PANDEMIC MODELLING IN SMALL ENVIRONMENTS

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Abstract

The Corona Virus Disease 2019 (COVID-19) discovered by China in December 2019 changes the whole world. With high infectiousness, high variation rate, as well as substantial severity and death rate, the virus spread around the world with unexpected speed and has greatly impacted the society and economy. However, with the aim of resuming the on-campus study, we propose a new compartmental model to figure out the transmission of COVID-19 in the University and to find the best control strategies based on optimal control techniques. Composed of two SEIQR structures and one environment compartment, the model includes staff-to-staff infections, student-staff cross infections, student-to-student infections, and individual-environment-individual infections. Analytical expression of $R_0$ is found and the system dynamics are analysed based on control methods. By defining and adding more variables, this sophisticated model could further simulate the effects of four control measures: mask-wearing, social-distancing, environment disinfection, and mandatory quarantine. By defining optimal control problems and simulating under four different scenarios, the significance and actual effectivenesses of the control measures are reflected based on the analysis of model parameters and derived optimal trajectories. Results reveal the particular importance of social-distancing and mask-wearing and also emphasize the necessity of isolating the infected students. Meanwhile, the output plots argue the dominant role of students in transmitting the virus. Thus, both the proposed model and the optimal control problems are reliable tools for University administrators to make decisions when they want to resume the on-campus teaching.
Acknowledgment

This is a quite different and difficult year. Due to COVID-19, I studied remotely all the time. I have too many thanks to express.

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Abbreviations

COVID-19  Corona Virus Disease 2019
WHO:  World Health Organisation
SARS-CoV-2:  Severe Acute Respiratory Syndrome Coronavirus 2
SIR  Susceptible-Infected-Recovered
SEIQR  Susceptible-Exposed-Infected-Quarantined-Recovered
DTMC  Discrete-Time Markov Chain
CTMC  Continuous-Time Markov Chain
SDE  Stochastic Differential Equation
Chapter 1

Introduction

This chapter will describe the influence of COVID-19 to the whole world, discuss the characteristics of the virus, and mention the challenges in tackling this pandemic. Then objectives of this project will be clearly explained.

1.1 Background

1.1.1 Global Situation

The unexpected outbreak of Corona Virus Disease 2019 (COVID-19) brings a dramatic change to the whole world. In December 2019, China first discovered this novel strain of coronavirus in Wuhan. Transmitting mainly by direct contact and droplets in the air, the virus spreads around the whole country within two months. The World Health Organisation (WHO) declared this contagious disease as a pandemic on March 11, 2020 [1]. Then the virus further travelled all over the world with an unstoppable speed. The year 2020 witnessed huge outbreaks of the disease in many countries and now in 2021 the pandemic still does not stop. India becomes the country that endures the most serious impact of the pandemic this year. By August 1st, 2021, WHO has reported over 195 million cases with more than 4.1 million deaths around the world [1]. The global distribution of cases is graphed in Figure 1.1. It can be explicitly seen that the epidemics in Asia, Europe, North America as well as South America are particularly severe. Countries with
huge populations, such as the USA, Brazil, Russia and India, have a cumulative number of cases even greater than 5 million. This count is horribly large, indicating that numerous people’s lives have been threatened or lost and economy of the world has been greatly impacted.

![Global Distribution of Cases](image1)

**Figure 1.1: Global Distribution of Cases [1]**

Almost all countries have conducted various national measures, such as mandatory quarantine, stay-at-home order and even lockdown, to contain the epidemic. Vaccines have also been invented, manufactured, and wildly applied to citizens. However, completely cutting off the transmission path of the virus is extremely hard. The pandemic is still proceeding and evolving.

### 1.1.2 Features and Challenges of COVID-19

This epidemic is ascribed to the virus called Severe Acute Respiratory Syndrome Coronavirus 2 (SARS-CoV-2). As its name indicates, the virus generally causes respiratory problems such as cough, sneezing, shortness of breath, along with other symptoms including fever and headache [2]. Furthermore, over one-third of patients suffer from olfactory or gustatory dysfunctions [3]. Since direct contact and aerosol transmissions are two important ways of infection [4], symptomatic individuals could be a huge spreader in the crowd
1.1 Background

because sneezing and coughing will release infinitely many viruses to the air. Meanwhile, the virus can also survive on the surface and then invades the human body through eyes, nose or mouth via touching [5, 6]. Moreover, some infected subjects are asymptomatic but they can still infect other susceptible individuals [7, 8]. This characteristic makes it impossible to distinguish all infected ones from the crowd without detecting all people by nucleic acid test. Therefore, it is extremely easy for the virus to spread from person to person. Detecting all carriers is a challenging task.

Two more challenges also cause higher difficulty in dealing with the epidemic. As an RNA virus, COVID-19 is easy to evolve genetically to adapt itself to different living environments [9, 10]. Therefore, it is extremely difficult to completely detect and remove the virus. Cascella et al. [10] has summarised the main variants and their features, four of which are mainly concerned globally: Alpha (B.1.1.7), Beta (B.1.351), Gamma (P.1) and Delta (B.1.617.2). These variants are more infectious or more severe, leading to the second or third waves in UK, India and South Africa, threatening people’s lives and impacting the economy. Furthermore, many countries have promoted vaccination. The advent of vaccines could provide an effective support for mitigating the decease but vaccines seem to be less effective against the new variants [11]. No one can guarantee that COVID-19 can be stopped by vaccination. In addition, under the context of globalization, no country can completely isolate itself from the transmission chain. This provides the virus with a larger chance of evolution and better ways of spreading widely. The most contagious variants originated from one local area can therefore spread all over the world. Referring to Figure 1.2 which displays the trends of cases and deaths since last year, the world has faced two explicit waves on December 2020 and April 2021 respectively. They are the result of combined effects of both new variants appearing in some countries and the fast spread of the more contagious variants due to globalization. Thus, it is still hard to witness end of this battle in the short run.
1.1.3 Necessity of Small-Environment Pandemic Modelling

Modelling the epidemics is an essential tool that helps the government to predict the trend and assist policy-makers to decide ways of prevention and control [12]. The mathematical model could indicate different development stages of the pandemic. By observing the estimated tendency, experts could evaluate the best moment of implementing the intervention and find the most powerful measure to stop the spread. Simultaneously, how to coordinate epidemic prevention and control with economic and social development is the most difficult topic. Mathematical models can also help find suitable measures that balances this trade-off.

Furthermore, under the circumstance where end of COVID-19 pandemic is currently not in sight, it is significant to prepare for the long-term co-existence with the virus. Universities and companies are two types of important places that guarantee the regular development of society. Therefore, analysing the spread of virus in these two confined spaces is particularly necessary. A suitable model can help the decision-maker in the college or company to better tackle the impact of epidemic. In this project, modelling the University case is the main scope.

1.2 Objectives

This project aims to establish a new mathematical model to simulate the spread of COVID-19 in the university. Due to the fact that most Universities are conducting remote study
and on-campus delivery is not promoted, suitable data do not exist. With this consideration, the detailed specifications are described below:

- The model should take different stages of infection such as incubation period, asymptomatic and symptomatic period into consideration.
- The model should involve effects of different control measures such as wearing masks, mandatory quarantine, vaccination.
- The model should consider the case in the confined space with people having different daily routines.
- Since no data can be used for regression or fitting, the model parameter values should be determined in other reliable ways. For example, study the COVID-19 properties from other available data or utilize similar parameter values in other literature.
- Determine and suggest suitable ways of controlling the epidemic by defining and solving the optimal control problem.

### 1.3 Thesis Outline

This chapter has described the current world situation under influence of the pandemic and inferred the necessity of modelling. Objectives of this project have also been outlined.

Chapter 2 will elaborate on different pandemic modelling strategies and justify the use of compartmental method in this project. The past researches and progresses on modelling COVID-19 will also be reviewed, implying the necessity of modelling the small-environment case.

Chapter 3 will firstly introduce the procedure of designing and formulating the University model and then analyse the system dynamics based on the next-generation matrix method and control methods. Model parameter meanings and values will also be discussed. After this, optimal control problems will be designed to represent and study the effectiveness of different control measures.
Chapter 4 will show the simulation result of the un-controlled model and investigation results on different model parameters. The optimal trajectories produced by solving the designed problems will also be displayed. Inference from the results and suggestions on pandemic control in the University are then discussed.

Chapter 5 will summarize the main contributions made by this project and express the potential improvements and relative research directions that can be explored in the future.
Chapter 2

Literature Review

There are various pandemic modelling methods and choosing an appropriate approach is crucial. Thus, this chapter will firstly analyse all pandemic modelling scenarios and then justify the feasibility of designing a compartmental model for this University case. All types of models are compared and two main criteria are particularly important in choosing the modelling method. After that, previous efforts in modelling COVID-19 by this method are also summarised, along with various deficiencies. This finally leads us to the necessity and main aims of this project.

2.1 Pandemic Modelling Strategies

Mathematical models designed by researchers to analyse epidemics can be classified into three categories: phenomenological model, deterministic model and stochastic model [13–15]. The methods are briefly discussed below.

2.1.1 Phenomenological Model

Phenomenological models are not directly related to fundamental theories. With available data, researchers apply regression or fitting techniques to directly formulate the trend of the epidemics. Therefore, different phenomenological models generally present completely different equations. Regarding modelling COVID-19, Hojman and Asenjo [16] estimated
variations of the infected and recovered populations of various regions based on tanh() functions. Meanwhile, Attanayake et al. [17] fitted the epidemiological curve of confirmed cases by exponential-based functions and found suitable parameter values by meta-analysis technique. The constructed models can be used to reflect and predict the tendency of epidemics when data is available. However, since the phenomenological models are not derived from basic principles of the epidemics, it is also difficult to apply the models to analyse and predict the circumstance when other interventions or policies are conducted by the government.

2.1.2 Deterministic Model

The deterministic model is generally constructed based on the compartmental modelling strategy. The overall population is generally separated into groups according to different epidemiological stages. Variations of compartment sizes are expressed by differential equations or difference equations. SIR model is the simplest compartmental model in epidemiology. Consisting of three compartments, ”susceptible, infected, recovered”, the model is expressed by three ordinary differential equations [18]:

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI - \nu I \\
\frac{dR}{dt} &= \nu I 
\end{align*}
\] (2.1)

where \(\beta\) is transmission rate while \(\nu\) is recovery rate. Note that the average recovery duration is just \(1/\nu\) days. In similar ways, researchers could design and include more compartments to describe more details. Meanwhile, consisting of differential equations, equilibrium points and stability analysis could be easily accomplished.

The model can be further modified to better simulate development of the pandemic in the real life. In the basic compartmental model, the total population is assumed to be constant and the overall population is assumed to be homogeneously mixing [15], which is generally not true in small regions. Consequently, this kind of model is generally used to analyse the epidemics in the large population. To improve its applicability, researchers
have recently put more network details into the compartmental model. A typical example is the SEIQR model created by Kang et al. [19]. The deterministic model includes scale-free network information and therefore can simulate the spread of virus in the real-world social network where the uniform-mixing hypothesis is not true.

2.1.3 Stochastic Model

The stochastic model is exhibited by stochastic processes to describe the epidemic dynamics. Classifications such as “S”, “I”, “R” are still used but they are random variables depending on infection and recovery probabilities. Thus, output of the model is no longer deterministic, but the probability distribution of each variable. Allen [20] reported and elaborated on the three typical methods of formulating the stochastic pandemic models: discrete-time Markov chain (DTMC), continuous-time Markov chain (CTMC), and stochastic differential equation (SDE). With different designated class, the stochastic compartmental models are popular in modelling the COVID-19. Various versions have been built based on data in different regions and of different epidemic evolution phases [21–25]. It is argued that this kind of model can involve and reflect both demographic and environmental variability under impact of the pandemic [26]. Therefore, compared with the deterministic version, the stochastic model is more feasible for the case with small populations.

2.2 Justification of Modelling Method

Among these discussed models, the compartmental model is the most suitable option for this project. To model this small-environment case, two major aspects need to be considered:

- Data availability
  Since the outbreak of COVID-19, most Universities have switched to online study mode and most staffs have the awareness of wearing masks under the intervention of governments. It is difficult to acquire suitable data which reflects the epidemic
evolution when the whole University operates normally. As a completely data-driven model, the phenomenological model cannot be applied in this case. Simultaneously, although the probabilistic models do not greatly depend on data, it generally has more parameters than the compartmental type. It is far more difficult to decide the parameter values when no data can be used.

- Model Complexity

Complexity of the model should also be considered. The phenomenological model is extremely simpler but it is not derived theoretically. The model cannot predict any effect of interventions and therefore does not allow us to discover appropriate measures the University can conduct to mitigate the epidemic. Meanwhile, the stochastic model is rather more complex than the compartmental model, which means it contains more parameters. However, when no accurate data can be used, choosing correct values for all parameters seems to be impossible. As the model is very complex, influence of having wrong values in model parameters could make a huge difference. In addition, Brauer [27] pointed out that the trade-off between complexity of the model and difficulty of analysis is an inevitable question. Although the stochastic model could capture more details in variability, it will be extremely difficult to analyse the model if more epidemic evolution stages and more control effectiveness are to be considered.

Therefore, considering both aspects, the compartmental model is a better tool in this project. Independent of data, the model can have fewer parameters to describe more complicated scenarios. Most parameter values can also be obtained from properties of the virus or previous researches on COVID-19. Furthermore, expressed by a system of differential equations, it is easier to find ways of analysing and controlling the model dynamics.

Most arguments assert that the compartmental model is not suitable for the small-population case because the homogeneity hypothesis does not apply. However, in this University, all students have similar daily routines in confined spaces and therefore viruses can easily spread to all individuals inside via droplets or aerosol. The same applies to
2.3 Summary of Previous Research

Since the outbreak of COVID-19, researchers have tried various compartmental modelling strategies to figure out the spread pattern of the disease. Cooper et al. [28] tried the simplest SIR model with various parameters to estimate and predict the spread of disease in large regions such as China, India, South Korea, Australia and the USA. Calafiore et al. [29] proposed a modified SIRD model to simulate the epidemics in Italy. However, these models do not describe the incubation period and asymptomatic cases. Therefore, various novel models with more compartments and therefore of higher orders were proposed. For example, Hu and Nie [30] involved effects of the environment and animals in their compartmental model. Giordano et al. [31] created a more complicated SIDARTHE model to demonstrate different illness stages in detail. However, the small-environment case is not the main scope of those paper. All these compartmental models focused on epidemics in the national level but they cannot provide any useful instructions in controlling the infections in Universities or companies.

Under this circumstance, Lopman et al. [32] constructed a compartmental model for COVID-19 in Emory University. They also considered incubation period, self-quarantine, and influence of testing. Nevertheless, the environment plays a dominant role in infection between students and staffs, which is not involved in their model. The model dynamics is also not analysed. Effectiveness of different measures is studied only by varying corresponding parameter values. This is not a systematic way and cannot present the optimal solution.

In addition, these previous researches analyse and suggest ways of controlling the epidemic merely based on numerical trials of different parameter values. However, these trials can only reflect a limited number of situations and cannot reveal the combined effect

staffs/faculties. Consequently, it is reasonable to have the assumption on uniform mixing if students and staffs are specifically classified into two compartments. Effect of the environment should be particularly considered.
of different measures. They do not have a systematic way of finding the optimal strategy.

Thus, this project aims to exhibit a novel compartmental model for spread of COVID-19 in the University. The effect of virus spread into the environment should also be involved. After deriving the system dynamics, extra control variables that are corresponding to effectiveness of different control and prevention measures should be defined. Then the optimal control technique can be applied to figure out the optimal solution. If the Universities or companies is to reopen after lockdown, findings of this project are essential for rule-making and potential risk analysis.
Chapter 3

University Model Design, Analysis, and Control

This chapter will describe structure and formulation of the model, analyse the system by some control methods and construct the optimal control problem that can study the effectiveness of different control measures. Having 11 compartments, the designed University compartmental model is comprised of one environment compartment and two SEIQR structures for students and staffs respectively. The basic reproduction number, equilibrium points, and stability are studied while values of the model parameters are assigned according to various researches on COVID-19. Finally, the control variables corresponding to effectiveness of different measures are defined. With a complete system, the optimal control problem is designed to derive optimal solutions of different strategies.

3.1 University Model Design

3.1.1 Overview

University is a special place where members are studying and working following specific regularity. To better model the pandemics in this small environment, people should be firstly split into different categories according to different tracks of daily life. For example, students of same year tend to attend lectures together, whereas those of different years have
less chance of contacting each other. Therefore, the overall population in the university is separated into four groups: year 1, year 2, year 3, postgraduate, and staffs. PhD students are involved in the staff group. Furthermore, considering the long incubation period of coronavirus as well as mandatory quarantine measures taken by Universities, a modified SEIQR (susceptible, exposed, infected, quarantined, recovered) model is constructed for each group. In addition, since COVID-19 could remain contagious in droplets or on the surface for more than 24 hours \cite{33, 34}, another variable \(C(t)\) is used to represent the concentration of virus in this confined environment.

Accordingly, different stages of COVID-19 are depicted in Figure 3.1.

\begin{figure} [h]
\centering
\includegraphics[width=\textwidth]{figure3.1.png}
\caption{Different stages of COVID-19 in the University}
\end{figure}

The overall model is expected to have five SEIQR structures and one compartment denoting the concentration of virus in the environment. Among the five groups, staffs are significant as they have direct contact with students of all levels. Lectures or seminars are the most common route. On the contrary, direct infection is far less likely to happen between students of different years. Universities generally do not exhibit activities that gather students of different groups and thus no direct path is included in this flow chart. Nevertheless, Since students of different years usually share the same lecture room, the virus can survive in the air or on the surface and then catch the susceptible students of
the other year. Thus, in confined environment, both staffs and students are vulnerable.

3.1.2 Model Simplification

It is feasible to construct this whole model with 26 compartments. However, considering the trade-off between model complexity and difficulty in analysis, it is necessary to simplify the whole model and reduce its order.

The main idea is to ignore the difference between different student groups and to all students into one SEIQR structure. Since there is no direct infections between students of different years, this cross-group infection can happen only through the environment. Living in the same confined space, all students have the same possibility of getting infected by the environment. Therefore, if we do not investigate the transmission between different years, seeing all students as a whole is reasonable. Then the model only remains two groups: student group and staff group.

3.1.3 Model Structure and Formulation

The resulting flow chart of the simplified model is shown in Figure 3.2 and 3.3.

![Figure 3.2: Epidemic Evolution stages of the Student Group](image)

As can be seen from the figures, a student or a staff can catch the disease from direct contact with infected students or staffs as well as from the contagious environment. Once infected, an individual will undergo an incubation period. Whether symptoms will develop after that depends on the carriers’ own body immunity. Strong immunity can keep
individuals asymptomatic until recovery, but they are still infectious in this period. Note that the infection rate from infected student/staff to the susceptible individuals are the same: $\beta_{sy} = \beta_{yy}$ and $\beta_{ys} = \beta_{ss}$.

According to this structure, the model could be formulated by a system of differential equations. Note that all SEIQR compartments are the number of corresponding subjects in the proportion of the total population.

\[
\begin{align*}
\frac{dS_y(t)}{dt} &= -\{\beta_{yy}[E_y(t) + kI_y(t) + E_s(t) + kI_s(t)] + \beta_{cy}C(t)\}S_y(t) \\
\frac{dS_s(t)}{dt} &= -\{\beta_{ss}[E_s(t) + kI_s(t) + E_y(t) + kI_y(t)] + \beta_{sa}C(t)\}S_s(t) \\
\frac{dE_y(t)}{dt} &= \{\beta_{yy}[E_y(t) + kI_y(t) + E_s(t) + kI_s(t)] + \beta_{cy}C(t)\}S_y(t) - (\epsilon_y + \xi_y)E_y(t) \\
\frac{dE_s(t)}{dt} &= \{\beta_{ss}[E_s(t) + kI_s(t) + E_y(t) + kI_y(t)] + \beta_{sa}C(t)\}S_s(t) - (\epsilon_s + \xi_s)E_s(t) \\
\frac{dI_y(t)}{dt} &= \epsilon_y E_y(t) - \eta_y I_y(t) \\
\frac{dI_s(t)}{dt} &= \epsilon_s E_s(t) - \eta_s I_s(t) \\
\frac{dQ_y(t)}{dt} &= \eta_y I_y(t) - \varphi_y Q_y(t) \\
\frac{dQ_s(t)}{dt} &= \eta_s I_s(t) - \varphi_s Q_s(t) \\
\frac{dR_y(t)}{dt} &= \xi_y E_y(t) + \varphi_y Q_y(t) \\
\frac{dR_s(t)}{dt} &= \xi_s E_s(t) + \varphi_s Q_s(t) \\
\frac{dC(t)}{dt} &= \mu_y(E_y(t) + kI_y(t)) + \mu_s(E_s(t) + kI_s(t)) - \delta C(t)
\end{align*}
\]

(3.1)
3.2 University Model Analysis

The biological meanings of the model parameters are elaborated in Table 3.1.

Table 3.1: Model Parameter Definitions

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Definitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{yy}, \beta_{ss}$</td>
<td>Infection rate for asymptomatic students/staffs to susceptible students/staffs</td>
</tr>
<tr>
<td>$k\beta_{yy}, k\beta_{ss}$</td>
<td>Infection rate for symptomatic students/staffs to susceptible students/staffs</td>
</tr>
<tr>
<td>$\beta_{cy}, \beta_{cs}$</td>
<td>Infection rate for contagious environment to susceptible students/staffs</td>
</tr>
<tr>
<td>$\varepsilon_{y}, \varepsilon_{s}$</td>
<td>Probability that an asymptomatic student/staff becomes symptomatic</td>
</tr>
<tr>
<td>$\xi_{y}, \xi_{s}$</td>
<td>Probability that an asymptomatic student/staff directly recovers</td>
</tr>
<tr>
<td>$\eta_{y}, \eta_{s}$</td>
<td>Isolation rate of symptomatic student/staffs</td>
</tr>
<tr>
<td>$\varphi_{y}, \varphi_{s}$</td>
<td>Recovery rate of quarantined students/staffs</td>
</tr>
<tr>
<td>$\mu_{y}, \mu_{s}$</td>
<td>Shedding rate by asymptomatic students/staffs</td>
</tr>
<tr>
<td>$k\mu_{y}, k\mu_{s}$</td>
<td>Shedding rate by symptomatic students/staffs</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Decaying rate of virus in the campus environment</td>
</tr>
</tbody>
</table>

3.2 University Model Analysis

3.2.1 Equilibrium Point

Denote $x = [S_y, S_s, E_y, E_s, I_y, I_s, Q_y, Q_s, R_y, R_s, R_{\text{c}}]^{\top}$. By equating all derivatives to zero, it is easy to spot the equilibrium $\bar{x} = [\bar{S}_y, \bar{S}_s, 0, 0, 0, 0, 0, \bar{R}_y, \bar{R}_s, 0]$, where

$$
\bar{S}_y + \bar{S}_s + \bar{R}_y + \bar{R}_s = 1, \bar{S}_y \geq 0, \bar{S}_s \geq 0, \bar{R}_y \geq 0, \bar{R}_s \geq 0
$$

(3.2)

The equilibrium means that at the end of pandemic, the remaining populations are either susceptible or recovered from the disease.

3.2.2 Basic Reproduction Number

Basic reproduction number is a crucial factor measuring the average number of susceptible people that could potentially be infected by a primary case [35]. This parameter is highly dependent on the fraction of the susceptible and it can infer the potential of breakout of
disease. If \( R_0 < 1 \), the infected cases can not generate significant impact on all susceptible individuals and the disease will gradually disappear. However, if \( R_0 > 1 \), increasingly more people will be infected and breakout of the disease is likely to occur.

Derivation of the basic reproduction number is based on the next generation matrix method described by [36–38]. The University model shown by Equation 3.1 has five compartments that can infect susceptible individuals: \( E_y, E_s, I_y, I_s \) and environment \( C \). Let \( F \) denote the rate of increase of secondary cases and \( V \) denote the progression rate. Accordingly, with \( x_{if} = [E_y, E_s, I_y, I_s, C]^\top \):

\[
\dot{x}_{if} = F - V
\]  

(3.3)

\( F \) and \( V \) are formulated as:

\[
F = \begin{bmatrix}
S_y (C \beta_{cy} + \beta_{yy} (E_s + E_y + I_s k + I_y k)) \\
S_s (C \beta_{cs} + \beta_{ss} (E_s + E_y + I_s k + I_y k)) \\
0 \\
0 \\
\mu_s (E_s + I_s k) + \mu_y (E_y + I_y k)
\end{bmatrix}, \quad V = \begin{bmatrix}
E_y (\varepsilon_y + \xi_y) \\
E_s (\varepsilon_s + \xi_s) \\
I_y \eta_y - E_y \varepsilon_y \\
I_s \eta_s - E_s \varepsilon_s \\
C \delta
\end{bmatrix}
\]  

(3.4)

Conducting linearization by taking Jacobian of \( F \) and \( V \) at the equilibrium, matrices \( F \) and \( V \) are derived such that:

\[
\dot{x}_{if} = (F - V)x_{if}
\]  

(3.5)

where

\[
F = \begin{bmatrix}
\bar{S}_y \beta_{yy} & \bar{S}_y \beta_{yy} & \bar{S}_y \beta_{yy} k & \bar{S}_y \beta_{yy} k & \bar{S}_y \beta_{cy} \\
\bar{S}_s \beta_{yy} p & \bar{S}_s \beta_{yy} p & \bar{S}_s \beta_{yy} k p & \bar{S}_s \beta_{yy} k p & \bar{S}_s \beta_{cy} p \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mu_y & \mu_s & k \mu_y & k \mu_s & 0
\end{bmatrix}
\]  

(3.6)
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and

\[ V = \begin{bmatrix}
\varepsilon_y + \xi_y & 0 & 0 & 0 \\
0 & \varepsilon_s + \xi_s & 0 & 0 \\
-\varepsilon_y & 0 & \eta_y & 0 \\
0 & -\varepsilon_s & 0 & \eta_s \\
0 & 0 & 0 & 0 & \delta
\end{bmatrix} \quad (3.7) \]

The next generation matrix K is therefore derived as:

\[
K = FV^{-1} = \begin{bmatrix}
\bar{S}_y \beta_{yy} (\eta_y + \varepsilon_y k) & \bar{S}_y \beta_{yy} (\eta_s + \varepsilon_s k) & \bar{S}_y \beta_{yy} k & \bar{S}_y \beta_{yy} k & \bar{S}_y \beta_{cy} \\
\eta_y (\varepsilon_y + \xi_y) & \eta_s (\varepsilon_y + \xi_y) + \bar{S}_s \beta_{yy} p (\eta_s + \varepsilon_s k) & \eta_s (\varepsilon_y + \xi_y) & \eta_s (\varepsilon_y + \xi_y) & \eta_s (\varepsilon_y + \xi_y) \\
0 & 0 & 0 & 0 & 0 \\
\mu_y (\eta_y + \varepsilon_y k) & \mu_s (\eta_s + \varepsilon_s k) & k \mu_y & k \mu_s & 0 \\
\eta_y (\varepsilon_y + \xi_y) & \eta_s (\varepsilon_y + \xi_y) & \eta_y (\varepsilon_y + \xi_y) & \eta_s (\varepsilon_y + \xi_y) & \eta_s (\varepsilon_y + \xi_y)
\end{bmatrix} \quad (3.8)
\]

Since the second row is \( \frac{\bar{S}_y}{S_y} \) times larger than the first row, the obtained matrix \( K \) is nonnegative and has rank of 2. Therefore, it has three zero eigenvalues and two positive eigenvalues. According to [36], \( R_0 \) is the spectral radius of \( K \), which means the largest eigenvalue of the next generation matrix. By computing \( \det(\lambda I - K) \) with elementary transformation techniques, the characteristic polynomial is derived as:

\[
p_k(\lambda) = \lambda^3 \left[ \lambda^2 - \left( \frac{\bar{S}_y \beta_{yy} (\eta_y + \varepsilon_y k)}{\eta_y (\varepsilon_y + \xi_y)} + \frac{\bar{S}_s \beta_{yy} p (\eta_s + \varepsilon_s k)}{\eta_s (\varepsilon_y + \xi_y)} \right) \lambda - \frac{1}{\delta} \left( \frac{\bar{S}_y \beta_{cy} \mu_y (\eta_y + \varepsilon_y k)}{\eta_y (\varepsilon_y + \xi_y)} + \frac{\bar{S}_s \beta_{cy} p \mu_s (\eta_s + \varepsilon_s k)}{\eta_s (\varepsilon_y + \xi_y)} \right) \right] \quad (3.9)
\]

Here we assume that \( \mu_y = \mu_s = \mu \), which means students and staffs have equal rates of spreading the virus into the environment. This assumption is reasonable. Omitting three zero eigenvalues, the remaining eigenvalues can be derived by solving the polynomial:

\[
p_{k2}(\lambda) = \lambda^2 - \beta_{yy} \left( \frac{\bar{S}_y (\eta_y + \varepsilon_y k)}{\eta_y (\varepsilon_y + \xi_y)} + \frac{\bar{S}_s p (\eta_s + \varepsilon_s k)}{\eta_s (\varepsilon_y + \xi_y)} \right) \lambda - \frac{\mu \beta_{cy}}{\delta} \left( \frac{\bar{S}_y (\eta_y + \varepsilon_y k)}{\eta_y (\varepsilon_y + \xi_y)} + \frac{\bar{S}_s p (\eta_s + \varepsilon_s k)}{\eta_s (\varepsilon_y + \xi_y)} \right) \quad (3.10)
\]
Denote:
\[
a = \beta_{yy} \left( \bar{S}_y \left( \eta_y + \varepsilon_y \right) + \bar{S}_s p \left( \eta_s + \varepsilon_s \right) \right), \quad c = \frac{\mu \beta_{cy}}{\delta \beta_{yy}}
\] (3.11)

Then:
\[
p_2(\lambda) = \lambda^2 - a\lambda - ac
\] (3.12)

Since constants \(a\) and \(c\) are both positive, the quadratic equation has two real roots and \(R_0\) should be the larger one, which is:
\[
R_0 = \frac{a(1 + \sqrt{1 + 4c/a})}{2} \approx \frac{a}{2} + \sqrt{ac}
\] (3.13)

If \(R_0 < 1\):
\[
\frac{a}{2} - 1 < -\sqrt{ac}, \quad a < 2
\] (3.14)

Note that \(\sqrt{ac} > 0\) because \(a > 0\) and \(c > 0\). Therefore, this inequality is true if and only if:
\[
\frac{a}{2} - 1 < 0
\] (3.15)

and
\[
\left(\frac{a}{2} - 1\right)^2 > ac
\] (3.16)

Then, re-arranging Equation 3.16, we get:
\[
\frac{1}{4} a^2 - (c + 1)a + 1 > 0
\] (3.17)

which leads to the result:
\[
a < f(c) = 2 \left[ c + 1 - \sqrt{c^2 + 2c} \right]
\] (3.18)

At \(c > 0\), function \(f(c)\) is monotonically decreasing. In general, \(c\) is larger than 1, which indicates that \(a < 0.5\). By this result, denoting \(r_1 = \eta_y + \varepsilon_y\) and \(r_2 = \eta_s + \varepsilon_s\), one can get
3.2 University Model Analysis

the relationship as:

\[
\bar{S}_y \beta_{yy} \eta_s r_2 (\eta_y + \varepsilon_y k) + \bar{S}_s \beta_{yy} \eta_y r_1 (\eta_y + \varepsilon_s k) < 0.5 \eta_y \eta_s r_1 r_2
\]  

This inequality is particularly useful when studying the stability.

3.2.3 Stability Analysis

The overall model can be reformulated into a feedback system. Compartments \(E_y, E_s, I_y, I_s, Q_y, Q_s, C\) can form a positive linear subsystem with output feedback topology. Defining \(x_l = [E_y, E_s, I_y, I_s, Q_y, Q_s, C]^\top\), \(y_s = [S_y, S_s, C]^\top\), \(y_R = [R_y, R_s]^\top\), the subsystem can be formulated as:

\[
\dot{x}_l(t) = Ax_l(t) + Bu(t)
\]  

\[
y_s(t) = C_s x_l(t) = \begin{bmatrix}
\beta_{yy} & \beta_{yy} & \beta_{yy} & \beta_{yy} & k & \beta_{yy} & k & 0 & 0 & \beta_{cy} \\
\beta_{ss} & \beta_{ss} & \beta_{ss} & \beta_{ss} & k & \beta_{ss} & k & 0 & 0 & \beta_{cs} \\
\mu_y & \mu_y & k & \mu_y & k & \mu_s & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
x_l(t)
\end{bmatrix}
\]  

\[
y_R(t) = C_R x_l(t) = \begin{bmatrix}
\xi_y & 0 & 0 & 0 & \varphi_y & 0 & 0 \\
0 & \xi_s & 0 & 0 & \varphi_s & 0
\end{bmatrix} \begin{bmatrix}
x_l(t)
\end{bmatrix}
\]
where \( r_1 = \varepsilon_y + \eta_y, \ r_2 = \varepsilon_s + \eta_s \). The output feedback part is:

\[
u(t) = K_s(t) y_s(t) = \begin{bmatrix} S_y(t) & 0 & 0 \\ 0 & S_s(t) & 0 \\ 0 & 0 & C(t) \end{bmatrix} y_s(t) \tag{3.23}\]

Derivatives of the other compartments can be calculated as:

\[
\begin{bmatrix} \dot{S}_y(t) \\ \dot{S}_s(t) \end{bmatrix} = - \begin{bmatrix} S_y(t) & 0 & 0 \\ 0 & S_s(t) & 0 \end{bmatrix} y_s(t) \tag{3.24}
\]

\[
\begin{bmatrix} \dot{R}_y(t) \\ \dot{R}_s(t) \end{bmatrix} = y_R(t) \tag{3.25}
\]

Note that the system has a time-varying feedback \( K_s(t) \). Since the equilibrium analysis indicates that the system could finally reach \( \bar{x} \), the system behaviour is figured out by using the constant feedback term \( \bar{K}_s \) as:

\[
\bar{K}_s = \begin{bmatrix} \bar{S}_y & 0 & 0 \\ 0 & \bar{S}_s & 0 \\ 0 & 0 & 0 \end{bmatrix} y_s(t) \tag{3.26}
\]

Therefore, the closed-loop system matrix \( A_{cl} \) can be derived as:

\[
A_{cl} = A + BK_s C_s = \begin{bmatrix}
S_y \beta_{yy} - \varepsilon_y & S_y \beta_{yy} & S_y \beta_{yy} k & S_y \beta_{yy} k & 0 & 0 & S_y \beta_{cy} \\
S_s \beta_{yy} p & S_s \beta_{yy} p - \varepsilon_y & S_s \beta_{yy} k p & S_s \beta_{yy} k p & 0 & 0 & S_s \beta_{cy} p \\
\varepsilon_y & 0 & -\eta_y & 0 & 0 & 0 & 0 \\
0 & \varepsilon_s & 0 & -\eta_s & 0 & 0 & 0 \\
\xi_y & 0 & \eta_y & 0 & -\varphi_y & 0 & 0 \\
0 & \xi_s & 0 & \eta_s & 0 & -\varphi_s & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\delta
\end{bmatrix} \tag{3.27}
\]
To determine its closed-loop poles, the characteristic equation is computed by \( \text{det}(\lambda I - A_{cl}) \). By elementary transformation and further simplifications, the polynomial becomes:

\[
p_{cl}(\lambda) = (\lambda + \varphi_y)(\lambda + \varphi_s)(\lambda + \delta)
\]

\[
\begin{vmatrix}
\lambda - (\bar{S}_y \beta_{yy} - r_1) & -\bar{S}_y \beta_{yy} & 0 & 0 \\
-\bar{p}(\lambda + r_1) & \lambda + r_2 & -k(\lambda + r_2) & 0 \\
-\varepsilon_y & 0 & \lambda + \eta_y & -(\lambda + \eta_y) \\
0 & -\varepsilon_s & k\varepsilon_s & \lambda + \eta_s \\
\end{vmatrix}
\]

\[
= (\lambda + \varphi_y)(\lambda + \varphi_s)(\lambda + \delta)p_4(\lambda)
\]  

(3.28)

where \( r_1 = \varepsilon_1 + \xi_1, r_2 = \varepsilon_2 + \xi_2, \) and \( \bar{p} = \frac{\bar{S}_p}{\bar{S}_y} \). Since all parameters are positive, \( A_{cl} \) must have three negative eigenvalues at \(-\varphi_y, -\varphi_s, -\delta\). The remaining four eigenvalues are roots of the polynomial \( p_4(\lambda) \) where

\[
p_4(\lambda) = \lambda^4 + \lambda^3 (\eta_s + \eta_y + r_1 + r_2) + \lambda^2 (\eta_s \eta_y + (r_1 + r_2)(\eta_s + \eta_y) + r_1 r_2 - \bar{S}_y \beta_{yy} \varepsilon_y k + \bar{S}_s \beta_{yy} \varepsilon_s p k \lambda^2 + ((r_1 + r_2) \eta_s \eta_y + r_1 r_2 (\eta_s + \eta_y) - \bar{S}_s \beta_{yy} \varepsilon_y \eta_s k + \bar{S}_y \beta_{yy} \varepsilon_s \eta_y k p - \bar{S}_s \beta_{yy} \varepsilon_s \eta_y \bar{S}_s \beta_{yy} \varepsilon_s \eta_y k p r_1) \lambda + \eta_s \eta_y r_1 r_2 - \bar{S}_y \beta_{yy} \varepsilon_s \eta_s k p r_1
\]  

(3.29)
Writing the polynomial as:

\[ p_4(\lambda) = \lambda^4 + \sigma_3 \lambda^3 + \sigma_2 \lambda^2 + \sigma_1 \lambda + \sigma_0 \]  
(3.30)

where

\[
\sigma_3 = \eta_s + \eta_y + r_1 + r_2 \\
\sigma_2 = \eta_s \eta_y + (r_1 + r_2)(\eta_s + \eta_y) + r_1 r_2 - \bar{S}_y \beta_{yy} \varepsilon_y k - \bar{S}_s \beta_{yy} \varepsilon_s k p \\
\sigma_1 = (r_1 + r_2) \eta_s \eta_y + r_1 r_2 (\eta_s + \eta_y) - \bar{S}_y \beta_{yy} \varepsilon_y \eta_s k - \bar{S}_y \beta_{yy} \varepsilon_y k r_2 - \bar{S}_s \beta_{yy} \varepsilon_s \eta_y k p \\
- \bar{S}_s \beta_{yy} \varepsilon_s k p r_1 \\
\sigma_0 = \eta_s \eta_y r_1 r_2 - \bar{S}_y \beta_{yy} \varepsilon_y \eta_s k r_2 - \bar{S}_s \beta_{yy} \varepsilon_s \eta_y k p r_1 \\
\tag{3.31}
\]

Stability of the equilibrium can be proved by applying Routh-Hurwitz stability criteria:

\begin{table}[h]
\centering
\caption{Routh Table}
\begin{tabular}{|c|c|c|c|}
\hline
\lambda^4 & 1 & \sigma_2 & \sigma_0 \\
\hline
\lambda^3 & \sigma_3 & \sigma_1 & 0 \\
\hline
\lambda^2 & b_3 = -\frac{1}{\sigma_3} (\sigma_1 - \sigma_2 \sigma_3) & \sigma_0 & 0 \\
\hline
\lambda^1 & b_2 = -\frac{1}{b_3} (\sigma_3 \sigma_0 - \sigma_1 b_3) & 0 & 0 \\
\hline
\lambda^0 & b_1 = \sigma_0 & 0 & 0 \\
\hline
\end{tabular}
\end{table}

Apparently, \( \sigma_3 > 0 \). Meanwhile, by Equation 3.19:

\[
\eta_s \eta_y r_1 r_2 > 2\bar{S}_y \beta_{yy} \eta_s r_2 (\eta_y + \varepsilon_y k) + 2\bar{S}_s p \beta_{yy} \eta_y r_1 (\eta_s + \varepsilon_s k) \\
> \bar{S}_y \beta_{yy} \varepsilon_y \eta_s k r_2 + \bar{S}_s \beta_{yy} \varepsilon_s \eta_y k p r_1 \\
\tag{3.32}
\]

Therefore, \( \sigma_0 > 0 \). Furthermore, expressions of \( b_3 \) and \( b_2 \) are derived by MATLAB. Although they are complicated, it is still possible to prove \( b_3 > 0 \), \( b_2 > 0 \) in the same way with Equation 3.19. Thus, according to the Routh-Hurwitz criteria, the equilibrium \( \bar{x} \) is asymptotically stable if \( R_0 < 1 \).
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3.2.4 Determination of Model Parameter Values

Deeper meanings and estimated values of different parameters are discussed in the following part. Parameter values are chosen to model the circumstance where no protection and restriction measurements are conducted.

- Individual infection rates $\beta_{yy}$, $\beta_{ss}$, $k\beta_{yy}$, $k\beta_{ss}$

The individual infection rates represent the average number of susceptible people who can be infected by a virus carrier via direct contacts in unit time. $\beta$ in this model expresses the infection rate of asymptomatic compartments while $k\beta$ describes the rate of symptomatic group. Since sneezing and coughing play a major role in virus direct transmission, the symptomatic infected subjects generally have larger infection rates than asymptomatic carriers. Therefore $k$ is the factor describing this increment. Here it is assumed that this ratio has the same value among students and staffs. These infection rates can be reduced by keeping social distance and wearing masks. According to the study conducted by Alexandros et al. [39], the general infection rate $\beta$ is 0.1466. However, this value should become larger in confined space. Meanwhile, considering the age difference, it is reasonable to see that $\beta_{yy} < \beta_{ss}$ because staffs are more likely to get infected. Therefore, in this model we set $k\beta_{yy} = 0.27$ and $k\beta_{ss} = 0.35$. Then $p = 0.35/0.27 = 1.2963$. Meanwhile, referring to the scenario 3 of pandemic planning concluded by CDC [40], $k$ is generally 4. However, in this confined environment case, $k$ should be smaller because asymptomatic subjects can spread the virus more easily. Thus, $k = 1.5$, rendering $\beta_{yy} = 0.27/1.5 = 0.18$ and $\beta_{ss} = 0.35/4 = 0.2333$.

- Environmental infection rates $\beta_{cy}$, $\beta_{cs}$

These parameters show how many susceptible people are infected by the contaminated environment in unit time. They are properties of the virus in the environment and cannot be reduced by any control methods. Since little literature has investigated their values, $\beta_{cy}$ is chosen to be 0.03 by numerical testing. Moreover, it is reasonable to see that ratio $\beta_{ss}/\beta_{ss}$ equals the ratio $\beta_{cy}/\beta_{cs}$. Then $\beta_{cs} = p\beta_{cy} = 0.0492$. 

• Probability of becoming symptomatic $\varepsilon_y, \varepsilon_s$

They are the inverse of the average incubation period. According to [41], this average period among all cases is 5 days. Considering that students are young adults while staffs are at higher age, $\varepsilon_y = 1/4 = 0.25$ and $\varepsilon_s = 1/10 = 0.1$.

• Probability of recovery from asymptomatic state $\xi_y, \xi_s$

Similarly, inverses of them denote the average number of days spent by exposed/asymptomatic subjects to directly recover. There exists a small proportion of people who have strong body immunity. We combine this asymptomatic group with individuals who are in the incubation period. Referring to [40], this portion accounts for 15%. Since $\varepsilon_y = 0.25$ and $\varepsilon_s = 0.1$, which occupy 85%, $\xi_y = 0.0441$ and $\xi_s = 0.0221$.

• Isolation rates $\eta_y, \eta_s$

These parameters denote the proportion of symptomatic individuals who are isolated due to serious illness or mandatory quarantine. Since initially we model the unrestricted situation, infected individuals are isolated mainly due to severe symptoms. According to the survey conducted and reported by [32], $\eta_y = 0.06$ and $\eta_s = 0.106$.

• Recovery rates $\varphi_y, \varphi_s$

$\varphi^{-1}_y$ and $\varphi^{-1}_s$ indicate the average time for infected people in quarantine to recover. A recent study conducted in India [42] reported that the shortest recovery time is around 5 days while the longest lasts 68 days. In our case, with better body immunity systems, students generally can recover faster. On the contrary, staffs/faculties may need a longer period for recovery. Thus, $\varphi_y = 1/10$ and $\varphi_s = 1/30$.

• Virus shedding rates to environment $\mu_y, \mu_s, k\mu_y, k\mu_s$

They measure spread of the virus from asymptomatic/symptomatic individuals to the environment. Similar to infection rates, the effects brought by symptomatic subjects are higher. However, their values are not studied by any reliable literature. In this confined-space model, we set $\mu_y = \mu_s = \mu = 0.3$.

• Virus decaying rate in environment
It measures speed of decay of the virus in the confined space. Since the airborne virus could stay in aerosol for up to 1 day and survive on the surface for longer, this rate \( \delta \) is set at 0.7.

### 3.3 Control of the University Model

#### 3.3.1 Possible Control Measures

Measures that Universities can conduct to restrain the pandemic include three intervention actions as well as the mandatory quarantine on both students and staffs. The variable \( \kappa \) is defined to represent the effectiveness of each measure. Ranging between 0 and 1, it indicates strength of each measure. Details are described below:

1. **Wear Masks**
   
   Asking individuals to wear a mask can greatly mitigate all infection rates \( \beta_{yy}, \beta_{ss}, \beta_{cy}, \beta_{cs} \) in this case. Furthermore, this measure also reduces the influence of infected subjects on the environment: shedding rates \( \mu_y \) and \( \mu_s \). Its effectiveness is expressed by a factor \( \kappa_m \).

2. **Keep Safe Social Distance**

   University could also conduct a social distancing policy. Students in the lecture room, library or lab should keep a safe distance from each other. This measure could help reduce the person-to-person infection, represented by \( \beta_{yy}, \beta_{ss} \). In the model, the influence of this measure can be reflected by the coefficient \( \kappa_d \).

3. **Environment Disinfection**

   Another effective action to suppress the spread of virus is to sterilize the air and surfaces. For example, the University could ask staffs to disinfect the whole lecture room after each class. The rate of disinfection is denoted by a factor \( \kappa_e \).

4. **Mandatory Quarantine**

   To prevent further spread of the virus, it is significant to isolate the infected subjects. Since contact tracing and identification of asymptomatic individuals are not easy, it
is recommended for the University to restrict track of symptomatic students/staffs. 

\( \kappa_{qy} \) and \( \kappa_{qs} \) are used to describe the isolation rates of students and staffs respectively.

### 3.3.2 Control System Formulation

A group of five variables is initially defined to explicitly represent the reductions on infection rates, shedding rates and the enhancement of quarantine rates. Denoting control variables \( u = [u_r, u_m, u_e, u_{qy}, u_{qs}]^T \), the control system at this stage is expressed as:

\[
\begin{align*}
\frac{dS_y(t)}{dt} &= -\{(1 - u_p) \beta_{yy} [E_y(t) + kI_y(t) + E_s(t) + kI_s(t)] - (1 - u_m) \beta_{cy} C(t)\} S_y(t) \\
\frac{dS_s(t)}{dt} &= -\{(1 - u_p) p \beta_{yy} [E_s(t) + kI_s(t) + E_y(t) + kI_y(t)] - (1 - u_m) p \beta_{cy} C(t)\} S_s(t) \\
\frac{dE_y(t)}{dt} &= \{(1 - u_p) \beta_{yy} [E_y(t) + kI_y(t) + E_s(t) + kI_s(t)] + (1 - u_m) \beta_{cy} C(t)\} S_y(t) \\
&\quad - (\varepsilon_y + \xi_y) E_y(t) \\
\frac{dE_s(t)}{dt} &= \{(1 - u_p) p \beta_{yy} [E_s(t) + kI_s(t) + E_y(t) + kI_y(t)] + (1 - u_m) p \beta_{cy} C(t)\} S_s(t) \\
&\quad - (\varepsilon_s + \xi_s) E_s(t) \\
\frac{dI_y(t)}{dt} &= \varepsilon_y E_y(t) - u_{qy} \eta_y I_y(t) \\
\frac{dI_s(t)}{dt} &= \varepsilon_s E_s(t) - u_{qs} \eta_s I_s(t) \\
\frac{dQ_y(t)}{dt} &= u_{qy} \eta_y I_y(t) - \varphi_y Q_y(t) \\
\frac{dQ_s(t)}{dt} &= u_{qs} \eta_s I_s(t) - \varphi_s Q_s(t) \\
\frac{dR_y(t)}{dt} &= \xi_y E_y(t) + \varphi_y Q_y(t) \\
\frac{dR_s(t)}{dt} &= \xi_s E_s(t) + \varphi_s Q_s(t) \\
\frac{dC(t)}{dt} &= (1 - u_m) \mu (E_y(t) + kI_y(t) + E_s(t) + kI_s(t)) - u_e \delta C(t)
\end{align*}
\]

(3.33)

Note that \( u_r, u_m, u_e, u_{qy} \eta_y, \) and \( u_{qs} \eta_s \) should be in the range \([0, 1]\).

Since one reduction factor can be influenced by multiple measures, identifying the relationship between control variables \( u \) and measure effectiveness factors \( \kappa \) is the next step. How control variables can be affected by each measure should be analysed.

1. Reduction of interpersonal infection rates \( \beta_{yy} \) and \( \beta_{ss} \)
The person-to-person infection rates are directly influenced by two measures: wearing masks and keeping social distance. Wearing masks could reduce the probability of infection in each contact and social distancing could greatly reduce the number of contacts between individuals. Study conducted by Karaivanov et al. [43] argued that the mandatory mask-wearing policy in confined spaces could reduce the number of infected cases up to 40% weekly. Furthermore, Jarvis et al. [44] conducted a survey which proved that the physical distancing policy could reduce the number of direct contact by 74% every day. However, since this survey might have selection and recall bias, the actual result should be lower than 74%. In this case, we set the maximum reduction at 65%. Hence, 

\[ 1 - u_p = (1 - 0.4 \kappa_m)(1 - 0.65 \kappa_m) \]

2. Reduction of shedding and environment-to-person infection rates: \( \mu_y, \mu_s, \beta_{cy}, \beta_{cs} \)
Shedding rates as well as infections due to environment can be reduced by application of masks. According to laboratory-based investigations implemented by [45,46], masks could block approximately 50% to 70% droplets and aerosol, greatly alleviating airborne transmission of virus. In this case, we set the maximum reduction of shedding rate to be 60%. Therefore, 

\[ 1 - u_m = 1 - 0.6 \kappa_m. \]

3. Enhancement of virus environmental decaying rate \( \delta \)
The environmental decaying rate could be magnified by disinfection of the confined space. However, it is hard to derive the maximum cleaning rate. We set this limit at 70%. Therefore, when \( \kappa_e = 0, \delta = 0.7 \). When \( \kappa_e = 1 \), the 70% disinfection rate is exerted to the remaining 30% virus in the environment. Only 9% viruses remain alive at the end of each day. 

\[ u_e \delta = 1 - 0.09 = 0.91 \] in this situation. The linear relationship could be finally defined as: 

\[ u_e = 1 + 0.3 \kappa_e. \]

4. Resulting mandatory quarantine rates
Referring to the previous definition of \( \kappa_{qy} \) and \( \kappa_{qs} \), it is easy to see that 

\[ u_{qy} \eta_y = \kappa_{qy} \]

and 

\[ u_{qs} \eta_s = \kappa_{qs}. \]

\( \kappa_{qy}^{-1} \) and \( \kappa_{qs}^{-1} \) also indicate the average time that an un-isolated symptomatic subject is not quarantined. The minimum period is two days, inferring that it minimally costs the University two days in average to detect and isolate the infected individuals after their symptom develops. Thus, the maximum values of \( \kappa_{qy} \)
and \( \kappa_{qs} \) are both 0.5, while their minimum values are \( \eta_y \) and \( \eta_s \) respectively.

Accordingly, the relationship can be concluded as:

\[
1 - u_p = (1 - 0.4\kappa_m)(1 - 0.65\kappa_d)
\]
\[
1 - u_m = 1 - 0.6\kappa_m
\]
\[
u_c = 1 + 0.3\kappa_e \tag{3.34}
\]
\[
u_{qy}\eta_y = \kappa_{qy}
\]
\[
u_{qs}\eta_s = \kappa_{qs}
\]

Then we denote the new control variables \( \kappa = [\kappa_m, \kappa_d, \kappa_e, \kappa_{qy}, \kappa_{qs}]^\top \). Substituting this relationship into Equation 3.35, the overall control system is completed as:

\[
\begin{align*}
\frac{dS_y(t)}{dt} &= -\{(1 - 0.4\kappa_m)(1 - 0.65\kappa_d)\beta_{yy}[E_y(t) + kI_y(t) + E_s(t) + kI_s(t)] \\
&\quad - (1 - 0.6\kappa_m)\beta_{cy}C(t)\}S_y(t) \\
\frac{dS_s(t)}{dt} &= -\{(1 - 0.4\kappa_m)(1 - 0.65\kappa_d)p\beta_{yy}[E_s(t) + kI_s(t) + E_y(t) + kI_y(t)] \\
&\quad - (1 - 0.6\kappa_m)p\beta_{cy}C(t)\}S_s(t) \\
\frac{dE_y(t)}{dt} &= \{(1 - 0.4\kappa_m)(1 - 0.65\kappa_d)\beta_{yy}[E_y(t) + kI_y(t) + E_s(t) + kI_s(t)] \\
&\quad + (1 - 0.6\kappa_m)\beta_{cy}C(t)\}S_y(t) - (\varepsilon_y + \xi_y)E_y(t) \\
\frac{dE_s(t)}{dt} &= \{(1 - 0.4\kappa_m)(1 - 0.65\kappa_d)p\beta_{yy}[E_s(t) + kI_s(t) + E_y(t) + kI_y(t)] \\
&\quad + (1 - 0.6\kappa_m)p\beta_{cy}C(t)\}S_s(t) - (\varepsilon_s + \xi_s)E_s(t) \tag{3.35}
\end{align*}
\]

\[
\begin{align*}
\frac{dI_y(t)}{dt} &= \varepsilon_yE_y(t) - \kappa_{qy}I_y(t) \\
\frac{dI_s(t)}{dt} &= \varepsilon_sE_s(t) - \kappa_{qs}I_s(t) \\
\frac{dQ_y(t)}{dt} &= \kappa_{qy}I_y(t) - \varphi_yQ_y(t) \\
\frac{dQ_s(t)}{dt} &= \kappa_{qs}I_s(t) - \varphi_sQ_s(t) \\
\frac{dR_y(t)}{dt} &= \xi_yE_y(t) + \varphi_yQ_y(t) \\
\frac{dR_s(t)}{dt} &= \xi_sE_s(t) + \varphi_sQ_s(t) \\
\frac{dC(t)}{dt} &= (1 - 0.6\kappa_m)\mu(E_y(t) + kI_y(t) + E_s(t) + kI_s(t)) - (1 + 0.3\kappa_e)\delta C(t)
\end{align*}
\]
Although the relationship is completely derived from the survey conducted by various research, the consideration is still ideal. In the University, we still cannot guarantee 100% control effects, which means the maximum values of control variables cannot be one. In this model, we set the upper bounds of the three intervention effectivenesses, $\kappa_m$, $\kappa_d$, and $\kappa_e$, to be 0.7. The maximum of resulting quarantine rates is still 0.5. Instead of setting the limit by defining constraints, the above control variables are directly normalized in the system dynamics. Then they can all keep the same scale within the range 0 to 1. Thus, substitute $\kappa = [\kappa_m, \kappa_d, \kappa_e, \kappa_qy, \kappa_qs]^{\top}$ by $\kappa = [0.7\kappa_m, 0.7\kappa_d, 0.7\kappa_e, 0.5\kappa_qy, 0.5\kappa_qs]^{\top}$, the previous system dynamics finally becomes:

$$
\begin{align*}
\frac{dS_y(t)}{dt} &= -\{(1 - 0.28\kappa_m)(1 - 0.455\kappa_d)\beta_{yy}[E_y(t) + kI_y(t) + E_s(t) + kI_s(t)]
- (1 - 0.42\kappa_m)\beta_{cy}C(t)\}\}
\frac{dS_s(t)}{dt} &= -\{(1 - 0.28\kappa_m)(1 - 0.455\kappa_d) p \beta_{yy} [E_s(t) + kI_s(t) + E_y(t) + kI_y(t)]
- (1 - 0.42\kappa_m) p \beta_{cy}C(t)\}\}
\frac{dE_y(t)}{dt} &= \{(1 - 0.28\kappa_m)(1 - 0.455\kappa_d)\beta_{yy}[E_y(t) + kI_y(t) + E_s(t) + kI_s(t)]
+ (1 - 0.42\kappa_m)\beta_{cy}C(t)\}S_y(t) - (\varepsilon_y + \xi_y)E_y(t)
\frac{dE_s(t)}{dt} &= \{(1 - 0.28\kappa_m)(1 - 0.455\kappa_d) p \beta_{yy} [E_s(t) + kI_s(t) + E_y(t) + kI_y(t)]
+ (1 - 0.42\kappa_m) p \beta_{cy}C(t)\}S_s(t) - (\varepsilon_s + \xi_s)E_s(t)
\frac{dI_y(t)}{dt} &= \varepsilon_y E_y(t) - 0.5\kappa_qyI_y(t)
\frac{dI_s(t)}{dt} &= \varepsilon_s E_s(t) - 0.5\kappa_qsI_s(t)
\frac{dQ_y(t)}{dt} &= 0.5\kappa_qyI_y(t) - \varphi_yQ_y(t)
\frac{dQ_s(t)}{dt} &= 0.5\kappa_qsI_s(t) - \varphi_sQ_s(t)
\frac{dR_y(t)}{dt} &= \xi_y E_y(t) + \varphi_yQ_y(t)
\frac{dR_s(t)}{dt} &= \xi_s E_s(t) + \varphi_sQ_s(t)
\frac{dC(t)}{dt} &= (1 - 0.42\kappa_m) \mu(E_y(t) + kI_y(t) + E_s(t) + kI_s(t)) - (1 + 0.21\kappa_e) \delta C(t)
\end{align*}
$$(3.36)
3.3.3 Optimal Control Problem Design

At this stage, the main aim is to formulate the optimal control problem and derive the numerical optimal solution which presents instructions on how to contain the epidemic in the University. Form of the optimal problem is defined as below.

$$J^* = \min_{x(\cdot), \kappa(\cdot)} J = x_f^T H x_f + \int_{t_0}^{t_f} x(\tau)^T Q x(\tau) + \kappa(\tau)^T R \kappa(\tau) \, d\tau$$

s.t. \hspace{1cm} \dot{x}(t) = f(x(t), \kappa(t), t), \hspace{1cm} (3.37)

$$x_{\text{min}} \leq x(t) \leq x_{\text{max}},$$

$$\kappa_{\text{min}} \leq \kappa(t) \leq \kappa_{\text{max}},$$

$$x(0) = x_{14}$$

The cost function is in quadratic form. It formulates both the terminal cost and Lagrangian cost. Square matrices, $H$, $Q$, $R$ contains weights of final states, path states and path control variables respectively. Higher weights mean stronger minimization on the corresponding variable. This formulation allows more intuitive adjustments of the cost on states and control variables.

Constraints on both states and control variables are necessary. The basic requirement on states is that all states should lie between 0 and 1. Meanwhile, referring to previous explanation of measure effectiveness and control variable normalization, lower bounds of the control variables should be $[0, 0, 0, \eta_y/0.5, \eta_s/0.5]^T$ while their upper constrains are $[1, 1, 1, 1, 1]^T$. Simultaneously, without extra requirements, the state constraints are $x_{\text{min}} = [0, 0, 0, 0, 0, 0, 0, 0, 0]^T$, $x_{\text{max}} = [1, 1, 1, 1, 1, 1, 1, 1, 1]^T$.

The initial condition is also significant as it sets the starting point of the system. We assume that the University spots outbreak of the epidemic two weeks after the exposed subjects appear among students or staffs. Thus MATLAB will firstly run the simulation of the original un-controlled model and get the resulting state values on the 14th day, $x_{14}$. This is the initial condition of states in the optimal control problem: $x(0) = x_{14}$. In addition, the epidemic is expected to be eliminated within 120 days so $t_0 = 0$ and $t_f = 120$. 
3.3 Control of the University Model

3.3.4 Implementation of Four Different Scenarios

Balancing the trade-off between controlling the spread of COVID-19 and resuming the normal campus life is the main question to be considered by the University. According to the different degrees of measure strength, four scenarios are studied in this project: minimum case, minimum norm, minimum intervention and minimum quarantine enforcement. Each scenario corresponds to different weight matrices and different path constraints.

As for the minimum-case scenario, University tries all possible measures against the epidemic without considering the sacrifice of student learning experience. This could lead to the fastest mitigation of the epidemic. Mathematically, the controller is designed to maximise the number of susceptible people during the whole period. Therefore, weight matrices are set as follows:

\[
H = \text{diag}([-100; -100; 0; 0; 0; 0; 0; 0; 0; 0])
\]

\[
Q = \text{diag}([-10; -10; 0; 0; 0; 0; 0; 0; 0; 0])
\]

\[
R = \text{diag}([0; 0; 0; 0])
\]

The constraints on both states and control variables remain the same as the basic requirements discussed before. Note that the un-controlled model should be firstly simulated. Variations of states are then used as guesses to activate the solver.

The other three scenarios involve the minimization of control variables. Nevertheless, if all weight matrices are non-zero in the cost function, it is difficult to compare the magnitudes of weights of states and control variables. Therefore, instead of maximizing the susceptible proportions in the meantime, it is better to directly put lower limits on the corresponding states while keeping \(H = 0\) and \(Q = 0\). In this project, we require the number of susceptible students and staffs should not exceed 90% in proportion to the total number of students and staffs respectively. Additionally, to ensure that the epidemic has been completely restrained, the number of exposed and infected subjects should be zero. In this case, we set this upper terminal bound at \(10^{-4}\). Denoting \(N_y\), \(N_s\), and \(N_t\) as the total number of student, staffs, and the whole population, the path constrains on states...
should be:

\[
x_{\text{min}} = \begin{bmatrix} 0.6 \frac{N_y}{N_t} & 0.6 \frac{N_s}{N_t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T
\]

\[
x_{\text{max}} = \begin{bmatrix} 1 & 1 & 10^{-4} & 10^{-4} & 10^{-4} & 10^{-4} & 1 & 1 & 1 & 1 \end{bmatrix}^T
\]

(3.39)

Then, the only difference in the formulation of the rest three scenarios is the value of weight matrix \( R \). In the minimum norm case, the solver aims to minimize the norm of control signal while obeying the 90% requirement. It presents the least overall effort in combating COVID-19. The matrix \( R \) is:

\[
R = \text{diag}(\begin{bmatrix} 10; 10; 10; 10 \end{bmatrix})
\]

(3.40)

Regarding the minimum intervention strategy, the effectiveness of three intervention methods (wearing masks, keeping social distance and disinfecting the environment) are minimised, without assigning any expectations to the quarantine enforcement. In this situation, the matrix \( R \) becomes:

\[
R = \text{diag}(\begin{bmatrix} 10; 10; 10; 0 \end{bmatrix})
\]

(3.41)

The minimum quarantine enforcement case is to figure out the least norm of quarantine rates without avoiding the mitigation requirement. Since mandatory quarantine will greatly impact students’ learning experience, this scenario aims to find the possibility of reducing the quarantine enforcement in controlling the spread of disease. Hence, the matrix \( R \) is:

\[
R = \text{diag}(\begin{bmatrix} 0; 0; 0; 10 \end{bmatrix})
\]

(3.42)

In addition, to guarantee the derivation of feasible solutions and fast convergence to the optimal point, the initial guesses assigned to the solver must be inside the constraints. However, in the last three strategies, output of the original model does not satisfy the state boundaries any more. The susceptible proportions do not remain higher than 90%. On this occasion, the best way to tackle this issue is to apply outputs of the minimum-case scenario to the solver. The structure is shown in Figure 3.4.
This operation means that the solver firstly finds the largest possible value of susceptible proportion. Then in the second problem, we try to reduce the input while setting a lower limit on susceptible proportion. With the quadratic cost, this structure will guarantee that the solver could start from a feasible point and find a feasible optimal solution. The trapezoidal collocation method is used in this case and the library proposed by [47] is deployed to solve the optimization problem.
Chapter 4

Result and Discussion of Simulation and Optimisation

This chapter displays a series of simulation results of the University model, which illustrate the effect of the model parameters. Furthermore, based on the designed optimal control problems, the optimal trajectories are computed by MATLAB and shown in different plots. They could better reflect the importance of different control measures and suggest the appropriate time to increase or decrease the strength of each measure. Discussions on all these results could infer better ways of tackling the trade-off between quality of delivery and prevention of COVID-19 in the University.

4.1 University Model Simulation

4.1.1 Original Model

With the parameter values assigned at the design stage, the model is constructed in MATLAB. It can simulate the evolution of the COVID-19 epidemic when no control measures are conducted. The time range is set at 150 days. The total number of students is 15000 while the number of staffs/faculties is 2500. Initially, 10 students and 5 staffs carry the virus. The result could be seen in Figure 4.1, 4.2 and 4.3.
Figure 4.1: Trajectory of Student Group

Figure 4.2: Trajectory of Staff/Faculty Group
4.1 University Model Simulation

The figure indicates a huge outbreak of the pandemic in the University. Its peak appears at around 47 days when the number of infected people and the virus concentration in the environment reaches the maximum. Most infections occur during the first 70 days and the number of susceptible people decreases dramatically. At the end of this epidemic, only around 1% students will not get infected while all staffs/faculties have suffered from COVID-19. By Equation 3.13, $R_0$ in this case is 4.

4.1.2 $R_0$ Testing

As $R_0$ is extremely important in analysing the epidemics, it is necessary to prove its availability. Previous simulations on the uncontrolled model have revealed that $R_0$ greater than one will imply the advent of a pandemic. The number of infected people will increase and reach a maximum point, such as the curve displayed in Figure 4.1, 4.2. On the contrary, when $R_0 < 0$, it is expected to witness a monotonic decrease in the number of exposed individuals. The disease will gradually disappear without any interventions or prevention measures.

Since only one result cannot powerfully verify the correctness of $R_0$ computation, the value of $\beta_{yy}$ and initial number of exposed subjects are changed to generate the model.
with different $R_0$. The resulting curves of exposed compartments are plotted in Figure 4.4.

![R_0 Testing Result - Exposed Students](image)

![R_0 Testing Result - Exposed Staffs](image)

**Figure 4.4: $R_0$ Testing: Exposed Students and Staffs**

It can be seen that the number of exposed individuals keeps decreasing when $R_0 < 1$. The curve will gradually converge to zero and spread of the virus will not expand. However, when $R_0 > 1$, after the sudden drop at the beginning due to small infection rates, the curve goes up, inferring a wider spread of COVID-19 occurs at this moment. These phenomena prove that the $R_0$ derivation is correct. It successfully labels whether the epidemic will break out in the University.

### 4.1.3 Effects of Individual Infection Rates $\beta_{yy}$ and $\beta_{ss}$

Individual infection rates are significant factors in every pandemic model. It directly determines the magnitude and duration of the epidemic. For this University model, the impact of two individual infection rates $\beta_{yy}$ and $\beta_{ss}$ are investigated. The value of $\beta_{yy}$ was initially varied from 0.08 to 0.16 while keeping $\beta_{ss} = p \beta_{yy}$ at 0.2333. Then in the second experiment, $\beta_{yy}$ was 0.1467 with different $\beta_{ss}$ in the range from 0.12 to 0.24. Results of both implementations are displayed in Figure 4.1, 4.2.
The results demonstrate that higher individual infection rates lead to an earlier and greater boost in the number of cases. Meanwhile, the parameters vary over the same scale but plots in Figure 4.5 are more dispersed. It implies that the infection rate of students, $\beta_{yy}$, plays a primary role in the University epidemic evolution. Change in $\beta_{ss}$ does not make a large difference. This result is reasonable because the University contains more students than staffs. Increment in $\beta_{yy}$ can accelerate the spread by larger degrees.
4.1.4 Effects of Environmental Parameter Values

Environmental infection rates $\beta_{cy}$, $\beta_{cs}$, shedding rate $\mu$ as well as the virus decaying rate $\delta$ are parameters reflecting the effect of environment on the spread of COVID-19. By following the same procedure as before, each parameter is varied and studied separately. Results are displayed in the following figures.

![Figure 4.7: $\beta_{cy}$ Testing Result](image1)

![Figure 4.8: $\beta_{cs}$ Testing Result](image2)
All tests output similar shapes of curves but reflect the importance of each parameter. Comparison between the results in Figure 4.7 and 4.8 indicates that students are more vulnerable to infection in this confined environment. A larger environment-to-student infection rate will bring apparent exacerbation of the COVID-19 pandemic in the University. Similarly, between $\delta$ and $\mu$, the shedding rate seems to be more significant. From the practical perspective, it demonstrates that controlling the spread of virus to the
environment is a more powerful mitigation method than disinfecting the environment.

### 4.1.5 Effects of Quarantine Rates $\eta_y$, and $\eta_s$

The effects of quarantine is finally analysed. Similarly, various values of each parameter are tested respectively. Plots are shown in Figure 4.11 and 4.12.

**Figure 4.11: $\eta_y$ Testing Result**

**Figure 4.12: $\eta_s$ Testing Result**
According to the graphs, change in $\eta_y$ results in greater variations of the exposed and infected compartment curves. This also reflects the fact that student carriers are more likely to be super-spreaders of COVID-19. Quarantine of the infected staffs, however, only mitigates the epidemic by a small degree.

4.2 Solution of Optimal Control Problems

4.2.1 Minimum-Case Scenario

Implementing the optimal control problem defined in the prior chapter, the obtained trajectories are plotted in Figure 4.13. Since we assume that the University takes 14 days to spot the potential outbreak and we expect the epidemic is tackled in 120 days, the overall timeline in this simulation is 134 days.

![Optimal Trajectories of Minimum-Case Scenario](image)

Figure 4.13: Optimal Trajectories of Minimum-Case Scenario

In this case, the epidemic is completely ended at around the 100th day when the University only remains two compartments: susceptible and recovered. At this moment, it reaches an stable equilibrium point of the system because now $R_0$ is around 0.6. Plots of the control
variable $\kappa$ indicate that all control measures are conducted unreservedly in this scenario. The trajectories all reach the maximum value across the whole period. After 100 days, execution of control measures is eased because the epidemic has been contained.

### 4.2.2 Minimum-Norm Scenario

The derived optimal trajectories of the minimum-norm case are depicted in Figure 4.14.

![Figure 4.14: Optimal Trajectories of Minimum-Norm Scenario](image)

In this scenario, the fractions of susceptible compartments are successfully kept higher than 90%. The number of exposed and infected subjects become zero at the end of the simulation, which means that the epidemic is handled 120 days and no further infections will occur in the future. However, the system still does not attain its equilibrium point because the number of quarantined subjects is still greater than zero. While completing this goal, we can also witness the obvious reduction in the norm of control variables. $\kappa_e$ is the smallest with the norm lower than 0.1. $\kappa_d$ still remains at the maximum point most of the time while $\kappa_m$ increases from 0.6 to 0.8. The normalized student quarantine rate $\kappa_{qy}$ reaches the upper limit at the beginning of control. Then it is always greater than $\kappa_{qs}$.
which remains around 0.5.

4.2.3 Minimum-Intervention Scenario

The optimal control problem that aims to minimize the norm of $\kappa_m$, $\kappa_d$, and $\kappa_e$ was then solved by MATLAB. The optimal trajectories can be seen in the following figure.

In this case, norm of the three target control variables are decreased dramatically but the control requirements are still satisfied. The final value of state $S_s$ is right at 0.9 while the student counterpart $S_y$ is higher. The exposed and infected groups are all eliminated and there will not be further infections after 134 days. Regarding the control effectiveness, quarantine rates $\kappa_{qy}$ and $\kappa_{qs}$ are kept at the highest level across the whole period. This is normal when the other three variables are reduced. Compared to results in the minimum-case scenario, all three intervention variables are reduced but $\kappa_d$ is still greatly higher than $\kappa_e$ and lower than $\kappa_d$. 

![Optimal Trajectories of Minimum-Intervention Scenario](image-url)
4.2.4 Minimum-Quarantine Scenario

The minimum quarantine scenario, on the contrary, tries to see the possibility of reducing the execution strength of the mandatory quarantine measure. Results are displayed in Figure 4.16.

![Figure 4.16: Optimal Trajectories of Minimum-Quarantine Scenario](image)

The norms of both quarantine rates are further decreased when compared to the plots in Figure 4.14 and 4.15. Similar to the previous scenarios, the trajectories of states still satisfy the expected constraints. Three intervention effectivenesses are all at the maximum level, reflecting the controller is trying to minimize the quarantine rates. Smaller than outputs in Figure 4.14, $\kappa_{qy}$ and $\kappa_{qs}$ vary around 0.6 and 0.4 respectively. They correspond to the 0.3 student quarantine rate and 0.2 staff quarantine rate.

4.2.5 Inference from the Result

The optimal trajectories plotted in four scenarios suggest the priority of control measures. Keeping social distance is the most effective measure among the intervention choices. Wearing masks are ranked second. If both measures can be promoted and executed
strongly, environmental disinfection seems to be less necessary. As for $\kappa_{qy}$ and $\kappa_{qs}$, the result yields that quarantine of symptomatic students is more significant. Along the timeline, the $40^{th}$ day is a vital point, at which all implementation of measures should be further improved to restrain the infections.

From the practical perspective, the University could emphasize social-distancing and mask-wearing more, especially among students. To maintain a good quality of on-campus teaching, exposed/infected lecturers could deliver the contents in person but distancing and masks are pre-requisite, whereas the infected students in the University should be asked to comply with mandatory quarantine. Environmental disinfection could be conducted once per day. When all deliveries are finished in the lecture room, disinfecting the environment is helpful in combating the spread of COVID-19.
Chapter 5

Conclusion and Future Development

This chapter will conclude what has been achieved in the project and what can be improved in future research. Main contributions will be summarized and potential of further explorations will be suggested.

5.1 Conclusion

In conclusion, this project has designed and simulated a University model on COVID-19 epidemic and applied the optimal control techniques to suggest methods of handling the trade-off between resuming the delivery of programmes and preventing the spread of COVID-19. Considering the current COVID-19 situation, this project makes contributions on modelling the COVID-19 in the University and suggesting suitable ways of containing this epidemic.

In this project, a new model has been constructed to analyse the COVID-19 pandemic in the University. The model is compartmental but the definition of an extra environment compartment enhances its applicability to the confined-space case. Consisting of 11 differential equations, it describes staff-to-staff infections, student-staff cross infections, and student-to-student infections. Different evolution stages of COVID-19 are also consid-
ered and effects of carriers under different stages are formulated in detail. Since we focus on the small environment, the special “environment” effects are specifically modelled by the extra environment compartment. Although the model is of a high order, the analytical expression of $R_0$ is found and stability of the equilibrium points are studied, which help analyse the evolution of this COVID-19 pandemic in such case. In addition, due to lack of data, model parameters cannot be derived by fitting methods. Nevertheless, since the model and parameters are well-defined with specific biological meanings, all parameter values are assigned based on reliable information and previous research on COVID-19. This new model is reliable to study the COVID-19 epidemics in the University.

With the pandemic model, defining control variables corresponding with effectiveness of control measures and deploying the optimal control methods to obtain the optimal trajectories is a novel and rigorous systematic way of finding the best strategy of combating the COVID-19 epidemics. Both simulation results and optimal solutions reflect the potential consequence of different measures, including mask-wearing, social-distancing, environment disinfection, and mandatory quarantine. Their combined effects are also shown in the four different assumed scenarios in this project. Results suggest that keeping social distance and wearing masks are far more important than environmental disinfection. Students play a major role in virus transmission and therefore mandatory quarantine of infected students is particularly significant. These results can assist University administrators to deal with the trade-off between maintaining the good quality of on-campus teaching and preventing the spread of COVID-19.

5.2 Future Development

The University model in this project can still be further improved. Since there are many variants of COVID-19 spreading worldwide, the protection rate of vaccines cannot be estimated precisely. Under this circumstance, we did not include the effectiveness of vaccination. Another factor that the model ignores is the infection brought from outside the campus, which is extremely difficult to estimate when we do not have precise data on it. However, if it is possible to obtain reliable information or data and the policy-maker
would like to include these factors, it is easy to achieve in this ODE-based compartmental
model. Moreover, the University model produced in this project provides a great prototype
in modelling the pandemics in small environment. For example, the same model structure
could be applied to model COVID-19 in the company. The whole population could be
separated into leaders and employees.

Data can bring crucial improvement to this area. In this project, model parameter
values are set according to various literature. This method still cannot guarantee 100%
precision when compared with the actual case. Furthermore, lack of data also limits the
choice of modelling method. The compartmental model is the most flexible and can be
explored without reliable data. However, if data can be obtained and used for parameter
fitting in the future, the compartmental model proposed in this project could be modified
accordingly, and other types of pandemic models such as SDE-based stochastic model or
deterministic model combined with scale-free networks can be tried to better imitate the
epidemic evolution.

Regarding the optimal control part, the greatest issue is that output of this optimal
control problem is continuous. Although we can get the optimal trajectories representing
the expected effectivenesses of different measures, it is difficult to distinguish between the
extent at 0.6 and 0.8, especially for the three intervention measures. Therefore, a desirable
way to tackle this problem is to discretize the input into several specifically defined levels
at the initial stage. However, due to lack of time, this work was not accomplished in this
project.

In addition, formulation of the control problem has great flexibility. This project
only considers four particular scenarios. However, referring to the actual circumstance and
expectations, policy-makers could easily change the weights and obtain the corresponding
optimal trajectories in their particular scenario.
Bibliography


