A bounding circle for the attractor of an IFS

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Abstract

Given an iterated function system consisting of a finite number of contracting affine maps on the plane and given any point of the plane, we obtain a circle centred at that point which contains the attractor of the IFS. We then find the point on the plane such that the bounding circle centred at that point has minimum radius.

Let $f_1, f_2 \ldots, f_N : \mathbb{R}^2 \to \mathbb{R}^2$ be contracting maps on the plane, and assume that f_i has contractivity factor s_i for $i = 1, \ldots, N$.

Given any point in the plane, we will obtain a bounding circle centred at this point for the area occupied by the attractor of the IFS generated by f_i's.

More precisely, let $u \in \mathbb{R}^2$ be any point. We obtain a circle centred at u which contains the attractor [Eda93a]. Put

$$R(u) = \max_{1 \le i \le N} \frac{|u - f_i(u)|}{1 - s_i}.$$

If $v \in \mathbb{R}^2$ is any point with $|u - v| \le R(u)$, then, for any $i = 1, \ldots, N$, we have

$$\begin{aligned} |u - f_i(v)| &\le |u - f_i(u)| + |f_i(u) - f_i(v)| \le (1 - s_i)R(u) + s_i|u - v| \\ &\le (1 - s_i)R(u) + s_i R(u) = R(u). \end{aligned}$$

In other words, the disk centred at u with radius $R(u)$ is mapped into itself by each of the maps f_i; it, therefore, contains the attractor of the IFS. In applications, we are usually interested in a square which contains the attractor. Clearly, for any point u in the plane, the square with centre u and sides of length $2R(u)$ parallel to the coordinate axes is a bounding square.

Assume that f_i's are affine transformations. In matrix notation, an affine map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is of the form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ h \end{pmatrix}.$$

The contractivity factor of f is the least number s such that

$$\sqrt{(x_1' - x_2')^2 + (y_1' - y_2')^2} \le s(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}).$$

It is given by [Eda93b]

$$s = \sqrt{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + \gamma^2}}$$

where $\alpha = (a^2 + c^2)/2$, $\beta = (b^2 + d^2)/2$, and $\gamma = ab + cd$.

Now, we would like to find $u = (x, y)$ with minimal value of $R(u)$. Let $N = 2$. Then,

$$R^2(u) =$$

$$\max\{\frac{(a_1x + b_1y + e_1 - x)^2 + (c_1x + d_1y + h_1 - y)^2}{(1 - s_1)^2}, \frac{(a_2x + b_2y + e_2 - x)^2 + (c_2x + d_2y + h_2 - y)^2}{(1 - s_2)^2}\}$$

The two elliptic cones

$$z_1(x, y) = \frac{(a_1x + b_1y + e_1 - x)^2 + (c_1x + d_1y + h_1 - y)^2}{(1 - s_1)^2}$$

and

$$z_2(x, y) = \frac{(a_2x + b_2y + e_2 - x)^2 + (c_2x + d_2y + h_2 - y)^2}{(1 - s_2)^2}$$

have their vertices at the fixed points of f_1 and f_2 respectively, and meet in a conic section. The point $u = (x, y)$ in their intersection which gives the minimum value of z is the required point. Using Lagrange's multipliers, this is obtained by minimising the function

$$g(x, y, \lambda) = z_1(x, y) + \lambda[z_1(x, y) - z_2(x, y)] =$$

$$\frac{(a_1x + b_1y + e_1 - x)^2 + (c_1x + d_1y + h_1 - y)^2}{(1 - s_1)^2} +$$

$$\lambda[\frac{(a_1x + b_1y + e_1 - x)^2 + (c_1x + d_1y + h_1 - y)^2}{(1 - s_1)^2} - \frac{(a_2x + b_2y + e_2 - x)^2 + (c_2x + d_2y + h_2 - y)^2}{(1 - s_2)^2}]$$

with respect to x, y, λ.

For $N = 3$, the required point is the intersection of the three cones z_1, z_2 and z_3 if indeed they meet. Otherwise, it is at one of the three minimums of their pairwise intersections. This easily generalises to $N > 3$ by recursion.

References

[Eda93a] A. Edalat. Dynamical systems, measures and fractals via domain theory (Extended abstract). In G. L. Burn, S. J. Gay, and M. D. Ryan, editors, *Theory and Formal Methods 1993*. Springer-Verlag, 1993. Full paper to appear in *Information and Computation*.

[Eda93b] A. Edalat. Power domain algorithms for fractal image decompression. Technical Report Doc 93/44, Department of Computing, Imperial College, 1993.