponds to $\Box$**false**) or I contains only balls $B_\varepsilon(T)$ with T non-empty (corresponding to $\Diamond$**true**). The result follows.]

7. Cauchy sequences

We continue to deal with a quasimetric space X.

Definition 7.1 [14] A sequence $(x_n)_{n\in\mathbb{N}}$ in X is *forward Cauchy* iff for every rational $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that whenever $N \le m \le n$ we have $d(x_m, x_n) < \varepsilon$.

(It is *backward Cauchy* iff it is forward Cauchy in X^{op}, i.e. in the above definition we have $d(x_n, x_m)$ instead of $d(x_m, x_n)$.)

For a geometric theory of Cauchiness, it has to be structure, not just property – the modulus of convergence (the dependence of N on ε) has to be supplied explicitly. We shall make do with a fixed, canonical rate of convergence:

A forward Cauchy sequence (x_n) has *canonical convergence* iff whenever $N \le m \le n$ we have $d(x_m, x_n) < 2^{-N}$; it suffices to have $d(x_m, x_{m+k}) < 2^{-m}$ for all m, k.

We write *Cauchy$_f$(X)* for the locale of forward Cauchy sequences in X with canonical convergence. Henceforth, we shall tacitly assume that all forward Cauchy sequences mentioned have canonical convergence.

Proposition 7.2 There is a map lim: $\mathrm{Cauchy}_f(X) \to \overline{X}$ such that if (x_n) is forward Cauchy, then lim x_n is defined by

$$(\lim x_n)_x = \inf_n (d(x, x_n)+2^{-n})$$

Proof To show that lim x_n is indeed flat, let us write $L(x) = (\lim x_n)_x$. First, $\inf_z L(z)$ is 0, for $L(x_k) \le d(x_k, x_k) + 2^{-k} = 2^{-k}$. Next, suppose we have x and y in X with $L(x) < \alpha$ and $L(y) < \beta$. Then we can find natural numbers k and l and rationals α' and β' such that

$$d(x, x_k) + 2^{-k} < \alpha' < \alpha \quad \text{and} \quad d(x, x_l) + 2^{-l} < \beta' < \beta$$

Now choose $n \ge \max(k, l)$ such that $\alpha'+2^{-n} < \alpha$, $\beta'+2^{-n} < \beta$. Then

$$\begin{aligned} d(x, x_n) + L(x_n) &\le d(x, x_k) + d(x_k, x_n) + 2^{-n} \\ &< d(x, x_k) + 2^{-k} + 2^{-n} < \alpha' + 2^{-n} < \alpha \end{aligned}$$

and similarly $d(y, x_n) + L(x_n) < \beta$.]

This would more normally be constructed as $\liminf_n d(x, x_n) = \sup_N \inf_{n\ge N} d(x, x_n)$. This is constructively difficult in $\overleftarrow{[0,\infty]}$ (we don't have sups), but they are classically equal. To show $\inf_n(d(x, x_n) + 2^{-n}) \ge \sup_N \inf_{n\ge N} d(x, x_n)$, we must show for every m and N that

$$d(x, x_m) + 2^{-m} \ge \inf_{n\ge N} d(x, x_n)$$

If $m \ge N$, then take $n = m$. If $m < N$, then

$$\text{RHS} \le d(x, x_N) \le d(x, x_m)+d(x_m, x_N) \le \text{LHS}.$$

For the reverse inequality, we must show that for every $\varepsilon > 0$ there is some m such that

$$d(x, x_m) + 2^{-m} < \sup_N \inf_{n\ge N} d(x, x_n) + \varepsilon$$

Choose N such that $2^{-N} < \varepsilon/2$, and then choose $m \ge N$ such that $d(x, x_m) < \inf_{n\ge N} d(x, x_n) + \varepsilon/2$. We then have that $d(x, x_m) + 2^{-m} < \inf_{n\ge N} d(x, x_n) + \varepsilon$.

Our aim now is to show that lim is a surjection. However, there is little point in showing merely that a map between locales is surjective, for surjective maps in generality are not well behaved – not preserved under pullback, for instance. One would normally hope to show that a surjection is either open or proper, but in general lim is neither.

Example 7.3 Consider $X = \mathbb{Q}\cap(-2,2)$. First, it is not too hard to show that its completion is homeomorphic to [-2,2]. (The proof is numerically slightly intricate; what is required is to show that for a real in [-2,2], its open neighbourhoods are determined by those centred on rationals in the interval (-2,2).)

Now consider the open sublocale of sequences (x_i) comprising those for which $x_0 = 0$. If such a sequence has limit x, then x is in [–1,1] – for

$$d(0,x) = \inf_i (d(0,x_i)+2^{-i}) = \inf_i (d(x_0,x_i)+2^{-i}) \le \inf_i (1+2^{-i}) = 1$$

But – classically at least – every real in the interval [–1,1] is the limit of such a sequence starting at 0, so [–1,1] is the direct image under lim of an open. It follows that lim is not an open map.

But neither is it proper, for inverse image under proper maps preserves compactness. [–2,2] is compact, but $\mathrm{Cauchy}_f(X)$ is not – it is covered by the open sets $(x_0 = q)$ for q in X, but there is no finite subcover. (This argument was shown me by Till Plewe.)]

Nonetheless, we shall show that lim is *triquotient.* This class of localic surjections was proposed by Plewe [13], who has proved that it is pullback stable, that it includes both open surjections and proper surjections, that triquotient maps have effective descent, and that any triquotient map is the coequalizer of its kernel pair. From this last property we see in effect that the completion is got from the locale of Cauchy sequences by factoring out an equivalence relation, though a direct construction this way would be problematic. (I conjecture too that triquotient maps have a key role to play in our synthetic reasoning [19], unifying the "lower" and "upper" flavours.)

Definition 7.4 [13] A map $f: X \to Y$ is *triquotient* iff there is a function $f_\#: \Omega X \to \Omega Y$ (a *triquotient assignment)* such that –

(i) $f_\#$ preserves directed joins

(ii) $f_\#(a \wedge \Omega f(b)) = f_\#(a) \wedge b \quad (a \in \Omega X, b \in \Omega Y)$

(iii) $f_\#(a \vee \Omega f(b)) = f_\#(a) \vee b \quad (a \in \Omega X, b \in \Omega Y)$

The usual special cases are open surjections ($f_\#$ is left adjoint to Ωf) and proper surjections ($f_\#$ right adjoint to Ωf). In any case, we see that a triquotient assignment $f_\#$ preserves **false** and **true** (put b = **false** in (ii), **true** in (iii)) and $f_\# \circ \Omega f(b) = b$ (put a = **true** in (ii)), showing that f is a surjection. In our case, where f is lim, we shall have an $f_\#$ that preserves all joins, and we see that a join-preserving function $f_\#$ is a triquotient assignment for f iff it preserves **true** and satisfies condition (ii), the Frobenius identity for $\wedge$. Note that a function $f_\#$ preserving all joins is equivalent to a map from Y to the lower powerlocale $P_L X$.

In the following Lemma we translate this sufficient condition into localic form so that we can apply the synthetic methods of [19]. These facilitate reasoning with powerlocales by allowing points of $P_L X$ (or, indeed, $P_U X$) to be treated like collections of points of X. We briefly recall some notation from there:

- $!: X \to 1$ is the unique map.
- $\downarrow: X \to P_L X$ is the unit of the monad P_L.
- If x and U are points of X and $P_L X$, then $x \in U$ iff $\downarrow x \sqsubseteq U$.
- $\times: P_L X \times P_L Y \to P_L(X \times Y)$ is the "Cartesian product map", $(x,y) \in U \times V$ iff $x \in U$ and $y \in V$.

Lemma 7.5 Let $f: X \to Y$ be a map of locales, and let $g: Y \to P_L X$. Then f is triquotient (with triquotient assignment $f_\# = \Diamond;\Omega g: \Omega X \to \Omega P_L X \to \Omega Y$) if

(i) $g;P_L! = !;\downarrow : Y \to P_L 1$

(ii) $g;P_L\langle Id_X, f\rangle = \langle g, Id_Y\rangle;(Id \times \downarrow);\times : Y \to P_L(X \times Y)$

Proof $f_\#$ is a suplattice homomorphism so by the above discussion it suffices to prove that it preserves 1 and that the Frobenius identity for $\wedge$ holds.

First, we apply the two sides of (i) to $\Diamond$**true** in $P_L 1$:

$\Omega(g;P_L!)(\Diamond \mathbf{true}) = \Omega g(\Diamond \mathbf{true}) = f_\# \mathbf{true}$
$\Omega(!;\downarrow)(\Diamond \mathbf{true}) = \mathbf{true}$

Next, we apply the two sides of (ii) to $\Diamond(a \otimes b)$ in $P_L(X \times Y)$:

$\Omega(g;P_L\langle Id_X, f\rangle)(\Diamond(a \otimes b)) = \Omega g(\Diamond \Omega\langle Id_X, f\rangle(a \otimes b)) = f_\#(a \wedge \Omega f(b))$
$\Omega(\langle g, Id_Y\rangle;(Id \times \downarrow);\times)(\Diamond(a \otimes b)) = \Omega\langle g, Id_Y\rangle(\Omega(Id \times \downarrow)(\Diamond a \otimes \Diamond b))$
$= \Omega\langle g, Id_Y\rangle(\Diamond a \otimes b) = f_\# a \wedge b$

]

The calculations arising from this Lemma involve comparing powerlocale points. The basic technique arising from [19] is that if K and L are points of $P_L D$, then $K \sqsubseteq L$ iff every $x \in K$ is also in L. We also have to consider powerlocale points of the form $P_L f(K)$ where $f: D \to E$ is a map. To show $P_L f(K) \sqsubseteq L$ it is equivalent to show that for every $x \in K$ we have $f(x) \in L$; however, the reverse direction $\sqsupseteq$ is trickier. We need that if $y \in L$ then $y \in P_L f(K)$, i.e. $\downarrow y \sqsubseteq P_L f(K)$. From the basic definition of the specialization order $\sqsubseteq$, this amounts to showing that if $y \vDash b \in \Omega E$ (it suffices to take b from a basis) then $P_L f(K) \vDash \Diamond b$, i.e. $K \vDash \Diamond \Omega f(b)$. (Notice that in classical point-set topology, the points of $P_L D$ are the closed subsets of D, and K $\models \Diamond a$ iff K meets a: so K $\models \Diamond\Omega f(b)$ iff K meets $f^{-1}(b)$, i.e. iff f(K) meets b. Interpreted classically, therefore, the reasoning shows that y is in $P_L f(K)$ iff every open neighbourhood of y meets f(K), and this is exactly what would be called for if $P_L f(K)$ were the closure of the direct image of K. It is remarkable that the constructively valid synthetic reasoning recreates a classical argument, even though the classical justification of the argument fails quite comprehensively.)

Proposition 7.6 The points of $P_L(\mathrm{Cauchy}_f(X))$ are the lower closed subsets U of $\mathcal{F}(\mathbb{N}\times X)$ such that if $S \in U$ then –

(i) if S contains (n,x) and (n,y) then x = y (in other words, S is a finite partial function from $\mathbb{N}$ to X);

(ii) if $n \in \mathbb{N}$ then $S\cup\{(n,z)\} \in U$ for some z;

(iii) if S contains (n,x) and (n+k,y) then $d(x,y) < 2^{-n}$.

Proof By the Suplattice Coverage Theorem. If we required U to be an ideal (closed under $\cup$), then we should just be describing the points of $\mathrm{Cauchy}_f(X)$: if the point is a sequence (x_i), then U is the set of finite subsets of the set $\{(i,x_i): i \in \mathbb{N}\}$. Dropping the closure under $\cup$ gives points of the lower powerlocale.]

Proposition 7.7 We can define a map g: $\overline{X} \to P_L(\mathrm{Cauchy}_f(X))$ by $S \in g(M)$ iff –

(i) $\forall(i,x) \in S.\ M_x < 2^{-i}$

(ii) $\forall(i,x), (j,y) \in S.\ (x = y \vee (i < j \wedge d(x,y) < 2^{-i})$
$\vee (j < i \wedge d(y,x) < 2^{-j}))$

Proof (Note that g(M) is a geometrically defined subset of $\mathcal{F}(\mathbb{N}\times X)$. This exploits the fact that universal quantification bounded over finite sets is geometric. Condition (ii) rewrites the geometric *axioms* (i) and (iii) of 7.6 as a geometric *formula*.) The only difficult part is (ii) in 7.6. Suppose $S \in g(M)$ and $n \in \mathbb{N}$. We have $\forall(i,x)\in S.\ (n \le i \vee i < n)$, and from the finiteness of S it follows [7] that either $\exists(i,x)\in S.\ n \le i$ or $\forall(i,x)\in S.\ i < n$.

In the first case, suppose we have $(k, x) \in S$ with $n \le k$; let k be the least such: so $\forall(i,y)\in S.\ (i < n \vee k \le i)$. Then $S\cup\{(n,x)\} \in g(M)$.

The second case is when $\forall(i,x)\in S.\ i < n$. Suppose $S = \{(n_i, x_i): 1 \le i \le m-1\}$, and choose x_m such that $M(x_m) < 2^{-n}$. By flatness we can find x such that

$$d(x_i, x) + M(x) < 2^{-n_i} \qquad (1 \le i \le m;\ \text{take } n_m = n)$$

It follows that $d(x_i, x) < 2^{-n_i}$ $(1 \le i \le m-1)$ and $M(x) < 2^{-n}$, so that $S\cup\{(n, x)\} \in g(M)$.]

Theorem 7.8 lim is triquotient.

Proof

We use Lemma 7.5. Let M be a flat left module. First we must show that $P_L!(g(M)) = {\downarrow}!$ in $P_L 1$. The troublesome direction is $\sqsupseteq$: we must show that $! \in P_L!(g(M))$, and for this we must show that g(M) contains a non-empty set. We can find x such that $M_x < 1$, and then $\{(0,x)\} \in g(M)$.

Next, we must show $P_L\langle \mathrm{Id}, \lim\rangle \circ g(M) = g(M)\times{\downarrow}M$ in $P_L(\mathrm{Cauchy}_f(X) \times \overline{X})$. For $\sqsubseteq$, following the remarks after Lemma 7.5, it suffices to show that if $(x_i) \in g(M)$ then $\langle \mathrm{Id}, \lim\rangle((x_i)) \in g(M)\times{\downarrow}M$, i.e. lim $(x_i) \sqsubseteq M$. If lim $(x_i) \models B_\varepsilon(y)$ then $d(y,x_n) + 2^{-n} < \varepsilon$ for some n, i.e. $B_\varepsilon(y) < B_{2^{-n}}(x_n)$. But from $(x_i) \in g(M)$ we know that $M \models B_{2^{-n}}(x_n)$ and hence also $B_\varepsilon(y)$.

For the reverse, $\sqsupseteq$, we show that if $(x_i) \in g(M)$ then $\langle (x_i), M\rangle \in P_L\langle Id,lim\rangle \circ g(M)$. A basic open neighbourhood of $\langle (x_i), M\rangle$ will be described by some $S \subseteq_{fin} \mathbb{N}\times X$, being a finite subsequence of (x_i), and $B_\varepsilon(y)$ with $M_y < \varepsilon$. Its inverse image under $\langle$Id, lim$\rangle$ is then

$$\bigvee\{\bigwedge(S\cup\{(n,x)\}): d(y,x)+2^{-n} < \varepsilon\}$$

It follows that we must find (n,x) so that the finite sequence $S\cup\{(n,x)\}$ has the Cauchy property 7.7 (ii), with $M_x < 2^{-n}$ and $d(y,x)+2^{-n} < \varepsilon$. Choose n so that $n > i$ for every (i,x_i) in S, and $M_y+2^{-n} < \varepsilon$, and then choose z such that $M_z < 2^{-n}$ and x such that

- $d(x_i,x)+M_x < 2^{-i}$ (for every (i,x_i) in S)
- $d(y,x)+M_x+2^{-n} < \varepsilon$
- $d(z,x)+M_x < 2^{-n}$]

8. Conclusions

Given a quasimetric space X, we have constructed a locale $\overline{X}$ that apparently enjoys many properties appropriate for a completion. In the metric case, its spatialization is classically homeomorphic to the usual completion.

The proposed completion is constructively robust, for its dependence solely on geometric constructions makes it stable under change of base. Nonetheless, it is hard to see what would be required in order to sustain a claim that it is "the right" notion of completion. For a start, it evades standard accounts based on any idea of complete quasimetric spaces as special kinds of more general quasimetric spaces. This is because of the different natures of the original space and the completion. The original space is considered to have its discrete topology and to try to construct its quasimetric topology would not be stable under change of base. On the other hand the topologized structure, the completion, does not in general have its own quasimetric, at least not in any straightforward way.

We therefore present the construction "as is", in the hope that its localic good behaviour will prove useful.

Moving on to "locales as topology-free spaces", I believe this must be the right way to handle locales (and, indeed, topological spaces), at least for certain considerations. On the other hand, I feel that the justification given in Section 2 is, ultimately, spurious. The essence of geometric logic is that it allows set-indexed disjunctions and coproducts, but this clearly depends on what sets are. The conventional topos approach would be to fix an elementary topos (with natural number object) as "the" category of sets, and build up a theory of Grothendieck toposes over it. These set-indexed infinities give a lot of structure in the geometric universes, including Cartesian closedness, subobject classifiers, natural number objects and free algebra constructions. However, what we see in the working in this paper is that not only do we have to be careful about the non-geometric constructions, but also to present theories we don't need the arbitrary infinities: the countable ones embodied in free algebra constructions suffice. This suggests that the correct approach is to use an even more restricted mathematics comprising finitary constructions together with free algebras (which also can be specified finitarily), and I conjecture that a good theory can be made by replacing the geometric universes by Joyal's arithmetic universes (pretoposes with free algebras). If one starts from this very basic mathematics as constructivist foundations, then there are severe limitations on what can be constructed as sets, and "locales as topology-free spaces" arises as a natural way – indeed, perhaps the only way – of handling anything like powersets, or function spaces, or the real line.

Many questions are left unanswered here. Some that perhaps merit further work are –

- Can quasimetric completions be given quasimetrics of their own in any sense? (The obvious sense – of a continuous map from $\overline{X}^2$ to $\overleftarrow{[0,\infty]}$ – is plainly not possible in general, for it would have to be contravariant in one argument with respect to the specialization order.) One approach that looks promising is to define a quasimetric on a locale X by using a map from X to $P_L(X\times\overleftarrow{[0,\infty]})$, conceptually mapping y to $\{(x,\delta): d(x,y) \leq \delta\}$. This has the right variances.
- What special properties are enjoyed by spaces for which the metric factors via $[0,\infty]$ (on its way to $\overleftarrow{[0,\infty]}$)?
- How does the theory appear when restricted to quasi*ultra*metrics? These can also be treated as enriched categories in a different way, enriched over $\overleftarrow{[0,\infty]}$ with max for its monoidal product instead of addition. Are the points of the completion still flat modules in the new setting?
- How can maps between quasimetric completions be expressed in terms of the original quasimetric spaces?
- Can one give criteria on the quasimetric spaces for their completions to have various properties – for instance, Hausdorff, stably locally compact, locally compact?
- Can one use the theory to "solve domain equations" for quasimetric completions? There is a topos [QMS] classifying quasimetric spaces, and it is local – it has an initial point, Ø. It follows from [24] that any geometric morphism F: [QMS] $\rightarrow$ [QMS] has an initial algebra X (take the colimit of $\emptyset \rightarrow F(\emptyset) \rightarrow F^2(\emptyset) \rightarrow \ldots$). If F on spaces corresponds to $\overline{F}$ on completions (as, for instance, $\mathcal{F}_L$ corresponded to P_L), then $\overline{X}$ is a solution to $\overline{F}(Y) \cong Y$.

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