Capacity Games with Supply Function Competition

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Abstract

We introduce a general model for suppliers competing for a buyer's procurement business. The buyer faces uncertain demand and there is a requirement to reserve capacity in advance of knowing the demand. Each supplier has costs that are two dimensional, with some capacity costs incurred prior to production and some production costs incurred at the time of delivery. These costs are general functions of quantity and this naturally leads us to a supply function competition framework in which each supplier offers a schedule of prices and quantities. We show that there is an equilibrium of a particular form: the buyer makes a reservation choice that maximizes the overall supply chain profit, each supplier makes a profit equal to their marginal contribution to the supply chain, and the buyer takes the remaining profit. This is a natural equilibrium for the suppliers to coordinate on since no supplier can do better in any other equilibrium. These results make use of a submodularity property for the supply chain optimal profits as a function of the suppliers available and build on the assumption that the buyer breaks a tie in favor of the solutions that give the largest supply chain profit. We demonstrate the applications of our model in three operations management problems: a newsvendor problem with unreliable suppliers, a portfolio procurement problem with supply options and a spot market, and a bundling problem with non-substitutable products.

Keywords: capacity game; supply function; option contract; submodularity; Nash equilibrium

1 Introduction

We deal with capacity reservation decisions that can also be thought of as purchasing an option for supply. Capacity reservation can be modeled in a two-stage framework. In the first stage, before knowing the demand, a buyer reserves a certain amount of capacity by paying a reservation price. In the second stage, after discovering information about the actual demand, the buyer decides the supply amount which cannot exceed the reserved

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amount. At this stage, the buyer pays an execution price only for the amount of capacity that is used.

In this paper, we develop a general framework to study a capacity setting in which multiple suppliers are competing against each other for a buyer's procurement business by offering both reservation and execution prices. From a modeling standpoint, our framework allows volume-dependent marginal costs (of capacity and production) for the suppliers, which are consistent with many practical situations despite being rarely studied in the literature [Cachon, 2003]. For example, capacity investment often involves a one-off setup cost [Luss, 1982, Van Mieghem, 2003], there may be (dis)economies of scale in production [Haldi and Whitcomb, 1967, Ha et al., 2011], or in cases where a supplier manages a portfolio of facilities with heterogeneous technologies the overall cost is far from linear. This framework includes constant marginal costs and reservation cost only (i.e., execution cost is zero) as special cases, so it is flexible enough to capture a range of applications as we will demonstrate in Section 4.

With constant marginal costs, it is plausible to focus on simple contract forms, but with general cost functions, more sophisticated contract formats may be worthwhile, such as bids specifying a schedule of prices and quantities (i.e., a supply function). This type of supply function bid often occurs in practice through the offer of some form of quantity discount. Competition with function bids is appropriate in a situation where a buyer does not stipulate a specific bidding format in its Request for Quote (RFQ), so that suppliers can bid in whatever format they like.

We model this capacity game in a Stackelberg framework, where the suppliers are leaders and the buyer is a follower. Each supplier has a good understanding of the information about all supplier costs and the distribution of the buyer's demand, but the buyer may not know the suppliers' costs. This assumption is appropriate for industries where the operating environments are more transparent and/or the production technologies are more mature such as electricity, electronics and semiconductors [Wu and Kleindorfer, 2005, Martínez-de Albéniz and Simchi-Levi, 2009, Anderson et al., 2017].

Within this framework, we first show that given knowledge of the other suppliers' bids, it is optimal for each supplier to set execution prices at execution costs and make profits only through the buyer's reservation payment. Second, we identify conditions under which there exists an equilibrium at which the buyer makes a reservation choice that maximizes the overall supply chain profit, each supplier makes a profit that is equal to their marginal contribution to the supply chain and the buyer takes the remaining profit. This equilibrium is of particular interest since it possesses some benevolent properties. For example, at this equilibrium there is no deadweight loss for the decentralized supply chain, and it is also the best equilibrium for the suppliers and thus a natural equilibrium for them to coordinate on. Third, we apply our general framework to a variety of problems in operations management and demonstrate that our equilibrium analysis holds in a range of circumstances.

Main contributions. First, we develop a general framework to study a broad class of capacity games in which the suppliers compete using supply functions and their costs are two-dimensional and nonlinear. Existing models have focused on problems where suppliers are restricted to a simple strategy space (i.e., scalar decision variables). To our knowledge, this paper is the first to study supply function competition in a capacity reservation setting. Moreover, the framework considered here encapsulates a wide range of applications in operations. Second, in this framework, we identify a natural property that is sufficient for a well-behaved equilibrium; that is, the *supply chain* optimal profits are submodular in terms of the set of suppliers available. While the equilibrium is not new, the result on sufficiency is. Third, we provide two different sets of conditions for the capacity games under which this submodularity property holds. This result is proved based on discrete convexity analysis and is of interest in its own right.

Related literature. Our paper studies supplier competition in a capacity reservation setting. It will be helpful to look in detail at three directly related papers in operations management. Wu and Kleindorfer [2005] consider an industrial buyer who purchases supply options from competing suppliers, while having access to an open spot market. No reservation cost at the suppliers is considered and the buyer's downward-sloping demand arises from utilization maximization. This demand modeling approach allows the buyer to rank suppliers by using a price index, which together with the zero reservation cost assumption leads to a Bertrand type of equilibrium for the suppliers. In contrast, our model considers a stochastic demand and that the suppliers have a two-dimensional nonlinear cost. In this case we obtain a Vickrey-Clarke-Groves (VCG) type of equilibrium. Martínez-de Albéniz and Simchi-Levi [2009] study a setting where marginal costs are constants and each supplier chooses a reservation price and an execution price for their limitless capacity. They show that there may be efficiency loss in equilibrium of up to 25% of the supply chain optimal profit. Our paper differs from theirs by considering general cost functions and allowing suppliers to submit function bids. By enlarging the strategy space of suppliers, we show that when the supply chain optimal profit is submodular (which happens under some commonly studied settings), there is no efficiency loss and each supplier makes a profit equal to their marginal contributions.

The third related paper is Anderson et al. [2017] (hereafter [ACS17]), which considers a discrete setting, where each supplier owns a block of capacity and the buyer must reserve all of a block or none. The general framework that we give here covers this model as a special case since we can recover their setting by restricting the buyer's capacity choice to be either zero or the block size. Equal block sizes are *necessary* and *assumed* in [ACS17] to ensure submodularity of the buyer's profit, which is then used to establish an equilibrium result. However, given the supply function bids considered in our model, equal block sizes are no longer applicable. On the other hand, our use of submodularity for the *supply chain* marks a significant difference from [ACS17]. Since the buyer's profit depends on supplier bids, which are endogenous decisions, it would not be appropriate to look for submodularity for the buyer's profit in our model given the supply function bids considered. We also note that the discrete framework of [ACS17] with each supplier having multiple blocks of capacity cannot be used to approximate the continuous functions here because costs are all tied to individual blocks and there is no way to restrict the buyer to use blocks in a specific order. Thus, if there are blocks with low costs then they will be preferred by the buyer. This independence of choice between blocks in their setting is fundamental in their proofs, so that even where our results appear similar, they are derived in a different way. Moreover, the equilibrium result they have is quite fragile and fails for unequal-size blocks (as well as in the extension with each supplier having multiple blocks), whereas our result holds with general cost functions as long as the optimal supply chain profit is submodular.

In our model suppliers compete by offering a pair of function bids, which resembles what is studied in the supply function equilibrium (SFE) literature [Klemperer and Meyer, 1989, Anderson and Philpott, 2002, Johari and Tsitsiklis, 2011, Sunar and Birge, 2019]. However, the game type is very different. The key difference is that the SFE literature does not involve a two-stage buyer decision, with capacity choices first. Another important difference is that the SFE literature has finite marginal costs, whereas our model allows a fixed cost that applies only when reservation or execution amounts are greater than zero. The consequence is that we need to consider the set of suppliers selected by the buyer, which is not a concern in the SFE literature.

Finally, we can compare our model with VCG models [Vickrey, 1961, Rothkopf et al., 1990]. The equilibrium profit allocation we obtain is of VCG style, but there is a fundamental difference: the VCG models take a mechanism design perspective in which the buyer selects bids that maximize system welfare, and pays according to each bidder's contribution. With this setup the equilibrium result is for bidders to bid truthfully and the auction result is efficient. The model we propose is a pay-as-bid auction with no freedom for the buyer to choose payment rules. We give conditions for an equilibrium in which the bidders mark up by the exact VCG amounts. We still get an efficient auction result, but without truthful bids.

2 Model setup

The structure of the model is illustrated in Figure 1. At the start the suppliers offer bids, following which a single buyer selects the capacity amount to reserve from each supplier. Then the buyer observes a signal (strictly a set of signals) about some intrinsic uncertain events (e.g., demand, spot price or supply). The signal is a set of random variables each with a known distribution. For example the signal might be either a forecast or the actual market demand, or it could contain information on the likelihood of a supplier being able to deliver. We do not need to impose any restrictions on signal distributions, and when there is more than one signal they can be correlated. In the next stage, the choice of amount to buy from each supplier is made by the buyer constrained by the capacity already reserved. No penalty is incurred by the buyer if there is a mismatch between the reserved capacity and the execution amount. After that the uncertain events are realized and then the buyer sells the product to consumers and collects the revenue. The expected revenue to the buyer is determined by the amounts bought and the signal received. On the other hand, the suppliers' bids and the buyer's reservation decision are based on the public information about the distribution of the signal. We are not considering risk aversion, so we can use the expected payoff as the objective for the buyer and the suppliers.

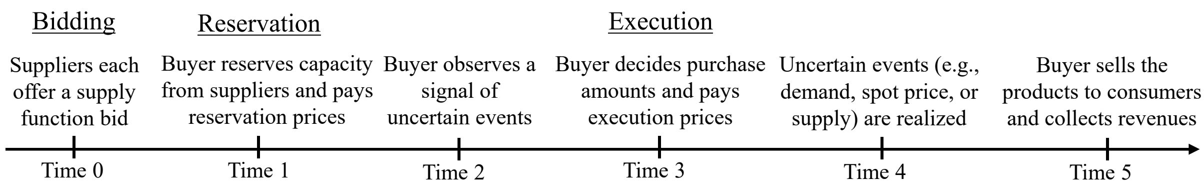

Figure 1: The timeline

Let $N = \{1, \dots, n\}$ denote the set of suppliers. The buyer makes capacity decision

$\mathbf{u} = (u_1, \ldots, u_n)$, where each $u_i \in U_i \subseteq \mathbb{R}_+$. We assume that each U_i is compact and contains 0. Typically, U_i is a closed interval $[0, \bar{d}_i]$ where $\bar{d}_i$ is the upper limit of the capacity amount. We may also have $U_i = \{0, W_i\}$, which corresponds to the case where the buyer either does not use this supplier, or uses the full amount W_i that the supplier has available. We write η for the random variable or set of variables, that are the signal received, with $\eta \in \mathcal{G} \subseteq \mathbb{R}^k$ for some positive integer k. After η has been received, the buyer chooses amounts $\mathbf{s} = (s_1 \ldots, s_n)$. These amounts depend on the capacities already purchased so that $\mathbf{s} \leq \mathbf{u}$. Supplier i offers a pair of payment functions $R_i(\cdot)$ and $P_i(\cdot)$, where $R_i(u_i)$ is the payment made to supplier i when the buyer makes a capacity reservation of u_i, and $P_i(s_i)$ is the payment made to supplier i given an execution amount s_i. In the formulation of the game there may be restrictions on the bids allowed, which we capture by specifying an allowed set of functions $\mathcal{A}$, and restrict the supplier's choice to $R_i(\cdot) \in \mathcal{A}$ and $P_i(\cdot) \in \mathcal{A}$. We will assume that all functions in $\mathcal{A}$ are lower semi-continuous and take the value zero at zero. Hence if the buyer decides not to include supplier i, then no payment is made.

Besides payments made by the buyer to the suppliers, we also have costs $E_i(u_i)$ and $C_i(s_i)$ incurred by supplier i who reserves a capacity u_i and then is required to produce an amount s_i. We assume that $E_i(\cdot) \in \mathcal{A}$ and $C_i(\cdot) \in \mathcal{A}$. The revenue to the buyer depends on the signal η and the execution amounts $\mathbf{s}$, and we write this as $V(\eta, \mathbf{s})$. Given that η is a signal, $V(\eta, \mathbf{s})$ may involve an expectation over other variables (as will be made specific in Section 4).

Since execution amounts will depend on the signal η, we can write a policy for the buyer as $(\mathbf{u}, \mathbf{s}(\cdot))$, where $\mathbf{u} \in \mathbf{U} = U_1 \times \cdots \times U_n$ and $\mathbf{s}(\cdot)$ is a function from the signal set $\mathcal{G}$ to $\mathbf{U}$, taking the value $\mathbf{s}(\eta)$ at η. For simplicity, we will often write $\mathbf{s}$ for $\mathbf{s}(\cdot)$. Given a set of supplier bids $\mathcal{B} = \{(P_i(\cdot), R_i(\cdot)) \in \mathcal{A} \times \mathcal{A} : i \in N\}$, the total expected buyer profit with a policy choice $(\mathbf{u}, \mathbf{s}(\cdot))$ is

$$\Pi_{\mathcal{B}}(\mathbf{u}, \mathbf{s}) = \mathbb{E}_\eta \left[V(\eta, \mathbf{s}(\eta)) - \sum_{i \in N} (P_i(s_i(\eta)) + R_i(u_i)) \right],$$

where we write $\mathbb{E}_\eta$ to indicate the expectation over η. The buyer's problem is to maximize its expected profit by making an optimal reservation choice $\mathbf{u}$ and a set of execution amounts $\mathbf{s}(\eta)$ for each possible signal $\eta \in \mathcal{G}$, i.e., to solve $\max_{\mathbf{u} \in \mathbf{U}} \Pi^\circ_{\mathcal{B}}(\mathbf{u})$, where

$$\Pi^\circ_{\mathcal{B}}(\mathbf{u}) = \max_{\mathbf{s}(\cdot) \leq \mathbf{u}} \Pi_{\mathcal{B}}(\mathbf{u}, \mathbf{s}). \tag{1}$$

With bids $\mathcal{B}$ and buyer choice $(\mathbf{u}, \mathbf{s}(\cdot))$, supplier i has an expected profit of

$$\pi_i(\mathbf{u}, \mathbf{s}) = \mathbb{E}_\eta \left[R_i(\mathbf{u}) - E_i(u_i) + P_i(s_i(\eta)) - C_i(s_i(\eta)) \right], \; i \in N. \tag{2}$$

The supply chain expected profit is

$$\Pi_{\mathcal{C}}(\mathbf{u}, \mathbf{s}) = \Pi_{\mathcal{B}}(\mathbf{u}, \mathbf{s}) + \sum_{i \in N} \pi_i(\mathbf{u}, \mathbf{s}) \tag{3}$$

$$= \mathbb{E}_\eta \left[V(\eta, \mathbf{s}(\eta)) - \sum_{i \in N} (C_i(s_i(\eta)) + E_i(u_i)) \right], \tag{4}$$

which is independent of the bids P_i and R_i. We define the set of bids where each supplier bids at cost as $\mathcal{C} = \{(C_i(\cdot), E_i(\cdot)) : i \in N\}$. Then the supply chain profit is the buyer profit with these bids, and hence the notation $\Pi_{\mathcal{C}}$ is consistent with $\Pi_{\mathcal{B}}$.

We write $I(\mathbf{u}) = \{i : u_i > 0\}$ for the support of a vector $\mathbf{u}$ and use $\Pi^*_{\mathcal{C}}(S)$, $S \subseteq N$, to denote the optimal supply chain profit when the capacity reserved is restricted to be zero outside the set S. Thus

$$\Pi^*_{\mathcal{C}}(S) = \max_{\mathbf{u} \in \mathbf{U},\, I(\mathbf{u}) \subseteq S} \Pi^\circ_{\mathcal{C}}(\mathbf{u}), \tag{5}$$

where $\Pi^\circ_{\mathcal{C}}(\mathbf{u})$ is defined in (1) with $\mathcal{B}$ replaced by $\mathcal{C}$. Similarly, given a set of supplier bids $\mathcal{B} = \{(P_i(\cdot), R_i(\cdot)) : i \in N\}$, we use $\Pi^*_{\mathcal{B}}(S)$, $S \subseteq N$, to denote the optimal buyer profit when the capacity reserved is restricted to be zero outside the set S. Thus

$$\Pi^*_{\mathcal{B}}(S) = \max_{\mathbf{u} \in \mathbf{U},\, I(\mathbf{u}) \subseteq S} \Pi^\circ_{\mathcal{B}}(\mathbf{u}).$$

Since the maximizations above are taken over $\mathbf{u} \in \mathbf{U}$ (a vector) and $\mathbf{s}(\cdot)$ (a function), to ensure that the maxima exist and are attained, we shall assume that buyer profit $\Pi_{\mathcal{B}}(\mathbf{u}, \mathbf{s})$ is upper semi-continuous in both arguments $\mathbf{u}, \mathbf{s} \in \mathbf{U}$.

3 Best response and equilibrium behavior

We now look at each supplier's best response problem. As a Stackelberg leader, each supplier is able to anticipate the buyer's reservation choice provided that the competitors' bids are observed. Our first result characterizes the best response for a supplier, given bids by the other suppliers. This result is more explicit than the related result on best response in [ACS17] and requires a different proof. Note that in their setting with blocks, the equivalent result fails when a single supplier offers more than one block.

Denote the marginal contribution of supplier $i \in N$ to the expected profit of the buyer under bids $\mathcal{B}$ by

$$Z_i(\mathcal{B}) = \Pi^*_{\mathcal{B}}(N) - \Pi^*_{\mathcal{B}}(N \setminus \{i\}).$$

In particular, we denote $\Delta_i = Z_i(\mathcal{C})$ as the marginal contribution of supplier i to the expected profit of the whole supply chain.

Theorem 1 (Best Response). *Given $\{(P_j(\cdot), R_j(\cdot)) : j \in N \setminus \{i\}\}$ for any $i \in N$, we have*

(a) *The profit for supplier i is no more than $Z_i = Z_i(\mathcal{B}_i)$, where $\mathcal{B}_i = \{(P_j(\cdot), R_j(\cdot)) : j \in N, j \neq i\} \cup \{(C_i(\cdot), E_i(\cdot))\}$.*

(b) *If $Z_i > 0$ then the profit Z_i is achieved by the offer $(\bar{P}_i(\cdot), \bar{R}_i(\cdot))$ defined by:*

$$\bar{P}_i(\mathbf{s}) = C_i(\mathbf{s}), \text{ and}$$
$$\bar{R}_i(\mathbf{u}) = E_i(\mathbf{u}) + Z_i \text{ when } u_i > 0 \text{ and } \bar{R}_i(\mathbf{u}) = 0 \text{ when } u_i = 0,$$

provided supplier i has preferred status under this set of bids.

(c) *If $Z_i > 0$, then for any $\epsilon > 0$, an offer of $\big(\bar{P}_i(\cdot), \bar{R}_i(\cdot) - \epsilon\big)$ will achieve within ϵ of the maximum possible supplier profit.*

Proof. (a) We consider any feasible offer $(\tilde{P}_i(\cdot), \tilde{R}_i(\cdot))$ from supplier i, giving a set of bids: $\tilde{\mathcal{B}}_i = \{(P_j(\cdot), R_j(\cdot)) : j \in N, j \neq i\} \cup \{(\tilde{P}_i(\cdot), \tilde{R}_i(\cdot))\}$. Then the buyer's profit from any policy $(\tilde{\mathbf{u}}, \tilde{\mathbf{s}}(\cdot))$ is

$$\begin{aligned}\Pi_{\tilde{\mathcal{B}}_i}(\tilde{\mathbf{u}}, \tilde{\mathbf{s}}(\cdot)) &= \mathbb{E}\left[V(\eta, \tilde{\mathbf{s}}(\eta)) - \sum_{j \neq i}(P_j(\tilde{\mathbf{s}}(\eta)) + R_j(\tilde{\mathbf{u}})) - (\tilde{P}_i(\tilde{\mathbf{s}}(\eta)) + \tilde{R}_i(\tilde{\mathbf{u}}))\right] \\ &= \Pi_{\mathcal{B}_i}(\tilde{\mathbf{u}}, \tilde{\mathbf{s}}(\cdot)) - \mathbb{E}\left[\tilde{P}_i(\tilde{\mathbf{s}}(\eta)) + \tilde{R}_i(\tilde{\mathbf{u}}) - C_i(\tilde{\mathbf{s}}(\eta)) - E_i(\tilde{\mathbf{u}})\right].\end{aligned}$$

The expectation term here is just the expected profit for supplier i. Since $\Pi_{\mathcal{B}_i}(\tilde{\mathbf{u}}, \tilde{\mathbf{s}}(\cdot)) \leq \Pi^*_{\tilde{\mathcal{B}}_i}(N)$, we can deduce that if supplier i has profit strictly greater than Z_i, then

$$\Pi_{\tilde{\mathcal{B}}_i}(\tilde{\mathbf{u}}, \tilde{\mathbf{s}}(\cdot)) < \Pi^*_{\tilde{\mathcal{B}}_i}(N) - Z_i = \Pi^*_{\tilde{\mathcal{B}}_i}(N \setminus \{i\}) = \Pi^*_{\mathcal{B}_i}(N \setminus \{i\}).$$

Thus buyer choices that deliver a profit more than Z_i to supplier i will not achieve the expected buyer profit $\Pi^*_{\mathcal{B}_i}(N \setminus \{i\})$, which is available to the buyer through not selecting supplier i (i.e. setting $u_i = 0$). Hence supplier i cannot achieve a profit of more than Z_i.

(b) Let $(\bar{\mathbf{u}}, \bar{\mathbf{s}}(\cdot))$ be optimal for the buyer with bids $\mathcal{B}_i$. Since $Z_i > 0$, we know that $\Pi^*_{\mathcal{B}_i}(N) > \Pi^*_{\mathcal{B}_i}(N \setminus \{i\})$, which implies that $\bar{u}_i > 0$. Consider the set of bids: $\bar{\mathcal{B}}_i = \{(P_j(\cdot), R_j(\cdot)) : j \in N, j \neq i\} \cup \{(\bar{P}_i(\cdot), \bar{R}_i(\cdot))\}$. Then

$$\Pi_{\bar{\mathcal{B}}_i}(\bar{\mathbf{u}}, \bar{\mathbf{s}}(\cdot)) = \Pi_{\mathcal{B}_i}(\bar{\mathbf{u}}, \bar{\mathbf{s}}(\cdot)) - Z_i = \Pi^*_{\mathcal{B}_i}(N) - Z_i.$$

But for the buyer choice $(\mathbf{u}, \mathbf{s}(\cdot))$ with $u_i > 0$, we have

$$\Pi_{\bar{\mathcal{B}}_i}(\mathbf{u}, \mathbf{s}(\cdot)) = \Pi_{\mathcal{B}_i}(\mathbf{u}, \mathbf{s}(\cdot)) - Z_i \leq \Pi^*_{\mathcal{B}_i}(N) - Z_i,$$

and, for any buyer choice with $u_i = 0$, we have

$$\Pi_{\bar{\mathcal{B}}_i}(\mathbf{u}, \mathbf{s}(\cdot)) = \Pi_{\mathcal{B}_i}(\mathbf{u}, \mathbf{s}(\cdot)) \leq \Pi^*_{\mathcal{B}_i}(N \setminus \{i\}) = \Pi^*_{\mathcal{B}_i}(N) - Z_i.$$

Thus $(\bar{\mathbf{u}}, \bar{\mathbf{s}}(\cdot))$ is optimal for $\bar{\mathcal{B}}_i$ and because of the preferred status of supplier i we know that this supplier is selected by the buyer, hence delivering the maximum profit Z_i for supplier i.

(c) In this case with the bid $(C_i(\mathbf{s}), \bar{R}_i(\mathbf{u}) - \epsilon)$ the profit for supplier i is $Z_i - \epsilon$ provided $u_i > 0$. Given this offer, the buyer by choosing $(\bar{\mathbf{u}}, \bar{\mathbf{s}}(\cdot))$, defined in part (b), will achieve a profit of $\Pi^*_{\mathcal{B}_i}(N) - Z_i + \epsilon = \Pi^*_{\mathcal{B}_i}(N \setminus \{i\}) + \epsilon$, which is therefore greater than any buyer profit available when $u_i = 0$. Hence the buyer's optimal choice must have $u_i > 0$. □

Theorem 1 shows that the maximum profit for a supplier is the increase in profit available to the buyer from including the supplier when his bids are made at cost. Moreover, when optimizing for the supplier, it is sufficient to consider supplier bids that are at cost for the execution component and make money only by adding a fixed amount to the reservation costs. However, we should note that the expected profit to the supplier is unaffected by parts of the function bids that are never selected by the buyer whatever demand occurs. The consequence is that there are a continuum of other best response function offers available.

Now let us consider the equilibrium strategies for suppliers. Since the buyer's optimization problem is embedded in the suppliers' best responses, we need to specify the buyer's choice when there are different options that give the same expected value to the buyer. The suppliers have an interest in raising prices to the point where the buyer is about to drop them from consideration. Therefore, a tie-breaking assumption is critical, otherwise, we can have a difficulty in defining optimal behavior for the suppliers, as a type of ϵ-optimality could occur when a supplier sets his price just below a benchmark value at which the buyer no longer wishes to select the supplier. Consequently, we make the following assumption.

Assumption 1 (Tie-Breaking Rule)**.** In case of multiple optimal solutions, the buyer will select one that gives the largest supply chain profit.

One reason why the buyer may select this Tie-Breaking Rule, in the case of multiple optimal solutions, is that by electing to give more profit to the suppliers, the buyer can potentially benefit in the future. It is also in line with that for the classic Bertrand competition model, where the firms have constant and different marginal costs. In this model, it is often assumed that the firm with the lowest cost prices at the second lowest cost or at the monopoly price, whichever is smaller, and wins the entire market [Vives, 2000, p. 123]. On the other hand, the Tie-Breaking Rule we have chosen is required if we want to ensure that an equilibrium exists as the following proposition demonstrates. Note that this result implies that there may not be an equilibrium if we use randomization to break ties. Thus, with the buyer breaking ties according to Assumption 1, the supply chain can be sustained in the long run, a desirable result for all supply chain members.

Proposition 1. *In any equilibrium between suppliers in which the buyer has multiple optimal solutions, the buyer will select one that gives the largest supply chain profit.*

We say that supplier $i \in N$ has *preferred status* under bids $\mathcal{B}$ if the buyer's preferred choice $\mathbf{u}$ under $\mathcal{B}$ satisfies $u_i > 0$. In particular, if these bids are all at costs (i.e., $\mathcal{B} = \mathcal{C}$), then supplier i is said to have *absolute preferred status*. It is evident that the absolute preferred status of a supplier is exogenous to suppliers' bidding decisions.

Note that the Tie-Breaking Rule stated above is sufficient for our equilibrium result when there is a unique *supply chain* optimal solution (as can be seen from our proof of Theorem 2), which is plausible in usual circumstances. In dealing with the unusual circumstance of multiple supply chain optimal solutions, we do not impose any *specific* restrictions on the buyer's preferred choice among a set of optimal solutions *after* application of the Tie-Breaking Rule. We only require that such a preference is public knowledge and satisfies the following assumption, which is related to the *Independence of Irrelevant Alternatives* axiom from decision theory.

Assumption 2 (Independence)**.** (a) Let X and X' be the sets of optimal solutions under two respective sets of bids after application of the Tie-Breaking Rule and let $(\mathbf{u}, \mathbf{s}(\cdot)) \in X' \subseteq X$. If $(\mathbf{u}, \mathbf{s}(\cdot))$ is the preferred buyer choice among alternatives of X, then it remains so among alternatives of X'. And (b) if supplier $i \in N$ has absolute preferred status with zero marginal contribution to the supply chain (i.e., $\Delta_i = 0$), then such a status will not change in the absence of another supplier.[1]

[1] Part (a) sometimes implies part (b), but not always.

With the Tie-Breaking Rule and Independence assumption, we will write $(\mathbf{u}_N^*, \mathbf{s}_N^*(\cdot))$ to denote the buyer's preferred choice for the supply chain problem with all bids at costs. We write $N^*(\mathcal{C})$ for the set of suppliers with absolute preferred status, i.e., $N^*(\mathcal{C}) = I(\mathbf{u}_N^*)$.

We are interested in a particular set of supplier bids, which we call the *supply chain contribution bids*. These are the bids:

$$\bar{\mathcal{B}} = \{(C_i(\cdot), \bar{R}_i(\cdot)) : i \in N^*(\mathcal{C})\} \cup \{(C_i(\cdot), E_i(\cdot)) : i \notin N^*(\mathcal{C})\},$$

where

$$\bar{R}_i(t_i) = \begin{cases} E_i(t_i) + \Delta_i, & \text{if } u_i > 0; \\ 0, & \text{if } u_i = 0. \end{cases}$$

Notice that they fit the pattern of the best response bids in Theorem 1, where the execution prices are offered at cost and the reservation prices involve a fixed additional markup on the reservation costs. The markup for supplier i is his marginal contribution Δ_i to the supply chain as defined earlier. Our theorem below establishes conditions under which the bids $\bar{\mathcal{B}}$ are a Nash equilibrium for the suppliers.

Theorem 2 (Nash equilibrium)**.** *If $\Pi_{\mathcal{C}}^*(\cdot)$ is submodular, then the supply chain contribution bids $\bar{\mathcal{B}}$ are a Nash equilibrium for the suppliers, at which (a) the buyer makes a supply chain optimal choice $(\mathbf{u}_N^*, \mathbf{s}_N^*)$; (b) the buyer makes profit $\Pi_{\mathcal{C}}^*(N) - \sum_{i=1}^n \Delta_i$; and (c) supplier i, $i \in N$, makes profit Δ_i. Moreover there is no equilibrium in which any of the suppliers makes a greater profit than they do with the supply chain contribution bids.*

Proof. Denote $I(\mathbf{u}_{N\setminus\{i\}}^*) = N_{-i}*(\mathcal{C})$. We start by establishing an intermediate result: if $\Pi_{\mathcal{C}}^*(\cdot)$ is submodular then (i) for any $S \subseteq N$,

$$\Pi_{\mathcal{C}}^*(N) - \Pi_{\mathcal{C}}^*(S) \geq \sum_{i \in N\setminus S} \left(\Pi_{\mathcal{C}}^*(N) - \Pi_{\mathcal{C}}^*(N \setminus \{i\})\right); \tag{6}$$

and (ii) for any $i \in N$, we have $N^*(\mathcal{C}) \setminus \{i\} \subseteq N_{-i}^*(\mathcal{C})$.

Part (i) is implied by Proposition 2.1 in Nemhauser et al. [1978]. We prove part (ii) by contradiction. Since $\Pi_{\mathcal{C}}^*(\cdot)$ is submodular, by definition we obtain

$$\Pi_{\mathcal{C}}^*(N) + \Pi_{\mathcal{C}}^*(N \setminus \{i,j\}) \leq \Pi_{\mathcal{C}}^*(N \setminus \{i\}) + \Pi_{\mathcal{C}}^*(N \setminus \{j\}). \tag{7}$$

Suppose statement (b) is false and there exists $j \in N^*(\mathcal{C})$ and $j \neq i$ such that $j \notin N_{-i}^*(\mathcal{C})$. Then by definition, $\Pi_{\mathcal{C}}^*(N \setminus \{i\}) = \Pi_{\mathcal{C}}^*(N \setminus \{i,j\})$, which together with inequality (7) implies $\Pi_{\mathcal{C}}^*(N \setminus \{j\}) \geq \Pi_{\mathcal{C}}^*(N)$, which in turn implies that the marginal contribution of supplier j to the supply chain is zero, even in the absence of supplier i. However, supplier j has absolute preferred status (i.e., $j \in N^*(\mathcal{C})$) but has not in the absence of supplier i (i.e., $j \notin N_{-i}^*(\mathcal{C})$), which contradicts part (b) of the Independence assumption.

Now we prove the results in the theorem. First observe that for any buyer choice $(\mathbf{u}, \mathbf{s})$,

$$\Pi_{\bar{\mathcal{B}}}(\mathbf{u}, \mathbf{s}) = \mathbb{E}[V(\eta, \mathbf{s}(\eta)) - \sum_{i \in N}(C_i(\mathbf{s}(\eta)) + E_i(\mathbf{u}))] - \sum_{j \in I(\mathbf{u})} \Delta_j.$$

The expectation term is bounded above by $\Pi^*_{\mathcal{C}}(I(\mathbf{u}))$. Moreover from (6) we know that $\Pi^*_{\mathcal{C}}(N) - \Pi^*_{\mathcal{C}}(I(\mathbf{u})) \geq \sum_{i \in N \setminus I(\mathbf{u})} \Delta_i$. Putting these inequalities together we get

$$\Pi_{\bar{\mathcal{B}}}(\mathbf{u}, \mathbf{s}) \leq \Pi^*_{\mathcal{C}}(N) - \sum_{j \in N} \Delta_j. \tag{8}$$

On the other hand, recalling that we use $(\mathbf{u}^*_N, \mathbf{s}^*_N)$ to denote the buyer's preferred choice for the supply chain problem with all bids $\mathcal{C}$ at costs, we have

$$\Pi_{\bar{\mathcal{B}}}(\mathbf{u}^*_N, \mathbf{s}^*_N) = \mathbb{E}[V(\eta, \mathbf{s}^*_N(\eta)) - \sum_{j \in N} (C_j(\mathbf{s}^*_N(\eta)) + E_j(\mathbf{u}^*_N))] - \sum_{j \in I(\mathbf{u}^*_N)} \Delta_j.$$

Observe that the expectation term is just $\Pi^*_{\mathcal{C}}(N)$. Thus

$$\Pi_{\bar{\mathcal{B}}}(\mathbf{u}^*_N, \mathbf{s}^*_N) \geq \Pi^*_{\mathcal{C}}(N) - \sum_{j \in N} \Delta_j, \tag{9}$$

which implies that the upper bound in (8) is achieved at $(\mathbf{u}^*_N, \mathbf{s}^*_N)$. In other words, solution $(\mathbf{u}^*_N, \mathbf{s}^*_N)$ is not only optimal to the supply chain, but also optimal to the buyer. Therefore, under bids $\bar{\mathcal{B}}$, if the buyer still has multiple optimal solutions after application of the Tie-Breaking Rule, then these optimal solutions form a subset of supply-chain optimal solutions, which together with part (a) of the Independence assumption implies that $(\mathbf{u}^*_N, \mathbf{s}^*_N)$ is *the* preferred choice of the buyer under bids $\bar{\mathcal{B}}$. Moreover, the inequalities (8) and (9) imply that $\Pi^*_{\mathcal{C}}(N) = \Pi^*_{\mathcal{C}}(N \setminus \{j\})$ for any $j \notin I(\mathbf{u}^*_N)$, and hence each supplier i achieves the profit Δ_i.

Now let us establish that this is an equilibrium, i.e., no supplier can improve his profit with a (unilaterally) different offer. According to part (ii) of the intermediate result we already proved: for any $i \in N$, $I(\mathbf{u}^*_N) \subseteq I(\mathbf{u}^*_{N \setminus \{i\}}) \cup \{i\}$, dropping a supplier from the set available will not cause other suppliers previously selected to also be dropped. Then

$$\begin{aligned}
\Pi_{\bar{\mathcal{B}}}(\mathbf{u}^*_{N \setminus \{i\}}, \mathbf{s}^*_{N \setminus \{i\}}) &\geq \Pi^*_{\mathcal{C}}(N \setminus \{i\}) - \sum_{j \in N^*(\mathcal{C}) \setminus \{i\}} (\Pi^*_{\mathcal{C}}(N) - \Pi^*_{\mathcal{C}}(N \setminus \{j\})) \\
&= \Pi^*_{\mathcal{C}}(N \setminus \{i\}) - \sum_{j \in N, j \neq i} (\Pi^*_{\mathcal{C}}(N) - \Pi^*_{\mathcal{C}}(N \setminus \{j\})) \\
&= \sum_{j \in N} \Pi^*_{\mathcal{C}}(N \setminus \{j\}) - (|N| - 1)\Pi^*_{\mathcal{C}}(N),
\end{aligned}$$

which is exactly the upper bound on buyer profit in (8) for any feasible $(\mathbf{u}, \mathbf{s})$. Therefore, $(\mathbf{u}^*_{N \setminus \{i\}}, \mathbf{s}^*_{N \setminus \{i\}})$ is also optimal for the buyer under bids $\bar{\mathcal{B}}$. Thus

$$\Pi^*_{\bar{\mathcal{B}}}(N \setminus \{i\}) = \Pi^*_{\bar{\mathcal{B}}}(N) = \sum_{j \in N} \Pi^*_{\mathcal{C}}(N \setminus \{j\}) - (|N| - 1)\Pi^*_{\mathcal{C}}(N), \tag{10}$$

and there is no loss to the buyer from a restriction that supplier i is unavailable.

Now suppose that given bids $\bar{\mathcal{B}}_{-i}$, supplier i gives a different offer $(\tilde{P}_i(\cdot), \tilde{R}_i(\cdot))$, leading to bids $\tilde{\mathcal{B}}_i = \{(\bar{P}_j(\cdot), \bar{R}_j(\cdot)) : j \neq i\} \cup \{(\tilde{P}_i(\cdot), \tilde{R}_i(\cdot))\}$. Suppose that $(\tilde{\mathbf{u}}, \tilde{\mathbf{s}})$ is the optimal choice by the buyer given bids $\tilde{\mathcal{B}}_i$. Then

$$\Pi^*_{\tilde{\mathcal{B}}_i}(N) = \Pi_{\mathcal{C}}(\tilde{\mathbf{u}}, \tilde{\mathbf{s}}) - \sum_{j \neq i, \tilde{\mathbf{u}}_j > 0} \Delta_j - \tilde{\pi}_i.$$

So

$$
\begin{aligned}
\tilde{\pi}_i &= \Pi_{\mathcal{C}}(\tilde{\mathbf{u}}, \tilde{\mathbf{s}}) - \sum_{j \neq i, \tilde{\mathbf{u}}_j > 0} \Delta_j - \Pi^*_{\bar{\mathcal{B}}}(N \setminus \{i\}) \\
&\leq \Pi^*_{\mathcal{C}}(I(\tilde{\mathbf{u}})) - \sum_{j \in N, j \neq i} \Delta_j + \sum_{j \notin I(\tilde{\mathbf{u}}), j \neq i} (\Pi^*_{\mathcal{C}}(N) - \Pi^*_{\mathcal{C}}(N \setminus \{j\})) - \Pi^*_{\bar{\mathcal{B}}}(N \setminus \{i\}) \\
&\leq \Pi^*_{\mathcal{C}}(I(\tilde{\mathbf{u}})) - \sum_{j \in N, j \neq i} \Delta_j + \Pi^*_{\mathcal{C}}(N) - \Pi^*_{\mathcal{C}}(I(\tilde{\mathbf{u}})) - \Pi^*_{\bar{\mathcal{B}}}(N \setminus \{i\}) \\
&= \Pi^*_{\mathcal{C}}(N) - (|N| - 1)\Pi^*_{\mathcal{C}}(N) + \sum_{j \neq i} \Pi^*_{\mathcal{C}}(N \setminus \{i\}) - \Pi^*_{\bar{\mathcal{B}}}(N \setminus \{i\}),
\end{aligned}
$$

where the first inequality is from the optimality of $\Pi^*_{\mathcal{C}}(I(\tilde{\mathbf{u}}))$, and the second inequality follows from (6). We can use (10) and cancel terms to obtain $\tilde{\pi}_i \leq \Delta_i$, thus showing that no improvement in profit for supplier i is possible, and leading to an equilibrium.

For the final statement of the theorem, we let π^*_j and π^*_B be the equilibrium profits of supplier $j \in N$ and the buyer, respectively. Suppose that in an equilibrium set of offers $\mathcal{B} = \{(P_j(\cdot), R_j(\cdot)) : j \in N\}$, there is a supplier i with higher profit than with the supply chain contribution bids, i.e., $\pi^*_i > \Pi^*_{\mathcal{C}}(N) - \Pi^*_{\mathcal{C}}(N \setminus \{i\})$. Since $\pi^*_B + \sum_{j \in N} \pi^*_j \leq \Pi^*_{\mathcal{C}}(N)$, this implies that $\pi^*_B + \sum_{j \in N, j \neq i} \pi^*_j < \Pi^*_{\mathcal{C}}(N \setminus \{i\})$. Now choose any supplier $k \in N^*(\mathcal{B})$ and $k \neq i$, and consider supplier k deviating by offering the new bid $(P_k(\cdot), R_k(\cdot) + \epsilon)$ where $\epsilon < \Pi^*_{\mathcal{C}}(N \setminus \{i\}) - \pi^*_B - \sum_{j \in N, j \neq i} \pi^*_j$ is a small positive number. If we can show that k is still chosen by the buyer, then this will improve his profit and this is enough to show that $\mathcal{B}$ is not an equilibrium.

If the buyer chooses supplier k, then the buyer's profit is at least $\Pi^*_{\mathcal{C}}(N \setminus \{i\}) - \sum_{j \in N, j \neq i} \pi^*_j - \epsilon$, which is strictly greater than π^*_B from the choice of ϵ. Whereas if the buyer does not choose supplier k, then the buyer will choose among suppliers in $N \setminus \{k\}$, and her profit cannot be greater than π^*_B which is the buyer's profit without the deviation by supplier k, given a free choice among suppliers in N. Thus supplier k is still chosen by the buyer with his new bid. □

The above theorem characterizes a Nash equilibrium based on the supply chain contribution offers. In this equilibrium, each supplier's profit is equal to his marginal contribution to the overall supply chain, a result that can be seen as allowing a fair allocation of profits. It is straightforward that each supplier makes a nonnegative profit. Suppliers in $N^*(\mathcal{C})$ each make a strictly positive profit while the other suppliers each make zero profits. On the buyer's side, the profit is nonnegative, which is a direct result of the submodularity property of $\Pi^*_{\mathcal{C}}(S)$ and is also implied by the fact that the buyer can achieve zero profit if it does not purchase from any supplier.

In equilibrium the buyer makes a choice that maximizes the supply chain profit, i.e., the supply chain is coordinated in equilibrium. This is a desired result since there is no deadweight loss for the decentralized supply chain. Much of the supply chain coordination literature [Cachon, 2003] focuses on designing contracts (e.g., revenue sharing or buy back) to achieve the chain optimal profit in a dyadic supply chain. In contrast, in our model the supply chain optimality arises at equilibrium as a result of supplier competition (rather than by way of design). In the same way, the profit allocation in the equilibrium of Theorem 2 is a VCG result, but this is not from a mechanism design that sets prices paid in a particular way, but arises as an equilibrium from our pay-as-bid capacity game.

Since no supplier can do better at equilibrium than by using supply chain contribution bids, this is a natural equilibrium for suppliers to coordinate on. However, there are other possible equilibria, which also achieve the maximum supplier profits and involve different offers away from the amounts that are selected by the buyer. In particular, it is possible to construct equilibria that have continuous functions for the reservation bids R_i, "smoothing out" the discontinuities that occur at zero for the supply chain contribution bids. Further details are given in the Electronic Companion.

As shown in Theorem 2 an important role in establishing the equilibrium is played by the submodularity property for the *supply chain* optimal profit. This distinguishes us from [ACS17] in which it is shown that the *buyer* optimal profit is submodular for the all-or-nothing setting with equal-size blocks. With submodularity applying to the supply chain profit, the scope of our framework is significantly expanded as will be shown in Section 4. Theorem 3 below gives two conditions under each of which the supply chain optimal profit is submodular as a set function.

Theorem 3 (Submodularity). $\Pi_{\mathcal{C}}^*(\cdot)$ *is submodular as a set function if either*

(a) $n = 2$ *and* $V(\eta, \mathbf{s})$ *is subadditive in* $\mathbf{s}$*; or*

(b) *the function* $\mathbb{E}_\eta\left[\max_{\mathbf{s}\le\mathbf{u}}\left(V(\eta,\mathbf{s}) - \sum_{i=1}^n C_i(s_i)\right)\right]$ *is* $M^\natural$*-concave in* $\mathbf{u} \in \mathbf{U}$*, where* $\mathbf{U} = [0, \bar{d}]^n$ *or* $\mathbf{U} = \{0, \bar{d}\}^n$ *for some constant* $\bar{d} > 0$*, and* $\{E_i(\cdot) : i \in N\}$ *are convex functions.*

Proof. (a) For submodularity with $n = 2$ we simply need to show that $\Pi_{\mathcal{C}}^*(\{1,2\}) \le \Pi_{\mathcal{C}}^*(\{1\}) + \Pi_{\mathcal{C}}^*(\{2\})$. Let $(\mathbf{u}^*, \mathbf{s}^*(\cdot))$ be the optimal supply chain choice. Then

$$\begin{aligned}\Pi_{\mathcal{C}}^*(\{1,2\}) &= \mathbb{E}[V(\eta, \mathbf{s}^*(\eta)) - \textstyle\sum_i (C_i(\mathbf{s}^*(\eta)) + E_i(\mathbf{u}^*))] \\ &= \mathbb{E}[V(\eta, \textstyle\sum_j [\mathbf{s}^*(\eta)]_j) - \sum_i (C_i(\sum_j [\mathbf{s}^*(\eta)]_j) + E_i(\sum_j [\mathbf{u}^*]_j))],\end{aligned}$$

where for any vector $\mathbf{x}$ we define $[\mathbf{x}]_1 := (x_1, 0)$ and $[\mathbf{x}]_2 := (0, x_2)$, and the summations are all over $i, j \in \{1, 2\}$. Using subadditivity of V we have

$$\begin{aligned}\Pi_{\mathcal{C}}^*(\{1,2\}) &\le \mathbb{E}[\textstyle\sum_i V(\eta, [\mathbf{s}^*(\eta)]_i)) - \sum_i (C_i([\mathbf{s}^*(\eta)]_i) + E_i([\mathbf{u}^*]_i))] \\ &= \textstyle\sum_i \Pi_{\mathcal{C}}([\mathbf{u}^*]_i, [\mathbf{s}^*(\eta)]_i) \le \Pi_{\mathcal{C}}^*(\{1\}) + \Pi_{\mathcal{C}}^*(\{2\}).\end{aligned}$$

(b) Note that the supply chain optimal profit as a set function can be expressed as follows:

$$\Pi^*(\sigma) = \max_{\mathbf{u}\le\sigma\bar{d}} \max_{\mathbf{s}(\cdot)\le\mathbf{u}} \Pi_{\mathcal{C}}(\mathbf{u}, \mathbf{s}(\cdot)), \sigma \in \{0,1\}^n.$$

According to (4) and noticing that concavity implies $M^\natural$-concavity, with the conditions of the theorem, $\Pi_{\mathcal{C}}^*(\mathbf{u}) \equiv \max_{\mathbf{s}\le\mathbf{u}} \Pi_{\mathcal{C}}(\mathbf{u}, \mathbf{s})$ is $M^\natural$-concave in $\mathbf{u} \in \mathbf{U}$, which implies that $\tilde{\Pi}_{\mathcal{C}}(\mathbf{u}) = \Pi_{\mathcal{C}}^*(\mathbf{u}\bar{d})$ is $M^\natural$-concave in $\mathbf{u} \in \mathbf{U}_0$, where $\mathbf{U}_0 = [0,1]^n$ if $\mathbf{U} = [0, \bar{d}]^n$ and $\mathbf{U}_0 = \{0,1\}^n$ if $\mathbf{U} = \{0, \bar{d}\}^n$. Let

$$\tilde{\Pi}(\bar{\sigma}) = \max\left\{\tilde{\Pi}_{\mathcal{C}}(\mathbf{u}) : \mathbf{u} \in \mathbf{U} \text{ and } \mathbf{u} \le \bar{\sigma}\right\},\ \bar{\sigma} \in \mathbf{U}_0.$$

We now show that $\tilde{\Pi}(\cdot)$ is $M^\natural$-concave over $\mathbf{U}_0$, which implies that $\Pi^*(\cdot)$ (which is the same as $\tilde{\Pi}(\cdot)$ if $\mathbf{U} = \{0, \bar{d}\}^n$ and is the restriction of $\tilde{\Pi}(\cdot)$ to $\{0,1\}^n$ if $\mathbf{U} = [0, \bar{d}]^n$) is submodular over $\{0,1\}^n$.

We assume $\mathbf{U} = [0, \bar{d}]^n$. In the case that $\mathbf{U} = \{0, \bar{d}\}^n$, we can use the same proof by simply replacing $\mathbb{R}^n$ by $\mathbb{Z}^n$ and replacing $[0,1]^n$ by $\{0,1\}^n$. Consider the infimal convolution function

$$\bar{\Pi}(\bar{\sigma}) = \sup\left\{\bar{\Pi}_{\mathcal{C}}(\mathbf{u}) + \alpha(\bar{\mathbf{u}}) : \mathbf{u} + \bar{\mathbf{u}} = \bar{\sigma},\ \mathbf{u}, \bar{\mathbf{u}} \in \mathbb{R}^n\right\},\ \bar{\sigma} \in \mathbb{R}^n;$$

where $\bar{\Pi}_{\mathcal{C}}(\mathbf{u}) = \tilde{\Pi}_{\mathcal{C}}(\mathbf{u}) - \sum_{i=1}^n \delta_{[0,1]}(u_i)$ and $\alpha(\bar{\mathbf{u}}) = -\sum_{i=1}^n \delta_{[0,1]}(\bar{t}_i)$, with $\tilde{\Pi}_{\mathcal{C}}(\mathbf{u}) = -\infty$ if $\mathbf{u} \notin [0,1]^n$ and $\delta_{[0,1]}(\cdot)$ is a function taking the value zero on the set $[0,1]$ and infinity elsewhere (i.e., $\delta_{[0,1]}(x) = 0$ if $x \in [0,1]$ and $\delta_{[0,1]}(x) = +\infty$ otherwise). It is clear that both $\bar{\Pi}_{\mathcal{C}}(\cdot)$ and $\alpha(\cdot)$ are $M^\natural$-concave over $\mathbb{R}^n$. According to Murota [2009, Section 4.2], $\bar{\Pi}(\bar{\sigma})$ is $M^\natural$-concave over $\mathbb{R}^n$. Note that the restriction of $\bar{\Pi}(\bar{\sigma})$ on $[0,1]^n$ is exactly $\tilde{\Pi}(\bar{\sigma})$, which therefore is $M^\natural$-concave over $[0,1]^n$ with straightforward direct verification according to the definition of $M^\natural$-concavity. We then conclude that $\tilde{\Pi}(\cdot)$ is submodular over $[0,1]^n$ [Murota and Shioura, 2004]. □

As discussed earlier, our framework covers the model in [ACS17] as a special case. Indeed, condition (b) can be satisfied in their equal-block case for which $\mathbf{U} = \{0, \bar{d}\}^n$. While the conditions in Theorem 3 may seem complicated, we provide simpler conditions for specific applications in Section 4. Finally, we remark that the second condition uses a property of $M^\natural$-concavity from discrete convexity, which has seen growing applications in operations [Chen and Li, 2020].

4 Applications and examples

The framework is general enough to encompass many different specific circumstances and we give three examples in this section. Numerical illustrations can be found in the Electronic Companion.

4.1 Newsvendor problem with unreliable suppliers

In a typical newsvendor problem, each supplier has a one-dimensional cost incurred before demand D is realized. This is equivalent to the special case of our model where execution costs are zero (i.e., $C_i(\cdot) = 0$). We will consider the case of unreliable suppliers, who may not be able to deliver the amount requested in full. We consider stochastically proportional yield, that is, an order for an amount s_i from supplier $i \in N$ will result in a delivered amount given by $\epsilon_i s_i$, where ϵ_i is the yield from supplier i. This assumption has been used in other studies such as Federgruen and Yang [2008]. In the special case where ϵ_i follows a Bernoulli distribution, we will have an all-or-nothing type of supply disruption model [Babich et al., 2007]. Denote by $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ the vector of yields from all the suppliers. Consistent with the newsvendor literature, any shortfall in demand is lost. The salvage value is $\beta \geq 0$ for each unit of overstock, and the penalty is $\alpha \geq 0$ for each unit of understock. The product is sold by the buyer to final customers at a retail price $\rho > \beta - \alpha$.

At the time when the buyer determines the reservation quantities, only information about the distributions of the suppliers' yields and the demand are known. Before the buyer determines the execution amounts a signal η is observed. The expected buyer revenue is given by

$$V(\eta, \mathbf{s}) = \mathbb{E}_{D,\epsilon}\left[(\rho - \beta + \alpha)\min\left\{\textstyle\sum_{i\in N}\epsilon_i s_i, D\right\} + \beta\textstyle\sum_{i\in N}\epsilon_i s_i - \alpha D \,\middle|\, \eta\right], \tag{11}$$

that is, the expectation is taken over D and ϵ conditional on the signal η. Since the execution costs $C_i(\cdot)$ are zero, the buyer chooses $\mathbf{s} \leq \mathbf{u}$ to maximize $V(\eta, \mathbf{s})$. Since both $(\rho - \beta + \alpha)$ and β are positive the maximum occurs at $\mathbf{s} = \mathbf{u}$. We have the following result, which shows that the convexity of the cost functions is a key requirement for the equilibrium results of Theorem 2 in this newsvendor application.

Proposition 2. *In the newsvendor model, if $\mathbf{U} = [0, \bar{d}]^n$ or $\mathbf{U} = \{0, \bar{d}\}^n$ for some constant $\bar{d} > 0$, and the cost functions E_i are (weakly) convex, then the supply chain optimal profit is submodular and the conclusions of Theorem 2 apply.*

4.2 Portfolio procurement problem

This application involves a buyer procuring a homogeneous product by using supply options and a spot market. Similar to Wu and Kleindorfer [2005] and [ACS17], we consider an open spot market so that the suppliers who participate in the options market cannot manipulate the spot price. With a random demand D, if the buyer orders amounts given by the vector $\mathbf{s}$, any shortfall in demand is met by purchases in a spot market at a price P_0. Both the final demand D and the spot market price P_0 are unknown at the time when $\mathbf{s}$ is chosen, but have distributions that are determined by the signal η received. For example η may be a pair of forecasts for D and P_0. Note that at the time when the reservation decision is made, only the distribution of the signal is known to the buyer. The products bought from the suppliers and the spot market are sold by the buyer to final customers at a price $\rho > P_{\max}$, where $P_{\max}$ is the maximum spot price. If demand turns out to be less than the execution amounts $\mathbf{s}$, then the excess can be sold at a salvage value $\beta \geq 0$ for each unit of overstock. To avoid arbitrage by the buyer, we assume that $\beta \leq P_{\min}$, where $P_{\min}$ is the minimum spot price. Then the expected buyer revenue given signal η and the execution quantities $\mathbf{s}$ is

$$V(\eta, \mathbf{s}) = \mathbb{E}_{D,P_0}\left[(\rho - P_0)D + (P_0 - \beta)\min\left\{\textstyle\sum_{i\in N} s_i, D\right\} + \beta\textstyle\sum_{i\in N} s_i \,\middle|\, \eta\right], \tag{12}$$

that is, the expectations over D and P_0 are taken conditional on the signal η.

In the case where there are just two suppliers, we will have an equilibrium of the standard type without any additional conditions.

Proposition 3. *In the portfolio procurement model, if there are just two suppliers, then the supply chain optimal profit is submodular and the conclusions of Theorem 2 apply.*

In moving to more than two suppliers, we need to assume that the execution costs are linear (i.e., each supplier has a constant marginal execution cost) as well as the reservation costs being convex. Moreover, we require that the signal η gives the actual demand D and spot price P_0. Thus there may be a shortfall in the amount supplied if the reserved quantities are too small, but there will not be an excess since otherwise costs can be saved by reducing the reservation quantities. Given the signal η (or equivalently, the realized demand and spot price) as well as the execution quantities $\mathbf{s}$, the buyer revenue is

$$V(\eta, \mathbf{s}) = (\rho - P_0)D + (P_0 - \beta)\min\left\{\sum_{i\in N} s_i, D\right\} + \beta\sum_{i\in N} s_i. \tag{13}$$

In addition, with linear execution costs we can write $C_i(s_i) = c_i s_i$ $(i \in N)$, where c_i is a constant. Thus, we can formulate the execution problem at the second stage as follows:

$$\max_{\mathbf{s}\leq\mathbf{u}}\left(V(\eta, \mathbf{s}) - \sum_{i\in N} c_i s_i\right). \tag{14}$$

Solving the above problem gives the optimal execution amounts. We have the following result.

Proposition 4. *In the portfolio procurement model, with* $\eta = (D, P_0)$*, if* $\mathbf{U} = [0, \bar{d}]^n$ *or* $\mathbf{U} = \{0, \bar{d}\}^n$ *for some constant* $\bar{d} > 0$*, the execution costs* C_i *are linear, and the reservation costs* E_i *are (weakly) convex, then the supply chain optimal profit is submodular and the conclusions of Theorem 2 apply.*

4.3 Non-substitutable products

Although our applications are primarily for the case where all suppliers provide the same product, our model also deals with situations in which different products are supplied. As an example, consider a buyer and two suppliers, where products 1 and 2 can be sold separately or bundled together, with a price premium for the bundled product. There is a substantial literature of bundling in an operations context (see, e.g., Geng et al. [2005] and the references therein). Let ρ_i be the price for product i, $i = 1, 2$, and $\rho > \rho_1 + \rho_2$ be the price for the bundle having equal quantities of the two products. Let D be the demand for the bundle and assume demand for each of the individual products is unlimited. If signal $\eta = D$ and $w = \min\{s_1, s_2, D\}$, then $V(\eta, \mathbf{s}) = \rho w + \sum_{i=1}^{2}\rho_i(s_i - w)$. Now suppose that execution costs are linear with $C_i(s_i) = c_i s_i$ and $c_i < \rho_i$, $i = 1, 2$. In this case the execution amounts will always equal the reserved amounts u_1 and u_2. Thus

$$\max_{\mathbf{s}\leq\mathbf{u}}(V(\eta, \mathbf{s}) - C_1(s_1) - C_2(s_2)) = (\rho - \rho_1 - \rho_2)\min\{u_1, u_2, D\} + \sum_{i=1}^{2}(\rho_i - c_i)t_i,$$

which is concave in $\mathbf{u} = (u_1, u_2)$. Thus from Theorem 2 we obtain the usual equilibrium solution provided that E_1 and E_2 are convex functions.

5 Conclusions and discussion

In this paper, we have developed a general framework to study a broad class of capacity games. This framework allows us to examine supplier competition in a capacity market where suppliers' costs may be nonlinear. In this setting, suppliers each submit a bid consisting of a reservation price function and an execution price function. The buyer decides how much capacity to reserve from each supplier and then, after obtaining further information about the demand, determines how much capacity to use.

When the competitors' bids are observed, we have shown that an optimal strategy for each supplier is to set the execution price to be the execution cost and add a margin on the reservation cost. This implies that suppliers make profits only from the buyer's reservation payments. This result does not hold when bids are constrained to have constant marginal costs (considered by [Martínez-de Albéniz and Simchi-Levi, 2009]). We have also given conditions under which the supply chain optimal profit is submodular in the set of suppliers, which allows us to identify an equilibrium in which the buyer's reservation choice is first best, each supplier's profit equals his marginal contribution to the supply chain and the buyer takes the remaining profit. We finally demonstrate how the model can be applied specifically to three important problems in operations.

We have chosen a relatively simple modelling framework, but the equilibrium results extend directly to cases where the costs and bids for one supplier depend on the capacity and execution amounts of all suppliers, rather than just those for this supplier. In fact our proofs are written for this more general case. An example of this occurs when players can partially collaborate by sharing warehouse facilities or transport links.

This paper can be extended in several directions. First we assume, as in many other supplier competition models, that supplier costs are known to the other suppliers. However, in some settings a model that considers cost uncertainties may be more appropriate. Second our equilibrium analysis builds on the submodularity condition. When this condition fails, the VCG strategies may not be a Nash equilibrium, as we demonstrate in Example 5 given in the Electronic Companion. It would therefore be interesting to investigate further what the equilibria look like without the submodularity. Finally, we should note our assumption that the random demand is exogenous. An important extension is to the case where the buyer can influence demand through setting a price.

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Electronic Companion to "Capacity Games with Supply Function Competition"

In this electronic companion we give some definitions in Section A.1, provide proofs for the propositions in Section A.2, and show examples of applications in Section A.3.

A.1 Some definitions

The set of real numbers is denoted by $\mathbb{R}$, and $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. The set of integers is denoted by $\mathbb{Z}$, and $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$ and $\underline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty\}$. We use $\mathbb{D}$ to denote *either* $\mathbb{Z}$ *or* $\mathbb{R}$. Denote $[n] = \{1, \ldots, n\}$ for any positive number n. The characteristic vector of $S \subseteq [n]$ is denoted by $\chi_S \in \{0,1\}^n$. For $i \in [n]$, we write χ_i for $\chi_{\{i\}}$, which is the ith unit vector, and $\chi_0 = \mathbf{0}$ (zero vector).

A.1.1 $M^\natural$-convexity

For a function $f\colon \mathbb{D}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$, the set

$$\mathrm{dom}_{\mathbb{D}} f = \{x \in \mathbb{D}^n : f(x) \in \mathbb{R}\}$$

is called the *effective domain* of f. For a vector $z \in \mathbb{R}^n$, define the *positive* and *negative supports* of z as

$$\mathrm{supp}^+(z) = \{i \in [n] : z_i > 0\}, \quad \mathrm{supp}^-(z) = \{i \in [n] : z_i < 0\}.$$

A function $f\colon \mathbb{Z}^n \to \bar{\mathbb{R}}$ is said *$M^\natural$-convex* if for any $x, y \in \mathrm{dom}_{\mathbb{Z}} f$ and any $i \in \mathrm{supp}^+(x-y)$, there exists $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that the following *exchange property* is satisfied:

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

Similarly, a function $f\colon \mathbb{R}^n \to \bar{\mathbb{R}}$ is said *$M^\natural$-convex* if for any $x, y \in \mathrm{dom}_{\mathbb{R}} f$ and any $i \in \mathrm{supp}^+(x-y)$, there exist $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ and $\lambda_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \lambda(\chi_i - \chi_j)) + f(y + \lambda(\chi_i - \chi_j))$$

for all $\lambda \in \mathbb{R}^n$ with $0 \leq \lambda \leq \lambda_0$. A function $f\colon \mathbb{D}^n \to \underline{\mathbb{R}}$ is said *$M^\natural$-concave* if $(-f)$ is $M^\natural$-convex.

A.1.2 Laminar convexity

A non-empty set $\mathcal{L} \subseteq 2^{[n]}$ is called a *laminar family* if for any $A, B \in \mathcal{L}$, we have $A \cap B = \emptyset$, or $A \subseteq B$, or $B \subseteq A$. A function $f\colon \mathbb{D}^n \to \bar{\mathbb{R}}$ is said *laminar convex* if it can be represented as

$$f(x) = \sum_{S \in \mathcal{L}} f_S(x(S)),$$

where $\{f_S\}$ are univariate convex functions, $\mathcal{L}$ is a laminar family, and $x(S) = \sum_{i \in S} x_i$. A function $f\colon \mathbb{D}^n \to \underline{\mathbb{R}}$ is said *laminar concave* if $(-f)$ is laminar convex.

A.2 Proofs of Propositions

In all the proofs we write cost and bid functions for individual suppliers as functions of the complete vectors. For example we write $R_i(\mathbf{u})$ even though we have assumed that R_i depends only on u_i. In fact our results hold in the more general case where costs for one supplier can depend on the quantities for other suppliers.

A.2.1 Proof of Proposition 1

Suppose that under bids $\mathcal{B} = \{(R_i(\cdot), P_i(\cdot)) : i \in N\}$, the buyer selects an optimal solution $(\mathbf{u}, \mathbf{s})$ that gives a lower expected supply chain profit than another optimal solution $(\mathbf{u}', \mathbf{s}')$, i.e., $\Pi_{\mathcal{C}}(\mathbf{u}, \mathbf{s}) < \Pi_{\mathcal{C}}(\mathbf{u}', \mathbf{s}')$ and $\Pi_{\mathcal{B}}(\mathbf{u}, \mathbf{s}) = \Pi_{\mathcal{B}}(\mathbf{u}', \mathbf{s}')$. Denote by π_i and π_i' the corresponding expected profits of supplier $i \in N$ under these two solutions given the bids $\mathcal{B}$. Since $\sum_{i\in N} \pi_i < \sum_{i\in N} \pi_i'$ according to (3), there is a supplier $k \in I(\mathbf{u}')$ such that $\pi_k < \pi_k'$. We claim that given the other suppliers' bids in $\mathcal{B}$ are unchanged, supplier k will have an incentive to deviate from his current pair of bids $(R_k(\cdot), P_k(\cdot))$ to a new pair of bids $(R_k'(\cdot), P_k'(\cdot))$, where

$$\begin{aligned} P_k'(x) &= P_k(x) - \epsilon(P_k(x) - C_k(x))/\pi_k', \\ R_k'(x) &= R_k(x) - \epsilon(R_k(x) - E_k(x))/\pi_k', \end{aligned} \tag{A-1}$$

and $0 < \epsilon < \pi_k' - \pi_k$. Under this new set $\mathcal{B}'$ of bids, it is evident that the buyer improves her expected profit by exactly $\epsilon(\pi_k/\pi_k')$ with solution $(\mathbf{u}, \mathbf{s})$, which is (strictly) smaller than ϵ, the expected profit improvement she can achieve by selecting solution $(\mathbf{u}', \mathbf{s}')$. On the other hand, suppose that under bids $\mathcal{B}'$ the buyer selects an optimal solution $(\mathbf{u}'', \mathbf{s}'')$ and the corresponding expected profit for player k is π_k''. Denote $\tilde{\pi}_k = \pi_k(\mathbf{u}'', \mathbf{s}'')$, which is defined as in (2). Then according to (A-1) the buyer's expected profit is

$$\Pi_{\mathcal{B}'}(\mathbf{u}'', \mathbf{s}'') = \Pi_{\mathcal{B}}(\mathbf{u}'', \mathbf{s}'') + \epsilon(\tilde{\pi}_k/\pi_k'),$$

which implies that her expected profit improvement is at most $\epsilon(\tilde{\pi}_k/\pi_k')$. Therefore, we have $\tilde{\pi}_k/\pi_k' \geq 1$ due to optimality of her selection, and thus according to (A-1) supplier k's new expected profit $\pi_k'' = (\pi_k' - \epsilon)(\tilde{\pi}_k/\pi_k') \geq \pi_k' - \epsilon > \pi_k$. We conclude that, under bids $\mathcal{B}'$, the buyer will deviate from her current solution, which results in both the buyer and supplier k being better off. In other words, $(\mathbf{u}, \mathbf{s})$ under bids $\mathcal{B}$ is not at equilibrium.

A.2.2 Proof of Proposition 2

We will check the conditions in Part 1(b) of Theorem 2. According to (11), and using the fact that $V(\eta, \mathbf{s})$ increases in $\mathbf{s}$ and $C_i(\cdot) = 0$, we have

$$\begin{aligned} &\mathbb{E}_\eta \left[\max_{\mathbf{s}\leq\mathbf{u}} \left(V(\eta, \mathbf{s}) - \sum_{i=1}^{n} C_i(s_i) \right) \right] \\ &= \mathbb{E}_\eta[V(\eta, \mathbf{u})] = \mathbb{E}_\eta \left[\mathbb{E}_{D,\epsilon} \left[(\rho - \beta + \alpha) \min \left\{ \sum_{i\in N} \epsilon_i u_i, D \right\} + \beta \sum_{i\in N} \epsilon_i u_i - \alpha D \,\middle|\, \eta \right] \right], \end{aligned}$$

which is concave in each u_i because $\rho - \beta + \alpha \geq 0$, and hence is $M^\natural$-concave in $\mathbf{u} \in \mathbf{U}$. Thus we have established that all the conditions Part 1(b) of Theorem 2 are satisfied, given the conditions of Proposition 2. Hence the conclusions follow.

A.2.3 Proof of Proposition 3

Given there are just two suppliers, we show that $V(\eta, \mathbf{s})$ is subadditive in $\mathbf{s}$, which leads to satisfaction of conditions Part 1(a) of Theorem 2 and we are done. According to (12), $V(\eta, \mathbf{s})$ is the expectation of the sum of the following three functions of $\mathbf{s}$:

$$(\rho - P_0)D, \quad (P_0 - \beta)\min\left\{\sum_{i\in N} s_i, D\right\}, \text{ and } \beta\sum_{i\in N} s_i.$$

The first function is a non-negative constant (with respect to $\mathbf{s}$) and thus clearly subadditive. The last function is clearly additive and hence also subadditive. The middle non-negative function can be easily verified to be subadditive. Therefore, as the expectation of the sum of three subadditive functions, $V(\eta, \mathbf{s})$ is subadditive.

A.2.4 Proof of Proposition 4

According to the conditions of the proposition, we assume that, for each supplier $i \in N$, the execution cost $C_i(s_i) = c_i s_i$ for some constant $c_i \geq 0$. Also assume without loss of generality that $\beta = c_0 \leq c_1 \leq \cdots \leq c_n \leq \rho$. For $\eta = (D, P_0)$, according to (13) and (14), we wish to solve the following maximization problem:

$$\max_{\mathbf{s}\leq\mathbf{u}} \left((\rho - P_0)D + (P_0 - \beta)\min\left\{D, \textstyle\sum_{i\in N} s_i\right\} - (c_i - \beta)\textstyle\sum_{i\in N} s_i\right). \tag{A-2}$$

It is optimal to use reserved capacities $\mathbf{u}$ as much as possible unless it is more expensive than to purchase from the spot market. Therefore, given $\beta \leq P_0 \leq \rho$, it is not difficult to see that, if we let $k = k(P_0) = \max\{i : c_i \leq P_0\}$, then the optimal value of (A-2) is given by

$$\sum_{i=1}^{k+1} \bar{\rho}_i b_i(\mathbf{u} \,|\, \eta), \tag{A-3}$$

where $\bar{\rho}_i = \rho - c_i$ and $b_i(\mathbf{u} \,|\, \eta) = \min\left\{(D - \sum_{j=1}^{i-1} u_j)^+, u_i\right\}$ $(1 \leq i \leq k)$, $\bar{\rho}_{k+1} = \rho - P_0$ and $b_{k+1}(\mathbf{u} \,|\, \eta) = \left(D - \sum_{i=1}^{k} u_i\right)^+$. Denote $a_i = D - \sum_{j=1}^{i} u_j$ for $i = 0, 1, \ldots, k$. Then $b_{k+1}(\mathbf{u} \,|\, \eta) = a_k^+$ and

$$b_i(\mathbf{u} \,|\, \eta) = \min\{a_{i-1}^+, u_i\} = a_{i-1}^+ - (a_{i-1}^+ - u_i)^+ = a_{i-1}^+ - a_i^+, \; i = 1, \ldots, k.$$

which leads to

$$\sum_{i=1}^{k+1} \bar{\rho}_i b_i(\mathbf{u} \,|\, \eta) = \bar{\rho}_1 D - \sum_{i=1}^{k} (\bar{\rho}_i - \bar{\rho}_{i+1}) a_i^+ = \bar{\rho}_1 D - \sum_{i=1}^{k} (\bar{\rho}_i - \bar{\rho}_{i+1}) \left(D - \sum_{j=1}^{i} u_j\right)^+.$$

Noticing that $(D - x)^+$ is convex in $x \geq 0$ and $\bar{\rho}_i - \bar{\rho}_{i+1} \geq 0$, we conclude that the above function of $\mathbf{u}$ is laminar concave with the corresponding laminar family $\mathcal{L} = \{\{1, \ldots, i\} : i \in N\}$, which implies that the expression in (A-3) as a function of $\mathbf{u} \in \mathbf{U}$ is laminar concave and thus $M^\natural$-concave. Therefore, Part 1(b) and Part 2 of Theorem 2 apply.

A.3 Examples

In this section we give some concrete numerical examples to illustrate the results obtained in Section 4. Some counter-examples are also provided to demonstrate the limitation of our results. We use the following example to illustrate Proposition 2.

Example 1. Consider an example with three suppliers. The buyer faces a random demand D, which follows a uniform distribution over the support $[0, 100]$. The supplier costs are $E_1(x) = 4.5x$, $E_2(x) = 0.1x^2$, $E_3(x) = 5.5x$, and $C_i(\cdot) = 0$ for $i = 1, 2, 3$. The random yield from each supplier follows a Bernoulli distribution. Specifically, there is a probability p_i that supplier i can deliver the full amount of the buyer's order, where we set $p_1 = 0.7$, $p_2 = 0.8$ and $p_3 = 0.85$. We assume the yields and demand are independent of each other. The retail price, salvage value and penalty cost are respectively $\rho = 10$, $\beta = 3$ and $\alpha = 1$ per unit of the product. We can carry out the detailed calculations to find the supply chain optimal solutions for different sets of available suppliers. For example, the values for u_1 and u_2 when $S = \{1, 2\}$ are chosen to maximize the following expression:

$$\begin{aligned}
(\rho - \beta + \alpha)\big(p_1 p_2\,\mathbb{E}[\min\{u_1 + u_2, D\}] + p_1(1 - p_2)\,\mathbb{E}[\min\{u_1, D\}]& \\
+ (1 - p_1)p_2\,\mathbb{E}[\min(u_2, D)]\big) + \beta \sum_{i=1}^{2} p_i u_i - \alpha\,\mathbb{E}[D]& \\
= 8\left(0.56\left(u_1 + u_2 - \frac{(u_1 + u_2)^2}{200}\right) + 0.14\left(u_1 - \frac{u_1^2}{200}\right)\right.& \\
\left. + 0.24\left(u_2 - \frac{u_2^2}{200}\right)\right) + 3(0.7u_1 + 0.8u_2) - 50.&
\end{aligned}$$

The optimal reservation choices and profits are summarized in Table 1.

Suppliers S	$(\mathbf{u}_S^*)_1$	$(\mathbf{u}_S^*)_2$	$(\mathbf{u}_S^*)_3$	$\Pi_{\mathcal{C}}^*(S)$
$\{1, 2, 3\}$	14.71	25.50	25.92	135.62
$\{1, 2\}$	35.26	27.35	0	126.76
$\{1, 3\}$	22.27	0	41.03	64.61
$\{2, 3\}$	0	25.94	35.86	133.19
$\{1\}$	57.14	0	0	41.43
$\{2\}$	0	33.33	0	96.67
$\{3\}$	0	0	56.62	58.99

Table 1: Supply chain optimal reservation choices and profits

Proposition 2 implies that $\Pi_{\mathcal{C}}^*(\cdot)$ is submodular as can be checked from the table. Thus, the equilibrium in Theorem 2 applies: we have zero execution prices, and the reservation price is the sum of the reservation cost $E_i(x)$ and the supply chain marginal contribution Δ_i for $x > 0$. Here these contributions are 2.43 for supplier 1, 71.01 for supplier 2, and 8.86 for supplier 3. In this equilibrium the buyer receives the remainder of the total supply chain profit: $135.62 - 2.43 - 71.01 - 8.86 = 56.32$. □

To illustrate the results in Proposition 3, we draw an example from Martínez-de Albéniz and Simchi-Levi [2009]. In this example, there are two suppliers, and we will give an example with three suppliers later.

Example 2. This example matches Example 1 in Martínez-de Albéniz and Simchi-Levi [2009] in which there is no spot market. The demand is uniformly distributed over $[0,1]$, so $F(x) = x$ or $\bar{F}(x) = 1 - F(x) = 1 - x$ for $x \in [0,1]$. There are two suppliers and their marginal costs are $(c_1, e_1) = (0, 60)$ and $(c_2, e_2) = (75, 5)$. The retail price is $\rho = 100$.

We begin with the supply chain optimal problems. When both suppliers are available, the supply chain problem is:

$$\max_{u_1,\, u_2 \in [0,1]} \left(\int_0^{u_1} \big((\rho - c_1)\bar{F}(x) - e_1\big)\, dx + \int_0^{u_2} \big((\rho - c_2)\bar{F}(x + u_1) - e_2\big)\, dx \right).$$

We can calculate that the optimal solution is $\mathbf{u}_N^* = (4/15, 8/15)$. The supply chain optimal profit is $\Pi_{\mathcal{C}}^*(N) = 32/3$. When only supplier 1 is available, the optimal solution is $\mathbf{u}_{\{1\}}^* = 2/5$ and the supply chain optimal profit is $\Pi_{\mathcal{C}}^*(\{1\}) = 8$. When the buyer is allowed to choose supplier 2 only, the optimal solution is $\mathbf{u}_{\{2\}}^* = 4/5$ and the supply chain optimal profit is $\Pi_{\mathcal{C}}^*(\{2\}) = 8$.

If suppliers are restricted to each offering a pair of reservation price and execution price, MS show that in equilibrium these two suppliers bid infinitesimally close to each other, and the following is a continuum of ϵ-equilibria, $(p_1^*, r_1^*) = (p_2^*, r_2^*) = \left(p, 60 - \frac{55}{75}p\right)$, which are parameterized with $p \in [50, 75]$. In equilibria, the buyer's reservation choice is $\mathbf{u}^* = \left(\frac{4}{15}, \frac{40}{3(100-p)}\right)$. The profit split amongst players is

$$\Pi_B^* = \frac{8(150-p)^2}{225(100-p)}, \quad \pi_1^* = \frac{8p}{225}, \quad \text{and} \quad \pi_2^* = \frac{800(75-p)}{9(100-p)^2}, \tag{A-4}$$

and the resulting supply chain profit is $\Pi_{\mathcal{C}}^* = \frac{32(225-2p)}{9(100-p)} + \frac{800(75-p)}{9(100-p)^2}$. Note that none of these equilibria is supply chain optimal except the one with $p = 75$.

We now demonstrate that the above strategies do not form an equilibrium if suppliers are allowed to offer function bids. Fix supplier 1's bid $(p_1^*, r_1^*) = (p, 60 - \frac{55}{75}p)$, and we examine supplier 2's best response in choosing a function bid. Using Theorem 1, we can show that the following offer for supplier 2 improves his profit: setting execution price to be execution cost, and for the reservation price, charging a fixed payment of $\frac{32(75-p)}{9(100-p)}$ on top of reservation cost. Given this offer (and supplier 1's offer (p_1^*, r_1^*)), the buyer's reservation amounts are $\tilde{\mathbf{u}} = \left(\frac{4}{15}, \frac{8}{15}\right)$, which gives her a profit equal to $8 - \frac{32(75-p)}{9(100-p)}$. Also if the buyer purchases from only supplier 2, she makes the same profit. According to the tie-breaking rule, the buyer will select $\tilde{\mathbf{u}}$. Then supplier 2's profit becomes $\tilde{\pi}_2 = \frac{32(75-p)}{9(100-p)}$, which is strictly greater than π_2^* in (A-4) for any $p < 75$.

From Proposition 3 we know there is an equilibrium with supply chain contribution bids, where there are fixed markups applied to the reservation costs of $\Pi_{\mathcal{C}}^*(N) - \Pi_{\mathcal{C}}^*(\{2\}) = \frac{8}{3}$ and $\Pi_{\mathcal{C}}^*(N) - \Pi_{\mathcal{C}}^*(\{1\}) = \frac{8}{3}$. At this equilibrium, the buyer's optimal reservation choice is $\mathbf{u}^* = (4/15, 8/15)$. The profit split amongst players is

$$\Pi_B^* = 16/3 \quad \text{and} \quad \pi_1^* = \pi_2^* = 8/3. \tag{A-5}$$

Thus, each supplier's profit equals his contribution to the supply chain system and the buyer takes the remaining profit. Moreover, the reservation choice by the buyer is supply chain optimal (in contrast with the case with constant prices).

An important message of this example is that imposing a restriction that each supplier submits a pair of reservation price and an execution price rather than a function bid leads to a higher buyer profit. On the other hand, the suppliers are better off if they submit function bids. This can be easily seen by comparing the profit splits in (A-4) and (A-5). □

For the equilibrium in Theorem 2, the capacity reservation price is obtained by simply adding a fixed markup to the reservation cost. Thus, it cannot be represented by defining marginal prices for capacity reservation: to do so would require an infinite marginal price for the first ϵ amount of capacity. However, the supply chain contribution bids are not the only equilibrium strategies. We can construct an alternative form of equilibria with finite marginal prices from the capacity offers of the supply chain contribution bids by smoothing out the beginning part of each of these bids. The example below demonstrates this with power functions.

Example 3 (Example 2 continued)**.** We revisit Example 2. As shown earlier, this example satisfies the submodularity property. Besides the equilibrium with supply chain contribution bids, we can show that the following bids with power functions (parameterized by β_1 and β_2) form a Nash equilibrium:

$$P_1^*(x) = 0,\ 0 \le x \le 1,\ \ R_1^*(x) = \begin{cases} 60x + \frac{8}{3} - \frac{8}{3}\left(\frac{15}{4}\right)^{\beta_1+1}\left(\frac{4}{15} - x\right)^{\beta_1+1}, & 0 \le x \le \frac{4}{15}, \\ 60x + \frac{3}{8}, & \frac{4}{15} < x \le 1; \end{cases}$$

$$P_2^*(x) = 75x,\ 0 \le x \le 1,\ \ R_2^*(x) = \begin{cases} 5x + \frac{8}{3} - \frac{8}{3}\left(\frac{15}{8}\right)^{\beta_2+1}\left(\frac{8}{15} - x\right)^{\beta_2+1}, & 0 \le x \le \frac{8}{15}, \\ 5x + \frac{8}{3}, & \frac{8}{15} < x \le 1. \end{cases}$$

where $\beta_1, \beta_2 \ge 2$.

Given the above bids, the buyer's optimal reservation choice is $(4/15, 8/15)$, matching the supply chain optimal solution. In addition, it can be shown that neither supplier has an incentive to unilaterally deviate from their power function bids. A detailed proof of this result can be found in an earlier version of this paper (Anderson et al. 2019)[2]. The profit split in equilibrium is that each supplier makes a profit of 8/3 and the buyer's profit is 16/3.

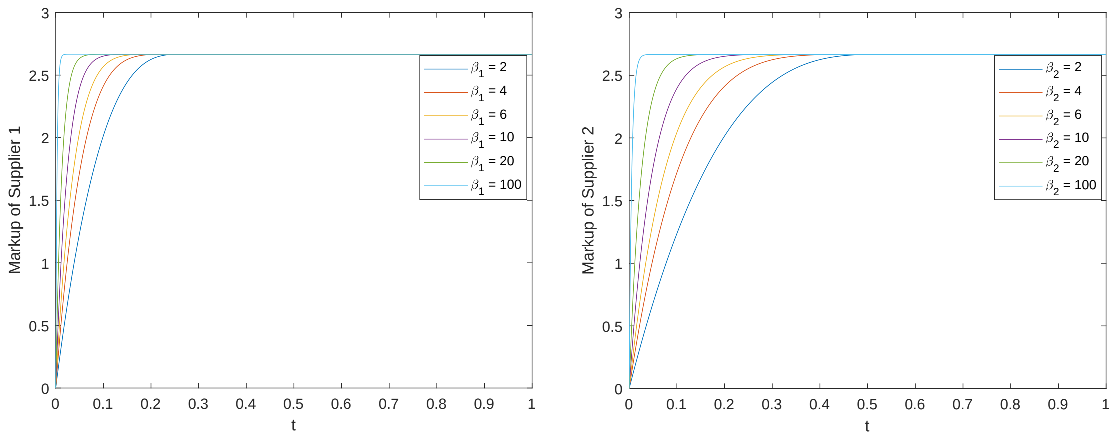
Compared with the supply chain contribution bids, the only difference of the above power function bids is that suppliers 1 and 2 respectively lower their reservation prices by $\frac{8}{3}\left(\frac{15}{4}\right)^{\beta_1+1}\left(\frac{4}{15} - x\right)^{\beta_1+1}$ for $x \in \left[0, \frac{4}{15}\right]$ and $\frac{8}{3}\left(\frac{15}{8}\right)^{\beta_2+1}\left(\frac{8}{15} - x\right)^{\beta_2+1}$ for $x \in \left[0, \frac{8}{15}\right]$. With these power functions, the reservation prices become differential and thus can be represented with their marginal prices.

The power functions are chosen so that the buyer will never choose to reserve an amount below 4/15 from supplier 1 and an amount below 8/15 from supplier 2. This is done by setting the values of β_1 and β_2 large enough (both are at least 2 for this example) to ensure that the price drops at the beginning part of each offer are inconsequential in terms of the buyer's optimal choice and the equilibrium for the suppliers.

[2]E.J. Anderson, B. Chen and L. Shao (2019). *Capacity games with supply function competition.* https://arxiv.org/abs/1905.11084

It is easy to see that when β_1 and β_2 approach positive infinity, the power function bids reduce to the supply chain contribution bids. Moreover, one can construct an equilibrium at which one supplier uses supply chain contribution bids (with an infinitely large β_i value), while the other supplier uses power function bids (with a finite β_i value). Despite the multiplicity of equilibrium bidding strategies, all these equilibria lead to the same profit allocation.

A final point about these power-function bids is that a pair of larger values of β_1 and β_2 will imply that the corresponding markups increase more steeply at the beginning of the offers. Figure 2 illustrates this with different β values. □



(a) Supplier 1 (b) Supplier 2

Figure 2: Illustration of supplier markups with different values of β_1 and β_2

To complement Example 2 in which there are only two suppliers, the following example considers three suppliers.

Example 4. Consider the case where there is no spot market. Suppose the demand follows a uniform distribution over $[0, 1]$. There are three suppliers whose costs are: $C_1(x) = x, E_1(x) = 3x$; $C_2(x) = 2.5x, E_2(x) = 2x$; and $C_3(x) = 5x, E_3(x) = x$. The retail price is $\rho = 10$. We can carry out the detailed calculations to find the supply chain optimal solutions for different sets of available suppliers. One way to do these calculations is to use the screening curve approach that is common in calculation of optimal generation mix in electricity markets (see Green, 2005)[3]. We summarize the optimal reservation choices and profits in Table 2.

Using the results in the table, we can easily check the submodularity of $\Pi^*_{\mathcal{C}}(S)$. We can now construct an equilibrium set of offers where the suppliers offer at cost and in addition require a fixed reservation payment of Δ_i, which then becomes supplier i's profit. Here these amounts are 0.08333 for supplier 1, 0.0333 for supplier 2, and 0.0333 for supplier 3. In this equilibrium the buyer receives the remainder of the total supply chain profit: $2.1333 - 0.15 = 1.9833$. □

The submodularity property (and hence the Nash equilibrium) in Proposition 4 may not hold when suppliers have decreasing marginal reservation costs as we demonstrate with the following example.

[3]R. Green (2005). Electricity and markets. *Oxford Review of Economic Policy* 21(1), 67–87.

Suppliers S	$(\mathbf{u}_S^*)_1$	$(\mathbf{u}_S^*)_2$	$(\mathbf{u}_S^*)_3$	$\Pi_{\mathcal{C}}^*(S)$
$\{1,2,3\}$	1/3	4/15	1/5	2.1333
$\{1,2\}$	1/3	2/5	0	2.1
$\{1,3\}$	1/2	0	3/10	2.1
$\{2,3\}$	0	3/5	1/5	2.05
$\{1\}$	2/3	0	0	2
$\{2\}$	0	11/15	0	2.0167
$\{3\}$	0	0	4/5	1.6

Table 2: Supply chain optimal reservation choices and profits

Example 5. Suppose the demand is fixed with $D = 10$ and the retail price is $\rho = 20$. There are three suppliers with $N = \{1, 2, 3\}$. Supplier 1 and supplier 2 have the same marginal costs with $c_1(x) = c_2(x) = e_1(x) = e_2(x) = 0$, for $x \in [0, 5]$ (and an infinite cost for any larger amount). Supplier 3's marginal costs are $c_3(x) = 0$ and $e_3(x) = 10 - x$ for $x \in [0, 10]$. So both supplier 1 and supplier 2 have the capacity of 5 and supplier 3's capacity is 10.

We now look at the supply chain optimal problems. If all the three suppliers are available, the buyer will choose 5 units from each of supplier 1 and supplier 2. The supply chain optimal profit is $\Pi_{\mathcal{C}}^*(\{1, 2, 3\}) = 200$. If only suppliers 3 and 1 (or 2) are available, the buyer will choose 5 units from each of 3 and 1 (or 2). The supply chain optimal profit is $\Pi_{\mathcal{C}}^*(\{1, 3\}) = \Pi(\{2, 3\}) = 162.5$. If supplier 3 is the sole supplier, the buyer will choose 10 units from supplier 3 and the supply chain optimal profit is $\Pi_{\mathcal{C}}^*(\{3\}) = 150$. Therefore, we have $\Pi_{\mathcal{C}}^*(\{1, 2, 3\}) + \Pi_{\mathcal{C}}^*(\{3\}) = 350 > 325 = \Pi_{\mathcal{C}}^*(\{1, 3\}) + \Pi_{\mathcal{C}}^*(\{2, 3\})$, which contradicts the submodularity property.

We can also show that the proposed equilibrium in Theorem 2 will not apply in this case. If each of suppliers 1 and 2 asks for a fixed payment of $200 - 162.5 = 37.5$, and the buyer makes the supply chain optimal choice of selecting these two suppliers, then the buyer profit is $200 - 75 = 125$. However, this is less than the profit available to the buyer from selecting supplier 3 alone, which gives the buyer a profit of 150. So this is not an equilibrium. We can check that an equilibrium exists where both suppliers 1 and 2 ask for a fixed payment of 25, and the buyer chooses both of these offers. □