K-THEORETIC DESCENDENT SERIES FOR HILBERT SCHEMES OF POINTS ON SURFACES

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ABSTRACT. We study the holomorphic Euler characteristics of tautological sheaves on Hilbert schemes of points on surfaces. In particular, we establish the rationality of K-theoretic descendent series.

Our approach is to control equivariant holomorphic Euler characteristics over the Hilbert scheme of points on the affine plane. To do so, we slightly modify a Macdonald polynomial identity of Mellit.

1. INTRODUCTION

1.1. **K-theoretic descendent series.** Let S be a nonsingular projective algebraic surface. For $n \geq 0$, let $S^{[n]}$ denote the Hilbert scheme of n points on S.

The Hilbert schemes carry natural K-theory classes induced from classes on S. Let $\Sigma_n \subset S^{[n]} \times S$ denote the universal family and $\pi_{S^{[n]}}$ and π_S denote the projections from $S^{[n]} \times S$ onto the corresponding factors. A class $\alpha \in K(S)$ induces a tautological class $\alpha^{[n]} \in K(S^{[n]})$ given by

$$\alpha^{[n]} = \pi_{S^{[n]}*}(\mathcal{O}_{\Sigma_n} \otimes \pi_S^* \alpha). \tag{1.1}$$

In this paper, we study the structure of holomorphic Euler characteristics of tautological classes. Namely, we consider the following *K-theoretic descendent series*: for classes $\alpha_1, \ldots, \alpha_l \in K(S)$ and integers $k_1, \ldots, k_l \geq 0$, set

$$Z_S(\alpha_1, \ldots, \alpha_l \,|\, k_1, \ldots, k_l) = \sum_{n \geq 0} q^n \chi\Big(S^{[n]}, \wedge^{k_1} \alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_l} \alpha_l^{[n]}\Big) \in \mathbb{Q}[[q]]. \tag{1.2}$$

Our first result is a positive answer to Question 5 of [AJLOP].

Theorem 1.1. *The series $Z_S(\alpha_1, \ldots, \alpha_l \,|\, k_1, \ldots, k_l)$ is the Laurent expansion of a rational function in q with denominator $(1-q)^{\chi(\mathcal{O}_S)}$ and numerator of degree at most $k_1 + \cdots + k_l$.*

In anticipation of applications to descendent series for Quot schemes on simply-connected surfaces of geometric genus 0 (see [AJLOP, Sec. 3.4],) we also record the following partial generalization in the case where one of the α_i is the class of a line bundle.

Theorem 1.2. *If α_1 is the class of a line bundle and $k_2, \ldots, k_l \in \mathbb{Z}$ are fixed, then the series*

$$\sum_{k_1=0}^{\infty} (-m)^{k_1} \cdot Z_S(\alpha_1, \ldots, \alpha_l \,|\, k_1, \ldots, k_l) \tag{1.3}$$

is the Laurent expansion of a rational function in q and m whose denominator is of the form $(1-q)^{\chi(\mathcal{O}_S)}(1-qm)^r$ for some r.

1.2. **Examples.** The simplest instance

$$Z_S(\varnothing \,|\, \varnothing) = \sum_{n\geq 0} q^n \chi(S^{[n]}, \mathcal{O}_{S^{[n]}}) = \frac{1}{(1-q)^{\chi(\mathcal{O}_S)}}$$

of Theorem 1.1 is computed in [G, Prop. 3.3(b)]. The second simplest example

$$Z_S(\alpha \,|\, 1) = \sum_{n\geq 0} q^n \chi(S^{[n]}, \alpha^{[n]}) = \frac{\chi(\alpha)\cdot q}{(1-q)^{\chi(\mathcal{O}_S)}}$$

is a consequence of [EGL, Prop. 5.6].

The complexity of the numerators of the rational functions (1.2) can grow as l, k_i and the ranks of α_i increase. For example, if $\mathcal{L} \in K(S)$ is a line bundle, then by [Sc2, Thm 5.25],

$$\begin{aligned} Z_S(-\mathcal{L} \,|\, 3) &= -\sum_{n\geq 0} q^n \chi(S^{[n]}, \mathrm{Sym}^3 \mathcal{L}^{[n]}) \\ &= \frac{\chi(\mathcal{L}^{\otimes 3})\cdot(q^2-q) + \big(\chi(\mathcal{L}^{\otimes 2})\chi(\mathcal{L}) - \chi(\mathcal{T}^*S\otimes\mathcal{L}^{\otimes 3})\big)\cdot(q^3-q^2) - \binom{\chi(\mathcal{L})+2}{3}\cdot q^3}{(1-q)^{\chi(\mathcal{O}_S)}}. \end{aligned}$$

Other computations of examples of K-theoretic descendent series (1.2) can be found in [A, Sec. 6], [D, Thm 1.1], [K, Cor. 8.11], [Sc1, Sec. 5], [Sc2, Thm 5.25] and [Z, Sec. 7].

Now, let $\mathcal{L}$ be a line bundle on S. The simplest example

$$\sum_{k=0}^{\infty}(-m)^k Z_S(\mathcal{L} \,|\, k) = \sum_{k,n\geq 0} q^n(-m)^k \chi(S^{[n]}, \wedge^k \mathcal{L}^{[n]}) = \frac{(1-qm)^{\chi(\mathcal{L})}}{(1-q)^{\chi(\mathcal{O}_S)}}$$

of Theorem 1.2 is a consequence of [Sc1, Thm 5.2.1]. The next simplest example

$$\begin{aligned} \sum_{k=0}^{\infty}(-m)^k Z_S(\mathcal{L},\alpha \,|\, k, 1) &= \sum_{n,k\geq 0} q^n(-m)^k \chi\big(S^{[n]}, (\wedge^k\mathcal{L}^{[n]})\otimes\alpha^{[n]}\big) \\ &= \frac{(1-qm)^{\chi(\mathcal{L})}}{(1-q)^{\chi(\mathcal{O}_S)}} \sum_{n=0}^{\infty} q^{n+1}m^n\chi(\mathcal{L}^{\otimes n}\otimes\alpha) - q^{n+1}m^{n+1}\chi(\mathcal{L}^{\otimes(n+1)}\otimes\alpha) \end{aligned} \tag{1.4}$$

is computed in [AJLOP, Prop. 20]. Hirzebruch-Riemann-Roch implies that the series (1.4) is of the form predicted by Theorem 1.2.

1.3. Comparison with cohomological descendents and other geometries.

1.3.1. *Cohomological descendent series.* The rationality of (1.2) contrasts with the expected behavior of descendent series in cohomology. Cohomological descendent integrals are often packaged as follows: for classes $\alpha_1, \ldots, \alpha_l \in K(S)$ and integers $k_1, \ldots, k_l \geq 0$, form the series

$$\sum_{n\geq 0} q^n \int_{S^{[n]}} \mathrm{ch}_{k_1}(\alpha_1^{[n]}) \cdot \ldots \cdot \mathrm{ch}_{k_l}(\alpha_l^{[n]}) c(\mathcal{T}S^{[n]}). \tag{1.5}$$

The series (1.5) have a different flavor than their K-theoretic counterparts (1.2). For example, by Göttsche's formula [G, Thm 0.1],

$$\sum_{n\geq 0} q^n \int_{S^{[n]}} c_{2n}(\mathcal{T}S^{[n]}) = \prod_{m>0} \frac{1}{(1-q^m)},$$

and by [CO, Cor. 3],

$$\sum_{n\geq 0} q^n \int_{S^{[n]}} \mathrm{ch}_1(\mathcal{O}^{[n]}) c_{2n-1}(\mathcal{T}S^{[n]}) = \frac{c_1(S)^2}{2} \cdot \frac{E_2(q) - E_3(q)}{\prod_{m>0}(1-q^m)};$$

here,

$$E_k(q) = \sum_{n>0} n^{k-1} \frac{q^n}{1-q^n}.$$

Instead, [O, Conj. 2] conjectures that a suitably normalized version of the series (1.5) belongs to a distinguished algebra of q-series called q-multiple zeta values. One result in this direction is [C, Thm. 2], in which a $\mathbb{C}^*$-equivariant version of (1.5) for the affine plane $\mathbb{C}^2$ is proved to be a quasimodular form.

1.3.2. *Curves.* Analogs of series (1.2) and (1.5) can be studied for integrals over Hilbert schemes of points on curves. Given a nonsingular projective curve C, classes $\beta_1, \ldots, \beta_l \in K(C)$ and integers $k_1, \ldots, k_l \geq 0$, both the K-theoretic descendent series

$$\sum_{n\geq 0} q^n \chi(C^{[n]}, \wedge^{k_1}\beta_1^{[n]} \otimes \ldots \otimes \wedge^{k_l}\beta_l^{[n]}) \tag{1.6}$$

and the cohomological descendent series

$$\sum_{n\geq 0} q^n \int_{C^{[n]}} \mathrm{ch}_{k_1}(\beta_1^{[n]}) \cdot \ldots \cdot \mathrm{ch}_{k_l}(\beta_l^{[n]}) c(\mathcal{T}C^{[n]}) \tag{1.7}$$

are Laurent expansions of rational functions. This rationality follows from the methods of [OP, Sec. 2.3]. Namely, the induction scheme of [EGL] (which we recall for Hilbert schemes on surfaces in Section 1.4) reduces the problem to cases where C is $\mathbb{P}^1$. The series (1.6) and (1.7) can be explicitly computed for this geometry using the relation

$$\mathcal{O}(d)^{[n]} = (d+1)\mathcal{O} - (d+1-n)\mathcal{O}(-1) \in K((\mathbb{P}^1)^{[n]}) \cong K(\mathbb{P}^n)$$

from the proof of [MOP1, Thm 2].

1.3.3. *Quot schemes.* Tautological integrals over Quot schemes parametrizing quotients (of dimension at most 1) of vector bundles on surfaces have been studied in [AJLOP, B1, B2, JOP, St1, St2]. Such Quot schemes are typically singular but carry perfect obstruction theories; see, for example, [St2, Sec. 4]. Descendent series of Quot schemes can therefore be defined through virtual structures.

Hilbert schemes of points on surfaces, in particular, can be regarded as Quot schemes parametrizing finite length quotients

$$\mathcal{O}_S \twoheadrightarrow \mathcal{Z}.$$

The associated virtual structure sheaves and virtual fundamental classes differ from the ordinary structure sheaves and fundamental classes of Hilbert schemes, and in fact give rise to more easily understood invariants. The virtual structures have explicit descriptions:

$$\mathcal{O}^{\mathrm{vir}}_{S^{[n]}} = \sum_{k=0}^{n} (-1)^k \wedge^k \mathcal{K}_S^{[n]}, \quad [S^{[n]}]^{\mathrm{vir}} = (-1)^n e(\mathcal{K}_S^{[n]}) \cap [S^{[n]}].$$

Given classes $\alpha_1, \dots, \alpha_l \in K(S)$ and integers $k_1, \dots, k_l \geq 0$, one can form the *virtual cohomological descendent series*

$$\sum_{n\geq 0} q^n \int_{[S^{[n]}]^{\mathrm{vir}}} \mathrm{ch}_{k_1}(\alpha_1^{[n]}) \cdot \ldots \cdot \mathrm{ch}_{k_l}(\alpha_l^{[n]}) c(\mathcal{T}^{\mathrm{vir}} S^{[n]}) \tag{1.8}$$

and the *virtual K-theoretic descendent series*

$$\sum_{n\geq 0} q^n \chi\big(S^{[n]}, \wedge^{k_1} \alpha_1^{[n]} \otimes \ldots \otimes \wedge^{k_l} \alpha_l^{[n]} \otimes \mathcal{O}^{\mathrm{vir}}\big). \tag{1.9}$$

In contrast to (1.5), the series (1.8) is proved in [JOP, Thm. 2] to be the Laurent expansion of a rational function. The series (1.9) is proved in [AJLOP, Thm. 1] to be the Laurent expansion of a rational function with denominator $(1-q)^{2(k_1+\ldots+k_l)}$. We remark that the rationality of (1.9) also follows from Theorem 1.2; in (1.3), one specializes $\alpha_1 = \mathcal{K}_S$ and $m = 1$. Unlike in (1.2), the order of the pole of (1.9) at $q = 1$ need not depend on S. For example, by [AJLOP, Ex. 7] or by substituting $m = 1$ and $\mathcal{L} = \mathcal{K}_S$ in (1.4),

$$\sum_{n\geq 0} q^n \chi\big(S^{[n]}, \alpha^{[n]} \otimes \mathcal{O}^{\mathrm{vir}}\big) = -c_1(\alpha) \cdot \mathcal{K}_S \cdot \frac{q}{1-q} - \mathrm{rk}(\alpha) \cdot \mathcal{K}_S^2 \cdot \frac{q^2}{(1-q)^2}.$$

The following observation accounts for the similarity in the behavior of descendent series for Hilbert schemes on curves and virtual descendent series for Hilbert schemes on surfaces: if S is a surface with a canonical curve C then the virtual structures $\mathcal{O}^{\mathrm{vir}}_{S^{[n]}}$ and $[S^{[n]}]^{\mathrm{vir}}$ localize onto $C^{[n]}$. To be precise, if $\iota : C \subset S$ is the inclusion of the canonical curve, then, by [OP, Eq. (33)]

$$[S^{[n]}]^{\mathrm{vir}} = (-1)^n \iota_*[C^{[n]}].$$

Similarly, if $\Theta \in K(C)$ is a theta characteristic, then by [AJLOP, Thm. 15],

$$\mathcal{O}^{\mathrm{vir}}_{S^{[n]}} = (-1)^n \iota_* \det \Theta^{[n]}.$$

Virtual descendent integrals on $S^{[n]}$ can therefore be written in terms of tautological integrals on $C^{[n]}$.

1.4. **Universal series.** For fixed $\alpha_1, \dots, \alpha_l \in K(S)$, important extra structure emerges when the k_i are allowed to vary and all series $Z_S(\alpha_1, \dots, \alpha_l \,|\, k_1, \dots, k_l)$ are considered together.

Namely, set

$$\hat{Z}_S(\alpha_1,\ldots,\alpha_l) = \sum_{\substack{n\geq 0\\ k_1,\ldots,k_l\geq 0}} q^n(-m_1)^{k_1}\cdots(-m_l)^{k_l}\chi\Big(S^{[n]},\wedge^{k_1}\alpha_1^{[n]}\otimes\ldots\otimes\wedge^{k_l}\alpha_l^{[n]}\Big). \quad (1.10)$$

Now, fix an r-tuple $\mathbf{r}=(r_1,\ldots,r_l)$ of integers. By [EGL, Thm 4.2], there exist *universal series*

$$\mathsf{A}^{\mathbf{r}},\mathsf{B}^{\mathbf{r}},\mathsf{C}_i^{\mathbf{r}},\mathsf{D}_i^{\mathbf{r}},\mathsf{E}_{i,j}^{\mathbf{r}}\in\mathbb{Q}[q,m_1,\ldots,m_l] \quad (1.11)$$

for which, given any surface S and any collection $\alpha_1,\ldots,\alpha_l$ of classes in $K(S)$ satisfying $\mathrm{rank}(\alpha_i)=r_i$, one has

$$\hat{Z}_S(\alpha_1,\ldots,\alpha_l) = (\mathsf{A}^{\mathbf{r}})^{\chi(\mathcal{O}_S)}(\mathsf{B}^{\mathbf{r}})^{\mathcal{K}_S^2}\prod_{i=1}^{l}(\mathsf{C}_i^{\mathbf{r}})^{\mathcal{K}_S\cdot c_1(\alpha_i)}(\mathsf{D}_i^{\mathbf{r}})^{c_2(\alpha_i)}\prod_{1\leq i\leq j\leq l}(\mathsf{E}_{i,j}^{\mathbf{r}})^{c_1(\alpha_i)\cdot c_1(\alpha_j)}. \quad (1.12)$$

More generally, such a factorization into universal series exists for generating series formed from integrals of multiplicative characteristic classes of tautological bundles and the tangent bundle. A precise statement can be found in [EGL, Thm 4.2].

One such series of interest is the *Verlinde series*

$$V_S(\alpha)=\sum_n q^n\chi(S^{[n]},\det(\alpha^{[n]})). \quad (1.13)$$

The series $V_S(\alpha)$ is formed from a subset of terms of (1.10) when α has positive rank. For α of negative rank, Serre duality implies a close relationship between $V_S(\alpha)$ and $V_S(-\alpha)$; see [EGL, Thm 5.3]. The series $V_S(\alpha)$ for α of rank $-1,0$ or 1 are explicitly computed in [EGL, Thm 5.3]. A relationship between the universal series appearing in a factorization of $V_S(\alpha)$ and those appearing in a factorization of the *Segre series*

$$\sum_n q^n\int_{S^{[n]}} s_{2n}(\alpha'^{[n]})$$

for α' of rank $-\mathrm{rk}(\alpha)-1$ was proposed by Johnson in [J] and further explicated in [MOP1, Conj. 1]. This relationship is used in [MOP1, MOP2] to obtain conjectural formulas for $V_S(\alpha)$ for α of rank $-3,-2,2$ or 3. It is expected that $V_S(\alpha)$ is an algebraic function of q for any α.

For general α_i, not much is known or conjectured about $\hat{Z}_S(\alpha_1,\ldots,\alpha_l)$ and their constituent universal series (1.11); one framework using vertex operators can be found in [Z]. Our approach yields a new combinatorial expression for the series $\hat{Z}_S(\alpha_1,\ldots,\alpha_l)$. However, it seems difficult to extract concisely stated consequences for the entire series, or even the Verlinde series. The individual coefficients in the m-variables $Z_S(\alpha_1,\ldots,\alpha_l\,|\,k_1,\ldots,k_l)$ studied in Theorem 1.1 seem more tractable from this perspective.

1.5. **Outline.** In Section 2, we introduce an equivariant affine analog $\hat{Z}_{\mathbb{C}^2}$ of the series $\hat{Z}_S$ and demonstrate that Theorems 1.1 and 1.2 follow from their equivariant analogs, Propositions 2.2 and 2.4. In Section 3.1, we obtain Propositions 2.2 and 2.4 via the combinatorial identity Proposition 3.1. This identity is a minor modification of a result obtained in [M] from a result of [GHT].

1.6. **Acknowledgments.** I thanks Anton Mellit and Richard Thomas for related conversations, as well as Drew Johnson, Woonam Lim, Dragos Oprea and Rahul Pandharipande for discussions and correspondence regarding Hilbert and Quot schemes. In particular, I thank Dragos Oprea for his suggestion to study the series (1.3). This work was supported by the EPSRC through grant EP/R013349/1 and the NSF through grant DMS-1902717.

2. Descendent series from equivariant descendents

2.1. **Equivariant descendents.** The following notation will be useful: for a variable or constant m and a K-theory class α set

$$\wedge^{\bullet}_{m}\alpha = \sum_{k=0}^{\infty}(-m)^k \wedge^k \alpha.$$

Consider $\mathbb{C}^2$ equipped with the action of a torus $T = \mathrm{diag}(t_1, t_2)$ scaling the coordinate axes with weights t_1^{-1} and t_2^{-1}. The action of T on $\mathbb{C}^2$ lifts to an action on the Hilbert schemes $(\mathbb{C}^2)^{[n]}$. The definition (1.1) is also valid in the equivariant setting; in this way, a class $\gamma \in K_T(\mathbb{C}^2)$ induces a tautological class $\gamma^{[n]} \in K_T((\mathbb{C}^2)^{[n]})$.

Definition (1.10) can be extended to the equivariant setting. Given $\gamma_1, \ldots, \gamma_l \in K_T(\mathbb{C}^2)$, define

$$\hat{Z}_{\mathbb{C}^2}(\gamma_1, \ldots, \gamma_l)(t_1, t_2) = \sum_{n\geq 0} q^n \chi\Big((\mathbb{C}^2)^{[n]}, \wedge^{\bullet}_{m_1}\gamma_1^{[n]} \otimes \ldots \otimes \wedge^{\bullet}_{m_l}\gamma_l^{[n]}\Big) \in \mathbb{Q}(t_1, t_2)[[q, m_1, \ldots, m_l]].$$

Here, each term on the right-hand side is an equivariant Euler characteristic and can be regarded as a rational function on T.

2.2. **Localization on the Hilbert scheme.** We recall the following special case of K-theoretic equivariant localization. Let M be a smooth complex variety equipped with an action of a complex torus $\mathbb{T}$ such that the fixed locus $M^{\mathbb{T}}$ is a nonempty finite set of points and let $\mathscr{F} \in K_{\mathbb{T}}(M)$.

Proposition 2.1. [T, Thm 3.5] *There is an equality of* $\mathbb{T}$*-equivariant Euler characteristics*

$$\chi(M, \mathscr{F}) = \sum_{p\in M^{\mathbb{T}}} \chi\Big(p, \frac{\mathscr{F}|_p}{\wedge^{\bullet}_{1}\mathcal{T}^* M|_p}\Big) \in \mathbb{Q}(\mathbb{T}). \tag{2.1}$$

When applied to the Hilbert scheme of points, (2.1) yields a combinatorial description of $\hat{Z}_{\mathbb{C}^2}(\gamma_1, \ldots, \gamma_l)$. We record this description.

A *Young diagram* λ is a finite subset of $\mathbb{Z}^2_{\geq 0}$ satisfying the following property: if $(c_1, c_2) \in \lambda$, then for any $(c_1', c_2') \in \mathbb{Z}^2_{\geq 0}$ such that $c_1' \leq c_1$ and $c_2' \leq c_2$, one also has $(c_1', c_2') \in \lambda$.

We can associate to a Young diagram λ the point $p_\lambda \in (\mathbb{C}^2)^{[|\lambda|]}$ cut out by the monomial ideal

$$\mathrm{Span}\,\{x_1^{b_1} x_2^{b_2} \mid (b_1, b_2) \notin \lambda\} \subset \mathbb{C}[x_1, x_2] = H^0(\mathcal{O}_{\mathbb{C}^2}).$$

The T-fixed locus of $(\mathbb{C}^2)^{[n]}$ consists of the points p_λ with λ of size n.

For $\gamma \in K_T(\mathbb{C}^2)$, let $\chi(\gamma|_0) \in \mathbb{Z}[t_1^\pm, t_2^\pm]$ denote the T-character of the fiber of γ over the origin. For λ of size n, the fiber $\gamma^{[n]}|_{p_\lambda}$ has T-character

$$\chi(\gamma|_0) \cdot \sum_{(c_1,c_2)\in\lambda} t_1^{c_1} t_2^{c_2}.$$

In particular, if

$$\chi(\gamma|_0) = \sum_i v_i - \sum_j w_j,$$

where each v_i and w_j is a T-weight (a Laurent monomial in t_1 and t_2,) then the fiber $\wedge_m^\bullet \gamma^{[n]}|_{p_\lambda}$ has T-character

$$\mathrm{Exp}\Big[-m\chi(\gamma|_0) \cdot \sum_{(c_1,c_2)\in\lambda} t_1^{c_1} t_2^{c_2}\Big] = \prod_{(c_1,c_2)\in\lambda} \frac{\prod_i (1 - m v_i t_1^{c_1} t_2^{c_2})}{\prod_j (1 - m w_j t_1^{c_1} t_2^{c_2})};$$

the definition of the plethystic exponential Exp is recalled in (3.1).

Let λ be a Young diagram. Given $(c_1, c_2) \in \lambda$, define the *leg length* $l((c_1, c_2))$ to be the largest integer k for which $(c_1 + k, c_2) \in \lambda$ and the *arm length* $a((c_1, c_2))$ to be the largest integer k for which $(c_1, c_2 + k) \in \lambda$.

For λ of size n, the T-character of the fiber of the cotangent bundle $\mathcal{T}^*(\mathbb{C}^2)^{[n]}$ at p_λ is computed in [ES, Lem. 3.2] to be

$$\sum_{\square\in\lambda} t_1^{l(\square)+1} t_2^{-a(\square)} + t_1^{-l(\square)} t_2^{a(\square)+1}.$$

Set C_λ to be the T-character of $\wedge_1^\bullet \mathcal{T}^*(\mathbb{C}^2)^{[n]}|_{p_\lambda}$; explicitly,

$$C_\lambda = \mathrm{Exp}\Big[-\chi\big(p_\lambda, \mathcal{T}^*(\mathbb{C}^2)^{[n]}|_{p_\lambda}\big)\Big] = \prod_{\square\in\lambda} (1 - t_1^{l(\square)+1} t_2^{-a(\square)})(1 - t_1^{-l(\square)} t_2^{a(\square)+1}).$$

By (2.1), we conclude that

$$\hat{Z}_{\mathbb{C}^2}(\gamma_1, \ldots, \gamma_l)(t_1, t_2) = \sum_\lambda \frac{q^{|\lambda|}}{C_\lambda} \mathrm{Exp}\Big[-\Big(\sum_{j=1}^l m_j \cdot \chi(\gamma_j|_0)\Big) \cdot \sum_{(c_1,c_2)\in\lambda} t_1^{c_1} t_2^{c_2}\Big]. \tag{2.2}$$

In mathematical physics, series of the form (2.2) arise as rank 1 Nekrasov partition functions of 5-dimensional supersymmetric gauge theories with fundamental matter.

2.3. From $\mathbb{C}^2$ to a general surface.

2.3.1. *Arbitrary descendents.* Fix $\gamma_1,\dots,\gamma_l \in K_T(\mathbb{C}^2)$. As $\mathbf{a}=(a_1,\dots,a_l)$ ranges over $\mathbb{Z}^l_{\geq 0}$, let

$$g_{\mathbf{a}}(q) \in \mathbb{Q}(t_1,t_2)[[q]]$$

be the collection of series for which

$$\hat{Z}_{\mathbb{C}^2}(\gamma_1,\dots,\gamma_l)(t_1,t_2) = \mathrm{Exp}\Big[\frac{q}{(1-t_1)(1-t_2)}\Big]\sum_{\mathbf{a}} m_1^{a_1}\cdots m_l^{a_l} g_{\mathbf{a}}(q). \tag{2.3}$$

We remark that

$$\mathrm{Exp}\Big[\frac{q}{(1-t_1)(1-t_2)}\Big] = \hat{Z}_{\mathbb{C}^2}(\varnothing)(t_1,t_2),$$

which can be seen shown by Proposition 3.1 or otherwise.

We formulate the following equivariant analog of Theorem 1.1.

Proposition 2.2. *The series $g_{\mathbf{a}}(q)$ are polynomials in q. Moreover,*

$$\deg_q g_{\mathbf{a}} \leq a_1+\ldots+a_l.$$

Proposition 2.2 will be proved in Section 3.

Returning to our original problem, let S be a projective surface and let $\alpha_1,\dots,\alpha_l$ be classes in $K(S)$. Let $f_{\mathbf{a}}(q)\in\mathbb{Q}[[q]]$ be the collection of series for which

$$\hat{Z}_S(\alpha_1,\dots,\alpha_l) = \frac{1}{(1-q)^{\chi(\mathcal{O}_S)}}\sum_{\mathbf{a}} m_1^{a_1}\cdots m_l^{a_l} f_{\mathbf{a}}(q). \tag{2.4}$$

Proposition 2.2 implies the following rephrasing of Theorem 1.1.

Corollary 2.3. *The series $f_{\mathbf{a}}(q)$ appearing in (2.4) are polynomials in q. Moreover,*

$$\deg_q f_{\mathbf{a}} \leq a_1+\ldots+a_l.$$

Proof. By the factorization (1.12), it suffices to prove the corollary when S is toric and $\alpha_1,\dots,\alpha_l\in K(S)$ are torus-equivariant.

Suppose $T=\mathrm{diag}(t_1,t_2)$ acts on S with finitely many fixed points s_i. For each s_i, let w_{i_1} and w_{i_2} denote the cotangent weights at s_i, let U_i denote the toric chart centered at s_i, and set

$$(\alpha_j)_i = \alpha_j|_{U_i} \in K_T(U_i)\cong K_T(\mathbb{C}^2).$$

The action of T on S lifts to an action on $S^{[n]}$. By (2.1), there is the following equality of T-equivariant Euler characteristics

$$\sum_{n\geq 0} q^n\chi\Big(S^{[n]},\wedge^\bullet_{m_1}\alpha_1^{[n]}\otimes\ldots\otimes\wedge^\bullet_{m_l}\alpha_l^{[n]}\Big) = \prod_i \hat{Z}_{\mathbb{C}^2}((\alpha_1)_i,\dots,(\alpha_l)_i)(w_{i_1},w_{i_2}).$$

So, the (nonequivariant) series $\hat{Z}_S$ can be recovered from $\hat{Z}_{\mathbb{C}^2}$ as follows:

$$\hat{Z}_S(\alpha_1,\dots,\alpha_l) = \Big(\prod_i \hat{Z}_{\mathbb{C}^2}((\alpha_1)_i,\dots,(\alpha_l)_i)(w_{i_1},w_{i_2})\Big)\Big|_{t_1=1,t_2=1}. \tag{2.5}$$

Moreover, by T-equivariant localization on S, one has

$$\chi(\mathcal{O}_S) = \Big(\sum_i \frac{q}{(1-w_{i_1})(1-w_{i_2})}\Big)\Big|_{t_1=1,t_2=1}$$

so that

$$\begin{aligned}\Big(\prod_i \mathrm{Exp}\Big[\frac{q}{(1-w_{i_1})(1-w_{i_2})}\Big]\Big)\Big|_{t_1=1,t_2=1} &= \mathrm{Exp}\Big[\Big(\sum_i \frac{q}{(1-w_{i_1})(1-w_{i_2})}\Big)\Big|_{t_1=1,t_2=1}\Big] \\ &= \mathrm{Exp}[q\chi(\mathcal{O}_S)] \\ &= \frac{1}{(1-q)^{\chi(\mathcal{O}_S)}}. \qquad (2.6)\end{aligned}$$

By (2.6), the specialization at $t_1 = 1, t_2 = 1$ of the product of prefactors of each term on the right-hand side of (2.5) matches the denominator of the right-hand side of (2.4). Applying Proposition 2.2 to each factor in (2.5), we obtain the corollary. □

2.3.2. *Descendents with α_1 a line bundle.* We now turn to Theorem 1.2, which we will also deduce from an equivariant analog, Propostion 2.4. The argument is a slightly more intricate version of that of Section 2.3.1.

Now, fix $\gamma_1, \ldots, \gamma_l \in K_T(\mathbb{C}^2)$ such that γ_1 is the class of a equivariant line bundle. Set

$$u = \chi(\gamma_1|_0) \in T^\vee.$$

As $\tilde{\mathbf{a}} = (a_2, \ldots, a_l)$ ranges over $\mathbb{Z}_{\geq 0}^{l-1}$, let

$$\tilde{g}_{\tilde{\mathbf{a}}}(q, m_1) \in \mathbb{Q}(t_1, t_2)[[q, m_1]]$$

be the series such that

$$\hat{Z}_{\mathbb{C}^2}(\gamma_1, \ldots, \gamma_l) = \mathrm{Exp}\Big[\frac{q - qm_1u}{(1-t_1)(1-t_2)}\Big] \sum_{\tilde{\mathbf{a}}} m_2^{a_2} \cdots m_l^{a_l} \tilde{g}_{\tilde{\mathbf{a}}}. \qquad (2.7)$$

Proposition 2.4. *The series $\tilde{g}_{\tilde{\mathbf{a}}}(q, m_1)$ are Laurent expansions of rational functions in q and m_1 of the form*

$$\frac{s(t_1^\pm, t_2^\pm, q, m_1)}{\prod_w (1-w) \prod_{w'} (1 - m_1 w')},$$

where s is a polynomial and w and w' range over finitely many T-weights.

Proposition 2.4 will also be proved in Section 3.

Now, let S be a projective surface, and fix $\alpha_1, \ldots, \alpha_l \in K(S)$ such that α_1 is the class of a line bundle. Let $\tilde{f}_{\tilde{\mathbf{a}}}(q, m_1) \in \mathbb{Q}[[q, m_1]]$ be the collection of series for which

$$\hat{Z}_S(\alpha_1, \ldots, \alpha_l) = \frac{(1-qm)^{\chi(\alpha_1)}}{(1-q)^{\chi(\mathcal{O}_S)}} \sum_{\tilde{\mathbf{a}}} m_2^{a_2} \cdots m_l^{a_l} \tilde{f}_{\tilde{\mathbf{a}}}(q, m_1).$$

Corollary 2.5. *The series $f_{\tilde{\mathbf{a}}}(q)$ is the expansion in q and m_1 of a rational function whose denominator is a power of $(1 - qm_1)$.*

Proof. Again, it suffices to prove the corollary for toric S and torus equivariant $\alpha_1, \dots, \alpha_l \in K(S)$. Let $s_i, w_{i_1}, w_{i_2}, U_i$ and $(\alpha_j)|_i$ be as in the proof of Corollary 2.3. By equivariant localization on S, we have

$$\begin{aligned}\Big(\prod_i \mathrm{Exp}\Big[\frac{q - qm(\alpha_1)|_i}{(1-w_{i_1})(1-w_{i_2})}\Big]\Big)\Big|_{t_1=1,t_2=1} &= \mathrm{Exp}\Big[\Big(\sum_i \frac{q - qm(\alpha_1)|_i}{(1-w_{i_1})(1-w_{i_2})}\Big)\Big|_{t_1=1,t_2=1}\Big] \\ &= \mathrm{Exp}[q\chi(\mathcal{O}_S) - qm\chi(\alpha_1)] \\ &= \frac{(1-qm)^{\chi(\alpha_1)}}{(1-q)^{\chi(\mathcal{O}_S)}}. \end{aligned} \tag{2.8}$$

Now, plug equation (2.7) into (2.5) and use (2.8) to combine prefactors. Putting the remaining expressions over a common denominator as needed, given some $\tilde{\mathbf{a}}$ we may write

$$\tilde{f}_{\tilde{\mathbf{a}}} = \frac{r(t_1^{\pm}, t_2^{\pm}, q, m_1)}{\prod_v (1-v) \prod_{v'}(1-qm_1v')}\Big|_{t_1=1,t_2=1}$$

where r is a polynomial and v and v' range over finitely many T-weights. In particular, if the rational function

$$\frac{r(t_1^{\pm}, t_2^{\pm}, q, m_1)}{\prod_v (1-v) \prod_{v'}(1-qm_1v')}$$

is expanded in positive powers of m_1 and q, then each $q^k m_1^{k'}$-coefficient of the resulting series is well defined under the specialization $t_1 = 1, t_2 = 1$. We conclude that each $q^k m_1^{k'}$ coefficient has no poles of the form $1 - v$. By induction on the degree $k + k'$, it follows that the quotient

$$\frac{r(t_1^{\pm}, t_2^{\pm}, q, m_1)}{\prod_v (1-v)}$$

has no poles in t_1 and t_2. We conclude that

$$\begin{aligned}\tilde{f}_{\tilde{\mathbf{a}}} &= \frac{r(t_1^{\pm}, t_2^{\pm}, q, m_1)}{\prod_v (1-v)}\Big|_{t_1=1,t_2=1} \cdot \frac{1}{\prod_{v'}(1-qm_1v')}\Big|_{t_1=1,t_2=1} \\ &= \frac{r(t_1^{\pm}, t_2^{\pm}, q, m_1)}{\prod_v (1-v)}\Big|_{t_1=1,t_2=1} \cdot \frac{1}{\prod_{v'}(1-qm_1)}.\end{aligned}$$

□

3. A Macdonald identity

3.1. **Plethystic notation.** Proposition 2.2 follows from a slight modification of a Macdonald polynomial identity obtained in [M, Sec. 7], where the identity is applied to find symmetries among conjectural expressions for mixed Hodge polynomials of certain character varieties. We recall this identity following the presentation in [M].

Let p_n denote the n-th power sum and let Sym denote the completion of the ring $\mathbb{Q}(t_1, t_2)[p_1, p_2, \cdots]$ of symmetric functions over $\mathbb{Q}(t_1, t_2)$ with respect to degree. We use the following "plethystic notation:" given a Laurent series

$$X = \sum_{\mathbf{k}} c_{\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

where each $c_{\mathbf{k}} \in \mathbb{Q}$ and each $\mathbf{x}^{\mathbf{k}}$ is a Laurent monomial with coefficient 1 in t_1, t_2 and some additional variables, set

$$p_n[X] = \sum_{\mathbf{k}} c_{\mathbf{k}} (\mathbf{x}^{\mathbf{k}})^n.$$

For arbitrary $F \in \mathrm{Sym}$, the value of the expression $F[X]$ is defined by stipulating that the assignment $F \mapsto F[X]$ is a ring homomorphism. The plethystic exponential Exp is defined as

$$\mathrm{Exp}[X] = \exp\Big(\sum_{n=0}^{\infty} \frac{p_n}{n}[X]\Big) = 1 + p_1[X] + \frac{p_2 + p_1^2}{2}[X] + \cdots . \tag{3.1}$$

In particular, if x_i and y_j are Laurent monomials with coefficient 1, then

$$\mathrm{Exp}\Big[\sum_i x_i - \sum_j y_j\Big] = \frac{\prod_j (1 - y_j)}{\prod_i (1 - x_i)},$$

where infinite products are taken in a suitable completion.

3.2. **Macdonald polynomials.** For a Young diagram λ, let $H_\lambda \in \mathrm{Sym}$ denote the corresponding Macdonald polynomial as normalized, for example, in [GH, I.11]. Equivalent definitions may be found in [GH, Thm 2.2] and [M, Def. 4.1]; note that the Macdonald polynomials are denoted $\tilde{H}_\lambda$ in [GH] and [GHT].

We summarize the relevant properties of the symmetric polynomials H_λ.

- The polynomial H_λ is homogeneous of degree $|\lambda|$.
- The polynomials H_λ form a $\mathbb{Q}(t_1, t_2)$-basis of the space of symmetric functions.
- By [GH, Cor. 2.1], they take the following special values:

$$H_\lambda[1 - x] = \mathrm{Exp}\Big[-x \cdot \sum_{(c_1,c_2)\in\lambda} t_1^{c_1} t_2^{c_2}\Big] = \prod_{(c_1,c_2)\in\lambda} (1 - x t_1^{c_1} t_2^{c_2}). \tag{3.2}$$

  In particular,

$$H_\lambda[1] = 1 \qquad H_\lambda[-1] = (-1)^{|\lambda|} \prod_{(c_1,c_2)\in\lambda} t_1^{c_1} t_2^{c_2}. \tag{3.3}$$

- The ring Sym carries a scalar product $\langle\,,\rangle_*$, called the Macdonald scalar product, for which

$$\langle H_\lambda, H_\mu \rangle_* = \delta_{\lambda,\mu} \cdot H_\lambda[-1] \cdot C_\lambda.$$

  In particular, the scalar product $\langle\,,\rangle_*$ has reproducing kernel

$$\sum_\lambda \frac{H_\lambda[X] H_\lambda[Y]}{H_\lambda[-1] \cdot C_\lambda}. \tag{3.4}$$

3.3. **A plethystic symmetry.** We will deduce Proposition 2.2 from a special case of the following symmetry.

Proposition 3.1. *cf.* [M, Sec. 7]. *For Laurent series X and Y, the expression*

$$\mathrm{Exp}\Big[\frac{Y}{(1-t_1)(1-t_2)}\Big]\sum_\lambda \frac{H_\lambda[X]}{C_\lambda}\mathrm{Exp}\Big[-Y\cdot\sum_{(c_1,c_2)\in\lambda} t_1^{c_1}t_2^{c_2}\Big] \tag{3.5}$$

is symmetric under exchange of X and Y.

Proof. We slightly modify the proof from [M], which uses the Macdonald polynomial identity [GHT, Thm I.3]. Denote by U, U^* and $\nabla : \mathrm{Sym}\to\mathrm{Sym}$ the operators

$$(UF)[X] = F[1+X],\quad (U^*F)[X] = \mathrm{Exp}\Big[-\frac{X}{(1-t_1)(1-t_2)}\Big]F[X],\quad \nabla H_\lambda = H_\lambda[-1]\cdot H_\lambda.$$

Then, [GHT, Thm I.3] states that

$$(\nabla U^* U)H_\lambda[X] = \mathrm{Exp}\Big[\frac{X}{(1-t_1)(1-t_2)}\Big]\mathrm{Exp}\Big[-X\cdot\sum_{(c_1,c_2)\in\lambda} t_1^{c_1}t_2^{c_2}\Big]. \tag{3.6}$$

In other words, the composition $\nabla U^* U$ sends Macdonald polynomials to normalized T-weights of tautological bundles at fixed points on the Hilbert scheme.

By [GHT, Prop. 1.11b], the operators U and U^* are adjoint with respect to $\langle\,,\rangle_*$. Moreover, the operator ∇ is self-adjoint. So, the operator $\nabla U^* U\nabla$ is self-adjoint. As (3.4) is the reproducing kernel for $\langle\,,\rangle_*$, we have

$$(\nabla U^* U\nabla)_Y\Big[\sum_\lambda \frac{H_\lambda[X]H_\lambda[Y]}{H_\lambda[-1]C_\lambda}\Big] = (\nabla U^* U\nabla)_X\Big[\sum_\lambda \frac{H_\lambda[X]H_\lambda[Y]}{H_\lambda[-1]C_\lambda}\Big], \tag{3.7}$$

where the subscript X or Y denotes action on symmetric functions taking that argument.

By (3.6), the left-hand side of (3.7) equals

$$\mathrm{Exp}\Big[\frac{Y}{(1-t_1)(1-t_2)}\Big]\sum_\lambda \frac{H_\lambda[X]}{C_\lambda}\mathrm{Exp}\Big[-Y\cdot\sum_{(c_1,c_2)\in\lambda} t_1^{c_1}t_2^{c_2}\Big].$$

As (3.4) is symmetric under exchange of X and Y, the proposition follows. □

It would be interesting to have a geometric interpretation of Proposition 3.1.

3.4. **Specialization.** To prove Proposition 2.2 and Proposition 2.4, we apply Proposition 3.1 to control the series $\hat{Z}_{\mathbb{C}^2}(\gamma_1,\ldots,\gamma_l)$. With (2.2) in mind, for $j=1,\ldots,l$, we set

$$u_j = \chi(\gamma_j|_0)\in\mathbb{Z}[t_1^\pm, t_2^\pm]$$

to be the T-character of the fiber of γ_j at the origin $0\in\mathbb{C}^2$.

3.4.1. *Arbitrary descendents.* Apply the specialization

$$X = q, \quad Y = \sum_{j=1}^{l} m_j u_j$$

in expression (3.5). By (3.3, 3.2) and (2.2), this specialization equals

$$\begin{aligned}
&\mathrm{Exp}\Big[\frac{\sum_{j=1}^{l} m_j u_j}{(1-t_1)(1-t_2)}\Big]\sum_{\lambda}\frac{q^{|\lambda|}}{C_\lambda}\mathrm{Exp}\Big[-\big(\sum_{j=1}^{l} m_j u_j\big)\cdot\sum_{(c_1,c_2)\in\lambda} t_1^{c_1}t_2^{c_2}\Big]\\
&= \Big[\frac{\sum_{j=1}^{l} m_j u_j}{(1-t_1)(1-t_2)}\Big]\hat{Z}_{\mathbb{C}^2}(\gamma_1,\ldots,\gamma_l)(t_1,t_2). \qquad (3.8)
\end{aligned}$$

Applying Proposition 3.1, we find that the series (3.8) equals

$$\mathrm{Exp}\Big[\frac{q}{(1-t_1)(1-t_2)}\Big]\sum_{\lambda}\frac{H_\lambda\big(\sum_{j=1}^{l} m_j u_j\big)}{C_\lambda}\prod_{(c_1,c_2)\in\lambda}(1-qt_1^{c_1}t_2^{c_2}).$$

Recall the definition of $g_{\mathbf{a}}$ from (2.3). We conclude that

$$\begin{aligned}
&\sum_{\mathbf{a}} m_1^{a_1}\cdots m_l^{a_l} g_{\mathbf{a}}(q)\\
&= \mathrm{Exp}\Big[-\frac{\sum_{j=1}^{l} m_j u_j}{(1-t_1)(1-t_2)}\Big]\sum_{\lambda}\frac{H_\lambda\big(\sum_{j=1}^{l} m_j u_j\big)}{C_\lambda}\prod_{(c_1,c_2)\in\lambda}(1-qt_1^{c_1}t_2^{c_2}). \qquad (3.9)
\end{aligned}$$

As H_λ is homogeneous of degree $|\lambda|$, only partitions λ of size at most $a_1+\ldots+a_l$ can contribute $m_1^{a_1}\cdots m_l^{a_l}$-terms to the right-hand side of (3.9). As the largest power of q that can appear in the λ-summand of (3.9) is $|\lambda|$, Proposition 2.2 follows.

We remark that for any fixed $\mathbf{a}$, equation (3.9) yields a closed form expression for $g_{\mathbf{a}}$.

3.4.2. *Descendents with γ_1 a line bundle.* When γ_1 is the class of a line bundle, Proposition 3.1 yields extra information about the structure of $\hat{Z}_{\mathbb{C}^2}(\gamma_1,\ldots,\gamma_l)$.

By the homogeneity of H_λ and (3.2), we have

$$H_\lambda[q-qm] = q^{|\lambda|}H_\lambda[1-m] = q^{|\lambda|}\cdot\mathrm{Exp}\Big[-m\sum_{(c_1,c_2)\in\lambda} t_1^{c_1}t_2^{c_2}\Big].$$

As γ_1 is a line bundle, the T-character $u_1 = \chi(\gamma_1|_0)$ is a Laurent monomial (with coefficient 1) in t_1 and t_2. So, applying the specialization

$$X = q - qm_1u_1, \quad Y = \sum_{j=2}^{l} m_j u_j$$

in expression (3.5), we obtain

$$\mathrm{Exp}\Big[\frac{\sum_{j=2}^{l} m_j u_j}{(1-t_1)(1-t_2)}\Big]\sum_{\lambda}\frac{q^{|\lambda|}}{C_\lambda}\mathrm{Exp}\Big[-\big(\sum_{j=1}^{l} m_j u_j\big)\cdot\sum_{(c_1,c_2)\in\lambda} t_1^{c_1}t_2^{c_2}\Big]$$
$$=\mathrm{Exp}\Big[\frac{\sum_{j=2}^{l} m_j u_j}{(1-t_1)(1-t_2)}\Big]\hat{Z}_{\mathbb{C}^2}(\gamma_1,\ldots,\gamma_l)(t_1,t_2). \tag{3.10}$$

By Proposition 3.1, the series (3.10) equals

$$\mathrm{Exp}\Big[\frac{q-qm_1u_1}{(1-t_1)(1-t_2)}\Big]\sum_{\lambda}\frac{H_\lambda\big(\sum_{j=2}^{l} m_j u_j\big)}{C_\lambda}\prod_{(c_1,c_2)\in\lambda}\frac{1-qt_1^{c_1}t_2^{c_2}}{1-qm_1u_1t_1^{c_1}t_2^{c_2}}.$$

Recall the definition of $\tilde{g}_{\tilde{\mathbf{a}}}$ from (2.7). We conclude that

$$\sum_{\mathbf{a}} m_2^{a_2}\cdots m_l^{a_l}\tilde{g}_{\tilde{\mathbf{a}}}(q,m_1)$$
$$=\mathrm{Exp}\Big[-\frac{\sum_{j=2}^{l} m_j u_j}{(1-t_1)(1-t_2)}\Big]\sum_{\lambda}\frac{H_\lambda\big(\sum_{j=2}^{l} m_j u_j\big)}{C_\lambda}\prod_{(c_1,c_2)\in\lambda}\frac{1-qt_1^{c_1}t_2^{c_2}}{1-qm_1u_1t_1^{c_1}t_2^{c_2}}. \tag{3.11}$$

Note that only partitions of size at most $a_2+\ldots+a_l$ can contribute $m_2^{a_2}\cdots m_l^{a_l}$-terms to the left hand side of 3.11. Proposition 2.4 follows.

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