Functional calculus for càdlàg paths and applications to model-free finance

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Abstract

This thesis synthesise my research on analysis and control of path-dependent random systems under uncertainty. In the first chapter, we revisit Föllmer's concept of pathwise quadratic variation for a càdlàg path and show that his definition can be reformulated in terms of convergence of quadratic sums in the Skorokhod topology. This new definition is simpler and amenable to define higher order variation for a càdlàg path.

In the second chapter, we introduced a new topology for functionals and adopted an abstract formulation of Functional calculus on generic domain based on the differentials introduced by Dupire (2009), Cont & Fournié (2010). Our aim is not to generalise an existing rich theory for irregular paths e.g. Lyons (1998), Friz & Hairer (2014) but to introduce a bespoke and yet versatile calculus for causal random system in general and mathematical finance in particular, in order to solve problems practically as well as bring in new aspects under uncertainty.

In the final chapter, we apply functional calculus to study mathematical finance under uncertainty. We first show that every self-financing portfolio can be represented by a pathwise integral and that every generic market is arbitrage free, a fundamental property that is linked to the solution, which is characterised by a fully non-linear path dependent equation, to the optimal hedging problem under uncertainty. In particular, we obtain explicit solution for the Asian option.

To my family and friends and to those who lost their loved ones during the pandemic.

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Contents

Overview

Uncertainty without measures

In his classic book [12], Knight (1921) distinguishes *risk* and *uncertainty*. As Föllmer & Schied [9] put it succinctly: risk corresponds to ”known unknowns”, while uncertainty corresponds to ”unknown unknowns”. That is, if we were even in the position to set the odds in the first place. There exists many examples in which one cannot construct a reasonable probabilistic model, most notably De Finetti's lottery (see Bingham [3, §3] for an insightful exposition).

Föllmer & Schied further pointed out two approaches to incorporate uncertainty into continuous-time finance: the probabilistic, quasi sure approach and the probability free, pathwise approach, which may be further classified into measure theoretic, e.g. Vovk [20, 21] (2015 & '17) and Lochowski, Perkowski & Prömel [13] (2018) and non-measure theoretic e.g. Bick & Willinger [1] (1994), Lyon [14] (1995) and Schied & Voloshchenko [18] (2016).

Our approach here will be pathwise and non-measure theoretic (hence non-probabilistic), in favor of transparency and constructiveness. We will further incorporate (*de facto*) uncertainties on the continuous evolution and the variation order of price paths. The aim is to gain an impartial understanding on the inter-relationships among the fundamentals: (1) variation order of price paths, (2) arbitrage, (3) market incompleteness and (4) the optimal hedging strategy under uncertainty.

We introduce a new version of causal functional calculus [5] based on Dupire's functional derivatives* [7] and a generic approach to pathwise integration, without a priori knowledge of the variation index of the underlying paths.

*Obtained by a vertical perturbation in space (resp. a horizontal extension in time) of a path, see § 2.4 for details.

Our approach establishes an equivalence relation between the value of a self-financing portfolio (mod initial capital) on one hand and a sub-class of path-wise integrals on the other.

On generic pathwise integration

Chapter 1[†] shows that Föllmer 's quadratic variation of a càdlàg path x may be defined (equivalently) by the convergence of sums of square increments:

$$\sum_{\pi_n \ni t_i \leq t} (x(t_{i+1}) - x(t_i))^{\otimes p},$$

for $p = 2$ in the Skorokhod topology along a sequence of time partitions π_n vanishing on compacts. Clearly, this may be extended to pth order variation for a càdlàg path in the direction of Cont & Perkowski [6] (2019) for $p \in 2\mathbb{N}$ and a pth order change of variable formula may be obtained. However, we did not pursue this direction for reasons that shall be clear in the sequel.

The one-form approach

For decades, Föllmer 's theory of pathwise integration [8] (1981) and its extensions may best be understood by one-form, that is if

$$\phi \in \mathcal{L} := \{\nabla_x f | f \in C^{1,2}\},$$

then $\int \phi dx$ is defined as (uncompensated) limit of left Riemann sums i.e.

$$\int_0^t \phi dx := \lim_n \sum_{\pi_n \ni t_i \leq t} \phi(t_i, x(t_i)) \left(x(t_{i+1}) - x(t_i) \right), \tag{1}$$

along a vanishing sequence of time partitions π_n of which x admits finite quadratic variation. This integration approach has been extended in multiple directions (under the uniform topology). Most notably, the extension for $C^{1,p}$ functionals driven by continuous paths of pth order variation [6].

[†]See [4] (2018) for the published version

The notion of an integral as path-by-path limit of left Riemann sums is fundamental in mathematical finance due to its interpretation as the change in one's financial position after a series of bets $\phi(t_i, x(t_i))$ placed against price movements $x(t_{i+1}) - x(t_i)$. This interpretation of (1) would be compromised if the convergence in (1) must be compensated for. This would in general, be the case if one were to extend the results of [6] to càdlàg paths of higher order variation. An important question remains however: How to characterise the class of uncompensated pathwise integals?

The generic approach

Looking at (1) from a different perspective, we observe that Föllmer 's integral, i.e. $\phi \in \mathcal{L}$;

$$F(t,x) := \int_0^t \phi(s, x(s-))dx(s)$$

is a *functional*, well defined everywhere at which x admits finite quadratic variation. Further, there exists a bespoke topology defined on a suitable domain which makes F a continuous functional. In particular, Föllmer 's integral possesses the following differential characteristics (unbeknown to the existing literature):

$$\mathcal{D}_t F = \nabla_x^{(p)} F = 0 \tag{2}$$

for all $p \geq 2$. The financial meaning is clear, the bet $\nabla_x F(t,x) = \phi(t, x(t-))$ placed, is interpreted here as a "prevision" and is therefore "blind" to a perturbation of future price movement. On the other hand, there won't be any change in the value of the portfolio without price variation, hence $\mathcal{D}_t F$ vanishes.

In fact, all continuous functionals that satisfy (2) are limits of left Riemann sums as well as canonical solutions (though discontinuous in the uniform topology) to the path-dependent heat equation. This class of functional is called $\mathcal{M}$. The most crucial observation is that such a characterisation can be extended to other generic domains that include, but not limited to, paths of $p-$th order variation.

In order to show that all (uncompensated) left Riemann sums that may possibly be characterised by the one-form approach (i.e. associated with a p-th order change of variable formula for $p \geq 2$) are in class $\mathcal{M}$, we deliberately start with an explicit definition of a pathwise integral on a generic domain, *unassociated* with any change of variable formula and shows that it belongs to class $\mathcal{M}_0$ (i.e. $\mathcal{M}$ mod initial value) and vice versa. In doing so, we also recover the corresponding analogues of the classical fundamental theorem of calculus I & II, which facilitates the evaluation of a pathwise integral. Most importantly, we are able to show that this generic notion of pathwise integral embodies the notion of a self-financing portfolio (mod initial capital) in mathematical finance.

From martingale to Isaacs' equation

What is a martingale?

On an occasion, Rama Cont spoke about the notion of a *probability free* martingale and suspected its relation with harmonic functionals (defined earlier in [5]). Upon investigation, we reckoned that the notion of a martingale preceded that of a probability measure. As the progress on the abstract formulation of functional calculus is made and further investigations into the origin of martingale is conducted, tracing back to the early literature[‡] on collectives by e.g. von Mises, Ville, de Finetti among others, we have come to an understanding of the generic notion of a martingale from a non-measure theoretic[§] point of view, in the context of functional calculus.

The term *martingale* was originally introduced to prove the impossibility of successful betting strategies in a game. The term *martingale* is synonymous to the term *fair game*. A game is fair if one cannot formulate a betting strategy to make sure profit. The value of a self-financing portfolio V is the value of a game (against the financial market). This game is fair on the set

[‡]See Bienvenu, Shafer & Shen [2] (2009) for an insightful exposition on the subject.
[§]The reformulation of Ville's by Shafer & Vovk [19] (2001) is measure theoretic.

of outcomes Ω if whenever there exists $T > 0$;

$$V(T,x) - V(0,x) \geq 0 \Longrightarrow V(T,x) - V(0,x) = 0 \tag{3}$$

for all $x \in \Omega$.

Since the value of a self-financing portfolio is a functional that belongs to a sub-class[¶] of $\mathcal{M}$ and that all functionals of class $\mathcal{M}$ satisfies (Thm. 2.5.13) the no arbitrage condition (3), thus the sub-class are *martingales* on generic domains.

In contrast to related results established using the measure-theoretic, game approach of Vovk et al. [19, 20, 13, 21], we are able to work with the classical notion of arbitrage, rather than passing to an asymptotic relaxation (i.e. vanishing risk) that may not necessarily be implementable by a *self-financing* trading strategy. This makes it possible to define a pathwise analogue of the concept of martingale on a space of paths of *arbitary regularity*.

Isaacs' equation under uncertainty

In the last chapter, we study mathematical finance under uncertainty on generic domains. We show that every self-financing portfolio is a functional of class $\mathcal{M}$ and hence is free of arbitrage in every generic market. For a non-linear payoff, a perfect hedge does not exist. We adopt a more general approach of (super)hedging on bounded subsets of paths.

We apply Isaacs principle of transition [11, p3] for differential games to formulate the (super)hedging problem by optimal control. This is accompanied by a verification theorem for the optimal solution, which solves a fully non-linear path-dependent equation, the Isaacs' equation under uncertainty (3.22). For the Asian option, the optimal solution is obtained explicitly.

Martingale is key in deriving Isaacs' equation and is best illustrated by the following causal relations:

[¶]A class $\mathcal{M}$ functional whose derivative admits right limits, we refer to Thm. 3.4.3 & Prop. 3.4.4 for details.

martingale→ no arbitrage → minimax theorem → Isaacs' equation.

Our final remark is that this problem is similar to that of (probabilistic) quasi sure analysis [16, 17]. However, our approach is much more robust (holds for all generic domains). Also, we are able to by-pass many technical difficulties (e.g. not having to deal with duality gap and polar set). Finally, our optimal hedging strategy (delta) comes as a by-product, whereas in the quasi-sure approach, it will not be straightforward to compute the optimal hedge.

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1

On pathwise quadratic variation for càdlàg functions

Overview

We revisit Föllmer 's notion of quadratic variation along a sequence of time partitions for multidimensional càdlàg functions and investigate its connection with the Skorokhod topology. To obtain a robust notion of pathwise quadratic variation applicable to sample paths of càdlàg processes, we reformulate the definition of pathwise quadratic variation as a limit in Skorokhod topology of discrete approximations along the partition. One then obtains an equivalent but simpler definition which implies the Lebesgue decomposition of quadratic variation, rather than requiring it as an extra condition.

This chapter is based on

H Chiu, R Cont (2018) On pathwise quadratic variation for càdlàg functions, *Electronic Communications in Probability*, Volume 23, paper no. 85.

1.1 Introduction

Quadratic variation is a fundamental object of study in stochastic analysis, if only because it sits in the celebrated Itô's formula. It was later discovered by Hans Föllmer that the Itô calculus can be applied pathwise to $\mathcal{C}^2$ functions of a càdlàg path without using any probabilistic machinery. To do so, it is required that the càdlàg path be of *finite quadratic variation*, with the additional requirements that the quadratic variation satisfies a specific *Lebesgue decomposition* (LD). The extension of this notion to the multidimensional setting is subtle [3, Rem.(1)], since, as noted by Schied [5], the space of such paths is not a vector space, leading to a different definition from the one dimensional case.

This chapter investigates Föllmer 's notion of quadratic variation along a sequence of time partitions for multidimensional càdlàg functions and investigate its connection with the Skorokhod topology. To obtain a robust notion of pathwise quadratic variation applicable to sample paths of càdlàg processes, we reformulate the definition of pathwise quadratic variation as a limit in the Skorokhod topology of discrete approximations along the partition.

In particular, we prove that quadratic variation is the limit in Skorokhod topology of the quadratic sums of a càdlàg function and vice versa without imposing the Lebesgue decomposition condition (LD) on the limit. This results in a simpler definition of quadratic variation for a càdlàg path on $[0, \infty)$ taking values in $\mathbb{R}^n$.

Outline In section 1.2, we introduce Föllmer's notion of pathwise quadratic variation and show how quadratic variation is naturally connected to the Skorokhod topology (Theorems A& B) and obtain an equivalent definition. In section 1.3, we extend our results to the multidimensional case.

Throughout this article, $\pi := (\pi_n)_{n\geq 1}$ is a fixed sequence of partitions $\pi_n = (t_0, ..., t_{n+1})$ of $[0, \infty)$ into intervals $0 \leq t_0 < ... < t_{n+1} < \infty$; $t_{n+1} \uparrow \infty$ with vanishing mesh $|\pi_n| \downarrow 0$ on compacts.

1.2 Quadratic variation for real càdlàg functions

We denote $\mathcal{D} := \mathcal{D}([0,\infty),\mathbb{R})$ to be the Skorokhod space and $\mathcal{C} := \mathcal{C}([0,\infty),\mathbb{R})$ the subspace of real-valued continuous functions. $\mathcal{D}$ shall be equipped with a metric d which induces the Skorokhod (a.k.a J_1) topology [2, VI]. Denote $\mathcal{D}_0^+ \subset \mathcal{D}$ to be the subset of non-negative increasing càd functions null at 0.

Definition 2 (Föllmer 1981) We say that $x \in \mathcal{D}$ has finite quadratic variation $[x]$ along π if the discrete measures

$$\mu_n := \sum_{t_i \in \pi_n} (x_{t_{i+1}} - x_{t_i})^2 \delta_{t_i} \tag{1.1}$$

converges vaguely to a Radon measure μ on $[0,\infty)$ and such that the distribution function $[x]$ of μ admits the Lebesgue decomposition (2), i.e.

$$[x]_t = [x]_t^c + \sum_{s \le t} (\Delta x_s)^2. \tag{1.2}$$

Denote $\mathcal{Q}_0^\pi$ to be the subset of those $x \in \mathcal{D}$ with finite quadratic variation.

Before we proceed, let us draw a link between vague and weak convergence of Radon measures on $[0,\infty)$, the link of which, is well known in the special case where the measures are sub-probability measures:

Lemma 3 Let v_n and v be non-negative Radon measures on $[0,\infty)$ and $J \subset [0,\infty)$ be the set of atoms of v, the followings are equivalent:

(i) $v_n \to v$ vaguely on $[0,\infty)$.
(ii) $v_n \to v$ weakly on $[0,T]$ for every $T \notin J$.

Proof. Let $f \in \mathcal{C}_K([0,\infty))$ be a compactly supported continuous function. Since J is countable, $\exists\ T \notin J$; $\mathrm{supp}(f) \subset [0,T]$. Now (ii) $\Rightarrow \int_0^\infty f dv_n = \int_0^T f dv_n \longrightarrow \int_0^T f dv = \int_0^\infty f dv \Rightarrow$ (i). Suppose (i) holds, let $T \notin J$ and

$f \in \mathcal{C}([0,T], \|\cdot\|_\infty)$. Since $f = (f)^+ - (f)^-$, we may take $f \geq 0$ and define the following extensions:

$$\begin{aligned}
\overline{f}^\epsilon(t) &:= f(t)1\!\!1_{[0,T]}(t) + f(T)\left(1 + \frac{T-t}{\epsilon}\right)1\!\!1_{(T,T+\epsilon]}(t) \\
\underline{f}^\epsilon(t) &:= f(t)1\!\!1_{[0,T-\epsilon]}(t) + f(T)\left(\frac{T-t}{\epsilon}\right)1\!\!1_{(T-\epsilon,T]}(t),
\end{aligned}$$

then $\overline{f}^\epsilon$, $\underline{f}^\epsilon \in \mathcal{C}_K([0,\infty))$, $0 \leq \underline{f}^\epsilon \leq f1\!\!1_{[0,T]} \leq \overline{f}^\epsilon \leq \|f\|_\infty$ and we have

$$\int_0^\infty \underline{f}^\epsilon dv_n \leq \int_0^T f dv_n \leq \int_0^\infty \overline{f}^\epsilon dv_n.$$

Since $v_n \to v$ vaguely and $T \notin J$, thus, as $n \to \infty$ we obtain

$$\begin{aligned}
0 &\leq \limsup_n \int_0^T f dv_n - \liminf_n \int_0^T f dv_n \leq \int_0^\infty \overline{f}^\epsilon - \underline{f}^\epsilon dv \\
&\leq \|f\|_\infty v((T-\epsilon, T+\epsilon]) \xrightarrow{\epsilon} 0,
\end{aligned}$$

hence by monotone convergence

$$\lim_n \int_0^T f dv_n = \lim_\epsilon \int_0^\infty \underline{f}^\epsilon dv = \int_0^T f dv$$

and (ii) follows. □

Proposition 4 If $x \in \mathcal{Q}_0^\pi$, then the pointwise limit s of

$$s_n(t) := \sum_{t_i \in \pi_n} (x_{t_{i+1}\wedge t} - x_{t_i \wedge t})^2 \tag{1.3}$$

exists and s admits the Lebesgue decomposition (2):

$$s(t) = s^c(t) + \sum_{s \leq t} (\Delta x_s)^2. \tag{1.4}$$

In this case $s = [x]$.

Proof. If $x \in Q_0^\pi$, define

$$q_n(t) := \sum_{\pi_n \ni t_i \le t} (x_{t_{i+1}} - x_{t_i})^2,$$

the distribution function of μ_n in (4). Since $\mu_n \to \mu$ vaguely, we have $q_n \to [x]$ pointwise at all continuity points of $[x]$ (Lemma 3 & [4, X.11]). Let I be the set of continuity points of $[x]$. Observe (q_n) is monotonic in $[0, \infty)$ and I is dense in $[0, \infty)$, if $t \notin I$, it follows [4, X.8] that

$$[x]_{t-} \le \liminf_n q_n(t) \le \limsup_n q_n(t) \le [x]_{t+} = [x]_t.$$

Thus, we may take any sub-sequence (n_k) such that $\lim_k q_{n_k}(t) =: q(t)$. Since $x \in Q_0^\pi$ and the Lebesgue decomposition (5) holds on $[x]$, we have

$$([x]_{t+\epsilon} \overset{\ge 0}{-} q(t)) + (q(t) \overset{\ge 0}{-} [x]_{t-\epsilon}) = [x]_{t+\epsilon} - [x]_{t-\epsilon} \overset{\epsilon}{\longrightarrow} (\Delta x_t)^2. \tag{1.5}$$

If $t \pm \epsilon \in I$, $\tilde{\pi}_k := \pi_{n_k}$ and $t_j^{(k)} := \max\{\tilde{\pi}_k \cap [0, t)\}$, the second sum in (8) is

$$\lim_k \sum_{\substack{t_i \in \tilde{\pi}_k; \\ t-\epsilon < t_i \le t}} (x_{t_{i+1}} - x_{t_i})^2 = \lim_k \sum_{\substack{t_i \in \tilde{\pi}_k; \\ t-\epsilon < t_i < t_j^{(k)}}} (x_{t_{i+1}} \overset{\ge 0}{-} x_{t_i})^2 + (\Delta x_t)^2 \ge (\Delta x_t)^2$$

by the fact that x is càdlàg and that $t \notin I$.

We see immediately from (8) that $q(t) = [x]_t$ as $\epsilon \to 0$. Since the choice of the sub-sequential limit $q(t)$ is arbitrary, we conclude that $q_n \to [x]$ pointwise on $[0, \infty)$. Observe that the pointwise limits of (s_n) and (q_n) coincide (11) by the right-continuity of x and that $x \in Q_0^\pi$, (Prop. 4) follows. □

Denote $\mathcal{Q}_1^\pi$ to be the subset of those $x \in \mathcal{D}$ such that the pointwise limit s of the quadratic sums (s_n) in (6) exists and the Lebesgue decomposition in (7) holds on s. Then $\mathcal{Q}_0^\pi \subset \mathcal{Q}_1^\pi$ and we have:

Proposition 5 If $x \in \mathcal{Q}_1^\pi$, then the pointwise limit q of

$$q_n(t) := \sum_{\pi_n \ni t_i \le t} (x_{t_{i+1}} - x_{t_i})^2 \tag{1.6}$$

exists and q admits the Lebesgue decomposition (2):

$$q(t) = q^c(t) + \sum_{s \le t} (\Delta x_s)^2. \tag{1.7}$$

In this case $q = s$.

Proof. Define $t_i^{(n)} := \max\{\pi_n \cap [0,t]\}$. Since the pointwise limits of (s_n) and (q_n) coincide i.e.

$$|s_n(t) - q_n(t)| = (x_{t_{i+1}^{(n)}} - x_t)^2 + 2(x_{t_{i+1}^{(n)}} - x_t)(x_t - x_{t_i^{(n)}}) \tag{1.8}$$

converges to 0 by the right-continuity of x. (Prop. 5) now follows from $x \in \mathcal{Q}_1^\pi$. □

Denote $\mathcal{Q}_2^\pi$ to be the subset of those $x \in \mathcal{D}$ such that the pointwise limit q of the quadratic sums (q_n) in (9) exists and the Lebesgue decomposition in (10) holds on q. Then $\mathcal{Q}_0^\pi \subset \mathcal{Q}_1^\pi \subset \mathcal{Q}_2^\pi$ and we have:

Theorem A If $x \in \mathcal{Q}_2^\pi$, then $q_n \to q$ in the Skorokhod topology.

Proof. Since $x \in \mathcal{Q}_2^\pi$, we have $q_n \to q$ pointwise on $[0, \infty)$ and that (q_n), q are elements in $\mathcal{D}_0^+$. By [2, Thm.VI.2.15], it remains to show that

$$\sum_{s \le t} (\Delta q_n(s))^2 \xrightarrow{n} \sum_{s \le t} (\Delta q(s))^2$$

on a dense subset of $[0, \infty)$. Let $t > 0$, define $J^\epsilon := \{s \ge 0 | (\Delta X_s)^2 \ge \frac{\epsilon}{2}\}$,

$J_n^\epsilon := \{t_i \in \pi_n | \exists s \in (t_i, t_{i+1}]; (\Delta X_s)^2 \geq \frac{\epsilon}{2}\} \subset \pi_n$ and observe that

$$\begin{aligned} \sum_{s \leq t} (\Delta q_n(s))^2 &= \sum_{\pi_n \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^4 \\ &= \sum_{J_n^\epsilon \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^4 + \sum_{(J_n^\epsilon)^c \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^4. \qquad (1.9) \end{aligned}$$

Since x is càdlàg and that $|\pi_n| \downarrow 0$ on compacts, the first sum in (12) converges to $\sum_{J^\epsilon \ni s \leq t} (\Delta x_s)^4$ and the second sum in (12)

$$\sum_{(J_n^\epsilon)^c \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^4 \leq \sup_{(J_n^\epsilon)^c \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^2 \sum_{(J_n^\epsilon)^c \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^2 \leq \epsilon q(t)$$

for sufficiently large n [5, Appendix A.8] hence

$$\lim_n \sum_{s \leq t} (\Delta q_n(s))^2 = \sum_{J^\epsilon \ni s \leq t} (\Delta x_s)^4 + \limsup_n \overbrace{\sum_{(J_n^\epsilon)^c \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^4}^{\leq \epsilon q(t)} .$$

By the Lebesgue decomposition (10), we observe $\sum_{J^\epsilon \ni s \leq t} (\Delta x_s)^4 \leq q(t)^2$ and that

$$\lim_n \sum_{s \leq t} (\Delta q_n(s))^2 = \sum_{s \leq t} (\Delta x_s)^4 = \sum_{s \leq t} (\Delta q(s))^2$$

as $\epsilon \to 0$. □

Denote $\mathcal{Q}^\pi$ to be the subset of those $x \in \mathcal{D}$ such that the limit $\tilde{q}$ of (q_n) exists in $(\mathcal{D}, d)$. Then $\mathcal{Q}_0^\pi \subset \mathcal{Q}_1^\pi \subset \mathcal{Q}_2^\pi \subset \mathcal{Q}^\pi$ and we have:

Theorem B $\mathcal{Q}^\pi \subset \mathcal{Q}_0^\pi$ and $\tilde{q} = [x] = s = q$.

Proof. Let $x \in \mathcal{Q}^\pi$ and I be the set of continuity points of $\tilde{q}$. [2, VI.2.1(b.5)] implies that $q_n \to \tilde{q}$ pointwise on I. Since $q_n \in \mathcal{D}_0^+$ and I is dense on $[0, \infty)$, it follows $\tilde{q} \in \mathcal{D}_0^+$. Denote μ to be the Radon measure of $\tilde{q}$ on $[0, \infty)$, observe the set of atoms of μ is $J := [0,\infty)\backslash I$ and that (q_n) are the distribution functions of the discrete measures (μ_n) in (4). Thus, by (Lemma 3 & [4, X.11]), we see that $\mu_n \longrightarrow \mu$ vaguely on $[0,\infty)$.

If $t > 0$, put $t_i^{(n)} := \max\{\pi_n \cap [0,t)\}$. Since $|\pi_n| \downarrow 0$ on compacts, we have $t_i^{(n)} < t$, $t_i^{(n)} \uparrow t$ and $t_{i+1}^{(n)} \downarrow t$. Observe that

$$\Delta q_n(t) = \begin{cases} \left(x_{t_{i+1}} - x_{t_i}\right)^2, & \text{if } t = t_i \in \pi_n. \\ 0, & \text{otherwise.} \end{cases} \tag{1.10}$$

If $\Delta\tilde{q}(t) = 0$, [2, VI.2.1(b.5)] implies that $\Delta q_n(t_i^{(n)}) \to \Delta\tilde{q}(t)$. Hence, by the fact that x is càdlàg, $(\Delta x_t)^2 = \lim_n \Delta q_n(t_i^{(n)}) = \Delta\tilde{q}(t)$. If $\Delta\tilde{q}(t) > 0$, there exists [2, VI.2.1(a)] a sequence $t_n' \to t$ such that $\Delta q_n(t_n') \to \Delta\tilde{q}(t) > 0$. Using the fact that x is càdlàg, $t_n' \to t$ and (13), we deduce that (t_n') must coincide with $(t_i^{(n)})$ for all n sufficiently large, else we will contradict $\Delta\tilde{q}(t) > 0$. Thus, $(\Delta x_t)^2 = \lim_n \Delta q_n(t_i^{(n)}) = \lim_n \Delta q_n(t_n') = \Delta\tilde{q}(t)$ and the Lebesgue decomposition (2) holds on $\tilde{q}$.

By (Def. 2), we have $\tilde{q} = [x]$ hence $\mathcal{Q}^\pi \subset \mathcal{Q}_0^\pi$. $[x] = s = q$ now follows from $\mathcal{Q}_0^\pi \subset \mathcal{Q}_1^\pi \subset \mathcal{Q}_2^\pi$ and (Prop. 4 & 5). □

Backed by Theorems A and B, we arrive at the following *equivalent*

Definition 6 We say that $x \in \mathcal{D}$ has finite quadratic variation $[x]$ along π if the following quadratic sums:

$$q_n(t) := \sum_{\pi_n \ni t_i \le t} (x_{t_{i+1}} - x_{t_i})^2$$

converge to $[x]$ in $(\mathcal{D}, d)$.

We immediately see that the two defining properties of $[x]$ in (Def. 2) are consequences per Theorem B. The following corollary treats the special case when the underlying space is $\mathcal{C}$. Observe that if (n_k) is *any* sub-sequence, $\tilde{\pi}_k := \pi_{n_k}$ then obviously $\mathcal{Q}^{\pi} \subset \mathcal{Q}^{\tilde{\pi}}$. Thus, we prove instead the following starker

Corollary 7 Let $\tilde{\pi} \subset \pi$ and $x \in \mathcal{Q}^{\tilde{\pi}}$, then

(i) $q_k \to [x]$ (local) uniformly on $[0, \infty)$ if and only if $x \in \mathcal{C}$.
(ii) If $q_k \to [x]$ (local) uniformly on $[0, \infty)$, so would (s_k) in (3).
(iii) $\mathcal{Q}^{\tilde{\pi}}_{\infty} \subset \mathcal{Q}^{\tilde{\pi}}$, $\mathcal{C} \cap \mathcal{Q}^{\tilde{\pi}}_{\infty} = \mathcal{C} \cap \mathcal{Q}^{\tilde{\pi}}$.

Proof. (i): It is an immediate consequence of (Thm. B), (5) and [2, VI.1.17(b)].
(ii): Let $T > 0$, $\|\cdot\|_T$ the supremum norm on $\mathcal{D}([0, T])$ and observe that

$$\|s_k - [x]\|_T \leq \|q_k - [x]\|_T + \|s_k - q_k\|_T.$$

Since (i) implies $x \in \mathcal{C}$, (ii) now follows from uniform continuity of x and (11).
(iii): By (Prop. 1) and that (local) uniform convergence implies pointwise convergence, we have $\mathcal{Q}^{\tilde{\pi}}_{\infty} \subset \mathcal{Q}^{\tilde{\pi}}_{1} \subset \mathcal{Q}^{\tilde{\pi}}$. (iii) now follows from (i) & (ii). $\square$

Remark 8
1.) The reverse of (ii) is in general not true. To see this, take any semimartingale X whose paths live a.s. in $\mathcal{D} \backslash \mathcal{C}$. Then a.s. paths of X actually live in $\mathcal{Q}^{\tilde{\pi}}_{\infty} \backslash \mathcal{C} \neq \emptyset$ (Section 2 & 3). If $x \in \mathcal{Q}^{\tilde{\pi}}_{\infty} \backslash \mathcal{C}$ then $s_k \to [x]$ (local) uniformly per definition of $\mathcal{Q}^{\tilde{\pi}}_{\infty}$ (Section 3) while $q_k \not\to [x]$ (local) uniformly according to (i).

2.) We note that a third form (instead of s_n and q_n) of quadratic sums

$$\bar{q}_n(x, t) := \sum_{\pi_n \ni t_{i+1} \leq t} (x_{t_{i+1}} - x_{t_i})^2$$

has sometimes been "mistakenly" adopted in the literature in the definition of $[x]$ (i.e. by the pointwise limit of $\bar{q}_n$ on $[0, \infty)$ that admits LD). To see the problem, take any $t_0 \notin \pi$, put $x_t := \mathbb{1}_{[t_0, \infty)}(t)$ then obviously $[x](t_0) =$

$\lim s_n(t_0) = \lim q_n(t_0) = 1$ but $\lim \bar{q}_n(t_0) = 0$. The same conclusion can be drawn in the probabilistic setting e.g. if one puts the entire probability mass on one path $1\!\!1_{[t_0,\infty)}$ in $\mathcal{D}$, the resulting canonical process X on $\mathcal{D}$, strictly speaking, is a semimartingale whose $[X](t_0) = 1$ a.s. but now $\lim \bar{q}_n(X, t_0) = 0$ a.s..

1.3 Quadratic variation for multidimensional functions

Denote $\mathcal{D}^n := \mathcal{D}([0,\infty), \mathbb{R}^n)$ and $\mathcal{D}^{n\times n} := \mathcal{D}([0,\infty), \mathbb{R}^{n\times n})$ to be the Skorokhod spaces, each of which equipped with a metric d which induces the corresponding Skorokhod (a.k.a J_1) topology [2, VI]. $\mathcal{C}^n := \mathcal{C}([0,\infty), \mathbb{R}^n)$ the subspace of continuous functions in $\mathcal{D}^n$. We recall (4. Thm. A & B) from the one dimensional case $n = 1$ that (Def. 2) and (Def. 6) are equivalent.

As is well known, if $x, y \in \mathcal{Q}^\pi$, it does not, in general, imply $x + y \in \mathcal{Q}^\pi$. An example can be found in [6]. Therefore, the notion of quadratic variation in the multidimensional setting was originally defined in [1, Rem.(1)] as follows:

Definition 9 (Föllmer 1981) We say that $\mathbf{x} := (x^1, \ldots, x^n)^T \in \mathcal{D}^n$ has finite quadratic variation along π if all x^i, $x^i + x^j$ $(1 \le i, j \le n)$ have finite quadratic variation.

In this case, note that the quadratic covariation $[x^i, x^j]$ of x^i and x^j may be defined by

$$[x^i, x^j]_t := \frac{1}{2}\left([x^i + x^j]_t - [x^i]_t - [x^j]_t\right), \tag{1.11}$$

which admits the following *specific* form of Lebesgue decomposition:

$$[x^i, x^j]_t = [x^i, x^j]^c_t + \sum_{s\le t} \Delta x^i_s \Delta x^j_s. \tag{1.12}$$

We shall call $[\mathbf{x}] := ([x^i, x^j])_{1\le i\le j\le n}$ the quadratic variation of $\mathbf{x}$.

Be aware that in the special case $n = 1$, the above definition becomes a tautological statement i.e. it is not possible to work with the above definition *alone* without the company of a one dimensional definition, for example (Def. 2) or equivalently (Def. 6). The following definition is the natural extension of (Def. 6) to the multidimensional setting and we shall prove its equivalence to that of Föllmer's.

Definition 10 We say that $\mathbf{x} \in \mathcal{D}^n$ has finite quadratic variation $[\mathbf{x}]$ along π if the following quadratic sums:

$$\mathbf{q}_n(t) := \sum_{\pi_n \ni t_i \leq t} (\mathbf{x}_{t_{i+1}} - \mathbf{x}_{t_i})(\mathbf{x}_{t_{i+1}} - \mathbf{x}_{t_i})^T$$

converges to $[\mathbf{x}]$ in $(\mathcal{D}^{n\times n}, d)$.

For $u, v, w \in \mathcal{D}$, let us write

$$q_n^{(u,v)}(t) := \sum_{\pi_n \ni t_i \leq t} (u_{t_{i+1}} - u_{t_i})(v_{t_{i+1}} - v_{t_i})$$

and $q_n^{(w)} := q_n^{(w,w)}$. Note that the Skorokhod topology on $(\mathcal{D}^n, d)$ is strictly finer than the product topology on $(\mathcal{D}, d)^n$ [2, VI.1.21] and that $(\mathcal{D}, d)$ is not a topological vector space [2, VI.1.22], hence the following would be an essential

Lemma 11 Let $t > 0$, there exists a sequence $t_n \to t$ such that

$$\lim_n \left(\Delta q_n^{(u,v)}(t_n)\right) = \Delta\left(\lim_n q_n^{(u,v)}\right)(t),$$

$\forall u, v \in \mathcal{D}$; $(q_n^{(u,v)})$ converges in $(\mathcal{D}, d)$.

Proof. Define $t_i^{(n)} := \max\{\pi_n \cap [0, t)\}$. Since $|\pi_n| \downarrow 0$ on compacts, we have $t_i^{(n)} < t$, $t_i^{(n)} \uparrow t$ and $t_{i+1}^{(n)} \downarrow t$. Observe that

$$\Delta q_n^{(u,v)}(t) = \begin{cases} \left(u_{t_{i+1}} - u_{t_i}\right)\left(v_{t_{i+1}} - v_{t_i}\right), & \text{if } t = t_i \in \pi_n. \\ 0, & \text{otherwise.} \end{cases} \tag{1.13}$$

Put $\tilde{q} := \lim_n q_n^{(u,v)}$. If $\Delta\tilde{q}(t) = 0$, [2, VI.2.1(b.5)] implies that $\Delta q_n^{(u,v)}(t_i^{(n)}) \to \Delta\tilde{q}(t)$. If $\Delta\tilde{q}(t) > 0$, there exists [2, VI.2.1(a)] a sequence $t'_n \to t$ such that $\Delta q_n^{(u,v)}(t'_n) \to \Delta\tilde{q}(t) > 0$. Using the fact that u, v are càdlàg, $t'_n \to t$ and (16), we deduce that (t'_n) must coincide with $(t_i^{(n)})$ for all n sufficiently large, else we will contradict $\Delta\tilde{q}(t) > 0$. Put $t_n := t_i^{(n)}$. □

Proposition 12 Let $x, y \in \mathcal{Q}^\pi$, then $(q_n^{(x+y)})$ converges in $(\mathcal{D}, d)$ if and only if $(q_n^{(x,y)})$ does. In this case, $x+y \in \mathcal{Q}^\pi$ and $\lim_n q_n^{(x,y)} = \frac{1}{2}([x+y] - [x] - [y])$.

Proof. Since

$$q_n^{(x+y)} = q_n^{(x)} + q_n^{(y)} + 2q_n^{(x,y)}$$

and that $x, y \in \mathcal{Q}^\pi$, we obtain (Prop. 12) immediately from (Lemma 11) and [2, VI.2.2(a)]. □

Proposition 13 $(\mathbf{q}_n)$ converges in $(\mathcal{D}^{n\times n}, d)$ if and only if it converges in $(\mathcal{D}, d)^{n\times n}$.

Proof. Since the Skorokhod topology on $(\mathcal{D}^{n\times n}, d)$ is strictly finer than the product topology on $(\mathcal{D}, d)^{n\times n}$ [2, VI.1.21], we have $(\mathcal{D}^{n\times n}, d)$ convergence implies $(\mathcal{D}, d)^{n\times n}$ convergence. The other direction follows immediately from the observation that

$$\mathbf{q}_n = \left(q_n^{(x^i,x^j)}\right)_{1\le i\le j\le n}, \tag{1.14}$$

(Lemma 11) and [2. VI.2.2(b)]. □

Theorem C (Def. 9) and (Def. 10) are equivalent.

Proof. It is an immediate consequence of (14), (17), (Prop. 12 & 13) and (4. Thm. A & B). □

Corollary 14 If $\mathbf{x} \in \mathcal{D}^n$ has finite quadratic variation, then

(i) $\mathbf{q_n} \to [\mathbf{x}]$ locally uniformly on $[0, \infty)$ if and only if $\mathbf{x} \in \mathcal{C}^n$.
(ii) $F(\mathbf{q_n}) \to F([\mathbf{x}])$ for all functionals F continuous at $[\mathbf{x}]$.

Proof. It is an immediate consequence of (Thm. C), (15) and [2, VI.1.17(b)]. □

Remark 15
A handful of functionals F, continuous *at points*, are well catalogued in [2, VI.2].

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2

Functional calculus and pathwise integration

Overview

The causal functional calculus, proposed in Cont & Fournié [7] for functionals of paths with finite quadratic variation is based on regularity assumptions for functionals which may fail to be satisfied for many interesting classes of functionals, most notably pathwise integrals. Also, its application to càdlàg paths involves additional assumptions linking the sequence of time-partitions with the jump times of the path.

In this chapter we present a new functional calculus for càdlàg paths, which overcomes these difficulties and extends previous results to a much larger class of functionals, including Föllmer integrals. We construct a bespoke topology on path spaces and introduce a formulation of functional calculus on generic domains, without any assumption on the variation order of a path. The Föllmer integral is shown to be a continuous functional with respect to this topology.

We adopt a generic approach to pathwise integration and obtain functional change of variable formulas which extend the results of Cont & Fournié [7] to a larger class of functionals, including pathwise integrals. In particular, we remove the previous assumptions of [7] relating the partition sequence to the jump times of the path. Finally, we study a class of functionals (class $\mathcal{M}$) which may be understood as the analytic analogue of the concept of martingale.

2.1 Introduction

2.1.1 Motivation

Let $\pi := (\pi_n)_{n\geq 1}$ be a sequence of interval partitions of $[0,\infty)$ and denote Q^π the set of càdlàg paths with finite quadratic variation along π in the sense of Föllmer [9]. Then for any $f \in C^2(\mathbb{R}^d)$, the Itô formula holds pathwise for every $x \in Q^\pi$ [9]:

$$f(x(T)) = f(x(0)) + \int_0^T \nabla f(x(t-))dx(t) + \frac{1}{2}\int_0^T \nabla^2 f(x(t)).d[x](t) \quad (2.1)$$
$$+ \sum_{0\leq s\leq t} f(x(s)) - f(x(s-)) - \nabla f(x(s-)).\Delta x(s) - \frac{1}{2}\Delta x(s).\nabla^2 f(x(s))\Delta x(s)$$

where the Föllmer integral $\int_0^T f'(x(t-))dx(t)$ is defined as a pointwise limit of left Riemann sums:

$$\int_0^T \nabla f(x(t-)).dx(t) := \lim_{n\to\infty} \sum_{\pi_n \ni t_i \leq T} \nabla f(x(t_i))(x(t_{i+1}) - x(t_i)), \qquad (2.2)$$

without resorting to any probabilistic notion of convergence. Based on the key observation that, for every semi-martingale X, there exists a sequence of partitions π such that the sample paths of X lie almost surely in Q^π, Föllmer showed [9] that for any integrand of the form $\nabla f \circ X$, where $f \in C^2(\mathbb{R}^d)$, the pathwise integral (2.2) coincides with probability one with the Itô integral, thus providing a pathwise interpretation of the Itô stochastic integral.

The extension of this result to path-dependent functionals has been the focus of several recent works [1, 7, 8, 20]. In particular, a functional change of variable formula for càdlàg paths was obtained by Cont & Fournié [7, Thm. 4] for a class of regular functionals denoted $\mathbb{C}^{1,2}(\Lambda_T)$. Moreover, [7] (see also [1, Thm 3.2]) establish that, for $F \in \mathbb{C}^{1,2}(\Lambda_T)$, one may define a pathwise integral $\int_0^T \nabla_x F(t, x_{t-}).d^\pi x$ as a pointwise limit of Riemann sums as in (2.2).

The key idea behind these results [5, 6, 7] can be summarised as follows [5, s4.1,s5.2]. First, one constructs a calculus for continuous functionals F on piecewise constant paths. Second, this calculus is extended to all càdlàg paths using a density argument, using piecewise-constant approximations of paths. This second step is where topology plays a role. The original construction of the functional Itô calculus was based on the uniform topology [6, 7, 7]. As is well known, piece-wise constant approximation of a càdlàg path under the uniform topology requires *exact* knowledge of all points of discontinuity, which leads to a requirement [7, Rem.7] that the sequence of partitions exhausts the set $J(x)$ of discontinuity points of the path x:

$$J(x) := \{t \in [0, \infty), \quad x(t-) \neq x(t)\} \subset \liminf_n \pi_n. \tag{2.3}$$

This condition, which links the partition with the path, is not required for Föllmer 's [9] results, but plays a key role in the proof of [7, Thm. 4].

The following result, whose proof is given in Section § 2.7, shows that this condition (2.3) is restrictive and need not be satisfied even for semimartingales:

Proposition 2.1.1. *There exists a semi-martingale X such that for any partition sequence π, $\mathbb{P}(J(X) \subset \liminf_n \pi_n) = 0$.*

A related issue is the differentiability and regularity of the pathwise integral. The Föllmer integral $\mathbb{I} : (t, x) \mapsto \int_0^t \nabla_x F.d^\pi x$, which is a central object in the pathwise Itô calculus, is not continuously differentiable in the sense of [7], even for $F \in \mathbb{C}^{1,2}(\Lambda_T)$.

To address these issues one needs to replace the uniform topology with another topology. Unfortunately, the usual topologies on the Skorokhod space D [18, s5] do not fit this purpose. Consider the identity map $I_d(u) := u$ on

$\mathbb{R}^d$, the pointwise evaluation map

$$F(x) := I_d(x(t))$$

is not J_1 continuous on D [8, VI. 2.3] and the same applies to all weaker topologies. Since $I_d \in C^2(\mathbb{R}^d)$, it may be a lost cause to obtain a functional calculus built on top of weak topologies on D.

In this work we circumvent these obstacles by introducing a new topology on the space D of càdlàg paths. The Föllmer pathwise integral and the pathwise quadratic variation functional are shown to be continuous functionals with respect to this topology. We define a class of continuously differentiable functionals with respect to this topology and derive change of variable formulas for such functionals without requiring the restrictive condition (2.3). In the case of paths with finite quadratic variation along a partition sequence, our change of variable formula extends results [1, 7, 9, 16] on the Föllmer-Ito calculus and relaxes previous assumptions relating the partition sequence to the discontinuities of the underlying path. In particular we obtain a pathwise Itô 's identity (Theorem 2.6.4) in the spirit of Beiglböck and Siorpaes' pathwise Burkholder-Davis-Gundy inequality [2].

2.1.2 Outline

After introducing some preliminary definitions and notations in Section 2.2 we prove, in section 2.2.2, a new limit theorem which allows us to treat functionals involving quadratic variation. In section 2.3, we introduce a new topology the space of càdlàg paths, discuss its relation with other well-known topologies and give examples of continuous functionals for this topology. In section 2.4, we introduce classes of smooth causal functionals and discuss their properties, in particular class $\mathcal{M}$ functionals, which are shown to possess a pathwise analogue of the martingale property (Theorem 2.5.13).

Section 2.5 discusses pathwise integration and functional change of variable formulas. We show in particular that pathwise integrals may be defined for class $\mathcal{M}$ functionals without any condition on the variation index (p-

variation) of the underlying path. Section 2.6 discusses in more detail the case of functionals of càdlàg paths with finite quadratic variation and the relation of class $\mathcal{M}$ functionals to path-dependent PDEs.

2.2 Preliminaries

2.2.1 Notations

Denote by D_m the Skorokhod space of $\mathbb{R}^m$-valued càdlàg functions

$$t \longmapsto x(t) := (x_1(t), \ldots, x_m(t))'$$

on $\mathbb{R}_+ := [0, \infty)$. Denote S_m (resp. BV_m) the subset of step functions (resp. locally bounded variation functions) in D_m. For $m = 1$, we will omit the subscript m. By convention, $x(0-) := x(0)$ and $\Delta x(t) := x(t) - x(t-)$. we denote by by $x_t \in D_m$ (resp. $x_{t-} \in D_m$) the path $x \in D_m$ stopped at t (resp. $t-$):

$$x_t(s) = x(s \wedge t), \qquad x_{t-}(s) = x(s)1_{s<t} + x(t-)x(s)1_{s\geq t}.$$

We equip $(D_m, \mathfrak{d}_{J_1})$ with a metric $\mathfrak{d}_{J_1}$ which induces the Skorokhod (a.k.a. J_1) topology.

Let $\pi := (\pi_n)_{n\geq 1}$ be a fixed sequence of partitions $\pi_n = (t_0^n, ..., t_{k_n}^n)$ of $[0, \infty)$ into intervals $0 = t_0^n < ... < t_{k_n}^n < \infty$ such that $t_{k_n}^n \to \infty$, with vanishing mesh $|\pi_n| = \sup_{i=1..k_n} |t_i^n - t_{i-1}^n| \to 0$ on compacts. By convention, $\max(\emptyset) := 0$.

We denote

$$t_n' := \max\{t_i < t | t_i \in \pi_n\}, \qquad x^n \ := \sum_{t_i \in \pi_n} x(t_{i+1}) \mathbb{I}_{[t_i, t_{i+1})}$$

and by $x^{(n)}$ the (continuous) piecewise linear approximations of x along π_n.

We denote $Q_m^\pi \subset D_m$ the subset of càdlàg paths with finite quadratic variation along π, defined as follows:

Definition 2.2.1 (Quadratic variation along a sequence of partitions)**.** We say that $x \in D_m$ has finite quadratic variation along π if the sequence of step functions:

$$q_n(t) := \sum_{\pi_n \ni t_i \le t} (x(t_{i+1}) - x(t_i))(x(t_{i+1}) - x(t_i))'$$

converges in the Skorokhod topology. The limit $[x]_\pi := ([x_i, x_j]_\pi)_{1 \le i,j \le m} \in D_{m \times m}$ is called the quadratic variation of x along π.

In the sequel we fix such a sequence of partitions π and drop the subscript π unless we want to emphasize the dependence on π.

As shown in [4, Thm. 3.6], Definition 2.2.1 is equivalent to the one given by Föllmer [9]:

Proposition 2.2.2 ([4]). *Let $x \in D_m$, then $x \in Q^\pi_m$ if and only if $x_i, x_i + x_j \in Q^\pi$. If $x \in Q^\pi_m$, then we have the polarisation identity*

$$\begin{aligned} [x_i, x_j](t) &= \frac{1}{2}\left([x_i + x_j] - [x_i] - [x_j]\right)(t) \in BV \\ &= [x_i, x_j]^c(t) + \sum_{s \le t} \Delta x_i(s) \Delta x_j(s) \end{aligned} \tag{2.4}$$

We set $\lim_n a_n := \infty$ whenever a real sequence (a_n) does not converge. For real-valued matrices of equal dimension, we write $\langle \cdot, \cdot \rangle$ to denote the Frobenius inner product and $| \cdot |$ to denote the Frobenius norm. If f (resp. g) are $\mathbb{R}^{m \times m}$-valued functions on $[0, \infty)$, we write

$$\int_0^t f dg := \sum_{i,j} \int_0^t f_{i,j}(s-) dg_{i,j}(s)$$

whenever the RHS makes sense. If $x \in Q^\pi_m$ and $f \in C^2(\mathbb{R}^m)$, we write

$$\int_0^t (\nabla f \circ x) d^\pi x := \int_0^t \nabla f(x(s-)) d^\pi x(s)$$

to denote the Föllmer integral [9], defined as a pointwise limit of left Riemann sums along π. The superscript π may be dropped in the sequel as π is fixed throughout.

2.2.2 Quadratic Riemann sums

In this section, we focus on paths with finite quadratic variation along a sequence of partitions and extend certain limit theorems obtained in [7] for the convergence of 'quadratic Riemann sums' (in particular [7, Lemma 12]) to a more general setting. The main result of this section is Theorem 2.2.7, which is a key ingredient in the proof of change of variable formula for functionals of paths with quadratic variation.

The following result [4, Lemma 2.2] will be useful in the sequel:

Lemma 2.2.3. *Let v_n, v be non-negative Radon measures on $\mathbb{R}_+$ and J be the set of atoms of v, then $v_n \to v$ vaguely on $\mathbb{R}_+$ if and only if $v_n \to v$ weakly on $[0,T]$ for every $T \notin J$*

Lemma 2.2.4. *Let $x \in Q^\pi$, $\mu = d[x]$ be the Radon measure associated with $[x]$. For every $[0,T]$, $T_n := \max\{t_i < T | t_i \in \pi_n\}$, $T_{n+1} := \min\{t_i \geq T | t_i \in \pi_n\}$ and define a sequence of non-negative Radon measures on $\mathbb{R}_+$ by:*

$$\mu_n([0,T]) \;\; := \;\; \sum_{t_i \in \pi_n} (x(t_{i+1}) - x(t_i))^2 \delta_{t_{i+1}}([0,T)) + (x(T_{n+1}) - x(T_n))^2.$$

Then

(i) $\xi_n := \sum_{t_i \in \pi_n} (x(t_{i+1}) - x(t_i))^2 \delta_{t_i} \longrightarrow \mu$ *vaguely on* $\mathbb{R}_+$,

(ii) $\mu_n \longrightarrow \mu$ *vaguely on* $\mathbb{R}_+$.

Proof. (i) follows from [4, Thm. 2.7]. By Lemma 2.2.3, we may assume T to be a continuity point of $d[x]$. Let f be a continuous function on $[0,T]$. If $T = 0$, then $\mu_n(\{0\}) \equiv d[x](\{0\}) = 0$. If $T > 0$, observe that $\xi_n([0,T)) \longrightarrow d[x]([0,T))$ (by (i)), f is uniform continuous on $[0,T]$ and that x is right-continuous, put $T'_{n+1} := \min\{t_i > T | t_i \in \pi_n\}$, it follows that for sufficiently large n

$$\begin{aligned}\left|\int_0^T f d\xi_n - \int_0^T f d\mu_n\right| &\leq \sum_{\pi_n \ni t_i < T} |f(t_i) - f(t_{i+1} \wedge T)|(x(t_{i+1}) - x(t_i))^2 \\ &+ f(T)(x(T'_{n+1}) - x(T_{n+1}))^2 \\ &\leq \sup_{t_i \in \pi_n \cap [0,T]} |f(t_i) - f(t_{i+1} \wedge T)|\xi_n([0,T)) \\ &+ \|f\|_T (x(T'_{n+1}) - x(T_{n+1}))^2 \longrightarrow 0.\end{aligned}$$

□

Lemma 2.2.5. *Let v_n, v be non-negative Radon measures on $\mathbb{R}_+$ with $v_n \longrightarrow v$ vaguely on $\mathbb{R}_+$ and J be the set of atoms of v. If for every $T \in J$, there exists a sequence (T_n) in $\mathbb{R}_+$, $T_n \uparrow T$ such that*

$$v_n(\{T_n\}) \longrightarrow v(\{T\}), \tag{2.5}$$

then $v_n \longrightarrow v$ weakly on $[0,T]$ for all $T \geq 0$.

Proof. For every $T \geq 0$, $\tilde{v}_n([0,T]) := v_n([0,T]) - v_n(\{T_n\})$ and $\tilde{v}([0,T]) := v([0,T]) - v(\{T\})$. If $T \notin J$, the claim follows immediately from Lemma 2.2.3. Thus, we may assume $T \in J$. If $T = 0 \in J$, then $T_n \equiv 0$. Let $T > 0$ and $f \in C([0,T], \|\cdot\|_\infty)$. Since $f = (f)^+ - (f)^-$, we may take $f \geq 0$ and for sufficiently small $\epsilon > 0$, we define the following extensions:

$$\begin{aligned}\overline{f}^\epsilon(t) &:= f(t)\mathbb{1}_{[0,T]}(t) + f(T)\left(1 + \frac{T-t}{\epsilon}\right)\mathbb{1}_{(T,T+\epsilon]}(t) \\ \underline{f}^\epsilon(t) &:= f(t)\mathbb{1}_{[0,T-\epsilon]}(t) + f(T)\left(\frac{T-t}{\epsilon}\right)\mathbb{1}_{(T-\epsilon,T]}(t),\end{aligned}$$

then $\overline{f}^\epsilon$, $\underline{f}^\epsilon \in \mathcal{C}_K([0,\infty))$, $0 \leq \underline{f}^\epsilon \leq f\mathbb{1}_{[0,T]} \leq \overline{f}^\epsilon \leq \|f\|_\infty$. and we have

$$\int_0^\infty \underline{f}^\epsilon d\tilde{v}_n \leq \int_0^T f d\tilde{v}_n \leq \int_0^\infty \overline{f}^\epsilon d\tilde{v}_n.$$

Since $v_n \to v$ vaguely and (2.5) holds, we obtain

$$\begin{aligned} 0 &\leq \limsup_n \int_0^T f d\tilde{v}_n - \liminf_n \int_0^T f d\tilde{v}_n \leq \int_0^\infty \overline{f}^\epsilon - \underline{f}^\epsilon d\tilde{v} \\ &\leq f(T) \left(v\left([T-\epsilon, T+\epsilon]\right) - v(\{T\})\right) \stackrel{\epsilon}{\longrightarrow} 0, \end{aligned}$$

hence by monotone convergence

$$\lim_n \int_0^T f d\tilde{v}_n = \lim_\epsilon \int_0^\infty \underline{f}^\epsilon d\tilde{v} = \int_0^T f d\tilde{v}.$$

By (2.5), it follows $\lim_n \int_0^T f dv_n = \int_0^T f dv$. □

Lemma 2.2.6. *Let v_n, v be non-negative Radon measures on $\mathbb{R}_+$ such that $v_n \longrightarrow v$ vaguely. Let f_n, f be real-valued left-continuous functions on $\mathbb{R}_+$ and J be the set of atoms of v. If*

(i) for every $T \in J$ there exists a sequence $(T_n) \in [0, T)$ with $T_n \uparrow T$ such that $v_n(\{T_n\}) \longrightarrow v(\{T\})$, and

(ii) (f_n) is locally bounded and converges pointwise to f,

then for every $T \geq 0$,

$$\int_0^T f_n dv_n \longrightarrow \int_0^T f dv.$$

Proof. Let $v = v^c + v^d$ be the Lebesgue decomposition of v into an absolutely continuous part v^c and a singular (discrete) measure v^d. By (i) and Lemma 2.2.5, we immediately see that $(v_n - v^d) \longrightarrow v^c$ weakly for every $[0, T]$. Since v^c has no atoms, by an application of [7, Lemma 12] we have

$$\int_0^T f_n d(v_n - v^d) \longrightarrow \int_0^T f dv^c.$$

By (ii) and dominated convergence, the proof is complete. □

Theorem 2.2.7. *Let $x \in Q^\pi$, f_n, f be real-valued left-continuous functions on $\mathbb{R}_+$ such that (f_n) is locally bounded and converges pointwise to f on $\mathbb{R}_+$.*

Then for any $T > 0$,

$$(i) \sum_{\pi_n \ni t_i \le T} f_n(t_i)(x(t_{i+1}) - x(t_i))^2 \longrightarrow \int_0^T f d[x].$$

$$(ii) \sum_{\pi_n \ni t_i \le T} f_n(t_{i+1} \wedge T)(x(t_{i+1}) - x(t_i))^2 \longrightarrow \int_0^T f d[x].$$

$$(iii) \sum_{\pi_n \ni t_i < T} f_n(t_i)(x(t_{i+1}) - x(t_i))^2 \longrightarrow \int_0^T f d[x].$$

$$(iv) \sum_{\pi_n \ni t_i < T} f_n(t_{i+1} \wedge T)(x(t_{i+1}) - x(t_i))^2 \longrightarrow \int_0^T f d[x].$$

Proof. If $T = 0$, then by (2.4) and the right-continuity of x its continuity at $T = 0$, the claims follow. If $T > 0$, put $T_n := \max\{t_i < T | t_i \in \pi_n\}$, $T_{n+1} := \min\{t_i \ge T | t_i \in \pi_n\}$, $T'_{n+1} := \min\{t_i > T | t_i \in \pi_n\}$, then $T_n \uparrow T$ and by Lemma 2.2.4, we observe that

$$\xi_n(\{T_n\}) = (x(T_{n+1}) - x(T_n))^2 \longrightarrow d[x](\{T\}),$$
$$\mu_n(\{T\}) = (x(T_{n+1}) - x(T_n))^2 \longrightarrow d[x](\{T\}),$$

and that

$$\sum_{\pi_n \ni t_i < T} f_n(t_i)(x(t_{i+1}) - x(t_i))^2 = \int_0^T f_n d\xi_n - f(T_{n+1})(x(T'_{n+1}) - x(T_{n+1}))^2,$$

$$\sum_{\pi_n \ni t_i \le T} f_n(t_{i+1} \wedge T)(x(t_{i+1}) - x(t_i))^2 = \int_0^T f_n d\mu_n + f(T)(x(T'_{n+1}) - x(T_{n+1}))^2.$$

By the right continuity of x, Lemma 2.2.4 and Lemma 2.2.6, the proof is complete. □

As a consequence of Prop. 2.2.2 and Thm. 2.2.7 we have:

Corollary 2.2.8 (Multidimensional paths)**.** *Let* $x \in Q_m^\pi$, $f_n, f : \mathbb{R}_+ \mapsto \mathbb{R}^{m \times m}$ *be left-continuous functions with* (f_n) *locally bounded and converging* *pointwise to f on $\mathbb{R}_+$. Then*

$$(i) \sum_{\pi_n \ni t_i \leq T} \langle f_n(t_i), (x(t_{i+1}) - x(t_i))(x(t_{i+1}) - x(t_i))' \rangle \longrightarrow \int_0^T f d[x]$$

$$(ii) \sum_{\pi_n \ni t_i \leq T} \langle f_n(t_{i+1} \wedge T), (x(t_{i+1}) - x(t_i))(x(t_{i+1}) - x(t_i))' \rangle \longrightarrow \int_0^T f d[x]$$

for every $T \geq 0$. In particular, the convergence also holds if the sum is replaced by $\sum_{\pi_n \ni t_i < T}$.

Remark 2.2.9. $t \longmapsto \int_0^t f d[x]$ is in BV and has Lebesgue decomposition:

$$\int_0^t f d[x] = \int_0^t f d[x]^c + \sum_{s \leq t} \langle f(s-), \Delta x(s) \Delta x(s)' \rangle.$$

2.3 Continuous functionals

In this section, we shall construct a topology on suitable subsets of

$$E := \mathbb{R}_+ \times D_m,$$

for which Föllmer integrals will be continuous functionals of the integrator.

2.3.1 Domains for causal functionals

We are interested in *causal* (non-anticipative) functionals [12], whose natural domain of definition is of the form

$$\{(t, x_t) | t \in \mathbb{R}_+, x \in \Omega\} \subset E,$$

for a suitable set of paths $\Omega \subset D_m$ [7].

In order to deploy our functional calculus on such functionals we require $\Omega \subset D_m$ to be closed under certain operations:

- stopping: $x \in \Omega \Longrightarrow \forall t \geq 0,\ x_t = x(t \wedge .) \in \Omega$.

- vertical perturbations, in order to define the vertical (Dupire) derivative:

$$x \in \Omega \Longrightarrow x_t + e1\!\!1_{[t,\infty)} \in \Omega,$$

- piecewise constant approximation along π.

We will call a set of paths *generic* if it is stable under these operations:

Definition 2.3.1 (Generic sets of paths)**.** A non-empty subset $\Omega \subset D_m$ is called *generic* if it satisfies:

i) For every $x \in \Omega$, $T > 0$, $\exists N \in \mathbb{N}$; $x_T^n \in \Omega, \quad \forall n \geq N$.

ii) For every $x \in \Omega, t \geq 0$, there exists a convex neighbourhood $\mathcal{U}$ of 0 such that

$$-\Delta x(t) \in \mathcal{U} \qquad \text{and} \quad x_t + e1\!\!1_{[t,\infty)} \in \Omega, \quad \forall e \in \mathcal{U}.$$

We will call a *domain* a set Λ of stopped paths of the form

$$\Lambda := \{(t, x_t) | t \in \mathbb{R}_+, x \in \Omega\}$$

where $\Omega \subset D_m$ is generic.

Remark 2.3.2. Def. 2.3.1(ii) implies that $-\mathcal{U}$ is a convex neighbourhood of 0 containing $\Delta x(t)$;

$$x_{t-} + e1\!\!1_{[t,\infty)} \in \Omega, \quad \forall e \in -\mathcal{U}.$$

Example 2.3.3. S_m, BV_m, Q_m^π, ${Q_m^\pi}^+$ (i.e. positive paths in Q_m^π) and D_m are all generic sets. If Ω is generic, then

$$\Omega_a^b := \{x \in \Omega | a < x_i < b\}$$

for all constants a, b are all generic. Subsets of continuous paths are not generic.

Example 2.3.4. Let Ω be generic. Then $\Omega \cap Q^\pi_m$ is generic.

Proof. We observe $S_m \subset Q^\pi_m$ and if $x \in Q^\pi_m$, then $x + S_m \in Q^\pi_m$. □

Two well known (product) topologies may be associated with E, generated by the metric Euclidean topology on $\mathbb{R}_+$ and local uniform (resp. the Skorokhod J_1) topology on D_m. On a domain $\Lambda \subset E$, we define the uniform (U) and J_1 topologies as the corresponding topology induced on Λ.

Remark 2.3.5. Every J_1-continuous functional is U-continuous: the local uniform topology is strictly finer than the J_1 topology on D_m [8, VI].

We will now show that, if Ω is 'rich enough' to contain a path with non-zero quadratic variation as well as its piecewise-linear approximations along π, then important examples of functionals such as quadratic variation or the Föllmer integral fail to be continuous on Ω in the uniform topology. We use the following assumption:

Assumption 2.1. Ω is a generic subset and contains a path $x \in Q^\pi_m$ with $[x]$ continuous and non-vanishing and

$$\exists N \in \mathbb{N}, \forall n \geq N, x^{(n)} \in \Omega,$$

where $x^{(n)}$ denotes the piecewise-linear approximation of x along π_n.

Example 2.3.6. Q^π_m and ${Q^\pi_m}^+$ satisfy Assumption 2.1, S_m and BV_m do not.

Lemma 2.3.7. *Let Ω satisfy Assumption 2.1 and $\Lambda = \{(t, x_t) | t \in \mathbb{R}_+, x \in \Omega\}$. Then the functionals*

$$F(t, x_t) := |[x](t)| \qquad G(t, x_t) := \int_0^t 2x dx$$

are not U-continuous on Λ.

Proof. If Ω satisfies Assumption 2.1, there exists $T > 0$, continuous $x, x^{(n)} \in \Omega$ such that $|[x](T)| > 0$. Since $x^{(n)}_T \longrightarrow x_T$ in the local uniform topology on $[0, \infty)$, it follows that

$$(T, x^{(n)}_T) \xrightarrow{\mathrm{U}} (T, x_T)$$

on Λ. We observe that each $x_T^{(n)}$ is a continuous function of bounded variation on $[0, \infty)$, it follows that

$$|[x^{(n)}](T)| = 0, \quad \forall n \geq 1$$

so F is not U-continuous. Using the above and the fact that $x, x^{(n)} \in Q_m^\pi$, we obtain by an application of the pathwise Itô formula [9]:

$$\begin{aligned}
&\lim_n \left| \int_0^T 2x dx - \int_0^T 2x^{(n)} dx^{(n)} \right| \\
=&\lim_n \left| |x(T)|^2 - |x(0)|^2 - tr\left([x](T)\right) - \left(|x^{(n)}(T)|^2 - |x^{(n)}(0)|^2\right) \right| \\
=&tr\left([x](T)\right) > 0,
\end{aligned}$$

hence G is not U-continuous on Λ. □

We shall now define a new topology on a domain Λ for which these examples of functionals will be continuous.

2.3.2 The $\pi-$topology

Definition 2.3.8 (The $\pi-$topology)**.**
For every $t \in \mathbb{R}_+, x \in \Omega$, we define $t'_n := \max\{t_i < t | t_i \in \pi_n\}$ and

$$x^n \quad := \sum_{t_i \in \pi_n} x(t_{i+1}) \mathbb{1}_{[t_i, t_{i+1})} \tag{2.6}$$

for n sufficiently large. Denote $\mathfrak{X}$ the set of functionals $F : \Lambda \longmapsto \mathbb{R}$ satisfying:

$$
\begin{aligned}
1.(a) \quad & \lim_{s\uparrow t; s\le t} F(s, x_{s-}) = F(t, x_{t-}),\\
(b) \quad & \lim_{s\uparrow t; s< t} F(s, x_{s}) = F(t, x_{t-}),\\
(c) \quad & t_n \longrightarrow t; t_n \le t'_n \Longrightarrow F(t_n, x^n_{t_n-}) \longrightarrow F(t, x_{t-}),\\
(d) \quad & t_n \longrightarrow t; t_n < t'_n \Longrightarrow F(t_n, x^n_{t_n}) \longrightarrow F(t, x_{t-}),
\end{aligned}
$$

$$
\begin{aligned}
2.(a) \quad & \lim_{s\downarrow t; s\ge t} F(s, x_{s}) = F(t, x_{t}),\\
(b) \quad & \lim_{s\downarrow t; s> t} F(s, x_{s-}) = F(t, x_{t}),\\
(c) \quad & t_n \longrightarrow t; t_n \ge t'_n \Longrightarrow F(t_n, x^n_{t_n}) \longrightarrow F(t, x_{t}),\\
(d) \quad & t_n \longrightarrow t; t_n > t'_n \Longrightarrow F(t_n, x^n_{t_n-}) \longrightarrow F(t, x_{t}),
\end{aligned}
$$

for all $(t, x_t) \in \Lambda$. The initial topology generated by $\mathfrak{X}$ on Λ is called the $\pi-$topology.

We note that the definition of this topology depends on the partition sequence π.

Remark 2.3.9. Every U-continuous functional satisfies Def. 2.3.8.1(a),(b) and 2(a),(b).

Definition 2.3.10 (Continuous functionals)**.**
A functional $F : \Lambda \to \mathbb{R}$ is called *continuous* if it is continuous with regard to the $\pi-$topology. We denote $C_\pi(\Lambda)$ the set of functionals $F : \Lambda \longmapsto \mathbb{R}$ that are continuous with regard to the $\pi-$topology.

F is called *left- (resp. right-) continuous* if it satisfies property 2.3.8.1 (resp. property 2.3.8.2).

Remark 2.3.11. Since

$$z_n \xrightarrow{\Lambda} z \Longleftrightarrow F(z_n) \to F(z) \quad \forall F \in \mathfrak{X},$$

hence $C_\pi(\Lambda) \subset \mathfrak{X}$ so in fact $C_\pi(\Lambda) = \mathfrak{X}$.

The following concept was introduced in [7] under the name 'predictable functional'; we redefine it here without any reference to measurability considerations:

Definition 2.3.12 (Strictly causal functionals)**.**
For $F : \Lambda \to \mathbb{R}^d$ denote $F_-(t, x_t) = F(t, x_{t-})$. F is *strictly causal* if $F = F_-$.

The following lemma follows from Def. 2.3.8.1(a) and (b) and Def. 2.3.8.2(a) and(b).

Lemma 2.3.13 (Pathwise regularity)**.** *Let* $F : \Lambda \to \mathbb{R}^d$*.*

(i) If F *is left-continuous, then* $t \longmapsto F_-(t, x_t)$ *is left-continuous and* $t \longmapsto F(t, x_t)$ *has left limits.*

(ii) If F *is right-continuous, then* $t \longmapsto F(t, x_t)$ *is right-continuous and* $t \longmapsto F_-(t, x_t)$ *has right limits.*

(iii) If F *is continuous, then* $t \longmapsto F_-(t, x_t)$ *(resp.* $t \longmapsto F(t, x_t)$*) is càglàd (resp. càdlàg) and its jump at time* t *is equal to* $\Delta F(t, x_t)$*.*

Example 2.3.14. Let $\Omega \subset Q_m^\pi$, then the functionals

(i) $F(t, x_t) := f(x(t));$ $\quad f \in C(\mathbb{R}^m),$

(ii) $F(t, x_t) := f([x](t));$ $\quad f \in C(\mathbb{R}^{m\times m}),$

(iii) $F(t, x_t) := \int_0^t (f \circ x) d[x];$ $\quad f \in C(\mathbb{R}^m, \mathbb{R}^{m\times m}),$

(iv) $F(t, x_t) := \int_0^t (\nabla f \circ x) dx;$ $\quad f \in C^2(\mathbb{R}^m),$

are continuous.

Proof. In the light of Remark 2.3.11, F is continuous if and only if F satisfies Def. 2.3.8.1 and 2 for all $(t, x_t) \in \Lambda$. Since conditions Def. 2.3.8.1(a),(b) and 2(a),(b) are easy to verify, we will put our focal on Def. 2.3.8.1(c),(d) and 2(c),(d). (i) is trivial. For (ii), we first remark from Def. 2.2.1 and (2.4) that

$$q_n \xrightarrow{\mathrm{J}_1} [x];$$
$$\Delta q_n(t'_n) = \Delta x^n(t'_n)\Delta x^n(t'_n)' \longrightarrow \Delta x(t)\Delta x(t)' = \Delta[x](t). \tag{2.7}$$

Since $[x^n](t) = q_n(t)$ and by (2.7), if $t_n \longrightarrow t$, the limits of $q_n(t_n)$ and $q_n(t_n-)$ are readily determined according to the rules laid down in [4, s4.2] and (ii) immediately follows from the continuity of f.

To show (iii) and (iv), it is suffice to assume $t_n \longrightarrow t$; $t_n \geq t'_n$ (i.e. the other criteria follow similar lines of proof, see [4, s4.2]). By (2.7) and [4, s4.2]

$$|q_n(t_n) - q_n(t'_n)| \longrightarrow 0. \tag{2.8}$$

A closer look at (iii), combined with Corollary 2.2.8, leads to

$$\begin{aligned} F(t_n, x^n_{t_n}) &= \int_0^{t_n} (f \circ x^n) d[x^n] \\ &= \sum_{\pi_n \ni t_i < t} \langle f(x(t_i)), (x(t_{i+1}) - x(t_i))(x(t_{i+1}) - x(t_i))' \rangle \longrightarrow F(t, x_t) \\ &+ \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \langle f(x(t_i)), (x(t_{i+1}) - x(t_i))(x(t_{i+1}) - x(t_i))' \rangle. \end{aligned}$$

By (2.8) and that $f \circ x$ is locally bounded on $\mathbb{R}_+$, we see that the absolute value of the last term is bounded by $\mathrm{const}|q_n(t_n) - q_n(t'_n)| \longrightarrow 0$.

For (iv), from the properties of the Föllmer integral [9], we first observe that

$$\begin{aligned} F(t_n, x^n_{t_n}) &= \int_0^{t_n} \nabla(f \circ x^n) dx^n \\ &= \sum_{\pi_n \ni t_i < t} \nabla f(x(t_i)) \cdot (x(t_{i+1}) - x(t_i)) \longrightarrow F(t, x_t) \\ &+ \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \nabla f(x(t_i)) \cdot (x(t_{i+1}) - x(t_i)). \end{aligned}$$

Define $\underline{t_n} := \min\{t_i > t'_n | t_i \in \pi_n\}$, $\overline{t_n} := \min\{t_i > t_n | t_i \in \pi_n\}$ and note that $\overline{t_n} \geq \underline{t_n} \geq t$, hence

$$|f(x(\overline{t_n})) - f(x(\underline{t_n}))| \longrightarrow 0.$$

Applying a second order Taylor expansion to f and by (2.8), we obtain

$$\left|\sum_{\pi_n \ni t_i \in (t'_n, t_n]} \nabla f(x(t_i)) \cdot (x(t_{i+1}) - x(t_i))\right| \leq |f(x(\overline{t_n})) - f(x(\underline{t_n}))| + \text{const}|q_n(t_n) - q_n(t'_n)| \longrightarrow 0.$$

□

Remark 2.3.15. If $x \in D_m$, so are x_T and x_{T-} and the corresponding piecewise constant approximation(s) in (2.6) shall be denoted by $(x_T)^n$ and $(x_{T-})^n$.

The following property may be derived from [3, Lemma 12.3] and [8, VI]:

Lemma 2.3.16. *Let $T \geq 0$, $x \in D_m$, then $(x_T)^n \xrightarrow{\mathrm{J}_1} x_T$.*

Lemma 2.3.17. *Let $(t, x) \in \Lambda$, $t_n \longrightarrow t$ and denote $t'_n := \max\{t_i < t | t_i \in \pi_n\}$. Then*

$$\begin{aligned}
&(i) && t_n \leq t'_n \Longrightarrow x^n_{t_n-} \xrightarrow{\mathrm{J}_1} x_{t-},\\
&(ii) && t_n < t'_n \Longrightarrow x^n_{t_n} \xrightarrow{\mathrm{J}_1} x_{t-},\\
&(iii) && t_n \geq t'_n \Longrightarrow x^n_{t_n} \xrightarrow{\mathrm{J}_1} x_t,\\
&(iv) && t_n > t'_n \Longrightarrow x^n_{t_n-} \xrightarrow{\mathrm{J}_1} x_t.
\end{aligned}$$

Proof. Let $t_n \leq t'_n$, by Lemma 2.3.16, we have $(x_{t-})^n \xrightarrow{\mathrm{J}_1} (x_{t-})$. Since x is càdlàg we observe

$$\|x^n_{t_n-} - (x_{t-})^n\|_\infty \leq \sup_{s \in [t_n, t'_n]} |x(t_n) - x(s)| + |x(t_n) - x(t-)| \longrightarrow 0,$$

and (i) follows immediately from [8, VI.1.23]. (ii)-(iv) follow similar lines of proof. □

Theorem 2.3.18. *Let Ω satisfy Assumption 2.1. Then:*

(i) Every J_1*-continuous functional is continuous.*

(ii) There exists a continuous functional which is not U*-continuous.*

(iii) There exists U*-continuous functionals which are not continuous.*

Proof. If F is J_1-continuous, then F satisfies Def. 2.3.8.1(a),(b) and 2(a),(b) due to Rem. 2.3.5 and 2.3.9. (i) now follows immediately from Lemma 2.3.17. (ii) is due to Example 2.3.14 and Lemma 2.3.7.

It remains to show (iii). We first note that the U topology on Λ is metrisable, hence sequential continuity is equivalent to continuity. Let us fix a $t_0 > 0$; $t_0 \notin \cup_n \pi_n$, define

$$F(t, x_t) := |\Delta x_t(t_0)|$$

on Λ. Observe that if $x_n \xrightarrow{\mathrm{U}} x$ in D_m then it is well known that:

$$\Delta x_n(s) \longrightarrow \Delta x(s) \tag{2.9}$$

for $s \geq 0$. In particular, if $t_n \longrightarrow t$; $x_n(\cdot \wedge t_n) \xrightarrow{\mathrm{U}} x_t$ then (2.9) implies $\Delta x_n(\cdot \wedge t_n)(s) \longrightarrow \Delta x_t(s)$ for $s \geq 0$, hence F is U-continuous on Λ.

On the other hand, we take an $x \in \Omega_0$; $\Delta x(t_0) \neq 0$, it follows from our choice of t_0 that

$$F(t_0, x^n_{t_0}) = |\Delta x^n(t_0)| \equiv 0,$$

hence by Def. 2.3.8.2(c), F is not continuous on Λ and (iii) follows. □

So, if Ω satisfies Assumption 2.1, Theorem 2.3.18 and Remark 2.3.5 imply that

- the $\pi-$topology is strictly finer than the J_1 topology.
- the $\pi-$topology and the U topology are non-comparable.

2.4 Smooth functionals

We recall that if f is a continuous function on a locally compact metric space X, then one also obtains from the continuity of f, (local) uniform continuity which implies (local) *local boundedness* and the existence of a *modulus of continuity*, two notions (esp. *modulus of continuity*) that were used to establish the change of variable formulas in [9].

In this section, we shall introduce weaker notions of *boundedness* and *modulus of continuity* for function defined on a domain Λ (Def. 2.3.1) and define the corresponding notion of a $C^{1,2}$ functional on Λ. We then introduce $\mathcal{S}(\Lambda)$ and $\mathcal{M}(\Lambda)$, two important subspaces of $C^{1,2}(\Lambda)$.

For $\Omega \subset Q_m^\pi$, it is shown that functionals such as quadratic variation and Föllmer integrals are not only $C^{1,2}$ but also belong to class $\mathcal{M}$, a sub-class of *infinitely differentiable* functionals.

We recall the definition of the horizontal and vertical derivatives [7, 6, 7]:

Definition 2.4.1 (Horizontal derivative)**.** $F : \Lambda \longmapsto \mathbb{R}$ is called differentiable in time or *horizontally differentiable* if the following limit exists for all $(t, x_t) \in \Lambda$:

$$\mathcal{D}F(t, x_t) := \lim_{h \downarrow 0} \frac{F(t+h, x_t) - F(t, x_t)}{h}.$$

Definition 2.4.2 (Vertical derivative)**.** $F : \Lambda \longmapsto \mathbb{R}$ is called *vertically differentiable* if for every $(t, x_t) \in \Lambda$, the map $f : \mathcal{U}_t(x) \longmapsto \mathbb{R}$:

$$e \longmapsto F\left(t, x_t + e\mathbb{1}_{[t,\infty)}\right)$$

is differentiable at 0. $\nabla_x F(t, x_t) := \nabla_e f(0)$ is called the vertical derivative of F at $(t, x_t) \in \Lambda$.

F is called differentiable if it is vertically and horizontally differentiable. We extend the above definitions to vector-valued maps $F : \Lambda \to \mathbb{R}^{d\times n}$ whose components $F_{i,j}$ satisfy the respective conditions.

Proposition 2.4.3. *A causal functional $F : \Lambda \to \mathbb{R}$ is strictly causal if and only if it is vertically differentiable with vanishing vertical derivative.*

Proof. The if part follows from the mean value theorem. The only if part:

Let $x \in \Omega$ and put $z := x_t + e1\!\!1_{[t,\infty)}$ then $z_{t-} = x_{t-}$ and

$$F(t, x_t + e1\!\!1_{[t,\infty)}) = F(t, z_t) = F_-(t, z_t) = F_-(t, x_t) = F(t, x_t),$$

by the strict causality of F (Def. 2.3.12). □

Definition 2.4.4 (Locally bounded functional)**.**
A function F on Λ is called *locally bounded* if for every $x \in \Omega$ and $T \geq 0$, there exists $n_0 \geq N_T(x)$ such that the family of maps

$$(t \longmapsto F(t, x_t^n), n \geq n_0)$$

is locally bounded on $[0, T]$.

Proposition 2.4.5. *Every continuous function on Λ is locally bounded.*

Proof. Let F be continuous; if F is not locally bounded, there exists $x \in \Omega$, $T \geq 0$, and a sub-sequence (n_k);

$$|F(t_{n_k}, x_{t_{n_k}}^{n_k})| > k, \quad \forall k \geq 1; \tag{2.10}$$

(t_{n_k}) is bounded on $[0, T]$.

For ease of notation, let $t_{n_k} \longrightarrow t \in [0, T]$ without actually passing through to the sub-sequence. Observe that one can always choose another sub-sequence, bounded (either above or below) by $t'_{n_k} = \max\{t_i < t | t_i \in \pi_{n_k}\}$. Since F is continuous, if $t_{n_k} < t'_{n_k}$ (resp. $t_{n_k} \geq t'_{n_k}$), then Def. 2.3.8.1(d) (resp. 2(c)) would contradict (2.10) as $k \uparrow \infty$. □

Lemma 2.4.6. *Let F be locally bounded;*

(i) F is left-continuous then F_- is locally bounded.

(ii) F is left-continuous then $t \longmapsto F_-(t, x_t)$ is locally bounded.

(iii) F is right-continuous then $t \longmapsto F(t, x_t)$ is locally bounded.

Proof. Since F is locally bounded, there exists a constant $K > 0$ such that

$$|F(t, x_t^n)| \leq K$$

for all $t \leq T$ and all n sufficiently large. If F is left-continuous, then Def. 2.3.8.1(b) implies

$$K \geq \lim_{s \uparrow t; s<t} |F(s, x_s^n)| = |F(t, x_{t-}^n)|,$$

(i) follows. If $t_n \longrightarrow t; t_n < t'_n$, then by the left-continuity of F (i.e. Def. 2.3.8.1(d)),

$$K \geq |F(t_n, x_{t_n}^n)| \longrightarrow |F(t, x_{t-})|,$$

(ii) follows. If F is right-continuous, then by Def. 2.3.8.2(c),

$$K \geq |F(t'_n, x_{t'_n}^n)| \longrightarrow |F(t, x_t)|,$$

(iii) follows. □

Definition 2.4.7 (Modulus of vertical continuity)**.**
We say that a function F on Λ admits a modulus of vertical continuity if for every $x \in \Omega$, $T \geq 0$ and $r > 0$ there exists a monotonic increasing function $\omega : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ with $\omega(0+) = 0$;

$$|F(t, x_{t-}^n + a\mathbb{1}_{[t,\infty)}) - F(t, x_{t-}^n + b\mathbb{1}_{[t,\infty)})| \leq \omega(|a-b|). \tag{2.11}$$

for all $a, b \in \mathcal{U}_{t-}(x^n) \cap \overline{B}_r(0)$, $t \leq T$ and sufficiently large n.

Example 2.4.8. Let $f \in C(\mathbb{R}_+ \times \mathbb{R}^m)$. Then $F : \Lambda \to \mathbb{R}$ defined by $F(t, x_t) := f(t, x(t))$ admits a modulus of vertical continuity.

Proof. For a given $x \in \Omega$ and $T \geq 0$, $r > 0$, put $\|x\|_T := \sup_{t \leq T} |x(t)|$, $r_0 := \alpha \|x\|_T + r$; $\alpha > 1$, then f is uniform continuous on $[0, T] \times \overline{B}_{r_0}(0)$ and a modulus of continuity of f on $[0, T] \times \overline{B}_{r_0}(0)$ is given by

$$\omega(\delta) := \sup_{|t-s|+|u-v| \leq \delta} |f(t, u) - f(s, v)|$$

which satisfies (2.11). □

Remark 2.4.9. If F, G admit moduli of vertical continuity, then $\alpha F + \beta G$ admits a modulus. If in addition, F_-, G_- are locally bounded, then FG admits a modulus of vertical continuity.

Lemma 2.4.10. *Let F be vertically differentiable and $(\nabla_x F)_-$ be locally bounded, if $\nabla_x F$ admits a modulus of vertical continuity then so does F.*

Proof. Since F is vertically differentiable and $\nabla_x F$ admits a modulus of vertical continuity ω, by the mean value theorem and the local boundedness of $(\nabla_x F)_-$, we obtain

$$|F(t, x^n_{t-} + a1\!\!1_{[t,\infty)}) - F(t, x^n_{t-} + b1\!\!1_{[t,\infty)})| \leq (\omega(r) + \text{const})\, |a - b|.$$

□

Definition 2.4.11 ($C^{1,2}$ functionals)**.**
We define $C^{1,2}(\Lambda)$ as the set of continuous functional F such that $\mathcal{D}F, \nabla_x F$ and $\nabla_x^2 F$ are defined on Λ and

(i) $\mathcal{D}F$ is right-continuous and locally bounded.

(ii) $(\nabla_x F)_-$ is left-continuous,

(iii) $(\nabla_x^2 F)_-$ is left-continuous, locally bounded and admits a modulus of vertical continuity.

If in addition, $(\nabla_x F)_-$ is locally bounded, then we denote $F \in C_b^{1,2}(\Lambda)$.

We now introduce two classes of functionals which, as we will observe later, play a special role in the context of stochastic analysis:

Definition 2.4.12 (Class $\mathcal{S}$)**.**
A continuous and differentiable functional F is of *class* $\mathcal{S}$ if $\mathcal{D}F$ is right-continuous and locally bounded, $\nabla_x F$ is left-continuous and strictly causal. We denote by $\mathcal{S}(\Lambda)$ the vector space of class $\mathcal{S}$ functionals.

Definition 2.4.13 (Class $\mathcal{M}$)**.**
A functional $F \in \mathcal{S}(\Lambda)$ is of *class* $\mathcal{M}$ if $\mathcal{D}F = 0$. We denote $\mathcal{M}(\Lambda)$ the set

of *class* $\mathcal{M}$ functionals and $\mathcal{M}_b(\Lambda)$ the set of functionals $F \in \mathcal{M}(\Lambda)$ whose vertical derivative $\nabla_x F$ is locally bounded.

Remark 2.4.14. Every functional of class $\mathcal{M}$ is infinitely differentiable by Prop. 2.4.3.

From Remarks 2.4.9, Lemma 2.4.6 and 2.4.10, we observe that $C^{1,2}(\Lambda)$, $\mathcal{S}(\Lambda)$, $\mathcal{M}(\Lambda)$, $\mathcal{M}_b(\Lambda)$ are vector spaces; $C_b^{1,2}(\Lambda)$ is an algebra.

Lemma 2.4.15.
Let $\Omega \subset Q_m^\pi$. If $\phi : \Lambda \longmapsto \mathbb{R}^{m\times m}$ is such that ϕ_- is left-continuous and locally bounded, then

$$(t, x_t) \in \Lambda \mapsto F(t, x_t) := \int_0^t \phi(t, x_{t-}) d[x]$$

is a continuous functional.

Proof. Since $t \longmapsto \phi(t, x_{t-})$ is left-continuous and locally bounded (Lemma 2.3.13(i)) and that $t \longmapsto [x_i, x_j](t)$ is in BV, càdlàg with $\Delta[x_i, x_j] \equiv \Delta x_i \Delta x_j$ (Prop. 2.2.2), it follows F is a finite sum of Lebesgue-Stieltjes integrals and satisfies conditions Def. 2.3.8.1(a),(b) and 2(a),(b). For the other conditions in Def. 2.3.8, it is suffice to assume $t_n \longrightarrow t$; $t_n \geq t'_n$ (i.e. the other criteria follow similar lines). Define

$$\phi_n(s) := \phi(t_0, x^n_{t_0-})\mathbb{1}_{\{0\}}(s) + \sum_{t_i \in \pi_n} \phi(t_i, x^n_{t_i-})\mathbb{1}_{(t_i, t_{i+1}]}(s),$$

which is a $\mathbb{R}^{m\times m}$-valued left-continuous function on $\mathbb{R}_+$. By the local boundedness of ϕ_-, we see that $\exists n_0 \geq N(x)$; $(\phi_n)_{n \geq n_0}$ is locally bounded on $\mathbb{R}_+$ and converges pointwise to $s \longmapsto \phi(s, x_{s-})$ on $\mathbb{R}_+$. By Cor. 2.2.8(ii), we obtain

$$\begin{aligned} F(t_n, x^n_{t_n}) &= \int_0^{t_n} \phi(s, x^n_{s-}) d[x^n] \\ &= \sum_{\pi_n \ni t_i < t} \langle \phi(t_i, x^n_{t_i-}), (x(t_{i+1}) - x(t_i)(x(t_{i+1}) - x(t_i)' \rangle \longrightarrow F(t, x_t) \\ &+ \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \langle \phi(t_i, x^n_{t_i-}), (x(t_{i+1}) - x(t_i)(x(t_{i+1}) - x(t_i)' \rangle. \end{aligned}$$

Since $q_n \xrightarrow{J_1} [x]$ and by [4, s4.2], the last term is bounded by

$$\text{const}|q_n(t_n) - q_n(t'_n)| \longrightarrow 0.$$

$\square$

As we shall see in the following two examples, in the path-independent case functionals of class $\mathcal{M}$ are simply affine functions. However, in the general case, they are no longer simple and include Föllmer integrals.

Example 2.4.16.
Let $S_m \subset \Omega$, $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m)$ and

$$F(t, x_t) := f(t, x(t)),$$

then F is of class $\mathcal{M}$ iff $f(t, u)$ is independent of t and affine linear in u.

Proof. For the if part: We can write $f(t, u) = \alpha + \beta \cdot u$ and hence

$$F(t, x_t) = \alpha + \beta x(t)$$

on Λ for some contants $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^m$. By Example 2.3.14(i) and computing the derivatives of F, we see that F is of class $\mathcal{M}$. The only if part: By Def. 2.4.13 and Prop. 2.4.3, we first obtain

(i) $\partial_t f(t, x(t)) = \mathcal{D}F(t, x_t) = 0$,

(ii) $\nabla^2 f(t, x(t)) = \nabla_x^2 F(t, x_t) = 0$,

$\forall\ t \geq 0$, $x \in \Omega$. Since $S_m \subset \Omega$, we have

$$R := \{(t, x(t))|t \in \mathbb{R}_+, x \in \Omega\} = \mathbb{R}_+ \times \mathbb{R}^m,$$

hence $\partial_t f \equiv \nabla^2 f \equiv 0$ on $\mathbb{R}_+ \times \mathbb{R}^m$. By the mean value theorem, we deduce that $\nabla f \equiv \beta$ on R, for some $\beta \in \mathbb{R}^m$. $\square$

Remark 2.4.17. The condition $S_m \subset \Omega$ may be weaken to simply requiring that R be a convex region in $\mathbb{R}_+ \times \mathbb{R}^m$, in this case, the only if part holds up to R.

Example 2.4.18 (Path-dependent examples)**.** Let $\Omega \subset Q^\pi_m$, $\phi : \Lambda \longmapsto \mathbb{R}^{m\times m}$ such that ϕ_- is left-continuous and locally bounded, $f \in C^2(\mathbb{R}^m)$, $f_i \in C^1(\mathbb{R})$. Then the functionals

(i) $F(t, x_t) := \int_0^t \phi(s, x_{s-})d[x]$,

(ii) $F(t, x_t) := \int_0^t (\nabla f \circ x)dx$,

(iii) $F(t, x_t) := \sum_i \left(\int_0^t (x_i(t) - x_i)f_i \circ x_i dx_i - \int_0^t f_i \circ x_i d[x_i]\right)$

belong to $C^{1,2}_b(\Lambda)$ and (ii) and (iii) are of class $\mathcal{M}_b$.

Proof. The functional in (iii) is well defined, since

$$F(t, x_t) = \sum_i \left(x_i(t)\int_0^t f_i \circ x_i dx_i - \int_0^t x_i f_i \circ x_i dx_i - \int_0^t f_i \circ x_i d[x_i]\right). \tag{2.12}$$

The first two integrals in (2.12) are Föllmer integrals, defined as a limit of Riemann sums along π, while the last one is a Lebesgue-Stieltjes integral. Continuity of F in (i), (ii) and (iii) follows from Lemma 2.4.15 and Example 2.3.14. Since $\mathcal{D}F \equiv 0$ in all cases, let us first compute $\nabla^k_x F$ for $k = 1, 2$ and demonstrate that F possesses the required properties. In case of (i), we have

$$\nabla_x F(t, x_t) = (\phi + \phi')(t, x_{t-})\Delta x(t), \quad \nabla^2_x F(t, x_t) = (\phi + \phi')(t, x_{t-}),$$

which are left-continuous, locally bounded and $\nabla^2_x F(t, x_t)$ is strictly causal, by Prop. 2.4.3, Lemma 2.4.6(ii) and 2.4.10, F is $C^{1,2}_b$. In case of (ii), we obtain

$$\nabla_x F(t, x_t) = \nabla f(x(t-)),$$

which is left-continuous, locally bounded and strictly causal, hence F is of class $\mathcal{M}_b$. In case of (iii), we apply ∇_x to (2.12) and verify that

$$\begin{aligned}\nabla_{x_i} F(t, x_t) &= \int_0^t f_i \circ x_i dx_i - f_i(x_i(t-))\Delta x_i(t) \\ &= \left(\int f_i \circ x_i dx_i \right)(t-). \end{aligned} \tag{2.13}$$

Applying $f(x) := \int_0^{x_i} f_i(\lambda)d\lambda;\ x \in \mathbb{R}^m$ to (ii) and by Prop. 2.4.5 and Lemma 2.4.6(i), we see that each $\nabla_{x_i} F$ is left-continuous and locally bounded and so is $\nabla_x F$. Since $\nabla_x F$ is strictly causal, F is of class $\mathcal{M}_b$. □

2.5 Pathwise integration and change of variable formulas

In this section, we discuss pathwise integration for causal functionals along paths in a generic domain. In contrast to rough integration theory [14], we define integrals as limits of (left) Riemann sums, when such limits exist. We obtain change of variable formulas for such integral and an analogue of the classical Fundamental theorem of calculus for functionals of class $\mathcal{M}$. For paths that possess quadratic variation, we obtain a functional Föllmer -Itô formula which extends [7, Theorem 4].

In particular, we show that pathwise integral is of class $\mathcal{M}$ and that functionals of class $\mathcal{M}$ are *primitives* i.e. are representable as pathwise integrals, a fact that facilitates the *computation* of pathwise integrals, as in classical calculus.

Lemma 2.5.1.
Let F be a left-continuous functional, differentiable in time, if $\mathcal{D}F$ is right-continuous and locally bounded, then

$$F(t, x_s) - F(s, x_s) = \int_s^t \mathcal{D}F(u, x_u)du, \tag{2.14}$$

for all $x \in \Omega$, $t \geq s \geq 0$.

Proof. Put $z := x_s \in \Omega$, then $z_t = x_s$ for $t \geq s$ and $z_{t-} = x_s$ for $t > s$. Define $f(t) := F(t, x_s)$ for $t \geq s$, then $f(t) = F(t, z_t)$ on $[s, \infty)$ and $f(t) = F(t, z_{t-})$ on (s, ∞). Since F is differentiable in time, f is right differentiable (hence right-continuous) on $[s, \infty)$ and the right derivative $f'(t)$ is $\mathcal{D}F(t, x_s)$ on $[s, \infty)$. Since F is left-continuous, it follows from Lemma 2.3.13 that $f(t) = F(t, z_{t-})$ is left-continuous on (s, ∞), hence we have first established that f is continuous on $[s, \infty)$. Next, we observe that

$$f'(u) = \mathcal{D}F(u, x_s) = \mathcal{D}F(u, z_u)$$

on $[s, \infty)$. The right continuity of $\mathcal{D}F$ and Lemma 2.3.13 implies that f' is right-continuous on $[s, \infty)$. Since $\mathcal{D}F$ is right-continuous and locally bounded, it follows from Lemma2.4.6(ii) that

$$u \longrightarrow \mathcal{D}F(u, z_u)$$

is locally bounded. Hence, f' is right-continuous and bounded on $[s, T]$, hence Riemann integrable. We can conclude using a stronger version [10] of the Fundamental theorem of calculus. □

Lemma 2.5.2. *Let ϕ be a right-continuous and locally bounded on Λ, then*

$$\sum_{\pi_n \ni t_i \leq T} \int_{t_i}^{t_{i+1}} \phi(t, x^n_{t_i}) dt \longrightarrow \int_0^T \phi(t, x_t) dt,$$

for all $x \in \Omega$, $T \geq 0$.

Proof. Define

$$\phi_n(t) := \sum_{\pi_n \ni t_i \leq T} \phi(t, x^n_{t_i}) \mathbb{1}_{[t_i, t_{i+1})}(t) = \sum_{\pi_n \ni t_i \leq T} \phi(t, x^n_t) \mathbb{1}_{[t_i, t_{i+1})}(t).$$

By the local boundedness of ϕ, we see that $\exists n_0 \geq N(x)$; $(\phi_n)_{n \geq n_0}$ is locally bounded on $[0, T]$. Since ϕ is right-continuous, it follows from Lemma 2.3.13 that $t \longmapsto \phi_n(t)$ is right-continuous (hence measurable) on $[0, T]$ and from Def.2.3.8.2(c) that ϕ_n converges to $t \longmapsto \phi(t, x_t)$ pointwise on $[0, T]$. and (i) follows from dominated convergence. □

Corollary 2.5.3. *Let ϕ be a right-continuous and locally bounded Λ, then*

$$(t, x_t) \longmapsto \int_0^t \phi(s, x_s) ds$$

is continuous.

Proof. The path $t \longmapsto \int_0^t \phi(s, x_s) ds$ is continuous. The rest follows from the local boundedness of ϕ and Lemma 2.5.2. □

Definition 2.5.4 (Pathwise integrability)**.**
Let $\phi : \Lambda \longmapsto \mathbb{R}^m$ such that ϕ_- is left-continuous. For every $x \in \Omega$, define

$$\mathbf{I}(t, x_t^n) := \sum_{\pi_n \ni t_i \leq t} \phi(t_i, x_{t_i-}^n) \cdot (x(t_{i+1}) - x(t_i)). \tag{2.15}$$

ϕ is said to be $\Lambda-$*integrable* if

- the limit $\mathbf{I}(t, x_t) := \lim_n \mathbf{I}(t, x_t^n)$ exists for each $(t, x_t) \in \Lambda$, and
- the map $\mathbf{I} : \Lambda \mapsto \mathbb{R}$ is continuous.

Note that the pathwise integral is defined as a limit of (left) Riemann sums, and not compensated Riemann sums as in rough path theory [13, 14]. One case in which such Riemann sums are known to converge is for gradients of C^2 functions along paths of finite quadratic variation:

Example 2.5.5. Let $\Omega = Q_m^\pi$. Then by the results of [9], for any $f \in C^2(\mathbb{R}^m)$, $\phi : \Lambda \longmapsto \mathbb{R}^m$ defined by $\phi(t, x) = \nabla_x f(t, x_t)$ is Λ-integrable and $\mathbf{I}(t, x)$ is the Föllmer integral [5]. Note that the continuity property of $\mathbf{I}$ is a consequence (and indeed, the main motivation) of the construction of the $\Lambda-$topology in Section 2.3.

Theorem 2.5.6. *Let $\phi : \Lambda \longmapsto \mathbb{R}^m$ such that ϕ_- is left-continuous and $\mathbf{I}$ the integration map defined as in (2.15). If for every $x \in \Omega$, $T > 0$ the sequence of step functions on $[0, T]$*

$$g_n(t) := \mathbf{I}(t, x_t^n),$$

is Cauchy in $(D[0,T], \mathfrak{d}_{\mathrm{J}_1})$, *then* ϕ *is* Λ*-integrable.*

Proof. Since (g_n) is Cauchy and $(D[0,T], \mathfrak{d}_{\mathrm{J}_1})$ is complete, there exists a $G \in D$ such that $g_n \overset{\mathrm{J}_1}{\longmapsto} G$. Hence $g_n(t) \mapsto G(t)$ for every continuity point of G on $[0,T]$. Observe that

$$\Delta g_n(t) = \begin{cases} \phi(t_i, x^n_{t_i-}) \cdot (x(t_{i+1}) - x(t_i)), & \text{if } t = t_i \in \pi_n. \\ 0, & \text{otherwise.} \end{cases} \tag{2.16}$$

If $\Delta G(t) > 0$, there exists [8, VI.2.1(a)] a sequence $t^*_n \to t$; $\Delta g_n(t^*_n) \to \Delta G(t)$. Using the fact that ϕ_- is left-continuous, x is càdlàg and (2.16), we see that

$$\lim_n \Delta g_n(t^*_n) = \phi(t, x_{t-}) \cdot \Delta x(t) = \lim_n \phi(t'_n, x^n_{t'_n-}) \cdot \Delta x^n(t'_n) = \lim_n \Delta g_n(t'_n), \tag{2.17}$$

else we will contradict $\Delta G(t) > 0$. Applying [8, VI.2.1(b)], we deduce that (t^*_n) must coincide with (t'_n) for all n sufficiently large and by [8, VI.2.1(b.3)], we have established that

$$g_n(t) \longrightarrow G(t), \tag{2.18}$$

hence we can define $\mathbf{I}(t, x_t) := G(t)$ on $[0,T]$.

Put $t''_n := \min\{t_i > t'_n | t_i \in \pi_n\}$, $z := x_{t-} \in \Omega$, it follows from (2.15), (2.17) and (2.18) that

$$\mathbf{I}(t, x_{t-}) = \lim_n \mathbf{I}(t, z^n_t) = \lim_n \big(\mathbf{I}(t, x^n_t) - \phi(t'_n, x^n_{t'_n-}) \cdot (x(t''_n) - x(t-))\big) = G(t-),$$

hence $t \longmapsto \mathbf{I}(t, x_t)$ is càdlàg and its jump at time t is $\mathbf{I}(t, x_t) - \mathbf{I}(t, x_{t-})$. If $t_n \longrightarrow t$, the limits of $g_n(t_n)$ and $g_n(t_n-)$ are readily determined according to (2.17) and [8, VI.2.1(b)]. The continuity criteria in Def. 2.3.8 are all satisfied. $\square$

Proposition 2.5.7. *Let* ϕ *be* Λ*-integrable. Then* $\mathcal{D}\mathbf{I} = 0$ *and* $\nabla_x \mathbf{I} = \phi_-$ *on* Λ.

Proof. Let $(t, x_t) \in \Lambda$ and $z := x + e\mathbb{1}_{[t,\infty)} \in \Lambda$. Then

$$\begin{aligned}\mathbf{I}(t, z_t) - \mathbf{I}(t, x_t) &= \lim_n \left(\mathbf{I}(t, z_t^n) - \mathbf{I}(t, x_t^n)\right) \\ &= \lim_n \phi(t'_n, z^n_{t'_n-}) \cdot e \\ &= \lim_n \phi(t'_n, x^n_{t'_n-}) \cdot e = \phi(t, x_{t-}) \cdot e,\end{aligned}$$

by the continuity of $\mathbf{I}$ and left-continuity of ϕ_-. □

Theorem 2.5.8 (Change of variable formula for class $\mathcal{S}$ functionals)**.** *Let $F \in \mathcal{S}(\Lambda)$. Then for any $(T, x_T) \in \Lambda$,*

$$F(T, x_T) = F(0, x_0) + \int_0^T \mathcal{D}F(t, x_t)dt + \int_0^T \nabla_x F(t, x_{t-})dx,$$

where

$$\int_0^T \nabla_x F(t, x_{t-})dx := \lim_n \sum_{\pi_n \ni t_i \le T} \nabla_x F(t_i, x^n_{t_i-}) \cdot (x(t_{i+1}) - x(t_i)) \tag{2.19}$$

exists.

Proof. Appendix § 2.7. □

Remark 2.5.9. By Prop. 2.5.7, we see that all pathwise integrals are functionals of class $\mathcal{M}$, hence by Thm. 2.5.8, we can write

$$\mathbf{I}(t, x_t) = \int_0^t \phi dx. \tag{2.20}$$

As we shall see, the converse is also true, all integrals that may be defined by (2.19) are pathwise integral in the sense of Def. 2.5.4:

Corollary 2.5.10 (Decomposition)**.** *Let $F \in \mathcal{S}(\Lambda)$. Then $M : \Lambda \to \mathbb{R}$ defined by*

$$M(t, x_t) := F(t, x_t) - F(0.x_0) - \int_0^t \mathcal{D}F(s, x_s)ds$$

is of class $\mathcal{M}$ and $\nabla_x M = \nabla_x F$. In particular, M may be represented as a pathwise integral: there exists a $\Lambda-$integrable functional $\phi : \Lambda \to \mathbb{R}^m$ such that $M = \mathbf{I}$:

$$\forall (t,x) \in \Lambda, \qquad M(t,x) = \int_0^t \phi.dx$$

Proof. By differentiating M, we obtain $\mathcal{D}M = 0$ and $\nabla_x M = \nabla_x F$. Continuity of M follows from Corollary 2.5.3 and Theorem 2.5.8, hence by (2.19), M is a pathwise integral. □

In fact, all functionals of class $\mathcal{M}$ have an integral representation. We obtain as a corollary a Fundamental theorem of calculus for Föllmer integrals:

Corollary 2.5.11.

(i) Let ϕ be Λ-integrable. Then the map $\mathbf{I} : (t, x_t) \in \Lambda \mapsto \int_0^t \phi.dx$ is continuous, differentiable and

$$\nabla_x \mathbf{I} = \phi_-.$$

(ii) Let $\phi : \Lambda \to \mathbb{R}$. If $F \in \mathcal{M}(\Lambda)$ such that $\nabla_x F = \phi_-$, then ϕ is Λ-ntegrable and

$$\int_0^t \phi dx = F(t, x_t) - F(0, x_0).$$

Proof. (i) is due to Prop. 2.5.7 and Rem. 2.5.9. (ii) is due to (2.19) and Cor. 2.5.10. □

Example 2.5.12. Let $\Omega \subset Q^\pi_m$, $f_i \in C^1(\mathbb{R})$, then

$$\int_0^T \left(\int f_1 \circ x_1 dx_1, \ \ldots, \int f_m \circ x_m dx_m \right)' dx = \sum_i \left(\int_0^T (x_i(T) - x_i) f_i \circ x_i dx_i - \int_0^T f_i \circ x_i d[x_i] \right), \tag{2.21}$$

by an application of Cor. 2.5.11(ii) to the RHS of (2.21), Example. 2.4.18(iii) and (2.13).

An important consequence of Theorem 2.5.8 is to relate class $\mathcal{M}$ functionals with a pathwise analogue of the *martingale* prpoperty. The concept of martingale was originally introduced to model the outcome of a *fair game* [22], which cannot lead to a profit across all scenarios. The following result, which does not make use of any probabilistic notion, shows that a class $\mathcal{M}$ functional represents the outcome of a 'fair game':

Theorem 2.5.13 (Fair game)**.**
Let $M \in \mathcal{M}$. If there exists $T > 0$ such that

$$M(T, x_T) - M(0, x_0) \geq 0, \quad \forall x \in \Omega,$$

then

$$\forall x \in \Omega, \quad M(T, x_T) = M(0, x_0).$$

This result suggests that class $\mathcal{M}$ functionals may be considered pathwise analogues of martingales.

Proof. Since $\mathcal{D}M$ vanishes, by Lemma 2.5.1 we obtain

$$M(t, x_t) = M(t, x_t) + \int_t^T \mathcal{D}M(s, x_t)ds = M(T, x_t) \geq 0 \tag{2.22}$$

for all $t \leq T$, where the last inequality is due to $x_t \in \Omega$. Suppose there exists $z \in \Omega$; $M(T, z_T) > 0$. By Thm. 2.5.8 and the continuity of M, it follows

$$M(T, z_T^n) = \sum_{\pi_n \ni t_i \leq T} \nabla_x M(t_i, z_{t_i-}^n)(z(t_{i+1}) - z(t_i)) > 0 \tag{2.23}$$

for all n sufficiently large. Define $t_n^* := \min\{t_i \in \pi_n | M(t_i, z_{t_i}^n) > 0\}$, then $t_n^* \leq T$. By (2.22), (2.23), the left-continuity of M and the fact that $z^n \in \Omega$, we obtain

$$M(t_n^*, z_{t_n^*}^n) > M(t_n^*, z_{t_n^*-}^n) = 0,$$

hence $M(t_n^*, z^n_{t_n^*}) = \nabla_x M(t_n^*, z^n_{t_n^*-})\Delta z(t_n^*) > 0$. Def. 2.3.1(ii) implies that there exists $\epsilon > 0$;

$$z^* := z^n_{t_n^*-} - \epsilon \Delta z(t_n^*) \mathbb{1}_{[t_n^*,\infty)} \in \Omega,$$

hence $M(t_n^*, z^*_{t_n^*}) = \nabla_x M(t_n^*, z^n_{t_n^*-})(-\epsilon\Delta z(t_n^*)) < 0$, which contradicts (2.22). □

The following change of variable formula for causal functionals extends [7, Theorem 4] to $C^{1,2}(\Lambda)$, removing the condition linking the partition sequence π with the jump times of a path:

Theorem 2.5.14 (Change of variable formula for $C^{1,2}$ functionals)**.**
Let $x \in \Omega \cap Q_m^\pi$. For any $F \in C^{1,2}(\Lambda)$ the following Föllmer -Itô formula holds:

$$\begin{aligned} F(T, x_T) = F(0, x_0) + \int_0^T \mathcal{D}F(t, x_t)dt + \int_0^T \nabla_x F(t, x_{t-})dx \qquad (2.24) \\ + \frac{1}{2}\int_0^T \nabla_x^2 F(t, x_{t-})d[x]^c + \sum_{t\le T} \left(\Delta F(t, x_t) - \nabla_x F(t, x_{t-}) \cdot \Delta x(t)\right), \end{aligned}$$

where the series is absolute convergent and the pointwise limit

$$\int_0^T \nabla_x F(t, x_{t-})dx := \lim_{n\to\infty} \sum_{\pi_n \ni t_i \le T} \nabla_x F(t_i, x^n_{t_i-}) \cdot (x(t_{i+1}) - x(t_i)) \qquad (2.25)$$

exists.

Proof. Appendix § 2.7. □

Proposition 2.5.15. *Let $\Omega \subset Q_m^\pi$ and $F \in C^{1,2}(\Lambda)$. Then*

$$\begin{aligned} J : \Lambda &\longmapsto \mathbb{R} \\ (t, x) &\longmapsto J(t, x_t) := \int_0^t \nabla_x F(s, x_s)dx \end{aligned}$$

is continuous. In particular, $\nabla_x F$ is integrable and J is a pathwise integral (Def. 2.5.4).

Proof. We apply the functional change of variable formula (Thm. 2.5.14) to F. Rearranging the terms in (2.24) we observe that $t \longmapsto J(t, x_t)$ is càdlàg whose jump at time t is $J(t, x_t) - J(t, x_{t-})$. It remains to show that J satisfies the continuity criteria Def. 2.3.8.1(c),(d) and 2(c),(d). It is suffice to assume $t_n \longrightarrow t$; $t_n \geq t'_n$ (i.e. the other criteria follow similarly). By (2.25) and that x is right-continuous, we first obtain

$$\begin{aligned} J(t_n, x^n_{t_n}) &= \int_0^{t_n} \nabla_x F(t, x^n_{t-}) dx^n \\ &= \sum_{\pi_n \ni t_i < t} \nabla_x F(t_i, x^n_{t_i-}) \cdot (x(t_{i+1}) - x(t_i)) \longrightarrow I(t, x_t) \\ &+ \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \nabla_x F(t_i, x^n_{t_i-}) \cdot (x(t_{i+1}) - x(t_i)). \end{aligned} \tag{2.26}$$

We have to show that the rest term (2.26) vanishes as $n \uparrow \infty$. Applying (2.24) to the path x^n and by the local boundedness of $\mathcal{D}F$, we have

$$\begin{aligned} \left| \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \nabla_x F(t_i, x^n_{t_i-}) \cdot \Delta x^n(t_i) \right| \leq& |F(t_n, x^n_{t_n}) - F(t'_n, x^n_{t'_n})| \\ &+ \text{const}|t_n - t'_n| \\ &+ \left| \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \Delta F(t_i, x^n_{t_i}) - \nabla_x F(t_i, x^n_{t_i-}) \cdot \Delta x^n(t_i) \right|. \end{aligned}$$

Since $t_n \geq t'_n$; $t_n, t'_n \longrightarrow t$ and by the right continuity of F the first two terms vanish. Since $(\nabla_x^2 F)_-$ is locally bounded and $\nabla_x^2 F$ admits a modulus, applying a second order Taylor expansion to the third term, we obtain

$$\left| \sum_{\pi_n \ni t_i \in (t'_n, t_n]} \Delta F(t_i, x^n_{t_i}) - \nabla_x F(t_i, x^n_{t_i-}) \cdot \Delta x^n(t_i) \right| \leq \text{const}|q_n(t_n) - q_n(t'_n)| \longrightarrow 0,$$

by the fact that $q_n \xrightarrow{\text{J}_1} [x]$ and [4, §4.2]. □

2.6 Application to paths with finite quadratic variation

We now examine in more detail the case of paths of finite quadratic variation and apply the results developed in §.2.5 to the case $\Omega \subset Q_m^\pi$. As we have already shown, integration and differentiation are inverse operations (Cor. 2.5.11). Using functionals of class $\mathcal{M}$, we show that these operations may be viewed as *isomorphisms* between certain spaces. We also obtain a pathwise identity related to Itô's isometry (Theorem 2.6.4).

The key objects here are functionals of class $\mathcal{M}$, which are *primitives* (e.g. 2.5.12) and may be understood as the pathwise analogue of martingales (Thm. 2.5.13). In addition, we shall show that class $\mathcal{M}$ are canonical *solutions* to path dependent heat equations.

Let us introduce the following vector spaces of integrands:

$$L(\Lambda) := \{\nabla_x F | F \in C^{1,2}(\Lambda)\}, \qquad L_b(\Lambda) := \{\nabla_x F | F \in C_b^{1,2}(\Lambda)\},$$
$$\mathcal{L}(\Lambda) := \{\nabla_x F | F \in \mathcal{M}(\Lambda)\}, \qquad \mathcal{L}_b(\Lambda) := \{\nabla_x F | F \in \mathcal{M}_b(\Lambda)\}.$$

By Prop. 2.5.15, the integral operator

$$\int : \phi \in L(\Lambda) \longmapsto \mathbf{I} \in \mathbb{R}^\Lambda,$$

where $\mathbf{I}$ is given by (2.20), is a well defined linear operator.

Example 2.6.1 (Path-dependent 1-form)**.**
Let $f_i \in C^1(\mathbb{R})$, $i = 1, \ldots, m$ then

$$\phi(t, x_t) := \left(\left(\int f_1 \circ x_1 dx_1 \right)(t-), \ldots, \left(\int f_m \circ x_m dx_m \right)(t-) \right)'$$

defines an element of $\mathcal{L}_b(\Lambda)$.

Proof. See Example 2.4.18(2.13). □

Lemma 2.6.2.

(i) If $\phi \in L(\Lambda)$ *then* $\int \phi \in \mathcal{M}(\Lambda)$ *and* $\nabla_x(\int \phi) = \phi_-$.

(ii) If $\phi \in L_b(\Lambda)$ *then* $\int \phi \in \mathcal{M}_b(\Lambda)$ *and* $\nabla_x(\int \phi) = \phi_-$.

(iii) If $\phi \in \mathcal{L}(\Lambda)$ *then* $\int \phi \in \mathcal{M}(\Lambda)$ *and* $\nabla_x(\int \phi) = \phi$.

(iv) If $\phi \in \mathcal{L}_b(\Lambda)$ *then* $\int \phi \in \mathcal{M}_b(\Lambda)$ *and* $\nabla_x(\int \phi) = \phi$.

Proof. It is due to Prop. 2.5.15 and Cor.2.5.11(i). □

Corollary 2.6.3. *Define*

$$\mathcal{M}_0(\Lambda) := \{F \in \mathcal{M}_b(\Lambda) | F(0, x_0) \equiv 0\},$$

then the integral operator

$$\int : \mathcal{L}_b(\Lambda) \longmapsto \mathcal{M}_0(\Lambda)$$

is an isomorphism and the inverse of $\int$ *is the differential operator* ∇_x.

Proof. Injectivity follows from Lemma 2.6.2(iv). Surjectivity is due to Cor. 2.5.11(ii). □

We now obtain a pathwise Itô 's identity, in the spirit of the pathwise Burkholder-Davis-Gundy inequality [2] and give an example of application. For $\phi, \psi \in \mathcal{L}_b(\Lambda)$ define $\{\phi, \psi\} \in \mathcal{L}_b(\Lambda)$ by

$$\begin{aligned} \{\phi, \psi\} : \Lambda \quad &\mapsto \quad \mathbb{R}^d \\ (t, x) \quad &\to \quad \left(\psi \int_0^{\cdot} \phi . dx + \phi \int_0^{\cdot} \psi . dx\right)(t, x_{t-}). \end{aligned}$$

Theorem 2.6.4. *For all* $\phi, \psi \in \mathcal{L}_b(\Lambda)$, $\{\phi, \psi\} \in \mathcal{L}_b(\Lambda)$ *and*

$$\left(\int \phi dx\right)\left(\int \psi dx\right) = \int \phi \psi' d[x] + \int \{\phi, \psi\} dx.$$

Proof. $C_b^{1,2}$ is an algebra. Let $\phi, \psi \in \mathcal{L}_b(\Lambda)$, put $F := \int \phi dx, G := \int \psi dx$, then $F, G \in \mathcal{M}_b$ by Lemma 2.6.2(iv). Since $\mathcal{M}_b \subset C_b^{1,2}$, it follows $FG \in C_b^{1,2}$.

Apply the change of variable formula (Thm. 2.5.14) to FG and by Lemma 2.6.2(ii), the proof is complete. $\square$

Corollary 2.6.5.
Let $\mathcal{E} \subset \mathcal{L}_b(\Lambda)$ be a subspace such that

$$\forall \phi, \psi \in \mathcal{E}, \quad \{\phi, \psi\} \in \mathcal{E}$$

and denote $\mathbf{I}(\mathcal{E})$ the image of $\mathcal{E}$ under $\int$. If $\mathbb{E}$ is any positive element of the algebraic dual $C^*(\Lambda)$ such that $\mathbf{I}(\mathcal{E}) \subset ker(\mathbb{E})$, then

$$\left\langle \int \phi dx, \int \psi dx \right\rangle_{\mathbf{I}(\mathcal{E})} := \mathbb{E}\left(\int \phi dx \int \psi dx \right) = \mathbb{E}\left(\int \phi \psi' d[x] \right) =: \langle \phi, \psi \rangle_{\mathcal{E}}$$

holds for all $\phi, \psi \in \mathcal{E}$.

In particular, the bracket $\langle .,. \rangle_{\mathcal{E}}$ induces a semi-norm on $\mathcal{E}$. Denoting $\tilde{\mathcal{E}}$ the quotient space induced by the semi-norm, the integral operator

$$\begin{aligned} \tilde{\int} : \tilde{\mathcal{E}} &\longmapsto \mathbf{I}(\tilde{\mathcal{E}}) \\ \tilde{\phi} &\longmapsto \tilde{\int} \tilde{\phi} := \int \phi \end{aligned}$$

is an isometric isomorphism between the pre-Hilbert spaces $\tilde{\mathcal{E}}$ and $\mathbf{I}(\tilde{\mathcal{E}})$. The inverse of $\tilde{\int}$ is the differential operator

$$\begin{aligned} \tilde{\nabla}_x : \mathbf{I}(\tilde{\mathcal{E}}) &\longmapsto \tilde{\mathcal{E}} \\ \tilde{F} &\longmapsto \tilde{\nabla}_x \tilde{F} := \nabla_x F \end{aligned}$$

Proof. The result is a consequence of Cor. 2.6.3 and Thm. 2.6.4. $\square$

We conclude with a discussion on the relation between class $\mathcal{M}(\Lambda)$ and harmonic functionals, which illustrates the universal nature of class $\mathcal{M}(\Lambda)$ as canonical solutions to path-dependent heat equations. Let Σ be a right-continuous function on Λ taking values in positive-definite symmetric $m \times m$ matrices and

$$\Omega_\Sigma := \{x \in \Omega | \frac{d[x]}{dt} = \Sigma\} \subset \Omega$$

the set of paths with absolutely continuous quadratic variation with Lebesgue density Σ.

Definition 2.6.6. $F \in C^{1,2}(\Lambda)$ is called Σ-harmonic if it satisfies

$$\mathcal{D}F(t, x_t) + \frac{1}{2}\langle \nabla_x^2 F(t, x_t), \Sigma(t, x_t)\rangle = 0 \tag{2.27}$$

for all $t \geq 0$ and $x \in \Omega_\Sigma$.

If F is Σ-harmonic, then the change of variable formula (Theorem 2.5.14) gives

$$F(t, x_t) = F(0, x_0) + \int_0^t \nabla_x F(s, x_{s-}) dx \tag{2.28}$$

for all $t \geq 0$ and $x \in \Omega_\Sigma$. Equality in (2.28) then holds on Ω_Σ. Every functional of class $\mathcal{M}$ satisfies (2.27), hence is Σ-harmonic for all Σ.

Theorem 2.6.7 (Representation of Σ-harmonic functionals)**.** *If F is Σ-harmonic, then there exists a class $\mathcal{M}$ functional M such that*

$$M|_{\Omega_\Sigma} \equiv F.$$

In particular, M is uniquely determined by (2.28) on Ω_Σ.

Proof. Let $F \in C^{1,2}(\Lambda)$ be Σ-harmonic, we can define a new functional on Λ

$$M(t, x_t) := F(0, x_0) + \int_0^t \nabla_x F(s, x_{s-}) dx. \tag{2.29}$$

By Lemma 2.6.2(i), we see that M is of class $\mathcal{M}$ and $\nabla_x M = (\nabla_x F)_-$. By (2.28) and (2.29), the proof is complete. □

2.7 Technical proofs

Proof of Prop. 2.1.1. For $\alpha \in \mathbb{R}_+$, define $w_\alpha(t) := \mathbb{1}_{[\alpha,\infty)}(t) \in D =: \Omega$, where D denotes the Skorokhod space. We assign to the collection $(w_\alpha)_{\alpha\in\mathbb{R}_+}$, a normalized Lebesgue measure

$$\mathbb{P}(\{w_\alpha | \alpha \in A\}) := \sum_{n\geq 1} \frac{\lambda(A \cap [0,n])}{2^{n+1}},$$

then $\mathbb{P}(\{w_\alpha | \alpha \in \mathbb{R}_+\})) = 1$ and $X_t(w) := w(t)$ is a finite variation process (i.e. a semi-martingale) under $\mathbb{P}$. Now let $\pi = (\pi_n)_{n\geq 1}$ be any sequence of time partitions and denote

$$Q_0^\pi := \{x \in Q^\pi | J(x) \subset \liminf_n \pi_n\}.$$

Since $\liminf_n \pi_n$ is countable, it follows that $\mathbb{P}(\{w_\alpha | \alpha \in \liminf_n \pi_n\}) = 0$ and therefore $\mathbb{P}(\{\omega \in \Omega | X_\cdot(\omega) \in Q_0^\pi\}) = 0$. □

Proof of Theorems 2.5.8 and 2.5.14. By the right continuity of F (Def. 2.3.8.2(d)), we have

$$F(T, x_T) - F(0, x_0) = \lim_n \sum_{\pi_n \ni t_i \leq T} F(t_{i+1}, x^n_{t_{i+1}-}) - F(t_i, x^n_{t_i-}), \qquad (2.30)$$

where for all n sufficiently large, we can decompose each increments

$$\begin{aligned} &F(t_{i+1}, x^n_{t_{i+1}-}) - F(t_i, x^n_{t_i-}) \\ =&F(t_{i+1}, x^n_{t_{i+1}-}) - F(t_i, x^n_{t_{i+1}-}) + F(t_i, x^n_{t_{i+1}-}) - F(t_i, x^n_{t_i-}) \\ =&\underbrace{\big(F(t_{i+1}, x^n_{t_i}) - F(t_i, x^n_{t_i})\big)}_{\text{time}} + \underbrace{\big(F(t_i, x^n_{t_i}) - F(t_i, x^n_{t_i-})\big)}_{\text{space}} \end{aligned}$$

into the sum of a time ('horizontal') and a space ('vertical') increment.

Since F is left-continuous and differentiable in time, $\mathcal{D}F$ is right-continuous and locally bounded, by Lemma 2.5.1 each time increment may be expressed as

$$F(t_{i+1}, x^n_{t_i}) - F(t_i, x^n_{t_i}) = \int_{t_i}^{t_{i+1}} \mathcal{D}F(t, x^n_{t_i}) dt.$$

By Lemma 2.5.2, we obtain

$$\lim_n \sum_{\pi_n \ni t_i \le T} F(t_{i+1}, x^n_{t_i}) - F(t_i, x^n_{t_i}) = \int_0^T \mathcal{D}F(t, x_t)dt,$$

which in light of (2.30), implies that the sum of space increments converges to

$$\lim_n \sum_{\pi_n \ni t_i \le T} \underbrace{F(t_i, x^n_{t_i}) - F(t_i, x^n_{t_i-})}_{\Delta F(t_i, x^n_{t_i})} = F(T, x_T) - F(0, x_0) - \int_0^T \mathcal{D}F(t, x_t)dt \quad (2.31)$$

If $F \in \mathcal{S}(\Lambda)$ then $\nabla_x F$ is strictly causal and by Prop. 2.4.3, $\nabla_x^2 F$ is vanishing everywhere. Thus, by a second order Taylor expansion, the remainder term vanishes, so

$$F(t_i, x^n_{t_i}) - F(t_i, x^n_{t_i-}) = \nabla_x F(t_i, x^n_{t_i-}) \cdot (x(t_{i+1}) - x(t_i))$$

and Thm. 2.5.8 follows. If $F \in C^{1,2}(\Lambda)$ then, by Taylor's Theorem, each space increment admits the following second order expansion

$$\begin{aligned}\Delta F(t_i, x^n_{t_i}) &= F\left(t_i, x^n_{t_i-} + \Delta x^n(t_i)\mathbb{1}_{[t_i,\infty)}\right) - F(t_i, x^n_{t_i-}) \\ &= \nabla_x F(t_i, x^n_{t_i-}) \cdot \Delta x^n(t_i) + \frac{1}{2}\langle \nabla_x^2 F(t_i, x^n_{t_i-}), \Delta x^n(t_i)\Delta x^n(t_i)' \rangle \\ &+ R^n_{t_i}, \qquad (2.32)\end{aligned}$$

where $\Delta x^n(t_i) = (x(t_{i+1}) - x(t_i))$ and

$$R^n_{t_i} = \frac{1}{2}\langle \nabla_x^2 F(t_i, x^n_{t_i-} + \alpha^n_i \Delta x^n(t_i)\mathbb{1}_{[t_i,\infty)}) - \nabla_x^2 F(t_i, x^n_{t_i-}), \Delta x^n(t_i)\Delta x^n(t_i)' \rangle$$

where $\alpha^n_i \in (0,1)$. Since $x \in \Omega_2 \subset Q^\pi_m$, by Cor. 2.2.8 and Rem. 2.2.9

$$\begin{aligned}\lim_n \sum_{\pi_n \ni t_i \le T} \langle \nabla_x^2 F(t_i, x^n_{t_i-}), \Delta x^n(t_i)\Delta x^n(t_i)' \rangle &= \int_0^T \nabla_x^2 F(t, x_{t-}) d[x] \\ = \int_0^T \nabla_x^2 F(t, x_{t-}) d[x]^c + \sum_{t \le T} \langle \nabla_x^2 F(t, x_{t-}), \Delta x(t)\Delta x(t)' \rangle. \qquad (2.33)\end{aligned}$$

Let $\delta > 0$, $r := \sup_{t\in[0,T]} |\Delta x(t)|$, $r_\delta := \delta + \sup_{t\in[0,T+\delta]} |\Delta x(t)|$. Using a result on càdlàg functions [7, Lemma 8], we see that $|\Delta x^n(t_i)| \le r_\delta$ for n sufficiently large. By Rem. 2.3.2, we see that $\alpha_i^n \Delta x^n(t_i) \in \mathcal{U}_{t_i-}(x^n) \cap \overline{B}_{r_\delta}(0)$. Since $\nabla_x^2 F$ admits a modulus of vertical continuity, it follows from Def. 2.4.7 that there exists a modulus of continuity ω such that

$$|R_{t_i}^n| \le \frac{1}{2}\omega(r_\delta)|\Delta x^n(t_i)\Delta x^n(t_i)'|$$

for n sufficiently large, hence by an application of Cor. 2.2.8(i), we obtain

$$\limsup_n \sum_{\pi_n \ni t_i \le T} |R_{t_i}^n| \le \frac{1}{2}\omega(r_\delta) \le \omega(r_\delta) tr\left([x](T)\right).$$

Send $\delta \downarrow 0$, and by the right continuity of x, we have established that

$$\limsup_n \sum_{\pi_n \ni t_i \le T} |R_{t_i}^n| \le \frac{1}{2}\omega(r+) tr\left([x](T)\right). \tag{2.34}$$

Let $0 < \epsilon < r$, define the following finite sets on $[0,T]$

$$J(\epsilon) := \{t \le T \,|\, |\Delta x(t)| > \epsilon\},$$
$$J_n(\epsilon) := \{\pi_n \ni t_i \le T \,|\, \exists t \in (t_i, t_{i+1}], |\Delta x(t)| > \epsilon\}.$$

We can decompose

$$\sum_{\pi_n \ni t_i \le T} R_{t_i}^n = \sum_{t_i \in J_n(\epsilon)} R_{t_i}^n + \sum_{t_i \in (J_n(\epsilon))^c} R_{t_i}^n. \tag{2.35}$$

into two partial sums. By (2.32), the right continuity (resp. left-continuity) of F (resp. $(\nabla_x F)_-$,$(\nabla_x^2 F)_-$) and that x is càdlàg we obtain

$$\begin{aligned}\sum_{t_i \in J_n(\epsilon)} \left(R_{t_i}^n\right)^{\pm} \xrightarrow{n} \sum_{t \in J(\epsilon)} \Bigg(&\Delta F(t, x_t) - \nabla_x F(t, x_{t-}) \cdot \Delta x(t) \\ &-\frac{1}{2}\langle \nabla_x^2 F(t, x_{t-}), \Delta x(t)\Delta x(t)'\rangle\Bigg)^{\pm} \\ &\le \frac{1}{2}\omega(r+) tr\left([x](T)\right),\end{aligned} \tag{2.36}$$

where the inequality follows from (2.34) and (2.35). Observe that $J(\epsilon) \uparrow J(0)$ as $\epsilon \downarrow 0$, by monotone convergence, we obtain

$$\lim_n \sum_{t_i \in J_n(\epsilon)} \left(R^n_{t_i}\right)^{\pm} \xrightarrow{\epsilon} \sum_{t \le T} \Bigg(\Delta F(t, x_t) - \nabla_x F(t, x_{t-}) \cdot \Delta x(t) \\ - \frac{1}{2} \langle \nabla_x^2 F(t, x_{t-}), \Delta x(t) \Delta x(t)' \rangle \Bigg)^{\pm} \\ \le \frac{1}{2} \omega(r+) tr\left([x](T)\right). \tag{2.37}$$

On the other hand, since w is monotonic, by (2.34) and (2.35), it follows that

$$\left| \limsup_n \sum_{t_i \in (J_n(\epsilon))^c} R^n_{t_i} - \liminf_n \sum_{t_i \in (J_n(\epsilon))^c} R^n_{t_i} \right| \le \omega(\epsilon) tr\left([x](T)\right), \tag{2.38}$$

and by (2.31)-(2.33), (2.35),(2.36) and (2.38), so is

$$\left| \limsup_n \sum_{\pi_n \ni t_i \le T} \nabla_x F^n_{t_i} \cdot \Delta x^n(t_i) - \liminf_n \sum_{\pi_n \ni t_i \le T} \nabla_x F^n_{t_i} \cdot \Delta x^n(t_i) \right| \\ \le \omega(\epsilon) tr\left([x](T)\right),$$

where we have denoted $\nabla_x F^n_{t_i} := \nabla_x F(t_i, x^n_{t_i-})$. Send $\epsilon \downarrow 0$, we obtain

$$\int_0^T \nabla_x F(t, x_{t-}) dx := \lim_n \sum_{\pi_n \ni t_i \le T} \nabla_x F(t_i, x^n_{t_i-}) \cdot (x(t_{i+1}) - x(t_i)). \tag{2.39}$$

Upon a second look at (2.31)-(2.33), (2.35),(2.36) and in light of (2.39), we immediately see that

$$\lim_n \sum_{t_i \in (J_n(\epsilon))^c} R^n_{t_i} =: o(\epsilon)$$

also exists and by (2.34), $|o(\epsilon)| \le \frac{1}{2}\omega(\epsilon) tr\left([x](T)\right) \xrightarrow{\epsilon} 0$ which, combined with (2.35) and (2.37) implies

$$\lim_n \sum_{\pi_n \ni t_i \leq T} R^n_{t_i} = \sum_{t \leq T} \Bigg(\Delta F(t, x_t) - \nabla_x F(t, x_{t-}) \cdot \Delta x(t) \\ - \frac{1}{2} \langle \nabla^2_x F(t, x_{t-}), \Delta x(t) \Delta x(t)' \rangle \Bigg) . \tag{2.40}$$

In view of (2.31)-(2.33), (2.39) and (2.40), it remains to show that

$$\sum_{t \leq T} \left(\Delta F(t, x_t) - \nabla_x F(t, x_{t-}) \Delta x(t) - \frac{1}{2} \langle \nabla^2_x F(t, x_{t-}), \Delta x(t) \Delta x(t)' \rangle \right) \\ = \sum_{t \leq T} \Big(\Delta F(t, x_t) - \nabla_x F(t, x_{t-}) \Delta x(t) \Big) - \frac{1}{2} \sum_{t \leq T} \langle \nabla^2_x F(t, x_{t-}), \Delta x(t) \Delta x(t)' \rangle, \tag{2.41}$$

and that the series are absolute convergent.

Since $(\nabla^2_x F)_-$ is left-continuous and locally bounded, we see from Lemma 2.4.6(ii) that the map $t \longmapsto \nabla^2_x F(t, x_{t-})$ is also bounded on $[0, T]$, hence by (2.4)

$$\frac{1}{2} \sum_{t \leq T} |\nabla^2_x F(t, x_{t-})| |\Delta x(t) \Delta x(t)'| \leq \text{const} \sum_i \left(\sum_{t \leq T} (\Delta x_i(t))^2 \right) \\ \leq \text{const} \cdot tr\left([x](T)\right),$$

which, combined with (2.37) implies (2.41))and the absolute convergence of the series, hence Theorem 2.5.14 is proven. □

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3

Mathematical finance without probability

Abstract

We present a non-probabilistic, pathwise approach to continuous-time finance based on functional calculus [5]. In the presence of complete model uncertainty (including the continuous evolution of price paths), we first obtain the analytical analogues of the classical probabilistic notions in mathematical finance and show that generic domains of functional calculus is inherently arbitrage free. We then demonstrate how one may use this fundamental property to obtain the optimal hedging strategy by a fully non-linear path dependent equation and solved the case for Asian option explicitly.

3.1 Introduction

In an insightful expository [10, §5], Föllmer and Schied sketched a non-probabilistic, pathwise trading framework (see also [3]) as a mean to model Knightian uncertainty. In their formulation, if price paths were to evolve continuously, then they cannot be of bounded variation, since this gives rise to very simple arbitrage opportunities (free lunches), e.g.

$$(x(T) - x(0))^2 - [x](T) = \int_0^T 2(x(t) - x(0))dx(t), \tag{3.1}$$

where (3.1) would become non-negative for all continuous paths of bounded variation (i.e. $[x] = 0$) and strictly positive for all paths meeting the condition

$$x(T) \neq x(0).$$

However, if we are *uncertain* that price paths would evolve continuously, then paths of bounded variations would no longer give rise to such arbitrage opportunities. Since in this case, for all (càdlàg) paths of bounded variation that admits at least one single discontinuity, we have

$$[x](T) \geq (\Delta x(t))^2 > 0,$$

for some $t \leq T$ and it is now possible for (3.1) to go negative. In this work, we shall relax the continuity hypothesis and first investigate, rigorously, the relationship between price variation and arbitrage, that is, we do not assume price paths must possess variation a priori.

We introduce the abstract formulation of functional calculus [5] on generic path space (the minimal domain that support a functional calculus) and first obtain the analytical analogue of the (probabilistic) classical notions in mathematical finance. In particular, we show that every self-financing porfolio, (formed by left continuous strategies with right limit), can be represented as a pathwise integral (i.e. limit of left Riemann sums) and proved that every generic path space is arbitrage free, a fundamental property that we shall use in order to obtain the optimal hedging strategies.

This arbitrage free result stands for all generic domains that include, but not limited to, paths of pth order variation, for any $p \in 2\mathbb{N}$. In contrast to related results established using the measure-theoretic, game approach of Vovk [18], Lochowski, Perkowski & Prömel [12] , we are able to work with the classical notion of arbitrage, rather than passing to an asymptotic relaxation that may not necessarily be implementable by a *self-financing* trading strategy.

For non-linear payoffs, we showed that a perfect hedge does not exist in general. We adopt a primal approach to superhedging on bounded generic subset i.e.

$$\{x \in \Omega | a < x(t) < b\}.$$

In particular, we solve the model-free superhedging problem over the above set of scenarios using a minimax approach, in the spirit of Isaacs's tenet of transition [11], and provide a verification theorem for the optimal cost-to-go functional. As an example, we study the case of Asian options.

Related superhedging problems have been studied using probabilistic approaches or so-called robust approaches based on quasi-sure analysis based on a family of probability measures [2, 12, 15, 16]. In contrast to these approaches, our approach is purely pathwise and does not appeal to any probabilistic assumptions. Also, we are able to by-pass many technical difficulties (e.g. not having to deal with duality gap and polar set). Finally, our optimal hedging strategy (delta) comes as a by-product, whereas in the quasi-sure approach, it will not be straightforward to compute the optimal delta, see for instance [15].

3.2 Notations

Denote D to be the Skorokhod space of $\mathbb{R}^m$-valued positive càdlàg functions

$$t \longmapsto x(t) := (x_1(t), \ldots, x_m(t))'$$

on $\mathbb{R}_+ := [0, \infty)$. Denote C, S, BV respectively, the subsets of continuous functions, step functions, locally bounded variation functions in D. By convention, $x(0-) := x(0)$ and $\Delta x(t) := x(t) - x(t-)$. The path $x \in D$ stopped at $(t, x(t))$ (resp. $(t, x(t-))$)

$$s \longmapsto x(s \wedge t)$$

shall be denoted by $x_t \in D$ (resp. $x_{t-} := x_t - \Delta x(t)\mathbb{1}_{[t,\infty)} \in D$). We write $(D, \mathfrak{d}_{\mathrm{J}_1})$ when D is equipped with a complete metric $\mathfrak{d}_{\mathrm{J}_1}$ which induces the Skorokhod (a.k.a. J_1) topology.

Let $\pi := (\pi_n)_{n\geq 1}$ be a fixed sequence of partitions $\pi_n = (t_0^n, ..., t_{k_n}^n)$ of $[0, \infty)$ into intervals $0 = t_0^n < ... < t_{k_n}^n < \infty$; $t_{k_n}^n \uparrow \infty$ with vanishing mesh $|\pi_n| \downarrow 0$ on compacts. By convention, $\max(\emptyset \cap \pi_n) := 0$, $\min(\emptyset \cap \pi_n) := t_{k_n}^n$.

For any $p \in 2\mathbb{N}$, we say that $x \in D$ has finite p-th order variation $[x]_p$ if

$$\sum_{\pi_n \ni t_i \leq t} (x(t_{i+1}) - x(t_i))^{\otimes p}$$

converges to $[x]_p$ in the Skorokhod J_1 topology in $D(\mathbb{R}_+, \mathbb{R}^m \otimes^p)$. In light of [4], we remark that in the special case $p = 2$, this definition is equivalent to that of Föllmer [9]. We refer to [6] for the origin of p-th order variation for a continuous path. We denote V_p the set of càdlàg paths of finite p-th order variations.

$$t_n' := \max\{t_i < t | t_i \in \pi_n\}, \tag{3.2}$$

and the following piecewise constant approximations of x by

$$x^n := \sum_{t_i \in \pi_n} x(t_{i+1})\mathbb{1}_{[t_i, t_{i+1})}. \tag{3.3}$$

We let $\Omega \subset D$ be *generic* (Def. 2.3.1) and define our *domain* as

$$\Lambda := \{(t, x_t) | t \in \mathbb{R}_+, x \in \Omega\}.$$

3.3 Functional calculus

In this section, we briefly recall from the previous chapter, key concepts and results that are indispensable in the sequel. Reader is advised to bypass the section if she is already familiar with the materials.

Definition 3.3.1 (generic)**.** A non-empty subset $\Omega \subset D$ is called *generic* if Ω satisfies the following closure properties under operations:

i For every $x \in \Omega$, $T > 0$, $\exists N \in \mathbb{N}$; $x^n \in \Omega, \quad \forall n \geq N$.

ii For every $x \in \Omega, t \geq 0$, $\exists$ convex neighbourhood $\Delta x(t) \in \mathcal{U}$ of 0;

$$x_{t-} + e\mathbb{1}_{[t,\infty)} \in \Omega, \quad \forall e \in \mathcal{U}. \tag{3.4}$$

Example 3.3.2. Examples of generic subsets include S, BV,D and $V_p(\pi)$ for $p \in 2\mathbb{N}$. Generic subsets are closed under finite intersections. All subsets of C are not generic.

Proposition 3.3.3 (continuous functional)**.** *For every $t \in \mathbb{R}_+, x \in \Omega$, we define t'_n resp. x^n by (3.2) resp. (3.3). There exists a topology on Λ such that a functional F is continuous with regard to this topology if and only if (for sufficiently large n) F satisfies*

$$1.(a) \lim_{s\uparrow t; s\leq t} F(s, x_{s-}) = F(t, x_{t-}),$$

$$(b) \lim_{s\uparrow t; s<t} F(s, x_s) = F(t, x_{t-}),$$

$$(c) t_n \longrightarrow t; t_n \leq t'_n \Longrightarrow F(t_n, x^n_{t_n-}) \longrightarrow F(t, x_{t-}),$$

$$(d) t_n \longrightarrow t; t_n < t'_n \Longrightarrow F(t_n, x^n_{t_n}) \longrightarrow F(t, x_{t-}),$$

$$2.(a) \lim_{s\downarrow t; s\geq t} F(s, x_s) = F(t, x_t),$$

$$(b) \lim_{s\downarrow t; s>t} F(s, x_{s-}) = F(t, x_t),$$

$$(c) t_n \longrightarrow t; t_n \geq t'_n \Longrightarrow F(t_n, x^n_{t_n}) \longrightarrow F(t, x_t),$$

$$(d) t_n \longrightarrow t; t_n > t'_n \Longrightarrow F(t_n, x^n_{t_n-}) \longrightarrow F(t, x_t),$$

for all $(t, x_t) \in \Lambda$. *In particular, the set* $C_\pi(\Lambda)$ *of continuous functionals is an algebra. A functional is called* left (resp. right) continuous *if it satisfies 1.(a)-(d) (resp. 2.(a)-(d)).*

Proof. [5, §3] □

Definition 3.3.4 (Strictly causal functionals)**.** Let $F : \Lambda \to \mathbb{R}$ and denote $F_-(t, x_t) = F(t, x_{t-})$. F is called *strictly causal* if $F = F_-$.

Definition 3.3.5 (Regulated functionals)**.** A functional $F : \Lambda \to \mathbb{R}$ is *regulated* if there exists $\widetilde{F} \in C_\pi(\Lambda)$ such that $\widetilde{F}_- = F_-$. $\widetilde{F}$ is then unique by Prop. 3.3.3.2(b).

Remark 3.3.6. Since $C_\pi(\Lambda)$ is an algebra, we remark the set of all regulated functionals forms an algebra.

Definition 3.3.7 (Horizontal differentiability)**.** $F : \Lambda \longmapsto \mathbb{R}$ is called *differentiable in time* if

$$\mathcal{D}F(t, x_t) := \lim_{h \downarrow 0} \frac{F(t+h, x_t) - F(t, x_t)}{h}$$

exists $\forall (t, x_t) \in \Lambda$.

Definition 3.3.8 (Vertical differentiability)**.** $F : \Lambda \longmapsto \mathbb{R}$ is called *vertically differentiable* if for every $(t, x_t) \in \Lambda$, the map

$$e \longmapsto F\left(t, x_t + e\mathbb{1}_{[t,\infty)}\right)$$

is differentiable at 0. We define $\nabla_x F(t, x_t) := (\nabla_{x_1} F(t, x_t), \ldots, \nabla_{x_m} F(t, x_t))'$;

$$\nabla_{x_i} F(t, x_t) := \lim_{\epsilon \to 0} \frac{F\left(t, x_t + \epsilon \mathbf{e}_i \mathbb{1}_{[t,\infty)}\right) - F(t, x_t)}{\epsilon}.$$

Definition 3.3.9 (differentiable)**.** A functional is called *differentiable* if it is differentiable in time and in space.

Remark 3.3.10. All definitions above are extended to multidimensional function on Λ whose components satisfy the respective conditions.

Lemma 3.3.11. *A function on Λ is strictly causal if and only if it is differentiable in space with vanishing derivative.*

Proof. We refer to [5, §4]. □

Definition 3.3.12 (Classes $\mathcal{S}$ and $\mathcal{M}$)**.** A continuous and differentiable functional F is of *class* $\mathcal{S}$ if $\mathcal{D}F$ is right continuous and locally bounded, $\nabla_x F$ is left continuous and strictly causal. If in addition, $\mathcal{D}F$ vanishes then F is of *class* $\mathcal{M}$.

Denote $\mathcal{M}(\Lambda)$ the set of all functionals of class $\mathcal{M}$ and $\mathcal{M}_0(\Lambda)$ the subset of $\mathcal{M}(\Lambda)$ with vanishing initial values.

Definition 3.3.13 (Pathwise integral)**.** Let $\phi : \Lambda \longmapsto \mathbb{R}^m$; ϕ_- be left continuous. For every $x \in \Omega$, define

$$\mathbf{I}(t, x_t^n) := \sum_{\pi_n \ni t_i \leq t} \phi(t_i, x_{t_i-}^n) \cdot (x(t_{i+1}) - x(t_i)) \tag{3.5}$$

for all n sufficiently large and $\mathbf{I}(t, x_t) := \lim_n \mathbf{I}(t, x_t^n)$. If $\mathbf{I}$ is continuous, then ϕ is called *integrable* and $\mathbf{I} := \int \phi dx$ is called the *pathwise integral.*

Theorem 3.3.14 (Representation theorem)**.** *A functional $F : \Lambda \to \mathbb{R}$ is a pathwise integral if and only if $F \in \mathcal{M}_0(\Lambda)$:*

$$F \in \mathcal{M}_0(\Lambda) \iff \exists \phi : \Lambda \to \mathbb{R}^m, \phi_-\text{left} - \text{continuous};$$

$$F(t, x_t) = \int_0^t \phi_- dx, \quad \forall (t, x_t) \in \Lambda.$$

Proof. We refer to [5, §5] □

3.4 Market scenarios, self-financing strategies and arbitrage

We consider a frictionless market with $d > 0$ tradable assets, and one numeraire whose price is identically 1. We denote x to be the price paths of tradable assets and $x \in \Omega$, where Ω is generic Def. 2.3.1.

The number of shares in assets ϕ and the numeraire ψ held immediately before the portfolio revision at time t will be denoted by ϕ_- and ψ_-.

We aim to address the following fundamental questions:

- What is *self-financing*? Since there is not a priori that the value of a portfolio V must be expressible as $dV = \phi dx$.
- What is *no-arbitrage*? Is it necessary that price paths must possess variation of some sort?

A trading strategy is a pair (ϕ, ψ) of regulated functionals $\phi : \Lambda \mapsto \mathbb{R}^d$ and $\psi : \Lambda \mapsto \mathbb{R}$. The value V of the portfolio is given by

$$V(t, x_t) := \phi(t, x_{t-}) \cdot x(t) + \psi(t, x_{t-}). \tag{3.6}$$

A key concept is the concept of *self-financing* strategy [3, §2]. This concept is usually defined in a probabilistic setting, by equating the changes in the portfolio value V with a *gain process* defined as a stochastic integral $\int \phi_- .dS$. We propose a new approach to this concept based on *local* properties, without involving any use of (pathwise or stochastic) integration notions:

Definition 3.4.1 (Self-financing strategy**).**
A trading strategy (ϕ, ψ) is called *self-financing* if for every $(t, x) \in \Lambda$

(i) $\Delta\widetilde{\phi}(t, x_t) \cdot x(t) + \Delta\widetilde{\psi}(t, x_t) = 0$,

(ii) $\left(\widetilde{\phi}(t+h, x_t) - \widetilde{\phi}(t, x_t)\right) \cdot x(t) + \widetilde{\psi}(t+h, x_t) - \widetilde{\psi}(t, x_t) = 0$.

Both conditions correspond to the property that the proceeds from any change in the asset position is reflected in the change in the cash position. However the important point is that we only require this in two situations:

i an instantaneous change in the asset position, and

ii a change in the asset/cash position while the asset prices remain constant.

As we will show, through piecewise constant approximation these two situations cover the case of all continuous-time strategies under minimal regularity properties.

Remark 3.4.2. If (ϕ, ψ) is self-financing, then the value of the portfolio may also be expressed as

$$V(t, x_t) = \widetilde{\phi}(t, x_t) \cdot x(t) + \widetilde{\psi}(t, x_t). \tag{3.7}$$

Theorem 3.4.3 (Gain of a self-financing strategy as a pathwise integral)**.** *Let V be the portfolio value associated with the trading strategy (ϕ, ψ). Then (ϕ, ψ) is self-financing if and only if $V \in \mathcal{M}(\Lambda)$, $\nabla_x V = \phi_-$. In that case*

$$V(t, x_t) = V(0, x_0) + \int_0^t \phi(s, x_{s-}) dx. \tag{3.8}$$

Proof. If (ϕ, ψ) is self-financing, we may first use (3.6) to deduce that $\nabla_x V = \phi_-$, which is left continuous and strictly causal. From (3.7) and the fact that $C_\pi(\Lambda)$ is an algebra (i.e. Prop. 3.3.3), we see that V is continuous. We then apply (3.7) and Def. 3.4.1(ii) to deduce that $\mathcal{D}V$ is vanishing. Hence, $V \in \mathcal{M}(\Lambda)$ and (3.8) follows from Thm. 3.3.14.

On the other hand, if $V \in \mathcal{M}(\Lambda)$, then V is continuous. By the continuity of V, (3.6) and Prop. 3.3.3.2(b), we first obtain (3.7), hence Def. 3.4.1(i). Since $\mathcal{D}V$ vanishes, by [5, Lem.5.1], we obtain

$$V(t+h, x_t) - V(t, x_t) = \int_t^{t+h} \mathcal{D}V(s, x_t) ds = 0.$$

Resorting once again to (3.7), we also obtain Def. 3.4.1.(ii), hence (ϕ, ψ) is self-financing. □

Proposition 3.4.4. *Let $V \in \mathcal{M}(\Lambda)$, then the following properties are equivalent:*

(i) V is the portfolio value of a self-financing trading strategy (ϕ, ψ).

(ii) $\nabla_x V$ is regulated.

Proof. (i) implies (ii) follows from Def. 3.4.1, Def. 3.3.5 and Thm. 3.4.3. Assume (ii) holds, let ϕ be the continuous version of $\nabla_x V$ and put

$$\psi(t, x_t) := V(t, x_t) - \phi(t, x_t) \cdot x(t), \tag{3.9}$$

then is continuous (i.e. $C_\pi(\Lambda)$ is an algebra). Taking Δ from (3.9), we obtain

$$\Delta V - \nabla_x V \Delta x = x \cdot \Delta\phi + \Delta\psi. \tag{3.10}$$

By Thm. 3.3.14, we deduce the LHS of (3.10) vanishes and obtain (3.6) i.e. V is the portfolio value associated with the trading strategy (ϕ, ψ), which is self-financing by Thm. 3.4.3. □

Definition 3.4.5 (Arbitrage)**.**
A self-financing strategy (ϕ, ψ) with value V is called an *arbitrage* on $[0, T]$ if

$$\forall x \in \Omega, \qquad V(T, x_T) - V(0, x_0) \geq 0 \tag{3.11}$$

and there exists $x \in \Omega$ such that $V(T, x_T) - V(0, x_0) > 0$.

Lemma 3.4.6. *Let $M \in \mathcal{M}_0(\Lambda)$. If there exists $T > 0$;*

$$M(T, x_T) \geq 0$$

$\forall x \in \Omega$*, then for every* $x \in \Omega$*, the map*

$$t \longmapsto M(t, x_t)$$

is non-negative for $t \leq T$*.*

Proof. If $M \in \mathcal{M}_0(\Lambda)$, then $\mathcal{D}M$ vanishes, by [5, Lem.5.1] we obtain

$$M(t, x_t) = M(t, x_t) + \int_t^T \mathcal{D}M(s, x_t)ds = M(T, x_t) \geq 0$$

for all $t \leq T$, where the last inequality is due to $x_t \in \Omega$. □

Theorem 3.4.7 (Fair game). *Let $M \in \mathcal{M}_0(\Lambda)$. If there exists $T > 0$;*

$$M(T, x_T) \geq 0$$

for all $x \in \Omega$, then $M(T, x_T) \equiv 0$.

Proof. Let $T > 0$; $M(T, x_T) \geq 0 \; \forall \; x \in \Omega$. By Lem. 3.4.6, we first obtain

$$M(t, x_t) \geq 0 \tag{3.12}$$

for all $t \leq T$, $x \in \Omega$. Suppose there exists $\omega \in \Omega$;

$$M(T, \omega_T) > 0.$$

By the continuity of M and Thm. 3.3.14, it follows

$$M(T, \omega^n_T) = \sum_{\pi_n \ni t_i \leq T} \nabla_x M(t_i, \omega^n_{t_i-})(\omega(t_{i+1}) - \omega(t_i)) > 0 \tag{3.13}$$

for all n sufficiently large. Define

$$t^*_n := \min\{t_i \in \pi_n | M(t_i, \omega^n_{t_i}) > 0\},$$

then $t^*_n \leq T$. By (2.22), (3.13), the left continuity of M and the fact that $\omega^n \in \Omega$, we obtain

$$M(t^*_n, \omega^n_{t^*_n}) > M(t^*_n, \omega^n_{t^*_n-}) = 0,$$

hence

$$M(t^*_n, \omega^n_{t^*_n}) = \nabla_x M(t^*_n, \omega^n_{t^*_n-}) \Delta\omega(t^*_n) > 0.$$

Def. 2.3.1(ii) implies that there exists $\epsilon > 0$;

$$\omega^* := \omega^n_{t^*_n-} - \epsilon \Delta\omega(t^*_n) \mathbb{1}_{[t^*_n, \infty)} \in \Omega,$$

hence

$$M(t_n^*, \omega_{t_n^*}^*) = \nabla_x M(t_n^*, \omega_{t_n^*-}^n)(-\epsilon \Delta\omega(t_n^*)) < 0,$$

which is a contradiction to (2.22). □

Using these results we can now show that if the set of market scenarios is a generic set of paths, arbitrage in the sense of Def. 2.3.1 may not exist:

Corollary 3.4.8. *Arbitrage does not exist in a generic market.*

Proof. It is a direct consequence of Def. 3.4.5, Thm. 3.4.3 and Thm. 3.4.7. □

Remark 3.4.9. As previously discussed, the set S of piecewise-constant paths, the space $D([0,\infty), \mathbb{R}_+^m)$ of positive càdlàg paths or the space V_p of càdlàg paths with finite p-th order variation for $p \in 2\mathbb{N}$ are examples of generic sets of paths, to which the above result applies. However, unlike the results of [12, 17, 18] the proof of the above result does not involve any assumption on the variation index of the path.

3.5 Market incompleteness: When does a payoff admit a perfect hedge?

In this section, we first uncover, from examples such as Asian, lookback and passport options, the mathematical properties of a *payoff* in the context of functional calculus.

We define what a (non-) *linear* payoff is and prove that a payoff can be perfectly hedged in a generic market if and only if it is linear such as the Asian option with 0 strike.

For $u, v \in \mathbb{R}^l$, we write $u > v$ if $u_i > v_i$ for all i. We call v positive if $v > 0$. Let Ω be a generic set of paths. In order for the operation

$$x_{t-} + \mathfrak{e}(t, x_{t-})\mathbb{1}_{[t,\infty)} \in \Omega$$

to be closed, $\mathfrak{e}(t, x_{t-})$ may not take arbitrary values, this motivates the following

Definition 3.5.1 (regular)**.** A regulated function $\mathfrak{e} : \Lambda \to \mathbb{R}^d$-valued $\mathfrak{e}$ on Λ is called *regular* if for every $x \in \Omega$, $t \geq 0$,

$$x_{t-} + \mathfrak{e}(t, x_{t-})1\!\!1_{[t,\infty)} \in \Omega$$

We denote $\mathcal{E}$ to be the set of all regular functions on Λ.

Example 3.5.2. $\mathfrak{e} := 0$ is regular. If Ω is either S, V_p or D, then every $\mathbb{R}^d$-valued regulated function $\mathfrak{e}$ satisfying

$$\mathfrak{e}(t, x_{t-}) > -x(t-),$$

for all $x \in \Omega$, $t \geq 0$ is regular.

Definition 3.5.3 (non-degenerate)**.** A subset Ω is called *non-degenerate*, if there exists $\mathfrak{e}^1, \ldots, \mathfrak{e}^d \in \mathcal{E}$ where

$$\mathfrak{e}^i_j(t, x_{t-}) \begin{cases} \neq 0, & \text{if } i = j. \\ = 0, & \text{otherwise;} \end{cases} \tag{3.14}$$

for every $x \in \Omega$, $t \geq 0$.

Remark 3.5.4. If Ω is either S, V_p or D, then Ω is non-degenerate. In the sequel, we shall assume that Ω is non-degenerate.

Definition 3.5.5 (Payoff)**.**
A *payoff* with maturity $T > 0$ is a functional $H : \Omega \to \mathbb{R}$ such that

(i) $H(x) = H(x_T)$ for all $x \in \Omega$.

(ii) For every $x \in \Omega$, $t \geq 0$ and the map

$$e \longmapsto H(x_{t-} + e1\!\!1_{[t,\infty)})$$

is continuous on every convex neighborhood $\mathcal{U} \subset \mathbb{R}^d$ of 0 satisfying (3.4).

(iii) The functional$(t, x_t) \longmapsto H(x_t)$ is continuous on Λ and for every $\mathfrak{e} \in \mathcal{E}$, the functional

$$(t, x_t) \in \Lambda \longmapsto H(x_{t-} + \mathfrak{e}(t, x_{t-})1\!\!1_{[t,\infty)}),$$

is regulated.

Example 3.5.6. Let $d = 1$, $T > 0$, $K \geq 0$ and let V be the value of a self-financing portfolio. Then

(i) $H(x) := \left(\frac{1}{T}\int_0^T x(t)dt - K\right)^+$,

(ii) $H(x) := \left(\sup_{s\leq T} x(s) - x(T)\right)^+$,

(iii) $H(x) := (V(T, x_T) - K)^+$,

satisfy Definition 3.5.5.

Proof. (i) is obvious. We first compute $H(x_{t-} + e1\!\!1_{[t,\infty)})$ and obtained the followings:

$$\begin{aligned}
&(i) && \left(\frac{1}{T}\left(\int_0^{t\wedge T} xds + (T-t)(x(t-)+e)1\!\!1_{[0,T]}\right) - K\right)^+, \\
&(ii) && \left(\sup_{s<t} x_T(s) - x_T(t-) - e1\!\!1_{[0,T]}\right)^+, \\
&(iii) && \left(V(t, x_{(t\wedge T)-}) + \nabla_x V(t, x_{t-})e1\!\!1_{[0,T]} - K\right)^+,
\end{aligned}$$

which are all continuous in e and we obtain Def. 3.5.5(ii). If we replace e with $\Delta x(t)$ and observe in (ii) that

$$\left(\sup_{s<t}\ x_T(s) - x_T(t)\right)^+ = \left(\sup_{s\leq t}\ x_T(s) - x_T(t)\right)^+,$$

we see that $(t, x_t) \mapsto H(x_t)$ is continuous. If we replace e with $\mathfrak{e} \in \mathcal{E}$, by the regularity of $\mathfrak{e}$ and Rem. 3.3.6, we obtain Def. 3.5.5(iii). $\square$

Definition 3.5.7 (Vertically affine functionals)**.** A payoff $H : \Omega \to \mathbb{R}$ is called *vertically affine* if for every $x \in \Omega$, $t \geq 0$ and convex neighborhood $\mathcal{U} \subset \mathbb{R}^d$ of 0 satisfying (3.4), the map

$$e \longmapsto H(x_{t-} + e\mathbb{1}_{[t,\infty)})$$

is affine on $\mathcal{U}$.

Remark 3.5.8. If $K = 0$, the payoffs in Example 3.5.6(i)&(iii) are vertically affine.

Definition 3.5.9 (Perfect hedge)**.** A payoff $H : \Omega \to \mathbb{R}$ with maturity $T > 0$ is said to admit a *perfect hedge* on Ω if there exists a self-financing portfolio with value V such that

$$\forall x \in \Omega, \qquad V(T, x_T) = H(x_T).$$

Theorem 3.5.10. *Every vertically affine payoff admits a perfect hedge on Ω.*

Proof. If H is vertically affine, then $e \longmapsto H(x_{t-} + e\mathbb{1}_{[t,\infty)})$ is an affine map. Since Ω is generic, it follows there exists a convex neighborhood $\Delta x(t) \in \mathcal{U} \subset \mathbb{R}^d$ of 0 satisfying (3.4) and we obtain a constant c and a $\phi \in \mathbb{R}^d$;

$$H(x_{t-} + e\mathbb{1}_{[t,\infty)}) = c(t, x_{t-}) + \phi(t, x_{t-}) \cdot e, \tag{3.15}$$

on $\mathcal{U}$, hence

$$H(x_t) - H(x_{t-}) = \phi(t, x_{t-}) \cdot \Delta x(t). \tag{3.16}$$

Since it holds for every $x \in \Omega$ and $t \geq 0$, it follows from Def. 3.5.5(iii) that

$$V(t, x_t) = H(x_t),$$

is continuous on Λ, $\mathcal{D}V$ vanishes and by (3.16) and Lem. 3.3.11, $\nabla_x V(t, x_t) = \phi(t, x_{t-})$ which is strictly causal. Since Ω is non-degenerate, there exists everywhere non-vanishing $\mathfrak{e}^i \in \mathcal{E}$, $i = 1, \dots, d$;

$$H(x_{t-} + \mathfrak{e}^i(t, x_{t-})\mathbb{1}_{[t,\infty)}) - H(x_{t-}) = \phi(t, x_{t-}) \cdot \mathfrak{e}^i(t, x_{t-}). \tag{3.17}$$

By assumption the matrix $(\mathfrak{e}^i_j)$ is non-singular so (3.17) uniquely defines the hedging strategy $\phi(t, x_{t-})$. It follows Def. 3.5.5(iii) and Rem. 3.3.6 that ϕ is regulated. By Prop. 3.4.4, V is self-financing and the claim follows. □

Corollary 3.5.11. *A payoff admits a perfect hedge on a generic set of paths Ω if and only if it is vertically affine.*

Proof. The if part follows from Thm. 3.5.10. If H admits a perfect hedge then there exists $V \in \mathcal{M}(\Lambda)$; $H(x_T) = V(T, x_T)$ on Ω. It follows that

$$H(t, x_{t-} + e\mathbb{1}_{[t,\infty)}) = V(t, x_{t-}) + \nabla_x V(t, x_{t-})e.$$

□

Example 3.5.12 (Asian with $K = 0$)**.** The Asian payoff with $K = 0$ i.e. Eg. 3.5.6(i) is linear and the perfect hedge is computed as:

$$\begin{aligned} \nabla_x V(t, x_t) &= \frac{T - t}{T}, \\ V(t, x_t) &= \frac{1}{T}\left(\int_0^{t\wedge T} x(s)ds + (T - t)x(t)\right), \\ V(0, x_0) &= x(0). \end{aligned} \tag{3.18}$$

We remark here that the perfect hedge is model independent.

3.6 Optimal strategy for non-linear payoffs: Asian option

In the last section, we have established a very important fact, i.e. that a perfect hedge does not exist for non-linear payoffs, thereby justifying the search for an alternative approach. A well-studied paradigm for valuation in the absence of perfect replicating strategies is super-hedging on a given subset of paths [1, 13].

Let Ω be generic and define, for $0 \leq a < b$,

$$\Omega_a^b := \{x \in \Omega | a < x(t) < b\}, \quad \bar{\Omega}_a^b := \{x \in \Omega | a \leq x(t) \leq b\}$$

Observe that Ω_a^b is again generic, hence is itself free of arbitrage in the sense of Def. 3.4.5. The superhedging price obtained on this set is a proper arbitrage-free price from the standpoint of Ω. We denote

$$\bar{\Omega}_a^b(t, x_t) := \{z \in \bar{\Omega}_a^b | z_t = x_t\} \qquad \mathcal{L} := \{\nabla_x V | V \text{ is self-financing}\}$$

Definition 3.6.1 (Superhedging cost)**.**
Let H be a payoff defined on Ω with maturity $T > 0$, if there exists a self-financing portfolio V satisfying

$$\forall x \in \Omega_{a,b}, \qquad V(T, x_T) \geq H(x_T), \tag{3.19}$$

such that the value W of any self-financing portfolio meeting condition (3.19), satisfies

$$W(0, x_0) \geq V(0, x_0), \tag{3.20}$$

, then $V(0, x_0)$ is called the superhedging cost of H and $\nabla_x V$ is a superhedging strategy for the payoff H.

A superhedging strategy, if it exists, may be non-unique. We will first develop the notion of optimal strategy (if exists, will be unique), in the spirit of Isaacs's tenet of transition [11, p3]. Our approach here to solve the superhedging problem is to construct a functional $U \in \mathcal{S}$ (Def.2.4.13) such that for all $0 \leq s \leq t \leq T$ and $x \in \Omega_a^b$ we have

$$\begin{aligned} U(s, x_s) &= \min_{\phi \in \mathcal{L}} \max_{z \in \bar{\Omega}_a^b(s, x_s)} \left\{ U(t, z_t) - \int_s^t \phi dz \right\}, \\ U(T, x_T) &= H(x_T). \end{aligned} \tag{3.21}$$

Lemma 3.6.2. *Let $U \in \mathcal{S}$ be a functional that satisfies (3.21). Then the* *map*

$$h \longmapsto U(s+h, x_s)$$

is monotonic decreasing in $[0, \infty)$.

Proof. We have

$$U(s, x_s) \geq \min_{\phi \in \mathcal{L}} \left\{ U(t, z_t) - \int_s^t \phi dz \right\}$$

for all $z \in \Omega_a^b(s, x_s)$, this holds, in particular for all z stopped at s. It follows

$$U(s, x_s) \geq \min_{\phi \in \mathcal{L}} U(t, z_s) = U(t, x_s).$$

□

Thus if U satisfies (3.21), then $V(T, x_T) := U_0 + \int_0^T \nabla_x U dx$ solves (3.19), meeting condition (3.20) and the solution is unique up to Ω_a^b due to

$$U_1(t, x_t) = \min_{\phi \in \mathcal{L}} \max_{z \in \bar{\Omega}_a^b(x_t)} \left\{ H(T, z_T) - \int_t^T \phi dz \right\} = U_2(t, x_t).$$

Then $U(t_0, x_{t_0})$ is the superhedging cost to hedge starting at time $0 \leq t_0 < T$. The relationship with the value of the hedging portfolio V is

$$V(t, x_t) = U(t_0, x_{t_0}) + \int_{t_0}^t \nabla_x U dx = u(t, x_t) - \int_{t_0}^t \mathcal{D} U ds,$$

hence at maturity time T, we have

$$V(T, x_T) = U(t_0, x_{t_0}) + \int_{t_0}^T \nabla_x U dx = H(T, x_T) - \int_{t_0}^T \mathcal{D} U ds,$$

and the final portfolio value is

$$H(T, x_T) - \int_{t_0}^T \mathcal{D} U ds \geq H(T, x_T),$$

where the last inequality is due to Lemma 3.6.2.

We shall use the following Minimax Theorem to prove the verification theorem.

Theorem 3.6.3 (Minimax)**.** *If $M \in \mathcal{M}$; then*

$$\min_{\phi\in\mathcal{L}} \max_{z\in\bar{\Omega}_a^b(x_s)} \left\{ \int_s^t (\nabla_x M - \phi) dz \right\} = 0 = \max_{z\in\bar{\Omega}_a^b(x_s)} \min_{\phi\in\mathcal{L}} \left\{ \int_s^t (\nabla_x M - \phi) dz \right\}$$

Proof. We first have

$$\begin{aligned} c := & \min_{\phi\in\mathcal{L}} \max_{z\in\bar{\Omega}_a^b(x_s)} \left\{ \int_s^t (\nabla_x M - \phi) dz \right\} \\ \leq & \max_{z\in\bar{\Omega}_a^b(x_s)} \left\{ \int_s^t (\nabla_x M - \nabla_x M) dz \right\} = 0. \end{aligned}$$

If $c < 0$, then we have

$$\int_s^t (\phi - \nabla_x M) dz \geq -c > 0.$$

It follows from Theorem 3.4.7 that $c = 0$. The case of minimax follows a similar line of proof. □

We obtain as a corollary, yet another look at functionals of class $\mathcal{M}$, in reminiscent to their probabilistic counterparts:

Corollary 3.6.4. *Define, for any bounded non-anticipative functional $H : \Lambda \to \mathbb{R}$,*

$$\mathbb{E}(H(t, x_t)|x_s) := \inf_{\phi\in\mathcal{L}} \sup_{z\in\Omega_a^b(s,x_s)} \left\{ H(t, z_t) - \int_s^t \phi dz \right\}.$$

Then for $M \in \mathcal{M}(\Lambda)$ we have

$$\forall (t, x) \in \Lambda, \qquad \mathbb{E}\left(M(t, x_t)|x_s \right) = M(s, x_s).$$

Theorem 3.6.5 (Verification theorem)**.** *Let $U \in \mathcal{S}(\Lambda)$ such that $\nabla_x U \in \mathcal{L}$*

satisfies

$$\max_{x\in\bar{\Omega}_a^b} \mathcal{D}U(t,x_t) = 0 \tag{3.22}$$
$$U(T,x_T) = H(x_T),$$

for all $t \leq T$ and $x \in \Omega_a^b$. Then $\phi = \nabla_x U$ is a superhedging strategy for H on Ω_a^b and achieves the optimum in (3.21)*.*

Proof. We first obtain

$$\begin{aligned} c :&= \min_{\phi\in\mathcal{L}} \max_{z\in\bar{\Omega}_a^b(x_s)} \left\{ \int_s^t \mathcal{D}U(r,z_r)dr + \int_s^t (\nabla_x U - \phi)dz \right\} \\ &\leq \min_{\phi\in\mathcal{L}} \max_{z\in\bar{\Omega}_a^b(x_s)} \left\{ \int_s^t (\nabla_x U - \phi)dz \right\} = 0, \end{aligned}$$

due to Lem.3.6.2 & Lem.3.6.3 (Minimax). It remains to show that $c \geq 0$.

$$\begin{aligned} c &\geq \max_{z\in\bar{\Omega}_a^b(x_s)} \min_{\phi\in\mathcal{L}} \left\{ \int_s^t \mathcal{D}U(r,z_r)dr + \int_s^t (\nabla_x U - \phi)dz \right\} \\ &\geq \max_{z\in\bar{\Omega}_a^b(x_s)} \left\{ \int_s^t \mathcal{D}U(r,z_r)dr \right\} + \max_{z\in\bar{\Omega}_a^b(x_s)} \min_{\phi\in\mathcal{L}} \left\{ \int_s^t (\nabla_x U - \phi)dz \right\} = 0, \end{aligned}$$

by (3.22) and Lem. 3.6.3 (Minimax). □

Example 3.6.6 (Asian option)**.** Let $\Omega = V_p$ for any $p \in 2\mathbb{N}$. The optimal cost-to-go functional is

$$U(t,x_t) = H^+(t,x_t)p(x(t)) + H^-(t,x_t)(1 - p(x(t)) \tag{3.23}$$

where

$$
\begin{aligned}
H^+(t, x_t) &= \left(\frac{1}{T}\left(\int_0^t x(s)ds + b(T-t)\right) - K\right)^+ \\
H^-(t, x_t) &= \left(\frac{1}{T}\left(\int_0^t x(s)ds + a(T-t)\right) - K\right)^+ \\
p(x(t)) &= \frac{x(t)-a}{b-a}
\end{aligned}
$$

and the optimal strategy is

$$
\nabla_x U(t, x_t) = \frac{H^+(t, x_t) - H^-(t, x_t)}{b-a}.
$$

Proof. We first see that U is of class $\mathcal{S}$, $U(T, X_T) = H(X_T)$ with

$$
\mathcal{D}U(t, x_t) = \mathcal{D}H^+(t, x_t)p(x) + \mathcal{D}H^-(t, x_t)(1 - p(x)),
$$

where

$$
\begin{aligned}
\mathcal{D}H^+(t, x_t) &= \frac{x(t)-b}{T}\mathbb{I}_{\{H^+>0\}}, \\
\mathcal{D}H^-(t, x_t) &= \frac{x(t)-a}{T}\mathbb{I}_{\{H^->0\}}.
\end{aligned}
$$

Since $H^+ = 0$ implies $H^- = 0$ and that $H^- > 0$ implies $H^+ > 0$, it follows

$$
\mathcal{D}U(t, x_t) = \frac{(x(t)-b)(x(t)-a)}{T(b-a)}\mathbb{I}_{\{H^+>0\}}\mathbb{I}_{\{H^-=0\}} \leq 0,
$$

hence $\max_{x \in \bar{\Omega}_a^b} \mathcal{D}U(t, x_t) = 0$ and we obtained (3.22) in Thm.3.6.5. □

Remark 3.6.7. Note that if $K = 0$, we obtain the perfect hedge in Example 3.5.12 (3.18) as a special case. If we set $a = 0$ and let $b \uparrow \infty$, then (3.23) converges to the (model independent) superhedging price of the Asian option

$$U(t, x_t) = \left(\frac{1}{T}\int_0^t x(s)ds - K\right)^+ + x(t)\frac{T-t}{T}.$$

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