Adaptive Control of Linear Systems with Time-Varying Parameters*

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***Abstract*— A new method for the adaptive control of linear systems with time-varying parameters is proposed. The method does not require any restriction on the rates of parameter variations. For linear systems in parametric strict-feedback form a state feedback adaptive backstepping controller with nonlinear damping terms is proposed and stability properties are proved. For systems in observable canonical form an ISS Kreisselmeier filter and an adaptive observer backstepping controller with an additional linear damping term are proposed: these guarantee asymptotic output regulation and bounded states. Simulation results show that the proposed controllers have superior performance over the standard controllers in the presence of varying parameters.**

I. Introduction

In the past 30 years extensive research has been performed on adaptive control (see [1], [2], [3], and [4]). Nevertheless, only a few works focus on systems with time-varying parameters. In early works on adaptive control for time-varying systems, e.g. [5], parameter identification is a prerequisite for controller design: convergence of the parameter estimation is required to guarantee stability. These methods need assumptions on *persistency of excitation*, or its variants, which do not typically hold.

Persistency of excitation is shown to be no longer necessary for stability in [6], [7], where it is assumed that the parameter variations are bounded and their derivatives are bounded in the average (integral) sense (occasional jumps are allowed). The parameter estimates are updated by a gradient or a least square law, along with *projection operations* [8] to limit the parameter estimates inside the compact set where the parameters belong, thus guaranteeing boundedness of the estimation error.

More recent works mainly belong to two trends. One of them is based on the *robust adaptive law* (RAL) [3], which applies the so-called *σ-modification*: a continuously switching parameter update law to adjust the adaptation rate according to the norm of the parameter estimates. In [9], the tracking error is guaranteed to converge into a residual set of size related to the rates of the parameter variations. Asymptotic tracking is achieved when the parameters are constant. In [10] and [11] adaptive backstepping controllers are introduced to cope with unknown parameters which do not satisfy the *matching condition* [12]. The parameter variations are modeled into known structured parameter variations and unknown unstructured variations. The tracking error is only related to the rate of the unstructured parameter variations and the structured parameter variations can be arbitrarily fast.

The other trend is based on *filtered transformations* [13]. [14] and [15] deal with the regulator and tracking problems, respectively, by applying filtered transformations on the state-space observer form. These methods do not require *a priori* knowledge on parameter variations, and the tracking error does not depend on the rates of parameter variations. Asymptotic tracking is achieved as long as the parameters are bounded in a compact set, their derivatives are $\mathscr{L}_1$ and an additive disturbance on the state evolution is $\mathscr{L}_2$.

Besides methods that put restrictions on the rates of parameter variations, the results in [16] only require parameters to vary inside a compact set. No restriction is imposed to the derivatives of the parameters. This relaxation is achieved by constructing a Lyapunov function without the varying parameters, and the estimation error is defined with respect to the origin instead of the true parameters. This method is based on systems satisfying the *matching condition* and relies on state feedback. Global boundedness of all states is guaranteed.

A new method called *congelation of variables* is proposed in this paper: it is conceptually similar to the method in [16], but the design and implementation are completely different. By using an adaptive backstepping controller with strengthened damping design, stability and output regulation are achieved and common restrictions on the derivatives of the unknown parameters are removed. To achieve this we only need the following natural assumption.

Assumption 1: The vector of unknown time-varying parameters $\theta(t)$ satisfies $\forall t \geq 0$ the box constraint

$$\underline{\theta} \leq \theta(t) \leq \bar{\theta}, \quad \underline{\theta}, \bar{\theta} \in \mathbb{R}^p, \tag{1}$$

where the sign "$\leq$" is defined element-wise. Only the "radius" of the compact set $\delta = \frac{1}{2}|\bar{\theta} - \underline{\theta}|$ is assumed to be known. $\diamond$

II. A Simple Example

In preparation for the general case we first consider the scalar linear system

$$\dot{x} = -\theta x + u, \tag{2}$$

where $x(t) \in \mathbb{R}$ is the state, $u(t) \in \mathbb{R}$ is the input and $\theta(t) \in \mathbb{R}$ is the unknown parameter.

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In a standard adaptive control scheme θ is assumed to be constant and a quadratic Lyapunov function candidate that incorporates the estimation error is considered: for example the function $V = \frac{1}{2}x^2 + \frac{1}{2\gamma}(\theta - \hat{\theta})^2$, with $\gamma > 0$. Eliminating the θ-related terms in $\dot{V}$ by properly designing $\dot{\hat{\theta}}$ yields

$$\begin{aligned} u &= \hat{\theta}x - kx, \\ \dot{\hat{\theta}} &= -\gamma x^2, \end{aligned} \tag{3}$$

hence $\dot{V} = -kx^2 \leq 0$, $\forall k > 0$. By LaSalle-Yoshizawa theorem, both x and $\hat{\theta}$ are globally uniformly bounded and all trajectories are such that $\lim_{t\to\infty} x(t) = 0$. As well known, there is no guarantee that $\hat{\theta}$ converges to the true parameter θ.

If θ is time-varying and satisfies Assumption 1, $\dot{V}$ becomes

$$\dot{V} = -kx^2 - (\theta - \hat{\theta})(\gamma^{-1}\dot{\hat{\theta}} - \gamma^{-1}\dot{\theta} + x^2). \tag{4}$$

Since $\dot{\theta}$ is unknown, we cannot design an update law to eliminate the θ-related terms. Most methods in the literature solve this problem by 1) adding restrictions on $|\dot{\theta}|$ or the integral of $|\dot{\theta}|$ (assumption of slow parameter variations), and 2) developing an update law that confines $\hat{\theta}$ inside the bound of parameter variations (projection operation) [7] [9] [14]. These methods are driven by two observations: 1) if the parameters vary sufficiently slowly, the problem of adaptive control with varying parameters is reduced to an adaptive control problem with constant parameters, which can be easily solved; 2) the parameter estimates should be close to the true parameters to guarantee stability. However, neither of these observations allows concluding stability. Thus a more direct approach to achieve stabilization under parameter variations should be pursued: this relies on eliminating the effects of varying parameters from the Lyapunov function.

In this paper a new approach is proposed. Since $\dot{\theta}$ derives from the parameter estimation error term $\frac{1}{2}(\theta - \hat{\theta})^2$ in V, if we remove θ from the parameter estimation error term, we also remove $\dot{\theta}$ from $\dot{V}$. This can be done by replacing θ with a constant ℓ to be determined, a process which we call the *congelation of variables*. After congealing θ, the Lyapunov function candidate becomes

$$V = \frac{1}{2}x^2 + \frac{1}{2\gamma}(\ell - \hat{\theta})^2, \tag{5}$$

the derivative of which along the trajectories of the system is

$$\dot{V} = -kx^2 - (\theta - \ell)x^2 - (\ell - \hat{\theta})(\gamma^{-1}\dot{\hat{\theta}} + x^2). \tag{6}$$

The first term in (6) is the stabilizing term that also appears in the standard adaptive scheme. The second term is a new parametric perturbation term caused by the *congelation of variables*. The third term can be cancelled by the same update law used in the standard scheme (3). Applying Young's inequality yields

$$\dot{V} = -kx^2 - \Delta x^2 \leq -(k - \frac{1}{2\varepsilon})x^2 + \frac{1}{2}\varepsilon^2|\Delta|^2 x^2, \tag{7}$$

where $\Delta = \theta - \ell$ and $\varepsilon > 0$. As (5) defines a family of Lyapunov function candidates, indexed by $\ell \in \mathbb{R}$, we can select the most favorable ℓ to find the minimum $|\Delta|$. Setting $\ell = \frac{1}{2}(\bar{\theta} + \underline{\theta})$, i.e. the "center" of $\theta(t)$, yields $|\Delta(t)|^2 \leq \delta^2$, $\forall t \geq 0$. As a result, $\dot{V} \leq -\bar{k}x^2$ provided k is such that

$$k = \bar{k} + \frac{1}{2\varepsilon} + \frac{1}{2}\varepsilon^2\delta^2. \tag{8}$$

We conclude that the effect of parameter variations can be regarded as a parametric perturbation and counteracted by a strengthened damping design. The above design can be easily extended to general linear systems satisfying the *matching condition*: it is sufficient to replace (8) with an algebraic Riccati equation. The extension to general systems is however non-trivial and it is studied in the next sections.

III. State Feedback

Systems which do not satisfy the *matching condition* can be adaptively stabilized using *adaptive backstepping* [2], provided they are in *parametric strict-feedback* form. For linear systems this form is given by the equation

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \bar{\phi}_{1,1}^\top\theta & 1 & 0 & \cdots & 0 \\ \bar{\phi}_{2,1}^\top\theta & \bar{\phi}_{2,2}^\top\theta & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ \bar{\phi}_{n,1}^\top\theta & \cdots & & \cdots & \bar{\phi}_{n,n}^\top\theta \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ &= Sx + \bar{\Phi}^\top(x \otimes I_p)\theta + e_n u, \end{aligned} \tag{9}$$

where $x(t) \in \mathbb{R}^n$, $\theta(t) \in \mathbb{R}^p$, $S \in \mathbb{R}^{n\times n}$ is the upper shift matrix, and

$$\bar{\Phi}^\top(x) = \begin{bmatrix} \bar{\phi}_{1,1}^\top & 0 & 0 & \cdots & 0 \\ \bar{\phi}_{2,1}^\top & \bar{\phi}_{2,2}^\top & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \bar{\phi}_{n,1}^\top & \cdots & & \cdots & \bar{\phi}_{n,n}^\top \end{bmatrix}, \tag{10}$$

where $\phi_{i,j} \in \mathbb{R}^p$, $j = 1,\ldots,n$, $i \leq j$, and $\Phi^\top(x) = \bar{\Phi}^\top(x \otimes I_p) = [\phi_1(x_1), \phi_2(x_1,x_2), \ldots, \phi_n(x)]^\top$.

For this system we consider a regulation problem with reference $x_r = 0$. The assumption on parameter variations is as in Assumption 1 with $\theta(t) \in \mathbb{R}^p$. Using adaptive backstepping with tuning functions, for each step i, $i = 1,\ldots,n$, define the error variables

$$z_0 = 0, \tag{11}$$

$$z_i = x_i - \alpha_{i-1}, \tag{12}$$

the regressor vectors

$$w_i(x_1,\ldots,x_i,\hat{\theta}) = \phi_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}\phi_j, \tag{13}$$

the tuning functions [17]

$$\tau_i(x_1,\ldots,x_i,\hat{\theta}) = \tau_{i-1} + w_i z_i = \sum_{j=1}^{i} w_i z_i, \tag{14}$$

and the virtual control laws

$$\alpha_0 = 0, \tag{15}$$

$$\alpha_i(x_1,\dots,x_i,\hat\theta) = -z_{i-1} - c_i z_i + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} - w_i^\top \hat\theta + \frac{\partial \alpha_{i-1}}{\partial \hat\theta}\Gamma\tau_i + \mu_i + \zeta_i. \tag{16}$$

In the virtual control laws α_i the terms μ_i are the correction terms to maintain the skew symmetry of the dynamics of z and are given by

$$\mu_i(x_1,\dots,x_i,\hat\theta) = -\sum_{j=2}^{i-1} \sigma_{j,i} z_j, \quad \mu_1 = \mu_2 = 0, \tag{17}$$

where $\sigma_{j,i} = -\frac{\partial \alpha_{j-1}}{\partial \hat\theta}\Gamma w_i$. The nonlinear damping term ζ_i is a function of z, the goal of which is to "improve damping" to counteract the effect of parameter variations. This term is not present in the standard adaptive backstepping scheme, and thus if we set $\zeta_i = 0$, the virtual control laws reduce to the one in [2]. Note that each ζ_i can be written as

$$\zeta_i(z_1,\dots,z_i,\hat\theta) = \bar\zeta_{i,1} z_1 + \bar\zeta_{i,2} z_2 + \dots + \bar\zeta_{i,i} z_i, \tag{18}$$

where each $\bar\zeta_{i,j}$ is a function of $z_1,\dots,z_i$, $i \le j$ and $\hat\theta$.

Using the above definitions and setting

$$\begin{aligned} u &= \alpha_n, \\ \dot{\hat\theta} &= \Gamma W(z,\hat\theta) z, \end{aligned} \tag{19}$$

where $W(z,\hat\theta) = [w_1, w_2, \dots, w_n]^\top$ is the matrix of regressors, yields

$$\dot z = \big(A_z(z,\hat\theta) + A_\zeta(z,\hat\theta)\big) z + W(z,\hat\theta)^\top(\theta - \hat\theta), \tag{20}$$

where

$$A_z = \begin{bmatrix} -c_1 & * & * & \cdots & * \\ -1 & -c_2 & * & \cdots & * \\ 0 & -1-\sigma_{2,3} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & -\sigma_{2,n} & \cdots & -1-\sigma_{n-1,n} & -c_n \end{bmatrix}, \tag{21}$$

with each $*$ representing a skew-symmetric term, and the matrix

$$A_\zeta = \begin{bmatrix} \bar\zeta_{1,1} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ \bar\zeta_{n,1} & \cdots & \cdots & \bar\zeta_{n,n} \end{bmatrix} \tag{22}$$

contains the nonlinear damping terms.

Consider the Lyapunov function candidate with congealed θ given by $V = \frac{1}{2} z^\top z + \frac{1}{2}(\ell - \hat\theta)^\top \Gamma^{-1}(\ell - \hat\theta)$, with $\Gamma = \Gamma^\top \succ 0$. Taking its time derivative along the trajectories of the closed-loop system yields

$$\begin{aligned} \dot V &= z^\top (A_z + A_\zeta) z + z^\top W^\top \Delta + (\ell - \hat\theta)^\top (\Gamma^{-1}\dot{\hat\theta} - Wz) \\ &= z^\top (A_z + A_\zeta) z + z^\top W^\top \Delta. \end{aligned} \tag{23}$$

To establish stability properties we need to exploit the linearity of the original system, which implies that the regressor matrix $W(z,\hat\theta)$ has a special factorizable structure.

Lemma 1: Consider the linear parametric strict-feedback system (9). The matrix of regressors $W(z,\hat\theta)$ can be written as

$$W^\top(z,\hat\theta) = \bar W^\top(z,\hat\theta)(z \otimes I_p), \tag{24}$$

where

$$\bar W^\top(z,\hat\theta) = \begin{bmatrix} \bar w_{1,1}^\top & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ \bar w_{n,1}^\top & \cdots & \cdots & \bar w_{n,n}^\top \end{bmatrix}, \tag{25}$$

and each $w_{i,j} \in \mathbb{R}^p$ is a function of $z_1,\dots,z_i$, $i \le j$ and $\hat\theta$. ◇

Exploiting Lemma 1, we can derive the following result.

Proposition 1: Consider system (9) with the adaptive backstepping controller (19). Assume the coefficients $c_1,\dots,c_n$ in A_z are positive. Let the nonlinear damping terms be defined as $\zeta = [\zeta_1,\dots,\zeta_n]^\top = A_\zeta z$, where A_ζ satisfies the condition

$$A_\zeta + A_\zeta^\top = -\frac{1}{\varepsilon^2}\bar W^\top \bar W - \varepsilon^2 \delta^2 I, \tag{26}$$

with $\varepsilon > 0$. Then all trajectories of the closed-loop system are bounded and $\lim_{t\to\infty} |x(t)| = 0$. ◇

Note that we have to design the nonlinear damping terms recursively: this is feasible since A_ζ is lower triangular and therefore the design of ζ_i only requires the terms computed in the previous steps.

IV. Output Feedback

In this section we focus on the output feedback adaptive stabilization problem. A SISO linear system wtih relative degree ρ in observable canonical form can be written as

$$\begin{aligned} \dot x &= Sx + F^\top(y,u)\theta, \\ y &= e_1^\top x, \end{aligned} \tag{27}$$

where S is the $n \times n$ upper shift matrix, $\theta(t) = [b^\top(t), a^\top(t)]^\top$, $a(t) = [a_{n-1}(t),\dots,a_0(t)]^\top \in \mathbb{R}^n$, $b(t) = [b_m(t),\dots,b_0(t)]^\top \in \mathbb{R}^{m+1}$, and $F^\top(y,u) = \big[[0_{(m+1)\times(\rho-1)}, I_{m+1}]^\top u(t), -I_n y(t)\big]$.

Typically, a *Kreisselmeier filter* (*K-filter*) [18] is applied to re-parametrize the system and overcome the difficulty caused by unmeasured states. This filter is given by the equations

$$\dot\eta = A_f \eta + e_n y, \tag{28}$$

$$\dot\lambda = A_f \lambda + e_n u, \tag{29}$$

$$\xi = -A_f^n \eta, \tag{30}$$

$$\Omega^\top = [v_m,\dots,v_0,\Xi], \tag{31}$$

where

$$v_i = A_f^i \lambda, \quad i = 0,\dots,m, \tag{32}$$

$$\Xi = -[A_f^{n-1}\eta,\dots,A_f^0\eta], \tag{33}$$

and $A_f = S - ke_1^\top$, with $k \in \mathbb{R}^n$ the vector of injection gains. This set of filters can be rewritten as

$$\dot{\xi} = A_f \xi + ky, \tag{34}$$
$$\dot{\Omega} = A_f \Omega^\top + F^\top(y,u). \tag{35}$$

A non-implementable state estimation in the standard adaptive schemes is given by

$$\hat{x} = \xi + \Omega^\top \theta. \tag{36}$$

If θ is constant the dynamics of the estimation error $\tilde{x} = x - \hat{x}$ is given by

$$\dot{\tilde{x}} = A_f \tilde{x}. \tag{37}$$

Then a Hurwitz A_f guarantees that the estimation error decays exponentially, which means that the new parametrization based on the stable *K-filter* is equivalent to the original system. If θ is varying the error dynamics becomes

$$\dot{\tilde{x}} = A_f \tilde{x} - \Omega^\top \dot{\theta}. \tag{38}$$

Whenever $\Omega^\top \dot{\theta} \neq 0$, the estimated state does not converge to the actual state, therefore the parametrization based on such a *K-filter* is not equivalent to the original system.

Existing methods in the literature include: 1) decomposing the varying parameters into known components and unknown components, and then use the known information to build additional filters [10], [11]; 2) assuming $\dot{\theta}$ belongs to a compact set and build a filter to estimate the upper bound of the estimation error [19]. These methods cannot guarantee asymptotic tracking or regulation in the presence of unknown varying parameters.

Since a *K-filter* which guarantees a stable and autonomous estimation error system is difficult to achieve, we seek an input-to-state stable (ISS) *K-filter* and determine stability properties of the whole system by jointly designing the filter and the controller. Applying the *congelation of variables* on (36) yields

$$\hat{x} = \xi + \Omega^\top \ell, \tag{39}$$

and

$$\dot{\tilde{x}} = A_f \tilde{x} + F^\top \Delta, \tag{40}$$

where $\tilde{x} = x - \hat{x}$ and $\Delta(t) = \theta(t) - \ell$.

Theorem 1 (ISS K-filter): The error dynamics of the modified K-filter (40) is ISS with inputs u and y, if the injection gain $k = \frac{1}{2} X e_1$, and $X = X^\top \succ 0$ satisfies the Riccati inequality

$$SX + XS^\top - X(e_1 e_1^\top - r^{-2} I)X + Q \preceq 0, \tag{41}$$

where $Q = \mathrm{diag}\big(\frac{1}{\varepsilon_a^2} I_{\rho-1}, (\frac{1}{\varepsilon_b^2} + \frac{1}{\varepsilon_a^2}) I_{m+1}\big)$, and $r > 0$, $\varepsilon_b > 0$, $\varepsilon_a > 0$. The ISS Lyapunov function is $V_{\tilde{x}} = r^2 \tilde{x}^\top P \tilde{x}$, $P = X^{-1}$, and

$$\dot{V}_{\tilde{x}} \leq -\tilde{x}^\top \tilde{x} + r^2 \varepsilon_b^2 \delta_b^2 u^2 + r^2 \varepsilon_a^2 \delta_a^2 y^2, \tag{42}$$

where δ_a and δ_b are the "radii" of $b(t)$ and $a(t)$, respectively. $\diamond$

Equation (41) can be written as a linear matrix inequality (LMI) and solved by standard LMI techniques. Alternatively one could solve the equation

$$SX + XS^\top - X(e_1 e_1^\top - r^{-2} I)X + Q = 0, \tag{43}$$

which is equivalent to the algebraic Riccati equation arising in the $\mathscr{H}_\infty$ filtering problem [20]. To see this note that the algebraic Riccati equation of the filtering problem is

$$AY + YA^\top - Y(C_2^\top C_2 - \gamma^{-2} C_1^\top C_1^\top)Y + B_1 B_1^\top = 0, \tag{44}$$

which is equivalent to (43) setting $A = S$, $Y = X$, $C_1 = I$, $C_2 = e_1^\top$, $B_1 = \mathrm{diag}\big(\frac{1}{\varepsilon_a} I_{\rho-1}, \sqrt{\frac{1}{\varepsilon_b^2} + \frac{1}{\varepsilon_a^2}} I_{m+1}\big) = Q^{\frac{1}{2}}$, and $\gamma = r$. It is easy to check that (A, C_2) is observable and (A, B_1, C_1) is minimal hence, for a sufficiently large r, (43) has a solution $X = X^\top \succ 0$.

Consider now the problem of finding an adaptive controller for (27) such that $\lim_{t\to\infty} y = 0$ under the following assumptions.

Assumption 2: $a(t)$ is unknown, time-varying, and bounded, $\underline{a} \leq a(t) \leq \bar{a}$. Only $\delta = \delta_a = \frac{1}{2}|\bar{a} - \underline{a}|$ is known. b is unknown, constant, and $b_m s^m + \cdots + b_0$ is a Hurwitz polynomial. $\diamond$

Assumption 3: The sign of the high-frequency gain b_m is known. $\diamond$

Due to Assumption 2, applying the *congelation of variables* on b is not necessary, therefore $\ell = [b^\top, \ell_a^\top]^\top$, $\Delta_b = 0$ and we can simplify Theorem 1 by canceling ε_b-related terms and reducing the LMI problem to an algebraic Riccati equation problem.

Applying the standard adaptive observer backstepping procedures in [2] with slight modifications yields the error variables

$$z_1 = y, \tag{45}$$
$$z_i = v_{m,i} - \alpha_{i-1}, \quad i = 2, \ldots, \rho, \tag{46}$$

the tuning functions

$$\tau_1 = (\omega - \hat{g}\bar{\alpha}_1 e_1) z_1, \tag{47}$$
$$\tau_i = \tau_{i-1} - \frac{\partial \alpha_{i-1}}{\partial y} \omega z_i, \quad i = 2, \ldots, \rho, \tag{48}$$

the virtual control laws

$$\alpha_1 = \hat{g}\big(-(c_1 + d_1) z_1 + \zeta_1 - \xi_2 - \bar{\omega}^\top \hat{\theta}\big) = \hat{g}\bar{\alpha}_1, \tag{49}$$
$$\alpha_2 = -\hat{b}_m z_1 - \big(c_2 + d_2 (\frac{\partial \alpha_1}{\partial y})^2\big) z_2 + \beta_2 + \frac{\partial \alpha_1}{\partial y} \Gamma \tau_2, \tag{50}$$
$$\alpha_i = -z_{i-1} - \big(c_i + d_i (\frac{\partial \alpha_{i-1}}{\partial y})^2\big) z_i + \beta_i + \frac{\partial \alpha_{i-1}}{\partial y} \Gamma \tau_i + \mu_i, \quad i = 3, \ldots, \rho, \tag{51}$$
$$\beta_i = \frac{\partial \alpha_{i-1}}{\partial y}(\xi_2 + \omega^\top \hat{\theta}) + \frac{\partial \alpha_{i-1}}{\partial \eta}(A_f \eta + e_n y) + k_i v_{m,1} + \sum_{j=1}^{m+i-1} \frac{\partial \alpha_{i-1}}{\partial \lambda_j}(-k_j \lambda_1 + \lambda_{j+1}) + \frac{\partial \alpha_{i-1}}{\partial \hat{g}} \dot{\hat{g}}, \quad i = 2, \ldots, \rho, \tag{52}$$

the control law

$$u = \alpha_\rho - v_{m,\rho+1}, \tag{53}$$

and the update laws

$$\dot{\hat{g}} = -\gamma \mathrm{sgn}(b_m)\bar{\alpha}_1 z_1, \tag{54}$$

$$\dot{\hat{\theta}} = \Gamma \tau_\rho. \tag{55}$$

In (45)-(55), $\hat{g}$ is an estimate of $\frac{1}{b_m}$ to avoid the singularity of $\frac{1}{\hat{b}_m}$; $\omega = [v_{m,2}, \ldots, v_{0,2}, \Xi_{(2)} - ye_1^\top]^\top = \bar{\omega} + v_{m,2}e_1$, where $\Xi_{(2)}$ is the second row of Ξ; $\zeta_1 = \bar{\zeta}_{1,1}z_1$ ($\bar{\zeta}_{1,1} \in \mathbb{R}$) is an additional linear damping term to dominate the perturbation term in the ISS K-filter; and μ is the correction term for the recursive tuning function design, which is defined as

$$\mu_i = -\sum_{j=2}^{i-1} \sigma_{j,i} z_j, \quad \mu_1 = \mu_2 = 0, \tag{56}$$

where $\sigma_{j,i} = \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega$.

Using the above definitions, the dynamics of the error system becomes

$$\dot{z} = \big(A_z(z,\hat{\theta}) + A_\zeta\big)z + W_{\tilde{x}}(z,\hat{\theta})\tilde{x}_2 + W_\theta(z,\hat{\theta})^\top(\ell - \hat{\theta}) - b_m\bar{\alpha}_1 e_1\Big(\frac{1}{b_m} - \hat{g}\Big), \tag{57}$$

where

$$A_z = \begin{bmatrix} -\kappa_1 & * & * & \cdots & * \\ -\hat{b}_m & -\kappa_2 & * & \cdots & * \\ 0 & -1-\sigma_{2,3} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & -\sigma_{2,\rho} & \cdots & -1-\sigma_{\rho-1,\rho} & -\kappa_\rho \end{bmatrix}, \tag{58}$$

$\kappa_i = -c_i - d_i(\frac{\partial \alpha_{i-1}}{\partial y})^2$, $A_\zeta = \mathrm{diag}(\bar{\zeta}_{1,1}, 0_{\rho-1})$, $W_{\tilde{x}} = [1, -\frac{\partial \alpha_1}{\partial y}, \ldots, -\frac{\partial \alpha_{\rho 1}}{\partial y}]^\top$, and $W_\theta^\top = W_{\tilde{x}}\omega^\top - \hat{g}\bar{\alpha}_1 e_1 e_1^\top$.

Proposition 2: Consider system (27) with the adaptive observer backstepping controller (53)-(55). Assume the coefficients $c_1, \ldots, c_\rho$ and $d_1, \ldots, d_\rho$ in A_z are all positive. Let the additional damping be defined as $\zeta = A_\zeta z = \bar{\zeta}_{1,1}z_1$, where

$$\bar{\zeta}_{1,1} = -\frac{1}{2} r^2 \varepsilon^2 \delta^2 \sum_{i=1}^{\rho} \frac{1}{4d_i}, \tag{59}$$

and $\varepsilon = \varepsilon_a > 0$. Then all trajectories of the closed-loop system are globally uniformly bounded and $\lim_{t\to\infty} y(t) = 0$. $\diamond$

V. Simulations

Consider the two-dimensional linear system in observable canonical form, which is also a special case of the *parametric strict-feedback* form, given by the equation

$$\dot{x} = \begin{bmatrix} -\theta_1 & 1 \\ -\theta_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Sx + \Phi^\top(x)\theta + e_2 u, \tag{60}$$

where $\Phi(x) = [\phi_1(x), \phi_2(x)]$, $\phi_1 = [-x_1, 0]^\top$, $\phi_2 = [0, -x_1]^\top$. To consider an open-loop unstable plant we set the means of $\theta_1(t)$ and $\theta_2(t)$ to -3 and +2, respectively.

To show that the proposed approach does not require any restriction on $\theta(t)$, the parameter variations in Fig. 1 intentionally include 1) fast varying components, to remove the restriction on $|\dot{\theta}|$, and 2) slowly but persistently varying components, to remove the restrictions on the integral of $|\dot{\theta}|$. Now, consider two state feedback controllers: Controller 1 is the proposed controller with nonlinear damping terms, Controller 2 is the standard adaptive backstepping controller with $\zeta_1 = \zeta_2 = 0$. Simulating both closed-loop systems with the setting $c_1 = c_2 = 0.5$, $\varepsilon = 0.3$, $\Gamma = I$, and $x(0) = [2, -1]^\top$ for both controllers yields the results in Fig. 2.

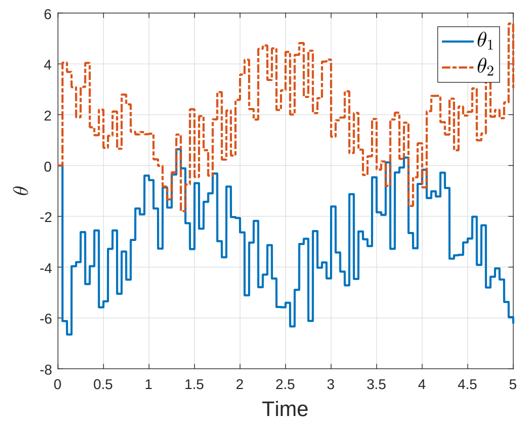
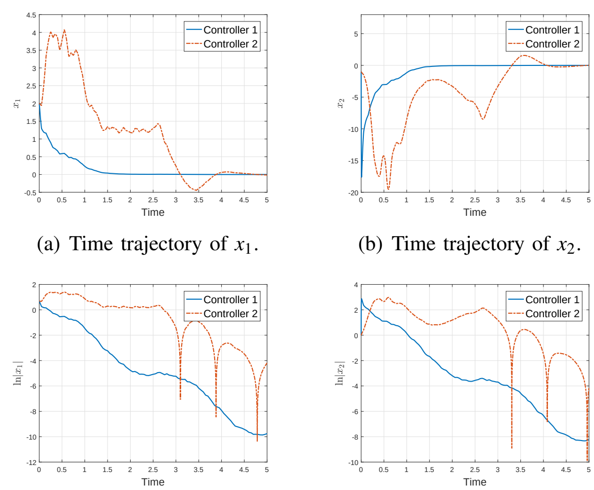


Fig. 1. Parameter variations.



(a) Time trajectory of x_1.

(b) Time trajectory of x_2.

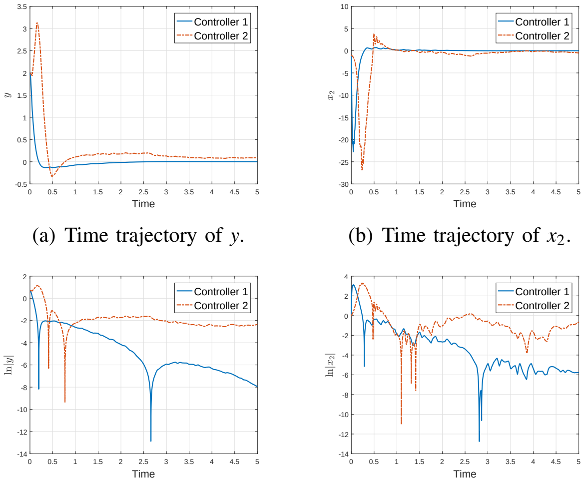
(c) Time trajectory of $\ln|x_1|$.

(d) Time trajectory of $\ln|x_2|$.

Fig. 2. Comparison between the proposed state feedback controller (Controller 1) and the standard state feedback controller (Controller 2).

We now consider again system (60) to be controlled via ouput feedback. To make a fair comparison both controllers are connected to two identical ISS *K-filters*, nevertheless in Controller 2, $\bar{\zeta}_{1,1} = 0$. Set $c_1 = c_2 = 0.8$, $d_1 = d_2 = 0.2$, $r = 5$, $\varepsilon = 0.1$, $\Gamma = I$, $\gamma = 1$, and $x(0) = [2, -1]^\top$ for both controllers. Using the same varying parameters as in the state feedback case, we obtain the results in Fig. 3.

In both the state feedback case and the output feedback case the proposed controllers show better performance in terms of smaller overshoot at the beginning, smoother response, and faster rate of convergence. These are due to the strengthened damping designs, which account for varying



(a) Time trajectory of y. (b) Time trajectory of x_2.

(c) Time trajectory of $\ln|y|$. (d) Time trajectory of $\ln|x_2|$.

Fig. 3. Comparison between the proposed output feedback controller (Controller 1) and the standard output feedback controller (Controller 2).

parameters.

VI. Conclusions and Future Work

In this paper a new method, called *congelation of variables*, for adaptive control of linear systems with time-varying parameters has been proposed. The state feedback adaptive backstepping controller with nonlinear damping terms designed by this method guarantees global boundedness and regulation of the state. In the output feedback case, an ISS *K-filter* is designed to replace the standard stable *K-filter*, and an output feedback adaptive backstepping controller with an additional linear damping term guarantees asymptotic output regulation and bounded states. All the results presented do not rely on any restriction on the derivatives of the unknown parameters.

In future work the minimum-phase assumption (Assumption 2) will be relaxed to allow time varying inverse dynamics, tracking problems will be considered and nonlinear systems will be studied.

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