A SHARP LIEB–THIRRING INEQUALITY FOR FUNCTIONAL DIFFERENCE OPERATORS

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Abstract. We prove sharp Lieb–Thirring type inequalities for the eigenvalues of a class of one-dimensional functional difference operators associated to mirror curves. We furthermore prove that the bottom of the essential spectrum of these operators is a resonance state.

To our friend and coauthor Leon Takhtajan on the occasion of his 70th birthday

1. Introduction

Let P be the self-adjoint quantum mechanical momentum operator on $L^2(\mathbb{R})$, i.e. $P = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ and for $b > 0$ denote by $U(b)$ the Weyl operator $U(b) = \exp(-bP)$. By using the Fourier transform

$$\widehat{\psi}(k) = (\mathcal{F}\psi)(k) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}kx}\psi(x)\, dx$$

we can write the domain of $U(b)$ as

$$\mathrm{dom}(U(b)) = \left\{\psi \in L^2(\mathbb{R}) :\ \mathrm{e}^{-2\pi bk}\widehat{\psi}(k) \in L^2(\mathbb{R})\right\}.$$

Equivalently, $\mathrm{dom}(U(b))$ consists of those functions $\psi(x)$ which admit an analytic continuation to the strip $\{z = x + \mathrm{i}y \in \mathbb{C} :\ 0 < y < b\}$ such that $\psi(x + \mathrm{i}y) \in L^2(\mathbb{R})$ for all $0 \le y < b$ and there is a limit $\psi(x + \mathrm{i}b - \mathrm{i}0) = \lim_{\varepsilon\to 0^+} \psi(x + \mathrm{i}b - \mathrm{i}\varepsilon)$ in the sense of convergence in $L^2(\mathbb{R})$, which we will denote simply by $\psi(x + \mathrm{i}b)$. The domain of the inverse operator $U^{-1}(b)$ can be characterised similarly.

For $b > 0$ we define the operator $W_0(b) = U(b)+U(b)^{-1} = 2\cosh(bP)$ on the domain

$$\mathrm{dom}(W_0(b)) = \left\{\psi \in L^2(\mathbb{R}) :\ 2\cosh(2\pi bk)\widehat{\psi}(k) \in L^2(\mathbb{R})\right\}.$$

The operator $W_0(b)$ is self-adjoint and unitarily equivalent to the multiplication operator $2\cosh(2\pi bk)$ in Fourier space. Its spectrum is thus absolutely continuous covering the interval $[2,\infty)$ doubly.

Let $V \ge 0, V \in L^1(\mathbb{R})$ now be a real-valued potential function. The scalar inequality $2\cosh(2\pi bk) - 2 \ge (2\pi bk)^2$ implies the operator inequality

$$W_0(b) - 2 \ge -b^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \tag{1}$$

on $\mathrm{dom}(W_0(b))$. By Sobolev's inequality, we can conclude that the operator

$$W_V(b) = W_0(b) - V.$$

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is symmetric and bounded from below on the common domain of $W_0(b)$ and V. We can thus consider its Friedrichs extension, which we continue to denote by $W_V(b)$. This operator acts as

$$(W_V(b)\psi)(x) = \psi(x+\mathrm{i}b) + \psi(x-\mathrm{i}b) - V(x)\psi(x)\,.$$

Furthermore, by an application of Weyl's theorem (in a version for quadratic forms) and Rellich's lemma together with the fact that the form domain of $W_0(b)$ is continuously embedded in $H^1(\mathbb{R})$ (as discussed at the beginning of Sect. 4) the spectrum of $W_V(b)$ consists of essential spectrum $[2,\infty)$ and discrete finite-multiplicity eigenvalues below. Details of this argument in the similar case of a Schrödinger operator can be found in the upcoming book [FLW, Proposition 4.14].

We will show that the discrete spectrum satisfies a version of Lieb–Thirring inequalities for 1/2-Riesz means. When formulating the main result of the paper it is convenient to parametrise the eigenvalues (repeated with multiplicities) as $\lambda_j = -2\cos(\omega_j)$, where $\omega_j \in [0,\pi]$ for $\lambda_j \in [-2,2]$ and $\omega_j \in \mathrm{i}\,[0,\infty)$ for $\lambda_j \le -2$. Note that in the latter case $\lambda_j = -2\cosh(|\omega_j|)$.

Theorem 1. *Let $V \ge 0$ and let $V \in L^1(\mathbb{R})$. If $W_V(b) \ge -2$, then the discrete eigenvalues $\lambda_j = -2\cos(\omega_j) \in [-2,2)$ (repeated with multiplicities) satisfy*

$$\sum_{j\ge1} \frac{\sin\omega_j}{\omega_j} \le \frac{1}{2\pi b}\int_{\mathbb{R}} V(x)\,\mathrm{d}x\,. \tag{2}$$

The constant in the inequality (2) *is sharp in the sense that there is a potential V such that* (2) *becomes equality.*

Remark 2. Note that Theorem 1 does not allow to estimate eigenvalues below -2. In fact, from the proof of this theorem, the case of one eigenvalue below -2 could be included in the inequality (2). We expect that the inequality holds true for all eigenvalues below -2. However, the method we use in the poroof prevents us from including all eigenvalues due to oscillating properties of the "free" resolvent $(W_0(b) - \lambda)^{-1}$ for $\lambda < -2$.

Lieb–Thirring inequalities were first established for Schrödinger operators in [LT]. For a one-dimensional Schrödinger operator $-\frac{\mathrm{d}}{\mathrm{d}x^2} - V$ on $L^2(\mathbb{R})$ with negative eigenvalues $\mu_1 \le \mu_2 \le \dots 0$, these bounds state that for any $\gamma \ge 1/2$ there is a constant $L_\gamma > 0$ such that

$$\sum_{j\ge1} |\mu_j|^\gamma \le L_\gamma \int_{\mathbb{R}} V(x)^{\gamma+1/2}\,\mathrm{d}x \tag{3}$$

for all $V \ge 0, V \in L^{\gamma+1/2}(\mathbb{R})$. The condition $\gamma \ge 1/2$ is optimal. Inequality (1) implies that

$$\sum_{j\ge1} |\lambda_j - 2|^\gamma \le \frac{L_\gamma}{b}\int_{\mathbb{R}} V(x)^{\gamma+1/2}\,\mathrm{d}x \tag{4}$$

for all eigenvalues $\lambda_j \leq 2$ of $W_V(b)$. Under the additional assumption $W_V(b) \geq -2$, our bound (2) presents an improvement of (4) for $\gamma = 1/2$ since the sharp constant $L_{1/2} = 1/2$ [HLT] and

$$|\lambda_j - 2|^{\frac{1}{2}} = |2\cos\omega_j + 2|^{\frac{1}{2}} \leq \frac{\pi \sin\omega_j}{\omega_j}\,.$$

The difference of the terms above vanishes as $\omega_j \to \pi$, implying that (4) is asymptotically optimal for small coupling. While the necessity of $\gamma \geq 1/2$ in the Lieb–Thirring inequality for Schrödinger operators does not allow us to conclude that (4) fails for $0 \leq \gamma < 1/2$, we will prove the following.

Theorem 3. *Let $b > 0$. If $V \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} V\,\mathrm{d}x > 0$, then $W_V(b)$ has at least one eigenvalue below 2. Furthermore, if $0 \leq \gamma < 1/2$, then there is no constant L_γ such that (4) holds for all compactly supported V. This conclusion holds even under the assumption that $W_V(b) \geq -2$.*

The study of different properties of functional difference operators $W_V(b)$ was considered before. In the case when $-V = V_0 = \mathrm{e}^{2\pi bx}$ is an exponential function, the operator $W_{V_0}(b)$ first appeared in the study of the quantum Liouville model on the lattice [FT1] and plays an important role in the representation theory of the non-compact quantum group $\mathrm{SL}_q(2,\mathbb{R})$. The spectral analysis of this operator was studied in [FT2]. In the case when $-V = 2\cosh(2\pi bx)$ the spectrum of $W_V(b)$ is discrete and converges to $+\infty$. Its Weyl asymptotics was obtained in [LST1]. This result was extended to a class of growing potentials in [LST2]. More information on spectral properties of functional difference operators can be found in papers [GHM, GKMR, KM, KMZ, T].

The proof method of Theorem 1 is similar to the proof of the sharp Lieb–Thirring inequality (3) for a one-dimensional Schrödinger operator in the case $\gamma = 1/2$ as presented in [HLW]. It relies on a property of convolutions of the resolvent kernels of the operator under consideration. Such a property was also recently established for Jacobi operators where it was again used to prove sharp Lieb–Thirring type inequalities [LLS]. With a different proof (not using the convolution property) the sharp inequalities for the Schrödinger operator and the Jacobi operator were first obtained in [HLT] and in [HS], respectively. Despite formal similarity to the case of Jacobi operators, it is still surprising that the proof method works for functional difference operators $W_V(b)$. These operators could be considered as differential operators of infinite order since the symbol $\cosh(2\pi bk)$ can be written as an infinite Taylor series of symbols of even degree w.r.t. the variable k.

2. Free resolvent

Since $W_0(b) \geq 2$ we conclude that $W_0(b) - \lambda$ is an invertible operator for $\lambda < 2$. Let $\lambda = -2\cos(\omega)$ with $\omega \in [0,\pi]$ if $\lambda \in [-2,2]$ and $\omega \in \mathrm{i}[0,\infty)$ if $\lambda < -2$. Then in Fourier space the inverse of $W_0(b) - \lambda$ is given by the multiplication operator $(2\cosh(2\pi bk) + 2\cos(\omega))^{-1}$.

Applying the inverse Fourier transform $\mathcal{F}^{-1}$ to $(2\cosh(2\pi bk) + 2\cos(\omega))^{-1}$ we find the kernel of the free resolvent $G_\lambda = (W_0(b) - \lambda)^{-1}$ that is

$$G_\lambda(x, y) = G_\lambda(x - y) = \frac{1}{2b\sin\omega}\,\frac{\sinh(\frac{\omega}{b}(x - y))}{\sinh(\frac{\pi}{b}(x - y))}. \tag{5}$$

Remark 4. Note that $G_\lambda(x - y)$ is an even and positive kernel for $\omega \in [0, \pi]$ and it becomes oscillating if $\omega \in \mathrm{i}(0, \infty)$. This fact is one of the reasons why we are able to study Lieb–Thirring inequalities only for the eigenvalues $\lambda_j \in [-2, 2]$. This interval contains all of the discrete spectrum if the potential V if "small" enough. However, if V generates eigenvalues lying in $(-\infty, -2)$, then the oscillating property of the Green's function prevents us from obtaining the desired inequality for all eigenvalues.

Note that the value of G_λ on the diagonal $x = y$ takes the form

$$G_\lambda(0) = \frac{1}{2\pi b}\,\frac{\omega}{\sin\omega} \tag{6}$$

and we can see the relation between the right-hand side of (6) and the expression in the left-hand side of (2). Due to our parameterisation of the spectral parameter, the convergence $\lambda \to 2^-$ implies $\omega \to \pi^-$ and thus

$$G_\lambda(0) \sim \frac{1}{2b}\,\frac{1}{\sqrt{1 - \cos^2\omega}} \sim \frac{1}{2b}\,\frac{1}{\sqrt{2 - \lambda}}, \quad \text{as} \quad \lambda \to 2^-.$$

If $\lambda \to -\infty$, then $\omega \to \mathrm{i}\infty$ and

$$G_\lambda(0) \sim \frac{1}{2b}|\lambda|^{-1}\log|\lambda|.$$

In [FT2] L. Faddeev and L. A. Takhtajan studied the resolvent in a slightly different form

$$G_\lambda(x, y) = \frac{\sigma}{\sinh(\frac{\pi\mathrm{i}\varkappa}{\sigma})}\left(\frac{\mathrm{e}^{-2\pi\mathrm{i}\varkappa(x-y)}}{1 - \mathrm{e}^{-4\pi\mathrm{i}\sigma(x-y)}} + \frac{\mathrm{e}^{2\pi\mathrm{i}\varkappa(x-y)}}{1 - \mathrm{e}^{4\pi\mathrm{i}\sigma(x-y)}}\right)$$

which coincides with (5) with $\sigma = \mathrm{i}/2b$, $\lambda = 2\cosh(2b\pi\varkappa)$ and $\varkappa = \frac{\omega - \pi}{2\pi\mathrm{i}b}$. It was pointed out that the free resolvent can be written using the analogues of the Jost solutions

$$f_-(x, \varkappa) = \mathrm{e}^{-2\pi\mathrm{i}\varkappa x} \quad \text{and} \quad f_+(x, \varkappa) = \mathrm{e}^{2\pi\mathrm{i}\varkappa x}$$

that appear in the theory of one-dimensional Schrödinger operators. Namely

$$G_\lambda(x - y) = \frac{2\sigma}{C(f_-, f_+)(\varkappa)}\left(\frac{f_-(x, \varkappa)f_+(y, \varkappa)}{1 - \mathrm{e}^{\frac{\pi\mathrm{i}}{\sigma'}(x-y)}} + \frac{f_-(y, \varkappa)f_+(x, \varkappa)}{1 - \mathrm{e}^{-\frac{\pi\mathrm{i}}{\sigma'}(x-y)}}\right)$$

where $\sigma'\sigma = -1/4$ and where $C(f, g)$ is the so-called Casorati determinant (a difference analogue of the Wronskian) of the solutions of the functional-difference equation

$$C(f, g)(x, \varkappa) = f(x + 2\sigma', \varkappa)g(x, \varkappa) - f(x, \varkappa)g(x + 2\sigma', \varkappa).$$

For the Jost solutions $C(f_-, f_+)(x, \varkappa) = 2\sinh(\frac{2\pi\kappa}{\sigma})$.

The equality $(W_0(b) - \lambda)G(x - y) = \delta(x - y)$ could be interpreted as an equation of distributions. Since the functions $f_\pm(x, k)$ are Jost solutions, the distribution defined by $(W_0(b)-\lambda)G(x-y)$ is supported only at $x=y$, and its singular part coincides with the singular part of the distribution

$$-\frac{\sigma\sigma'}{\pi \mathrm{i} C(f_-,f_+)(\varkappa)}\left(\frac{f_-(x+2\sigma',\varkappa)f_+(y,\varkappa)-f_-(y,\varkappa)f_+(x+2\sigma',\varkappa)}{x-y-\mathrm{i}0}\right.$$
$$\left.+\frac{f_-(x-2\sigma',\varkappa)f_+(y,\varkappa)-f_-(y,\varkappa)f_+(x-2\sigma',\varkappa)}{x-y+\mathrm{i}0}\right)$$

in the neighbourhood of $x=y$. This singular part is equal to

$$-\frac{2\sigma\sigma'}{\pi\mathrm{i}}\left(\frac{1}{x-y-\mathrm{i}0}-\frac{1}{x-y+\mathrm{i}0}\right)=\delta(x-y)$$

where the authors used the Sokhotski–Plemelj formula. This formula is similar to the respective formula for a Schrödinger operator when the Dirac δ-function appears by differentiating a step function.

3. Proof of inequality (2)

3.1. **Some auxiliary results.** We first collect some results from [HLW] verbatim. Let A be a compact operator on a Hilbert space $\mathcal{G}$ and let us denote

$$\|A\|_n=\sum_{j=1}^{n}\sqrt{\lambda_j(A^*A)}\,.$$

Then by Ky-Fan's inequality (see for example [GK], Lemma 4.2) the functionals $\|\cdot\|_n$, $n=1,2,...$, are norms and thus for any unitary operator Y in $\mathcal{G}$ we have

$$\|Y^*AY\|_n=\|A_n\|.$$

Definition 5. Let A,B be two compact operators on $\mathcal{G}$. We say that A majorises B or $B\prec A$, iff

$$\|B\|_n\prec\|A\|_n,\quad \text{for all}\quad n\in\mathbb{N}.$$

Lemma 6. *Let A be a nonnegative compact operator $\mathcal{G}$, $\{Y(k)\}_{k\in\mathbb{R}}$ be a family of unitary operators on $\mathcal{G}$, and let g be a probability measure on $\mathbb{R}$. Then the operator*

$$B=\int_{\mathbb{R}}Y(k)^*AY(k)\,g(k)\,\mathrm{d}k$$

is majorised by A.

Proof. This is a simple consequence of the triangle inequality

$$\|B\|_n\le\int_{\mathbb{R}}\|Y^*(k)AY(k)\|_n\,g(k)\,\mathrm{d}k=\|A\|_n\int_{\mathbb{R}}g(k)\,\mathrm{d}k=\|A\|_n.$$

□

Let $\lambda_j = -2\cos\omega_j \leq 2$ be the eigenvalues of $W_0(b) - V$ with $V \geq 0$. In order to slightly simplify the notations it is convenient to write

$$\lambda_j = -2\cos(\sqrt{\theta_j})$$

with $\theta_j \in (-\infty, \pi^2]$ and $\omega_j^2 = \theta_j$.

Let us denote by K_λ the Birman–Schwinger operator

$$K_\lambda = V^{1/2} G_\lambda V^{1/2}\,. \tag{7}$$

Let $\mu_j(K_\lambda)$ be the eigenvalues of the Birman–Schwinger operator K_λ defined in (7). Then due to the Birman-Schwinger principle we have

$$1 = \mu_j(K_{\lambda_j}). \tag{8}$$

Let us define the operator

$$L_\theta := \frac{1}{G_{-2\cos\sqrt{\theta}}(0)} K_{-2\cos\sqrt{\theta}}\,,$$

where $G_\lambda(0) = \frac{1}{2\pi b}\frac{\omega}{\sin\omega}$ is given in (6). Then from (8) we obtain

$$\sum_{j\geq 1} \frac{1}{G_{\lambda_j}(0)} = \sum_{j\geq 1} \frac{1}{G_{\lambda_j}(0)} \mu_j(K_{\lambda_j}) = \sum_{j\geq 1} \mu_j(L_{\theta_j}).$$

The integral kernel of this operator is given by $\sqrt{V(x)} g_{\pi^2,\theta}(x-y)\sqrt{V(y)}$ where

$$g_{\pi^2,\theta}(x) := \frac{\pi}{\sqrt{\theta}} \frac{\sinh(\frac{\sqrt{\theta}}{b}x)}{\sinh(\frac{\pi}{b}x)}\,.$$

Consider a more general function

$$g_{\varphi,\theta}(x) := \frac{\sqrt{\varphi}}{\sqrt{\theta}} \frac{\sinh(\frac{\sqrt{\theta}}{b}x)}{\sinh(\frac{\sqrt{\varphi}}{b}x)}\,.$$

Since $g_{\varphi,\theta}(0) = 1$ its Fourier transform $\widehat{g}_{\varphi,\theta} = \mathcal{F}(g_{\varphi,\theta})$ satislies the equation

$$\int_{\mathbb{R}} \widehat{g}_{\varphi,\theta}(k)\,\mathrm{d}k = 1.$$

Moreover, for any $-\infty < \theta \leq \varphi \leq \pi^2$ we have

$$\widehat{g}_{\varphi,\theta}(k) = \mathcal{F}\left(\frac{\sqrt{\varphi}}{\sqrt{\theta}} \frac{\sinh(\frac{\sqrt{\theta}}{b}x)}{\sinh(\frac{\sqrt{\varphi}}{b}x)} \right)(k) = \frac{2\pi \sin(\pi \frac{\sqrt{\theta}}{\sqrt{\varphi}})}{\sqrt{\theta}} \frac{b}{2\cosh(\frac{2\pi^2 bk}{\sqrt{\varphi}}) + 2\cos(\frac{\pi\sqrt{\theta}}{\sqrt{\varphi}})}.$$

The right hand side is positive for (θ, φ) such that $-\infty < \theta < \varphi$ and $0 < \varphi < \pi^2$. Thus $\widehat{g}_{\varphi,\theta}\,\mathrm{d}k$ is a probability measure for such values.

Note also that importantly

$$\frac{g_{\pi^2,\theta}(x)}{g_{\pi^2,\theta'}(x)} = \frac{\sqrt{\theta'}}{\sqrt{\theta}} \frac{\sinh(\frac{\sqrt{\theta}}{b}x)}{\sinh(\frac{\sqrt{\theta'}}{b}x)} = g_{\theta',\theta}(x)$$

and therefore

$$(\widehat{g}_{\pi^2,\theta'} * \widehat{g}_{\theta',\theta})(k) = \widehat{g}_{\pi^2,\theta}(k)$$

which is the convolution property mentioned in the introduction.

Lemma 7. *(Monotonicity)*
For (θ, θ') such that $-\infty < \theta \le \theta'$ and $0 < \theta' < \pi^2$ we have $L_\theta \prec L_{\theta'}$.

Proof. Let $Y(k) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the unitary multiplication operator

$$(Y(k)\psi)(x) = \mathrm{e}^{-2\pi \mathrm{i} kx}\psi(x)$$

and let T be the projection onto $V^{1/2}$, i.e.

$$(T\psi)(x) = V^{1/2}(y) \int_{\mathbb{R}} V^{1/2}(y)\, \psi(y)\, \mathrm{d}y.$$

Using $Y(k' + k'') = Y(k')Y(k'')$ and Lemma 6 we obtain

$$\begin{aligned} L_\theta &= \int_{\mathbb{R}} Y(k)^* T\, Y(k) \widehat{g}_{\pi^2,\theta}(k)\, \mathrm{d}k \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} Y(k)^* T\, Y(k)\, \widehat{g}_{\pi^2,\theta'}(k')\, \widehat{g}_{\theta',\theta}(k-k')\, \mathrm{d}k'\, \mathrm{d}k \\ &= \int_{\mathbb{R}} Y(k'')^* \Big(\int_{\mathbb{R}} Y(k')^* T\, Y(k')\, \widehat{g}_{\pi^2,\theta'}(k')\, \mathrm{d}k' \Big) Y(k'')\, \widehat{g}_{\theta',\theta}(k'')\, \mathrm{d}k'' \prec L_{\theta'}, \end{aligned}$$

where we have used that $\widehat{g}_{\theta',\theta}\, \mathrm{d}k$ is a probability measure. ☐

Remark 8. With a slight abuse of notations, Lemma 7 says that $L_\lambda \prec L_{\lambda'}$ for any $\lambda < 2$ as long as $\lambda \le \lambda'$ and $-2 < \lambda' < 2$.

3.2. **Proof of inequality** (2)**.** We now enumerate the eigenvalues of the operator $W_V(b)$ belonging to the interval $(-2, 2)$ such that $-2 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ repeated with multiplicity. By using the monotonicity established in Lemma 7 we have a sequence of inequalities

$$\begin{aligned} \frac{1}{G_{\lambda_1}(0)} &= 2\pi b \frac{\sin\omega_1}{\omega_1} = \mu_1(L_{\theta_1}) \le \mu_1(L_{\theta_2}), \\ \sum_{j=1}^{2} \frac{1}{G_{\lambda_j}(0)} &= 2\pi b \sum_{j=1}^{2} \frac{\sin\omega_j}{\omega_j} \le \sum_{j=1}^{2} \mu_j(L_{\theta_2}) \le \sum_{j=1}^{2} \mu_j(L_{\theta_3}), \\ \sum_{j=1}^{3} \frac{1}{G_{\lambda_j}(0)} &= 2\pi b \sum_{j=1}^{3} \frac{\sin\omega_j}{\omega_j} \le \sum_{j=1}^{3} \mu_j(L_{\theta_3}) \le \sum_{j=1}^{3} \mu_j(L_{\theta_4}), \quad \text{etc.} \end{aligned}$$

Note that we do not use any assumptions on the multiplicities of the eigenvalues, other than their finiteness. Furthermore, by Lemma 7 the same results also hold true if a single eigenvalue is below -2. Continuing the above process and noting that L_{π^2} is a rank-one operator with the single eigenvalue $\int_{\mathbb{R}} V\, \mathrm{d}x$, we finally obtain

$$\sum_{j\ge 1} \frac{\sin\omega_j}{\omega_j} \le \frac{1}{2\pi b} \int_{\mathbb{R}} V(x)\, \mathrm{d}x\,.$$

The proof is complete.

Remark 9. Note that $\frac{2\cosh(2\pi bk)-2}{b^2} \to (2\pi k)^2$ tends to the symbol of the second derivative as $b \to 0$ and that $W_{b^2V}(b) \geq -2$ for sufficiently small b. We thus expect that it should be possible to recover the Lieb–Thirring inequality (3) for a Schrödinger operator with the sharp constant $L_{1/2} = 1/2$ from Theorem 2.

4. SHARPNESS OF INEQUALITY (2)

Similarly to the case of Schrödinger operators, we aim to prove that the Lieb–Thirring inequality becomes an equality for Dirac-delta potentials. To this end let $c > 0$ and consider the potential $V_c(x) = c\delta(x)$. To properly define $W_{V_c}(b)$, we first note that the quadratic form $\langle\psi, (W_0(b) - 2)\psi\rangle$ can be written as

$$\langle\psi, (W_0(b) - 2)\psi\rangle = \int_{\mathbb{R}} 2\sinh(\pi b|k|)|\widehat{\psi}(k)|^2 \,\mathrm{d}k = \int_{\mathbb{R}} |\psi(x + \mathrm{i}b/2) - \psi(x - \mathrm{i}b/2)|^2 \,\mathrm{d}x \,. \tag{9}$$

This can be seen by introducing the self-adjoint operator $D(b) = U(b/2) - U(b/2)^{-1} = 2\sinh(bP/2)$ and checking that $D(b)^2 = W_0(b) - 2$ either directly or by means of the identity $\cosh(2\pi bk) - 1 = 2\sinh(\pi bk)$. The form domain of $W_0(b)$ is thus $\mathrm{dom}(D(b)) = \mathrm{dom}(W_0(b/2)) \subset H^1(\mathbb{R})$ and on this domain Sobolev's inequality yields that

$$\begin{aligned}|\psi(0)|^2 &\leq \varepsilon \int_{\mathbb{R}} |\psi'(x)|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}} |\psi(x)|^2 \,\mathrm{d}x \\ &\leq \frac{\varepsilon}{b}\int_{\mathbb{R}} 2\sinh(\pi b|k|)|\widehat{\psi}(k)|^2 \,\mathrm{d}k + \frac{1}{\varepsilon}\int_{\mathbb{R}} |\psi(x)|^2 \,\mathrm{d}x\end{aligned}$$

for any choice of $\varepsilon > 0$. The KLMN theorem thus allows us to define $W_0(b) - V_c$. As a rank one perturbation of the operator $W_0(b)$ the potential V_c generates one eigenvalue below the continuous spectrum $[2, \infty)$.

In Fourier space the eigenequation $(W_0(b) - c\delta)\psi_c = \lambda\psi_c$ becomes

$$2\cosh(2\pi bk)\widehat{\psi_c}(k) - c\widehat{\psi_c}(0) = \lambda\widehat{\psi_c}(k)$$

by means of the formal identity $\mathcal{F}(\delta\psi_c) = \psi_c(0)$. Writing again $\lambda = -2\cos\omega$ we obtain

$$\widehat{\psi_c}(k) = \frac{c\psi_c(0)}{2\cosh(2\pi bk) + 2\cos\omega} \tag{10}$$

and therefore

$$\psi_c(x) = c\psi_c(0)G_{-2\cos\omega}(x) = \frac{c\psi_c(0)}{2b\sin\omega}\frac{\sinh(\frac{\omega}{b}x)}{\sinh(\frac{\pi}{b}x)} \,. \tag{11}$$

Of course we could have seen this immediately by using the equation for the Green's function

$$(W_0(b) + 2\cos\omega)G_{-2\cos\omega}(x) = \delta(x) \,.$$

Letting $x \to 0$ in (11) we find

$$1 = \frac{c}{2b\sin\omega}\frac{\omega}{\pi}$$

or equivalently

$$\frac{\sin\omega}{\omega} = \frac{c}{2\pi b}\,. \tag{12}$$

Since $\frac{\sin\sqrt{\theta}}{\sqrt{\theta}}$ is a monotone decreasing function of $\theta = \omega^2 \in (-\infty, \pi^2]$ that takes all values in $[0, \infty)$, for any $c > 0$ there is a unique solution ω_c to (12) and vice versa. If $c/(2\pi b) < 1$ then $\omega_c \in (0, \pi)$ and otherwise $\omega_c \in \mathrm{i}[0, \infty)$. Since $\int V_c \, \mathrm{d}x = c$, the identity (12) can be rewritten as

$$\frac{\sin\omega}{\omega} = \frac{1}{2\pi b}\int_{\mathbb{R}} V(x)\,dx$$

showing that the Lieb–Thirring inequality is satisfied for potentials $-c\delta$ with a single eigenvalue that can be placed anywhere in $(-\infty, 2)$ by choosing $c > 0$ suitably.

Remark 10. If we choose the normalising constant $\psi(0) > 0$ then the eigenfunction defined in (10)

$$\psi_c(x) = \frac{c\psi(0)}{2b\sin\omega_c}\frac{\sinh(\frac{\omega_c}{b}x)}{\sinh(\frac{\pi}{b}x)}$$

is positive assuming that the coupling constant c is small enough satisfying the inequality $c/(2\pi b) \le 1$ and thus $\omega_c \in [0, \pi)$. Note that if $c/(2\pi b) = 1$ then $\omega_c = 0$ and

$$\psi_c(x) = \frac{\pi x}{2b\sinh(\frac{\pi}{b}x)} > 0.$$

However, if the coupling constant $c > 2\pi b$ then $\omega_c \in \mathrm{i}(0, \infty)$ and hence

$$\psi_c(x) = \frac{c\psi(0)}{2b\sin\omega_c}\frac{\sin(\frac{|\omega_c|}{b}x)}{\sinh(\frac{\pi}{b}x)}$$

is an oscillating function and in particular has an infinite number of zeros. This contradicts a possible conjecture that the eigenfunction for the lowest eigenvalue is strictly positive.

Open Problem. Assume that the discrete spectrum $\sigma_d(W_V(b))$ of the operator $W_V(b)$ satisfies the property $\sigma_d(W_V(b)) \subset [-2, 2)$. Is it true that the eigenfunction corresponding to the lowest eigenvalue could be chosen strictly positive?

5. NECESSITY OF $\gamma \geq 1/2$

The following argument is similar to that presented in the upcoming book [FLW, Proposition 4.41 and 4.42] for the proof in the case of a Schrödinger operator. For $\varepsilon > 0$ let $\psi_\varepsilon(x) = 1/\cosh(2\varepsilon x/b)$. If ε is sufficiently small, say $\varepsilon \le \varepsilon_0$, then $\psi_\varepsilon \in \mathrm{dom}(W_0(b))$. Using (9) we compute that

$$\langle\psi_\varepsilon, (W_0(b) - 2)\psi_\varepsilon\rangle = \frac{b\sin^2\varepsilon}{\varepsilon}\int_{\mathbb{R}}\left|\frac{2\sinh x}{\cos^2\varepsilon\cosh^2 x + \sin^2\varepsilon\sinh^2 x}\right|^2 \mathrm{d}x \le Cb\varepsilon$$

for a constant $C > 0$ independent of $\varepsilon \leq \varepsilon_0$. For any potential $V \in L^1(\mathbb{R})$ it holds that $\langle \psi_\varepsilon, V\psi_\varepsilon \rangle \to \int_{\mathbb{R}} V \,\mathrm{d}x$ as $\varepsilon \to 0$ by dominated convergence and thus for sufficiently small ε

$$\langle \psi_\varepsilon, (W_V(b) - 2)\psi_\varepsilon \rangle < 0\,.$$

By the Min–Max principle this proves the first part of Theorem 3.

For the second assertion of the theorem we choose more specifically the compactly supported potential $V(x) = c\chi_{[-1/2,1/2]}(x/b)$. By Sobolev's inequality $W_V(b) \geq -2$ for sufficiently small $c \leq c_0$ such that all the discrete eigenvalues of $W_V(b)$ are contained in $[-2, 2)$. Furthermore $\|\psi\|^2 = b/\varepsilon$ and

$$\langle \psi_\varepsilon, V\psi_\varepsilon \rangle = cb \int_{-1/2}^{1/2} |\cosh(2\varepsilon x)|^{-2} \,\mathrm{d}x = \frac{cb \tanh \varepsilon}{\varepsilon} \leq cb\,.$$

We now choose $\varepsilon = c\delta$. If $\delta \leq \varepsilon_0/c_0$, then the first bound established above holds and

$$\frac{\langle \psi_\varepsilon, (W_V(b) - 2)\psi_\varepsilon \rangle}{\|\psi\|^2} \leq C\varepsilon^2 - c\varepsilon = c^2\delta(C\delta - 1)\,.$$

Choosing $\delta < \min(\varepsilon_0/c_0, 1/C)$ we can conclude by the Min–Max principle that $W_V(b) - 2$ has a negative eigenvalue $\lambda_1 \leq -c^2\delta(1 - C\delta)$. If a Lieb–Thirring inequality (4) were to hold for $\gamma < 1/2$ then for some finite L_γ

$$c^{2\gamma}\delta^\gamma(1 - C\delta)^\gamma \leq \frac{L_\gamma}{b} \int_{\mathbb{R}} V(x)^{\gamma+\frac{1}{2}} \,\mathrm{d}x = L_\gamma c^{\gamma+\frac{1}{2}}$$

which is clearly a contradiction if $c \to 0$.

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