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Research

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Article submitted to journal

Subject Areas:

analysis, applied mathematics

Keywords:

thin films, two-layer Couette flow, surfactant effects, local and global existence

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Mathematical study of a system of multidimensional nonlocal evolution equations describing surfactant-laden two-fluid shear flows

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This paper studies a coupled system of model multidimensional partial differential equations (PDEs) that arise in the nonlinear dynamics of two-fluid Couette flow when insoluble surfactants are present on the interface. The equations have been derived previously, but a rigorous study of local and global existence of its solutions, or indeed solutions of analogous systems, has not been considered previously. The evolution PDEs are two-dimensional in space and contain novel pseudo-differential terms that emerge from asymptotic analysis and matching in the multiscale problem at hand. The one-dimensional surfactant-free case was studied previously, where travelling wave solutions were constructed numerically and their stability investigated; in addition the travelling wave solutions were justified mathematically. The present study is concerned with some rigorous results of the multidimensional surfactant system, including local well-posedness and smoothing results when there is full coupling between surfactant dynamics and interfacial motion, and global existence results when such coupling is absent. As far as we know such results are new for nonlocal thin film equations in either one or two dimensions.

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1. Introduction

Shear flows involving multiple immiscible viscous fluid regions are of considerable technological, geophysical and physiological interest - see for example Papageorgiou [1], Bertagni & Camporeale [2], Grotberg [3] and references therein. Couette flow, namely the viscous flow in a channel one of whose walls moves in its plane at constant speed, is one of the simplest paradigm exact solutions of the Navier-Stokes equations. It is known that Couette flow is linearly stable at all Reynolds numbers and for both two- and three-dimensional disturbances, in the sense that the allied Orr-Sommerfeld problem does not yield any non-decaying solutions - see Schmid & Henningson [4]. When two immiscible fluids are present, however, it was shown by Yih [5] that a long wave interfacial instability is possible for the right viscosity and layer thickness ratios; the instability disappears in the absence of inertia. Following Yih's pioneering findings, the linear stability of two-fluid Couette flow was investigated by Hooper & Boyd [6] who considered two viscous unbounded shear flows and showed that the absence of walls precludes a long wave instability but allows short waves to grow in the absence of surface tension (not that the growth rates tend to zero as the wavenumber tends to infinity). The same authors extended the results when one bounding wall is present [7], and show that long wave instabilities are then possible. The physical mechanisms of the instability were identified by Hinch [8] and Charru & Hinch [9], and their genesis from a local velocity discrepancy of the base-flow at the perturbed interface due to viscosity stratification was established. A fairly comprehensive introduction to the theory and applications of multi-fluid flows can be found in the textbooks of Joseph & Renardy [10,11]. More recent linear stability studies include a Newtonian phase sheared by a viscoelastic phase, Sahu et al. [12], as well as a nonlinear stability for viscoelastic phases when one layer is thin, Ray et al. [13]. A recent review that details the instabilities and applications of sheared two-layer flows has appeared recently, Govindarajan & Sahu [14], while some new interesting aspects have been reported recently, for example a new linear mode present when there is a large viscosity contrast between the fluids, see Mohammadi & Smits [15] and Salu [16] for channel and core-annular pipe flows respectively. It is pointed out that such viscosity disparities are relevant to prescient drag reduction technologies that involve superhydrophobic or liquid infused surfaces. The adverse role of surfactants in such flows has been investigated in recent years, e.g. Landel et al. [17] and references therein. The present study includes surfactant effects in a nonlinear study of the underlying phenomena described above.

The present work is concerned with the nonlinear dynamics of two-layer Couette flows in the physically relevant limit when one of the layers is thin relative to the channel thickness. The configuration involving a thin more viscous layer (termed the thin-layer effect) has been shown to support instabilities and hence is a natural limit to study nonlinearly (see [5], [10]). Briefly, taking the lower layer to be thin and to support waves with wavelengths scaling with the channel height, reduces the thin layer to a lubrication layer. An asymptotic analysis was carried out in this case - see Kalogirou & Papageorgiou [18], Kalogirou et al. [19], Kalogirou [20], Kalogirou et al. [21] - and for weakly nonlinear interfacial amplitudes a class of integro-differential partial differential equations were derived. The nonlocal terms arise due to matching the dynamics in the thin layer with the main part of the channel flow where the balance of terms produces linear Navier-Stokes dynamics. The studies [19], [21] restricted attention to two-dimensional perturbations (the interface is a function of the streamwise coordinate alone), but carried out comparisons between the model evolution equations and direct numerical simulations as well as experiments at moderate Reynolds numbers, with very encouraging agreement. Kalogirou & Papageorgiou [18] included insoluble surfactant effects in order to model physical phenomena encountered in industrial and physiological applications (the ideas are based on the earlier work of Kas-Danouche et al. [22] that studied surfactant effects in pressure-driven core-annular flows). Kalogirou & Papageorgiou [18] showed that if surfactants are present then the flow can be destabilized by the Marangoni forces that arise due to surface tension variations.

The purpose of the present work is to report some rigorous results of the coupled system of equations described next. (Rigorous results for single PDEs involving nonlocal terms arising in electrified falling film flows can be found in Tseluiko & Papageorgiou [23].) Global well-posedness is of physical relevance in order to determine if the approximations made in deriving the model equations remain valid or not for all times. We first prove local well-posedness and smoothing results for the general case with no restriction on α, the coupling constant between diffusion of surfactant concentration and thin-film motion. We then prove global existence using global energy bounds for the particular case when $\alpha = 0$, *i.e.* when surfactant effects on the motion of the thin-film can be ignored. We are unaware of any such results for this or similar non-local thin-film model, even in one space dimension.

2. Mathematical model

A schematic of the problem is given in figure 1. For a full derivation see [18] - here we provide a brief description for completeness. In the schematic the upper plate velocity U, say, is used to non-dimensionalise fluid velocities, and the separation distance D between the two plates is used to non-dimensionalise lengths. The fluids in regions 1 and 2 have viscosities μ_1, μ_2, each with density ρ. Time is scaled by D/U, and pressure by ρU^2. At the interface $y = S(x, z, t)$ insoluble surfactants are present - the clean value of the surface tension is denoted by σ_0. In the asymptotic analysis that follows the lower layer is assumed to be thin and of mean thickness h^*.

Non-dimensionalising as described above yields the Navier-Stokes equations in each region $i = 1, 2$ (a standard $\boldsymbol{x} = (x, y, z)$ Cartesian coordinate system is used):

$$\frac{\partial \boldsymbol{u}_i}{\partial t} + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i = -\nabla p_i + \frac{1}{R_i} \Delta \boldsymbol{u}_i, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u}_i = 0, \tag{2.2}$$

where $\boldsymbol{u}_i = (u_i, v_i, w_i)$, and the Reynolds numbers are

$$R_2 = \frac{\rho U d}{\mu_2} := R, \qquad R_1 = \frac{\rho U d}{\mu_1} = m\, R, \tag{2.3}$$

where the viscosity ratio has been introduced

$$m = \frac{\mu_2}{\mu_1}. \tag{2.4}$$

The boundary conditions are those of no slip at the walls (indicated in figure 1) and interfacial conditions at $y = S(x, z, t)$ - in what follows it is useful to denote the position vector of the interface by $\boldsymbol{X} = (x, S(x, z, t), z)$. The latter are: (i) a kinematic condition, (ii) continuity of velocities across the interface, (iii) continuity of tangential stresses across the interface, (iv) continuity of normal stresses, and (v) a conservation of surfactant equation:

$$\frac{\partial \boldsymbol{X}}{\partial t} = \boldsymbol{u} \cdot \boldsymbol{n}, \tag{2.5}$$

$$[\boldsymbol{u}]_2^1 = 0, \tag{2.6}$$

$$[\boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{t}_\ell]_2^1 = -\boldsymbol{t}_\ell \cdot \nabla_s \sigma, \qquad \ell = 1, 2, \tag{2.7}$$

$$[\boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{n}]_2^1 = \sigma \kappa, \tag{2.8}$$

$$\frac{\partial \Gamma}{\partial t} - \frac{\partial \boldsymbol{X}}{\partial t} \cdot \nabla_s \Gamma + \nabla_s \cdot (\Gamma \boldsymbol{u}_s) - \frac{1}{Pe_s} \nabla_s^2 \Gamma + \Gamma\, \kappa\, u_n = 0. \tag{2.9}$$

In (2.5)-(2.8), $\boldsymbol{T}$ denotes the hydrodynamic stress tensor (recall that $T_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ in dimensional terms), $\boldsymbol{n}$ is the unit normal to the interface pointing into region 2, $\boldsymbol{t}_\ell, \ell = 1, 2$ are the contravariant base vectors of the tangent plane whose normal is $\boldsymbol{n}$, and κ is the mean curvature of the interface $\kappa = \frac{(1+S_x^2)S_{zz} - 2S_x S_z S_{xz} + (1+S_z^2)S_{xx}}{(1+S_x^2+S_z^2)^{3/2}}$. The symbol ∇_s denotes the

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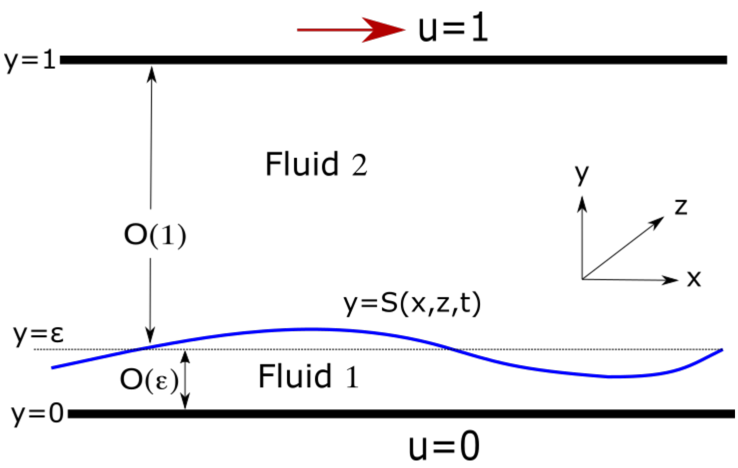

Figure 1. Schematic of the dimensionless problem. The upper wall moves with unit speed and the lower wall is stationary. The interface $y = S(x, z, t)$ must be determined.

gradient operator at the interface and it can be expressed in terms of the covariant tangent base vectors using elementary differential geometry [24,25]. Note that there is no ambiguity regarding the interfacial velocity to be used due to the continuity condition (2.6). The velocity $\boldsymbol{u}_I$ at the interface is written in terms of its contravariant form as $\boldsymbol{u}_I = \boldsymbol{u}_s + u_n\,\boldsymbol{n} = u^{(1)}\boldsymbol{t}_1 + u^{(2)}\boldsymbol{t}_2 + u_n\,\boldsymbol{n}$, and these quantities appear in the surfactant concentration equation (2.9). The surfactant diffusion at the interface is measured by the surface Peclet number $Pe_s = UD/D_s$ where D_s is the surfactant diffusivity at the interface.

The interfacial conditions (2.5)-(2.9) are in general nonlinear and depend on the amplitude of the interfacial waves. Conditions (2.7) contain Marangoni effects that induce tangential forces at the interface due to surfactant concentration gradients, while the normal stress condition (2.8) includes normal stresses from the fluid tensor as well as the jump in pressure due to surface tension. The surface tension σ is a decreasing function of the local surfactant concentration Γ^* - see [26]. We have used the dimensional linear equation of state $\sigma^* = \sigma_0 - \widehat{R}\widehat{T}\Gamma^*$, where $\widehat{R}$ and $\widehat{T}$ are the universal gas constant and absolute temperature of the system. In dimensionless terms this becomes $\sigma = 1 - \widehat{\beta}\Gamma$, where $\widehat{\beta} = \widehat{R}\widehat{T}\Gamma_\infty/\sigma_0$ with Γ_∞ being the maximum packing concentration of the surfactant on the interface. Note that due to the weakly nonlinear analysis that ensues, the linear isotherm used here would follow even if we utilized a nonlinear one. The following physical picture emerges: as flow sweeps surfactant along the interface it causes a higher Γ^* concentration downstream. The surface tension there is lower and hence a tangential Marangoni force is set up (proportional to $-\nabla_s\sigma(\Gamma^*)$). Asymptotic analysis for small $\varepsilon \ll 1$ enables considerable analytical progress.

Region 1 has dimensionless thickness $\varepsilon = h^*/D \ll 1$, and region 2 has $\mathcal{O}(1)$ thickness as shown in the figure. The interfacial waves have order one streamwise and spanwise lengths, i.e. $(x, z) = \mathcal{O}(1)$, and since $\varepsilon \ll 1$ we have a multiscale problem that is amenable to asymptotic analysis. The salient aspects include a lubrication theory in region 1 that allows for analytical solution of the flow field variables, coupled with a weakly nonlinear deflection of the interface whose size is selected to be large enough to retain nonlinear dynamics of the interface, and small enough to induce linear dynamics in the main region 2. More specifically, with appropriate non-dimensionalisation, the deflection (using the coordinate system of Figure 1) takes the form $y = \varepsilon - \varepsilon^2 H(x, z, t)$, and the analysis is carried out for trace amounts of surfactants. The result is a system of coupled evolution equations for the scaled interfacial height $H(x, z, t)$ and the scaled surfactant concentration $\Gamma(x, z, t)$ (note that in deriving the equations a Galilean frame is used in the weakly nonlinear analysis as well as a new slow time scale). In addition to the physical parameters, we consider the system on rectangular spatial domains that are doubly periodic of size $L \times \frac{L}{\beta}$. With appropriate spatial rescaling so that the wavelength in the x-direction is 2π, the model for the time evolution of H and Γ takes the form

$$H_t + HH_x + \nu\Delta^2 H + \sum_{k\in\mathbb{Z}_0^2} \mathcal{N}[k]\widehat{H}(k,t)e^{ik_1x+i\beta k_2 z} - \alpha\Delta\Gamma = 0, \tag{2.10}$$

$$\Gamma_t + (H\Gamma)_x - \gamma\Delta\Gamma = 0, \tag{2.11}$$

where

$$\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{(0,0)\}\,, \tag{2.12}$$

$\Delta = \partial_x^2 + \partial_z^2$, and $k_1, \beta k_2$ are wavenumbers in the x and z direction. In (2.10), $\alpha \geq 0$ is the coupling constant describing the effect of surfactant diffusion on thin-film evolution. The positive parameter $\nu = (\pi/L)^2$ is seen to decrease as the size L of the system increases, and $\gamma \geq 0$ is an inverse Peclet number measuring surfactant diffusion on the interface. It remains to define the kernel $\mathcal{N}$ that arises due to the coupling of the solutions in regions 1 and 2 at the scaled interface. Referring to [18] - see [27] also - we find

$$\mathcal{N}[k] = -\frac{i\Lambda}{\nu}\left(\frac{\kappa_1}{2}\right)F''(0;\kappa_1,\kappa_2), \qquad \kappa_1 = k_1\sqrt{\nu}, \quad \kappa_2 = \beta k_2\sqrt{\nu}, \tag{2.13}$$

where the function $F(y;\kappa_1,\kappa_2)$ is proportional to the perturbation velocity in region 2 and is found by solving the following two-point boundary value

$$\left(\frac{d^2}{dy^2} - (\kappa_1^2+\kappa_2^2)\right)^2 F(y;\kappa_1,\kappa_2) - i\kappa_1 R\, y\left(\frac{d^2}{dy^2} - (\kappa_1^2+\kappa_2^2)\right)F(y;\kappa_1,\kappa_2) = 0, \tag{2.14}$$

$$F(0;\kappa_1,\kappa_2) = 0, \quad F'(0;\kappa_1,\kappa_2) = 1, \quad F(1;\kappa_1,\kappa_2) = 0, \quad F'(1;\kappa_1,\kappa_2) = 0. \tag{2.15}$$

Equation (2.14) can be recognised as an Orr-Sommerfeld type equation for a Couette linear shear base flow $\overline{U} = y$; the parameter $R = \rho UD/\mu_2$ is the Reynolds number and we note that an unsteady term that typically appears in the Orr-Sommerfeld equation, is absent due to the asymptotic slow time-scale.

The properties of $F''(0,\kappa_1,\kappa_2)$ and therefore of the Kernel $\mathcal{N}[k]$, can be deduced in detail from earlier work [27] where the one-dimensional version of (2.14) that follows by setting $\kappa_2 = 0$, was studied. Crucially, [27] also determined the large wavenumber asymptotics of $\mathcal{N}[k]$. The asymptotic results needed here are reproduced in Appendix A (see Proposition A.1 and the Remark following it). These results for large κ in 1-D can be translated to the present case by noting that in what follows the inertial term in (2.15) may be re-expressed as a product:

$$\kappa_1 R = \operatorname{sgn}\kappa_1\sqrt{\kappa_1^2+\kappa_2^2}\left[\frac{R|\kappa_1|}{\sqrt{\kappa_1^2+\kappa_2^2}}\right].$$

Therefore, identifying $\operatorname{sgn} k_1\sqrt{k_1^2+\beta^2k_2^2}$ and $\frac{R|k_1|}{\sqrt{k_1^2+\beta^2k_2^2}}$ with k and R, respectively, in the 1-D expression (A 5) for $F''\left(0,k\sqrt{\nu}\right)$, determines $F''\left(0,\kappa_1,\kappa_2\right)$, and thereby yields the asymptotic behavior of $\mathcal{N}[k]$ when $\kappa_1^2+\kappa_2^2 \gg 1$, noting that the modified Reynolds number remains $O(1)$. Therefore, Proposition (A.1) in the Appendix, translates to

$$\mathcal{N}[k] = \frac{i\Lambda k_1\sqrt{k_1^2+\beta^2k_2^2}}{\left(1 - \frac{iRk_1}{4(k_1^2+\beta^2k_2^2)^{3/2}}\right)}\left(\frac{1+E_1}{(1+E_2)(1+E_A)}\right), \tag{2.16}$$

where E_1, E_2 are exponentially small for large $\sqrt{\nu(k_1^2+\beta^2k_2^2)}$ and $E_A = O\left(\frac{R^2k_1^2}{(k_1^2+\beta^2k_2^2)^3}\right)$. In particular, as in 1-D, we still have upper bounds (A 6) for $-\Re\mathcal{N}[k]$ and $\left|\Im\mathcal{N}[k]\right|$, though the bounds may be different in 2-D.

3. Global existence for the model thin film equation with surfactants

We will study (2.10)-(2.11) in a right-moving frame of reference having speed C and will decompose Γ into a mean component and a zero-mean perturbation, i.e. we transform $\Gamma(x,z,t) = b + \tilde{\Gamma}(x,z,t)$ where $b \geq 0$. With slight abuse of notation, we drop the $\tilde{}$ in what follows. The equations for the initial value problem become

$$H_t - CH_x + HH_x + \nu\Delta^2 H + \sum_{k\in\mathbb{Z}_0^2} \mathcal{N}[k]\widehat{H}(k,t)e^{ik_1x+i\beta k_2 z} - \alpha\Delta\Gamma = 0, \quad H(\cdot,0) = H^0, \tag{3.1}$$

$$\Gamma_t - C\Gamma_x + bH_x + (H\Gamma)_x - \gamma\Delta\Gamma = 0, \quad \Gamma(\cdot,0) = \Gamma^0. \tag{3.2}$$

It is easy to see that the spatial averages of H and Γ over the periodic rectangle $[0,2\pi]\times[0,2\pi/\beta]$ given by $\langle H\rangle = \frac{\beta}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi/\beta} H(x,z,t)dxdz$ (with a similar expression for $\langle\Gamma\rangle$), are conserved in t. Each of these may be taken to be zero without loss of generality, since any nonconstant average corresponds to a shift in constants C and b. Hence, starting with initial conditions H^0, Γ^0 having zero spatial mean, this remains so for all t, and hence the zero Fourier modes satisfy $\widehat{H}(0,0,t) = 0 = \widehat{\Gamma}(0,0,t)$.

We prove the following two theorems.

Theorem 1. *(Local existence and uniqueness) Assume that the Fourier coefficients of the initial conditions H^0 and Γ^0 are in $\ell_1(\mathbb{Z}_0^2)$. Then, for T sufficiently small, there exists a unique solution to the initial value problem (3.1), (3.2) for which the Fourier coefficients $\widehat{H}(k_1,k_2,t)$ and $\widehat{\Gamma}(k_1,k_2,t)$ are in $\ell_1(\mathbb{Z}_0^2)$ for $t\in[0,T)$. Moreover, the solution $(H(\cdot,t),\Gamma(\cdot,t))$ is smooth for any $t\in(0,T)$.*

Theorem 2. *(Global existence for $\alpha = 0$) The solution (H,Γ) in Theorem 1 can be extended globally in time for $\alpha = 0$.*

Remark 3.1. *In the following section we prove local existence. We do so in Fourier space by formulating an integral equation and employing the Banach Contraction theorem in an appropriate space for small T.*

Remark 3.2. *Theorem 2 follows from global energy bounds in Section 5 (see Remarks 4.1, 4.2).*

4. Local existence and uniqueness

In Fourier space, with $k = (k_1,k_2)\in\mathbb{Z}_0^2$, equation (3.1) becomes

$$\widehat{H}_t + \left(-iCk_1 + \nu(k_1^2+\beta^2k_2^2)^2 + \mathcal{N}[k]\right)\widehat{H} + \alpha\left(k_1^2+\beta^2k_2^2\right)\widehat{\Gamma} = -\frac{ik_1}{2}\widehat{H}*\widehat{H}, \quad \widehat{H}(\cdot,0) = \widehat{H}^0, \tag{4.1}$$

$$\widehat{\Gamma}_t + \left(-ik_1C + \gamma(k_1^2+\beta^2k_2^2)\right)\widehat{\Gamma} + ibk_1\widehat{H} = -ik_1\widehat{H}*\widehat{\Gamma}, \quad \widehat{\Gamma}(\cdot,0) = \widehat{\Gamma}^0. \tag{4.2}$$

In the analysis towards finding an integral equation representation suitable for the nonlinear initial value problem (4.1)-(4.2) above, we first formulate the linear initial value problem. We represent the linear version of (4.1)-(4.2) as an inhomogeneous system with the nonlinearities on the right hand side assumed known,

$$\begin{aligned}
\frac{d\widehat{H}}{dt} + a_{11}(k)\,\widehat{H} + a_{12}(k)\,\widehat{\Gamma} &= \widehat{r}_1(k,t), \qquad \widehat{H}(\cdot,0) = \widehat{H}^0, \\
\frac{d\widehat{\Gamma}}{dt} + a_{21}(k)\,\widehat{H} + a_{22}(k)\,\widehat{\Gamma} &= \widehat{r}_2(k,t), \qquad \widehat{\Gamma}(\cdot,0) = \widehat{\Gamma}^0,
\end{aligned} \tag{4.3}$$

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where

$$a_{11}(k) = \nu(k_1^2 + \beta^2 k_2^2)^2 - iCk_1 + \mathcal{N}[k], \qquad a_{12}(k) = \alpha(k_1^2 + \beta^2 k_2^2),$$
$$a_{21}(k) = ik_1 b, \qquad a_{22}(k) = \gamma(k_1^2 + \beta^2 k_2^2) - iCk_1. \tag{4.4}$$

We define $\lambda_1(k), \lambda_2(k)$ to be the two distinct eigenvalues of the matrix (a_{ij}), given explicitly as the roots of the quadratic equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0,$$
$$\lambda_{1,2} = \frac{1}{2}(a_{11} + a_{22}) \pm \frac{1}{2}\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}. \tag{4.5}$$

Due to the dependence of $a_{11}, a_{12}, a_{21}, a_{22}$ on k shown in (4.4), it follows that for large k,

$$\lambda_1(k) = a_{11}(k) + o(1) = \nu(k_1^2 + \beta^2 k_2^2)^2 + \mathcal{N}[k] - iCk_1 + o(1), \tag{4.6}$$
$$\lambda_2(k) = a_{22}(k) + o(1) = \gamma(k_1^2 + \beta^2 k_2^2) - iCk_1 + o(1), \tag{4.7}$$

where the large k behaviour of $\mathcal{N}[k]$ is given by equation (2.16). Then, it follows that the solution to the initial value problem (4.3) may be written as

$$\begin{pmatrix} \widehat{H}(k,t) \\ \widehat{\Gamma}(k,t) \end{pmatrix} = \mathbf{U}(k,t) \begin{pmatrix} \widehat{H}^0 \\ \widehat{\Gamma}^0 \end{pmatrix} + \int_0^t \mathbf{U}(k, t-t') \begin{pmatrix} \widehat{r}_1(\cdot, t') \\ \widehat{r}_2(\cdot, t') \end{pmatrix} dt', \tag{4.8}$$

where $\mathbf{U}(k,t)$ is given by

$$\mathrm{e}^{-\lambda_2 t} \begin{pmatrix} \frac{(a_{11}-\lambda_1)-(a_{11}-\lambda_2)\mathrm{e}^{-(\lambda_1-\lambda_2)t}}{\lambda_2-\lambda_1} & \frac{a_{12}\left(1-\mathrm{e}^{-(\lambda_1-\lambda_2)t}\right)}{\lambda_2-\lambda_1} \\ \frac{(a_{11}-\lambda_1)(a_{11}-\lambda_2)\left(\mathrm{e}^{-(\lambda_1-\lambda_2)t}-1\right)}{a_{12}(\lambda_2-\lambda_1)} & \frac{-(a_{11}-\lambda_2)+(a_{11}-\lambda_1)\mathrm{e}^{-(\lambda_1-\lambda_2)t}}{\lambda_2-\lambda_1} \end{pmatrix}, \tag{4.9}$$

and each of $a_{11}, a_{22}, \lambda_1, \lambda_2$ are understood to depend on k. If for some $k = k_0 \in \mathbb{Z}_0^2$, we have $\lambda_1(k_0) = \lambda_2(k_0)$, then in (4.9), $\mathbf{U}(k_0, t)$ is replaced by the well-defined limit $\lambda_1 \to \lambda_2$.

Hence we may write the solution to the initial value problem

$$\begin{aligned} \partial_t \widehat{H} + a_{11}(k)\widehat{H} + a_{12}(k)\widehat{\Gamma} &= -\frac{ik_1}{2}\widehat{H} * \widehat{H}, \qquad \widehat{H}(\cdot, 0) = \widehat{H}^0, \\ \partial_t \widehat{\Gamma} + a_{21}(k)\widehat{H} + a_{22}(k)\widehat{\Gamma} &= -ik_1 \widehat{H} * \widehat{\Gamma}, \qquad \widehat{\Gamma}(\cdot, 0) = \widehat{\Gamma}^0, \end{aligned} \tag{4.10}$$

as the solution to the nonlinear system of integral equations

$$\begin{aligned} &\mathbf{Y} = \mathbf{M}[\mathbf{Y}], \quad \mathbf{Y} = \begin{pmatrix} \widehat{H}(\cdot, \tau) \\ \widehat{\Gamma}(\cdot, \tau) \end{pmatrix}, \\ &\mathbf{M}[\mathbf{Y}](\cdot, t) = \mathbf{U}(\cdot, t) \begin{pmatrix} \widehat{H}^0 \\ \widehat{\Gamma}^0 \end{pmatrix} + \int_0^t \mathbf{U}(\cdot, t-t') \begin{pmatrix} -\frac{ik_1}{2}\left[\widehat{H} * \widehat{H}\right](\cdot, t') \\ -ik_1 \left[\widehat{H} * \widehat{\Gamma}\right](\cdot, t') \end{pmatrix} dt'. \end{aligned} \tag{4.11}$$

We define

$$\delta = \min\left\{\frac{\nu}{3}, \frac{\gamma}{3}, \frac{1}{3}\nu\beta^4, \frac{1}{3}\gamma\beta^2\right\}. \tag{4.12}$$

By choice of δ in (4.12), and expressions (4.6)-(4.7), it follows that for large $|k|$,

$$\Re\lambda_2(k) > \frac{5}{2}\delta|k|^2, \quad \text{and} \quad \Re(\lambda_1(k) - \lambda_2(k)) > 0. \tag{4.13}$$

We define

$$M = \max\left\{\sup_{k \in \mathbb{Z}_0^2}\left(\frac{5}{2}\delta|k|^2 - \Re\lambda_2(k)\right), \sup_{k \in \mathbb{Z}_0^2}\left(\frac{5}{2}\delta|k|^2 - \Re\lambda_1(k)\right)\right\}, \tag{4.14}$$

which is finite because of (4.13). From the properties discussed above, we can choose C_U independent of k so that

$$\sup_{k\in\mathbb{Z}_0^2}\ \sup_{t\in[0,T)} e^{-(M-2\delta|k|^2)t}\left|\mathbf{U}(k,t)\right| \le C_U\,, \tag{4.15}$$

where $\left|.\right|$ for 2×2 matrices or 2×1 vectors is interpreted in the ∞-norm sense. With *a priori* restriction $T\le 1$, C_U may be chosen independent of T. In particular, this implies that

$$e^{-Mt+\omega|k|t}\left|\mathbf{U}(k,t)\right| \le C_U e^{-\delta|k|^2 t}. \tag{4.16}$$

For any choice of $\omega\in[0,\delta]$, we introduce norms $\|\mathbf{Y}\|$ in the space $\mathcal{Y}$ of vector-valued functions $\mathbf{Y}(k,t)$ that are continuous in $t\in[0,T)$ for each $k\in\mathbb{Z}_0$. The norm $\|\mathbf{Y}\|$ as defined below is finite for $\mathbf{Y}\in\mathcal{Y}$:

$$\|\mathbf{Y}\| = \max\left\{\sum_{k\in\mathbb{Z}_0}\sup_{t\in[0,T)} e^{\omega|k|t-Mt}\left|\widehat{Y}_1(k,t)\right|\,,\ \sum_{k\in\mathbb{Z}_0}\sup_{t\in[0,T)} e^{\omega|k|t-Mt}\left|\widehat{Y}_2(k,t)\right|\right\}. \tag{4.17}$$

Lemma 4.1.

$$\mathbf{Y}_0(\cdot,t) = \mathbf{M}\left[0\right](\cdot,t) = \mathbf{U}(\cdot,t)\begin{pmatrix}\widehat{H}^0\\ \widehat{\Gamma}^0\end{pmatrix} \tag{4.18}$$

satisfies the following condition

$$\|\mathbf{Y}_0\| \le C_U \max\left\{\|\widehat{H}^0\|_{l^1}, \|\widehat{\Gamma}^0\|_{l_1}\right\}. \tag{4.19}$$

Proof. From (4.12) and (4.15), it follows that for $\omega\in[0,\delta]$ we have

$$\sup_{t\in[0,T]}\left|U(k,t)\right| e^{\omega|k|t-Mt} \le C_U, \tag{4.20}$$

and hence from definition of the norm, the result follows from (4.18) on inspection.

□

Lemma 4.2. *If we define*

$$\mathbf{Z}(k,t) = \begin{pmatrix}\frac{1}{2}\widehat{H}*\widehat{H}\\ \widehat{\Gamma}*\widehat{H}\end{pmatrix}, \tag{4.21}$$

then with $\mathbf{Y}=\left(\widehat{H},\widehat{\Gamma}\right)$, *we have*

$$\|\mathbf{Z}\| \le e^{MT}\|\mathbf{Y}\|^2. \tag{4.22}$$

Furthermore for two different $\mathbf{Y}^{(1)}=\left(\widehat{H}^{(1)},\widehat{\Gamma}^{(1)}\right)$ *and* $\mathbf{Y}^{(2)}=\left(\widehat{H}^{(2)},\widehat{\Gamma}^{(2)}\right)$ *we have corresponding* $\mathbf{Z}^{(1)}$ *and* $\mathbf{Z}^{(2)}$ *satisfying*

$$\|\mathbf{Z}^{(1)}-\mathbf{Z}^{(2)}\| \le e^{MT}\left(\|\mathbf{Y}^{(1)}\|+\|\mathbf{Y}^{(2)}\|\right)\|\mathbf{Y}^{(1)}-\mathbf{Y}^{(2)}\|. \tag{4.23}$$

Proof. We define $v(k)=\sup_{t\in[0,T)} e^{-Mt+\omega|k|t}\left|\widehat{H}(k,t)\right|$ and $w(k)=\sup_{t\in[0,T)} e^{-Mt+\omega|k|t}\left|\widehat{\Gamma}(k,t)\right|$. We have the first component of the vector $\mathbf{Z}$ satisfying

$$\begin{aligned} e^{-Mt+\omega|k|t}\left|\widehat{Z}_1(k,t)\right| &\le \frac{e^{MT}}{2}\sum_{j\in\mathbb{Z}_2^0}\left(e^{-Mt+\omega|k-j|t}\left|\widehat{H}(k-j,t)\right|\right)\left(e^{-Mt+\omega|j|t}\left|\widehat{H}(j,t)\right|\right)\\ &\le \frac{1}{2}[v*v](k), \end{aligned} \tag{4.24}$$

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while the second component of $\mathbf{Z}$ satisfies

$$e^{-Mt+\omega|k|t}\left|\widehat{Z}_2(k,t)\right| \le e^{MT} \sum_{j\in\mathbb{Z}_2^0} \left(e^{-Mt+\omega|k-j|t}\left|\widehat{H}(k-j,t)\right|\right)\left(e^{-Mt+\omega|j|t}\left|\widehat{\Gamma}(j,t)\right|\right)$$
$$\le [v*w](k). \tag{4.25}$$

Therefore, taking sup over $t\in(0,T)$ of the left hand side, and then taking the l^1 norm we obtain the desired result by noting that $\|\mathbf{Y}\| = \max\{\|v\|_{l^1}, \|w\|_{l^1}\}$. In the second case, we repeat the same argument by first noting that

$$\widehat{H}^{(1)} * \widehat{H}^{(1)} - \widehat{H}^{(2)} * \widehat{H}^{(2)} = \left(\widehat{H}^{(1)} + \widehat{H}^{(2)}\right) * \left(\widehat{H}^{(1)} - \widehat{H}^{(2)}\right), \tag{4.26}$$

while

$$\widehat{H}^{(1)} * \widehat{\Gamma}^{(1)} - \widehat{H}^{(2)} * \widehat{\Gamma}^{(2)} = \widehat{H}^{(1)} * \left(\widehat{\Gamma}^{(1)} - \widehat{\Gamma}^{(2)}\right) + \widehat{\Gamma}^{(2)} * \left(\widehat{H}^{(1)} - \widehat{H}^{(2)}\right). \tag{4.27}$$

□

Definition 4.1. *We define the linear operator $\mathcal{J}$ acting on a two component vector function $\mathbf{Z}(k,t)$ by*

$$\mathcal{J}\left[\mathbf{Z}\right](k,t) = \int_0^t ik_1 \mathbf{U}(k,t-t')\mathbf{Z}(k,t')dt'. \tag{4.28}$$

Lemma 4.3.

$$\|\mathcal{J}\left[\mathbf{Z}\right]\| \le \gamma\delta^{-1/2}T^{1/2}C_U\|\mathbf{Z}\|. \tag{4.29}$$

Proof. We define

$$\zeta(k) = \sup_{t\in[0,T)} e^{-Mt}e^{\omega|k|t}\left|\hat{Z}(k,t)\right|, \tag{4.30}$$

then

$$\begin{aligned} e^{-Mt+\omega|k|t}\int_0^t |k_1|\left|\mathbf{U}(t-t')\hat{Z}(k,t')\right|dt' &\le C_U \int_0^t |k_1| e^{-\delta|k|^2(t-t')}e^{-Mt'+\omega|k|t'}\left|\hat{Z}(k,t')\right|dt' \\ \le \left(\int_0^t e^{-\delta|k|^2(t-t')}|k_1|dt'\right) C_U\zeta(k) &\le \delta^{-1/2}\gamma T^{1/2}C_U\zeta(k). \end{aligned} \tag{4.31}$$

By taking the sup over $t\in[0,T)$ followed by the l^1 norm in k gives the desired result. □

Proposition 4.1. *There exists a unique solution of the integral equation (4.11) in a ball $\mathcal{B}_{2\|\mathbf{Y}_0\|}\subset\mathcal{Y}$ for T small enough to satisfy*

$$4\delta^{-1/2}\gamma T^{1/2}e^{MT}C_U\|\mathbf{Y}_0\| < 1. \tag{4.32}$$

Proof. We note that

$$\mathbf{M}\left[\mathbf{Y}\right] = \mathbf{Y}_0 + \mathcal{J}\left[\mathbf{Z}\right], \tag{4.33}$$

where $\mathbf{Z}$ is defined in accordance to (4.21). Using Lemmas 4.2 and 4.3, if $\mathbf{Y}\in\mathcal{B}_{2\|\mathbf{Y}_0\|}$, then

$$\|\mathbf{M}\left[\mathbf{Y}\right]\| \le \|\mathbf{Y}_0\| + 4\gamma\delta^{-1/2}T^{1/2}C_U e^{MT}\|\mathbf{Y}_0\|^2 \le 2\|\mathbf{Y}_0\|, \tag{4.34}$$

and

$$\|\mathbf{M}\left[\mathbf{Y}^{(1)}\right] - \mathbf{M}\left[\mathbf{Y}^{(2)}\right]\| \le 4\gamma\delta^{-1/2}T^{1/2}C_U e^{MT}\|\mathbf{Y}_0\|\|\mathbf{Y}^{(1)} - \mathbf{Y}^{(2)}\|. \tag{4.35}$$

Therefore, $\mathbf{M}:\mathcal{B}_{2\|\mathbf{Y}_0\|}\to\mathcal{B}_{2\|\mathbf{Y}_0\|}$ contractively, and the Proposition follows from the Banach contraction theorem. □

Remark 4.1. *Though we have proved the solution to (4.11) to be unique in the ball $\mathcal{B}_{2\|\mathbf{Y}_0\|}\subset\mathcal{Y}$, this is the only solution to (4.11) in $\mathcal{Y}$ since continuity in t implies that if we were to choose T small enough, then it must be in the ball $\mathcal{B}_{2\|\mathbf{Y}_0\|}$ containing $\mathbf{Y}_0$; uniqueness implies that this is the solution guaranteed by*

Proposition 4.1. Furthermore, since the integrand on the right hand side of (4.11) is continuous in time for each k, it follows that the integral and therefore $\mathbf{M}[\mathbf{Y}]$ is a C^1 function in time, implying from (4.11) that the fixed point $\mathbf{Y}$ is C^1 in t. On differentiation of (4.11), it follows that $\mathbf{Y}$ in Proposition 4.1 satisfies the initial value problem (4.10) for $t \in [0,T]$.

Remark 4.2. *By choosing $\omega = \frac{\delta}{2} > 0$, it is clear that the solution instantly smooths out in $x - z$ space, since in the Fourier space, for any $T > 0$, $\sup_{k \in \mathbb{Z}_0^2} |k|^j e^{-\frac{\omega}{2}|k|T} \|\mathbf{Y}\|$ exists for any $j > 0$. This implies that if we restart the problem at $t = T$, we can assume an infinitely regular initial condition. Because the system (4.1)-(4.2) is autonomous in time, this is equivalent to starting with smooth initial conditions at $t = 0$. This also implies that on inverse Fourier Transform the pair $(H(x,z,t), \Gamma(x,z,t))$ is a classical solution of (3.1)-(3.2).*

5. Global existence

If we were to choose $\omega = 0$ in Proposition 4.1, then it is clear that since $\|\mathbf{Y}(\cdot, T)\|_{l^1(\mathbb{Z}_0^2)} < \infty$, we can continue the same argument for the initial value problem (4.1)-(4.2) and formulate an integral equation for $t \in [T, T']$ for some T' and thereby prove existence. Define $T_0 = \sup T$ so that the solution exists for $t \in (0, T_0)$. If $T_0 < \infty$, it would follow that

$$\lim_{t \to T_0^-} \|\widehat{H}(\cdot,t)\|_{l_1(\mathbb{Z}_0^2)} = \infty \qquad \text{or} \qquad \lim_{t \to T_0^-} \|\widehat{\Gamma}(\cdot,t)\|_{l_1(\mathbb{Z}_0^2)} = \infty,$$

as otherwise we can continue the solution beyond T_0. In what follows we rule out this possibility, at least for the case $\alpha = 0$, by showing that

$$\sum_{k \in \mathbb{Z}_0^2} (k_1^2 + \beta^2 k_2^2)^2 \left|\widehat{H}(k_1,k_2,t)\right|^2 \le M_0(T) < \infty, \qquad \text{and}$$
$$\sum_{k \in \mathbb{Z}_0^2} (k_1^2 + \beta^2 k_2^2)^2 \left|\widehat{\Gamma}(k_1,k_2,t)\right|^2 \le M_0(T) < \infty,$$

for $t \in (0,T)$ for any finite T, which by Lemma A.1 in Appendix A, rules out $\lim_{t \to T_0^-} \|\widehat{H}(\cdot,t)\|_{l^1(\mathbb{Z}_0^2)} = \infty$ and $\lim_{t \to T_0^-} \|\widehat{\Gamma}(\cdot,t)\|_{l^1(\mathbb{Z}_0^2)} = \infty$, and therefore global existence in time of the initial value problem (3.1)-(3.2) follows.

(a) Energy Bounds

Given real valued functions u, v, we define the inner product

$$(u,v) = \frac{\beta}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi/\beta} v(x,z) u(x,z) dx dz, \tag{5.1}$$

with corresponding L_2 norm $\|u\|_{L_2} = (u,u)^{1/2}$. The norm has Fourier representation

$$\|u\|_{L^2}^2 = \sum_{k \in \mathbb{Z}_0^2} \left|\widehat{u}(k)\right|^2. \tag{5.2}$$

In this section $\|.\|$ will denote the L_2 norm. Taking the inner product of (3.1) with H gives

$$\frac{d}{dt}\frac{1}{2}\|H\|^2 + \nu \|\Delta H\|^2 = -\sum_{k \in \mathbb{Z}_0^2} \Re \mathcal{N}[k] \left|\widehat{H}(k,t)\right|^2 + \alpha\left(\Delta H, \Gamma\right). \tag{5.3}$$

Using the fact

$$M = \sup_{k \in \mathbb{Z}_0^2} \left\{-\Re \mathcal{N}[k]\right\}, \tag{5.4}$$

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along with the Cauchy-Schwartz inequality $\alpha(\Delta H, \Gamma) \leq \alpha\|\Delta H\|\,\|\Gamma\| \leq \frac{\alpha^2}{2\nu}\|\Gamma\|^2 + \frac{\nu}{2}\|\Delta H\|^2$, yields the inequality

$$\frac{d}{dt}\frac{1}{2}\|H\|^2 + \frac{\nu}{2}\|\Delta H\|^2 \leq M\|H\|^2 + \frac{\alpha^2}{2\nu}\|\Gamma\|^2. \tag{5.5}$$

Next, taking the inner product of (3.2) with Γ produces

$$\frac{d}{dt}\frac{1}{2}\|\Gamma\|^2 + \gamma\|\nabla\Gamma\|^2 = b\,(H, \Gamma_x) + (H\Gamma, \Gamma_x)\,, \tag{5.6}$$

which on use of the Cauchy-Schwartz inequality as above implies

$$\frac{d}{dt}\frac{1}{2}\|\Gamma\|^2 \leq \frac{b^2}{2\gamma}\|H\|^2 + \frac{\|H\|_\infty^2}{2\gamma}\|\Gamma\|^2. \tag{5.7}$$

(b) The case of $\alpha = 0$

Applying Gronwall's inequality to equation (5.5) (noting that $\alpha = 0$ and that we can disregard the term $\frac{\nu}{2}\|\Delta H\|^2$ in the calculation), we obtain

$$\|H(\cdot,t)\|^2 \leq \|H^0\|^2 e^{2Mt} =: C_0(t). \tag{5.8}$$

Integrating (5.5) between 0 and t and using (5.8) again, yields the bound

$$\nu\int_0^t \|\Delta H(\cdot,\tau)\|^2 d\tau \leq \|H^0\|^2 + 2M\|H^0\|^2\int_0^t e^{2M\tau}d\tau = C_0(t). \tag{5.9}$$

From the relation (5.7) and using Lemma A.1 we have

$$\frac{d}{dt}\frac{1}{2}\|\Gamma\|^2 \leq \frac{c_0^2}{2\gamma}\|\Delta H\|^2\|\Gamma\|^2 + \frac{b^2}{2\gamma}\|H\|^2. \tag{5.10}$$

We obtain

$$\|\Gamma(\cdot,t)\|^2 \leq \|\Gamma^0\|^2 \exp\left[\frac{c_0^2}{2\gamma}\int_0^t \|\Delta H(\cdot,\tau)\|^2 d\tau\right] + \frac{b^2}{2\gamma}\int_0^t C_0(\tau)\exp\left[\frac{c_0^2}{2\nu\gamma}C_0(\tau)\right]d\tau =: C_1(t) \tag{5.11}$$

Applying the Δ operator to equation (3.1), and taking the inner product with ΔH we have

$$\frac{d}{dt}\frac{1}{2}\|\Delta H\|^2 + \nu\|\Delta^2 H\|^2 = -\left(\Delta^2 H, HH_x\right) - \sum_{k\in\mathbb{Z}_0^2}\Re\mathcal{N}[k]\left(k_1^2 + k_2^2\beta^2\right)^2|\hat{H}(k)|^2. \tag{5.12}$$

On integration by parts and use of Lemma A.2 we have

$$\left|\left(\Delta^2 H, HH_x\right)\right| \leq 4c_0\|\Delta H\|^2\|\Delta H_x\| \leq 4c_0\|\Delta H\|^2\|\Delta^2 H\| \leq \frac{8c_0^2}{\nu}\|\Delta H\|^4 + \frac{\nu}{2}\|\Delta^2 H\|^2,$$

hence (5.12) provides the expression

$$\frac{d}{dt}\frac{1}{2}\|\Delta H\|^2 + \frac{\nu}{2}\|\Delta^2 H\|^2 \leq \left(\frac{8c_0^2}{\nu}\|\Delta H\|^2 + M\right)\|\Delta H\|^2. \tag{5.13}$$

This implies that

$$\|\Delta H(\cdot,t)\|^2 \leq \|\Delta H^0\|^2 \exp\left[2Mt + \frac{16c_0^2}{\nu^2}C_0(t)\right] =: C_2(t). \tag{5.14}$$

Next we take the Laplacian of the evolution equation (3.2) for Γ, and its inner product with $\Delta\Gamma$, to obtain

$$\frac{d}{dt}\frac{1}{2}\|\Delta\Gamma(\cdot,t)\|^2 + \gamma\|\Delta\nabla\Gamma\|^2 = (\Delta\Gamma_x, \Delta(\Gamma H)) + b\,(\Delta H, \Delta\partial_x\Gamma), \tag{5.15}$$

which on use of Lemma A.2 and the bound $\|\Delta\Gamma_x\| \leq \|\Delta\nabla\Gamma\|$ yields

$$\frac{d}{dt}\frac{1}{2}\|\Delta\Gamma(\cdot,t)\|^2 \leq \frac{8c_0^2}{\gamma}\|\Delta H\|^2\|\Delta\Gamma\|^2 + \frac{b^2}{2\gamma}\|\Delta H\|^2. \tag{5.16}$$

Therefore, it follows from (5.14) and Gronwall's inequality that

$$\frac{1}{2}\|\Delta\Gamma(\cdot,t)\|^2 \leq C_3(t), \tag{5.17}$$

where $C_3(t)$ is finite for finite t. From the above computation it follows that the Fourier coefficients $\widehat{\Gamma}$, $\widehat{H}$ have finite $l^1(\mathbb{Z}_0^2)$ norms globally in time, ruling out finite time blow up.

6. Conclusion

The effect of surfactants in interfacial flows is a problem of wide technological interest due to the generation of Marangoni stresses that can produce beneficial or detrimental results in different processes. For example, surfactants retard pinching in liquid jets resulting in larger drops; they stabilise foams used in manufacturing applications as well as fire prevention; they can induce interfacial drag in bubble motion used in distillation processes hence reducing efficiency; they can completely remove any drag reduction benefits found in superhydrophobic surfaces used in a variety of applications including thermal management in microelectronics and energy harvesting technologies. Hence, the fundamental interplay between fluid mechanics, moving interfaces and surfactant distributions on them, are at the centre of all such processes and their quantitative understanding, both rigorous and computational, is of great value. Rigorous results provide certainty for the local and global existence of solutions and hence underpin numerical analysis and computations. Global well-posedness and absence of finite time singularities shows that the physical assumptions leading to the model do not break down. The present work undertakes a mathematical study of the nonlinear dynamics of a fundamental surfactant-laden shear flow in a channel at arbitrary Reynolds numbers and in a three-dimensional setting. It is worth noting that in the simpler situation of a clean two-dimensional flow (no surfactants and a one-dimensional interface), it has been shown in Kalogirou et al. [19] that the predictions of models analogous to the ones studied here, are in very good agreement with both direct numerical simulations as well as the experiments of Barthelet et al. [28].

We studied mathematically the nonlocal thin film evolution PDEs (3.1), (3.2), and have reported two main results. First, the local existence and smoothing of solutions of the equations when there is a full coupling between the interfacial motion and the surfactant dynamics - this case has $\alpha \neq 0$ in (3.1). Second, we proved global existence of solutions for the case $\alpha = 0$ using energy estimates. Such proofs are novel, as far as we know, for classes of multidimensional nonlocal thin film PDEs and indeed for their one-dimensional analogues. As mentioned earlier, our energy estimates do not apply for $\alpha \neq 0$ and new methods are needed - this is left for future work.

Numerical solutions of the system (3.1), (3.2) by Kalogirou & Papageorgiou [18] indicate that the solutions remain smooth (at least for the parameters studied there) and further analysis and computations are warranted. Specific directions include the construction of three-dimensional travelling waves (two-dimensional interfaces) and their mathematical justification via quasi-solution techniques, for example, as well as their stability and bifurcation structure.

Competing Interests. We declare that we have no competing interests.

Funding. Partly funded by EPSRC grant no. EP/L020564/1 to D.T.P, and NSF-DMS-1515755 grant to S.T. S.T. also thanks the Isaac Newton Institute for hospitality and the Simons Foundation for support.

A. Appendix.

(a) Results for $\mathcal{N}[k]$ and $F''(0, k\sqrt{\nu})$ in one dimension

Proposition A.1. *(Papageorgiou & Tanveer) [27] Assume k is large enough to satisfy the restriction in (A 1) below*

$$\kappa = k\sqrt{\nu} \geq \max\left\{\sqrt{3}R, \kappa_r\right\}. \tag{A 1}$$

Then,

$$\mathcal{N}[k] = \frac{i\Lambda k^2}{\left(1 - \frac{iR}{4k^2\nu}\right)} \left(\frac{1+E_1}{(1+E_2)(1+E_A)}\right), \tag{A 2}$$

where E_1, E_2 have exponentially small bounds for large $\kappa = k\sqrt{\nu}$, and $E_A = O\left(\frac{R^2}{\kappa^4}\right)$ is algebraically small in κ. Specifically, if we make the choice $\kappa_r = 10$, then in the regime (A 1)

$$\left|E_1\right| \leq 1.05 \times 10^{-7}, \quad \left|E_2\right| \leq 1.14 \times 10^{-6}, \quad \left|E_A\right| \leq 0.0171. \tag{A 3}$$

On the other hand with choice $\kappa_r = 6$, in the regime (A 1),

$$\left|E_1\right| \leq 2.24 \times 10^{-4}, \quad \left|E_2\right| \leq 1.61 \times 10^{-3}, \quad \left|E_A\right| \leq 0.051. \tag{A 4}$$

Remark A.1. *Since $\mathcal{N}[k] = -\frac{i\Lambda k}{2\sqrt{\nu}} F''(0, k\sqrt{\nu})$, it follows that*

$$F''(0, k\sqrt{\nu}) = -\frac{2\sqrt{\nu}k}{\left(1 - \frac{iR}{4k^2\nu}\right)} \left(\frac{1+E_1}{(1+E_2)(1+E_A)}\right). \tag{A 5}$$

We also note that the results imply in particular the bounds

$$-\Re\mathcal{N}[k] \leq C_N, \qquad \left|\Im\mathcal{N}[k]\right| \leq C_I |k|^2, \tag{A 6}$$

for constants C_N ad C_I independent of k.

(b) Some inequalities

The results that follow are routine and are included here only to make the paper self-contained.

Lemma A.1. *We have*

$$\|H\|_\infty \leq \|\hat{H}\|_{l_1(\mathbb{Z}_0^2} \leq c_0 \|\Delta H\|, \tag{A 7}$$

where

$$c_0 = \left(\sum_{(k_1,k_2)\in\mathbb{Z}_0^2} \frac{1}{(k_1^2 + \beta^2 k_2^2)^2}\right)^{1/2}. \tag{A 8}$$

Proof. We have

$$\begin{aligned} \|H\|_\infty \leq \|\hat{H}\|_{l^1(\mathbb{Z}_0^2)} &= \sum_{(k_1,k_2)\in\mathbb{Z}_0^2} \left|\hat{H}_{k_1,k_2}\right| \\ &\leq \left(\sum_{(k_1,k_2)\in\mathbb{Z}_0^2} (k_1^2+\beta^2k_2^2)^2 \left|\hat{H}_{k_1,k_2}\right|^2\right)^{1/2} \left(\sum_{(k_1,k_2)\in\mathbb{Z}_0^2} \frac{1}{(k_1^2+\beta^2k_2^2)^2}\right)^{1/2}. \end{aligned} \tag{A 9}$$

□

Lemma A.2. *If H, G each have representation*

$$H(x,z) = \sum_{k\in\mathbb{Z}_0^2} \widehat{H}_{k_1,k_2} e^{ik_1x+i\beta k_2 z}, \qquad G(x,z) = \sum_{k\in\mathbb{Z}_0^2} \widehat{G}_{k_1,k_2} e^{ik_1x+i\beta k_2 z}, \tag{A 10}$$

then,

$$\|\Delta(HG)\| \leq 4c_0\|\Delta H\|\|\Delta G\|. \tag{A 11}$$

Proof. Using $k_1^2 \leq 2(k_1-j_1)^2 + 2j_1^2$ and $k_2^2 \leq 2(k_2-j_2)^2 + 2j_2^2$, we have

$$\|\Delta(HG)\| = \|(k_1^2+\beta^2k_2^2)\sum_{j\in\mathbb{Z}_0^2} \widehat{G}_j\widehat{H}_{k-j}\|_{l^2(\mathbb{Z}_0^2)} \leq 2\|\sum_{j\in\mathbb{Z}_0^2}\left((k_1-j_1)^2+\beta^2(k_2-j_2)^2\right)\widehat{G}_j\widehat{H}_{k-j}\|$$
$$+2\|\sum_{j\in\mathbb{Z}_0^2}\left(j_1^2+\beta^2j_2^2\right)\widehat{G}_j\widehat{H}_{k-j}\| \leq 2\sum_{j\in\mathbb{Z}_0^2}|\widehat{G}_j|\|\Delta H\| + 2\sum_{j\in\mathbb{Z}_0^2}|\widehat{H}_j|\|\Delta G\|. \tag{A 12}$$

But the computation in (A 9) shows that $\|\hat{G}\|_{l^1(\mathbb{Z}_0^2)} \leq c_0\|\Delta G\|$ and $\|\hat{H}\|_{l^1(\mathbb{Z}_0^2)} \leq c_0\|\Delta H\|$, hence the result follows. □

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