Optimal Market Making with Competition

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Abstract

Competition between market makers, which considers the impacts on trading strategy of individual and liquidity of whole market resulting from multiple market makers competing for order flow and market maker incentives, was not properly studied in the literature of optimal market making problem. This thesis is devoted to the optimal market making problem, with competition between market makers. Three main topics are studied in this thesis.

In the first topic, we consider the price competition between market makers. We discuss optimal market marking with price competition and incomplete information, which results in a looping dependence structure among market makers. We solve the problem with the non-zero-sum stochastic differential game approach and characterize the equilibrium value function with a coupled system of nonlinear ordinary differential equations. We prove, do not assume a priori, that the Issac condition is satisfied, which ensures the existence and uniqueness of Nash equilibrium. We also perform some numerical tests that show our model produces tighter bid/ask spread than a benchmark model without price competition and improves market liquidity.

In the second topic, we consider market makers competing for the market maker incentive reward proposed by exchange, which depends on their trading volume ranking. We model the competition as a stochastic mean field game, which can be further reduced to a finite state mean field game, whose equilibrium is characterized by a forward backward ODE systems. We numerically solve the equilibrium with the deep neural network approach proposed in our third topic, and perform some numerical tests to compare bid/ask spread under different types market maker incentive reward. It is suggested that the introduction of incentive can reduce the implicit trading cost, and rank-based reward, compared with the linear trading volume reward, can produce lower best bid/ask spread.

In the third topic, we discuss the deep neural network approach for solving the forward backward ODE system corresponding to a more general class of finite state mean field game, and the game in the second topic is just a special case of it. We prove that the error between true solution and our approximation is linear to the square root of loss function of our deep neural network.

Keywords: optimal market making, price competition, non-zero-sum stochastic differential game, Issac condition, existence and uniqueness of Nash equilibrium, forward backward ODE, mean field game, numerical method, deep neural network.

To my family.

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Introduction

In the financial market, market makers play an important role in providing liquidity for other market participants. They keep quoting bid and ask prices at which they stand ready to buy and sell for a wide variety of assets simultaneously. Without market makers, a buyer in the market might need to wait a long time until a seller appears, as demand and supply usually do not appear at the same time. Market makers act as an intermediary bridging demand and supply appearing from different timing, and hence are crucial to financial market's liquidity and stability. Exchanges or some other trading venues even appoint designated market makers and set up different market making incentive schemes for them to encourage liquidity provision for various securities trading in their venues. Designated market makers are obliged to quote for most of the time, but benefit from the market making incentive schemes.

Behaviours of market makers are different if they are trading in different markets, with different trading rules and market structure. It is interesting for both market makers and the trading venues to model and understand those market making behaviours. Market makers can find the optimal strategy to maximize their payoffs, while exchanges or trading venues, can then design better trading rules or market making incentive scheme to improve the liquidity.

Optimal Market Making Theory

The research that focus on market making evolves as the innovation of both trading mechanism and technology. Before 90s in twenty century, floor trading was the most common way of trading. Most of market makers are specialists, or dealers, hired by exchanges, to stand in the central of exchange and trade with different people in the market. They are considered as the traditional market makers. Their trading frequency are low, and each trade might be of a relatively large units compared with current trading behaviours. Market makers' common concern is the change of asset value, since they might inevitably have to take their positions overnight. Ho and Stoll (1981); Avellaneda and Stoikov (2008); Guéant et al. (2013) formulate market making for traditional market makers as a stochastic optimization problem that maximize the expected utility of terminal wealth. On the other hand, other research focus more on the price formation and adverse selection in market making (See Kyle (1984, 1985, 1989); Dennert (1993); Calcagno and Lovo (2006); Loertscher (2008); Ho and Stoll (1980); Bondarenko (2001)).

After 2000s, the innovation of technology, the appearance of more advanced computer with sophisticated algorithm and electronic trading platform totally change the landscape of market making business. Electronic trading platform becomes the mainstream for both exchange traded and OTC traded products. Trading automation boosts the trading frequency. Following this trend, market makers also become electronic. The old class of specialists has almost disappeared, and a modern version of designated market makers (DMM) emerges. DMMs have contracts with exchanges and are obliged to help provide liquidity to the market most of the trading time. They are usually equipped with co-location facilities, high speed connections, and fast computers, reducing the time for each transaction to milliseconds. Nowadays, market making is mostly controlled by these electronic market makers. There are quite some research and articles that support this view (See the second and third footnote at page 1 in Bellia et al. (2019)). In fact, after January 2016, all DMM on the NYSE are electronic market makers (See "High-frequency traders in charge at NYSE," Financial Times, January 26, 2016). Thanks to the boost of trading frequency, electronic market makers tend to quote in high frequency with low trade size per transaction, keep very low inventory and not carry overnight position, in order to avoid the risk from variation of asset price. Market makers' behaviours are different. Inventory management becomes their main concern. One of the most important problems for the modern version of market makers is how to determine their bid/ask spread optimally such that they can maximize their profits while remains in a low inventory level.

Inspired by current market makers' need to manage inventory, Cartea et al. (2015) adds an terminal inventory punishment term to market maker's payoff to extend the model of Avellaneda and Stoikov (2008); Guéant et al. (2013). Guéant (2017) further generalizes the model to a general intensity function and manage to reduce the high dimensional HJB PDE to a simpler form by Ansatz under CARA utility function. They also consider multi asset market making. Fodra and Labadie (2012) derive an analytical solution for exponential market order intensity function when there is no inventory punishment. With inventory punishment, they also provide a analytical sub-solution of the original problem. In these models, the arrivals of market order are all modelled as Poison process with controlled intensity, which provides more tractability, but is less realistic. In contrast, Cartea and Jaimungal (2015) extend the framework used in Cartea et al. (2014). They considered market impact and adopted a more sophisticated method to model the fill rate of agent's limit order. In order to capture the cluster feature of market order arrivals, intensity of the market order arrival is modeled as a self-excited process excited by informative market orders and news events. The resulting HJB PDE is more comprehensive with solution approximated by an asymptotic method. Meanwhile, Cartea et al. (2017) extend Avellaneda and Stoikov (2008); Guéant et al. (2013) and consider model uncertainty. Fodra and Pham (2015) uses a different framework from Avellaneda and Stoikov (2008). They use Markov Renewal Model to model mid price, which can be bumped up by market orders with size that is big enough to consume all the limit orders in the book. Abergel et al. (2020) propose a pure jump model for optimal market making on the limit order book. Mid price dynamic is simply the best bid/ask price that the last trade occurred at. They combine the limit order book dynamic with the optimal market making control problem, and solve it with Markov Decision Process technique conditioned on the jump time clock.

One common point of the existing literature mentioned above is that they only consider the optimal market making problem for single market maker. Every market maker in the model is independent to each other, and never get impacted by other market makers' strategies. However, we know in reality it is not the case. But the impact of interaction among market makers on liquidity provision is still not well understood.

Competition between Market Makers

Market makers face competition. There are usually multiple designated market makers for one security. They usually have two kinds of interaction. The first is the competitions among themselves, and the second is competition against voluntary market makers that make the market without signing any agreement with the exchange, informed traders and speculative traders etc.

There have been some empirical research analyzing the competition among market makers from data analysis view point (See Breckenfelder (2019); Bellia et al. (2019)), which suggests that competition among market makers have considerable impact on the market liquidity and should not be neglected. However, the question of how the competition among market makers themselves affects the market liquidity and market dynamic is largely unaddressed from a theoretical perspective, although there is mention of competition in a broad sense between market makers and other market participants, i,e the second kind of interaction. Kyle (1984, 1985, 1989); Dennert (1993); Calcagno and Lovo (2006); Loertscher (2008) try to model the price formation process and asymmetric competition of market makers against other market participants e.g informed traders, low frequency market makers and etc. But in their single period game setting, there is no need to manage inventory. Ho and Stoll (1980); Bondarenko (2001) do use a multi-period setting, but the need for inventory management is still not considered in their models. Hence we deem their models are better fitted to traditional market makers instead of the current electronic ones. Recently, Ait Sahalia and Saglam (2017) consider market maker competition in a limit order book setting and introducing competition feature into the traditional inventory control problems (See Guéant (2017)). But the two competing market makers in Ait Sahalia and Saglam (2017) are of different trading frequency and information advantage, while only the high frequency market maker can take strategic move.

To the best of our knowledge, the competition among designated market makers (which are symmetric) is largely neglected in existing literature. As the designated market makers tend to be the most influential liquidity providers in the market, it is crucial for both themselves and the exchanges to understand how the competition can impact their optimal strategies, as well as their liquidity provision. Our ambition in this thesis is to fill this gap.

Organization of the Thesis

The thesis is organized as follows. Since it is intended to be self-contained, we start by recalling in Chapter 2 some preliminaries in stochastic calculus and game theory.

There are usually two kinds of competition among designated market makers. One is price competition for order flows, another one is competition for market maker incentive reward. In Chapter 3, We are to discuss the first one, i.e the competition for order flow. Traders in the market can choose the best price to hit. Hence if a designated market maker quotes a price with less spread compared with others, it will be more possible for her price to be hit by market traders. The price competition among designated market makers is fierce since they are almost always providing quotes near the tightest spread in the market. Most of existing literature either fails to consider the inventory management concern of current designated market makers, or only considers the price competition among different market participants instead of among symmetric market makers like designated market makers. Then there comes the question on how designated market maker to optimally decide the bid/ask spread when other designated market makers' strategies are considered.

To answer this question, we build a optimal market making model with price competition and incomplete information, extending the market making model without competition from Guéant (2017). The arrival of market orders to certain market maker is modeled by jump processes with controlled intensity, but the intensity, or the arrival rate depends not only on her own quotes, but also on quotes from others, which results in a looping dependence structure among market makers. We study how market makers can optimally decide their optimal strategies to maximize their terminal wealth while keeping their inventory low. We solve the problem with the non-zero-sum stochastic differential game approach and characterize the equilibrium value function with a coupled system of nonlinear ordinary differential equations. The main contributions of this chapter are summarized by the following bullet points:

- We discuss price competition between market makers in a continuous time setting with inventory constraints and incomplete market information of competitors' inventory, and extend the classical optimal market making framework in Guéant (2017) with the game theoretic approach. As far as we are concerned, this is the first attempt to study the competition for order flow among symmetric designated market makers.

- We prove, do not assume a prior or solve explicitly (See Hamadene et al.

(1997); Buckdahn et al. (2004); Bensoussan et al. (2014); Lin (2015)), that the Issac condition is satisfied, which ensures the existence and uniqueness of Nash equilibrium.

- We perform some numerical tests to compute the equilibrium value function and equilibrium controls (bid/ask spreads). The results show that our model produces tighter bid/ask spread than the benchmark model without price competition from Guéant (2017).

In Chapter 4, we discuss the second kind of competition, i.e the competition for the market making incentive scheme. Designated market makers usually have contracts with exchanges and follow the market making incentive scheme designed by the exchanges. The purpose of appointing designated market makers and designing market making incentives scheme for them is to stimulate liquidity provision. The market making incentive scheme, on one hand, obliges designated market makers to provide liquidity for most of the trading time, and on the other hand, offers them various benefit including waives of commission fee, award of make-take fee, cash reward depending on one's absolute value of trading volume, or the relative rank of trading volume among all other designated market makers. The reward structure might vary as different exchanges offer different market making incentive schemes for different securities. Then there comes the question on how the different reward structures of market making incentive schemes can impact market makers' optimal strategies that determines their liquidity provision, and more specifically, whether the introduction of reward related to relative rank of trading volume can improve the liquidity provision.

To answer above question, we model market makers' competition for trading volume rank based reward as a mean field game problem.

2

Preliminaries

2.1 General Notations and Abbreviations

For any real numbers x, y, $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$.

$\mathbb{R}^d$ denotes the d–dimensional Euclidian space. $\mathbb{R} = \mathbb{R}^1$. For all $x = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$, we denote by $||\cdot||$ the norm. Without further specification, it is the Euclidian norm:

$$||x|| = \sqrt{\sum_{i=1}^{d} x_i^2}.$$

We denote by $B_\eta(x)$ (resp. $\bar{B}_\eta(x)$) the open (resp. closed) ball of center $x \in \mathbb{R}^d$, and radius $\eta > 0$, with corresponding norm in the corresponding paragraph.

$C(\mathcal{O})$ is the space of all real-valued continuous functions on $\mathcal{O} \subset \mathbb{R}^{d+1}$.

$C^{1,2,\ldots,2}(\mathcal{O})$ is the space of all real-valued continuous functions f on $\mathcal{O} \subset \mathbb{R}^{d+1}$

whose partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist and are continuous on $\mathcal{O}$. Note that sometimes for notation convenience, we also denote by f_1' the first order partial derivative of f to its first variable, f_{11}'' the second order partial derivative of f to its first variable, f_2' the first order partial derivative of f to its second variable, etc.

$f(x) = o(g(x))$ means that $\lim_{x \to 0} f(x)/g(x) = 0$.

$(\Omega, \mathcal{F}, \mathbb{P})$: probability space.

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$: filtered probability space.

$\mathbb{E}_t[X]$: expectation of random variable X given filtration $\mathcal{F}_t$ generated by the state processes of the model.

SDE: stochastic differential equation.

ODE: ordinary differential equation.

PDE: partial differential equation.

DPP: dynamic programming principle.

HJB: Hamilton-Jacobi-Bellman.

MFG: Mean field game.

u.s.c.: upper-semicontinuous.

l.s.c.: lower-semicontinuous.

2.2 Definitions and Theorems

Definition 2.2.1. (Nash Equilibrium) A Nash equilibrium of a strategic game is a profile of strategies $(s_1^*, ..., s_n^*)$, where s_i^* in S_i (S_i is the strategy set of player i), such that for each player i, $\forall s_i$ in S_i, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$, where $s_{-i}^* = (s_j^*)_{j \neq i, j \notin \mathbb{N}}$ and $u_i : \prod_{i=1}^n S_i \to \mathbb{R}$.

Definition 2.2.2. (Zero Sum Game) A zero-sum game is a mathematical representation of a situation in which each participant's gain or loss of utility is exactly balanced by the losses or gains of the utility of the other participants. If the total gains of the participants are added up and the total losses are subtracted, they will sum to zero.

Definition 2.2.3. (Non-zero Sum Game) Non-zero-sum game is the situation in which the interacting game players' aggregate gains and losses can be less than or more than zero.

Definition 2.2.4. (Mean Field Game) Mean Field Games are games with a very large number of agents interacting in a mean field manner in such a way that each agent has a very small impact on the outcome. As a result, the game can be analyzed in the limit of an infinite number of agents.

The following generalized Ito's formula, Gronwall's inequality and Lebesgue's Dominated Convergence Theorem are used throughout this thesis.

Theorem 2.2.5. *(Generalized Ito's formula) Let X be a semimartingale and let f be a C^2 real function. Then $f(X)$ is again a semimartingale, and the following formula holds:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s^C + \frac{1}{2}\int_0^t f''(X_{s-})d[X,X]_s^C + \sum_{0\leq s\leq t} \Delta f(X_s),$$

where X^C is the continuous part of process X and $\Delta f(X_s) := f(X_s) - f(X_{s-})$.

Lemma 2.2.6 (Gronwall's inequality)**.** *Let I denote an interval of the real line of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with $a < b$. Let α, β and u be real-valued functions defined on I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded sub-interval of I. If β is non-negative and if u satisfies the integral inequality*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds, \qquad \forall t \in I,$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_t^s \beta(r)dr}ds, \qquad t \in I.$$

If, in addition, the function α is non-decreasing, then

$$u(t) \leq \alpha(t)e^{\int_t^a \beta(s)ds}, \qquad t \in I.$$

Theorem 2.2.7 (Lebesgue's Dominated Convergence Theorem)**.** *Let (f_n) be a sequence of complex-valued measurable functions on a measure space (S, Σ, μ). Suppose that the sequence converges point-wise to a function f and is dominated by some integrable function g in the sense that*

$$|f_n(x)| \leq g(x),$$

for all numbers n in the index set of the sequence and all points $x \in S$. Then f is integrable and

$$\lim_{n\to\infty} \int_S |f_n - f| d\mu = 0,$$

which also implies

$$\lim_{n\to\infty} \int_S f_n d\mu = \int_S f d\mu.$$

The Schauder fixed point theorem, Berge Maximum Theorem, as well as the global implicit function theorem in (Galewski and Rădulescu, 2018, Theorem 4), are mainly used by the proof of Generalized Issac condition in Chapter 3. The Schaefer's fixed point theorem on the other hand, is used by the proof in Chapter 5.

Theorem 2.2.8 (Schauder Fixed Point Theorem)**.** *If K is a nonempty convex closed subset of a Hausdorff topological vector space V and T is a continuous mapping of K into itself such that $T(K)$ is contained in a compact subset of K, then T has a fixed point.*

Theorem 2.2.9 (Schaefer Fixed Point Theorem)**.** *Let T be a continuous and com-*

pact mapping of a Banach space X into itself, such that the set

$$\{x \in X : x = \lambda Tx \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then T has a fixed point.

Theorem 2.2.10 (Berge)**.** *Let X and Θ be metric spaces, $f : X \times \Theta \to \mathbb{R}$ be a function jointly continuous in its two arguments, and $C : \Theta \to X$ be a compact-valued correspondence. For x in X and θ in Θ, let*

$$f^*(\theta) = \max\{f(x, \theta)|x \in C(\theta)\},$$

and

$$x^*(\theta) = \arg\max\{f(x, \theta)|x \in C(\theta)\} = \{x \in C(\theta) \mid f(x, \theta) = f^*(\theta)\}.$$

If C is continuous at some θ, then f^* is continuous at θ and x^* is non-empty, compact-valued, and upper hemicontinuous at θ, that is, if $\theta_n \to \theta$ and $b_n \to b$ as $n \to \infty$ with $b_n \in x^*(\theta_n)$, then $b \in x^*(\theta)$.

Theorem 2.2.11 (Global Implicit Function Theorem)**.** *Assume $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz mapping such that*

- *For every $y \in \mathbb{R}^m$, the function $\phi_y : \mathbb{R}^n \to \mathbb{R}$, defined by $\phi_y(x) = \frac{1}{2}||F(x, y)||^2$, is coercive, i.e., $\lim_{||x|| \to \infty} \phi_y(x) = +\infty$.*
- *The set $\partial_x F(x, y)$ is of maximal rank for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$.*

Then there exists a unique locally Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}^n$ such that equations $F(x, y) = 0$ and $x = f(y)$ are equivalent in the set $\mathbb{R}^n \times \mathbb{R}^m$.

The next lemma shows that any single valued, bounded, upper hemicontinuous mapping is a continuous function.

Lemma 2.2.12. *Let A, B be two Euclidean spaces, $\Gamma : A \to B$ be a single-valued, bounded and upper hemicontinuous mapping, then Γ is a continuous function.*

Proof. For any sequence $a_n \to a$ and $b_n = \Gamma(a_n)$ (Γ is a single-valued mapping), if b_n tends to a limit b, then we must have $b = \Gamma(a)$ by the hemicontinuity of Γ and we are done. Assume the sequence b_n did not have a limit. Since b_n is a bounded sequence, there exist at least two subsequences b_{n_k} and $b_{n'_k}$ that converge to two different values b and b'. Since $a_n \to a$, we must have both a_{n_k} and $a_{n'_k}$ tend to a, the hemicontinuity of Γ would imply $b = \Gamma(a)$ and $b' = \Gamma(a)$, a contradiction to the assumption that $b \neq b'$. Therefore, Γ is continuous. □

The following Picard-Lindelof theorem (Theorem 2.2.13) in ODE theory, together with the direct extension from its proof (Lemma 2.2.14) are used mainly in Chapter 3 to provide global existence and uniqueness of solution to ODE system.

Theorem 2.2.13 (Picard-Lindelof theorem). *Consider the initial value problem in* $\mathbb{R}^M$*:*

$$y'(t) = F(t, y(t)),\ y(t_0) = y_0, \tag{2.1}$$

where $F : \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}^M$ *is uniformly Lipschitz continuous in* y *with Lipschitz constant* L *(independent of* t*) and continuous in* t*. Then, for some value* $\varepsilon > 0$*, there exists a unique solution* $y(t)$ *to the initial value problem on the interval* $[t_0-\varepsilon, t_0+\varepsilon]$*.*

The lemma is a direct conclusion from the proof of Theorem 2.2.13, see Teschl (2012). It helps us to extend the local existence and uniqueness of solution to the global existence and uniqueness.

Lemma 2.2.14. *Let* $C_{a,b} = [t_0 - a, t_0 + a] \times B_b(y_0)$*, where* $B_b(y_0)$ *is a closed ball in* $\mathbb{R}^M$ *with center* y_0 *and radius* b*. Define*

$$M = \sup_{(t,y)\in C_{a,b}} \|F(t, y)\|.$$

Then the solution to the ODE system (2.1) exists and is unique on interval $[t_0 -$

$\epsilon, t_0 + \epsilon]$, *if* ϵ *satisfies following:*

$$\epsilon < \min\{\frac{b}{M}, \frac{1}{L}, a\}.$$

To prove existence of fixed point, we need sometimes to verify the relatively compactness like Proposition 5.4.5 in Chapter 5. Hence we introduce following Arzela-Ascoli theorem from Dunford and Schwartz (1958).

Theorem 2.2.15 (Arzela-Ascoli theorem)**.** *Let* X *be a compact Hausdorff space, and* $C(X)$ *be space of real-valued continuous functions on* X*. Then a subset* F *of* $C(X)$ *is relatively compact in the topology induced by the uniform norm if and only if it is equi-continuous and point-wise bounded.*

3

Market Making with Price Competition

3.1 Introduction

Market makers play an important role in providing liquidity for other market participants. They keep quoting bid and ask prices at which they stand ready to buy and sell for a wide variety of assets simultaneously. One of the key challenges faced by market makers is to manage inventory risk. Market makers need to decide bid/ask prices which influence both their profit margins and accumulation of inventory. Many market makers compete for market order flows as their profits come from the bid/ask spread of each transaction. Traders choose to buy/sell at the most competitive prices offered in the market. Hence market makers face a complex optimization problem. In this chapter, we model market making for a single asset with price competition as a non-zero-sum stochastic differential game.

There has been active research on optimal market making in the literature with focus on inventory risk management. Stochastic control and Hamilton-Jacobi-Bellman (HJB) equation, a nonlinear partial differential equation (PDE), are used to derive the optimal bid/ask spread. Ho and Stoll (1981) give the first prototype model for the market making problem. Avellaneda and Stoikov (2008) propose a basic trading model in which the asset mid-price follows a Brownian motion, market buy/sell order arrivals follow a Poisson process with exponentially decreasing intensity function of bid/ask spread, and market makers optimally set the bid/ask spread to maximize the expected utility of the terminal wealth. Guéant et al. (2013) discuss a quote driven market and include the inventory penalty for terminal utility maximization. Guéant (2017) extends the model in Guéant et al. (2013) to a general intensity function and reduces the dimensionality of the HJB equation for CARA utility. Cartea and Jaimungal (2015) consider the market impact and capture the clustering effect of market order arrivals with a self-exciting process driven by informative market orders and news events, and solve the HJB equation by an asymptotic method. Cartea et al. (2017) study the model uncertainty, similar to Avellaneda and Stoikov (2008); Guéant et al. (2013), except for the self-exciting feature of market order arrivals. Fodra and Pham (2015) divide the market orders depending on the size which may bump up the mid-price that follows a Markov renewal process. Abergel et al. (2020) discusses a pure jump model for optimal market making on the limit order book with the Markov decision process technique conditioned on the jump time clock.

One common feature in the aforementioned papers is that market order arrivals follow a Poison process with controlled intensity. The probability that a market maker buys/sells a security at the bid/ask price she quotes is a function of her own bid/ask spread only. This setting provides tractability, but ignores the influence of prices offered by other market makers. The price competition between market makers in practice is an important trading factor and needs to be integrated in the model. Kyle adopts the game theoretic approach in a number of papers Kyle (1984, 1985, 1989) to study the price competition between market participants of informed traders, noisy traders and market makers, and finds the equilibrium explicitly and shows its impact on price formation and market liquidity. To the best knowledge of the authors there are no known results in the literature on price competition between market makers who keep trading to profit from bid/ask spread while minimize inventory risk and improve market liquidity. The primary motivation of this chapter is to fill this gap. Market making with price competition is the key difference of our model to that of Guéant et al. (2013) and others in the literature. The standard optimal stochastic control is not applicable to our model due to the looping dependence structure and the equilibrium control is used instead to solve the problem.

The main contributions of this chapter is the following: Firstly, we discuss price competition between market makers in a continuous time setting with inventory constraints and incomplete market information of competitors' inventory, and extend the classical optimal market making framework in Avellaneda and Stoikov (2008) with the game theoretic approach. Secondly, we prove the existence and uniqueness of Nash equilibrium for the game under linear quadratic payoff and prove the generalized Issac's condition is satisfied for a system of nonlinear ordinary differential equations (ODEs), rather than assuming it to hold a priori or solving it explicitly as in the most literature, see Hamadene et al. (1997); Buckdahn et al. (2004); Bensoussan et al. (2014); Lin (2015). Thirdly, we perform some numerical tests to compute the equilibrium value function and equilibrium controls (bid/ask spreads) and compare results with those from a benchmark model without price competition, and we find our model reduces the bid/ask spread and improves the asset liquidity in the market considerably.

The rest of the chapter is organized as follows. In Section 3.2 we introduce the model setup and notations. In Section 3.3 we state the main results on the existence and uniqueness of Nash equilibrium, the generalized Issac's condition, and the verification theorem for the equilibrium value function. In Section 3.4 we perform numerical tests to show the impact of price competition and compare the results with a benchmark model without price competition. In Section 3.5 we prove the main results (Theorems 3.3.3 and 3.3.4). Section 3.6 concludes.

3.2 Model Setting

Consider a market in a probability space $(\Omega, \mathcal{F}, P)$ with homogeneous market makers in a set Ω_{mm}. Choose one of them as a reference market maker, whose states include time variable $t \in [0, T]$, asset reference price S_t, cash position X_t and the inventory position q_t. S_t is public information known to all market makers, whereas X_t and q_t are each market maker's private information. The reference asset price S_t is assumed to follow a Gaussian process

$$dS_t = \sigma dW_t,$$

where W is a standard Brownian motion adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, generated by W and augmented with all P-null sets, and σ is a constant representing asset volatility. The terminal time T is small, normally a day, the probability that S_t becomes negative is negligible and we may assume S_t is always positive. Market makers do not buy/sell the asset at the reference price, but at bid and ask prices, and make profit from the bid/ask spread. Denote by a a buying order and b a selling order. The reference market maker's bid price S_t^b and ask price S_t^a are given by

$$S_t^b = S_t - \delta_t^b, \quad S_t^a = S_t + \delta_t^a,$$

where δ_t^b and δ_t^a are the bid and ask spreads controlled by the reference market maker.

At time t, other market makers also quote bid and ask prices simultaneously to compete with the reference market maker. Among their quotes there exist a lowest ask price and a highest bid price, which are the most competitive prices other than reference market maker's prices. Denote by $\mathbf{k_a}$ the market maker who provides the lowest ask price $S^a_{\mathbf{k_a},t}$ (other than the reference market maker), and $\mathbf{k_b}$ the market maker who provides the highest bid price $S^b_{\mathbf{k_b},t}$ (other than the reference market maker), in other words, $\delta^b_{\mathbf{k_b},t}$ and $\delta^a_{\mathbf{k_a},t}$ are the lowest bid and ask spreads among the reference market maker's competitors.

Traders tend to sell/buy at the most competitive bid/ask price, but may accept less competitive prices due to other factors such as liquidation of large quantities. From the reference market maker's perspective, the arrival timing of buying/selling orders is unpredictable, while the intensities of them depend on reference market maker's bid/ask spreads as well as the most competitive spreads other than hers. On one hand, the intensity depends on the absolute size of bid/ask spread. If bid/ask spread are large, the quoted prices will be more away from traders' expected fair value, which decreases their willingness of trading. On the other hand, the intensity also depends on the relative ranking of market maker's spread compared to the most competitive spread in the market. The lower her bid/ask spreads compared relatively to that most competitive spreads, the more likely her bid/ask quotes are to be hit by traders. Hence as a simplification of reality, we assume the arrival intensity is decreasing in terms of reference market maker's spread and increasing in the most competitive spread. The number of selling market order arrival is denoted by N^b_t and that of buying market order is denoted by N^a_t. Both of them are Poisson processes with controlled intensities λ^b_t and λ^a_t, defined by

$$\lambda^a_t = f(\delta^a_t, \delta^a_{\mathbf{k_a},t}), \quad \lambda^b_t = f(\delta^b_t, \delta^b_{\mathbf{k_b},t}),$$

where f is the intensity function. Denote by f'_1 the first order partial derivative of f to its first variable, f''_{11} the second order partial derivative of f to its first variable, etc.

Assumption 3.2.1. Assume f is twice continuously differentiable and for all $\delta, x, y \in \mathbb{R}$, $f(\delta, x) > 0$, $f_1'(\delta, x) < 0$, $f_2'(\delta, x) \geq 0$, $\lim_{\delta \to +\infty} -\frac{f_1'(\delta,\delta)}{f(\delta,\delta)} > 0$, and

$$f(\delta, x) f_{11}''(\delta, y) - 2 f_1'(\delta, x) f_1'(\delta, y) + |f_1'(\delta, x) f_2'(\delta, y) - f_{12}''(\delta, y) f(\delta, x)| < 0. \tag{3.1}$$

Furthermore, assume there exists a twice continuously differentiable function $\lambda : \mathbb{R} \to \mathbb{R}$ such that $f(\delta, x) \leq \lambda(\delta)$ for all $x \in R$, $\lim_{\delta \to +\infty} \lambda(\delta)\delta = 0$ and $\lambda(\delta)\lambda''(\delta) < 2(\lambda'(\delta))^2$.

Some conditions in Assumption 3.2.1 are technical and needed in the proof. Many functions satisfy these conditions, for example, $f(\delta, x) = \lambda(\delta) g(x)$, where λ is the one in Assumption 3.2.1 with negative first order derivative and $\lim_{\delta \to +\infty} -\frac{\lambda'(\delta)}{\lambda(\delta)} > 0$, and g is increasing, positive and bounded. Here is another example:

$$f(\delta, x) := \frac{\Lambda e^{-a\delta}}{\sqrt{1 + 3e^{k(\delta - x)}}}, \tag{3.2}$$

where Λ is the magnitude of market order arrival rate, a the decay rate, k the dependence rate of the difference between reference market maker's price and the most competitive price in the market with $a \geq \frac{\sqrt{2}}{2} k > 0$. It is easy to check that f satisfies all conditions in Assumption 3.2.1. Some simple functions may not satisfy Assumption 3.2.1. For example, a constant function is excluded, if it were allowed, it would imply the size of bid/ask spread does not affect the arrival rate for market makers. In this case no matter how high the market makers set their bid/ask spread, there are always traders willing to trade with them. Then consequently every market maker would set their bid/ask spread as large as possible, which is clearly unrealistic.

Market makers are allowed to short sell the asset. It means the inventory can be negative. We assume there is an inventory position constraint for all market makers. Let $\mathbf{Q} = \{-Q, \cdots, Q\}$ be a finite set of integers with Q and $-Q$ the maximum and minimum positions a market maker may hold. Denote q_t by the reference market maker's inventory process and $q_t \in \mathbf{Q}$. When $q_t = Q$ (or $-Q$), market maker can not buy (or sell) any more. Denote by I^b and I^a the indicator functions of market maker's buying or selling capability:

$$I^b(q) := \mathbb{1}_{\{q+1\in\mathbf{Q}\}}, \quad I^a(q) := \mathbb{1}_{\{q-1\in\mathbf{Q}\}},$$

where $\mathbb{1}_A$ is an indicator that equals 1 if A is true and 0 if A is false. When market maker's bid price is hit by a market order (N_t^b increases by 1), her inventory q_t increases by 1 and she pays S_t^b for buying the asset. Similarly, when market maker's ask price is hit by a market order (N_t^a increases by 1), her inventory q_t decreases by 1 and she receives S_t^a for selling the asset. Market makers have inventory limit, due to reasons like desk risk limit. We assume when their inventory reach $-Q$ or Q, they can no longer increase their exposure, but only trade in opposite direction to decrease the exposure. It is equivalent to they will quote infinity for the corresponding ask (for $-Q$ inventory) or bid (for Q inventory) spread, which induces 0 intensity of order arrival. The dynamics of cash X_t and inventory q_t are given by

$$\begin{aligned} dX_t &= S_t^a I^a(q_t)dN_t^a - S_t^b I^b(q_t)dN_t^b \\ dq_t &= I^b(q_t)dN_t^b - I^a(q_t)dN_t^a \end{aligned}$$

with the initial condition $(X_0, q_0) = (x, q) \in \mathbb{R} \times \mathbf{Q}$.

The reference market maker does not have complete information on the whole market. Denote by $(\mathbf{x}_{k_b}, \mathbf{q}_{k_b})$ and $(\mathbf{x}_{k_a}, \mathbf{q}_{k_a})$ the states (cash and inventory) of market makers $\mathbf{k_b}$ and $\mathbf{k_a}$, respectively. They are random variables from the reference market maker's perspective, as her competitors' states are not public information. The reference market maker can only deduce the probability distribution for both $(\mathbf{x}_{k_b}, \mathbf{q}_{k_b})$ and $(\mathbf{x}_{k_a}, \mathbf{q}_{k_a})$ based on available public information. We assume their probability distributions are known and time-invariant. They are P_b for $(\mathbf{x}_{k_b}, \mathbf{q}_{k_b})$ and P_a for $(\mathbf{x}_{k_a}, \mathbf{q}_{k_a})$. This incomplete information assumption is a reasonable approximation of real market. We next use a heuristic example to illustrate the incomplete information setting and P_a and P_b.

Example 3.2.1. *Consider at time t there are 3 market makers quoting in the market including the reference market maker. They are independent and may have different states (cash and inventory). For each of them, all possible states and corresponding probability, bid/ask spread are assumed by following table.*

x	q	*Probability*	*Bid spread*	*Ask spread*
0	-1	$\frac{1}{3}$	*10 bps*	$+\infty$
0	0	$\frac{1}{3}$	*30 bps*	*30 bps*
0	1	$\frac{1}{3}$	$+\infty$	*10 bps*

For simplicity we assume they all have same cash position $x = 0$ and there are only three inventory possibilities $q = -1, 0, 1$. When market maker's inventory achieves maximum capacity, they simply quote $+\infty$ bid/ask spread to stop buying/selling ($\lim_{\delta \to +\infty} f(\delta, x) = 0$ for any x). Assume uniform probability on $q = -1, 0, 1$. When $q = -1$, market maker will prefer to buy than sell. Hence they will quote bid spread 10bps lower than their ask spread. For $q = 1$, it is the opposite. Denote the inventory of the reference market maker's two competitors as q_1 and q_2. We can derive P_a as following. $P_a(0, -1)$ is the probability that one of the two other market makers (other than reference market maker) who quotes the lowest ask spread has inventory -1. It implies both market makers have inventory $q_1 = q_2 = -1$, otherwise a lower ask spread 30 bps or 10 bps would be quoted if one of them had inventory 0 or 1.

$$P_a(0, -1) = P(q_1 = -1)P(q_2 = -1) = \frac{1}{9}.$$

Similarly, we can calculate $P_a(0, 0)$ and $P_a(0, 1)$ as following.

$$P_a(0, 0) = P(q_1 = -1)P(q_2 = 0) + P(q_1 = 0)P(q_2 = -1) + P(q_1 = 0)P(q_2 = 0) = \frac{1}{3}$$
$$P_a(0, 1) = 1 - (P_a(0, -1) + P_a(0, 0)) = \frac{5}{9}.$$

Note that above is just a extremely heuristic example to illustrate why usually P_a has higher density on states with positive inventory (For simplicity we even assume there is only one cash value $x = 0$). Market makers with positive inventory are more prone to sell than buy, hence they are more likely to be the one that quotes the lowest ask spread among all reference market maker's competitors. In reality, it might not be possible to calculate P_a and P_b as above example since the equilibrium bid/asks spread strategies have not yet been solved. But we assume market makers can still get P_a and P_b by empirical analysis on all public information.

Note that P_a and P_b can also be time dependent. It would not affect our results as the same proof still applies. For notation simplicity we only discuss the time invariant case. Moreover, after solving the equilibrium, one can also deduce the probability distributions for market makers' states at each time t under equilibrium. From our setting, they don't necessarily equal to P_a and P_b, since we assume market makers might quit or return to the market due to some unexpected interruption, like risk limit touched, trading system or algorithm upgrade, etc. These different factors all might make the states' distribution in reality differ from the one we solve from the equilibrium.

We assume market makers take closed loop feedback strategies that are deterministic functions of state variables at time t, that is, there exist functions δ^a and δ^b such that bid/ask spreads of market maker are given by

$$\delta_t^a = \delta^a(t, S, x, q), \quad \delta_t^b = \delta^b(t, S, x, q).$$

Definition 3.2.2. Denote by $\mathbf{A}^a$ and $\mathbf{A}^b$ the admissible strategy set. They are the sets of all δ^a and δ^b that are lower bounded square integrable measurable functions of t, S and x for all possible q.

$\boldsymbol{\delta} := (\delta^b, \delta^a) \in \mathbf{A}^b \times \mathbf{A}^a$ reference market maker's strategy, $\vec{\boldsymbol{\delta}}_\Omega := \{\boldsymbol{\delta}_m, m \in \Omega_{mm}\}$ the collection of all market makers' strategies, so reference market maker's strategy $\boldsymbol{\delta} \in \vec{\boldsymbol{\delta}}_\Omega$. Using the game theory convention, we may label the reference market maker as 0 and $\vec{\boldsymbol{\delta}}_{-0}$ the set of strategies of all other market makers in Ω_{mm} except the reference market maker, i.e., $\vec{\boldsymbol{\delta}}_{-0} := \{\boldsymbol{\delta}_m, m \neq 0, m \in \Omega_{mm}\}$.

All strategies in $\vec{\boldsymbol{\delta}}_{\Omega}$ can influence reference market maker's expected intensity of market order arrival, as everyone else in Ω_{mm} can be reference market maker's competitor when a market order arrives. Hence reference market maker's cash X_t and inventory q_t are determined by her own strategy $\boldsymbol{\delta}$ as well as implicitly by those in the set $\vec{\boldsymbol{\delta}}_{-0}$. Starting at time $t \in [0, T]$ with initial asset price S, cash x and inventory q, the reference market maker wants to maximize the following payoff function:

$$J(\boldsymbol{\delta}, \vec{\boldsymbol{\delta}}_{-0}, t, S, x, q) = \mathbb{E}_t[X_T + q_T S_T - l(|q_T|) - \frac{1}{2}\gamma\sigma^2 \int_t^T (q_s)^2 ds], \tag{3.3}$$

where $\mathbb{E}_t$ is the conditional expectation operator given $S_t = S$, $X_t = x$ and $q_t = q$. The reference market maker wants to maximize the expected value of terminal wealth, but holding inventory is penalized both at terminal time T (denote by l, an increasing convex function on R_+ with $l(0) = 0$), and throughout the time interval $[0, T]$ (denoted by the integral term in (3.3) with γ a positive constant representing the risk adverse level).

Market makers' payoffs depend on each other's strategy, which again depend on each other's payoff. Due to the circular dependence nature among market makers' strategies and payoffs, we use a game theoretic approach to solve the problem. We next define the Nash equilibrium.

Definition 3.2.3. We call the Nash equilibrium exists for a game G_{mm} if there exists an equilibrium control profile $\vec{\boldsymbol{\delta}}^*_{\Omega} = \{\boldsymbol{\delta}^*_m, m \in \Omega_{mm}\}$, such that for every reference player 0 in Ω_{mm}, given her strategy $\boldsymbol{\delta}^* \in \vec{\boldsymbol{\delta}}^*_{\Omega}$ and other players' strategy set $\vec{\boldsymbol{\delta}}^*_{-0}$, her payoff satisfies the following equilibrium condition:

$$J(\boldsymbol{\delta}^*, \vec{\boldsymbol{\delta}}^*_{-0}, t, S, x, q) = \max_{\boldsymbol{\delta} \in \mathbf{A}^b \times \mathbf{A}^a} J(\boldsymbol{\delta}, \vec{\boldsymbol{\delta}}^*_{-0}, t, S, x, q). \tag{3.4}$$

Moreover, the reference market maker's equilibrium control is $\boldsymbol{\delta}^*$ and the equilibrium value function is

$$V(t,S,x,q) := J(\boldsymbol{\delta}^*, \vec{\boldsymbol{\delta}}^*_{-0}, t, S, x, q). \tag{3.5}$$

3.3 Existence and Uniqueness of Dynamic Equilibrium

In this section, we prove the existence and uniqueness of Nash equilibrium for G_{mm} when price competition is in place. We first reduce the model's dimension by ansatz, then characterize the equilibrium value function by a system of nonlinear ODEs, and prove the verification theorem, finally show the existence and uniqueness of Nash equilibrium by an equivalent ODE system.

Writing the integral form of X_T and q_T in payoff function (3.3) with Ito's lemma, we can simplify the equilibrium value function V as

$$V(t,S,x,q) = x + qS + \theta_q(t), \tag{3.6}$$

where $\theta_q : [0,T] \to \mathbb{R}$ is defined by

$$\theta_q(t) = \sup_{\boldsymbol{\delta} \in \mathbf{A}^b \times \mathbf{A}^a} \mathbb{E}_t[\int_t^T [\delta_s^a f(\delta_s^a, \delta^a_{\mathbf{k_a},s}) + \delta_s^b f(\delta_s^b, \delta^b_{\mathbf{k_b},s}) - \frac{1}{2}\gamma\sigma^2 q_s^2] ds - l(|q_T|)] \tag{3.7}$$

with $\mathbb{E}_t$ being the conditional expectation operator given $q_t = q$. The reason why θ is independent on x and S is because each reference market maker is only optimizing against an expected intensity function that does not depend on other market makers' state, but only on their distributions, which are assumed to be known and given. Assume there is equilibrium such that market makers with same inventory have different value function (dependent on their x and S), then given the equilibrium strategies which are function of t, S, x and q, let's consider the optimization problem for certain reference market maker. In this case, it is simply equivalent to that the reference market maker is solving her own market making optimization problem similar to the one in (Guéant (2017)), but with a time dependent intensity function. In this case, the same dimension reduction techniques in (Guéant (2017)) can be applied and the unique optimal solution can be found. As suggested by (Guéant (2017)), the optimal bid/ask spread would not depend on x and S. It contradicts our assumption that equilibrium strategies depend on x and S. Hence we know both equilibrium strategies and θ defined in above are independent on x and S.

Since process q_t takes value in a finite set $\mathbf{Q}$, it is a Markov chain with $M = 2Q + 1$ states. Hence game G_{mm} is reduced to a continuous time finite state stochastic game. Define a function $\theta : [0, T] \to \mathbb{R}^M$ as

$$\theta(t) = (\theta_{-Q}(t), \cdots, \theta_Q(t)). \tag{3.8}$$

The equilibrium bid/ask spreads only depend on state q_t at time t. As market makers are homogeneous, under equilibrium at time t, any two market makers with the same state q quote the same bid/ask spread, denoted by $\pi_q^b(t)$ and $\pi_q^a(t)$ respectively. Note that $\pi_q^b(t)$ exists for every $q \in \mathbf{Q}$ except $q = Q$ when market maker reaches the maximum inventory and stops quoting bid price. $\pi_q^a(t)$ is similarly defined. We can define the equilibrium control as

$$\pi^a(t) = (\pi_{-Q+1}^a(t), \cdots, \pi_Q^a(t)), \quad \pi^b(t) = (\pi_{-Q}^b(t), \cdots, \pi_{Q-1}^b(t)).$$

The market maker's equilibrium control $\boldsymbol{\delta}^* = ((\delta^a)^*, (\delta^b)^*)$ is given by

$$(\delta^a)^*(t, S, x, q) = \pi_q^a(t), \quad (\delta^b)^*(t, S, x, q) = \pi_q^b(t). \tag{3.9}$$

When market order arrives at time t, the reference market maker expects her most competitive market maker in bid side to have inventory q with probability P_q^b and in ask side P_q^a. As there are only finite number of states, the most competitive market maker's state probability is given by:

$$P^a = (P^a_{-Q+1}, \cdots, P^a_Q), \quad P^b = (P^b_{-Q}, \cdots, P^b_{Q-1}).$$

Market makers with inventory on boundary values do not quote in the market, so $P^a_{-Q} = P^b_Q = 0$.

We next provide a characterization for the value function θ and the equilibrium controls π^a, π^b. Applying the dynamic programming principle, we get the following Hamilton Jacobi ODE system:

$$\begin{aligned}
&\theta'_q(t) = \frac{1}{2}\gamma\sigma^2 q^2 - \sup_\delta \eta^a(\theta(t), \delta, \pi^a(t), q) I^a(q) - \sup_\delta \eta^b(\theta(t), \delta, \pi^b(t), q) I^b(q) \\
&\theta_q(T) = -l(|q|) \\
&\pi^a_q(t) \in \operatorname*{argsup}_\delta \eta^a(\theta(t), \delta, \pi^a(t), q), \quad \forall q \in \{-Q+1, \cdots, Q\} \\
&\pi^b_q(t) \in \operatorname*{argsup}_\delta \eta^b(\theta(t), \delta, \pi^b(t), q), \quad \forall q \in \{-Q, \cdots, Q-1\},
\end{aligned} \tag{3.10}$$

where $\eta^a, \eta^b : \mathbb{R}^M \times \mathbb{R} \times \mathbb{R}^{M-1} \times \mathbf{Q} \to \mathbb{R}$ are defined by vectors $\theta = (\theta_{-Q}, \cdots, \theta_Q) \in \mathbb{R}^M$, $\pi^a = (\pi^a_{-Q+1}, \cdots, \pi^a_Q)$ or $\pi^b = (\pi^b_{-Q}, \cdots, \pi^b_{Q-1})$ as

$$\begin{aligned}
\eta^a(\theta, \delta, \pi^a, q) &:= \sum_{j=-Q+1}^{Q} P^a_j f(\delta, \pi^a_j)(\delta + \theta_{q-1} - \theta_q) \\
\eta^b(\theta, \delta, \pi^b, q) &:= \sum_{j=-Q}^{Q-1} P^b_j f(\delta, \pi^b_j)(\delta + \theta_{q+1} - \theta_q).
\end{aligned} \tag{3.11}$$

Note that $\sum_{j=-Q+1}^{Q} P^a_j f(\delta, \pi^a_j(t))$ and $\sum_{j=-Q}^{Q-1} P^b_j f(\delta, \pi^b_j(t))$ are market maker's expected intensity of buying/selling market order arrival when her spread is δ and other market makers take the equilibrium control. We can now characterize the Nash equilibrium.

Theorem 3.3.1. *Assume the Nash equilibrium of the game G_{mm} exists. Then the equilibrium value function V can be decomposed as (3.6) with function θ. Equilibrium* *control $\boldsymbol{\delta}^*$ can be written as (3.9) with two vectors $\pi^a(t)$ and $\pi^b(t)$. Moreover, θ, $\pi^a(t)$ and $\pi^b(t)$ satisfy the ODE system in (3.10).*

The optimality condition that $\pi^a(t)$ and $\pi^b(t)$ satisfy in (3.10) leads to the following generalized Issac condition, which is also defined in Cohen and Fedyashov (2017) to guarantee the existence of Nash equilibrium for non-zero-sum stochastic differential game. It is a natural extension of the standard assumptions (Issac condition) used in the zero-sum game to the non-zero-sum game.

Definition 3.3.2. We call the generalized Issac condition holds if there exist functions $w^a, w^b : \mathbb{R}^M \to \mathbb{R}^{M-1}$ such that for any vector $\mu \in \mathbb{R}^M$,

$$\begin{aligned} \eta^a(\mu, w_q^a(\mu), w^a(\mu), q) &= \sup_\delta \eta^a(\mu, \delta, w^a(\mu), q), \quad \forall q \in \{-Q+1, \cdots, Q\} \\ \eta^b(\mu, w_q^b(\mu), w^b(\mu), q) &= \sup_\delta \eta^b(\mu, \delta, w^b(\mu), q), \quad \forall q \in \{-Q, \cdots, Q-1\}, \end{aligned} \tag{3.12}$$

where $w_q^a, w_q^b : \mathbb{R}^M \to \mathbb{R}$ and w^a, w^b are defined by

$$w^a(\mu) := (w^a_{-Q+1}(\mu), \cdots, w^a_Q(\mu)), \quad w^b(\mu) := (w^b_{-Q}(\mu), \cdots, w^b_{Q-1}(\mu)).$$

If the generalized Issac condition is satisfied, we can substitute the function w^a, w^b into operators η^a, η^b, and the system (3.10) is reduced to the following ODE system:

$$\begin{aligned} &\theta_q'(t) = \frac{1}{2}\gamma\sigma^2 q^2 - \eta^a(\theta(t), w_q^a(\theta(t)), w^a(\theta(t)), q) I^a(q) \\ &- \eta^b(\theta(t), w_q^b(\theta(t)), w^b(\theta(t)), q) I^b(q) \\ &\theta_q(T) = -l(|q|). \end{aligned} \tag{3.13}$$

We next state the verification theorem.

Theorem 3.3.3. *Assume that f satisfies Assumption 3.2.1, that there exist bounded strategies π^a, π^b and function θ on $[0, T]$ satisfying the system (3.10). Then the Nash equilibrium of the game G_{mm} exists. The equilibrium value function is given by (3.6)* *and the equilibrium control by (3.9).*

From Theorems 3.3.1 and 3.3.3 we know the existence and uniqueness of Nash equilibrium for game G_{mm} are equivalent to the existence and uniqueness of equilibrium controls π^a, π^b and function θ that satisfy the ODE system (3.10). We now state the main result of this chapter.

Theorem 3.3.4. *Assume f satisfies Assumption 3.2.1. Then there exists a unique Nash equilibrium for game G_{mm}. Specifically, there exist unique locally Lipschitz continuous functions w^a, w^b that satisfy generalized Issac condition in Definition 3.3.2, and there exists unique classical solution θ to the ODE system (3.13), such that the equilibrium value function is given by (3.6) and the equilibrium controls by*

$$\pi^a(t) = w^a(\theta(t)),\ \pi^b(t) = w^b(\theta(t)),\ t \in [0, T]. \tag{3.14}$$

3.4 Numerical Test

In this section, we numerically find the Nash equilibrium value function and bid/ask spread when there is price competition with the intensity f defined in (3.2) and compare the numerical results with those derived using a benchmark model in Guéant (2017) without price competition and with the intensity $\tilde{f}(\delta) := 0.5\Lambda e^{-a\delta}$ and the liquidity penalty $l(q) := 0.1q^2$. To make two models comparable, we define parameters for f and $\tilde{f}$ in a way that when every market maker provides the same bid/ask spread, the intensity of market order arrivals is the same in both cases, which gives 0.5Λ in the definition of $\tilde{f}$. The parameters of both models are set as follows:

S	σ (daily)	γ	k	a	Λ	T (day)	N	Q
20.0	0.01	1.0	2.0	2.0	60.0	1.0	100	10

Here S is the initial asset value, N the number of time steps in discretization, T the period of one day, σ the daily volatility, a and Λ used in intensity functions, γ

inventory penalty coefficient, and Q the inventory maximum capacity. Furthermore, probabilities of the most competitive market makers' state P^a and P^b are assumed to be given by (see Example 3.2.1 for explanation of P^a and P^b)

$$\begin{aligned}
&P^a_{-10} = P^b_{10} = 0\\
&P^a_0 = P^b_0 = 0.2\\
&P^a_1 = P^b_{-1} = 0.4\\
&P^a_2 = P^b_{-2} = 0.3\\
&P^a_q = 1/170, \quad q \neq -10, 0, 1, 2\\
&P^b_q = 1/170, \quad q \neq 10, 0, -1, -2.
\end{aligned}$$

The rationale of designing above probability is given as following. Market makers tend to have neutral position. When they have positive position, they are more prone to sell them than buy further. That is why most market makers' position would focus on -1, 0, 1, with decreasing values for probability of larger size of (positive or negative) inventory.

Figures 3.1 and 3.2 plot the optimal bid/ask spreads of both models at time 0.5. We note that higher inventory leads to lower ask spread but higher bid spread, indicating the preference of market makers to sell rather than to buy in order to remain inventory neutral, and that the equilibrium bid/ask spreads of our model are tighter than those of the benchmark model, indicating improved market liquidity.

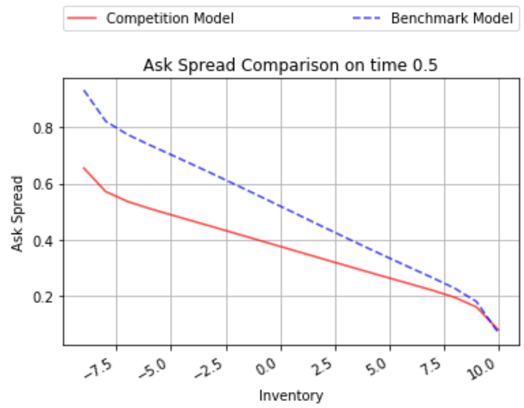


Figure 3.1: Ask spread strategy profile at time 0.5

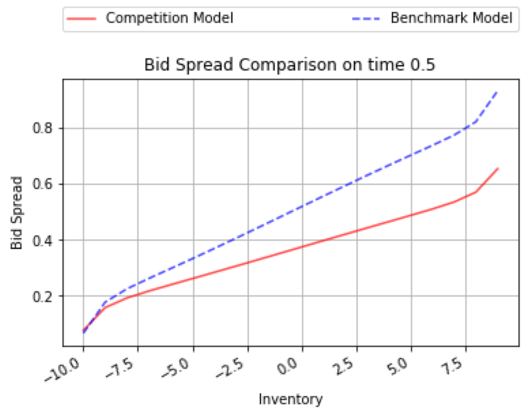


Figure 3.2: Bid spread strategy profile at time 0.5

Figure 3.3 plots the equilibrium ask spreads with different inventory levels on $[0, T]$. Market makers with positive inventory are more willing to sell and clear their positions due to the liquidity punishment at terminal time T, and this willingness increases as time nears T as the equilibrium ask spread is decreasing when t tends to T. They might even be willing to quote a negative ask spread, which means selling the asset under its fair value, in order to return to neutral position. That is because they deem they would suffer more if they hold the position and are forced to liquidate their position at time T, which incurs a liquidity punishment. For market makers with negative inventory, it is opposite. This explains empirical facts that trading volume increases at the end of the day.

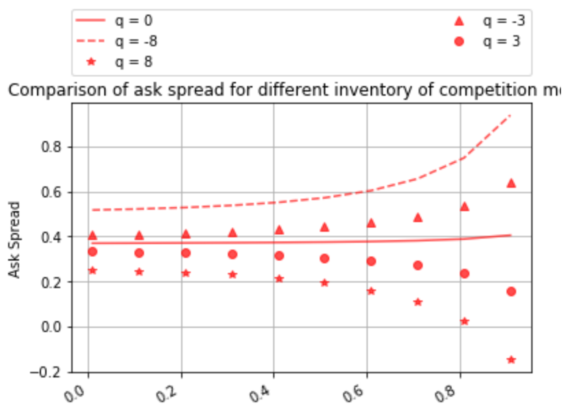


Figure 3.3: Optimal Ask Spread for Competition Model

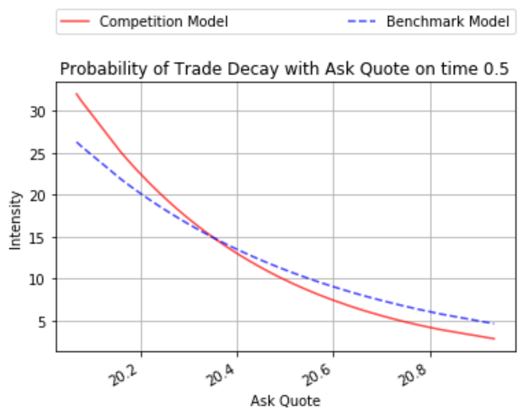


Figure 3.4: Intensity v.s ask quote at time 0.5

Figure 3.4 plots the expected intensity functions in terms of bid/ask spread at time 0.5, which are given by $G_b(\delta) = \tilde{f}(\delta)$ for the benchmark model and $G(\delta) = \sum_{j=-Q+1}^{Q} P_j^a f(\delta, \pi_j^a(t))$ for our model, respectively. The one from our model is derived endogenously from equilibrium (note that though the state distribution P_a and P_b are still exogenous, equilibrium strategies profile π^a and π^b are derived endogenously). But the one assumed by the benchmark model comes from Avellaneda and Stoikov (2008) in which the distribution of of market order size and the statistics of the market impact are used. When price competition is in place, the market order arrival intensity decays faster, indicating that when price competition is in place but market maker assumes there were not, they would tend to overestimate the market order arrival intensity and quote higher bid/ask spreads.

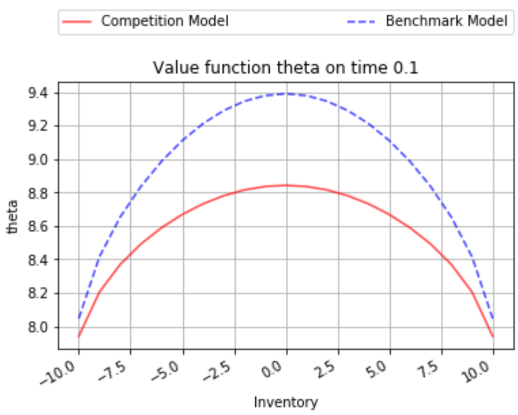


Figure 3.5: Value function θ at time 0.1

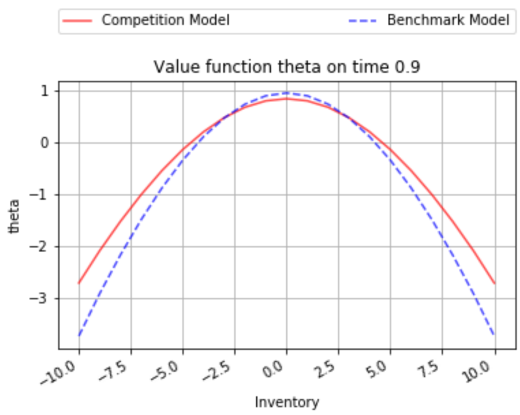


Figure 3.6: Value function θ at time 0.9

Figures 3.5 and 3.6 plot the equilibrium value function θ near the starting time 0 and the terminal time T, respectively. We notice that θ with price competition takes lower value than the one without at time 0.1 but performs better at time 0.9, especially when there are still large inventories to be liquidated. The reason is given following. Intuitively, the value function mainly depends on two factors. One is the profit from bid/ask spread, while the other one is the liquidity punishment at terminal time. At time 0.9 which is close to terminal time, the profit from bid/ask spread is less important since not much time left to collect profit from the spread. Market makers with non-zero inventory have larger chances to liquidate their positions than those in benchmark model, as their spreads are aggressive and lower than most of other market makers, as well as market makers with same inventory in benchmark model (see Figures 3.3 and 3.4). Comparing intensity functions from competition model and benchmark model, $\frac{\Lambda e^{-a\delta_1}}{\sqrt{1+3e^{k(\delta_1-x)}}} > 0.5\Lambda e^{-a\delta_2}$ when δ_1 is smaller than x and δ_2. Hence the order flow intensity that enable market makers in competition model with non-zero inventory to liquidate their positions are generally larger than those in benchmark model. Then they are exposed to less potential liquidity punishment, and higher value functions. On the other hand, when it is at time 0.1, the profit from bid/ask spread is more important, and market makers in competition model has lower value function since their bid/ask spreads are generally lower than the benchmark model (see Figures 3.3).

In summary, when price competition between market makers is in place, market maker tends to quote tighter bid/ask spreads and the market has better liquidity and lower transaction cost. However, the profit of market maker is reduced. The value function is lower when there is competition between market makers.

3.5 Proofs of Theorems 3.3.3 and 3.3.4

3.5.1 Proof of Theorem 3.3.3

Proof. To verify that $(\vec{\boldsymbol{\delta}}_\Omega)^*$ is the equilibrium control profile and V is the equilibrium value function, it is sufficient to check that they satisfy the equilibrium condition in (3.4). For any market maker in Ω_{mm}, given other market makers' strategies in $(\vec{\boldsymbol{\delta}}_\Omega)^*$ and any admissible strategy $\boldsymbol{\delta}$ defined in Definition 3.2.2 we should prove:

$$J(\boldsymbol{\delta}, (\vec{\boldsymbol{\delta}}_{-0})^*, t, S, x, q) \leq J((\boldsymbol{\delta})^*, (\vec{\boldsymbol{\delta}}_{-0})^*, t, S, x, q) = V(t, S, x, q).$$

Let the reference market maker takes the arbitrary admissible strategy $\boldsymbol{\delta}$, while every other market maker decide their bid/ask spread by

$$(\delta^a)^*(t, S_t, X_t, q_t) = \pi^a_{q_t}(t), \quad (\delta^b)^*(t, S_t, X_t, q_t) = \pi^b_{q_t}(t).$$

As π^a and π^b are bounded, they are also square integrable and hence admissible. Denote reference market maker's cash position at any time t as $X_t^{*,\boldsymbol{\delta}}$, while their inventory is $q_t^{*,\boldsymbol{\delta}}$. We want to prove V defined by (3.6) via the solution θ to the ODE system (3.13) is indeed the equilibrium value function. For any time $t \in [0, T]$, by the terminal condition of V and θ, as well as the Ito lemma with respect to function θ, we get following.

$$\begin{aligned} V(T, S_T, X_T^{*,\boldsymbol{\delta}}, q_T^{*,\boldsymbol{\delta}}) &= X_T^{*,\boldsymbol{\delta}} + q_T^{*,\boldsymbol{\delta}} S_T + \theta_{q_T^{*,\boldsymbol{\delta}}}(T) = x + qS + \theta_q(t) \\ &+ \int_t^T \delta_u^b I^b(q_u^{*,\boldsymbol{\delta}}) dN_u^b + \int_t^T \delta_u^a I^a(q_u^{*,\boldsymbol{\delta}}) dN_u^a + \int_t^T q_u^{*,\boldsymbol{\delta}} dS_u + \int_t^T \theta'_{q_u^{*,\boldsymbol{\delta}}}(u) du \\ &+ \int_t^T (\theta_{q_u^{*,\boldsymbol{\delta}}+1}(u) - \theta_{q_u^{*,\boldsymbol{\delta}}}(u)) I^b(q_u^{*,\boldsymbol{\delta}}) dN_u^b + \int_t^T (\theta_{q_u^{*,\boldsymbol{\delta}}-1}(u) - \theta_{q_u^{*,\boldsymbol{\delta}}}(u)) I^a(q_u^{*,\boldsymbol{\delta}}) dN_u^a. \end{aligned} \tag{3.15}$$

The first equality in (3.15) holds as V in this case is defined by (3.6) via θ. As $q_u^{*,\boldsymbol{\delta}}$ takes value in finite set $\mathbf{Q}$, and the solution for ODE exists on compact set $[0, T]$, we know both $\theta_q(u)$ and $\theta'_q(u)$ is uniformly bounded on $[0, T]$ for all $q \in \mathbf{Q}$ and:

$$\mathbb{E}[\int_t^T (q_u^{*,\boldsymbol{\delta}})^2 du] < +\infty, \quad \mathbb{E}[\int_t^T (\theta'_{q_u^{*,\boldsymbol{\delta}}}(u))^2 du] < +\infty.$$

Moreover, as θ is the unique solution to ODE system (3.13), it is bounded and hence integrable over $[0, T]$. From assumption of f and we assume admissible control, using similar argument to the verification theorem in Guéant (2017), we have,

$$\mathbb{E}[\sum_{j=-Q+1}^{Q} P_j^a \int_t^T f(\delta_u^a, \pi_j^a(t)) I^a(q_u^{*,\boldsymbol{\delta}}) |\delta_u^a + \theta_{q_u^{*,\boldsymbol{\delta}}-1}(u) - \theta_{q_u^{*,\boldsymbol{\delta}}}(u)| du] < +\infty$$

$$\mathbb{E}[\sum_{j=-Q}^{Q-1} P_j^b \int_t^T f(\delta_u^b, \pi_j^b(t)) I^b(q_u^{*,\boldsymbol{\delta}}) |\delta_u^b + \theta_{q_u^{*,\boldsymbol{\delta}}+1}(u) - \theta_{q_u^{*,\boldsymbol{\delta}}}(u)| du] < +\infty.$$

Take expectation on both side of (3.15), we have:

$$\mathbb{E}[V(T, S_T, X_T^{*,\boldsymbol{\delta}}, q_T^{*,\boldsymbol{\delta}})] = V(t, S, x, q) + \mathbb{E}[\int_t^T \eta^a(\theta(u), \delta_u^a, \pi^a(u), q_u^{*,\boldsymbol{\delta}}) I^a(q_u^{*,\boldsymbol{\delta}}) du]$$
$$+ \mathbb{E}[\int_t^T \eta^b(\theta(u), \delta_u^b, \pi^b(u), q_u^{*,\boldsymbol{\delta}}) I^b(q_u^{*,\boldsymbol{\delta}}) du] + \mathbb{E}[\int_t^T \theta'_{q_u^{*,\boldsymbol{\delta}}}(u) du].$$

where η^a and η^b are defined in (3.11). Hence we have:

$$\mathbb{E}[V(T, S_T, X_T^{*,\boldsymbol{\delta}}, q_T^{*,\boldsymbol{\delta}})] \leq V(t, S, x, q) + \mathbb{E}[\int_t^T \sup_{\delta_u^a} \eta^a(\theta(u), \delta_u^a, \pi^a(u), q_u^{*,\boldsymbol{\delta}}) I^a(q_u^{*,\boldsymbol{\delta}}) du]$$
$$+ \mathbb{E}[\int_t^T \sup_{\delta_u^b} \eta^b(\theta(u), \delta_u^b, \pi^b(u), q_u^{*,\boldsymbol{\delta}}) I^b(q_u^{*,\boldsymbol{\delta}}) du] + \mathbb{E}[\int_t^T \theta'_{q_u^{*,\boldsymbol{\delta}}}(u) du]. \tag{3.16}$$

As θ satisfies ODE system (3.10) for every $u \in [0, T]$. We substitute it into the corresponding part in (3.16) and have following.

$$J(\boldsymbol{\delta}, (\vec{\boldsymbol{\delta}}_{-0})^*, t, S, x, q) = \mathbb{E}[V(T, S_T, X_T^{*,\boldsymbol{\delta}}, q_T^{*,\boldsymbol{\delta}}) - \frac{1}{2}\gamma\sigma^2 \int_t^T (q_u^{*,\boldsymbol{\delta}})^2 du] \leq V(t, S, x, q).$$

On the other hand, if the reference market maker also take equilibrium control, her cash position and inventory are denoted by X_t^* and q_t^* respectively. And we have following.

$$\eta^a(\theta(t), \pi_q^a(t), \pi^a(t), q) = \sup_{\delta} \eta^a(\theta(t), \delta, \pi^a(t), q)$$
$$\eta^b(\theta(t), \pi_q^b(t), \pi^b(t), q) = \sup_{\delta} \eta^b(\theta(t), \delta, \pi^b(t), q).$$

Substituting the equilibrium control defined in (3.9) to (3.16) can conclude the proof as following:

$$J((\boldsymbol{\delta})^*, (\vec{\boldsymbol{\delta}}_{-0})^*, t, S, x, q) = \mathbb{E}[V(T, S_T, X_T^*, q_T^*) - \frac{1}{2}\gamma\sigma^2 \int_t^T (q_u^*)^2 du]$$
$$= V(t, S, x, q) \geq J(\boldsymbol{\delta}, (\vec{\boldsymbol{\delta}}_{-0})^*, t, S, x, q).$$

□

3.5.2 Proof of Theorem 3.3.4

The proof of Theorem 3.3.4 is made of three steps:

1. There exist functions w^a, w^b such that for any vector $\mu \in \mathbb{R}^M$, $w^a(\mu)$ and $w^b(\mu)$ satisfy equation (3.12).
2. w^a and w^b are unique and locally Lipschitz continuous, which guarantees RHS of the ODE system (3.13) are also locally Lipschitz continuous.
3. There exists unique classical solution to ODE system (3.13).

The key step for proving Steps 1 and 2 is to characterize the vectors $w^a(\mu)$ and $w^b(\mu)$ by the first order condition of Hamiltonian. They are the solution to some equation system. Then we can prove step 1 and 2 by discussing the zero point for the equation system. The key step for proving Step 3 is to obtain upper bound estimation for θ. It can be done by showing θ is also a solution to another system of ODE, which admits the comparison principle, and hence upper bound for its solution. Without confusion of notations, we write $w^a(\mu)$ and $w^b(\mu)$ as,

$$w^a(\mu) = w^a = (w^a_{-Q+1}, \cdots, w^a_Q), \quad w^b(\mu) = w^b = (w^b_{-Q}, \cdots, w^b_{Q-1}).$$

Proof of Step 1

We first show that w^a and w^b satisfy the optimal condition of the Hamiltonian. We provide some preliminary results for the existence and uniqueness of the maximum point for Hamiltonian $\eta^a(\mu, \delta, w, q)$ given any vector $\mu \in \mathbf{R}^M$ and $w \in \mathbf{R}^{M-1}$. We can prove the same result similarly for $\eta^b(\mu, \delta, w, q)$.

Lemma 3.5.1. *Assume intensity function f satisfies all the assumptions in Theorem 3.2.1. Then given any vectors $w = (w_{-Q+1}, \cdots, w_Q) \in \mathbb{R}^{M-1}$ and μ, the maximum point w.r.t δ exists and is unique for function $\eta^a(\mu, \delta, w, q)$ when $q = -Q+1, \cdots, Q$. Furthermore, the maximum point of $\eta^a(\mu, \delta, w, q)$ satisfies the first order condition:*

$$\frac{d\eta^a}{d\delta}(\mu, \delta, w, q) = 0.$$

Proof. Given any vector μ and w, the expected intensity function d is defined by

$$d(\delta) := \sum_{j=-Q+1}^{Q} P^a_{j2} f(\delta, w_j).$$

From Assumption 3.2.1, we know for any δ, x and y:

$$f(\delta, x) f''_{11}(\delta, y) + f(\delta, y) f''_{11}(\delta, x) < 4 f'_1(\delta, x) f'_1(\delta, y). \tag{3.17}$$

Simple calculation shows

$$d(\delta) d''(\delta) < 2(d'(\delta))^2,$$

which implies $\delta + \mu_{q-1} - \mu_q + d(\delta)/d'(\delta)$ is a strictly increasing function of δ. It means $\frac{d\eta^a}{d\delta}(\mu, \delta, w, q) = 0$ at most has one root. Assume it has no root, which means either $\eta^a(\mu, \delta, w, q)$ is strictly increasing or strictly decreasing w.r.t δ. As it can be easily deduced that $\lim_{\delta \to +\infty} \eta^a(\mu, \delta, w, q) = 0$ and for certain $\delta > -\mu$, $\eta^a(\mu, \delta, w, q) > 0$, $\eta^a(\mu, \delta, w, q)$ can only be decreasing function w.r.t δ. However, when $\delta < -\mu$, $\eta^a(\mu, \delta, w, q) < 0$. It suggests $\eta^a(\mu, \delta, w, q)$ can not be decreasing either. Therefore there is unique root for $\frac{d\eta^a}{d\delta}(\mu, \delta, w, q) = 0$. There exists a unique δ^* such that $\frac{d\eta^a}{d\delta}(\mu, \delta, w, q) = 0$ and $\eta^a(\mu, \delta, w, q)$ is strictly increasing for $\delta < \delta^*$ and strictly decreasing for $\delta > \delta^*$, that is, δ^* is the unique global maximum point of $\eta^a(\mu, \delta, w, q)$. □

Step 1 is equivalent to following theorem, which proves that generalized Issac condition in Definition 3.3.2 holds for any vector $\mu \in \mathbb{R}^M$. We only focus on w^a, as the proof of w^b is similar.

Theorem 3.5.2. *Assume the intensity function f satisfies Assumption 3.2.1. Then for any fixed vector $\mu = (\mu_{-Q}, \cdots, \mu_Q) \in \mathbb{R}^M$, there is vector $w^a = (w^a_{-Q+1}, \cdots, w^a_Q)$ such that for $q = -Q + 1, \cdots, Q$,*

$$w^a_q = \underset{\delta}{\operatorname{argmax}}\{\eta^a(\mu, \delta, w^a, q)\}. \tag{3.18}$$

Define a mapping $T : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ as

$$\begin{aligned} T_q(w) &= \underset{\delta \in \mathbb{R}}{\operatorname{argmax}}\{\eta^a(\mu, \delta, w, q)\}, \quad \forall q \in \{-Q + 1, \cdots, Q\} \\ T(w) &:= (T_{-Q+1}(w), \cdots, T_Q(w)), \end{aligned} \tag{3.19}$$

(3.18) is equivalent to $w^a = T(w^a)$, namely, w^a is a fixed point of mapping T. We need the Schauder Fixed Point Theorem 2.2.8 to prove the existence of w^a. To apply Theorem 2.2.8, we need to show the existence of K and the continuity of T. The next lemma confirms the first requirement.

Lemma 3.5.3. *Given any vector $\mu = (\mu_{-Q} \cdots, \mu_Q) \in \mathbb{R}^M$ and mapping T defined in (3.19), there exists a nonempty convex compact set $K \subset \mathbb{R}^{M-1}$ such that $T(K) \subset K$.*

Proof. Firstly, for any vector $w \in \mathbb{R}^{M-1}$, define $\vec{y} = (y_{-Q+1}, \cdots, y_Q) = T(w)$. There exist a uniform $\delta_{min} \in \mathbb{R}$ such that for every q,

$$y_q \geq \delta_{min}. \tag{3.20}$$

We can prove by contradiction. Assume there were no lower bound for y_q. Define $G_q^a(\delta) = \eta^a(\mu, \delta, y, q)$ for $q = -Q+1, \cdots, Q$, we know

$$y_q = \underset{\delta}{\operatorname{argmax}}\{G_q^a(\delta)\}.$$

Denote the uniform upper bound and lower bound of $\mu_{q-1} - \mu_q$ among all $q \in \mathbf{Q}$ as M_d and m_d. We have

$$y_q > -M_d.$$

Otherwise, $G_q^a(y_q) < 0$ and contradicts with the fact that $y_q = \operatorname{argmax}_\delta\{G_q^a(\delta)\}$, and $G_q^a(\delta) > 0$ as long as $\delta > -(\mu_{q-1} - \mu_q)$. Hence we can conclude that

$$y_q \geq \delta_{min} := -M_d.$$

Secondly, for any vector $w \in [\delta_{min}, +\infty)^{M-1}$, define $\vec{y} = (y_{-Q+1}, \cdots, y_Q) = T(w)$. There exists a uniform $\delta_{max} \in \mathbb{R}$ such that for every q,

$$y_q \leq \delta_{max}. \tag{3.21}$$

Define $\delta_0 := -m_d + 1$. By definition of m_d, for every q we have

$$\delta_0 + \mu_{q-1} - \mu_q \geq 1 > 0.$$

Hence for every $q \in \mathbf{Q}$, $G_q^a(\delta_0) > 0$. Moreover, as f is increasing to its second argument, for any vector $w \in [\delta_{min}, +\infty)^{M-1}$, we have:

$$G_q^a(\delta_0) \geq \sum_{j=-Q+1}^{Q} P_j^a f(\delta_0, \delta_{min}). \tag{3.22}$$

By assumption $\lim_{\delta \to +\infty} \lambda(\delta)\delta = 0$, there exists $\delta_{max} > \delta_0$ such that

$$\max_q \{ \sum_{j=-Q+1}^{Q} P_j^a \lambda(\delta_{max})(\delta_{max} + \mu_{q-1} - \mu_q) \} < \sum_{j=-Q+1}^{Q} P_j^a f(\delta_0, \delta_{min}). \tag{3.23}$$

As $f(\delta_{max}, \cdot)$ is bounded by $\lambda(\delta_{max})$ uniformly, (3.22) and (3.23) imply that for any vector $w \in [\delta_{min}, +\infty)^{M-1}$,

$$\max_q G_q^a(\delta_{max}) < G_q^a(\delta_0).$$

Since $\delta_{max} > \delta_0$ and $G_q^a(\delta_{max}) < G_q^a(\delta_0)$, we know that the maximum point δ^* of G_q^a cannot be in the interval (δ_{max}, ∞) as it would otherwise be a contradiction to $G_q^a(\delta)$ being a strictly increasing function of δ for $\delta < \delta^*$. Hence for any $q \in \mathbf{Q}$,

$$y_q \in [\delta_{min}, \delta_{max}],$$

which shows $T(K) \subset K$, where $K = [\delta_{min}, \delta_{max}]^{M-1}$. □

With the help of Berge Maximum Theorem (Theorem 2.2.10), we now can prove the mapping T defined in (3.19) is continuous on K.

Lemma 3.5.4 (Continuious Mapping T in $\mathbb{R}^M$)**.** *Given any vector μ and bounded set K defined in Lemma 3.5.3, mapping T defined in (3.19) is continuous on K.*

Proof. We prove that given vector μ, each element $T_q(w)$ of mapping T is continuous respect to each w_q. As the maximum point of $\eta^a(\mu, \cdot, w, q)$ exists and is unique for every $q \in \{-Q+1, \cdots, Q\}$, T_q is a well defined single value mapping. Moreover, $\eta^a(\mu, \delta, w, q)$ is jointly continuous w.r.t δ and w. By Berge's maximum theorem, T_q is upper hemicontinuous function of w on bounded set K. Therefore, by Lemma 2.2.12, for $q \in \{-Q+1, \cdots, Q\}$, T_q is also continuous w,r,t every w_q. We conclude that given vector μ, the mapping T is a continuous mapping from $K \to K$. □

Finally we can prove theorem 3.5.2, which concludes the proof of step 1.

Proof of Theorem 3.5.2. As the intensity function f satisfies Assumption 3.2.1, from the Lemma 3.5.1, the maximum point of $G_q^a(\delta)$ exists and is unique for every q. Fixed vector $\mu \in \mathbb{R}^M$, define mapping $T : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ as in (3.19). w^a is the fixed point of mapping T. To show the existence of fixed point to the mapping, Schauder fixed-point theorem is applied to T by following steps.

Firstly, by Lemma 3.5.3, there exists a bounded closed set $K \subset \mathbb{R}^{M-1}$ which is equivalently a compact set, such that $T(K) \subset K$. From the proof of Lemma 3.5.3, the compact set K is convex.

Secondly, from Lemma 3.5.1 and 3.5.4, T is a single value continuous mapping from K to K. By Theorem 2.2.8, T has a fixed point for every given μ, denoted by w^a, and

$$w_q^a = T(w^a) \in K. \tag{3.24}$$

This concludes the proof of Step 1. □

Proof of Step 2

With the help of global implicit function theorem (Theorem 2.2.11), we can show the local Lipschitz continuity of functions w^a and w^b.

Theorem 3.5.5. *Assume the intensity function f satisfies Assumption 3.2.1. Then there are single valued and locally Lipschitz continuous functions $w^a, w^b : \mathbb{R}^M \to \mathbb{R}^{M-1}$, such that they satisfy the generalized Issac condition (3.12) in Definition 3.3.2 for any given vector $\mu \in \mathbb{R}^M$.*

Proof. We provide the proof for w^a only. The proof for w^b is similar.

To begin with, from Assumption 3.2.1, we have (3.17) for all δ, x and y. From Lemma 3.5.1, the maximum point of $G_q^a(\delta) = \eta^a(\mu, \delta, w^a, q)$ is unique. From Remark 3.5.1, given any vector μ, w^a that satisfies the generalized Issac condition in Definition 3.3.2 is also the solution to the following first order condition for every q,

$$\sum_{j=-Q+1}^{Q} P_j^a[f(w_q^a, w_j^a) + f_1'(w_q^a, w_j^a)(w_q^a + \mu_{q-1} - \mu_q)] = 0.$$

For any vector μ and $\delta = (\delta_{-Q+1}, \cdots, \delta_Q)$, define function $F_q : \mathbb{R}^{M-1} \times \mathbb{R}^M \to \mathbb{R}$ for every $q \in \{-Q+1, \cdots Q\}$ as following:

$$F_q(\delta, \mu) := -\frac{\sum_{j=-Q+1}^{Q} P_j^a f(\delta_q, \delta_j)}{\sum_{j=-Q+1}^{Q} P_j^a f_1'(\delta_q, \delta_j)} - \delta_q - (\mu_{q-1} - \mu_q).$$

Define mapping $F : \mathbb{R}^{M-1} \times \mathbb{R}^M \to \mathbb{R}^{M-1}$ as

$$F(\delta, \mu) := (F_{-Q+1}(\delta, \mu), \cdots, F_Q(\delta, \mu)).$$

F is continuously differentiable and w^a is determined implicitly by $F(w^a, \mu) = 0$. From the proof of step 1, there exists a function $w^a : \mathbb{R}^M \to \mathbb{R}^{M-1}$ such that $F(w^a(\mu), \mu) = 0$ for any vector μ. If we can verify Theorem 2.2.11 holds in this case, the function w^a satisfying $F(w^a(\mu), \mu) = 0$ must be unique and continuously differentiable, which concludes our proof. Hence the next step is to verify Theorem 2.2.11.

Firstly, we prove that the Jacobian matrix of F never vanish. Denote Jacobian matrix of F with respect to δ as $\partial_\delta F$, a $2Q \times 2Q$ matrix, and its component at

(i, m) is $\frac{\partial F_i}{\partial \delta_m}(\delta, \mu)$ for $i, m = -Q+1, \ldots, Q$. Denote by, for $i \in \{-Q+1, \cdots, Q\}$,

$$D_i := (\sum_{m=-Q+1}^{Q} P_m^a f_1'(\delta_q, \delta_m))^2 > 0$$

$$A_i = \frac{1}{D_i} \sum_{m=-Q+1}^{Q} \sum_{j=-Q+1}^{Q} P_m^a P_j^a [f_{11}''(\delta_i, \delta_m) f(\delta_i, \delta_j) - f_1'(\delta_i, \delta_m) f_1'(\delta_i, \delta_j)]$$

$$I_{im} := \frac{1}{D_i} P_m^a \sum_{j=-Q+1}^{Q} P_j^a [f(\delta_i, \delta_j) f_{12}''(\delta_i, \delta_m) - f_1'(\delta_i, \delta_j) f_2'(\delta_i, \delta_m)].$$

For $m = i$, we have:

$$\frac{\partial F_i}{\partial \delta_i}(\delta, \mu) = -1 + A_i + I_{ii}.$$

From Assumption 3.2.1 we have (3.17), and simple calculation shows:

$$-1 + A_i = \frac{1}{D_i} \sum_{m,j=-Q+1}^{Q} P_m^a P_j^a [f_{11}''(\delta_i, \delta_m) f(\delta_i, \delta_j) - 2 f_1'(\delta_i, \delta_m) f_1'(\delta_i, \delta_j)] < 0.$$

Hence

$$|\frac{\partial F_i}{\partial \delta_i}(\delta, \mu)| \geq 1 - A_i - |I_{i,i}|.$$

For $i \neq m$, the non-diagonal element of the Jacobian matrix $\partial_\delta F$ is given by:

$$\frac{\partial F_i}{\partial \delta_m}(\delta, \mu) = I_{im}.$$

To compare the diagonal element with the sum of non-diagonal elements, we have:

$$|\frac{\partial F_i}{\partial \delta_i}(\delta, \mu)| - \sum_{m \neq i} |\frac{\partial F_i}{\partial \delta_m}(\delta, \mu)| \geq 1 - A_i - \sum_{m=-Q+1}^{Q} |I_{im}|. \tag{3.25}$$

From the definition of A_i and I_{im},

$$
\begin{aligned}
&1 - A_i - \sum_{m=-Q+1}^{Q} |I_{im}| \\
&= \frac{1}{D_i} \sum_{m=-Q+1}^{Q} P_m^a \{ \sum_{j=-Q}^{Q} P_j^a [2f_1'(\delta_i, \delta_m) f_1'(\delta_i, \delta_j) - f_{11}''(\delta_i, \delta_m) f(\delta_i, \delta_j)] \\
&- | \sum_{j=-Q+1}^{Q} P_j^a [f(\delta_i, \delta_j) f_{12}''(\delta_i, \delta_m) - f_1'(\delta_i, \delta_j) f_2'(\delta_i, \delta_m)] | \}.
\end{aligned} \tag{3.26}
$$

By the assumption of f in (3.1), we have

$$
\begin{aligned}
&\sum_{j=-Q+1}^{Q} P_j^a [2f_1'(\delta_i, \delta_m) f_1'(\delta_i, \delta_j) - f_{11}''(\delta_i, \delta_m) f(\delta_i, \delta_j)] \\
&\pm [\sum_{j=-Q+1}^{Q} P_j^a [-f_2'(\delta_i, \delta_m) f_1'(\delta_i, \delta_j) + f_{12}''(\delta_i, \delta_m) f(\delta_i, \delta_j)]] > 0.
\end{aligned} \tag{3.27}
$$

Therefore, as $D_i > 0$, from (3.25), (3.26) and (3.27), we conclude that

$$
|\frac{\partial F_i}{\partial \delta_i}(\delta, \mu)| - \sum_{m \neq i} |\frac{\partial F_i}{\partial \delta_m}(\delta, \mu)| > 0.
$$

The Jacobian matrix $\partial_\delta F(\delta, \mu)$ is strictly diagonally dominant, and is therefore a nonsingular matrix.

Secondly, we show that given any fixed vector μ, whenever $||\delta|| \to \infty$, $||F(\delta, \mu)|| \to \infty$. For any vector sequence $\vec{\delta}^k, k = 1, 2 \cdots$, $||\vec{\delta}^k|| \to \infty$. Then there exists sequence $n_k \in \{-Q+1, \cdots, Q\}, k = 1, 2 \cdots$, such that $|\delta_{n_k}^k| \to \infty$. $\delta_{n_k}^k$ is the n_kth element of vector $\vec{\delta}^k$. In the case that $\delta_{n_k}^k \to -\infty$, as we have

$$
L_{n_k}(\vec{\delta}^k) := \frac{\sum_{m=-Q+1}^{Q} P_m^a f(\delta_{n_k}^k, \delta_m^k)}{\sum_{m=-Q+1}^{Q} P_m^a f_1'(\delta_{n_k}^k, \delta_m^k)} < 0.
$$

Hence we know following when $k \to +\infty$:

$$F_{n_k}(\vec{\delta}^k, \mu) = -L_{n_k}(\vec{\delta}^k) - \delta^k_{n_k} - (\mu_{n_k-1} - \mu_{n_k}) > -\delta^k_{n_k} - (\mu_{n_k-1} - \mu_{n_k}) \to +\infty.$$

It means when $\delta^k_{n_k} \to -\infty$, $||F(\vec{\delta}^k, \mu)|| \to \infty$.

On the other hand, in the case that $\delta^k_{n_k} \to +\infty$, we can always assume $\delta^k_{n_k} = \max\{\delta^k_i\}_{i\in\mathbf{Q}, i>-Q}$. As $f'_1 < 0$, $f > 0$ and f is increasing function to its second variable, we have the following estimation on $F_{n_k}(\vec{\delta}^k, \mu)$:

$$\begin{aligned} F_{n_k}(\vec{\delta}^k, \mu) &= -\frac{\sum_{m=-Q+1}^{Q} P^a_m f(\delta^k_{n_k}, \delta^k_m)}{\sum_{m=-Q+1}^{Q} P^a_m f'_1(\delta^k_{n_k}, \delta^k_m)} - \delta^k_{n_k} - (\mu_{n_k-1} - \mu_{n_k}) \\ &\leq -\frac{\sum_{m=-Q+1}^{Q} P^a_m f(\delta^k_{n_k}, \delta^k_{n_k})}{P^a_{n_k} f'_1(\delta^k_{n_k}, \delta^k_{n_k})} - \delta^k_{n_k} - (\mu_{n_k-1} - \mu_{n_k}). \end{aligned}$$

From the assumption that $\lim_{\delta\to+\infty} -\frac{f'_1(\delta,\delta)}{f(\delta,\delta)} > 0$, we have:

$$0 < -\lim_{\delta^k_{n_k}\to+\infty} \frac{\sum_{m=-Q+1}^{Q} P^a_m f(\delta^k_{n_k}, \delta^k_{n_k})}{P^a_{n_k} f'_1(\delta^k_{n_k}, \delta^k_{n_k})} < +\infty.$$

Then by taking $\delta^k_{n_k} \to +\infty$, we finally have:

$$\lim_{\delta^k_{n_k}\to+\infty} F_{n_k}(\vec{\delta}^k, \mu) = -\infty.$$

Hence when fixed μ and $\delta^k_{n_k} \to +\infty$, we also get $||F(\vec{\delta}^k, \mu)|| \to \infty$. Moreover, if $\delta^k_{n_k}$ is consisted of two sub-sequences such that one converges to $+\infty$, another to $-\infty$, by combining above, we can still get $||F(\vec{\delta}^k, \mu)|| \to \infty$. We conclude that whenever $||\delta|| \to \infty$, $||F(\delta, \mu)|| \to \infty$.

Theorem 2.2.11 implies that there exists a function $w^a : \mathbb{R}^M \to \mathbb{R}^{M-1}$ such that $F(w^a(\mu), \mu) = 0$ and w^a is unique and locally Lipschitz continuous, which concludes the proof of Step 2. □

Proof of Step 3

We next prove there exists a unique classical solution θ to ODE system (3.13) on $[0, T]$. The proof is divided by two parts. Firstly, we show the solution to ODE system (3.13) is bounded if it exists. Secondly, we provide the proof for existence and uniqueness of the classical solution to ODE system (3.13).

Lemma 3.5.6. *Assume the intensity function f satisfies Assumption 3.2.1. If $\theta : [0, T] \to \mathbb{R}^M$ is a solution to the ODE system (3.13), then for all $q \in \mathbf{Q}$ we have*

$$-\frac{1}{2}\gamma\sigma^2 Q^2 T - l(Q) \leq \theta_q(t) \leq 2 \sup_\delta \lambda(\delta)\delta T.$$

Proof. We first prove the upper bound. From the assumption on f and the proof for the steps 1 and 2, the ODE system (3.13) is well defined. Since θ is assumed to be a solution, define functions d^0 and d^1 twice continuously differentiable w.r.t δ as

$$d^0(t, \delta) := \sum_{j=-Q}^{Q-1} P_j^b f(\delta, w_j^b(\theta(t))) \leq \lambda(\delta)$$

$$d^1(t, \delta) := \sum_{j=-Q+1}^{Q} P_j^a f(\delta, w_j^a(\theta(t))) \leq \lambda(\delta).$$

From Assumption 3.2.1, we have (3.17) for all δ, x and y. Simple calculation shows that d^0 and d^1 satisfy

$$d^\zeta(t, \delta) \leq \lambda(\delta), \quad \frac{\partial^2 d^\zeta}{\partial \delta^2}(t, \delta) d^\zeta(t, \delta) < 2(\frac{\partial d^\zeta}{\partial \delta}(t, \delta))^2, \quad \zeta = 0, 1.$$

On the other hand, θ is also the solution to ODE system for all $q \in \mathbf{Q}$:

$$\begin{aligned} \theta_q'(t) &= \frac{1}{2}\gamma\sigma^2 q^2 - \sup_\delta\{d^0(t, \delta)(\delta + \theta_{q+1}(t) - \theta_q(t))\} I^b(q) \\ &- \sup_\delta\{d^1(t, \delta)(\delta + \theta_{q-1}(t) - \theta_q(t))\} I^a(q) \\ \theta_q(T) &= -l(|q|). \end{aligned} \tag{3.28}$$

The comparison principle for ODE system (3.28) can be proved easily with similar argument in the proof of comparison principle in Guéant (2017). Define operator $H^\zeta : [0,T] \times \mathbb{R} \to \mathbb{R}$ for both $\zeta = 0, 1$ as

$$H^\zeta(t, \Delta\mu) := \sup_\delta \{d^\zeta(t,\delta)(\delta + \Delta\mu)\}.$$

Then from Guéant (2017), we know H^ζ is an increasing and non-negative function in $\Delta\mu$.

$$\max_{t\in[0,T],\zeta=0,1} H^\zeta(t,0) \le \sup_\delta\{\lambda(\delta)\delta\}.$$

Define $\bar{\theta} : [0,T] \to \mathbb{R}^M$ as following:

$$\bar{\theta}_q(t) = 2\sup_\delta \lambda(\delta)\delta(T-t).$$

Substituting $\bar{\theta}$ into ODE system (3.28), we have

$$\begin{aligned}
&-\bar{\theta}'_q(t) + \frac{1}{2}\gamma\sigma^2 q^2 - H^0(t, \bar{\theta}_{q+1}(t) - \bar{\theta}_q(t)) I^b(q) - H^1(t, \bar{\theta}_{q-1}(t) - \bar{\theta}_q(t)) I^a(q) \\
&= \sum_{\zeta=0}^{1} (\sup_\delta \lambda(\delta)\delta - H^\zeta(t,0)) + \frac{1}{2}\gamma\sigma^2 q^2 \ge 0 \\
&\bar{\theta}_q(T) = 0 \ge \theta_q(T) = -l(|q|).
\end{aligned}$$

Then by the comparison principle from Guéant (2017), we know for every $q \in \mathbf{Q}$,

$$\theta_q(t) \le \bar{\theta}_q(t) \le 2\sup_\delta \lambda(\delta)\delta T.$$

We next prove the lower bound. Let $\tilde{\theta} : [0,T] \to \mathbb{R}^M$ satisfy the following ODE system for all $q \in \mathbf{Q}$:

$$\begin{aligned}
&\tilde{\theta}'_q(t) - \frac{1}{2}\gamma\sigma^2 q^2 = 0 \\
&\tilde{\theta}_q(T) = -l(|q|).
\end{aligned} \tag{3.29}$$

The closed-form solution is given by

$$\tilde{\theta}_q(t) = \frac{1}{2}\gamma\sigma^2 q^2(t-T) - l(|q|).$$

Note we have estimation that for every vector $\mu \in \mathbb{R}^M$ and every $q \in \mathbf{Q}$,

$$\eta^a(\mu, w_q^a(\mu), w^a(\mu), q) \geq 0, \quad \eta^b(\mu, w_q^b(\mu), w^b(\mu), q) \geq 0.$$

Since $\tilde{\theta}_q(T) \leq \theta_q(T)$, $\tilde{\theta}_q'(t) \geq \theta_q'(t)$, then it can be proved similarly as the proof of the upper solution that for every $q \in \mathbf{Q}$:

$$\theta_q(t) \geq \tilde{\theta}_q(t) \geq -\frac{1}{2}\gamma\sigma^2 Q^2 T - l(Q).$$

□

Then with the help of Picard-Lindelof theorem (Theorem 2.2.13) and its extension (Lemma 2.2.14), we can prove the existence of a classical solution to the coupled ODE system (3.13).

Theorem 3.5.7. *Consider the terminal value ODE problem on* $[0,T]$*:*

$$\theta'(t) = F(t, \theta(t)),\ \theta(T) = \theta_0, \tag{3.30}$$

where $F : [0,T] \times \mathbb{R}^M \to \mathbb{R}^M$ *is a jointly locally Lipschitz continuous function. Assume that there exists a constant* K *such that if solution* θ *exists on any sub-interval of* $[0,T]$*,* $\theta(t) \in [-K, K]^M$*. Then there exists a unique solution to (3.30) on* $[0,T]$*.*

Proof. Define $A_{T,2\sqrt{M}K} := [0,T] \times [-2\sqrt{M}K, 2\sqrt{M}K]^M$. F is a continuous function.

Hence there exists uniform constant $C > 0$ such that

$$C := \sup_{(t,y)\in A_{T,2\sqrt{M}K}} \|F(t,y)\|. \tag{3.31}$$

Since F is jointly locally Lipschitz continuous, there exists a series of open set A_i such that F is Lipschitz continuous in A_i with Lipschitz coefficient L_i, and $A_{T,2\sqrt{M}K} \subset \cup_i A_i$. By Heine Borel theorem, there are finite set I of i such that $A_{T,2\sqrt{M}K} \subset \cup_{i\in I} A_i$. Define $L := \max_{i\in I} L_i$, we know F is Lipschitz continuous on the compact set $A_{T,2\sqrt{M}K}$ with uniform Lipschitz coefficient L.

As terminal value $\theta_0 \in [-K,K]^M$, we define $C^0_{T,\sqrt{M}K} := [0,T] \times B_{\sqrt{M}K}(\theta_0)$. Then $C^0_{T,\sqrt{M}K} \subset A_{T,2\sqrt{M}K}$. For $\epsilon := \min\{\frac{\sqrt{M}K}{C}, \frac{1}{L}, T\}$, the solution θ to ODE system (3.30) exists and is unique on $[T-\epsilon, T]$. If $\epsilon = T$, then we are done, otherwise, update the new terminal time as $\tilde{T} := T - \epsilon$. Since $\theta(\tilde{T}) \in [-K,K]^M$ by assumption, we can update a new terminal value $\theta_0 := \theta(\tilde{T})$. Define a new $C^1_{\tilde{T},\sqrt{M}K} := [0,\tilde{T}] \times B_{\sqrt{M}K}(\theta(\tilde{T})) \subset A_{T,2\sqrt{M}K}$. For $\epsilon := \min\{\frac{\sqrt{M}K}{C}, \frac{1}{L}, \tilde{T}\}$, solution θ to ODE system (3.30) exists and is unique on $[\tilde{T}-\epsilon, \tilde{T}]$, and hence exists and is unique also on $[\tilde{T}-\epsilon, T]$. Repeat this process and we can reach $\epsilon = \tilde{T}$ after finite number of steps, in which case we have proved the existence and uniqueness of solution θ to ODE system (3.30) on the whole time interval $[0,T]$. □

Combining Lemma 3.5.6, Theorem 3.5.5, and Theorem 3.5.7, we can finally proceed to show that the ODE system (3.13) has a unique classical solution.

Theorem 3.5.8. *There exists unique classical solution θ to ODE system (3.13) on $[0,T]$.*

Proof. According to Lemma 3.5.6, we know if the solution θ exists on any sub-interval of $[0,T]$, there exists constant $K \geq 0$ such that

$$-K \leq \theta_q(t) \leq K.$$

Define $F : [0, T] \times \mathbb{R}^M \to \mathbb{R}^M$ as

$$
\begin{aligned}
&F_q(t, \theta(t)) := \frac{1}{2}\gamma\sigma^2 q^2 - \eta^a(\theta(t), w_q^a(\theta(t)), w^a(\theta(t)), q)I^a(q) \\
&- \eta^b(\theta(t), w_q^b(\theta(t)), w^b(\theta(t)), q)I^b(q) \\
&F(t, \theta(t)) := (F_{-Q}(t, \theta(t)), \cdots, F_Q(t, \theta(t))).
\end{aligned}
$$

As q is finite, the original ODE system (3.13) can be rewritten in a vector form with F as in (3.30). Then F is a jointly locally Lipschitz continuous function, and if solution θ exists on any sub-interval of $[0, T]$, $\theta(t) \in [-K, K]^M$. By Theorem 3.5.7, the ODE system has unique solution on $[0, T]$. This concludes the proof of step 3. □

Completion of Proof of Theorem 3.3.4

From Steps 1, 2 and 3, we know there exist unique locally Lipschitz continuous functions w^a, w^b that satisfy generalized Issac condition in Definition 3.3.2, the ODE system (3.13) is well defined and equivalent to the ODE system (3.10).There exists a unique classical solution to ODE system (3.13). Define the equilibrium value function for G_{mm} by (3.6), and the equilibrium controls by (3.14). As θ is the classical solution to the ODE system (3.13), it is a continuous function on $[0, T]$ and hence bounded. Then both $\pi^a(t) = w^a(\theta(t))$ and $\pi^b(t) = w^b(\theta(t))$ are bounded on $[0, T]$. θ, $\pi^a(t)$ and $\pi^b(t)$ satisfy the ODE system (3.10). Hence from the verification Theorem 3.3.3, the equilibrium for game G_{mm} exists. On the other hand, as the solution to ODE system (3.10) is unique, by Theorem 3.3.1 we know the equilibrium point is also unique.

3.5.3 Proof of example intensity function (3.2) satisfying Assumption 3.2.1

Proof. Define function g as following.

$$g(\delta, x) := \frac{\Lambda}{\sqrt{1 + 3e^{k(\delta - x)}}}$$
$$f(\delta, x) = e^{-a\delta} g(\delta, x) \tag{3.32}$$

By simple calculation, we have following.

$$\begin{aligned}
f_1'(\delta, x) &= e^{-a\delta}(g_1'(\delta, x) - ag(\delta, x)) \\
f_2'(\delta, x) &= e^{-a\delta} g_2'(\delta, x) \\
f_{11}''(\delta, x) &= e^{-a\delta}(g_{11}''(\delta, x) - 2ag_1'(\delta, x) + a^2 g(\delta, x)) \\
f_{12}''(\delta, x) &= e^{-a\delta}(g_{12}''(\delta, x) - ag_2'(\delta, x))
\end{aligned} \tag{3.33}$$

Moreover, for the derivatives of g we have following.

$$\begin{aligned}
g_1'(\delta, x) &= -\frac{3\Lambda k}{2} \frac{e^{k(\delta - x)}}{(1 + 3e^{k(\delta - x)})^{\frac{3}{2}}} \\
g_2'(\delta, x) &= -g_1'(\delta, x) \\
g_{11}''(\delta, x) &= -\frac{3\Lambda k^2}{2} \frac{e^{k(\delta - x)}(1 - \frac{3}{2} e^{k(\delta - x)})}{(1 + 3e^{k(\delta - x)})^{\frac{5}{2}}} \\
g_{12}''(\delta, x) &= -g_{11}''(\delta, x)
\end{aligned} \tag{3.34}$$

To check the conditions above, we maker some change of variables as below.

$$\bar{x} := 3e^{k(\delta - x)} > 0, \quad \bar{y} := 3e^{k(\delta - y)} > 0 \tag{3.35}$$

Define A_1 and A_2 as following.

$$\begin{aligned} A_1 &:= f(\delta,x)f''_{11}(\delta,y) - 2f'_1(\delta,x)f'_1(\delta,y) + (f'_1(\delta,x)f'_2(\delta,y) - f''_{12}(\delta,y)f(\delta,x)) \\ A_2 &:= f(\delta,x)f''_{11}(\delta,y) - 2f'_1(\delta,x)f'_1(\delta,y) - (f'_1(\delta,x)f'_2(\delta,y) - f''_{12}(\delta,y)f(\delta,x)) \end{aligned} \tag{3.36}$$

Then we only need to show both A_1 and A_2 are negative. By simple calculation, we have the simplification of A_1 and A_2.

$$\begin{aligned} A_1 = & \frac{\Lambda^2 e^{-2a\delta}}{(1+\bar{x})^{\frac{3}{2}}(1+\bar{y})^{\frac{5}{2}}}\{[(\frac{k^2}{2} - a^2)\bar{y}^2 - (k^2+2a^2)\bar{y} - a^2] \\ & + \bar{x}[-(\frac{1}{2}k+a)^2\bar{y}^2 - (\frac{7}{4}k^2 + 2ak + 2a^2)\bar{y} - (a^2+ak)]\} \end{aligned} \tag{3.37}$$

Since $\frac{k^2}{2} < a^2$, we know $A_1 < 0$. On the other hand, for A_2 we have following representation.

$$A_2 = e^{-2a\delta}(-a^2 g(\delta,x)g(\delta,y) - g'_1(\delta,x)(g'_1(\delta,y) - 2ag(\delta,y))) \tag{3.38}$$

Since both $g(\delta,x)$ and $g(\delta,y)$ are positive, while both $g'_1(\delta,x)$ and $g'_1(\delta,y)$ are negative. Hence we can also know $A_2 < 0$. Therefore we have verified the following holds for any δ, x and y.

$$f(\delta,x)f''_{11}(\delta,y) - 2f'_1(\delta,x)f'_1(\delta,y) + |f'_1(\delta,x)f'_2(\delta,y) - f''_{12}(\delta,y)f(\delta,x)| < 0 \tag{3.39}$$

Furthermore, it is easy to check that

$$\begin{aligned} & f(\delta,x) > 0, \quad f'_1(\delta,x) < 0, \quad f'_2(\delta,x) \geq 0 \\ & \lim_{\delta\to+\infty} f(\delta,x)\delta = 0, \quad \lim_{\delta\to+\infty} -\frac{f'_1(\delta,\delta)}{f(\delta,\delta)} > 0 \end{aligned} \tag{3.40}$$

To conclude, f satisfies all the conditions mentioned. □

3.6 Conclusions

In this chapter we have modeled the price competition between market makers, proved the generalized Issac condition, which ensures the existence and uniqueness of Nash equilibrium for market making with price competition, and derived the equilibrium strategies and the equilibrium value function. We have also performed numerical tests to compare our model with a benchmark model in existing literature without price competition and found that the introduction of price competition reduces bid/ask spreads and improves market liquidity.

4

Market Making with Rank-based Trading Volume Reward Competition

4.1 Introduction

For market makers in exchange market, they can profit not only from the bid/ask spread, but also from exchange's market making incentive program if their business is large enough to be appointed as the designated market maker by the exchange. Exchange's purpose of appointing designated market maker is to stimulate stable liquidity provision for certain products, which can attract more trading on these products in its venue, and ultimately increase exchange's revenue, as commission fee income is the main source of exchanges' profit. In order to do so, exchange sets up contract known as market making incentive scheme with the designated market makers. This is quite common nowadays. Different market making incentive schemes can be found in different exchanges for different products (see the ones for LSE, ICE, Euronext and etc), though they may be called with a different name. In these schemes, designated market makers are obliged to keep providing bid and ask quotes in the market for certain percentage of the day. In some exchanges, designated market makers might also need to satisfy certain requirements on their trading volume, speed to respond request for quote, their size of bid/ask spreads, and etc. In return, they can receive various kinds of incentive reward, depending on the fee structure of the market making incentive scheme.

The detailed fee structures of market making inventive schemes from different exchanges might differ quite a lot (see different market makers incentive programs of LSE, ICE, Euronext etc), but we can still summarize two most commonly seen incentive types:

1. Make take fees: cash reward proportional to the absolute value of designated market makers' total trading volume.

2. Profit sharing pool: cash reward related to the relative ranking of designated market makers' total trading volume among all designated market makers.

To be more specific, make take fee is the commission waive or further cash reward provided by exchange every time when designated market maker's bid/ask quote is hit. It is a quite general approach and can be seen in the market incentive program from different exchanges. The purpose is to motivate designated market makers to lower their bid/ask spread in order to increase their trading volume and earn more make take fee. As a result, exchange's profit coming from the commission fee from the other side of the trade gets increased. On the other hand, certain exchange also tend to introduce trading volume competition among designated market makers to provide further trading motivation for designated market makers. One example of exchanges using this approach is the London Stock Exchange Derivatives Market (see London Stock Exchange Derivatives Market, market making obligations, version 6.4, 19 April 2018). Like in the IOB market in London Stock Exchange, at the beginning of certain period of time (month, quarter, etc), certain percentage of the revenue, called revenue sharing pool is set aside, and at the end of that period, designated market makers will be ranked in terms of their total trading volume during this period of time. Their ranking then will determine the amount of reward they could get. Most of the market making incentive schemes are mainly just the mixture of the two incentive types.

Understanding and comparing both incentive types are crucial for exchanges to design a more efficient market making incentive scheme. However, there is still not much existing literature discussing the impact of different market making incentive scheme. For the first type of incentive, i.e make take fee, there are some relevant studies that focus on the make take fee structure and its impact on the market welfare. However, they are either empirical, or adopting stylized models that might be too simple to characterize the market reality (See Foucault et al. (2013); Anand et al. (2016); Laruelle and Lehalle (2018); Colliard and Foucault (2012); Angel et al. (2011)). Recently, El Euch et al. (2018) adopt a more realistic model for market making from Cartea et al. (2015); Guéant (2017), and discuss the optimal make-take fee structure design by extending the optimal market making model to a principal-agent problem between market makers and exchange. But the assumption on make-take fee in El Euch et al. (2018) is not very practical. Their commission fee schedule is exchange's stochastic feed back control, while in reality it should be a constant stated in the contract between market makers and exchange at the initial time. On the other hand, for the second type of incentive, to the best of our knowledge, there is still no existing relevant literature, while it is actually commonly seen in the contracts of market making incentive scheme from various exchanges and products. Hence our work is to fill this gap and include both the two types of incentive reward into the optimal market making problem.

We extend the optimal market making model from Cartea et al. (2015); Guéant (2017). Similar to El Euch et al. (2018), we can easily include make take fee into our model, except that the our make take fee is a given constant parameter, not a stochastic control of exchange. The main obstacle is to model market makers' competition for the profit sharing pool. As far as we know, we are the first to model market making with reward related to trading volume ranking. We will show that market making with both make take fee and trading volume ranking related reward can be modelled as a mean field game problem, which can further be simplified to a finite state mean field game in continuous time. The existence, uniqueness, and convergence property of the game can be proved using results of Cecchin and Pelino (2019). As the equilibrium value function is characterized by a forward backward ODE system, which is generally difficult to solve even numerically. We apply a deep neural network approach from Sirignano and Spiliopoulos (2018) to numerically solve the ODE system. Then we compare and analyze market makers' different behaviour and the market liquidity under incentive schemes with and without make take fee or trading volume ranking competition.

We also find that our model belongs to a more general class of finite state mean field game, which can also be numerically solved by our deep neural network approach. We will provide a error estimation for this deep neural network numerical solution to the general finite state mean field game in the next section.

4.2 Model Setting

The model setting is similar to Guéant (2017), except that exchange provides incentive reward for market making. In this case the terminal payoffs of market makers also depend on their trading volume sizes and rankings among other market makers.

Consider a family of market makers Ω_{mm} in the market who keep quoting bid/ask limit orders in order to profit from the bid/ask spread. Select one of them as our reference market maker. Asset reference price S_t follows a simple Brownian motion with initial value S,

$$dS_t = \sigma dW_t,$$

where W_t is a standard Brownian adapted to a standard filtration $\{\mathcal{F}_t^W\}_{t\in\mathbb{R}_+}$. And reference market maker's bid price S_t^b and ask price S_t^a are defined by following:

$$S_t^b = S_t - \delta_t^b, \quad S_t^a = S_t + \delta_t^a,$$

where δ_t^b and δ_t^a are the bid and ask spreads respectively, and we use a and b to denote the type of limit order (bid or ask). Define N_t^b and N_t^a two Poisson processes, modelling the sell/buy market orders arrival to the reference market maker. The intensities of N_t^b, N_t^a are $\Lambda(\delta_t^b)$, $\Lambda(\delta_t^a)$ respectively. Similar to Guéant (2017), we assume $\Lambda : \mathbb{R} \to \mathbb{R}$ is a decreasing, continuously differentiable function satisfying:

$$\frac{\partial^2 \Lambda}{\partial \delta^2}(\delta)\Lambda(\delta) < 2(\frac{\partial \Lambda}{\partial \delta}(\delta))^2. \tag{4.1}$$

Market maker has state variables (X_t, q_t, v_t). q_t is her inventory with initial value q. We assume q_t can only take values in a finite set $\mathbf{Q} = \{-Q, \cdots, Q\}$. It means when $q = Q$, market maker achieves their maximal inventory capacity and can not buy anymore. It is similar when $q = -Q$. We use I^b and I^a to denote market maker's buying or selling capability.

$$I^b(q) := \mathbb{1}_{q+1\in\mathbf{Q}}, \quad I^a(q) := \mathbb{1}_{q-1\in\mathbf{Q}}.$$

Then the dynamic of q_t is

$$dq_t = I^b(q_t)dN_t^b - I^a(q_t)dN_t^a.$$

X_t is the cash account of the reference market maker with initial value x. Its dynamic is as following:

$$dX_t = (S_t^a - c)I^a(q_t)dN_t^a - (S_t^b + c)I^b(q_t)dN_t^b,$$

where c is the commission fee charged by the exchanges. When c is positive, exchange charges market maker commission fee, and when c is negative, exchange pays market maker for market making. We assume c is a constant determined by exchange at time 0. Meanwhile, as market maker's terminal payoff depend on her trading volume ranking, we use v_t to record the accumulated trading volume for the reference market maker. We assume $v_t \in \mathbf{V} := \{0, \cdots, v_{max}\}$, i.e trading volume above v_{max} is not counted in the reward calculation. This is simply a assumption needed by technical perspective of the proof. In reality, one can always set v_{max} high enough such that it can never be touched.

$$dv_t = (I^b(q_t)dN_t^b + I^a(q_t)dN_t^a)\mathbb{1}_{\{v_t < v_{max}\}}.$$

Every market maker wants to maximize the expected value of terminal wealth while being penalized for holding inventory at terminal time T and throughout the time interval $[0, T]$ with γ, a positive constant representing the risk adverse level and with l an increasing convex function on R_+ with $l(0) = 0$, denoting the liquidity penalty for holding inventory at T. Also, market maker is rewarded according to their trading volume at T by the market making incentive schemes set by exchange, denoted by R, an increasing function on Z^+.

$$\sup_{\delta^a, \delta^b} \mathbb{E}_0[X_T + q_T S_T - l(|q_T|) + R(v_T) - \frac{1}{2}\gamma\sigma^2 \int_0^T q_t^2 dt], \tag{4.2}$$

In this chapter, We focus on the rank based trading volume reward. So $R(v_T)$ is proportional to the number of market makers with trading volume less than v_T. If reference market maker's ranking of accumulated trading volume v_T among all market makers is higher, the reward $R(v_T)$ will also be higher. The optimization problems of different market makers are coupled. In reality, the total number of market makers participating in this competition is finite. So it is a stochastic game with finite number of players. However, since the total number of market makers is not small, the game problem is of high dimension. The equilibrium is difficult to solve even numerically due to the curse of dimension. To tackle it, we use a limiting case when there is infinitely many market makers to approximate the reality when there are only finite number of players. By taking the limit, it becomes a mean field game with finite states.

In the limiting mean field game, denote the probability measure on the mean field of discrete states (q_t, v_t) as $p(t, q_t, v_t)$. Then given p, if the reference market maker's accumulated trading volume is v, then the percentage of market makers in the market that the reference market maker exceeds w.r.t trading volume is $1 - \sum_{j=v}^{v_{max}} \sum_{i=-Q}^{Q} p(T, i, j)$. At terminal time T, exchange will reward every market maker according to this percentage. Given the maximum reward R, a constant set by the exchange, $R(\cdot)$ is defined by:

$$R(v_T) := R(1 - \sum_{j=v_T}^{v_{max}} \sum_{i=-Q}^{Q} p(T, i, j)). \tag{4.3}$$

We have the convergence between finite players game to the mean field game from following remark according to Theorem 6 in Cecchin and Pelino (2019). It will be explained in details by Remark 4.3.2 later.

Remark 4.3.2 suggests that mean field game is a good approximation of reality when there are finite number of players. We can use a mean field game to approximate the game between finite number of players.

4.3 Dimension reduction and Main Results

Using the martingale property, (4.2) can be reduced to

$$
\begin{aligned}
x + qS + \sup_{\delta^a,\delta^b} \mathbb{E}[\int_0^T & [(\delta_t^a - c)\Lambda(\delta_t^a) + (\delta_t^b - c)\Lambda(\delta_t^b) - \frac{1}{2}\gamma\sigma^2 q_t^2]dt \\
& - l(|q_T|) + R(1 - \sum_{j=v_T}^{v_{max}} \sum_{i=-Q}^{Q} p(T,i,j))].
\end{aligned}
$$

We can notice what market maker is actually maximizing does not depend on state X_t. We assume market maker takes closed loop feed back control, i.e when market maker has state (q, v),

$$
\delta_t^a = \delta^a(t,q,v), \quad \delta_t^b = \delta^b(t,q,v). \tag{4.4}
$$

Then given any p, the value function θ for market maker is defined as

$$
\begin{aligned}
\theta(t,q,v) := \sup_{\delta^a,\delta^b} \mathbb{E}_t[\int_t^T & [(\delta_s^a - c)\Lambda(\delta_s^a) + (\delta_s^b - c)\Lambda(\delta_s^b) - \frac{1}{2}\gamma\sigma^2 q_s^2]ds \\
& - l(|q_T|) + R(1 - \sum_{j=v_T}^{v_{max}} \sum_{i=-Q}^{Q} p(T,i,j))|q_t = q, v_t = v].
\end{aligned} \tag{4.5}
$$

As the only relevant states are q_t and v_t that both take values in finite sets, the problem can be reduced to a continuous time finite state mean field game discussed in Cecchin and Pelino (2019) by reformulating some notations as following. Define $K := (2Q+1)(v_{max}+1)$ and $\Sigma := \{1, \cdots, K\}$. There is a one to one mapping $Z : \mathbf{Q} \times \mathbf{V} \to \Sigma$. For every $(q,v) \in \mathbf{Q} \times \mathbf{V}$, there exists $z \in \Sigma$ such that

$$
z = Z(q,v). \tag{4.6}
$$

And for every $z \in \Sigma$, there exists $(q, v) \in \mathbf{Q} \times \mathbf{V}$ such that

$$(q, v) = Z^{-1}(z)$$

We further define inverse Z_1^{-1} and Z_2^{-1} as

$$q = Z_1^{-1}(z), \quad v = Z_2^{-1}(z). \tag{4.7}$$

The state (q, v) is then reformulated by state z. The value function θ and probability measure on mean field of state p are reformulated as $\theta, p : [0, T] \to \mathbb{R}^K$, where

$$\begin{aligned}
\theta(t) &:= (\theta_1(t), \cdots, \theta_K(t)), \quad \theta_z(t) = \theta(t, Z_1^{-1}(z), Z_2^{-1}(z)) \\
p(t) &:= (p_1(t), \cdots, p_K(t)), \quad p_z(t) = p(t, Z_1^{-1}(z), Z_2^{-1}(z))
\end{aligned}$$

Define λ as

$$\lambda(t, z) := (\lambda_1(t, z), \cdots, \lambda_K(t, z)),$$

where λ satisfy

$$\begin{aligned}
&\lambda_{\beta^a(z)}(t, z) := \Lambda(\delta_t^a) > 0; \quad \lambda_{\beta^b(z)}(t, z) := \Lambda(\delta_t^b) > 0; \\
&\lambda_z(t, z) := -\sum_{y \neq z} \lambda_y(t, z); \quad \lambda_y(t, z) := 0 \quad y \neq \beta^a(z), \beta^b(z), z.
\end{aligned} \tag{4.8}$$

Note that

$$\lambda_{\beta^a(z)}(t, z) = \Lambda(\delta_t^a) = \Lambda(\delta^a(t, q, v)) = \Lambda(\delta^a(t, Z_1^{-1}(z), Z_2^{-1}(z))).$$

$\beta^a(z)$ and $\beta^b(z)$ are defined as the two accessible states from state z,

$$
\begin{aligned}
\beta^a(z) &= \begin{cases} Z(Z_1^{-1}(z)-1, Z_2^{-1}(z)+1) & Z_1^{-1}(z) > -Q, Z_2^{-1}(z) < v_{max} \\ Z(Z_1^{-1}(z)-1, v_{max}) & Z_1^{-1}(z) > -Q, Z_2^{-1}(z) = v_{max} \\ z & Z_1^{-1}(z) = -Q \end{cases} \\
\beta^b(z) &= \begin{cases} Z(Z_1^{-1}(z)+1, Z_2^{-1}(z)+1) & Z_1^{-1}(z) < Q, Z_2^{-1}(z) < v_{max} \\ Z(Z_1^{-1}(z)+1, v_{max}) & Z_1^{-1}(z) < Q, Z_2^{-1}(z) = v_{max} \\ z & Z_1^{-1}(z) = Q \end{cases}
\end{aligned} \tag{4.9}
$$

Define F and G as

$$
\begin{aligned}
&F(t,z,\lambda(t,z)) := (\Lambda^{-1}(\lambda_{\beta^a(z)}(t,z)) - c)\lambda_{\beta^a(z)}(t,z) \\
&+ (\Lambda^{-1}(\lambda_{\beta^b(z)}(t,z)) - c)\lambda_{\beta^b(z)}(t,z) - \frac{1}{2}\gamma\sigma^2 Z_1^{-1}(z)^2 \\
&G(z,p) := -l(|Z_1^{-1}(z)|) + R(1 - \sum_{j=v}^{v_{max}} \sum_{i=-Q}^{Q} p_{Z(i,j)}).
\end{aligned} \tag{4.10}
$$

Fixed c and R, the optimal market making problem is reduced to a continuous time finite state mean field game discussed in both Cecchin and Pelino (2019) and section 2 of this chapter. Denote the game as $G_{c,R}$.

Proposition 4.3.1. *$G_{c,R}$ satisfies both Assumption 5.2.1.*

The detailed proof can be found in our proof section. Then according to Cecchin and Pelino (2019), both the Nash equilibrium of mean field game $G_{c,R}$ and that of game with finite number of players exist and are unique for every given c and R. Moreover, the game with N players converges to the limiting mean field game case in $O(\frac{1}{N})$ speed. It is given as following remark.

Remark 4.3.2. As for different initial condition $p(t_0)$, we will solve different solution θ. Hence we define $U(t_0, z, p(t_0)) := \theta(t_0, z)$ for corresponding to any given $p(t_0)$. When there are N players as mentioned in the Section 3 in Gomes et al. (2013) and characterized by (HJB) in Cecchin and Fischer (2018), the game also has unique

Nash equilibrium point. Denote the equilibrium value function for N players game as $\theta^{(N)}(t, z, p^N)$ for $p^N \in P^N(\Sigma) = \{(\frac{n_1}{N}, \cdots, \frac{n_K}{N}), \quad \sum_{z=0}^{K} n_z = N, n_z \geq 0\}$. Then there exists constant C such that

$$\sum_{z=1}^{K} p_z^N |\theta^{(N)}(t, z, p^N) - U(t, z, p^N)| \leq \frac{C}{N}.$$

Moreover, both θ and $\theta^{(N)}$ are bounded.

We can numerically solve the mean field game $G_{c,R}$ by solving the corresponding forward backward ODE system for the value function and probability of mean field.

4.4 Numerical Test

In this section, with deep neural network technique, we numerically solve forward backward ODE system corresponding to the finite state mean field game defined in (4.10). The market order arrival intensity function is defined as $\Lambda(\delta) := Ae^{-k\delta}$ and the liquidity penalty $l(q) := aq^2$. We assume that initial mean field of state p_0 is 0 for all components except $p_0(0, 0) = 1$. It means all market makers start at 0 inventory and 0 trading volume.

4.4.1 Rank based trading volume reward v.s No reward

The value function θ and optimal bid ask spread are solved with neural network, and compared with those derived from benchmark model in Avellaneda and Stoikov (2008) where $R = 0$ and there is no trading volume reward. When $R = 0$, the forward backward ODE system is decoupled and we can numerically solve the equation in a standard approach. The terminal value of value function in this case does not depend on the mean field of state. Hence we can first solve the backward ODE of value function by Euler scheme, and substitute the value function and optimal bid/ask spread solved to the forward ODE system to solve the mean field of state.

The parameters used are defined by Table 4.1.

S	σ (hours)	γ	T (hours)	Q	v_{max}	k	a	A	R	c
20.0	0.01	1.0	10.0	1	10	2.0	2.0	0.5	2.0	0.0

Table 4.1: Parameters

The training result for the deep neural network is fairly satisfactory and the average loss is lower than 0.003. We then present the value function θ comparison with different initial v and the same q.

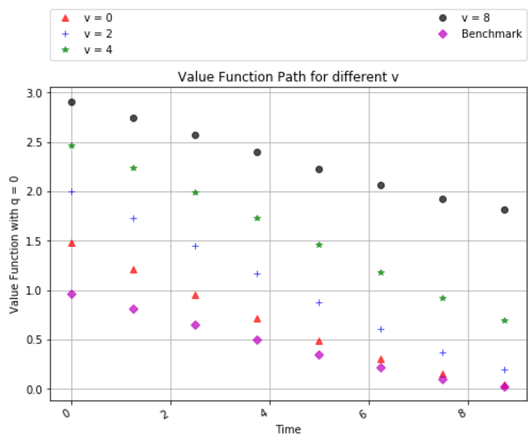


Figure 4.1: $\theta(t, 0, v)$ Path

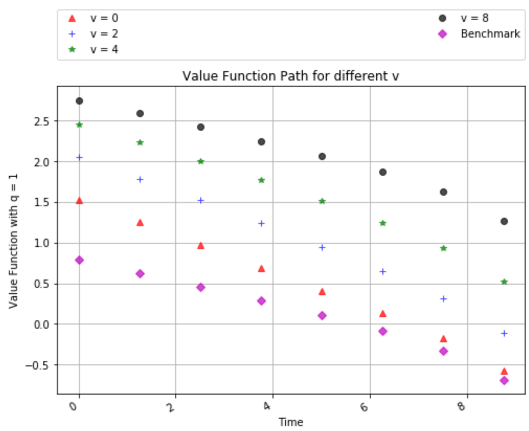
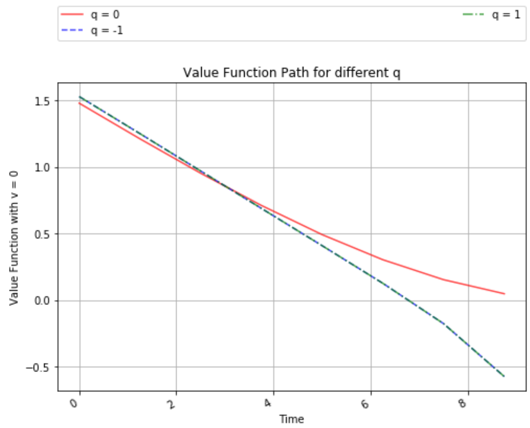


Figure 4.2: $\theta(t, 1, v)$ Path

By comparing our model ($R = 2$) and the benchmark model ($R = 0$) in Figure 4.1 and 4.2, we notice that the introduction of market incentive R increases the value functions for market makers. And the higher is the initial trading volume v (with same inventory q), the higher is the value function. Even for market makers with initial trading volume $v = 0$, their value functions are still higher than the one from benchmark as they benefit from their potential capacity of trading in the future and the corresponding potential market incentive gains. That is also why their value functions converge to the benchmark's one when $t \to T$. At the mean time, value functions for different initial q but the same v are presented in Figure 4.3 and 4.4.



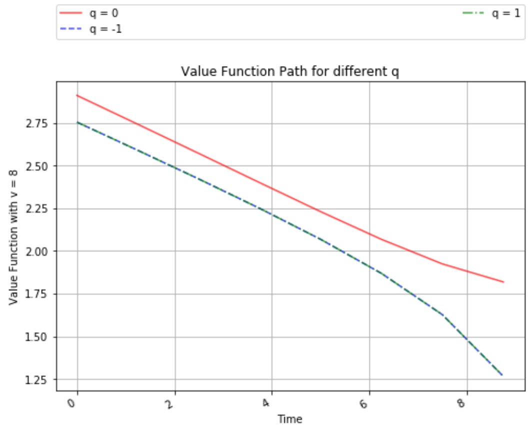
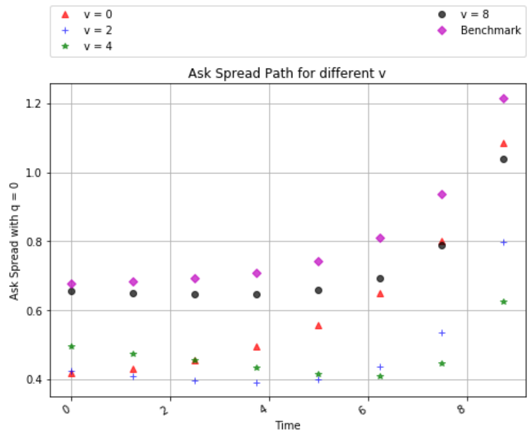


Figure 4.3: $\theta(t, q, 0)$ Path

Figure 4.4: $\theta(t, q, 4)$ Path

The value functions for $q = 1$ and $q = -1$ in Figure 4.3 and 4.4 coincide because of symmetry. Moreover, we present the comparison of optimal ask spread for our model and the benchmark model in Figure 4.5 and 4.6.



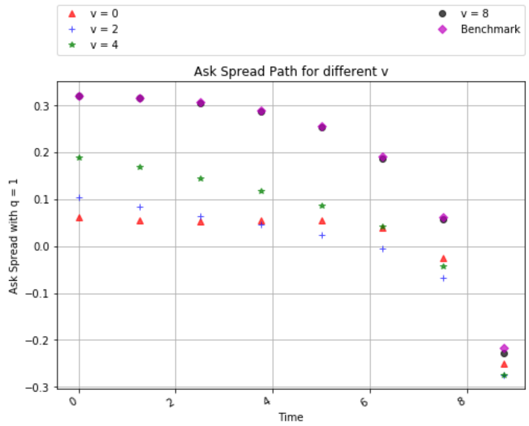


Figure 4.5: Ask Spread Comparison ($q = 0$)

Figure 4.6: Ask Spread Comparison ($q = 1$)

Figure 4.5 and 4.6 suggest that when the rank based trading volume market incentive is in place ($R > 0$), the lowest ask spread among all market makers is generally lower. However, the optimal ask spread is not monotone decreasing w.r.t the initial trading volume v. It is due to the nature of the rank based competition. Take the ask spread of $v = 0$ case as an example. At the beginning when most of market makers have not received any order flow yet, they have total trading volume $v = 0$. One market maker with $v = 0$ can easily get higher trading volume ranking if she succeeds to trade one unit of asset making her total trading volume to 1, which is higher than the total trading volume of most market makers at that time. Hence in this case market makers have stronger motivation to quote a lower ask spread to attract order flow. However, as time goes by and closed to T, most of other market makers already have trading volume v above 3 (See Figure 4.8), far above 0. In this case, for a market maker with $v = 0$, even if they are able to trade one more time, they can still not be able to improve their trading volume ranking, as it can only make their total trading volume equal to 1, but the relative ranking remains unchanged. And as time is quite closed to T, the market maker is quite unlikely to trade more than one unit. Hence they lack the motivation to reduce their ask spread and sacrifice their profit. That is why the ask spread for $v = 0$ is quite low at the beginning of time compared with others, while becomes relatively higher than others as time closed to T.

We plot the total probability paths among different q for trading volume v, i.e $p(t, v) = \sum_{q \in \mathbf{Q}} p(t, q, v)$.

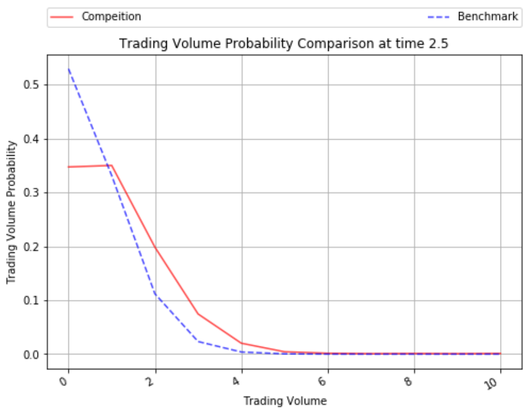


Figure 4.7: $p(2.5, v)$

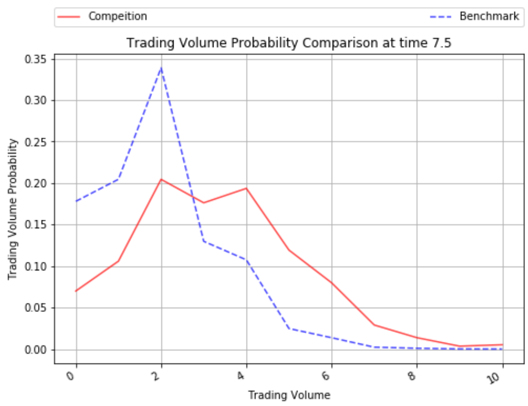


Figure 4.8: $p(7.5, v)$

From Figures 4.7 and 4.8, we can conclude that the introduction of trading volume rank based market making incentive increases the market trading volume, which in turn improves the market liquidity.

4.4.2 Rank based trading volume reward v.s Linear trading volume reward

It is suggested in last section that exchange can increase their market liquidity and reduce the implicit trading cost by introducing trading volume related reward for market makers. Different exchanges have proposed different forms of trading volume related reward in their market making incentive programs. There are two typical schemes of trading volume reward, and most of exchanges' incentive programs are just mixture of the two. One is the rank based trading volume reward, which is the focus of this chapter and market makers try to maximize their expected payoff in (4.2). Another is the linear trading volume reward. $R(v_T)$ in this case is defined below:

$$R(v_T) := R\frac{v_T}{v_{max}}. \tag{4.11}$$

Since $R(v_T)$ under linear trading volume reward scheme does not depend on the mean field of state, the forward backward ODE system that its value function and mean field of state satisfy is also decoupled. Similar to the case when $R = 0$, we can again apply the standard numerical scheme like Euler scheme, to solve its value function and mean field of state numerically. Then we can compare the value function, optimal bid/ask spread as well as the mean field of states under two different design of trading volume reward scheme.

To be comparable, the maximum reward constant R are set the same for the two schemes, and all parameters are the same as Table 4.1. But the rank based trading volume reward scheme introduces competition between market makers, while the linear trading volume reward scheme does not. The value function and optimal bid/ask spread for market makers with rank based trading volume reward scheme can be numerically solved by the MFG deep neural network scheme introduced by Chapter 5 in this thesis, while the one with linear trading volume reward can be obtained by solving the corresponding HJB equation numerically with Euler scheme, as it is a stochastic optimal control problem similar to the one in Avellaneda and Stoikov (2008). We will compare market makers' value functions, optimal bid/ask spreads as well as the probability distribution of their trading volumes between these two schemes.

We first present the value function θ comparison with different initial v and the same q.

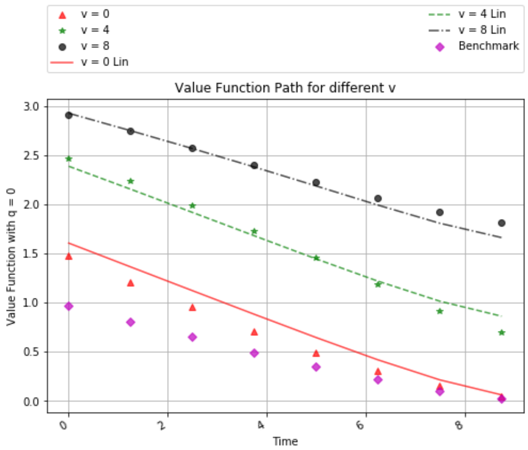


Figure 4.9: $\theta(t, 0, v)$ Path for two schemes

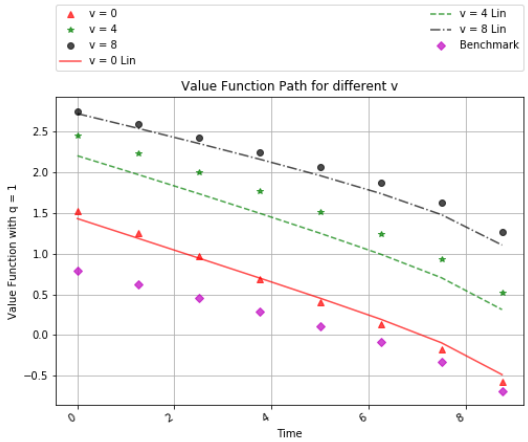


Figure 4.10: $\theta(t, 1, v)$ Path for two schemes

The '$v = 0$ lin' corresponds to the path of value function under linear trading volume reward scheme with initial trading volume $v = 0$, while '$v = 0$' corresponds to the one under rank based trading volume reward scheme. The benchmark model is still the one when $R = 0$. Other legends of this figure and the following figures are all defined similarly. There is not large gap between market makers' value functions under two scheme for different initial states. Depending on different initial state, value function under rank based trading volume reward scheme can be larger or lower than the one under linear trading volume reward scheme. But they are all larger than the benchmark one.

At each trade, traders need to pay implicit trading cost, the ask spread quoted by market makers. To compare the implicit trading cost under the two schemes, we compare market makers' optimal ask spreads for both schemes.

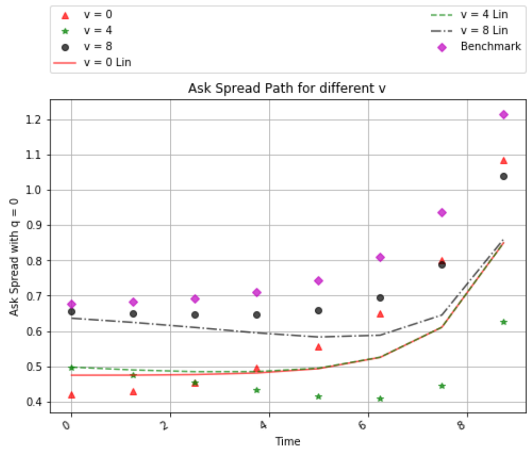


Figure 4.11: Schemes Ask Spread Comparison ($q = 0$)

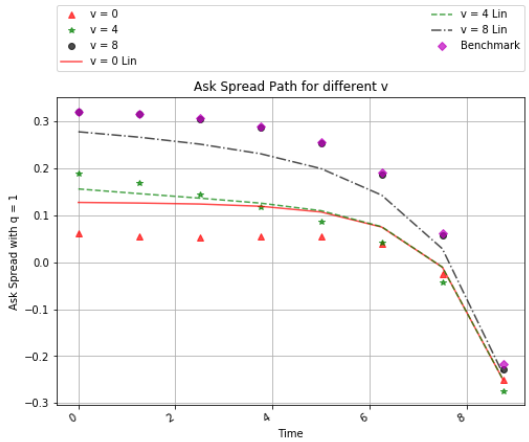


Figure 4.12: Schemes Ask Spread Comparison ($q = 1$)

From Figures 4.11 and 4.12, market makers with different states quote different optimal ask spreads under the two models. Nowadays, traders tend to use algorithm to split their large order into small pieces for lower market impact. Each time, they will only trade small unit. Hence when traders want to buy with market order, they will usually buy at the best ask price offered in the market, which corresponds to the lowest ask spread at each time shown in Figures 4.11 and 4.12. It is a proxy of the implicit trading cost under equilibrium of different models. From Figures 4.11 and 4.12 we find the lowest optimal ask spreads under rank based trading volume reward scheme are lower than the corresponding one under the linear scheme. In fact, the rank based reward scheme serves to diverge the difference between optimal ask spreads quoted by market makers with different trading volume.

Meanwhile, traders sometime might fail to trade at the best price in the market when they are asked to executed the trade in short time horizon. In this case, a better proxy of the implicit trading cost is the bid/ask spreads weighted by the market order arrival intensity on those spreads.

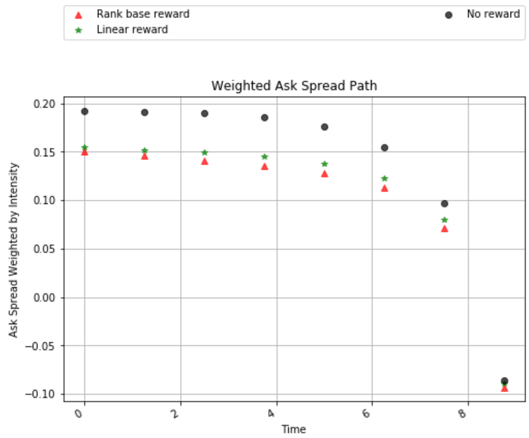


Figure 4.13: Weighted Ask Spread Comparison

For both proxies, we can conclude that rank based trading volume reward scheme is better in terms of reducing implicit trading cost.

Furthermore, market liquidity means the ease of buying or selling assets when needed. It can be evaluated by the trading volume of the whole market, which can be derived from the average trading volume for each market maker in the market. As exchanges profit from the commission fees that traders pay to trade in their venues, the average trading volume for each market maker also affects exchanges' revenue. To compare the liquidity in markets under the two schemes, we plot the total probability paths for trading volume v, i.e $p(t,v) = \sum_{q \in \mathbf{Q}} p(t,q,v)$ similar to last section.

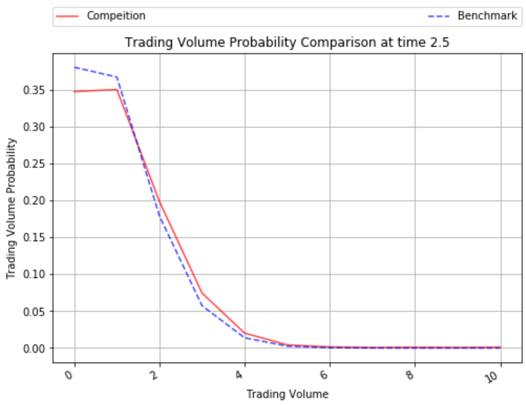


Figure 4.14: $p(2.5, v)$

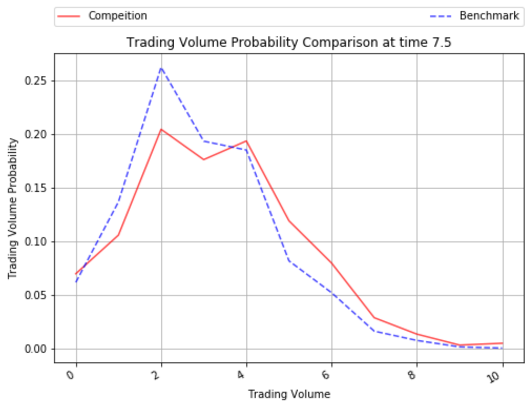


Figure 4.15: $p(7.5, v)$

From Figures 4.14 and 4.15, the trading volume distribution under rank based trading volume reward scheme has higher weights on higher trading volume.

Moreover, we also provide the expected trading volume path as following.

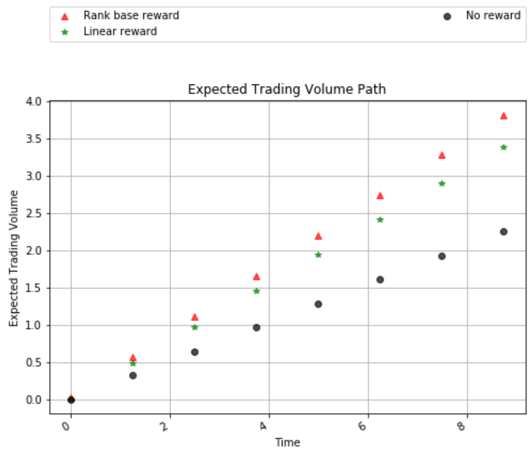


Figure 4.16: Expected Trading Volume Comparison

From Figures 4.16, the expected trading volume is higher under rank base trading volume reward scheme. It suggests that rank based trading volume reward scheme performs better in terms of providing liquidity to the market and increasing revenue for exchanges.

The numerical result suggests that introducing trading volume reward can increase market liquidity and reduce implicit trading cost. Among the two most frequently seen trading volume reward scheme, rank based trading volume reward scheme, which introduces competition among market makers, performs better in liquidity provision and trading cost reduction than the linear trading volume reward.

4.5 Conclusion

In this chapter, we discuss market makers' competition for the market making incentive reward scheme when it depends on market makers' trading volume ranking in the market. We use a mean field game approach to approximate the reality that is difficult to tackle due to curse of dimensionality. We numerically solve the equilibrium with deep neural network approach, and compare market makers' strategies under equilibrium with different types of market making incentive scheme. We find that introducing trading volume reward serves to increase market liquidity and reduce implicit trading cost. Among the two most frequently seen trading volume reward scheme, rank based trading volume reward scheme is better in terms of lowering the best spread in the market, comparing with no reward or linear trading volume reward.

4.6 Proof of Proposition 4.3.1

Proof. The proof is divided to several steps to prove the conditions for H and G respectively.

Step 1: proof of λ^* for Assumption 5.2.1.

Let's first write out the Hamilton operator H for $G_{c,R}$. Define $\mathcal{A}$ as the admissible control set for all λ that satisfy (4.8). Define $\delta^a := \Lambda^{-1}(\lambda_{\beta^a(z)}(t,z))$ and $\delta^b := \Lambda^{-1}(\lambda_{\beta^b(z)}(t,z))$, then we have

$$\begin{aligned} H(z,\mu) = &\sup_{\lambda\in\mathcal{A}}\{g(\Lambda^{-1}(\lambda_{\beta^a(z)}(t,z)),\mu_{\beta^a(z)}) + g(\Lambda^{-1}(\lambda_{\beta^b(z)}(t,z)),\mu_{\beta^b(z)}) \\ &- \frac{1}{2}\gamma\sigma^2 Z_1^{-1}(z)^2\} = \sup_{\delta^a\in\mathbb{R}}\{g(\delta^a,\mu_{\beta^a(z)})\} + \sup_{\delta^b\in\mathbb{R}}\{g(\delta^b,\mu_{\beta^b(z)})\} - \frac{1}{2}\gamma\sigma^2 Z_1^{-1}(z)^2, \end{aligned}$$

where

$$g(\delta,\mu) := \Lambda(\delta)(\delta - c + \mu).$$

From (4.1) and according to the proof of Lemma 3.1 in Guéant (2017), $\zeta(\mu) := \sup_\delta\{g(\delta,\mu)\}$ is increasing w.r.t μ. Moreover, the optimal δ^* exists and is unique, which is a continuously differentiable function of μ.

Step 2: proof of H satisfying Assumption 5.2.1.

We only need to prove that the second order derivative $\zeta''(\mu)$ is positive. From the proof of Lemma 3.1 in Guéant (2017), ζ is $\mathcal{C}^2$, $\zeta'(\mu) = \Lambda(\delta^*)$, and δ^* is strictly decreasing w.r.t μ. Hence $\Lambda(\delta^*)$ is strictly increasing w.r.t μ, which implies $\zeta''(\mu) > 0$. Then there exists constant C such that $\zeta''(\mu) > C$ when μ is bounded.

Step 3: proof of G satisfying Assumption 5.2.1.

From (4.10), the differentiablity and (5.7) of G are trivial. We then only need to prove (5.8). Notice that

$$\begin{aligned}&\sum_{z\in\Sigma}(G(z,p_z)-G(z,\bar{p}_z))(p_z-\bar{p}_z)\\&=\sum_{v=0}^{v_{max}}\sum_{q=-Q}^{Q}\sum_{i=v}^{v_{max}}(\bar{p}(T,i)-p(T,i))(p(T,q,v)-\bar{p}(T,q,v))R,\end{aligned} \tag{4.12}$$

where

$$p(t,v) := \sum_{q=-Q}^{Q} p(t,q,v), \quad \bar{p}(t,v) := \sum_{q=-Q}^{Q} \bar{p}(t,q,v).$$

Define further

$$\begin{aligned}x_i &:= p(T,i) - \bar{p}(T,i)\\ \vec{x} &= (x_0, \cdots, x_{v_{max}}),\end{aligned}$$

and reorganize the term in (4.12), we have

$$\sum_{z\in\Sigma}(G(z,p_z)-G(z,\bar{p}_z))(p_z-\bar{p}_z) = -\sum_{v=0}^{v_{max}}\sum_{i=v}^{v_{max}} x_v x_i R.$$

We have following:

$$2\sum_{v=0}^{v_{max}}\sum_{i=v}^{v_{max}} x_v x_i = (\sum_{v=0}^{v_{max}} x_v)^2 + \sum_{v=0}^{v_{max}} x_v^2 \geq 0.$$

This conclude the proof. □

5

Continuous Time Finite State Mean Field Game: A Deep Learning Approach

5.1 Introduction

Mean field game is introduced by Larsry and Lions in Lasry and Lions (2007) and by Huang in Huang et al. (2006) as a limit of symmetric non-zero sum non-cooperative N-player dynamic games when the number of players $N \to +\infty$. More detailed introduction to the topic can be seen in Carmona and Delarue (2013). Though there has been literature on different classes of mean field game, in this paper we focus on continuous time finite state mean field game, i.e mean field game in finite time horizon, with continuous time state dynamic of each agent taking values in a finite set under fully symmetric payoff and complete information. This finite state mean field game is first studied by Gomes, Mohr and Souza in Gomes et al. (2013). They prove both the existence and uniqueness of Nash equilibrium by looking into the coupled forward backward ODE system which characterizes the equilibrium. When the time horizon is small, they also prove the convergence of N-player game's Nash equilibrium to that of the limiting mean field game when $N \to +\infty$. In Cecchin and Fischer (2018), they analyze the mean field game with a probabilistic approach, which is also used by Carmona and Wang in Carmona and Wang (2018). Carmona and Wang use BSDE approach to prove the existence of equilibrium when both mean field of states and mean field of controls are in the model. They further prove uniqueness of equilibrium when the Hamiltonian does not depend on mean field of control. In Carmona and Wang (2016), they also analyze finite state mean field game between one major player and infinite number of minor players. Besides the existence and uniqueness of Nash equilibrium, the convergence result sees the breakthrough in Cardaliaguet et al. (2015), where Cardaliaguet et al studies mean field game in the diffusion case with common noise. They characterize the equilibrium with Master equation, and the convergence argument is based on the regularity of Master equation's solution. Cecchin and Pelino follow his approach and apply the Master equation to obtain the convergence of feedback Nash equilibrium in the finite state space scenario. It extends the convergence result in Gomes et al. (2013) without the need to assume time horizon is small.

However, though we can prove the existence, uniqueness and convergence for Nash equilibrium of the finite state mean field game, there is still obstacle in our way to approximate the N-player game (with curse of high dimension) with a simpler mean field game. The Nash equilibrium of finite state mean field game is characterized by a forward backward ODE system, half number of which only has initial conditions, and the other half has only terminal conditions. This initial-terminal value problem generally has no analytical solution. It is also difficult to solve it numerically, as the finite difference methods frequently used in solving single direction ODE system fails due to the non-classical forward backward feature and non-regular boundary conditions. One frequent used method for solving general forward backward ODE is the shooting method, but there is no guarantee for convergence. In Gomes and Saude (2017), Gomes proposes a numerical scheme to solve finite state mean field game. However, they need to assume the differential operators in the forward backward ODE system to satisfy some monotone conditions, which does not hold for many cases in application.

In light of the recent fast-growing research interest in applying deep neural network (DNN for short) to solve PDE, and given that the feature of forward backward ODE system is similar to a PDE, we are motivated to use DNN to numerically solve the forward backward ODE system that appears in the finite state mean field game problem. Quite some existing literature is about how to solve high dimensional PDEs by DNN. Lee and Kang (1990), Lagaris et al. (1998), Lagaris et al. (2000), Malek and Beidokhti (2006) and Rudd (2013) use neural networks to solve different kinds of PDEs and ODEs with different boundary conditions. Sirignano and Spiliopoulos (2018) focus on solving high dimensional PDEs with a mesh-free DNN. Their approach is similar in spirit to Galerkin methods, except that the solution is approximated by a neural network instead of a linear combination of basis functions. They also prove the convergence of approximation to the true solution of certain type of PDEs. However, to our best knowledge, there is still no result in existing literature that let us infer the error between the approximation and the true solution by the loss function. It means the approximation might not be accurate enough even if the loss function is already small. The approach in this chapter is similar to Sirignano and Spiliopoulos (2018), but we provide a error bound estimation to fill this gap.

The main contribution of this chapter is to provide a deep neural network approach to solve the forward backward ODE system arising from the finite state mean field game problem in Gomes et al. (2013) and Cecchin and Pelino (2019). We provide an estimation of error between true solution and our DNN approximation by inferring the loss function. It is crucial for us to estimate the accuracy of our numerical solution, without which, even when we have trained the DNN such that the loss function is very small, we are still not sure how close is our DNN approximation to the true solution.

The chapter is organized as follows. Our finite state mean field game model is presented in Section 5.2. Then in Section 5.3, we present the main results: convergence and error estimation of our deep neural network approach. And all detailed proof is in Section 5.4.

5.2 Model Setting

Define a finite state mean field game in continuous time with same setting as the one in Cecchin and Pelino (2019). The finite state space is $\Sigma = \{1, \cdots, K\}$, and the reference game player's state is denoted by z, which is a Markov chain. Game player at state z only can control the switching intensities of their own state process. Their controls $\lambda : [0, T] \times \Sigma \to (\mathbb{R}^+)^K$ are feedback in Σ, and take values in $(\mathbb{R}^+)^K$ as from z there are K possible directions to switch. However if there are some states that state z can not access, then we can simply force the corresponding components in the intensity vector to 0. The probability measure on mean field of state is a function $p : [0, T] \to P(\Sigma)$, where

$$P(\Sigma) = \{(p_1, \cdots, p_K), \quad s.t \quad \sum_{z=1}^{K} p_z = 1, \quad p_z \geq 0\}.$$

Starting at time $t \in [0, T]$, given any probability measure p on the mean field of state, game player with controlled state process Z_t that starts at state z solves the following optimization problem.

$$\theta_z(t) := \sup_{\lambda \in \mathcal{A}} \mathbb{E}_t[\int_t^T F(Z_t, \lambda(t, Z_t))dt + G(Z_T, p(T))], \tag{5.1}$$

where $\mathbb{E}_t$ is the conditional expectation operator given the initial state $Z_t = z$ at time t. And F is the running profit. We assume for any $z \in \Sigma$, $F(z, \lambda)$ is a upper bounded function which does not depend on λ_z, the zth component of λ. G is the terminal payoff, and $\mathcal{A}$ is admissible control set that contains all measurable function $\lambda : [0, T] \times \Sigma \to (\mathbb{R}^+)^K$. Define $\theta : [0, T] \to \mathbb{R}^K$ by

$$\theta(t) = (\theta_1(t), \cdots, \theta_K(t)).$$

According to Cecchin and Pelino (2019), in the equilibrium, value function θ and mean field probability p satisfy a forward backward ODE system. The backward equations come from the optimization problem (5.1) given p, while the forward equations come from the consistent condition for probability measure p on mean field of state when everyone follows equilibrium strategy.

$$\begin{aligned}
\frac{d\theta_z(t)}{dt} &= -H(z, \Delta^z \theta(t)), \quad \theta_z(T) = G(z, p(T)), \\
\frac{dp_z(t)}{dt} &= \sum_y p_y(t) \lambda_z^*(y, \Delta^y \theta(t)), \quad p_z(t_0) = p_{z,0},
\end{aligned} \tag{5.2}$$

where operator Δ^z is defined as:

$$\Delta^z \theta(t) := (\theta_1(t) - \theta_z(t), \cdots, \theta_K(t) - \theta_z(t)).$$

And the Hamilton operator $H : \Sigma \times \mathbb{R}^K \to \mathbb{R}$ is defined for any $\mu \in \mathbb{R}^K$ with $\mu_z = 0$ as:

$$H(z, \mu) := \sup_{\lambda \in (\mathbb{R}^+)^K} \{\lambda \cdot \mu + F(z, \lambda)\}.$$

And $\lambda^*(z,\mu) = (\lambda_1^*(z,\mu), \cdots, \lambda_K^*(z,\mu))$ is the optimizer of Hamiltonian $H(z,\mu)$ except for $\lambda_z^*(z,\mu)$, which can be any value since in the proof of our main result we always let $\mu_z = [\Delta^z\theta(t)]_z = \theta_z(t) - \theta_z(t) = 0$ and $F(z,\lambda)$ is independent to λ_z. For notation convenience, we define

$$\lambda_z^*(z,\mu) := -\sum_{y \neq z} \lambda_y^*(z,\mu). \tag{5.3}$$

According to (Gomes et al., 2013, Propostition 1), if H is differentiable and $\lambda^*(z,\mu)$ is positive except the zth element, for $y \neq z$, we have

$$\lambda_y^*(z,\mu) = [D_\mu H(z,\mu)]_y,$$

where $\lambda_y^*(z,\mu)$ is the intensity from state z to state y, and $[D_\mu H(z,\mu)]_y$ denotes the yth component of gradient $D_\mu H(z,\mu)$. As in the following proof of main results, we always have $\mu_z = 0$ when we use $H(z,\mu)$, $D_\mu H(z,\mu)$ or $D^2_{\mu\mu}H(z,\mu)$, for proof simplicity, with a little abuse of notation we can follow Cecchin and Pelino (2019) to define artificially that

$$[D_\mu H(z,\mu)]_z = \lambda_z^*(z,\mu). \tag{5.4}$$

Then we can conclude that

$$\lambda^*(z,\mu) = D_\mu H(z,\mu), \tag{5.5}$$

and the feed back control $\lambda(t,z) = \lambda^*(z, \Delta^z\theta(t))$ under equilibrium.

We next assume H, G and λ^* satisfy following assumptions.

Assumption 5.2.1. Assume under (5.3), $H(z,\mu)$ has unique optimizer $\lambda^*(z,\mu)$ for every μ. H is C^2 w.r.t μ on bounded set; H, $D_\mu H$ and $D^2_{\mu\mu}H$ are locally Lipschitz in μ, where $D_\mu H$ denotes the gradient of H w.r.t μ and $D^2_{\mu\mu}H$ denotes its Hessian matrix; the second derivatives is bounded away from 0 on bounded set, i.e. there exists a constant C such that for any μ in that bounded set satisfying $\mu_z = 0$, we have

$$
\begin{aligned}
\mu \cdot D^2_{\mu\mu} H(z, \mu) \cdot \mu &\geq C^{-1} \|\mu\|^2 \\
\mu \cdot D^2_{\mu\mu} H(z, \mu) \cdot \mu &\leq C \|\mu\|^2
\end{aligned}
\tag{5.6}
$$

Moreover, we assume the cost function G is differentiable, and its directional derivative w.r.t any vector w is Lipschitz in p when p is bounded, i.e there exists constant C such that

$$
|\frac{\partial G}{\partial w}(z, p + \Delta p) - \frac{\partial G}{\partial w}(z, p)| \leq C \|\Delta p\| \|w\|. \tag{5.7}
$$

Assume that G is monotone decreasing in p, i.e. for every $p, \bar{p} \in \mathbb{R}^K$,

$$
\sum_{z \in \Sigma} (G(z, p) - G(z, \bar{p}))(p_z - \bar{p}_z) \leq 0. \tag{5.8}
$$

Note that the assumptions are very similar to the assumptions in Cecchin and Pelino (2019) that guarantees not only the equilibrium's existence, uniqueness, and convergence, but also that it satisfies a well-posed Master equation. The only differences is that we assume G satisfies (5.7) and (5.8) for p in any bounded set, while in Cecchin and Pelino (2019) it is only for p that is probability measure on the state. Nevertheless, Cecchin and Pelino (2019) only assumes the first inequality in (5.6), while we assume both. But as we both assume $D^2_{\mu\mu} H(z, \mu)$ is Lipschitz continuous to μ when μ is bounded, the second inequality in (5.6) can actually be deduced from the Lipschitz continuity of $D^2_{\mu\mu} H(z, \mu)$ and the fact that μ is bounded. We will show in (5.9) that μ in our paper is indeed bounded.

Remark 5.2.2. For proof simplicity, we will only discuss the case when every state can be accessed from state z, but the proof will be very similar if state y is not accessible from state z. In this case, the running profit $F(z, \lambda)$ does not depend on λ_y either. We can always modify the definition of operator Δ^z such that the yth element of $\Delta^z \theta(t)$ always equals 0. Then in the proof for our main results, we will have $\mu_y = 0$ when we need to use $H(z,\mu)$, $D_\mu H(z,\mu)$ or $D^2_{\mu\mu} H(z,\mu)$. With a little abuse of notation, we can always artificially set the values of yth element of $\lambda^*(z,\mu)$ and $D_\mu H(z,\mu)$, the yth column and yth row of $D^2_{\mu\mu} H(z,\mu)$ to 0. We require H to satisfy (5.6) in Assumption 5.2.1 only when μ satisfies $\mu_y = 0$. Then every step in the proof for our main results is still applicable. Note that it applies similar to the case when there are multiple states that are not accessible from state z.

As λ is always non-negative, besides assumptions above, H also satisfies following property.

Remark 5.2.3. *H satisfies $H(z,\mu) \geq H(z,\bar{\mu})$ for any $z \in \Sigma$ if two vectors $\mu = (\mu_1, \cdots, \mu_K)$ and $\bar{\mu} = (\bar{\mu}_1, \cdots, \bar{\mu}_K)$ satisfy*

$$\mu_i \geq \bar{\mu}_i, \quad i \in \Sigma,$$

Then from (Gomes et al., 2013, Proposition 2), solution to (5.2) has a prior bound C_{GH} as long as H satisfies Remark 5.2.3 and G is bounded for all $p(T)$ in compact set $[0,1]^K$. C_{GH} is defined as,

$$\|\theta\| \leq C_{GH} := \max_{z\in\Sigma, p\in[0,1]^K} \{G(z,p)\} + 2 \max_{z\in\Sigma} H(z,0)T, \tag{5.9}$$

where the norm $\|\cdot\|$ is the ∞ norm. G is bounded because it is continuous and defined on a compact set. For given H and G, as θ satisfies ODE system (5.2), and both H is Lipschitz continuous in Assumption 5.2.1, $\frac{d\theta_z(t)}{dt}$ is also bounded. Similarly, as $D_\mu H$ and $\frac{d\theta_z(t)}{dt}$ are bounded, we can further see that $\frac{d^2\theta_z(t)}{dt^2}$ is bounded. From similar argument on p and λ^*, $\frac{dp_z(t)}{dt}$ and $\frac{d^2p_z(t)}{dt^2}$ are also bounded. It means for given H and G, there exists constants $C_{\theta GH}$ and C_{pGH}, such that

$$\begin{aligned} \|\frac{d\theta_z(t)}{dt}\| \leq C_{\theta GH}, \quad \|\frac{d^2\theta_z(t)}{dt^2}\| \leq C_{\theta GH}, \\ \|\frac{dp_z(t)}{dt}\| \leq C_{pGH}, \quad \|\frac{d^2p_z(t)}{dt^2}\| \leq C_{pGH}. \end{aligned} \tag{5.10}$$

We further summarize Theorem 2 in Gomes et al. (2013), Theorem 1 in Cecchin and Pelino (2019), and provide following theorem without proof.

Theorem 5.2.4. *Under Assumption 5.2.1, ODE system (5.2) has unique solution* (θ, p) *for any initial value* $p(t_0) \in P(\Sigma)$*. The mean field game has an unique Nash equilibrium point.*

In the following sections, we always assume Assumption 5.2.1, which guarantees the existence, uniqueness and convergence of the finite state mean field game. However, to find the equilibrium, we need to solve (5.2), which generally does not have analytical solution. As (5.2) is a forward backward ODE system, we can not solve it numerically by discretization. Hence we provide a deep learning approach to numerically solve (5.2).

5.3 Main Results

To solve (5.2) numerically, we apply the deep neural network approach in Sirignano and Spiliopoulos (2018). Define two sets of neural network functions as

$$\begin{aligned}
&\mathbf{\Theta}^n(\nu_1, \nu) := \{\tilde{\theta} : [0, T] \to \mathbb{R}^K; \quad \tilde{\theta}(t) \\
&= (\nu_1(\sum_{i=1}^{n} \beta_{1,i}\nu(\alpha_i t + c_i))), \cdots, \nu_1(\sum_{i=1}^{n} \beta_{K,i}\nu(\alpha_i t + c_i)))\}, \\
&\mathbf{P}^n(\nu_2, \nu) := \{\tilde{p} : [0, T] \to \mathbb{R}^{K-1}; \quad \tilde{p}(t) \\
&= (\nu_2(\sum_{i=n+1}^{2n} \beta_{1,i}\nu(\alpha_i t + c_i))), \cdots, \nu_2(\sum_{i=n+1}^{2n} \beta_{K-1,i}\nu(\alpha_i t + c_i)))\},
\end{aligned}$$

where $\nu : \mathbb{R} \to \mathbb{R}$ is the triple continuously differentiable activation function, and two strictly increasing triple continuously differentiable activation functions $\nu_1, \nu_2 : \mathbb{R} \to \mathbb{R}$ have twice continuously differentiable inverse functions ν_1^{-1} and ν_2^{-1}. They satisfy

$$\sup |\nu_1| = C_{GH} + e, \quad \inf \nu_2 = -e, \quad \sup \nu_2 = 1 + e, \tag{5.11}$$

where e is a small enough constant. Moreover, we assume the bounds on above inequalities are strict. We approximate the solution (θ, p) to (5.2) numerically by $(\tilde{\theta}^{(N)}, \tilde{p}^{(N)})$, which satisfy

$$\begin{aligned} &(\tilde{\theta}_1^{(N)}, \cdots, \tilde{\theta}_K^{(N)}) \in \mathbf{\Theta}^N(\nu_1, \nu) \\ &(\tilde{p}_1^{(N)}, \cdots, \tilde{p}_{K-1}^{(N)}) \in \mathbf{P}^N(\nu_2, \nu) \\ &\tilde{p}_K^{(N)} = 1 - \sum_{i \neq K} \tilde{p}_i^{(N)}. \end{aligned} \tag{5.12}$$

For fixed $n = N$, the structure of the neural network is determined, and it remains to train the neural network. By considering both the differential operator and boundary condition in (5.2), we define the loss function Ψ w.r.t any approximated solution $(\tilde{\theta}, \tilde{p})$ as

$$\begin{aligned} \Psi(\tilde{\theta}, \tilde{p}) := &\sum_{z \in \Sigma} \{ \int_{t_0}^{T} (\frac{d\tilde{\theta}_z(t)}{dt} + H(z, \Delta^z \tilde{\theta}(t)))^2 dt \\ &+ \int_{t_0}^{T} (\frac{d\tilde{p}_z(t)}{dt} - \sum_{y} \tilde{p}_y(t) \lambda_z^*(y, \Delta^y \tilde{\theta}(t)))^2 dt + \int_{t_0}^{T} (\sum_{z} (\tilde{p}_z(t))^-)^2 dt \\ &+ (\tilde{p}_z(t_0) - \tilde{p}_{z,0})^2 + (\tilde{\theta}_z(T) - G(z, \tilde{p}(T)))^2 \\ &+ \sum_{z \in \Sigma} (B_\theta - \max_{t \in [0,T]} |\frac{d^2 \tilde{\theta}_z(t)}{dt^2}|)^- + \sum_{z \in \Sigma} (B_p - \max_{t \in [0,T]} |\frac{d^2 \tilde{p}_z(t)}{dt^2}|)^- \}. \end{aligned} \tag{5.13}$$

where $(\tilde{p}_K(t))^- := -\tilde{p}_K(t) \mathbb{1}_{\{p_K(t) \leq 0\}}$ and B_θ, B_p can be any constants that satisfy

$$\begin{aligned} B_\theta > C_{\theta GH} \geq \max_{t \in [0,T]} |\frac{d^2 \theta_z(t)}{dt^2}|, \\ B_p > C_{pGH} \geq \max_{t \in [0,T]} |\frac{d^2 p_z(t)}{dt^2}|. \end{aligned}$$

where constants $C_{\theta GH}$ and C_{pGH} are from (5.10). Then it follows

$$\sum_{z \in \Sigma} (B_\theta - \max_{t \in [0,T]} |\frac{d^2 \theta_z(t)}{dt^2}|)^- + \sum_{z \in \Sigma} (B_p - \max_{t \in [0,T]} |\frac{d^2 p_z(t)}{dt^2}|)^- = 0.$$

Both the integral term and maximum term in (5.13) can be calculated via Monte Carlo simulation. Practically, we use similar approach as in Sirignano and Spiliopoulos (2018) to calculate these two to increase the robustness of training. Given N, the structure of the neural network has been determined. We train the network by finding the optimal values of $\{\beta_{j,i}\}_{i,j=1}^{2K-1,2n}$, $\{\alpha_i\}_{i=1}^{2n}$ and $\{c_i\}_{i=1}^{2n}$ that determine $(\tilde{\theta}^{(N)}, \tilde{p}^{(N)})$ such that they minimize Ψ. For the true solution (θ, p), $\Psi(\theta, p) = 0$. Since (θ, p) exists and is unique, Ψ has unique minimal point $\Psi(\theta, p) = 0$. We provide the convergence result Theorem 5.3.1 similar to the Theorem 7.1 in Sirignano and Spiliopoulos (2018).

Theorem 5.3.1. *There exists a sequence of $(\tilde{\theta}^{(N)}, \tilde{p}^{(N)})$ defined in (5.12) such that*

$$\lim_{N \to +\infty} \Psi(\tilde{\theta}^{(N)}, \tilde{p}^{(N)}) = 0.$$

The proof is given later. When the Loss function Ψ is smaller than certain value, because of the uniform bounds on the approximation function's first and second derivative, the maximum error on the time interval is also smaller than certain value. Hence besides the convergence, we also provide our main result as the following error estimation on the DNN approximation.

Theorem 5.3.2. *For every $t \in [t_0, T]$ and $z \in \Sigma$, assume $\tilde{\theta}(t)$ and $\tilde{p}(t)$ satisfy:*

$$\begin{aligned}
\frac{d\tilde{\theta}_z(t)}{dt} &= -H(z, \Delta^z \tilde{\theta}(t)) + \epsilon_1(t, z) \\
\tilde{\theta}_z(T) &= G(z, \tilde{p}(T)) + \epsilon_3(z), \\
\frac{d\tilde{p}_z(t)}{dt} &= \sum_y \tilde{p}_y(t) \lambda_z^*(y, \Delta^y \tilde{\theta}(t)) + \epsilon_2(t, z) \\
\tilde{p}_z(t_0) &= p_{z,0} + \epsilon_4(z),
\end{aligned} \tag{5.14}$$

where $p_0 \in P(\Sigma)$, $\tilde{p}_K(t) = 1 - \sum_{z \neq K} \tilde{p}_z(t)$ and $\tilde{p}_z(t) \in [0, 1]$ for $z < K$. Then there *exists uniform constant B and N_0, such that when $N > N_0$ and*

$$\sum_{i=1}^{2} |\epsilon_i(t,z)| + \sum_{i=3}^{4} |\epsilon_i(z)| + \sum_{z} (\tilde{p}_z(t))^- \le \frac{1}{N}, \quad \forall (t,z) \in [t_0, T] \times \Sigma,$$

we have for all $t \in [t_0, T]$ and $z \in \Sigma$,

$$|\theta_z(t) - \tilde{\theta}_z(t)| + |p_z(t) - \tilde{p}_z(t)| \le \frac{B}{N}.$$

It suggests that when loss function Ψ is smaller than certain value, which implies the maximum error on the ODE system is also smaller than certain value, the error between DNN approximation $(\tilde{\theta}, \tilde{p})$ and the true solution (θ, p) to (5.2) is linear to the maximum error on the ODE system. Note that the condition that all components of $\tilde{p}(t)$ sum up to 1 implicitly sets $\sum_{z\in\Sigma} \epsilon_2(t,z) = 0$ for all $t \in [t_0, T]$. The detailed proof is given in the proof section in the end of this chapter.

Theorem 5.3.3. *Sequences $\tilde{\theta}^{(N)}$ and $\tilde{p}^{(N)}$ in Theorem 5.3.1 converge uniformly for $t \in [0, T]$:*

$$\lim_{N\to+\infty} \tilde{\theta}^{(N)}(t) = \theta(t), \quad \lim_{N\to+\infty} \tilde{p}^{(N)}(t) = p(t).$$

As when Ψ converges to 0, the derivative of $\tilde{\theta}^{(N)}$ and $\tilde{p}^{(N)}$ are uniform bounded. Hence Ψ converge to 0 in Theorem 5.3.3 will guarantee that all the ϵ_i and $(\tilde{p}_K(t))^-$ also converge to 0 uniformly. Then the proof of Theorem 5.3.3 is trivial by combining Theorems 5.3.1 and 5.3.2.

Remark 5.3.4. Note that though we only prove Theorems 5.3.1 and 5.3.3 for a two layers neural network structure characterized by $\mathbf{\Theta}^n(\nu_1, \nu)$ and $\mathbf{P}^n(\nu_2, \nu)$, which is one of the simplest neural network structure, Theorems 5.3.1 and 5.3.3 can actually be applicable to other more sophisticated neural network structures (more layers, LSTM, etc) since this simple structure is just a special case of those more advanced network. By taking a certain set of parameter values, those advanced network can be reduced to a structure like $\mathbf{\Theta}^n(\nu_1, \nu)$ and $\mathbf{P}^n(\nu_2, \nu)$.

5.4 Proofs

5.4.1 Proof of Theorem 5.3.1

Proof. According to Theorem 5.2.4, there exists unique solution (θ, p) to ODE system (5.2), which is also the unique minimal point for Ψ such that

$$\Psi(\theta, p) = 0.$$

We use $(\nu_i^{-1})'$ to denote the first order derivative of ν_i^{-1} for $i = 1, 2$. From (5.9) we know θ is bounded by C_{GH}. Hence $\frac{d}{dt}\theta_z(t)$ is also bounded uniformly for t and z. Moreover, $p(t) \in P(\Sigma)$ for any $t \in [0, T]$ and hence is also bounded. From the assumption on ν_1, ν_2, we know

$$\begin{aligned} &\theta_z(t) < \sup \|\nu_1\|, \\ &\inf \nu_2 < p_z(t) < \sup \nu_2. \end{aligned}$$

It means θ_z's image is bounded and a strict subset of ν_1^{-1}'s domain. Similar for p_z and ν_2^{-1}. Combining with the continuously differentialability of ν_1^{-1} and ν_2^{-1}, we know $\nu_1^{-1}(\theta_z(t))$, $(\nu_1^{-1})'(\theta_z(t))$, $\nu_2^{-1}(p_z(t))$ and $(\nu_2^{-1})'(p_z(t))$ are bounded by some constant C uniformly for t and z. $(\nu_i)'$ and $(\nu_i)''$ are Lipschitz continuous on $[-2C, 2C]$ with coefficient L for $i = 1, 2$. Define $C^N(\nu) := \{\zeta : [0, T] \to \mathbb{R}; \quad \zeta(t) = \sum_{i=1}^N \beta_i \nu(\alpha_i t + c_i)\}$. According to the proof of Theorem 7.1 in Sirignano and Spiliopoulos (2018), for any $0 < \varepsilon < C$, there exists $N > 0$ and $y_z \in C^N(\nu)$ such that

$$\|y_z(t) - \nu_1^{-1}(\theta_z(t))\| + \|\frac{d}{dt}y_z(t) - \frac{d}{dt}\nu_1^{-1}(\theta_z(t))\| + \|\frac{d^2}{dt^2}y_z(t) - \frac{d^2}{dt^2}\nu_1^{-1}(\theta_z(t))\| \leq \varepsilon. \tag{5.15}$$

Hence we have

$$\|\nu_1(y_z(t)) - \theta_z(t)\| \leq C\|y_z(t) - \nu_1^{-1}(\theta_z(t))\| \leq C\varepsilon.$$

On the other hand,

$$\begin{aligned}
&\frac{d}{dt}\nu_1(y_z(t)) - \frac{d}{dt}\theta_z(t) = \frac{d}{dt}\nu_1(y_z(t)) - \frac{d}{dt}\nu_1(\nu_1^{-1}(\theta_z(t)))\\
&= (\nu_1)'(y_z(t))\frac{d}{dt}y_z(t) - (\nu_1)'(\nu_1^{-1}(\theta_z(t)))\frac{d}{dt}\nu_1^{-1}(\theta_z(t))\\
&= (\nu_1)'(y_z(t))[\frac{d}{dt}y_z(t) - \frac{d}{dt}\nu_1^{-1}(\theta_z(t))]\\
&+ \frac{d}{dt}\nu_1^{-1}(\theta_z(t))[(\nu_1)'(y_z(t)) - (\nu_1)'(\nu_1^{-1}(\theta_z(t)))].
\end{aligned}$$

As $y_z(t) \in [-2C, 2C]$, there exists constant C_1 such that $(\nu_1)'(y_z(t))$ is bounded by it uniformly. Moreover, we have

$$\|\frac{d}{dt}\nu_1^{-1}(\theta_z(t))\| \leq \|(\nu_1^{-1})'(\theta_z(t))\|\|\frac{d}{dt}\theta_z(t)\| \leq C^2.$$

Hence we have

$$\begin{aligned}
&\|\frac{d}{dt}\nu_1(y_z(t)) - \frac{d}{dt}\theta_z(t)\| \leq \|\frac{d}{dt}y_z(t) - \frac{d}{dt}\nu_1^{-1}(\theta_z(t))\|\|\nu_1'(y_z(t))\|\\
&+ \|\nu_1'(y_z(t)) - \nu_1'(\nu_1^{-1}(\theta_z(t)))\|\|(\nu_1^{-1})'(\theta_z(t))\|\|\frac{d}{dt}\theta_z(t)\|\\
&\leq C_1\|\frac{d}{dt}y_z(t) - \frac{d}{dt}\nu_1^{-1}(\theta_z(t))\|\\
&+ C^2L\|y_z(t) - \nu_1^{-1}(\theta_z(t))\| \leq (C_1 + C^2L)\varepsilon.
\end{aligned}$$

The first inequality above comes from the boundness and Lipshitz continuity of ν_1', as well as the boundness of $(\nu_1^{-1})'(\theta_z(t))$. Moreover, for second order derivatives, we have

$$\|\frac{d^2}{dt^2}\nu_1^{-1}(\theta_z(t))\| = \|(\nu_1^{-1})''(\theta_z(t))\frac{d}{dt}\theta_z(t) + (\nu_1^{-1})'(\theta_z(t))(\frac{d}{dt}\theta_z(t))^2\|.$$

As $\theta_z(t)$ and $\frac{d}{dt}\theta_z(t)$ are bounded and (ν_1^{-1}) is twice continuously differentiable, $\frac{d^2}{dt^2}\nu_1^{-1}(\theta_z(t))$ is bounded. To estimate the difference of second order derivatives between approximation function and true function, we have

$$\frac{d^2}{dt^2}\nu_1(y_z(t)) - \frac{d^2}{dt^2}\theta_z(t) = (\frac{d}{dt}y_z(t))^2\nu_1''(y_z(t)) + \nu_1'(y_z(t))\frac{d^2}{dt^2}y_z(t)$$
$$- (\frac{d}{dt}\nu_1^{-1}(\theta_z(t)))^2\nu_1''(\nu_1^{-1}(\theta_z(t))) - \nu_1'(\nu_1^{-1}(\theta_z(t)))\frac{d^2}{dt^2}\nu_1^{-1}(\theta_z(t)).$$

Define

$$a := \|(\frac{d}{dt}y_z(t))^2 - (\frac{d}{dt}\nu_1^{-1}(\theta_z(t)))^2\|\|\nu_1''(y_z(t))\|$$
$$b := \|\nu_1''(y_z(t)) - \nu_1''(\nu_1^{-1}(\theta_z(t)))\|\|(\frac{d}{dt}\nu_1^{-1}(\theta_z(t)))^2\|$$
$$c := \|\frac{d^2}{dt^2}y_z(t) - \frac{d^2}{dt^2}\nu_1^{-1}(\theta_z(t))\|\|\nu_1'(y_z(t))\|$$
$$d := \|\nu_1'(y_z(t)) - \nu_1'(\nu_1^{-1}(\theta_z(t)))\|\|\frac{d^2}{dt^2}\nu_1^{-1}(\theta_z(t))\|,$$

and we have

$$\|\frac{d^2}{dt^2}\nu_1(y_z(t)) - \frac{d^2}{dt^2}\theta_z(t)\| \leq a + b + c + d.$$

As $y_z(t)$ and $\frac{d}{dt}\nu_1^{-1}(\theta_z(t))$ are bounded from previous proof, and ν_1 is triple continuously diferentiable function by definition, $\nu_1''(y_z(t))$, $\nu_1'(y_z(t))$, $(\frac{d}{dt}\nu_1^{-1}(\theta_z(t)))^2$ and $\frac{d^2}{dt^2}\nu_1^{-1}(\theta_z(t))$ are also bounded. Moreover, $\|\frac{d}{dt}y_z(t)\| \leq \|\frac{d}{dt}\nu_1^{-1}(\theta_z(t))\| + \varepsilon$, hence bounded. According to the Lipschitz continuity of ν_1' and ν_1'', as well as (5.15), we know there exists constants C_2, such that

$$\|\frac{d^2}{dt^2}\nu_1(y_z(t)) - \frac{d^2}{dt^2}\theta_z(t)\| \leq a + b + c + d \leq C_2\varepsilon.$$

By making transformation on ε in above proof, we know for any $0 < \varepsilon < C$, there exists $N > 0$ and $y_z \in C^N(\nu)$ such that

$$\|\nu_1(y_z(t)) - \theta_z(t)\| + \|\frac{d}{dt}\nu_1(y_z(t)) - \frac{d}{dt}\theta_z(t)\| + \|\frac{d^2}{dt^2}\nu_1(y_z(t)) - \frac{d^2}{dt^2}\theta_z(t)\| \leq \varepsilon.$$

Hence we know there exists $N > 0$ and $y_z \in C^N(\nu)$ such that

$$\|\frac{d^2}{dt^2}\nu_1(y_z(t))\| \leq \varepsilon + \|\frac{d^2}{dt^2}\theta_z(t)\| \leq \varepsilon + C_{\theta GH} < B_\theta.$$

Similarly, we know $\|\frac{d}{dt}\nu_2(y_z(t))\| \leq C_{pGH} < B_p$. Then we get

$$(B_\theta - \max_{t\in[0,T]} |\frac{d\tilde{\theta}_z(t)}{dt}|)^- = 0,$$
$$(B_p - \max_{t\in[0,T]} |\frac{d\tilde{p}_z(t)}{dt}|)^- = 0$$

If we define

$$\hat{\mathbf{\Theta}}^N(\nu_1,\nu) := \{\zeta : [0,T] \to \mathbb{R}^K; \quad \zeta(t) = (\nu_1(\sum_{i=1}^{N} \beta_{1,i}\nu(\alpha_{1,i}t + c_{1,i}))), \cdots, \nu_1(\sum_{i=1}^{n} \beta_{K,i}\nu(\alpha_{K,i}t + c_{K,i})))\}$$

Then from proof above we know for any $0 < \varepsilon < C$, there exists $N > 0$ and $\tilde{\theta}^{(N)} \in \hat{\mathbf{\Theta}}^N(\nu_1,\nu)$ such that

$$\|\tilde{\theta}_z^{(N)}(t) - \theta_z(t)\| + \|\frac{d}{dt}\tilde{\theta}_z^{(N)}(t) - \frac{d}{dt}\theta_z(t)\| \leq \varepsilon.$$

On the other hand, notice that any function $f_N \in \hat{\mathbf{\Theta}}^N(\nu_1,\nu)$, there should exists $f_{KN} \in \mathbf{\Theta}^{KN}(\nu_1,\nu)$ such that $f_{KN} = f_N$, by letting some $\beta_{j,i} = 0$. It means $\hat{\mathbf{\Theta}}^N(\nu_1,\nu) \subset \mathbf{\Theta}^{KN}(\nu_1,\nu)$, and $\tilde{\theta}^{(N)} \in \mathbf{\Theta}^{KN}(\nu_1,\nu)$. For p and $\mathbf{P}^n(\nu_2,\nu)$, we can have similar argument. Hence we conclude that for any $0 < \varepsilon < C$, there exists $N > 0$ and $\tilde{\theta}^{(N)} \in \mathbf{\Theta}^N(\nu_1,\nu)$, $\tilde{p}^{(N)} \in \mathbf{P}^N(\nu_2,\nu)$ such that

$$\|\tilde{\theta}_z^{(N)}(t) - \theta_z(t)\| + \|\frac{d}{dt}\tilde{\theta}_z^{(N)}(t) - \frac{d}{dt}\theta_z(t)\| \leq \varepsilon$$
$$\|\tilde{p}_z^{(N)}(t) - p_z(t)\| + \|\frac{d}{dt}\tilde{p}_z^{(N)}(t) - \frac{d}{dt}p_z(t)\| \leq \varepsilon.$$

Then similar to the proof for Theorem 7.1 in Sirignano and Spiliopoulos (2018), we know there exists a uniform constant M which only depends on boundedness of θ, λ^* and Lipshcitz coefficient of λ^* and H, such that

$$\Psi(\tilde{\theta}^{(N)}, \tilde{p}^{(N)}) \leq M\varepsilon.$$

It concludes the proof. □

5.4.2 Proof of Theorem 5.3.2

To bridge θ and $\tilde{\theta}$, we use the Master equation for θ in Cecchin and Pelino (2019), and prove that $\tilde{\theta}$ also satisfy a similar equation. The general idea of the proof is similar to that of (Cecchin and Pelino, 2019, Theorem 6), while our purpose is different from Cecchin and Pelino (2019). Cecchin and Pelino (2019) want to prove the equilibrium of finite players finite state game converges to the one of corresponding mean field game. It is difficult to directly compare the two ODE systems characterizing the two equilibrium of finite players game and mean field game respectively, as one is backward ODE system, and one is forward backward ODE system. Hence Cecchin and Pelino (2019) prove that the forward backward ODE system is equivalent to a backward PDE (Master equation), which can then be compared with the backward ODE system. In contrast, we want to estimate the error between true solution and the DNN approximation to mean field game. Both of them satisfy forward backward ODE systems, while the one characterizing the DNN approximation having extra error term compared with the one for true solution. And we leverage the Master equations to bridge the two forward backward ODE systems. However, due to the perturbation term in the ODE system 5.14, $\tilde{p}$ can be negative. One of our key contribution is finding a new way to bypass the non-negative requirement which is required in the proof for (Cecchin and Pelino, 2019, Theorem 6).

Note that the general structure and idea of the proof might look similar to that in (Cecchin and Pelino, 2019, Proposition 5, 6, Theorem 7, Section 5.3.1 and 5.3.3). However, in Cecchin and Pelino (2019), their p satisfies a non-perturbed Kolmogorov forward equation, with initial value always sits in $P(\Sigma)$, hence each component of their p is always non-negative. In contrast, the Kolmogorov forward equation in our model in (5.14) is perturbed, and its initial value does not necessarily locates in $P(\Sigma)$ due to the perturbation term. Hence the $\tilde{p}$ in our proof can be negative. The key difference between our proof and theirs is due to this fact which makes some prior estimations proof in Cecchin and Pelino (2019) not applicable for our case. We need to provide extra modifications in the prior estimations stage in our proof, by adding and subtracting an extra term M_1 simultaneously such that $\tilde{p}(t) + M_1$ is non-negative. Moreover, we also need to modify every later step in the proof to estimate the extra terms introduced by M_1. As these modifications appear in most details of the proof, for the ease of readers' understanding, we still keep the whole proof in this chapter, though its idea might seem similar to that in Cecchin and Pelino (2019).

The proof is organized as following. Notice that the solution pair $(\tilde{\theta}, \tilde{p})$ to (5.14) is determined only by initial time t_0 and initial value $\tilde{p}(t_0)$. We first prove in Proposition 5.4.1 and 5.4.2 that $\tilde{\theta}$ when considered as function of t_0 and $\tilde{p}(t_0)$ is well defined and continuous w.r.t $\tilde{p}(t_0)$ on some neighbourhood. With the help of Proposition 5.4.5 and 5.4.6, we also prove that $\tilde{\theta}$ is continuously differentiable w.r.t $\tilde{p}(t_0)$ in Theorem 5.4.7 by discussing the linearized system (5.25). Then we finally prove in Theorem 5.4.8 that $\tilde{\theta}$, when considered as a function of t_0 and $\tilde{p}(t_0)$, satisfy a PDE similar to the Master equation in Cecchin and Pelino (2019). In order to compare the two Master equations, we prove in Proposition 5.4.9 that Master equation on some discrete grids of $P(\Sigma)$ can be approximated by a backward ODE with extra error term. By comparing the two backward ODE systems, we can finally estimate the difference between θ and $\tilde{\theta}$.

We first define norm $\|f\|$ for f in $\mathbb{R}^K$ or $\mathcal{C}^0([0,T];\mathbb{R}^K)$, where $\mathcal{C}^0([0,T];\mathbb{R}^K)$ contains all continuous functions with domain $[0,T]$ and images in $\mathbb{R}^K$.

$$\|f\| := \begin{cases} \max_{1 \le z \le K} |f_z|, & f \in \mathbb{R}^K \\ \max_{t \in [0,T]} \max_{1 \le z \le K} |f_z(t)|, & f \in \mathcal{C}^0([0,T];\mathbb{R}^K) \end{cases}$$

Due to the introduction of perturbation terms in ODE system (5.14), the existence and uniqueness of its solution can no longer be guaranteed for every initial value $\tilde{p}(t_0)$. However, under certain conditions on (5.14), we can have the existence and prior bound estimation of solution to (5.14).

Proposition 5.4.1. *Given constant $M > 0$, define $I_{p,M} := [-M, 1 + M]^K$ and*

$$C_G(M) := \max_{z\in\Sigma, p\in I_{p,M}} |G(z,p)| + \|\epsilon_3\| + 2\|\epsilon_1\|T + 2\max_{z\in\Sigma} H(z,0)T,$$
$$A_G(M) := [-2C_G(M), 2C_G(M)]^K, \quad \Lambda(M) := \max_{y,z\in\Sigma,\mu\in A_G(M)} |\lambda_y^*(z,\mu)|.$$

If functions $\epsilon_i, i = 2, 4$ satisfy

$$\|\epsilon_2\| + \|\epsilon_4\| < \frac{1}{N_0}, \tag{5.16}$$

where $\frac{1}{N_0} := \frac{1}{3}Me^{-\Lambda(M)T}$. Then for any initial time $t_0 \in [0, T]$ and $\tilde{p}(t_0) \in \bar{B}(P(\Sigma), \frac{1}{N_0})$, ODE system (5.14) has solution $(\tilde{\theta}, \tilde{p})$, where

$$\bar{B}(P(\Sigma), \frac{1}{N_0}) = \{\tilde{p} \in \mathbb{R}^K, \quad s.t \quad \min_{p\in P(\Sigma)} \|\tilde{p} - p\| \le \frac{1}{N_0}\}.$$

Moreover, $(\tilde{\theta}, \tilde{p})$ satisfy following on $[t_0, T]$ uniformly for any initial time $t_0 \in [0, T]$ and initial value $\tilde{p}(t_0)$.

$$\tilde{\theta}_z(t) \in [-C_G(M), C_G(M)], \quad \tilde{p}_z(t) \in [-M, 1 + M].$$

The main idea of the proof is to find a fixed point of the mapping that maps a given prior $\tilde{p}$ to a new $\tilde{p}$. The mapping is constructed by following. With prior $\tilde{p}$, we solve the backward equation in (5.14) to get $\tilde{\theta}$, and solve again the forward equation in (5.14) to get the solution $\tilde{p}$, the image of the mapping. The key step is to show the mapping is self contained in a compact set, then the existence of fixed point can be guaranteed by showing that the mapping is continuous.

Proof. Given a prior $\bar{p}$ such that $\bar{p}(t) \in [-M, 1 + M]^K$ for all $t \in [t_0, T]$, Lipschitz continuous with Lipschitz coefficient bounded by $L(M)$, where

$$\|\frac{d\tilde{p}}{dt}\| \leq L(M) = K(2M+1)\Lambda(M) + \frac{1}{N_0},$$

and starts with the same $\bar{p}(t_0) = \tilde{p}(t_0) \in \bar{B}(P(\Sigma), \frac{1}{N_0})$, with which we solve the backward ODE in (5.14):

$$\frac{d\tilde{\theta}_z(t)}{dt} = -H(z, \Delta^z\tilde{\theta}(t)) + \epsilon_1(t,z), \quad \tilde{\theta}_z(T) = G(z, \bar{p}(T)) + \epsilon_3(z).$$

We know function $\tilde{\theta}(t)$ is bounded by constant $C_G(M)$ following a similar proof as (Gomes et al., 2013, Proposition 2). Note that $C_G(M)$ is monotonically non-decreasing w.r.t M, hence $\Lambda(M)$ is also non-decreasing w.r.t M. Since $\bar{p}(t_0) \in \bar{B}(P(\Sigma), \frac{1}{N_0})$, there exists $p_0 \in P(\Sigma)$ such that $\bar{p}(t_0) - p_0 = \epsilon_4$ where $\|\epsilon_4\| \leq \frac{1}{N_0}$. Consider two functions $\tilde{p}$ and p satisfying

$$\frac{d\tilde{p}_z(t)}{dt} = \sum_y \tilde{p}_y(t)\lambda_z^*(y, \Delta^y\tilde{\theta}(t)) + \epsilon_2(t,z), \quad \tilde{p}_z(t_0) = p_{z,0} + \epsilon_4(z),$$
$$\frac{dp_z(t)}{dt} = \sum_y p_y(t)\lambda_z^*(y, \Delta^y\tilde{\theta}(t)), \quad p_z(t_0) = p_{z,0}$$

Integrating both side and subtracting $\tilde{p}$ and p, we get

$$\|\tilde{p}(t) - p(t)\| \leq \Lambda(M)\int_{t_0}^{t} \|\tilde{p}(s) - p(s)\| ds + \|\epsilon_2\| + \|\epsilon_4\|.$$

By Gronwall inequality, we have

$$\|\tilde{p}(t) - p(t)\| \leq (\|\epsilon_2\| + \|\epsilon_4\|)e^{\Lambda(M)T} < M. \tag{5.17}$$

As p is the solution to a Kolmogorov equation, $p(t) \in P(\Sigma)$. Hence the solution $\tilde{p}(t) \in [-M, 1+M]^K$ for all $t \in [t_0, T]$, and $\tilde{p}$ is also Lipshitz continuous with Lipshitz coefficient bounded by $L(M)$, as $\|\frac{d\tilde{p}}{dt}\| \leq L(M)$.

Let $\mathcal{F}([t_0,T])$ be the set of Lipshitz continuous functions defined on $[t_0,T]$, with Lipshitz coefficient bounded by $L(M)$, taking values in $[-M,1+M]^K$ and starting at the same initial value $\tilde{p}(t_0)$ at t_0. We can define mapping $\xi : \mathcal{F}([t_0,T]) \to \mathcal{F}([t_0,T])$ in the following way: given $\tilde{p} \in \mathcal{F}([t_0,T])$, let $\tilde{\theta}$ be the solution of terminal value problem in (5.14). Then $\tilde{\theta}(t)$ is bounded by $C_G(M)$. Let $\xi(\tilde{p})$ be the solution to the initial value problem in (5.14). $\xi(\tilde{p}) \in \mathcal{F}([t_0,T])$ from the above argument. Following the proof of (Gomes et al., 2013, Proposition 4), $\mathcal{F}([t_0,T])$ is a set of uniformly bounded and equicontinuous functions. Thus, by Arzela-Ascoli theorem, it is a relatively compact set. It is also clear that it is a convex set. Hence, by Brouwer fixed point Theorem, we know there exists fixed point for ξ, which proves the existence of solution to (5.14). □

From Proposition 5.4.1, for every t_0 and $\tilde{p}_0 \in \bar{B}(P(\Sigma), \frac{1}{N_0})$, the ODE system (5.14) has at least one solution $(\tilde{\theta}, \tilde{p})$, bounded by constants that only depends on the given constant M. Note that both these bounds are monotonically increasing w.r.t M. Also, both $\tilde{\theta}$ and $\tilde{p}$ are Lipshitz continuous functions with Lipshitz coefficients uniformly bounded. Define the bound as $L(M)$, since their Lipshitz coefficients only depend on the bounds on $\tilde{\theta}$ and $\tilde{p}$, which again only depend on the constant M. We next prove that under certain condition, $(\tilde{\theta}, \tilde{p})$ is unique and continuous w.r.t initial condition.

Proposition 5.4.2. *There exist positive constants N_0 and C, such that if we have condition (5.16), then for any $t_0 \in [0,T]$ and initial condition $\tilde{p}(t_0) \in \bar{B}(P(\Sigma), \frac{1}{N_0})$, the solution to (5.14) is unique. Moreover, let $(\tilde{\theta}, \tilde{p})$ and $(\hat{\theta}, \hat{p})$ be two solutions to ODE system (5.14) with different initial conditions $\tilde{p}(t_0), \hat{p}(t_0) \in \bar{B}(P(\Sigma), \frac{1}{N_0})$, then*

$$\begin{aligned} \|\tilde{\theta} - \hat{\theta}\| &\le C\|\tilde{p}(t_0) - \hat{p}(t_0)\| \\ \|\tilde{p} - \hat{p}\| &\le C\|\tilde{p}(t_0) - \hat{p}(t_0)\|. \end{aligned} \tag{5.18}$$

The general idea of the proof is similar to that in (Cecchin and Pelino, 2019, Proposition 5). But in (Cecchin and Pelino, 2019, Proposition 5), only the system's continuity to initial value in $P(\Sigma)$ is proved, while in our model our initial value does not necessarily locates in $P(\Sigma)$ due to the perturbation term. We apply Gronwall inequality to the forward ODE and backward ODE that $\tilde{p}-\hat{p}$ and $\tilde{\theta}-\hat{\theta}$ satisfy respectively, with the help of Lipschitz continuity of H and λ^*. And to combine the two coupled inequalities, we start with some prior estimations by differentiating $(\tilde{p}-\hat{p})(\tilde{\theta}-\hat{\theta})$. The key difference between our proof and theirs lies in this prior estimation, where we need to make some modifications and deal with some extra terms, since $\tilde{p}$ and $\hat{p}$ can potentially be negative due to the perturbation terms in (5.14).

Proof. Start with any M and the corresponding N_0 defined in Proposition 5.4.1. Then both $\tilde{\theta}$ and $\hat{\theta}$ uniform bounded by $C_G(M)$. Let's first assume $\tilde{p}_z(t), \hat{p}_z(t) \geq -M_1$ uniformly, and we will decide later the value for M_1 and prove the condition for it. Similarly to the proof for (Cecchin and Pelino, 2019, Proposition 5), we first try to obtain estimation on LHS of (5.21) given later. Define $\phi := \tilde{\theta}-\hat{\theta}$ and $\pi := \tilde{p}-\hat{p}$. Then the couple (ϕ, π) solves

$$\begin{aligned}
\frac{d\phi_z(t)}{dt} &= -H(z, \Delta^z\tilde{\theta}(t)) + H(z, \Delta^z\hat{\theta}(t)) \\
\phi_z(T) &= G(z, \tilde{p}(T)) - G(z, \hat{p}(T)), \\
\frac{d\pi_z(t)}{dt} &= \sum_y \{\tilde{p}_y(t)\lambda_z^*(y, \Delta^y\tilde{\theta}(t)) - \hat{p}_y(t)\lambda_z^*(y, \Delta^y\hat{\theta}(t))\} \\
\pi_z(t_0) &= \tilde{p}_z(t_0) - \hat{p}_z(t_0),
\end{aligned} \tag{5.19}$$

Integrating $\frac{d}{dt}\sum_{z\in\Sigma}\phi_z(t)\pi_z(t)$, we have

$$\begin{aligned}
&\sum_{z\in\Sigma}[\phi_z(T)\pi_z(T)-\phi_z(t_0)\pi_z(t_0)]\\
&=-\int_{t_0}^{T}\sum_{z\in\Sigma}[H(z,\Delta^z\tilde{\theta}(t))-H(z,\Delta^z\hat{\theta}(t))](\tilde{p}_z(t)-\hat{p}_z(t))dt\\
&+\int_{t_0}^{T}\sum_{z,y\in\Sigma}\{\tilde{p}_y(t)\lambda_z^*(y,\Delta^y\tilde{\theta}(t))-\hat{p}_y(t)\lambda_z^*(y,\Delta^y\hat{\theta}(t))\}(\tilde{\theta}_z(t)-\hat{\theta}_z(t))dt.
\end{aligned}$$

As $\sum_z\lambda_z^*=0$, we have $\sum_z\lambda_z^*(y,\Delta^y\tilde{\theta}(t))\phi_y(t)=0$, and

$$\begin{aligned}
&\int_{t_0}^{T}\sum_{z,y\in\Sigma}\{\tilde{p}_y(t)\lambda_z^*(y,\Delta^y\tilde{\theta}(t))-\hat{p}_y(t)\lambda_z^*(y,\Delta^y\hat{\theta}(t))\}(\tilde{\theta}_z(t)-\hat{\theta}_z(t))dt\\
&=\int_{t_0}^{T}\sum_{z,y\in\Sigma}\{\tilde{p}_y(t)\lambda_z^*(y,\Delta^y\tilde{\theta}(t))-\hat{p}_y(t)\lambda_z^*(y,\Delta^y\hat{\theta}(t))\}(\phi_z(t)-\phi_y(t))dt\\
&=\int_{t_0}^{T}\sum_{y\in\Sigma}\Delta^y\phi(t)\cdot\{\tilde{p}_y(t)\lambda^*(y,\Delta^y\tilde{\theta}(t))-\hat{p}_y(t)\lambda^*(y,\Delta^y\hat{\theta}(t))\}.
\end{aligned}$$

Substituting it back, we have equation:

$$\begin{aligned}
&\sum_{z\in\Sigma}\phi_z(t_0)\pi_z(t_0)=\sum_{z\in\Sigma}\phi_z(T)\pi_z(T)\\
&+\int_{t_0}^{T}\sum_{z\in\Sigma}[H(z,\Delta^z\tilde{\theta}(t))-H(z,\Delta^z\hat{\theta}(t))-\Delta^z\phi(t)\cdot\lambda^*(z,\Delta^z\tilde{\theta}(t))]\tilde{p}_z(t)dt\\
&+\int_{t_0}^{T}\sum_{z\in\Sigma}[H(z,\Delta^z\hat{\theta}(t))-H(z,\Delta^z\tilde{\theta}(t))+\Delta^z\phi(t)\cdot\lambda^*(z,\Delta^z\hat{\theta}(t))]\hat{p}_z(t)dt.
\end{aligned}$$

As $\lambda^*(z,\mu)=D_\mu H(z,\mu)$, by Taylor theorem, there exists point a on the line between $\Delta^z\tilde{\theta}(t)$ and $\Delta^z\hat{\theta}(t)$ such that

$$\begin{aligned}
&H(z,\Delta^z\tilde{\theta}(t))-H(z,\Delta^z\hat{\theta}(t))-\Delta^z\phi(t)\cdot\lambda^*(z,\Delta^z\tilde{\theta}(t))\\
&=-\Delta^z\phi(t)\cdot D^2_{\mu\mu}H(z,a)\cdot\Delta^z\phi(t).
\end{aligned}$$

Then from assumption (5.6), and do above similar on another way round, we have following estimations:

$$
\begin{aligned}
&H(z,\Delta^z\tilde{\theta}(t)) - H(z,\Delta^z\hat{\theta}(t)) - \Delta^z\phi(t)\cdot\lambda^*(z,\Delta^z\tilde{\theta}(t)) \le -C^{-1}\|\Delta^z\phi(t)\|^2\\
&H(z,\Delta^z\hat{\theta}(t)) - H(z,\Delta^z\tilde{\theta}(t)) + \Delta^z\phi(t)\cdot\lambda^*(z,\Delta^z\hat{\theta}(t)) \le -C^{-1}\|\Delta^z\phi(t)\|^2.
\end{aligned}
$$

However, unlike the proof for (Cecchin and Pelino, 2019, Proposition 5), both $\tilde{p}_z$ and $\hat{p}_z$ can be negative in our setting. The same technique in (Cecchin and Pelino, 2019, Proposition 5) that substitutes the inequality above back to obtain estimation of (5.21) is no longer applicable. Hence we rewrite the equation as following to cope with the possible negativeness of $\tilde{p}_z$ and $\hat{p}_z$:

$$
\begin{aligned}
&\sum_{z\in\Sigma}\phi_z(t_0)\pi_z(t_0) = \sum_{z\in\Sigma}\phi_z(T)\pi_z(T)\\
&+\int_{t_0}^T\sum_{z\in\Sigma}[H(z,\Delta^z\tilde{\theta}(t)) - H(z,\Delta^z\hat{\theta}(t)) - \Delta^z\phi(t)\cdot\lambda^*(z,\Delta^z\tilde{\theta}(t))](\tilde{p}_z(t)+M_1)dt\\
&+\int_{t_0}^T\sum_{z\in\Sigma}[H(z,\Delta^z\hat{\theta}(t)) - H(z,\Delta^z\tilde{\theta}(t)) + \Delta^z\phi(t)\cdot\lambda^*(z,\Delta^z\hat{\theta}(t))](\hat{p}_z(t)+M_1)dt\\
&+M_1\int_{t_0}^T\sum_{z\in\Sigma}\Delta^z\phi(t)\cdot[\lambda^*(z,\Delta^z\tilde{\theta}(t)) - \lambda^*(z,\Delta^z\hat{\theta}(t))]dt.
\end{aligned}
\tag{5.20}
$$

The following proof will be also similar to that in (Cecchin and Pelino, 2019, Proposition 5) except we need to pay extra effort to deal with the appearance of M_1. From (5.8), $\sum_{z\in\Sigma}\phi_z(T)\pi_z(T)\le 0$. As $\tilde{p}_z(t),\hat{p}_z(t) > -M_1$, we have

$$
\begin{aligned}
&\int_{t_0}^T\sum_{z\in\Sigma}\|\Delta^z\phi(t)\|^2(\tilde{p}_z(t)+\hat{p}_z(t)+2M_1)dt \le -C(\tilde{p}(t_0)-\hat{p}(t_0))\cdot(\tilde{\theta}(t_0)-\hat{\theta}(t_0))\\
&+CM_1\int_{t_0}^T\sum_{z\in\Sigma}\Delta^z\phi(t)\cdot[\lambda^*(z,\Delta^z\tilde{\theta}(t)) - \lambda^*(z,\Delta^z\hat{\theta}(t))]dt
\end{aligned}
$$

By Lipschitz continuity of λ^*, there exists C such that

$$
\left|\int_{t_0}^T\sum_{z\in\Sigma}\|\Delta^z\phi(t)\|^2(\tilde{p}_z(t)+\hat{p}_z(t)+2M_1)dt\right| \le C(\|\pi(t_0)\|\|\phi\| + M_1\|\phi\|^2). \tag{5.21}
$$

We next derive the bound for π. Integrating the second equation in (5.19) over $[t_0, t]$, we have

$$\pi_z(t) = \pi_z(t_0) + \int_{t_0}^{t} \sum_y \{\tilde{p}_y(s)\lambda_z^*(y, \Delta^y\tilde{\theta}(s)) - \hat{p}_y(s)\lambda_z^*(y, \Delta^y\hat{\theta}(s))\}ds.$$

As λ^* is both bounded and Lipschiz continuous, there exists C such that

$$\begin{aligned}
&\max_{z\in\Sigma}|\pi_z(t)| \le \max_{z\in\Sigma}|\pi_z(t_0)| + C\int_{t_0}^{t}\max_{z\in\Sigma}|\pi_z(s)|ds + C\int_{t_0}^{t}\sum_{z\in\Sigma}\|\Delta^z\phi(s)\||\tilde{p}_z(s)|ds \\
&\le \max_{z\in\Sigma}|\pi_z(t_0)| + C\int_{t_0}^{t}\max_{z\in\Sigma}|\pi_z(s)|ds + C\int_{t_0}^{t}\sum_{z\in\Sigma}\|\Delta^z\phi(s)\|(\tilde{p}_z(s)+M_1)ds \\
&+ M_1C\int_{t_0}^{t}\sum_{z\in\Sigma}\|\Delta^z\phi(s)\|ds,
\end{aligned}$$

where the second line holds because $\tilde{p}_z(s) + M_1 > 0$. Moreover, we have

$$\begin{aligned}
&\int_{t_0}^{t}\sum_{z\in\Sigma}\|\Delta^z\phi(s)\|(\tilde{p}_z(s)+M_1)ds \\
&= \int_{t_0}^{t}\sum_{z\in\Sigma}\sqrt{\|\Delta^z\phi(s)\|^2(\tilde{p}_z(s)+M_1)}\sqrt{\tilde{p}_z(s)+M_1}ds \\
&\le \sqrt{\int_{t_0}^{t}\sum_{z\in\Sigma}\|\Delta^z\phi(s)\|^2(\tilde{p}_z(s)+M_1)ds}\sqrt{\int_{t_0}^{t}\sum_{z\in\Sigma}(\tilde{p}_z(s)+M_1)ds}
\end{aligned}$$

Applying Gronwall inequality, as $\tilde{p}_z(s) \in [-M, 1+M]$, there exists C such that

$$\begin{aligned}
&\|\pi\| \le C\|\pi(t_0)\| + C\sqrt{\int_{t_0}^{T}\sum_{z\in\Sigma}\|\Delta^z\phi(t)\|^2(\tilde{p}_z(t)+M_1)dt} + M_1C\|\phi\| \\
&\le C\|\pi(t_0)\| + C\sqrt{\|\pi(t_0)\|\|\phi\| + M_1\|\phi\|^2} + CM_1\|\phi\| \\
&\le C\|\pi(t_0)\| + C\|\pi(t_0)\|^{\frac{1}{2}}\|\phi\|^{\frac{1}{2}} + C(M_1+\sqrt{M_1})\|\phi\|,
\end{aligned} \tag{5.22}$$

where C also only depends on M in Proposition 5.4.1.

We next derive the bound for ϕ. Integrating the first equation in (5.19) over $[t_0, t]$, from the Lipschitz continuity of G, H, there exists C such that

$$\max_{z\in\Sigma}\phi_z(t) \leq C\max_{z\in\Sigma}|\pi_z(T)| + C\int_t^T \max_{z\in\Sigma}|\phi_z(s)|ds.$$

Applying Gronwall inequality, there exists constant C such that

$$\|\phi\| \leq C\|\pi\| \tag{5.23}$$

By combining (5.22) and (5.23), using $AB \leq \varepsilon A^2 + \frac{1}{\varepsilon}B^2$ for $A, B > 0$, there exists C such that

$$\|\pi\| \leq C\|\pi(t_0)\| + [\frac{1}{4} + C^2(M_1 + \sqrt{M_1})]\|\pi\|.$$

As C only depend on the boundedness and Lipschitz coefficient of H, G, λ^* and the bound of $D^2_{\mu\mu}H$, $\tilde{\theta}$, $\hat{\theta}$, which depend on the M in Proposition 5.4.1. We only need to select M_1 such that

$$C^2(M_1 + \sqrt{M_1}) < \frac{1}{4},$$

and we can have (5.18). Then it remains to decide the new N_0 such that we have $\tilde{p}_z(t), \hat{p}(t) > -M_1$ uniformly as we assumed. From Proposition 5.4.1, $N_1 := \frac{3e^{\Lambda(M_1)}}{M_1}$ and we can simply define our new N_0 as $\max N_0, N_1$. On the other hand, the uniqueness of solution comes directly from (5.18). □

According to Proposition 5.4.1 and 5.4.2, take any $t \in [t_0, T]$ and $\bar{p}_0 \in \bar{B}(P(\Sigma), \frac{1}{N_0})$ as the initial value for ODE system (5.14), there exists an unique solution $(\bar{\theta}(s), \bar{p}(s))$ on $[t, T]$. Note that $\bar{\theta}(s)$ might not equal $\tilde{\theta}(s)$ stated as the solution to (5.14) in Theorem 5.3.2, since $\bar{\theta}$ depends on the values of initial time t and initial condition $\bar{p}_0$ chosen above. And we can define a function $\tilde{U}$ on $t \in [t_0, T]$ and $\tilde{p}_0 \in \bar{B}(P(\Sigma), \frac{1}{N_0})$ by the corresponding $\bar{\theta}(t)$ explained above.

$$\tilde{U}(t, z, \tilde{p}_0) := \bar{\theta}(t, z). \tag{5.24}$$

According to Proposition 5.4.1 and 5.4.2, $\tilde{U}$ is well defined and continuous w.r.t $\tilde{p}_0$. Moreover, for $(\tilde{\theta}, \tilde{p})$, the solution to (5.14) in Theorem 5.3.2 on $[t_0, T]$, which is the approximated solution we got from DNN and want to estimate error on, we have for all $t \in [t_0, T]$ that:

$$\tilde{U}(t, z, \tilde{p}(t)) := \tilde{\theta}(t, z).$$

It suggests $\tilde{U}$ has all information of $\tilde{\theta}$. If we can compare $\tilde{U}$ with the U defined similar in Cecchin and Pelino (2019) corresponding to the true solution to (5.2), we can estimate the error of $\tilde{\theta}$. To compare $\tilde{U}$ with the U, we need to prove that $\tilde{U}$ also satisfy the Master equation similar to U in Cecchin and Pelino (2019). To achieve this goal, we are to prove the continuously differentiability of $\tilde{U}$ in the following steps. We first define the derivative of $\tilde{U}$ w.r.t vector $\tilde{p}_0$ in a similar way to in Cecchin and Pelino (2019), Define operator D^y_p as following.

Definition 5.4.3. Define operator of a function $U : \mathbb{R}^K \to \mathbb{R}$ as $D^y U : \mathbb{R}^K \to \mathbb{R}^K$ for $y \in \Sigma$.

$$[D^y U(p)]_z := \lim_{s \to 0} \frac{U(p + s(\delta_z - \delta_y)) - U(p)}{s},$$

where $D^y U(p) = ([D^y U(p)]_1, \cdots, [D^y U(p)]_K)$, and $\delta_z \in \mathbb{R}^K$ such that all elements are 0 except the z element is 1.

By noticing that $\mu = \sum_{z \neq 1} \mu_z(\delta_z - \delta_1) + (\sum_{z=1}^K \mu_z)\delta_1$, if $\tilde{U}$ is differentiable, we have following lemma from the linearity of directional derivative.

Lemma 5.4.4. *Define the derivative of function $U(p)$ along the direction $\mu \in \mathbb{R}^K$ as a map $\frac{\partial}{\partial \mu} U : \mathbb{R}^K \to \mathbb{R}$,*

$$\frac{\partial}{\partial \mu} U(p) := \lim_{s \to 0} \frac{U(p + s\mu) - U(p)}{s}.$$

It satisfies

$$\frac{\partial}{\partial \mu} U(p) = D^1 U(p) \cdot \mu + \frac{\partial}{\partial \delta_1} U(p)(\sum_{z=1}^K \mu_z),$$

where $\frac{\partial}{\partial \delta_1}$ *is in fact the first component of the gradient of* $\tilde{U}$*, and* $DU(p) := D^1U(p)$ *for notation simplicity. When* $\sum_{z=1}^K \mu_z = 0$*, for any* $y \in \Sigma$*, we have*

$$D^yU(p) \cdot \mu = DU(p) \cdot \mu = \frac{\partial}{\partial \mu} U(p).$$

In order to characterize the directional derivative of $\tilde{U}$ w.r.t $\tilde{p}_0$, given $\tilde{\theta}$ and $\tilde{p}$, let's define a linear system of ODE for (u, ρ) similar to (Cecchin and Pelino, 2019, Equation (80)), which will be used quite a few times in the following.

$$\begin{aligned}
\frac{du_z(t)}{dt} &= -\lambda^*(z, \Delta^z\tilde{\theta}(t)) \cdot \Delta^z u(t) - b(t, z) \\
\frac{d\rho_z(t)}{dt} &= \sum_y \rho_y(t)\lambda_z^*(y, \Delta^y\tilde{\theta}(t)) + \sum_y \tilde{p}_y(t) D_\mu \lambda_z^*(y, \Delta^y\tilde{\theta}(t)) \cdot \Delta^y u(t) + c(t, z) \\
u_z(T) &= \frac{\partial G}{\partial \rho(T)}(z, \tilde{p}(T)) + u_{T,z} = \nabla G(z, \tilde{p}(T)) \cdot \rho(T) + u_{T,z} \\
\rho_z(t_0) &= \rho_{z,0}.
\end{aligned} \tag{5.25}$$

Similar to (Cecchin and Pelino, 2019, Equation (80)), $D_\mu \lambda_z^*(y, \Delta^y\tilde{\theta}(t))$ is the gradient of λ_z^* w.r.t its second variable in $\mathbb{R}^K$. The unknowns are u and ρ, while b, c, u_T, ρ_0 are given measurable functions, with c satisfying $\sum_{z=1}^K c(t, z) = 0$. In fact, (5.25) is generalization of (Cecchin and Pelino, 2019, Equation (80)). In (5.25), it is a general directional derivatives of any direction in the terminal condition of $u_z(T)$, while in (Cecchin and Pelino, 2019, Equation (80)), it is directional derivatives of specific directions.

We first prove in following Proposition 5.4.5 that the linear system (5.25) has a unique solution, which is linear bounded by its initial and boundary conditions.

Proposition 5.4.5. *There exist positive constants* N_0 *and* C*, such that if we have (5.16) and* $\tilde{p}(t_0) \in \bar{B}(P(\Sigma), \frac{1}{N_0})$*, then for any measurable function* b*,* c *and vector*

u_T, the linear system (5.25) has a unique solution (u, ρ). Moreover it satisfies

$$\begin{aligned} \|u\| &\leq C[\|u_T\| + \|\rho_0\| + \|b\| + \|c\|] \\ \|\rho\| &\leq C[\|u_T\| + \|\rho_0\| + \|b\| + \|c\|]. \end{aligned} \tag{5.26}$$

The proof shares similar idea to (Cecchin and Pelino, 2019, Proposition 6), except we need to make extra effort on dealing with the possible negativeness of $\tilde{p}$, $\hat{p}$ and the generalized terminal condition in (5.25). We use Schaefer's Fixed Point Theorem to prove the existence of solution. In order to verify one condition in Schaefer's Fixed Point Theorem, we provide the prior estimation on the solution following a similar trick as Proposition 5.4.2, i.e differentiating $u\rho$ before applying Gronwall inequality on the ODEs of u and ρ respectively. Then the uniqueness of solution comes naturally from the prior estimation.

Proof. We only discuss the case when $t_0 = 0$, as it can be extended to any $t_0 \in [0, T]$ by the same argument.

We first let N_0 bigger than the one in Proposition 5.4.2. And similar to the proof for Proposition 5.4.2, to cope with the potential negativeness, we first assume $\tilde{p}_z(t) \geq -M_1$ uniformly and $M_1 \leq M$, and we will decide later the value for M_1 small enough and find the N_0 such that it holds. As $\sum_{z\in\Sigma} \lambda_z^*(y, \Delta^y\tilde{\theta}(t)) = 0$, we have $\sum_{z,y\in\Sigma} \tilde{p}_y(t) D_\mu \lambda_z^*(y, \Delta^y\tilde{\theta}(t)) \cdot \Delta^y u(t) = 0$, and $\sum_{z\in\Sigma} \frac{d\rho_z(t)}{dt} = 0$. Hence for any $t \in [0, T]$, we have

$$\eta := \sum_{z\in\Sigma} \rho_z(t) = \sum_{z\in\Sigma} \rho_{z,0}. \tag{5.27}$$

Define set $P_\eta(\Sigma)$ as

$$P_\eta(\Sigma) := \{p \in \mathbb{R}^K, \quad s.t \quad \sum_{z=1}^{K} p_z = \eta\}.$$

We define map ξ from $\mathcal{C}^0([0, T]; P_\eta(\Sigma))$ to itself as following: for a fixed $\rho \in \mathcal{C}^0([0, T]; P_\eta(\Sigma))$, we consider the solution $u = u(\rho)$ to the backward ODE for u

in (5.25), and define $\xi(\rho)$ to be the solution to the forward ODE for ρ in (5.25) with $u = u(\rho)$. From (5.27), $\xi(\rho)$ is well defined as $\xi(\rho)(t) \in P_\eta(\Sigma)$ for any t.

Similar to the proof for (Cecchin and Pelino, 2019, Proposition 6), the solution to (5.25) is the fixed point of mapping ξ, and we prove its existence by Schaefer's Fixed Point Theorem, which asserts that a continuous and compact mapping ξ of a Banach space X into itself has fixed point if the set $\{\rho \in X : \rho = \omega\xi(\rho), \omega \in [0,1]\}$ is bounded. Firstly, ξ is continuous as the system (5.25) is linear in u and ρ. $\mathcal{C}^0([0,T]; P_\eta(\Sigma))$ is a convex subset of Banach space $\mathcal{C}^0([0,T]; \mathbb{R}^K)$. Moreover, from the linearity and bounded coefficients of system (5.25), ξ maps any bounded set of $\mathcal{C}^0([0,T]; P_\eta(\Sigma))$ into set of bounded and Lipshitz continuous functions with uniform Lipshitz coefficient in $\mathcal{C}^1([0,T]; P_\eta(\Sigma))$, which by Arzela–Ascoli theorem, is relatively compact. By compact map definition, ξ is a compact map. Hence to apply Schaefer's Fixed Point Theorem, it remains to prove that the set $\{\rho : \rho = \omega\xi(\rho)\}$ is uniform bounded for $\forall \omega \in [0,1]$. We can restrict to $\omega > 0$ since otherwise $\rho = 0$. Fix a ρ such that $\rho = \omega\xi(\rho)$, which means the couple $(u(\rho), \xi(\rho))$ solves (for notation simplicity we neglect their dependency on ρ)

$$
\begin{aligned}
&\frac{du_z(t)}{dt} = -\lambda^*(z, \Delta^z\tilde{\theta}(t)) \cdot \Delta^z u(t) - b(t,z) \\
&\frac{d\xi_z(t)}{dt} = \sum_y \xi_y(t)\lambda_z^*(y, \Delta^y\tilde{\theta}(t)) + \sum_y \tilde{p}_y(t) D_\mu \lambda_z^*(y, \Delta^y\tilde{\theta}(t)) \cdot \Delta^y u(t) + c(t,z) \\
&u_z(T) = \nabla G(z, \tilde{p}(T)) \cdot \omega\xi(T) + u_{T,z} \\
&\xi_z(t_0) = \rho_{z,0}.
\end{aligned}
\tag{5.28}
$$

We need to prove the solution (u, ξ) if existed, are bounded uniformly for any $\omega \in (0,1]$. For notation simplicity, we omit the dependence of λ^* on the second variable.

From (5.28),

$$\sum_{z\in\Sigma}\frac{d}{dt}(u_z(t)\xi_z(t)) = -\sum_{z,y\in\Sigma}\xi_z(t)\lambda_y^*(z)(u_y(t)-u_z(t)) + \sum_{z,y\in\Sigma}\xi_y(t)\lambda_z^*(y)u_z(t)$$
$$+\sum_{z,y\in\Sigma}u_z(t)\tilde{p}_y(t)D_\mu\lambda_z^*(y)\cdot\Delta^y u(t) + \sum_{z\in\Sigma}c(t,z)u_z(t) - \sum_{z\in\Sigma}\xi_z(t)b(t,z).$$

The first line is 0 by exchanging z and y in the second double sum and using (5.3). Integrating over $[0,T]$ and using the expression of $u_z(T)$ we have

$$\sum_{z\in\Sigma}\xi_z(T)[\nabla G(z,\tilde{p}(T))\cdot\omega\xi(T)+u_{T,z}] - u(0)\cdot\rho_0$$
$$= \int_0^T\sum_{z\in\Sigma}c(t,z)u_z(t)dt - \int_0^T\sum_{z\in\Sigma}\xi_z(t)b(t,z)dt$$
$$+\int_0^T\sum_{z,y\in\Sigma}\tilde{p}_y(t)D_\mu\lambda_z^*(y)\cdot\Delta^y u(t)(u_z(t)-u_y(t))dt$$

Reorganize the terms and we get

$$\int_0^T\sum_{z,y\in\Sigma}\tilde{p}_y(t)D_\mu\lambda_z^*(y)\cdot\Delta^y u(t)(u_z(t)-u_y(t))dt - \omega\sum_{z\in\Sigma}\xi_z(T)\nabla G(z,\tilde{p}(T))\cdot\xi(T)$$
$$= \int_0^T\sum_{z\in\Sigma}\xi_z(t)b(t,z)dt - \int_0^T\sum_{z\in\Sigma}c(t,z)u_z(t)dt + \sum_{z\in\Sigma}\xi_z(T)u_{T,z} - u(0)\cdot\rho_0.$$

From assumption on G in (5.8) and definition of directional derivative, we have

$$-\omega\sum_{z\in\Sigma}\xi_z(T)\nabla G(z,\tilde{p}(T))\cdot\xi(T) = -\omega\sum_{z\in\Sigma}\xi_z(T)\frac{\partial G}{\partial\xi(T)}(z,\tilde{p}(T)) \geq 0.$$

Moreover, as $\lambda^*(y) = D_\mu H(y)$ (we also neglect the dependence of H on the second variable),

$$\int_0^T\sum_{z,y\in\Sigma}\tilde{p}_y(t)D_\mu\lambda_z^*(y)\cdot\Delta^y u(t)(u_z(t)-u_y(t))dt$$
$$= \int_0^T\sum_{y\in\Sigma}\tilde{p}_y(t)\Delta^y u(t)\cdot D^2_{\mu\mu}H(y)\cdot\Delta^y u(t)dt.$$

Since $\tilde{p}$ and $\hat{p}$ can be negative, the same step in (Cecchin and Pelino, 2019, Proposition 6) to obtain estimation on RHS of above is not applicable. However, as $\tilde{p}_y(t) + M_1 \geq 0$ for all $y \in \Sigma$, from (5.6), we can rewrite the RHS of the equation and get following estimation instead.

$$
\begin{aligned}
&\int_0^T \sum_{y\in\Sigma} \tilde{p}_y(t)\Delta^y u(t) \cdot D^2_{\mu\mu}H(y) \cdot \Delta^y u(t)dt \\
&= \int_0^T \sum_{y\in\Sigma} (\tilde{p}_y(t) + M_1)\Delta^y u(t) \cdot D^2_{\mu\mu}H(y) \cdot \Delta^y u(t)dt \\
&- M_1 \int_0^T \sum_{y\in\Sigma} \Delta^y u(t) \cdot D^2_{\mu\mu}H(y) \cdot \Delta^y u(t)dt \\
&\geq C^{-1} \int_0^T \sum_{z\in\Sigma} (\tilde{p}_y(t) + M_1)\|\Delta^z u(t)\|^2 dt - M_1 C \int_0^T \sum_{z\in\Sigma} \|\Delta^z u(t)\|^2 dt.
\end{aligned}
$$

So there exists constant C and C_1 (C_1 only depends on the dimension of u) such that

$$
\begin{aligned}
&\int_0^T \sum_{z\in\Sigma} (\tilde{p}_z(t) + M_1)\|\Delta^z u(t)\|^2 dt \leq C(\int_0^T |c(t)\cdot u(t)|dt + \int_0^T |\xi(t)\cdot b(t)|dt \\
&+ \|\xi(T)\|\|u_T\| + \|u(0)\|\|\rho_0\| + M_1 C_1 \|u\|^2),
\end{aligned} \tag{5.29}
$$

where $b(t) := (b(t,1), \cdots, b(t,K))$, and $c(t)$ is defined similarly. As λ^* and $D_\mu \lambda^*$ is bounded by constant C, from ODE for ξ in (5.28) we have

$$
\begin{aligned}
&|\xi_z(t)| \leq |\rho_{0,z}| + C\int_0^t \sum_{y\in\Sigma} |\xi_y(s)|ds + C\int_0^t [\sum_{y\in\Sigma} (\tilde{p}_y(s) + M_1)\|\Delta^y u(s)\| + |c(s,z)|]ds \\
&+ CM_1 \int_0^t \sum_{y\in\Sigma} \|\Delta^y u(s)\| ds.
\end{aligned}
$$

So that by Gronwall's inequality, there exists constant C such that

$$\|\xi\| \le C(\|\rho_0\| + \|c\|) + C\int_0^T \sum_{y\in\Sigma} (\tilde{p}_y(t) + M_1)\|\Delta^y u(t)\|dt$$
$$+ CM_1 \int_0^T \sum_{y\in\Sigma} \|\Delta^y u(t)\|dt,$$

where there exists C such that $\sum_{y\in\Sigma}(\tilde{p}_y(t)+M_1) \le= K(M+1)+KM_1 \le K(2M+1) \le C^2$ and

$$\int_0^T \sum_{y\in\Sigma} (\tilde{p}_y(t) + M_1)\|\Delta^y u(t)\|dt = \int_0^T \sum_{y\in\Sigma} \sqrt{\tilde{p}_y(t) + M_1}\sqrt{\tilde{p}_y(t) + M_1}\|\Delta^y u(t)\|dt$$
$$\le \int_0^T \sqrt{\sum_{y\in\Sigma}(\tilde{p}_y(t) + M_1)}\sqrt{\sum_{y\in\Sigma}(\tilde{p}_y(t) + M_1)\|\Delta^y u(t)\|^2}dt$$
$$\le C\int_0^T \sqrt{\sum_{y\in\Sigma}(\tilde{p}_y(t) + M_1)\|\Delta^y u(t)\|^2}dt \le C\sqrt{\int_0^T \sum_{y\in\Sigma}(\tilde{p}_y(t) + M_1)\|\Delta^y u(t)\|^2 dt}.$$

From (5.29), there exist different constants C at each line such that

$$\|\xi\| \le C(\|\rho_0\| + \|c\|) + C\int_0^T \sum_{y\in\Sigma} (\tilde{p}_y(t) + M_1)\|\Delta^y u(t)\|dt$$
$$+ CM_1 \int_0^T \sum_{y\in\Sigma} \|\Delta^y u(t)\|dt$$
$$\le C(\|\rho_0\| + \|c\|) + CM_1\|u\| + C\int_0^T |c(t)\cdot u(t)| + |\xi(t)\cdot b(t)|dt$$
$$+ C(\|\xi(T)\|\|u_T\| + \|u(0)\|\|\rho_0\| + M_1C_1\|u\|^2)^{\frac{1}{2}}$$
$$\le C(\|\rho_0\| + \|c\|) + C(M_1 + \sqrt{M_1})\|u\|$$
$$+ C(\|c\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}} + \|\xi\|^{\frac{1}{2}}\|b\|^{\frac{1}{2}} + \|\xi(T)\|^{\frac{1}{2}}\|u_T\|^{\frac{1}{2}} + \|u(0)\|^{\frac{1}{2}}\|\rho_0\|^{\frac{1}{2}}),$$

Moreover, using Gronwall inequality on the backward ODE in (5.28) for function u, there exists C such that

$$\|u\| \le C[\|u_T\| + \omega\|\xi(T)\| + \|b\|] \le C[\|u_T\| + \|\xi(T)\| + \|b\|].$$

Then there exists C such that

$$\begin{aligned}&\|\xi\| \leq C(\|\rho_0\| + \|c\|)\\&+ C(M_1 + \sqrt{M_1})(\|u_T\| + \|\xi(T)\| + \|b\|) + C\|c\|^{\frac{1}{2}}(\|u_T\| + \|\xi(T)\| + \|b\|)^{\frac{1}{2}}\\&+ C(\|\xi\|^{\frac{1}{2}}\|b\|^{\frac{1}{2}} + \|\xi(T)\|^{\frac{1}{2}}\|u_T\|^{\frac{1}{2}} + (\|u_T\|^{\frac{1}{2}} + \|\xi(T)\|^{\frac{1}{2}} + \|b\|^{\frac{1}{2}})\|\rho_0\|^{\frac{1}{2}})\end{aligned}$$

As $\|\xi(T)\| \leq \|\xi\|$, using the inequality $AB \leq \varepsilon A^2 + \frac{1}{4\varepsilon}B^2$ for $A, B \geq 0$, there exists C such that

$$\|\xi\| \leq C(\|c\| + \|b\| + \|\rho_0\| + \|u_T\|) + (C(M_1 + \sqrt{M_1}) + \frac{1}{4})\|\xi\|.$$

Note that the constant C only depends on the boundedness of $\tilde{\theta}$, which depends on M in Proposition 5.4.1. If

$$C(M_1 + \sqrt{M_1}) \leq \frac{1}{4}.$$

Then we have

$$\|\xi\| \leq 2C(\|c\| + \|b\| + \|\rho_0\| + \|u_T\|),$$

Hence the solution pair (u, ξ) are bounded for all $\omega \in [0, 1]$, which means $\rho = \omega\xi(\rho)$ are also uniform bounded, and hence proves the existence of solution to (5.25). Meanwhile, let $\omega = 1$ leads to the uniform bound estimation for solution (u, ρ) to (5.25), and the uniqueness of it comes directly from (5.26). If $N_0 > \frac{3e^{\Lambda(M_1)}}{M_1}$, from Proposition 5.4.1, we have $\tilde{p}_y(t) > -M_1$ uniformly, which concludes our proof. Hence we can just update our N_0 set before to satisfy the inequality. □

Then we can prove the differentiablity of $\tilde{U}$ w.r.t $\tilde{p}_0$ in Proposition 5.4.6.

Proposition 5.4.6. *Let $(\tilde{\theta}, \tilde{p})$ and $(\hat{\theta}, \hat{p})$ be the solutions to ODE system (5.14) respectively starting from $(t_0, \tilde{p}(t_0))$ and $(t_0, \hat{p}(t_0))$, and (v, ζ) be the solution to (5.25) starting from $\rho_0 := \hat{p}(t_0) - \tilde{p}(t_0)$. There exist positive constants N_0 and C, such that* *if we have (5.16), then for any $t_0 \in [0,T]$ and $\tilde{p}(t_0), \hat{p}(t_0) \in \bar{B}(P(\Sigma), \frac{1}{N_0})$, we have*

$$\|\hat{\theta} - \tilde{\theta} - v\| + \|\hat{p} - \tilde{p} - \zeta\| \leq C\|\hat{p}(t_0) - \tilde{p}(t_0)\|^2.$$

The proof is straightforward as we already prove Proposition 5.4.5. $(v.\zeta)$ is characterized by a linear forward backward ODE system similar to (5.25), with $b, c = 0$. On the other hand, $(\hat{\theta} - \tilde{\theta} - v, \hat{p} - \tilde{p} - \zeta)$ also satisfies the same kind of linear forward backward ODE system with different b and c. Hence we can get the conclusion of Proposition 5.4.6 by applying the Proposition 5.4.5 on $(\hat{\theta} - \tilde{\theta} - v, \hat{p} - \tilde{p} - \zeta)$.

Proof. Without loss of generality, we assume $t_0 = 0$. Similar to the proof of (Cecchin and Pelino, 2019, Theorem 7), we can use results from Proposition 5.4.5 to prove our conclusion. Define N_0 as the one in Proposition 5.4.5. Then $\tilde{p}_y(t), \hat{p}_y(t) > -M_1$ uniformly on $(t, y) \in [0,T] \times \Sigma$. Define linearized system with $w := \hat{p}(0) - \tilde{p}(0)$:

$$\begin{aligned}
&\frac{dv_z(t)}{dt} = -\lambda^*(z, \Delta^z \tilde{\theta}(t)) \cdot \Delta^z v(t) \\
&\frac{d\zeta_z(t)}{dt} = \sum_y \zeta_y(t) \lambda_z^*(y, \Delta^y \tilde{\theta}(t)) + \sum_y \tilde{p}_y(t) D_\mu \lambda_z^*(y, \Delta^y \tilde{\theta}(t)) \cdot \Delta^y v(t) \\
&v_z(T) = \frac{\partial G}{\partial \zeta(T)}(z, \tilde{p}(T)) = D^1 G(z, \tilde{p}(T)) \cdot \zeta(T) + \frac{\partial G}{\delta_1}(z, \tilde{p}(T)) \sum_{z=1}^K w_z \\
&\zeta_z(0) = w_z.
\end{aligned} \tag{5.30}$$

From condition in Theorem 5.3.2, the sum of every component of $\tilde{p}$ equals 1 for all $t \in [0,T]$. Hence we know $\sum_{z \in \Sigma} \epsilon_2(t, z) = 0$, and define

$$S(\hat{p}, \tilde{p}) := \sum_{z \in \Sigma} (\hat{p}_z(0) - \tilde{p}_z(0)) = \sum_{z \in \Sigma} (\hat{p}_z(T) - \tilde{p}_z(T))$$

We know there exists C such that $|S(\hat{p}, \tilde{p})| \leq C\|\hat{p}(T) - \tilde{p}(T)\|$. Set $u := \hat{\theta} - \tilde{\theta} - v$

and $\rho := \hat{p} - \tilde{p} - \zeta$, they solve (5.25), where

$$b(t,z) := H(z, \Delta^z\hat{\theta}(t)) - H(z, \Delta^z\tilde{\theta}(t)) - \lambda^*(z, \Delta^z\tilde{\theta}(t)) \cdot (\Delta^z\hat{\theta}(t) - \Delta^z\tilde{\theta}(t))$$
$$c(t,z) := \sum_y \hat{p}_y(t)[\lambda^*_z(y, \Delta^y\hat{\theta}(t)) - \lambda^*_z(y, \Delta^y\tilde{\theta}(t))]$$
$$- \sum_y \tilde{p}_y(t) D_\mu \lambda^*_z(y, \Delta^y\tilde{\theta}(t)) \cdot (\Delta^y\hat{\theta}(t) - \Delta^y\tilde{\theta}(t)))$$
$$u_{T,z} := G(z, \hat{p}(T)) - G(z, \tilde{p}(T)) - D^1G(z, \tilde{p}(T))(\hat{p}(T) - \tilde{p}(T))$$
$$- \frac{\partial G}{\delta_1}(z, \tilde{p}(T))S(\hat{p}, \tilde{p}).$$

From (5.3), $\sum_{z\in\Sigma} c(t,z) = 0$. The existence and uniqueness of solution to (5.30) is guaranteed by Proposition 5.4.5. We can simplify b and c as

$$b(t,z) = \int_0^1 [D_\mu H(z, \Delta^z\tilde{\theta}(t) + s(\Delta^z\hat{\theta}(t) - \Delta^z\tilde{\theta}(t))) - D_\mu H(z, \Delta^z\tilde{\theta}(t))]$$
$$\cdot (\Delta^z\hat{\theta}(t) - \Delta^z\tilde{\theta}(t))ds$$
$$c(t,z) = \sum_y \hat{p}_y(t) \int_0^1 [D_\mu \lambda^*_z(y, \Delta^y\tilde{\theta}(t) + s(\hat{\theta}(t) - \Delta^y\tilde{\theta}(t))) - D_\mu \lambda^*_z(y, \Delta^y\tilde{\theta}(t))]$$
$$\cdot (\Delta^z\hat{\theta}(t) - \Delta^z\tilde{\theta}(t))ds + \sum_y (\hat{p}_y(t) - \tilde{p}_y(t)) D_\mu \lambda^*_z(y, \Delta^y\tilde{\theta}(t)) \cdot (\Delta^y\hat{\theta}(t) - \Delta^y\tilde{\theta}(t))). \tag{5.31}$$

Moreover, since

$$G(z, \hat{p}(T)) - G(z, \tilde{p}(T)) = \int_0^1 \frac{\partial G}{\partial(\hat{p}(T) - \tilde{p}(T))}(z, \tilde{p}(T) + s(\hat{p}(T) - \tilde{p}(T)))ds$$
$$= \int_0^1 D^1G(z, \tilde{p}(T) + s(\hat{p}(T) - \tilde{p}(T))) \cdot ((\hat{p}(T) - \tilde{p}(T))))ds$$
$$+ \int_0^1 \frac{\partial G}{\partial \delta_1}(z, \tilde{p}(T) + s(\hat{p}(T) - \tilde{p}(T)))S(\hat{p}, \tilde{p})ds$$

we have

$$u_{T,z} = \int_0^1 (D^1G(z, \tilde{p}(T) + s(\hat{p}(T) - \tilde{p}(T))) - D^1G(z, \tilde{p}(T))) \cdot ((\hat{p}(T) - \tilde{p}(T))))ds$$
$$+ \int_0^1 (\frac{\partial G}{\partial \delta_1}(z, \tilde{p}(T) + s(\hat{p}(T) - \tilde{p}(T))) - \frac{\partial G}{\delta_1}(z, \tilde{p}(T)))S(\hat{p}, \tilde{p})ds \tag{5.32}$$

From Proposition 5.4.1, $\tilde{\theta}$, $\tilde{p}$, $\hat{\theta}$ and $\hat{p}$ are bounded. From the Assumption 5.2.1, namely the Lipschitz continuity of $D_\mu H$, $D_\mu \lambda^*$, $\frac{\partial G}{\delta_1}$ and D^1G in their second variable, there exists constant C such that

$$\|b\| \leq C\|\tilde{\theta} - \hat{\theta}\|^2$$
$$\|u_{T,z}\| \leq C\|\tilde{p}(T) - \hat{p}(T)\|^2$$
$$\|c\| \leq C(\|\tilde{\theta} - \hat{\theta}\|^2 + \|\tilde{\theta} - \hat{\theta}\| \cdot \|\tilde{p} - \hat{p}\|).$$

Applying Proposition 5.4.5 and then Proposition 5.4.2, we have there exists C such that

$$\|u\| + \|\rho\| \leq C\|\hat{p}(0) - \tilde{p}(0)\|^2,$$

which concludes the proof. □

As (5.30) is a linear system. v and ζ in (5.30) can be viewed as a linear map of w. Hence by definition of differentiability for multivariate function, Proposition 5.4.6 suggests that $\tilde{U}$ is differntiable w.r.t $\tilde{p}_0$ and the directional derivative $\frac{\partial}{\partial w}\tilde{U}(t, z, \tilde{p})$ is the solution to ODE system (5.30), with $\tilde{\theta}_z(t) = \tilde{U}(t, z, \tilde{p}(t))$.

Theorem 5.4.7. *There exist positive constants N_0 and C large enough. If the error terms in our FBODE system (5.14) satisfy (5.16), i.e*

$$\|\epsilon_2\| + \|\epsilon_4\| < \frac{1}{N_0},$$

Then $\tilde{U}$ defined in (5.24) is differntiable on $B(P(\Sigma), \frac{1}{N_0})$, and for any vector w, $\frac{\partial}{\partial w}\tilde{U}(t, z, \tilde{p}(t))$ exists and is Lipschitz continuous w.r.t $\tilde{p}$, uniformly in t, z. $\frac{\partial}{\partial w}\tilde{U}(t, z, \tilde{p}(t))$

is also continuous w.r.t t.

The proof is divided into two parts. The first part we prove $\frac{\partial}{\partial w}\tilde{U}(t,z,\tilde{p}(t))$ is Lipshitz continuous. The main idea is similar to the regularity proof for Master equation's solution in (Cecchin and Pelino, 2019, Section 5.3.3), except for some special treatment due to the generalization of directional derivative. From Proposition 5.4.6, the directional derivatives of $\tilde{U}$ on different $\tilde{p}$, i.e $\frac{\partial}{\partial w}\tilde{U}(t,z,\tilde{p}(t))$ and $\frac{\partial}{\partial w}\tilde{U}(t,z,\hat{p}(t))$, can both be characterized by linear forward backward ODE systems similar to (5.30). Hence their difference also satisfies linear system like (5.30) with initial value equaling to 0. Then from Proposition 5.4.5 we can prove the continuity of $\frac{\partial}{\partial w}\tilde{U}(t,z,\tilde{p}(t))$ w.r.t $\tilde{p}$. On the other hand, the second part of our proof focus on proving $\frac{\partial}{\partial w}\tilde{U}(t,z,\tilde{p}(t))$ is continuous w.r.t t. It is not discussed in (Cecchin and Pelino, 2019, Section 5.3.3), and we think it is worth providing the proof as it is not as trivial as it is suggested.

Proof. Define N_0 as the one in Proposition 5.4.5. Let $(\tilde{\theta},\tilde{p})$ and $(\hat{\theta},\hat{p})$ be two solutions to (5.14), with initial conditions $\tilde{p}(t_0),\hat{p}(t_0)\in B(P(\Sigma),\frac{1}{N_0})$. Let also $(\tilde{v},\tilde{\zeta})$ and $(\hat{v},\hat{\zeta})$ characterize $\frac{\partial}{\partial w}\tilde{U}(t_0,z,\tilde{p}(t_0))$ and $\frac{\partial}{\partial w}\tilde{U}(t_0,z,\hat{p}(t_0))$ respectively. Then $(\tilde{v},\tilde{\zeta})$ satisfies following.

$$\begin{aligned}
&\frac{d\tilde{v}_z(t)}{dt}=-\lambda^*(z,\Delta^z\tilde{\theta}(t))\cdot\Delta^z\tilde{v}(t)\\
&\frac{d\tilde{\zeta}_z(t)}{dt}=\sum_y\tilde{\zeta}_y(t)\lambda^*_z(y,\Delta^y\tilde{\theta}(t))+\sum_y\tilde{p}_y(t)D_\mu\lambda^*_z(y,\Delta^y\tilde{\theta}(t))\cdot\Delta^y\tilde{v}(t)\\
&\tilde{v}_z(T)=\frac{\partial G}{\partial\tilde{\zeta}(T)}(z,\tilde{p}(T))\\
&\tilde{\zeta}_z(t_0)=w_z.
\end{aligned}\tag{5.33}$$

From Proposition 5.4.5, we know the uniform bound of both $\tilde{v}$ and $\tilde{\zeta}$ depend linearly on norm of w. Similar is for $(\hat{v},\hat{\zeta})$, except for replacing $(\tilde{\theta},\tilde{p})$ by $(\hat{\theta},\hat{p})$. Set $u:=\tilde{v}-\hat{v}$,

$\rho := \tilde{\zeta} - \hat{\zeta}$. They solve the linear system (5.25) with $\rho(t_0) = 0$ and

$$
\begin{aligned}
b(t,z) &:= (\lambda^*(z, \Delta^z\tilde{\theta}(t)) - \lambda^*(z, \Delta^z\hat{\theta}(t))) \cdot \Delta^z\hat{v}(t) \\
c(t,z) &:= \sum_{y\in\Sigma} \hat{\zeta}_y(t)(\lambda^*_z(y, \Delta^y\tilde{\theta}(t)) - \lambda^*_z(y, \Delta^y\hat{\theta}(t))) \\
&+ \sum_{y\in\Sigma} [\tilde{p}_y(t) D_\mu \lambda^*_z(y, \Delta^y\tilde{\theta}(t)) - \hat{p}_y(t) D_\mu \lambda^*_z(y, \Delta^y\hat{\theta}(t))] \cdot \Delta^z\hat{v}(t) \\
u_{T,z} &:= \frac{\partial G}{\partial\hat{\zeta}(T)}(z, \tilde{p}(T)) - \frac{\partial G}{\partial\hat{\zeta}(T)}(z, \hat{p}(T))
\end{aligned}
$$

Using the Lipschitz continuity of λ^*, $D_\mu\lambda^*$ and directional derivatives of G, applying the bounds (5.26) to $\hat{v}$ and $\hat{\zeta}$, and the estimation on $\|\tilde{\theta}-\hat{\theta}\|$, $\|\tilde{p}-\hat{p}\|$ from Proposition 5.4.2, there exists C such that

$$
\begin{aligned}
&\|b\| \le C\|\tilde{\theta} - \hat{\theta}\|\|\hat{v}\| \le C\|\tilde{p}(t_0) - \hat{p}(t_0)\|\|w\| \\
&\|c\| \le C\|\tilde{\theta} - \hat{\theta}\|\|\hat{\zeta}\| + C\|\tilde{\theta} - \hat{\theta}\|\|\hat{v}\| + C\|\tilde{p} - \hat{p}\|\|\hat{v}\| \\
&\le C\|\tilde{p}(t_0) - \hat{p}(t_0)\|\|w\| \\
&\|u_T\| \le C\|\tilde{p} - \hat{p}\|\|\hat{\zeta}\| \le C\|\tilde{p}(t_0) - \hat{p}(t_0)\|\|w\|
\end{aligned}
$$

From Proposition 5.4.5, we have

$$
\|u\| \le C(\|b\| + \|c\| + \|u_T\|) \le C\|\tilde{p}(t_0) - \hat{p}(t_0)\|\|w\|.
$$

From Proposition 5.4.6, we have

$$
\tilde{v}_z(t_0) = \frac{\partial\tilde{U}}{\partial w}(t_0, z, \tilde{p}(t_0)), \quad \hat{v}_z(t_0) = \frac{\partial\tilde{U}}{\partial w}(t_0, z, \hat{p}(t_0)).
$$

Therefore, $\frac{\partial\tilde{U}}{\partial w}$ is Lipschitz continuous, uniform w.r.t t and z.

On the other hand, for another initial time $t_1 > t_0$, we first compare $\frac{\partial}{\partial w}\tilde{U}(t_0, z, \tilde{p}(t_0))$ and $\frac{\partial}{\partial w}\tilde{U}(t_1, z, \tilde{p}(t_1))$, where $(t_1, \tilde{p}(t_1))$ is on the path $(t, \tilde{p}(t))$ start from t_0 to T. They are both characterized by system like (5.33), though we need to replace t_0 with t_1

for $\frac{\partial}{\partial w}\tilde{U}(t_1, z, \tilde{p}(t_1))$. Let $(\tilde{v}, \tilde{\zeta})$ satisfy (5.33). Then we know

$$\tilde{v}(t_0) = \frac{\partial}{\partial w}\tilde{U}(t_0, z, \tilde{p}(t_0)), \quad \tilde{v}(t_1) = \frac{\partial}{\partial \tilde{\zeta}(t_1)}\tilde{U}(t_1, z, \tilde{p}(t_1)).$$

$\frac{\partial}{\partial \tilde{\zeta}(t_1)}\tilde{U}(t_1, z, \tilde{p}(t_1))$ is also characterized by (5.33), except that t_0 and initial value need to be replaced by t_1 and $\tilde{\zeta}(t_1)$. It means $\frac{\partial}{\partial \tilde{\zeta}(t_1)}\tilde{U}(t_1, z, \tilde{p}(t_1)) - \frac{\partial}{\partial w}\tilde{U}(t_1, z, \tilde{p}(t_1))$ can also be characterized by (5.33) except that t_0 and initial value need to be replaced by t_1 and $\tilde{\zeta}(t_1) - w$. From Proposition 5.4.5, we have there exists constant C such that

$$|\frac{\partial}{\partial \tilde{\zeta}(t_1)}\tilde{U}(t_1, z, \tilde{p}(t_1)) - \frac{\partial}{\partial w}\tilde{U}(t_1, z, \tilde{p}(t_1))| \leq C|\tilde{\zeta}_z(t_1) - w_z|.$$

As λ^*, $D_\mu \lambda^*$ and the directional derivative of G are Lipschitz continuous and uniform bounded, as well as that both $\tilde{v}$ and $\tilde{\zeta}$ are uniformly bounded, we know hence both $\frac{d\tilde{v}_z(t)}{dt}$ and $\frac{d\tilde{\zeta}_z(t)}{dt}$ are also uniformly bounded by some constant C. We have

$$\|\tilde{\zeta}(t_1) - w\| = \|\tilde{\zeta}(t_1) - \tilde{\zeta}(t_0)\| \leq C|t_1 - t_0|,$$
$$|\frac{\partial}{\partial w}\tilde{U}(t_0, z, \tilde{p}(t_0)) - \frac{\partial}{\partial \tilde{\zeta}(t_1)}\tilde{U}(t_1, z, \tilde{p}(t_1))| = |\tilde{v}_z(t_0) - \tilde{v}_z(t_1)| \leq C|t_1 - t_0|.$$

Combine above, we know there exists constant C such that

$$|\frac{\partial}{\partial w}\tilde{U}(t_0, z, \tilde{p}(t_0)) - \frac{\partial}{\partial w}\tilde{U}(t_1, z, \tilde{p}(t_1))| \leq C|t_1 - t_0|.$$

Then by the continuity of $\frac{\partial}{\partial w}\tilde{U}$ w.r.t its third argument, as well as the continuity of $\tilde{p}$, we can also conclude that $\frac{\partial}{\partial w}\tilde{U}$ is continuous w.r.t t, its first argument. □

From Proposition 5.4.6 and Theorem 5.4.7, $\tilde{U}$ is $\mathcal{C}^1$ on compact set $\bar{B}(P(\Sigma), \frac{1}{N_0})$. Hence both $D\tilde{U}$ and the directional derivative of $\tilde{U}$ along any direction are well-defined, bounded, and Lipschitz continuous, uniformly for $t \in [0, T]$. Theorem 5.4.7 also suggests that the directional derivative of $\tilde{U}$ along any direction is continuous w.r.t t. Thanks to these properties, we can use similar idea of the proof for existence of solution to Master's equation in (Cecchin and Pelino, 2019, Section 5.3.1), to show that $\tilde{U}$ also satisfies the Master equation with some extra error terms.

Theorem 5.4.8. *Let $(\tilde{\theta}, \tilde{p})$ be the solution to ODE system (5.14). Define $\tilde{U}$ as (5.24). There exist positive constants N_0 and C, such that if we have condition (5.16) in Theorem 5.3.2, then $\tilde{U}$ satisfies following Master equation along the path $(t, \tilde{p}(t))$ on $[t_0, T]$, as long as $\tilde{p}(t) \in B(P(\Sigma), \frac{1}{N_0})$.*

$$\begin{aligned} &\frac{\partial \tilde{U}(t,z,\tilde{p}(t))}{\partial t} + H(z, \Delta^z \tilde{U}) + \sum_{y\in\Sigma} \tilde{p}_y(t)\lambda^*(y, \Delta^y \tilde{U}) \cdot D\tilde{U}(t,z,\tilde{p}(t)) = \epsilon(t,z) \\ &\tilde{U}(T,z,\tilde{p}(T)) = G(z,\tilde{p}(T)) + \epsilon_3(z), \end{aligned} \tag{5.34}$$

where $\Delta^z \tilde{U} := (\tilde{U}(t,1,\tilde{p}(t)) - \tilde{U}(t,z,\tilde{p}(t)), \cdots, \tilde{U}(t,K,\tilde{p}(t)) - \tilde{U}(t,z,\tilde{p}(t)))$ and $\|\epsilon\| < \frac{C+1}{N}$, where $N > N_0$ and C comes from the uniform bound coefficient in Proposition 5.4.2.

The main idea of the proof is similar to (Cecchin and Pelino, 2019, Section 5.3.1). We decompose $\frac{\partial \tilde{U}(t,z,\tilde{p}(t))}{\partial t}$ into two different limits. We reformulate them into other equivalent forms before taking the limits, such that after taking limit they can be represented by some terms in (5.34). We also prove the convergence, hence we can substitute them back to get (5.34).

Proof. From condition in Theorem 5.3.2, $\tilde{p}(t) \in B(P(\Sigma), \frac{1}{N_0})$ for every $t \in [t_0, T]$ where $B(P(\Sigma), \frac{1}{N_0})$ being the open neighbourhood of $P(\Sigma)$. Hence from Proposition 5.4.1, 5.4.2 and Theorem 5.4.7, $\tilde{U}$, $D\tilde{U}$ and $\frac{\partial}{\partial \delta_1}\tilde{U}$ are well-defined on $(t, \tilde{p}(t))$. Take t as initial time and $\tilde{p}(t)$ as initial value, there exists an unique solution to (5.14), and we can always choose h small enough such that this solution taking value on $t+h$, i.e $\tilde{p}(t+h) \in B(P(\Sigma), \frac{1}{N_0})$. Note that as $\sum_{z\in\Sigma} \epsilon_2(t,z) = 0$ for all $t \in [t_0, T]$, we have

$$\sum_{z\in\Sigma} \tilde{p}_z(t) = \sum_{z\in\Sigma} \tilde{p}_z(t+h).$$

Let's first compute limit of following when h tends to 0.

$$\frac{\tilde{U}(t+h,z,\tilde{p}(t))-\tilde{U}(t,z,\tilde{p}(t))}{h}=$$
$$\frac{\tilde{U}(t+h,z,\tilde{p}(t))-\tilde{U}(t+h,z,\tilde{p}(t+h))}{h}+\frac{\tilde{U}(t+h,z,\tilde{p}(t+h))-\tilde{U}(t,z,\tilde{p}(t))}{h} \tag{5.35}$$

For the first term in (5.35), we first define

$$W(s):=\tilde{U}(t+h,z,\tilde{p}(t)+s(\tilde{p}(t+h)-\tilde{p}(t))).$$

By definition, we derive the derivative of W as

$$W'(s)=\frac{\partial}{\partial(\tilde{p}(t+h)-\tilde{p}(t))}\tilde{U}(t+h,z,\tilde{p}(t)+s(\tilde{p}(t+h)-\tilde{p}(t)))$$

Then the first term in (5.35) can be reformulated as

$$\frac{\tilde{U}(t+h,z,\tilde{p}(t))-\tilde{U}(t+h,z,\tilde{p}(t+h))}{h}=\frac{W(0)-W(1)}{h}=-\frac{1}{h}\int_0^1 W'(s)ds.$$

From Lemma 5.4.4 and $c(h)=\sum_{z=1}^{K}(\tilde{p}_z(t+h)-\tilde{p}_z(t))=0$. We know

$$W'(s)=D\tilde{U}(t+h,z,\tilde{p}(t)+s(\tilde{p}(t+h)-\tilde{p}(t)))\cdot(\tilde{p}(t+h)-\tilde{p}(t))$$

Substitute above to the first term in (5.35), we get

$$\begin{aligned}
&\frac{\tilde{U}(t+h,z,\tilde{p}(t))-\tilde{U}(t+h,z,\tilde{p}(t+h))}{h}\\
&=-\frac{1}{h}\int_0^1 D\tilde{U}(t+h,z,\tilde{p}(t)+s(\tilde{p}(t+h)-\tilde{p}(t)))\cdot(\tilde{p}(t+h)-\tilde{p}(t))ds\\
&=-\frac{1}{h}\int_0^1 \tilde{U}(t+h,z,\tilde{p}(t)+s(\tilde{p}(t+h)-\tilde{p}(t)))ds\\
&\cdot\int_t^{t+h}(\sum_y \tilde{p}_y(u)\lambda^*(y,\Delta^y\tilde{\theta}(u))+\epsilon_2(u))du,
\end{aligned} \tag{5.36}$$

where $\epsilon_2(t) := (\epsilon_2(t,1), \cdots, \epsilon_2(t,z))$. From Theorem 5.4.7, we know for any $y \in \Sigma$,

$$\lim_{h\to 0}[D\tilde{U}(t+h, z, \tilde{p}(t) + s(\tilde{p}(t+h) - \tilde{p}(t)))]_y = [D\tilde{U}(t, z, \tilde{p}(t))]_y.$$

As $D\tilde{U}$ is uniform bounded, we have following with dominated convergence theorem:

$$\lim_{h\to 0}\int_0^1 D\tilde{U}(t+h, z, \tilde{p}(t) + s(\tilde{p}(t+h) - \tilde{p}(t)))ds = D\tilde{U}(t, z, \tilde{p}(t)).$$

On the other hand, dividing h and letting $h \to 0$, we have following:

$$\lim_{h\to 0}\frac{\int_t^{t+h}(\sum_y \tilde{p}_y(u)\lambda^*(y, \Delta^y\tilde{\theta}(u)) + \epsilon_2(u))du}{h}$$
$$= \sum_y \tilde{p}_y(t)\lambda^*(y, \Delta^y\tilde{\theta}(t)) + \epsilon_2(t) = \sum_y \tilde{p}_y(t)\lambda^*(y, \Delta^y\tilde{U}) + \epsilon_2(t).$$

The last equation comes from Definition of $\tilde{U}$, which suggests $\Delta^y\tilde{U} = \Delta^y\tilde{\theta}(t)$.

For the second term in (5.35), from definition of $\tilde{U}$, we know

$$\tilde{U}(t+h, z, \tilde{p}(t+h)) - \tilde{U}(t, z, \tilde{p}) = \frac{d\tilde{\theta}_z(t)}{dt}h + o(h).$$

and hence

$$\lim_{h\to 0}\frac{\tilde{U}(t+h, z, \tilde{p}(t+h)) - \tilde{U}(t, z, \tilde{p}(t))}{h} = \frac{d\tilde{\theta}_z(t)}{dt} = -H(z, \Delta^z\tilde{U}) + \epsilon_1(t,z).$$

Combining both the results from first and second term in (5.35), taking $h \to 0$, we have

$$\frac{\partial\tilde{U}(t, z, \tilde{p}(t))}{\partial t} = -H(z, \Delta^z\tilde{U}) - D\tilde{U}(t, z, \tilde{p}(t)) \cdot (\sum_{y\in\Sigma} \tilde{p}_y(t)\lambda^*(y, \Delta^y\tilde{U}) + \epsilon_2(t))$$
$$+ \epsilon_1(t,z),$$

As $\|D\tilde{U}(t,z,\tilde{p}(t))\| \leq C$ uniformly and $\|\epsilon_2(t)\| \leq \frac{1}{N}$, we know

$$|D\tilde{U}(t,z,\tilde{p}(t)) \cdot \epsilon_2(t)| \leq \frac{C}{N}$$

Hence defining $\epsilon(t,z) := \epsilon_1(t,z) - D\tilde{U}(t,z,\tilde{p}(t)) \cdot \epsilon_2(t)$ concludes the proof. □

Then the DNN approximation $(\tilde{\theta},\tilde{p})$ is characterized by (5.34), while the true solution (θ,p) of the MFG is characterized by similar one, except ϵ and ϵ_3 are 0. Although the two Master equations are now backward PDE, it is still difficult to directly compare their solutions. Hence we would like to approximate the two PDEs by two ODE systems on some discrete grids of $P(\Sigma)$.

Define $P^N(\Sigma) = \{(\frac{n_1}{N},\cdots,\frac{n_K}{N}), \quad \sum_{z=0}^{K} n_z = N, n_z \in \mathbb{Z}^+\}$. Then $P^N(\Sigma)$ is a discrete grid of $P(\Sigma)$. For any $p^N \in P^N(\Sigma)$, define operators:

$$\begin{aligned}
&\alpha^{N,i,j}(p^N) := \begin{cases} p^N + \frac{1}{N}(\delta_j - \delta_i) & p_i^N > 0, p_j^N < 1 \\ p^N & else \end{cases} \\
&\Delta^{N,y}\tilde{U}(t,z,p^N) := (\tilde{U}(t,z,\alpha^{N,y,1}(p^N)) - \tilde{U}(t,z,p^N), \\
&\cdots, \tilde{U}(t,z,\alpha^{N,y,K}(p^N)) - \tilde{U}(t,z,p^N)) \\
&\Delta^{N,z,z}\tilde{U}(t,z,p^N) := (\tilde{U}(t,1,\alpha^{N,z,1}(p^N)) - \tilde{U}(t,z,p^N), \\
&\cdots, \tilde{U}(t,K,\alpha^{N,z,K}(p^N)) - \tilde{U}(t,z,p^N)).
\end{aligned} \tag{5.37}$$

With the discrete grid and discrete operators defined above, we next show in Proposition 5.4.9 that the Master equation can be approximate by a backward ODE system.

Proposition 5.4.9. *There exists N_0 such that for $N > N_0$, every $p^N \in P^N(\Sigma)$ and*

$z \in \Sigma$, $\tilde{U}$ *solves*

$$\begin{aligned}
&\frac{\partial \tilde{U}}{\partial t}(t,z,p^N) = \tilde{\epsilon}^N(t,z,p^N) - H(z, \Delta^{N,z,z}\tilde{U}(t,z,p^N)) \\
&- \sum_{y\in\Sigma}(p^N_y - \frac{\mathbb{1}_{y=z}}{N})\lambda^*(y, \Delta^{N,y,y}\tilde{U}(t,y,p^N)) \cdot \Delta^{N,y}\tilde{U}(t,z,p^N) \qquad (5.38)\\
&\tilde{U}(T,z,p^N) = G(z,p^N) + \epsilon_3(z),
\end{aligned}$$

where $\tilde{\epsilon}^N \in \mathcal{C}^0([0,T]\times\Sigma\times P^N(\Sigma))$, $\|\tilde{\epsilon}^N\| \leq \frac{C}{N}$.

Note that when only limiting $\tilde{U}$ on grid points in $P^N(\Sigma)$, $\tilde{U}(t,z,p^N)$ satisfies ODE system. But we still use $\frac{\partial \tilde{U}}{\partial t}(t,z,p^N)$ instead of $\frac{d\tilde{U}}{dt}(t,z,p^N)$ to stay consistent with above. The key step of the proof is to estimate the difference between the discrete operator $\Delta^{N,y}$ and the differential operator D^y, which is obtained by the uniform continuity of $\tilde{U}$'s directional derivatives. With the estimation it is straight forward to estimate the difference between the Master equation and the backward ODE system on those discrete grid points.

Proof. From Theorem 5.4.8, there exists constant N_0, such that when $N > N_0$ and (5.16), $\tilde{U}$ satisfies (5.34) when taking value on point (t,z,p^N).

$$\frac{\partial \tilde{U}(t,z,p^N)}{\partial t} = -H(z, \Delta^z\tilde{U}) - \sum_{y\in\Sigma} p^N_y \lambda^*(y, \Delta^y\tilde{U}) \cdot D\tilde{U}(t,z,p^N) + \epsilon(t,z).$$

It looks similar to (5.38), except for the discrete operator $\Delta^{N,y}$ and the differential operator D^y. Hence we next compare the two operators similar to (Cecchin and Pelino, 2019, Proposition 3). We first discuss the first component $\delta_1 - \delta_y$ of

$\Delta^{N,y}\tilde{U}(t,z,p^N)$ defined in (5.37),

$$\tilde{U}(t,z,p^N+\frac{1}{N}(\delta_1-\delta_y))-\tilde{U}(t,z,p^N)=\int_0^{\frac{1}{N}}[D^y\tilde{U}(t,z,p^N+s(\delta_1-\delta_y))]_1 ds$$
$$=[D^y\tilde{U}(t,z,p^N)]_1+\int_0^{\frac{1}{N}}([D^y\tilde{U}(t,z,p^N+s(\delta_1-\delta_y))]_1-[D^y\tilde{U}(t,z,p^N)]_1)ds$$
$$=[D^y\tilde{U}(t,z,p^N)]_1+O(\frac{1}{N^2}).$$

where the last equality is derived by the Lipschitz continuity in $p^N\in P(\Sigma)$ of $D^y\tilde{U}$. As above can be applied to every component in $\Delta^{N,y}\tilde{U}(t,z,p^N)$, we conclude that there exists N_0 such that for $N>N_0$,

$$\Delta^{N,y}\tilde{U}(t,z,p^N)=D^y\tilde{U}(t,z,p^N)+\epsilon^{N,y}(t,z,p^N),$$

where $\epsilon^{N,y}\in\mathcal{C}^0([0,T]\times\Sigma\times P^N(\Sigma);\mathbb{R}^K)$, $\|\epsilon^{N,y}\|\le\frac{C}{N^2}$.

Hence we have

$$\frac{\partial\tilde{U}}{\partial t}(t,z,p^N)=-\sum_{y\in\Sigma}(p_y^N-\frac{\mathbb{1}_{y=z}}{N})\lambda^*(y,\Delta^{N,y,y}\tilde{U}(t,y,p^N))\cdot\Delta^{N,y}\tilde{U}(t,z,p^N)$$
$$-H(z,\Delta^{N,z,z}\tilde{U}(t,z,p^N))+\sum_{i=1}^{4}e_i(t,z),$$

where

$$e_1(t,z):=H(z,\Delta^{N,z,z}\tilde{U}(t,z,p^N))-H(z,\Delta^z\tilde{U})$$
$$e_2(t,z):=\sum_{y\in\Sigma}p_y^N\Delta^{N,y}\tilde{U}(t,z,p^N)\cdot(\lambda^*(y,\Delta^{N,y,y}\tilde{U}(t,y,p^N))-\lambda^*(y,\Delta^y\tilde{U}))$$
$$e_3(t,z):=\sum_{y\in\Sigma}p_y^N(\Delta^{N,y}\tilde{U}(t,z,p^N)-D\tilde{U}(t,z,p^N))\cdot\lambda^*(y,\Delta^y\tilde{U})$$
$$e_4(t,z):=-\frac{\mathbb{1}_{y=z}}{N}\lambda^*(y,\Delta^{N,y,y}\tilde{U}(t,y,p^N))\cdot\Delta^{N,y}\tilde{U}(t,z,p^N)+\epsilon(t,z).$$

From the Lipschitz continuity of H and λ^*, as well as that $\tilde{U}$ is bounded, there exists constant C such that

$$|e_1(t,z)| \leq C\|\Delta^{N,z,z}\tilde{U}(t,z,p^N) - \Delta^z\tilde{U}\|$$
$$|e_2(t,z)| \leq C\max_{z\in\Sigma}\|\Delta^{N,z,z}\tilde{U}(t,z,p^N) - \Delta^z\tilde{U}\|.$$

From Proposition 5.4.2, we know there exists constant C such that

$$|e_1(t,z)| + |e_2(t,z)| \leq \frac{C}{2N}.$$

From Lemma 5.4.4 and $\sum_{z\in\Sigma}\lambda^*_z(y,\Delta^y\tilde{U}) = 0$ for every $y \in \Sigma$, we have

$$D^y\tilde{U}(t,z,p^N)) \cdot \lambda^*(y,\Delta^y\tilde{U}) = D\tilde{U}(t,z,p^N)) \cdot \lambda^*(y,\Delta^y\tilde{U}).$$

It follows that

$$e_3(t,z) = \sum_{y\in\Sigma} p^N_y(\Delta^{N,y}\tilde{U}(t,z,p^N) - D^y\tilde{U}(t,z,p^N)) \cdot \lambda^*(y,\Delta^y\tilde{U}).$$

From the boundedness of λ^* and ϵ, there is constant C such that

$$|e_3(t,z)| \leq \sum_{y\in\Sigma} p^N_y(\sum_{i\in\Sigma}\frac{C}{N^2}) \leq \frac{C}{4N}$$
$$|e_4(t,z)| \leq \frac{C}{4N}.$$

We can conclude the proof by defining

$$\tilde{\epsilon}^N(t,z,p^N) := \sum_{i=1}^{4} e_i(t,z) < \frac{C}{N}.$$

□

Finally we can proceed to the proof of our main result. The main idea of the proof is to characterize both the DNN approximation $(\tilde{\theta},\tilde{p})$ and the true solution (θ,p) by their corresponding Master equations, which are further approximated by two backward ODE systems on certain discrete grid points. Then the error of the two can be directly estimated on these grid points using Gronwall inequality. As both of $(\tilde{\theta}, \tilde{p})$ and (θ, p) are uniformly Lipschitz continuous w.r.t their initial conditions, the error between the grid points can also be estimated.

Proof of Theorem 5.3.2. As ODE system (5.2) admits solution to any initial value $p_0 \in P(\Sigma)$, we can define

$$U(t, z, p) := \theta(t, z).$$

Then from Cecchin and Pelino (2019), U satisfy the Master equation for any $p \in P(\Sigma)$.

$$\begin{aligned} &\frac{\partial U(t,z,p)}{\partial t} + H(z, \Delta^z U) + \sum_{y\in\Sigma} p_y D\tilde{U}(t,z,p)\cdot\lambda^*(y, \Delta^y U) = 0 \\ &U(T,z,p) = G(z,p). \end{aligned} \tag{5.39}$$

Similar to the proof of Proposition 5.4.9, we know $U(t, z, p^N)$ satisfy ODE

$$\begin{aligned} &\frac{\partial U}{\partial t}(t,z,p^N) = \epsilon^N(t,z,p^N) - H(z, \Delta^{N,z,z}U(t,z,p^N)) \\ &- \sum_{y\in\Sigma}(p^N_y - \frac{\mathbb{1}_{y=z}}{N})\lambda^*(y, \Delta^{N,y,y}U(t,y,p^N))\cdot\Delta^{N,y}U(t,z,p^N) \\ &U(T,z,p^N) = G(z,p^N), \end{aligned} \tag{5.40}$$

where $\epsilon^N \in \mathcal{C}^0([0,T]\times\Sigma\times P^N(\Sigma))$, $\|\epsilon^N\| \le \frac{C}{N}$.

From (5.38) and (5.40), There exists N_0 such that when $N > N_0$ and (5.16) holds, we have

$$\tilde{U}(t,z,p^N) - U(t,z,p^N) = \epsilon_3(z) + e + A + \sum_{y\in\Sigma}(p_y^N - \frac{\mathbb{1}_{y=z}}{N})(B_y + C_y)$$

$$e := \int_t^T (\tilde{\epsilon}^N(s,z,p^N) - \epsilon^N(s,z,p^N))ds$$

$$A := \int_t^T (H(z,\Delta^{N,z,z}\tilde{U}(s,z,p^N)) - H(z,\Delta^{N,z,z}U(s,z,p^N)))ds$$

$$B_y := \int_t^T [\lambda^*(y,\Delta^{N,y,y}\tilde{U}(s,y,p^N)) - \lambda^*(y,\Delta^{N,y,y}U(s,y,p^N))]\cdot\Delta^{N,y}\tilde{U}(s,z,p^N)ds$$

$$C_y := \int_t^T \lambda^*(y,\Delta^{N,y,y}U(s,y,p^N))\cdot[\Delta^{N,y}\tilde{U}(s,z,p^N) - \Delta^{N,y}U(s,z,p^N)]ds.$$

From Proposition 5.4.1, both U and $\tilde{U}$ are bounded. Hence H and λ^* are Lipschitz continuous w.r.t their second variable. Define

$$d(t) := \max_{z\in\Sigma, p^N\in P^N(\Sigma)} |\tilde{U}(t,z,p^N) - U(t,z,p^N)|.$$

There exists a constant C such that

$$|A| + |B_y| + |C_y| \le C\int_t^T d(s)ds.$$

As $p_N \in P^N(\Sigma)$, there exists constant C such that

$$d(t) \le \max_{z\in\Sigma,p^N\in P^N(\Sigma)}\{\int_t^T |\tilde{\epsilon}^N(s,z,p^N) - \epsilon^N(s,z,p^N)|ds + \epsilon_3(z)\} + C\int_t^T d(s)ds$$
$$\le \frac{C}{N} + C\int_t^T d(s)ds.$$

By applying Gronwall inequality, there is constant C such that for every $t \in [0,T]$, $z \in \Sigma$ and $p^N \in P^N(\Sigma)$ we have

$$|\tilde{U}(t,z,p^N) - U(t,z,p^N)| \le \frac{C}{N}. \tag{5.41}$$

For $N > 2N_0$ where N_0 is defined in Proposition 5.4.9 above, if $\tilde{p} \in B(P(\Sigma), \frac{1}{N})$, there is $p \in P(\Sigma)$ such that $\tilde{p} = p + \epsilon_4$ and $\epsilon_4 < \frac{1}{N}$. And there exists $p^N \in P^N(\Sigma)$ such that

$$\|p - p^N\| < \frac{1}{N}$$
$$\|\tilde{p} - p^N\| \leq \|\tilde{p} - p\| + \|p - p^N\| < \frac{2}{N} < \frac{1}{N_0}.$$

Hence from Proposition 5.4.1, $\tilde{U}(t, z, \tilde{p})$ is well defined. From Proposition 5.4.2, there exists constant C independent to N and p, such that for every $t \in [0, T]$ and $z \in \Sigma$,

$$|U(t, z, p) - U(t, z, p^N)| \leq \frac{C}{N}, \quad |\tilde{U}(t, z, \tilde{p}) - \tilde{U}(t, z, p^N)| \leq \frac{2C}{N}.$$

Hence combining above with (5.41), there is constant C independent to N and p, such that

$$|\tilde{U}(t, z, \tilde{p}) - U(t, z, p)| \leq \frac{C}{N}.$$

It is equivalent to

$$\|\tilde{\theta} - \theta\| \leq \frac{C}{N}.$$

By using the uniform boundedness and Lipschitz continuity of λ^*, we can prove p and $\tilde{p}$ are Lipschitz continuous w.r.t θ and $\tilde{\theta}$ respectively, with the help of Gronwall inequality and technique similar to the proof of Proposition 5.4.2. Note also that the Lipschitz coefficient only depends on the the uniform bound and Lipschitz continuous coefficient of λ^*, which again only depend on the preliminary M given in Proposition 5.4.1. Hence we know there exists a uniform constant C independent on N such that

$$\|\tilde{p} - p\| \leq \frac{C}{N},$$

which concludes our proof. □

5.5 Conclusion

In this chapter, we discuss numerically solving a general finite state mean field game with deep neural network. The equilibrium of mean field game is characterized by a forward backward ODE system, which is generally difficult to tackle using traditional ODE numerical scheme. We provide a deep neural network approach to numerically solve the forward backward ODE system. Moreover, by using the Master equation techniques, we provide a error estimation on the numerical solution. We prove that the error between numerical solution and true solution is linear to the square root of neural network's loss, given that the loss is smaller than certain threshold.

6

Conclusions

In this thesis we discuss the impact of different kind of competition on market makers' behaviour and strategy, which in turn, changes the implicit transaction cost of the market.

In Chapter 3, we consider a optimal market making problem when price competition is in place between market makers. We assume market maker's order flow arrival intensity depend on bid/ask spreads of both their own and the competitors. We model the problem as a non-zero sum stochastic differential game in a continuous time setting, because of the looping dependence structure among market makers. We characterize the equilibrium by a coupled system of HJB PDE, which could be further reduced to an ODE system. Verification theorem is proved, and without assuming a priori, the Issac condition is shown satisfied. It ensures the existence and uniqueness of solution to the ODE system, and hence that of the equilibrium. Then we numerically solve the ODE system to get market makers' optimal bid/ask spreads under equilibrium. It is shown that the best optimal bid/ask spreads derived from our model, is tighter than those from a comparable benchmark model without price competition in place. Hence it is suggested that price competition tends to tighten the best spreads in the market, and consequently lower the implicit transaction cost.

In Chapter 4, we consider market makers' competition for the market making incentive reward, especially when the reward is based on their trading volume ranking that measures their liquidity provision contributions. By considering the limiting case when the number of market makers tends to infinity, We simplify the original high dimension stochastic differential game, suffering from the curse of dimension, to a finite state mean field game. With the existence, uniqueness and convergence of the equilibrium guaranteed, we design a neural network approach to numerically solve the forward backward ODE system that characterizes the mean field game equilibrium. By comparing best equilibrium bid/ask spread under different types market making incentive reward, we find that the introduction of incentive can reduce the implicit trading cost. Rank-based reward, compared with the linear reward, tends to produce lower best spread in the market.

In Chapter 5, we estimate the theoretical error bound for the deep neural network numerical method in Chapter 4. We show that the deep neural network approach can be adapted to solve the forward backward ODE system generated from a more general type of finite state mean field game. We proved that the numerical solution, which itself is the true solution to a perturbed forward backward ODE system, satisfies a Master equation with some extra perturbed terms. And by comparing the corresponding Master equations, we prove that the error between true solution and the numerical solution is bounded linearly by the loss of the deep neural network.

There are some open questions in this thesis. For example, how to consider mixed strategy, instead of pure strategy when solving the equilibrium in both types of competition, or how to integrate the models of both competition, remains unexplored. We would like to leave these questions into future research.

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A

Numerical Method Based on Deep Neural Network

In this section, we will present the detailed algorithm for the deep neural network in Chapter 5.

Consider the forward backward ODE system.

$$\frac{d\theta_z(t)}{dt} = -H(z, \Delta^z\theta(t)), \quad \theta_z(T) = G(z, p(T)),$$
$$\frac{dp_z(t)}{dt} = \sum_y p_y(t)\lambda_z^*(y, \Delta^y\theta(t)), \quad p_z(t_0) = p_{z,0},$$

We use a LSTM (long short term memory) neural network to approximate the solution (θ, p). Denote the function constructed by LSTM neural network as $(\tilde{\theta}(t, \beta), \tilde{p}(t, \beta))$, where β is the parameters set for neural network. Neural network is designed by following. Layer 0 is the input $t \in [0, T]$. Then for layer k with output denoted by

h_k, it is designed as:

$$
\begin{aligned}
f_k &= \sigma_g(W_f t + U_f h_{k-1} + b_f) \\
i_k &= \sigma_g(W_i t + U_i h_{k-1} + b_i) \\
o_k &= \sigma_g(W_o t + U_o h_{k-1} + b_o) \\
\tilde{c}_t &= \sigma_c(W_c t + U_c h_{k-1} + b_c) \\
c_k &= f_k \circ c_{k-1} + i_k \circ \tilde{c}_k \\
h_k &= o_k \circ \sigma_h(c_k).
\end{aligned}
$$

Note that the initial values $c_0 = h_0 = 0$ and the operator $\circ$ is the element-wise product. The detailed notation is explained following:

- $t \in [0, T]$: input to the LSTM network
- $f_k \in \mathbb{R}^h$: forget gate's activation vector
- $i_k \in \mathbb{R}^h$: input/update gate's activation vector
- $o_k \in \mathbb{R}^h$: output gate's activation vector
- $h_k \in \mathbb{R}^h$: hidden state vector also known as output vector of the LSTM unit
- $\tilde{c}_k \in \mathbb{R}^h$: cell input activation vector
- $c_k \in \mathbb{R}^h$: cell state vector
- $W \in \mathbb{R}^{h\times 1}, U \in \mathbb{R}^{h\times h}, b \in \mathbb{R}^h$: weight matrices and bias vector parameters which need to be learned during training

where h is the number of hidden units.

The advantage of this specific structure, compared with the traditional neural network is that it provides better approximation ability for more complicated functions. For our model, this specific structure performs better than traditional neural network. We use a LSTM type network as above with total 3 layers and 32 nodes per layer.

The network is trained by stochastic gradient approach. We train the network mesh-free by randomly sampling points in $[0, T]$. This randomness actually adds to the robustness of the network. The detailed training procedure is similar to that in Sirignano and Spiliopoulos (2018).