

# Adaptive Control for Systems with Time-Varying Parameters

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**Abstract**—This paper investigates the adaptive control problem for systems with time-varying parameters using the so-called *congelation of variables* method. First, two scalar examples to illustrate how to deal with time-varying parameters in the feedback path and in the input path, respectively, are discussed. The control problem for an  $n$ -dimensional lower triangular system via state feedback is then discussed to show how to combine the *congelation of variables* method with *adaptive backstepping* techniques. To achieve output regulation problem via output feedback, problem which cannot be solved directly due to the coupling between the input and the time-varying perturbation, the ISS of the inverse dynamics, referred to as strong minimum-phasesness, is exploited. This allows converting such coupling into the coupling between the output and the time-varying perturbation. A set of filters, resulting in ISS state estimation error dynamics, are designed to cope with the unmeasured state variables. Finally, a controller is designed based on a small-gain-like analysis that takes all subsystems into account. Simulation results show that the proposed controller achieves asymptotic output regulation and outperforms the classical adaptive controller, in the presence of time-varying parameters that are neither known nor asymptotically constant.

**Index Terms**—Adaptive control, Nonlinear systems, Output feedback.

## I. INTRODUCTION

Since the 1980s, adaptive control has undergone extensive research (see *e.g.* [1]–[5]), yet the works on systems with time-varying parameters appear not to be as voluminous as works that only consider time-invariant systems. Some pioneering works on adaptive control for time-varying systems (see *e.g.* [6]) exploit *persistence of excitation* to guarantee stability by ensuring that parameter estimates converge to the true parameters.<sup>1</sup> Subsequent works (see *e.g.* [9], [10]) have removed the

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<sup>1</sup>In recent literature, methods requiring that the parameter estimates converge to the true parameters are categorized as identification methods, see *e.g.* [7], [8]. In the rest of this paper we only consider control methods which do not require convergence of the parameter estimates.

restriction of *persistence of excitation* by requiring bounded and slow (in an average sense) parameter variations.

More recent works can be mainly categorized into two trends. One of them is based on the so-called *robust adaptive law* or *switching  $\sigma$ -modification*, see [3], a mechanism which adds leakage to the parameter update law if the parameter estimates drift out of a pre-specified reasonable region to guarantee boundedness of the parameter estimates. This approach achieves asymptotic tracking when the parameters are constant, otherwise the tracking error is nonzero and related to the rates of the parameter variations, see [11]. In [12] and [13] the parameter variations are modelled in two parts: known parameter variations and unknown variations, and the residual tracking error only depends on the rates of the unknown parameter variations.

The other trend exploits the so-called *filtered transformation*, which is essentially an adaptive observer described via a change of coordinates, and the *projection operation*, which confines the parameter estimates within a pre-specified compact set to guarantee the boundedness of the parameter estimates, see [14], [15] and [16]. These methods can guarantee asymptotic tracking provided that the parameters are bounded in a compact set, their derivatives are  $\mathcal{L}_1$  and the disturbance on the state evolution is additive and  $\mathcal{L}_2$ . Moreover, *a priori* knowledge on parameter variations is not needed and the residual tracking error is independent of the rates of the parameter variations.

The methods mentioned above cannot guarantee zero-error regulation when the unknown parameters are persistently varying. To achieve asymptotic state/output regulation when the time-varying parameters are neither known nor asymptotically constant, in [17] and [18] a method called the *congelation of variables* has been proposed and developed on the basis of the *adaptive backstepping* approach and the *adaptive immersion and invariance (I&I)* approach, respectively. In the spirit of the *congelation of variables* method each unknown time-varying parameter is treated as a nominal unknown constant parameter perturbed by the difference between the true parameter and the nominal parameter, which causes a time-varying perturbation term. The controller design is then divided into a classical adaptive control design, with constant unknown parameters, and a damping design via dominance to counteract the time-varying perturbation terms. This method is compatible with most adaptive control schemes using parameter estimates, as it does not change the original parameter update law designed for time-invariant systems.

Since full-state feedback is not always implementable, most

practical scenarios require an output-feedback adaptive control scheme. In the output-feedback design with the *congelation of variables* method, the major difficulty is caused by the coupling between the input and the time-varying perturbation. In this case simply strengthening damping terms in the controller alters the input (as well as the perturbation itself) and therefore causes a chicken-and-egg dilemma, which prevents stabilization via dominance. In [19] and [20], a special output-feedback case is solved on the basis of *adaptive backstepping* and *adaptive I&I*, respectively, by exploiting a modified minimum-phase property for time-varying systems and decomposing the coupling between the input and the time-varying perturbation into couplings between some output-related nonlinearities and some other time-varying perturbations, which enables the use of the dominance design again, though it is still restricted by a relative degree condition. This restriction is relaxed in this paper.

The paper is organized as follows. In Section II, two motivating examples of scalar systems to illustrate the use of the *congelation of variables* method are presented, and an  $n$ -dimensional lower-triangular system with unmatched uncertainties controlled by an adaptive state-feedback controller is discussed to elaborate on the combination of the *congelation of variables* method with *adaptive backstepping*. With these design tools, in Section III the paper revisits, integrates, and further develops the results in [19] and [20] on the decomposition of the perturbation coupled with the input, and proposes a controller design based on the scheme in [21] together with a more comprehensive small-gain-like analysis, when compared with the one in [19] and [20], that incorporates the filter subsystems into the analysis. These allow the output-feedback scheme proposed in this paper to achieve asymptotic output regulation and to guarantee boundedness of all closed-loop signals and, at the same time, remove the restriction of having relative degree 1 or constant high-frequency gain, as assumed in [19] and [20]. In Section IV a numerical example to highlight the performance improvement achievable with the proposed scheme is presented.

*Notation:* This paper uses standard notation unless stated otherwise. For an  $n$ -dimensional vector  $v \in \mathbb{R}^n$ ,  $|v|$  denotes the Euclidean 2-norm,  $|v|_M = \sqrt{v^\top M v}$ ,  $M = M^\top \succ 0$ , denotes the weighted 2-norm with weight  $M$ ,  $v_i \in \mathbb{R}^i$ ,  $1 \leq i \leq n$ , denotes the vector composed of the first  $i$  elements of  $v$ .  $e_i$  denotes the  $i$ -th unit vector of proper dimension. For an  $n \times m$  matrix  $M$ ,  $(M)_i$  denotes the  $i$ -th column,  $(M^\top)_i$  denotes the  $i$ -th row,  $(M)_{ij}$  denotes the  $i$ -th element on the  $j$ -th column,  $\text{tr}(M)$  denotes the trace, and  $|M|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^m (M)_{ij}^2}$  denotes the Frobenius norm.  $I$  and  $S$  denote the identity matrix and the upper-shift matrix with proper dimension, respectively. For an  $n$ -dimensional time-varying signal  $s : \mathbb{R} \rightarrow \mathbb{R}^n$ , the image of which is contained in a compact set  $\mathcal{S}$ ,  $\Delta_s : \mathbb{R} \rightarrow \mathbb{R}^n$  denotes the deviation of  $s$  from a constant value  $\ell_s$ , i.e.  $\Delta_s(t) = s(t) - \ell_s$ , and  $\delta_s \in \mathbb{R}$  denotes the supremum of the 2-norm of  $s$ , i.e.  $\delta_s = \sup_{t \geq 0} |s(t)| \geq 0$ .  $(\cdot)^{(n)} = \frac{d^n}{dt^n}$  denotes the  $n$ -th time derivative operator.  $\diamond$

In this paper the unknown time-varying system parameters  $\theta : \mathbb{R} \rightarrow \mathbb{R}^q$  and  $b_m : \mathbb{R} \rightarrow \mathbb{R}$  may verify one of the

assumptions below.

*Assumption 1 (Bounded parameters):* The parameter  $\theta$  is piecewise continuous and  $\theta(t) \in \Theta_0$ , for all  $t \geq 0$ , where  $\Theta_0$  is a compact set. The “radius” of  $\Theta_0$ , i.e.  $\delta_{\Delta_\theta}$ , is assumed to be known, while  $\Theta_0$  can be unknown (see Fig. 1).  $\diamond$

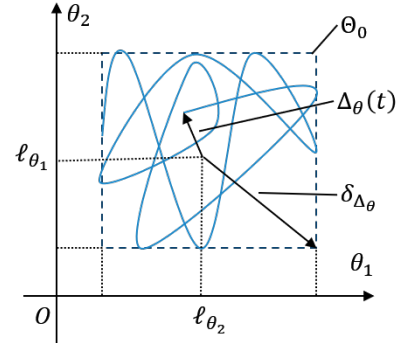


Fig. 1. Graphical illustration of the role of  $\Theta_0$ ,  $\ell_\theta$ ,  $\Delta_\theta(t)$ , and  $\delta_{\Delta_\theta}$ .

*Assumption 2 (Smooth bounded parameters):* The parameter  $\theta$  is smooth, that is,  $\theta^{(i)}(t) \in \Theta_i$ , for  $i \geq 0$ , for all  $t \geq 0$ , respectively, where  $\Theta_i$  are compact sets possibly unknown.  $\delta_{\Delta_\theta}$  is assumed to be known.  $\diamond$

*Assumption 3 (Sign-definite parameter):* The parameter  $b_m(t)$  is bounded away from 0 in the sense that there exists a constant  $\ell_{b_m}$  such that  $\text{sgn}(\ell_{b_m}) = \text{sgn}(b_m(t)) \neq 0$  and  $0 < |\ell_{b_m}| \leq |b_m(t)|$ , for all  $t \geq 0$ . The sign of  $\ell_{b_m}$  and  $b_m(t)$ , for all  $t \geq 0$ , is known and does not change.  $\diamond$

## II. MOTIVATING EXAMPLES AND PRELIMINARY RESULT

In this section two motivating examples are provided to briefly introduce the so-called *congelation of variables* method, which is the core idea of this paper for coping with time-varying parameters.

### A. Parameter in the Feedback Path

To begin with consider a scalar nonlinear system described by the equation

$$\dot{x} = \theta(t)x^2 + u, \quad (1)$$

where  $x(t) \in \mathbb{R}$  is the state,  $u(t) \in \mathbb{R}$  is the input, and  $\theta(t) \in \mathbb{R}$  is an unknown time-varying parameter satisfying Assumption 1. Assuming that we have an “estimate”  $\hat{\theta}$  of the parameter  $\theta(t)$ , we can rewrite (1) as

$$\dot{x} = \hat{\theta}x^2 + u + (\theta - \hat{\theta})x^2. \quad (2)$$

One way to design an update law for  $\hat{\theta}$  is to consider a Lyapunov function candidate of the form

$$V(x, \hat{\theta}, \theta) = \frac{1}{2}x^2 + \frac{1}{2\gamma_\theta}(\theta - \hat{\theta})^2. \quad (3)$$

Assuming  $\theta$  is differentiable with respect to time for the time being and taking the time derivative of  $V$  along the solutions of (2) yields

$$\dot{V} = \hat{\theta}x^3 + ux + (\theta - \hat{\theta})x^3 - (\theta - \hat{\theta})\frac{\dot{\theta}}{\gamma_\theta} + (\theta - \hat{\theta})\frac{\dot{\theta}}{\gamma_\theta}, \quad (4)$$

which means that the selection of the parameter update law

$$\dot{\hat{\theta}} = \gamma_\theta x^3 \quad (5)$$

cancel the effect of the unknown  $(\theta - \hat{\theta})x^3$  term. The constant  $\gamma_\theta > 0$  is known as the adaptation gain. In classical adaptive control problems one assumes that  $\theta$  is constant, that is  $\dot{\theta} = 0$  for all  $t \geq 0$ , and selects the control law

$$u = -kx - \hat{\theta}x^2, \quad (6)$$

with  $k > 0$ , which yields  $\dot{V} = -kx^2 \leq 0$ . We can conclude from this that  $x$  and  $\hat{\theta}$  are bounded, and  $x$  converges to 0 by invoking Barbalat's lemma. When  $\dot{\theta} \neq 0$ , one has to deal with the indefinite term  $(\theta - \hat{\theta})\frac{\dot{\theta}}{\gamma_\theta}$ . One way to do this is to modify (5) with the so-called *projection operation* (see e.g. [22], [23]), which confines the parameter  $\hat{\theta}$  inside a convex compact set and therefore guarantees the boundedness of  $(\theta - \hat{\theta})$ . It follows that the boundedness of  $\dot{\theta}$  guarantees the boundedness of  $x$  (either exact boundedness, e.g. in [24] or boundedness in an average sense, e.g. in [10]), and  $\dot{\theta} \in \mathcal{L}_1$  guarantees the convergence of  $x$  to 0 (e.g. in [15], [25], [16]). In some other works (e.g. in [11], [12], [13]), the boundedness of  $\hat{\theta}$  is guaranteed by the so-called *switching  $\sigma$ -modification*, which adds some leakage to the integrator (5) if the parameter estimate drifts outside a reasonable region, and it is often referred to as soft projection. All these schemes share the similarity that they treat  $\dot{\theta}$  as a disturbance. As a result some disturbance attenuation effort is made to guarantee that bounded  $\dot{\theta}$  causes bounded state/output regulation/tracking error, and sufficiently fast converging  $\theta$ , which means that  $\theta$  becomes constant sufficiently fast, guarantees the convergence of the error to 0. As a result, none of these methods can guarantee zero-error regulation/tracking when the unknown parameter is persistently time-varying, in which case  $\dot{\theta}$  is non-vanishing.

However, note that the reason why we cannot avoid  $\dot{\theta}$  in the analysis is the  $\theta - \hat{\theta}$  term in (3). This term is included only to guarantee the boundedness of  $\hat{\theta}$ , yet by no means guaranteeing the convergence of  $\hat{\theta}$  to  $\theta$ , no matter whether  $\theta$  is time-varying or constant, thus replacing  $\theta$  with a constant  $\ell_\theta$ , to be determined, can guarantee the same properties.  $\ell_\theta$  can be regarded as the average of  $\theta(t)$ , which is not necessarily known. In the light of this, consider the modified Lyapunov function candidate

$$V_\ell(x, \hat{\theta}, \ell_\theta) = \frac{1}{2}x^2 + \frac{1}{2\gamma_\theta}(\ell_\theta - \hat{\theta})^2. \quad (7)$$

Taking the time derivative of  $V_\ell$  along the trajectories of (2) yields

$$\dot{V}_\ell = \hat{\theta}x^3 + ux + (\ell_\theta - \hat{\theta})x^3 - (\ell_\theta - \hat{\theta})\frac{\dot{\theta}}{\gamma_\theta} + \Delta_\theta x^3, \quad (8)$$

where  $\Delta_\theta = \theta - \ell_\theta$ . Comparing (8) with (4) we see that the substitution of  $\ell_\theta$  for  $\theta$  eliminates the  $\dot{\theta}$  term, at the cost of adding a perturbation term  $\Delta_\theta x^3$  due to the inconsistency between  $\theta$  and  $\ell_\theta$ . Considering the same parameter update law as in (5) and a new control law

$$u = -\left(k + \frac{1}{2\epsilon_{\Delta_\theta}}\delta_{\Delta_\theta}\right)x - \frac{1}{2}\epsilon_{\Delta_\theta}\delta_{\Delta_\theta}x^3 - \hat{\theta}x^2, \quad (9)$$

where  $\epsilon_{\Delta_\theta} > 0$  is a constant, to balance the linear and the nonlinear terms, yields

$$\begin{aligned} \dot{V}_\ell &= -\left(k + \frac{1}{2\epsilon_{\Delta_\theta}}\delta_{\Delta_\theta}\right)x^2 - \frac{1}{2}\epsilon_{\Delta_\theta}\delta_{\Delta_\theta}x^4 + \Delta_\theta x^3 \\ &\leq -kx^2 \leq 0. \end{aligned} \quad (10)$$

Therefore we can conclude boundedness of all trajectories of the closed-loop system as well as convergence of  $x$  to 0 using the same argument as the one used in the classical constant parameter problem, without requiring a vanishing  $\dot{\theta}$ . The method of substituting the constant  $\ell_\theta$  for the time-varying  $\theta$  to avoid unnecessary time derivatives is called *congelation of variables* [17]<sup>2</sup>. Note that controllers designed via the *congelation of variables* method can be used for systems with fast-varying parameters, as the design does not rely on properties of  $\dot{\theta}$ .

*Remark 1:* The control law (9) and the parameter update law (5) do not depend on  $\ell_\theta$ , in the same way as classical adaptive controllers do not depend on  $\theta$ , thus showing the ‘‘adaptive’’ property of the proposed mechanism. One can interpret the proposed controller as a combination of an adaptive controller, to cope with the unknown parameter  $\ell_\theta$ , and a robust controller, to cope with the time-varying perturbation  $\Delta_\theta(t)$ . This fact can also be revealed by noting that, when  $\theta$  is a constant, one could select  $\ell_\theta = \theta$ , hence  $\delta_{\Delta_\theta} = 0$ , and the control law (9) is reduced to the classical control law (6).

It is also worth discussing the difference between the proposed adaptive control scheme and a pure robust control scheme in which  $\ell_\theta$  is treated as nominal parameter. To illustrate this consider a practical scenario in which we have a circuit that has to work with one of three resistors with values 50 $\Omega$ , 100 $\Omega$ , and 150 $\Omega$ , yet which one is used is unknown. In addition, due to temperature variations, the resistances have a fluctuation of  $\pm 10\Omega$ . In the spirit of the proposed method,  $\ell_\theta$  equals either 50 $\Omega$ , 100 $\Omega$ , or 150 $\Omega$ , which is unknown and not used in the controller design, as it is replaced by the dynamically updated  $\hat{\theta}$ , and  $\delta_{\Delta_\theta} = 10\Omega$ , which is known and used in the controller design. In the spirit of robust control, one has to determine the nominal resistance of the resistor before designing the controller, and according to the known information, the best guess is  $\ell_\theta = 100\Omega$ . In this case the maximum deviation from this nominal value is  $\delta_{\Delta_\theta} = 60\Omega$ , which is caused not only by the parameter variation but also

<sup>2</sup>Some works predating [17] exploit similar ideas to avoid involving  $\dot{\theta}$  in the analysis. For example, in [26] the unknown time-varying controller parameter in the Lyapunov function is replaced with a constant (0, as a matter of fact). In other works one first derives a constant parameter controller via dominance design (instead of directly using a time-varying parameter controller that cancels the time-varying parameter) and then estimates the constant parameter of the dominance controller, see e.g. [27], [28].

by the imperfect knowledge on the true resistance of the resistor used. This leads to a more conservative design that uses an unnecessarily high gain and may cause severe noise amplification issues. In an extreme case in which the nominal resistance of the used resistor is completely unknown, one cannot design a robust controller while one can still design an adaptive controller using the proposed method.  $\diamond$

*Remark 2:* The control law (9) depends on  $\delta_{\Delta_\theta}$ , which is assumed to be known by Assumption 1. Even if  $\delta_{\Delta_\theta}$  is unknown, one can easily overcome this by building an “estimate” for  $\delta_{\Delta_\theta}$  via classical adaptive control techniques, since  $\delta_{\Delta_\theta}$  is a constant and the control law is linearly parameterized, see Remark 2 of [21] for a brief example.  $\diamond$

*Remark 3:* It worth introducing a convention to clarify the spirit in which we treat unknown quantities. If an unknown indefinite term in the time derivative of the Lyapunov function vanishes as the system parameters become constant, then this term is to be dominated by a static damping design, like the  $\Delta_\theta$ -term in this case, and we do not aim at estimating  $\delta_{\Delta_\theta}$ , the bound of  $\Delta_\theta(t)$ ; if an unknown indefinite term is not vanishing even when all system parameters are constant, like the  $\ell_\theta$ -term in this case, then this term is to be compensated by a dynamically updated “estimate”, which is  $\hat{\theta}$  in this case. The reasons for this convention of design are, first, that we do not want to over-extend the dimension of the closed-loop system by adding too many dynamic estimates, and second, that we need the static damping terms to counteract fast parameter variations for better transient performance (for the same reason one can use nonlinear damping techniques even for system with constant parameters).  $\diamond$

*Remark 4:* Consider the classical adaptive control problem in which  $\theta$  is constant. The closed-loop dynamics can be described via a negative feedback loop consisting of two passive systems, namely

$$\Sigma_1 : \begin{cases} \dot{x}_1 &= -kx_1 + x_1^2 u_1, \\ y_1 &= x_1^3, \end{cases} \quad (11)$$

$$\Sigma_2 : \begin{cases} \dot{x}_2 &= \gamma_\theta u_2, \\ y_2 &= x_2, \end{cases} \quad (12)$$

where  $x_1 = x$ ,  $x_2 = \hat{\theta} - \theta$ ,  $u_1 = -y_2$ ,  $u_2 = y_1$ . The storage functions are  $S_1 = \frac{1}{2}x_1^2$  and  $S_2 = \frac{1}{2\gamma_\theta}x_2^2$ , respectively. It is well-known that the parameter update law (5) is neither designed to guarantee the convergence of  $\hat{\theta} - \theta$  to 0 nor to make  $\hat{\theta}$  estimate  $\theta$ , though  $\hat{\theta}$  is called the parameter estimate by convention, but to make  $\hat{\theta} - \theta$  an input/output signal to form a passive interconnection. When  $\theta$  is time-varying, the dynamics of  $\Sigma_2$  are described by

$$\Sigma_2 : \begin{cases} \dot{x}_2 &= \gamma_\theta u_2 - \dot{\theta}, \\ y_2 &= x_2, \end{cases} \quad (13)$$

which causes the loss of passivity from  $u_2$  to  $y_2$ . The *congelation of variables* method can therefore be interpreted as selecting a new signal  $\hat{\theta} - \ell_\theta$  that can yield a passive interconnection, while maintaining the passivity of  $\Sigma_1$  by strengthened damping. Within this framework, the two passive

systems are described by

$$\Sigma_1 : \begin{cases} \dot{x}_1 &= -a(x_1, t)x_1 + x_1^2 u_1, \\ y_1 &= x_1^3, \end{cases} \quad (14)$$

$$\Sigma_2 : \begin{cases} \dot{x}_2 &= \gamma_\theta u_2, \\ y_2 &= x_2, \end{cases} \quad (15)$$

where  $x_1 = x$ ,  $x_2 = \hat{\theta} - \ell_\theta$ ,  $u_1 = -y_2$ ,  $u_2 = y_2$  and  $a(x_1, t) = (k + \frac{1}{2\epsilon_{\Delta_\theta}}\delta_{\Delta_\theta}) + \frac{1}{2}\epsilon_{\Delta_\theta}\delta_{\Delta_\theta}x_1^2 - \Delta_\theta x_1 \geq k > 0$ .  $\diamond$

## B. Parameter in the input path

In what follows we show how to extend the idea of *congelation of variables* to systems in which a time-varying parameter is coupled with the input by considering the nonlinear system

$$\dot{x} = \theta(t)x^2 + b(t)u, \quad (16)$$

where  $\theta(t)$  satisfies Assumption 1 and  $b(t) \in \mathbb{R}$  satisfies Assumption 1 and Assumption 3. Equation (16) can be rewritten as

$$\begin{aligned} \dot{x} &= \hat{\theta}x^2 + \bar{u} + \Delta_\theta x^2 + \Delta_b \hat{\rho} \bar{u} \\ &+ (\ell_\theta - \hat{\theta})x^2 - \ell_b \left( \frac{1}{\ell_b} - \hat{\rho} \right) \bar{u}, \end{aligned} \quad (17)$$

where  $\Delta_b(t) = b(t) - \ell_b$ ,  $\hat{\rho}$  is an “estimate” of  $\frac{1}{\ell_b}$ , and  $u = \hat{\rho}\bar{u}$ . From classical adaptive control theory (see *e.g.* [2]) we know that the effect of the second line of (17) can be cancelled by selecting the parameter update laws (5) and

$$\dot{\hat{\rho}} = -\gamma_\rho \text{sgn}(\ell_b) \bar{u} x, \quad (18)$$

and considering the Lyapunov function candidate  $V(x, \hat{\theta}, \hat{\rho}) = \frac{1}{2}x^2 + \frac{1}{2\gamma_\theta}(\theta - \hat{\theta})^2 + \frac{|\ell_b|}{2\gamma_\rho}(\frac{1}{\ell_b} - \hat{\rho})^2$ , the time derivative of which along the trajectories of (17) satisfies

$$\dot{V} = \hat{\theta}x^3 + \bar{u}x + \Delta_\theta x^3 + \Delta_b \hat{\rho} \bar{u} x. \quad (19)$$

Note that the perturbation term  $\Delta_b \hat{\rho} \bar{u} x$  depends on  $\bar{u}$  explicitly, which means that we cannot dominate this term by simply adding damping terms to  $\bar{u}$ , as doing this also alters the perturbation term itself. Instead, we need to make  $\Delta_b \hat{\rho} \bar{u} x$  non-positive by designing  $\bar{u}$  and selecting  $\ell_b$ . Consider  $\bar{u}$  as a feedback control law with a non-positive nonlinear gain, that is

$$\begin{aligned} \bar{u} &= -\left( k + \frac{1}{2} \left( \frac{\delta_{\Delta_\theta}}{\epsilon_{\Delta_\theta}} + \frac{1}{\epsilon_\theta} \right) + \frac{1}{2} (\epsilon_{\Delta_\theta} \delta_{\Delta_\theta} + \epsilon_\theta \hat{\theta}^2) x^2 \right) x \\ &= -\kappa(x, \hat{\theta}) x, \end{aligned} \quad (20)$$

where  $\epsilon_\theta > 0$ . Note that  $\kappa(x, \hat{\theta}) > 0$  by construction. Substituting (20) into (18) yields  $\dot{\hat{\rho}} = \gamma_\rho \text{sgn}(\ell_b) \kappa x^2$ . When  $b(t) > 0$ , for all  $t \geq 0$ , due to Assumption 3, there exists a constant  $\ell_b$  such that  $0 < \ell_b \leq b(t)$ ,  $\Delta_b > 0$ ,  $\hat{\rho} \geq 0$ , which means that any initialization with  $\hat{\rho}(0) > 0$  guarantees that  $\hat{\rho}(t) > 0$ , for all  $t \geq 0$ , and therefore  $\Delta_b \hat{\rho} \bar{u} x = -\Delta_b \hat{\rho} \kappa x^2 \leq 0$ , for all  $t \geq 0$ . When  $b(t) < 0$ , for all  $t \geq 0$ , similarly there exists  $\ell_b$  such that  $b(t) \leq \ell_b < 0$ ,  $\Delta_b < 0$ ,  $\hat{\rho} \leq 0$ . Then selecting  $\hat{\rho}(0) < 0$  guarantees  $\hat{\rho}(t) < 0$ , for all  $t \geq 0$ , and

$\Delta_b \hat{\rho} \bar{u} x \leq 0$ . Recalling (19), (20), and noting that  $\Delta_b \hat{\rho} \bar{u} x \leq 0$  yields

$$\begin{aligned} \dot{V} \leq & -kx^2 - \left( \frac{\epsilon_{\hat{\theta}}}{2} \hat{\theta}^2 x^4 + \frac{1}{2\epsilon_{\hat{\theta}}} x - \hat{\theta} x^3 \right) \\ & - \left( \frac{\epsilon_{\Delta_{\theta}} \delta_{\Delta_{\theta}}}{2} x^4 + \frac{\delta_{\Delta_{\theta}}}{2\epsilon_{\Delta_{\theta}}} x^2 + \Delta_{\theta} x^3 \right) \leq -kx^2 \leq 0. \end{aligned} \quad (21)$$

Exploiting the same stability argument as before, boundedness of the system trajectories and convergence of  $x$  to zero follows.

*Remark 5:* This example highlights the flexibility of the *congelation of variables* method: the congealed parameter  $\ell_{(\cdot)}$  can be selected according to the specific usage. It can be a nominal value for robust design, or an ‘‘extreme’’ value to create sign-definiteness, as long as the resulting perturbation  $\Delta_{(\cdot)}$  is considered consistently. One can even make  $\ell_{(\cdot)}$  a time-varying parameter subject to some of the assumptions used in the literature (e.g.  $\dot{\ell}_{(\cdot)} \in \mathcal{L}_{\infty}$ ,  $\dot{\ell}_{(\cdot)} \in \mathcal{L}_1$ , see e.g. [10], [15]), and use the *congelation of variables* method to relax these assumptions. This is the reason why the proposed method is named ‘‘congelation’’<sup>3</sup> not ‘‘freeze’’.

*Remark 6:* Similarly to the effect of the selection of  $\ell_{\theta}$  in Remark 4, the selection of  $\ell_b$  makes  $\hat{\rho} - \frac{1}{\ell_b}$  a passivating input/output signal. In addition, note that the overall system is passive from  $-\Delta_b \hat{\rho} \kappa x$  to  $x$ . Our selection of  $\ell_b$  always guarantees that  $-\Delta_b \hat{\rho} \kappa$  is negative and therefore yields a negative feedback ‘‘control’’ (if regarding  $-\Delta_b \hat{\rho} \kappa x$  as the control law), which is well-known to possess an arbitrarily large gain margin in a passive system and robust against the variation of  $\Delta_b \hat{\rho} \kappa$ .

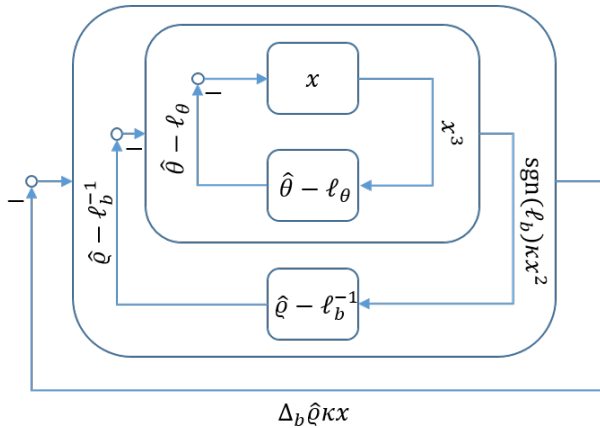


Fig. 2. Schematic representation of system (17), (5) and (18) as the interconnection of passive subsystems.

The examples discussed above are simple, yet illustrate the core ideas put forward in the paper: no matter if the time-varying parameters appear in the feedback path or in the input path. The readers will see that the following sections are essentially applying the same ideas in more sophisticated ways and to more complicated scenarios.

<sup>3</sup>The word ‘‘congelation’’ is polysemous: it means both ‘‘coagulation’’ and ‘‘freeze/solidification’’ [29].

### C. Preliminary Result: State-Feedback Design for Unmatched Parameters

In the examples of Section II-A and Section II-B the unknown parameter  $\theta(t)$  enters the system dynamics from the same integrator from which the input  $u$  enters, that is, the so-called *matching condition* holds. For a more general class of systems in which the unknown parameters are separated from the input by integrators, *adaptive backstepping* design [2] is needed. Consider an  $n$ -dimensional nonlinear system in the so-called *parametric strict-feedback* form, namely

$$\begin{aligned} \dot{x}_1 &= \phi_1^\top(x_1)\theta(t) + x_2, \\ &\vdots \\ \dot{x}_i &= \phi_i^\top(\underline{x}_i)\theta(t) + x_{i+1}, \\ &\vdots \\ \dot{x}_n &= \phi_n^\top(x)\theta(t) + b(t)u, \end{aligned} \quad (22)$$

where  $i = 2, \dots, n-1$ ,  $x(t) = [x_1, \dots, x_n]^\top \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}$  is the input,  $\theta(t) \in \mathbb{R}^q$  is the vector of unknown parameters satisfying Assumption 1, and  $b(t) \in \mathbb{R}$  is an unknown parameter satisfying Assumption 1 and Assumption 3. The regressors  $\phi_i : \mathbb{R}^i \rightarrow \mathbb{R}^q$ ,  $i = 1, \dots, n$ , are smooth mappings and satisfy  $\phi_i(0) = 0$ .

*Remark 7:* The condition  $\phi_i(0) = 0$  implies that  $\phi_i^\top(0)\theta(t) = 0$ , which allows zero control effort at  $x = 0$ . One can easily see that if  $\phi_i(0) \neq 0$ ,  $\phi_i^\top(0)\theta(t)$  becomes an unknown time-varying disturbance, yielding a disturbance rejection/attenuation problem not discussed here. By Hadamard’s lemma [30], one can express the regressors as  $\phi_i(x_i) = \bar{\Phi}_i(x_i)x_i$ , where  $\bar{\Phi}_i$  are smooth mappings.

We directly give the results below and omit the step-by-step procedures<sup>4</sup>. For each step  $i$ ,  $i = 1, \dots, n$ , define the error variables

$$z_0 = 0, \quad (23)$$

$$z_i = x_i - \alpha_{i-1}, \quad (24)$$

the new regressor vectors

$$w_i(x_i, \hat{\theta}) = \phi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \phi_j, \quad (25)$$

the tuning functions

$$\tau_i(x_i, \hat{\theta}) = \tau_{i-1} + w_i z_i = \sum_{j=1}^i w_j z_j, \quad (26)$$

and the virtual control laws

$$\alpha_0 = 0, \quad (27)$$

$$\begin{aligned} \alpha_i(x_i, \hat{\theta}) &= -z_{i-1} - (c_i + \zeta_i)z_i - w_i^\top \hat{\theta} \\ &\quad + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma_\theta \tau_i \\ &\quad + \sum_{j=2}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma_\theta w_j z_j, \quad i = 1, \dots, n-1, \end{aligned} \quad (28)$$

$$\alpha_n = \hat{\rho} \bar{\alpha}_n = -\hat{\rho} \kappa(x, \hat{\theta}) z_n, \quad (29)$$

<sup>4</sup>The classical procedures of *adaptive backstepping*, on which the following procedures are based, can be found in Chapter 4 of [2].

where  $c_i > 0$  are constant feedback gains,  $\zeta_i(\underline{x}_i, \theta)$  are nonlinear feedback gains to be defined later,  $\Gamma_\theta = \Gamma_\theta^\top > 0$  is the adaptation gain,  $\kappa(x, \hat{\theta})$  is a positive nonlinear feedback gain to be defined later, and similar to the one in Section II-B. To proceed with the analysis, select the control law and the parameter update laws as

$$u = \alpha_n, \quad (30)$$

$$\dot{\hat{\theta}} = \Gamma_\theta \tau_n, \quad (31)$$

$$\dot{\hat{\rho}} = -\gamma_\rho \text{sgn}(\ell_b) \bar{\alpha}_n z_n, \quad (32)$$

respectively, and consider the Lyapunov function candidate  $V(z, \hat{\theta}, \hat{\rho}) = \frac{1}{2}|z|^2 + \frac{1}{2}|\ell_\theta - \hat{\theta}|_{\Gamma^{-1}}^2 + \frac{|\ell_b|}{2\gamma_\rho} |\frac{1}{\ell_b} - \hat{\rho}|^2$ , where  $z = [z_1, \dots, z_n]^\top$ . Taking the time-derivative of  $V$  yields

$$\begin{aligned} \dot{V} = & -\sum_{i=1}^n (c_i + \zeta_i) z_i + z_n \bar{\alpha}_n + \Delta + z_n \psi \\ & + (\ell_\theta - \hat{\theta})^\top \left( \sum_{i=1}^{n-1} w_i z_i - \Gamma_\theta^{-1} \dot{\hat{\theta}} \right) \\ & + \ell_b \left( \frac{1}{\ell_b} - \hat{\rho} \right) \left( \bar{\alpha}_n z_n - \frac{\dot{\hat{\rho}}}{\gamma_\rho} \right), \end{aligned} \quad (33)$$

where

$$\Delta = \sum_{i=1}^{n-1} z_i w_i^\top \Delta_\theta + \Delta_b \hat{\rho} \bar{\alpha}_n z_n, \quad (34)$$

$$\begin{aligned} \psi = & z_{n-1} + w_n^\top \hat{\theta} - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} x_{j+1} \\ & - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \Gamma_\theta \tau_n - \sum_{j=2}^{n-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma_\theta w_n z_j. \end{aligned} \quad (35)$$

*Remark 8:* Recalling Remark 7 and implementing (23) to (29) recursively, it is not hard to see that  $z_i(\underline{x}_i, \hat{\theta})$ ,  $w_i(\underline{x}_i, \hat{\theta})$ ,  $\tau_i(\underline{x}_i, \hat{\theta})$ ,  $\alpha_i(\underline{x}_i, \hat{\theta})$  are smooth and  $z_i(0, \hat{\theta}) = 0$ ,  $w_i(0, \hat{\theta}) = 0$ ,  $\tau_i(0, \hat{\theta}) = 0$ ,  $\alpha_i(0, \hat{\theta}) = 0$ . Note also that the  $\hat{\theta}$ -dependent change of coordinates between  $z_i$  and  $x_i$  is smooth, invertible, and  $x_i = 0 \Leftrightarrow z_i = 0$ , thus we can directly express  $w_i$  as  $w_i = \bar{W}_i(\underline{x}_i, \hat{\theta}) z_i$  with  $\bar{W}_i$  smooth and, similarly,  $\psi$  as  $\psi = \bar{\psi}^\top(x, \hat{\theta}) z$  with  $\bar{\psi}$  smooth.  $\diamond$

The last two lines of (33) are eliminated by the parameter update laws (31) and (32), and the non-positivity of  $\Delta_b \hat{\rho} \bar{\alpha}_n z_n$  can be established in the same way as in Section II-B, thanks to the form<sup>5</sup> of  $\bar{\alpha}_n$ . The rest of the problem is to determine the nonlinear damping gains  $\zeta_i(\underline{x}_i, \hat{\theta})$  and  $\kappa(x, \hat{\theta})$  to dominate the  $\Delta_\theta$ -terms.

*Proposition 1:* Consider system (22) and the control law (30) with the nonlinear damping gains

$$\zeta_i(\underline{x}_i, \hat{\theta}) = \frac{1}{2} \left( (n-i+1) \frac{\delta_{\Delta_\theta}}{\epsilon_{\Delta_\theta}} + \epsilon_{\Delta_\theta} \delta_{\Delta_\theta} |\bar{W}_i|_{\mathbb{F}}^2 + \frac{1}{\epsilon_{\bar{\psi}}} \right), \quad (36)$$

$$\kappa(x, \hat{\theta}) = c_n + \zeta_n + \frac{1}{2} \epsilon_{\bar{\psi}} |\bar{\psi}|^2, \quad (37)$$

<sup>5</sup>This form of  $\bar{\alpha}_n$  is inspired by [31], which also designs a control law with a nonlinear negative feedback gain, albeit to achieve inverse optimality.

with  $c_n > 0$  and  $\epsilon_{(\cdot)} > 0$ , and the parameter update laws (31) and (32) with  $\text{sgn}(\hat{\rho}(0)) = \text{sgn}(b)$ . Then all closed-loop signals are bounded and  $\lim_{t \rightarrow +\infty} x(t) = 0$ .  $\diamond$

*Proof:* Recalling Remark 8 and invoking Young's inequality yields

$$\begin{aligned} z_i w_i^\top \Delta_\theta &= z_i \Delta_\theta^\top \bar{W}_i z_i \\ &\leq \frac{1}{2} \left( \frac{\delta_{\Delta_\theta}}{\epsilon_{\Delta_\theta}} + \epsilon_{\Delta_\theta} \delta_{\Delta_\theta} |\bar{W}_i|_{\mathbb{F}}^2 \right) z_i^2 + \frac{\delta_{\Delta_\theta}}{2\epsilon_{\Delta_\theta}} |z_{i-1}|^2, \end{aligned} \quad (38)$$

$$z_n \psi = z_n \bar{\psi} z = \frac{1}{2} \left( \frac{1}{\epsilon_{\bar{\psi}}} + \epsilon_{\bar{\psi}} |\bar{\psi}|^2 \right) z_n^2 + \frac{1}{2\epsilon_{\bar{\psi}}} |z_{n-1}|^2 \quad (39)$$

Consider now (33) and note that there exists  $\ell_b$  such that  $\Delta_b \hat{\rho} \bar{\alpha}_n z_n \leq 0$  provided that  $\text{sgn}(\hat{\rho}(0)) = \text{sgn}(b)$ , which yields

$$\dot{V} \leq -\sum_{i=1}^n (c_i + \zeta_i) z_i - \kappa z_n^2 + z_n \psi + \sum_{i=1}^{n-1} z_i w_i^\top \Delta_\theta. \quad (40)$$

Substituting (36) and (37) together with (38) and (39) into (40) yields  $\dot{V} \leq -\sum_{i=1}^n c_i z_i^2 \leq 0$ , which guarantees that  $z$ ,  $\hat{\theta}$  and  $\hat{\rho}$  are globally uniformly bounded. The global uniform boundedness of  $x$  is also guaranteed by Remark 8. Note that the exogenous input signals to the dynamics of  $z$  are  $\theta(t)$  and  $b(t)$ , which are bounded by Assumption 1 and Assumption 3, and therefore  $\dot{z}$  is also bounded. Hence invoking Barbalat's lemma one can conclude that  $\lim_{t \rightarrow +\infty} z(t) = 0$ , which further indicates that  $\lim_{t \rightarrow +\infty} x(t) = 0$ , by Remark 8.  $\blacksquare$

Although state-feedback is in general not available in practice, the result presented above indicates how to combine the *congelation of variables* method and *backstepping* to cope with the unmatched time-varying parameters. We will see that, once proper filters are built, the same techniques can be also applied to systems in which only the output is available for feedback.

### III. OUTPUT-FEEDBACK DESIGN

Consider now an  $n$ -dimensional system in output feedback form with relative degree  $\rho$  described by the equations

$$\begin{aligned} \dot{x}_1 &= x_2 + \phi_{0,1}(y) + \sum_{j=1}^q \phi_{1,j}(y) a_j(t), \\ &\vdots \\ \dot{x}_\rho &= x_{\rho+1} + \phi_{0,\rho}(y) + \sum_{j=1}^q \phi_{\rho,j}(y) a_j(t) + b_m(t) g(y) u, \\ &\vdots \\ \dot{x}_n &= \phi_{0,n}(y) + \sum_{j=1}^q \phi_{n,j}(y) a_j(t) + b_0(t) g(y) u, \\ y &= x_1, \end{aligned} \quad (41)$$

or, in compact form, by the equations

$$\begin{aligned} \dot{x} &= Sx + \phi_0(y) + F^\top(y, u)\theta, \\ y &= e_1^\top x, \end{aligned} \quad (42)$$

where  $x(t) = [x_1, \dots, x_n]^\top \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}$  is the input,  $y(t) \in \mathbb{R}$  is the output,  $\theta(t) = [b^\top(t), a^\top(t)]^\top$  is the vector of unknown time-varying parameters,  $a(t) =$

$[a_1(t), \dots, a_q(t)]^\top \in \mathbb{R}^q$ ,  $b(t) = [b_m(t), \dots, b_0(t)]^\top \in \mathbb{R}^{m+1}$ ,  $m = n - \rho$ ,

$$F^\top(y, u) = \left[ \begin{array}{c} 0_{(\rho-1) \times (m+1)} \\ I_{m+1} \end{array} \right] g(y)u, \Phi^\top(y), \quad (43)$$

$(\Phi^\top(y))_{ij} = \phi_{i,j}(y)$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth mapping and  $g(y) \neq 0$ , for all  $y \in \mathbb{R}$ . In addition,  $\theta(t)$  satisfies Assumption 2, and, in particular,  $b_m(t)$  also satisfies Assumption 3. The mappings  $\phi_{0,i} : \mathbb{R} \rightarrow \mathbb{R}$  and  $\phi_{i,j} : \mathbb{R} \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, q$ , are smooth and such that  $\phi_{0,i}(0) = 0$ ,  $\phi_{i,j}(0) = 0$ .

*Remark 9:* Similarly to what is discussed Remark 7, there exist smooth mappings  $\bar{\phi}_{0,i}$  and  $\bar{\phi}_{i,j}$  such that  $\phi_{0,i}(y) = \bar{\phi}_{0,i}(y)y$ ,  $\phi_{i,j}(y) = \bar{\phi}_{i,j}(y)y$ .  $\diamond$

### A. System Reparameterization

Due to the presence of unmeasured state variables we use *Kreisselmeier filters* (*K-filters*) [32] to reparameterize the system with the filter state variables (which are known) into a new form that is favorable for the *adaptive backstepping* design [2]. The filters are given by the equations

$$\dot{\xi} = A_k \xi + ky + \phi_0(y), \quad (44)$$

$$\dot{\Xi}^\top = A_k \Xi^\top + \Phi^\top(y), \quad (45)$$

$$\dot{\lambda} = A_k \lambda + e_n g(y)u, \quad (46)$$

where  $A_k = S - ke_1^\top$  and  $k \in \mathbb{R}^n$  is the vector of filter gains. These filters are equivalent, see [2], to the filters

$$\dot{\xi} = A_k \xi + ky + \phi_0(y), \quad (47)$$

$$\dot{\Omega}^\top = A_k \Omega^\top + F^\top(y, u), \quad (48)$$

where

$$\Omega^\top = [v_m, \dots, v_0, \Xi^\top], \quad (49)$$

$$v_i = A_k^i \lambda, \quad i = 0, \dots, m. \quad (50)$$

Define now the non-implementable state estimate

$$\hat{x} = \xi + \Omega^\top \ell_\theta. \quad (51)$$

The state estimation error dynamics are then described by the equation

$$\begin{aligned} \dot{\varepsilon} &= A_k \varepsilon + F^\top(y, u) \Delta_\theta \\ &= A_k \varepsilon + \Phi^\top(y) \Delta_a + \begin{bmatrix} 0_{(\rho-1) \times 1} \\ \Delta_b \end{bmatrix} g(y)u, \end{aligned} \quad (52)$$

where  $\varepsilon = x - \hat{x}$ . We now show that after using the *K-filters* (44)–(46) with the *congelation of variables* method the original  $n$ -dimensional system with time-varying parameters can be reparameterized as a  $\rho$ -dimensional system with constant parameters  $\ell_\theta$  and some auxiliary systems to be defined. The substitution of  $\ell_\theta$  for  $\theta(t)$  prevents  $\dot{\theta}$  from appearing in the  $\varepsilon$ -dynamics. For  $\rho > 1$  one has the problem described by the equations

$$\begin{aligned} \dot{y} &= \omega_0 + \bar{\omega}^\top \ell_\theta + \varepsilon_2 + \ell_{b_m} v_{m,2} \\ \dot{v}_{m,i} &= -k_i v_{m,1} + v_{m,i+1}, \quad i = 2, \dots, \rho - 1, \\ \dot{v}_{m,\rho} &= -k_\rho v_{m,1} + v_{m,\rho+1} + g(y)u, \end{aligned} \quad (53)$$

and, for  $\rho = 1$  one has

$$\dot{y} = \omega_0 + \omega^\top \ell_\theta + \varepsilon_2 + \ell_{b_m} g(y)u, \quad (54)$$

where  $\omega_0 = \phi_{0,1} + \xi_2$ ,  $\bar{\omega} = [0, v_{m-1,2}, \dots, v_{0,2}, (\Phi)_1^\top + (\Xi)_2^\top]^\top$ ,  $\omega = \bar{\omega} + e_1 v_{m,2}$ .

Similarly to the classical adaptive backstepping scheme we consider the  $\rho$ -th order system (53) (or (54) if  $\rho = 1$ ) to exploit its lower triangular form yet (53) and (54) are useful only if the estimation error  $\varepsilon_2$  is converging to 0. In classical schemes this is not a problem since there are no  $\Delta_a(t)$  or  $\Delta_b(t)$  terms and  $\varepsilon$  converges to 0 exponentially provided that  $A_k$  is Hurwitz. The effect of  $\Delta_a(t)$  can be dominated via a strengthened damping design, as proposed in [17]. However, the dominance method cannot be directly applied to (52) since  $\Delta_b(t)$  is coupled with the input  $u$ , which causes a chicken-and-egg dilemma if we add additional damping terms to the controller without further modifications. To this end, in the next section we revisit the ideas of [19] and [20] to see how we can decouple  $\Delta_b(t)$  and  $u$  with the help of the inverse dynamics of system (41).

### B. Inverse Dynamics

To study the inverse dynamics of (41) pretend that the system is “driven” by  $y$ ,  $\phi_{0,i}(y)$ ,  $\phi_i(y)$ , and their time derivatives. Then one could write

$$\begin{aligned} x_2 &= y^{(1)} - (\phi_1^\top a + \phi_{0,1}), \\ &\vdots \\ x_\rho &= y^{(\rho-1)} - (\phi_1^\top a + \phi_{0,1})^{(\rho-2)} - \dots \\ &\quad - (\phi_{\rho-1}^\top + \phi_{0,\rho-1}). \end{aligned} \quad (55)$$

Setting  $y_i = \phi_i^\top a + \phi_{0,i}$ ,  $i = 1, \dots, n$  and  $u_g = g(y)u$ , yields

$$u_g = \frac{1}{b_m} (-x_{\rho+1} + y^{(\rho)} - y_1^{(\rho-1)} - \dots - y_\rho). \quad (56)$$

The resulting inverse dynamics are then described by

$$\begin{aligned} \dot{x}_{\rho+1} &= -\frac{b_{m-1}}{b_m} x_{\rho+1} + x_{\rho+2} + y_{\rho+1} \\ &\quad + \frac{b_{m-1}}{b_m} (y^{(\rho)} - y_1^{(\rho-1)} - \dots - y_\rho), \\ &\vdots \\ \dot{x}_n &= -\frac{b_0}{b_m} x_{\rho+1} + y_n \\ &\quad + \frac{b_0}{b_m} (y^{(\rho)} - y_1^{(\rho-1)} - \dots - y_\rho). \end{aligned} \quad (57)$$

Since it is difficult to use backstepping techniques to establish stability, or convergence, properties for the time derivatives of  $y$  or  $y_i$ , we need to perform a change of coordinates to remove the derivative terms from the inverse dynamics. Note that for any pair of smooth signals  $s_1(t)$  and  $s_2(t)$  the equation

$$s_1 s_2^{(i)} = (-1)^i s_1^{(i)} s_2 + \left( \sum_{j=0}^{i-1} (-1)^j s_1^{(j)} s_2^{(i-1-j)} \right)^{(1)} \quad (58)$$

**Algorithm 1** Change of coordinates  $x_{\rho+1}, \dots, x_n$ .**Input:**  $x_{\rho+1}, \dots, x_n, \dot{x}_{\rho+1}, \dots, \dot{x}_n$ .**Output:**  $\bar{x}_{\rho+1}, \dots, \bar{x}_n, \dot{\bar{x}}_{\rho+1}, \dots, \dot{\bar{x}}_n$ .

- 1: **while** time derivatives of  $y$  appear in the expression of  $\dot{x}_{\rho+1}, \dots, \dot{x}_n$  **do**  $\triangleright$  This while-loop iterates for  $\rho$  times as it reduces the order of  $y^{(\rho)}$  by one each iteration.
- 2:   **for**  $i = n \rightarrow \rho + 2$  **do**
- 3:     Update  $\bar{x}_i$  and  $\dot{\bar{x}}_i$  using (58).
- 4:     Rewrite  $x_i$  in terms of  $\bar{x}_i$  in the expression of  $\dot{x}_{i-1}$  and leave the feedback term  $-\frac{b_{n-i}}{b_m}x_{\rho+1}$  unchanged.
- 5:   **end for**
- 6:   Update  $\bar{x}_{\rho+1}$  and  $\dot{\bar{x}}_{\rho+1}$  using (58).
- 7:   Rewrite  $x_{\rho+1}$  in terms of  $\bar{x}_{\rho+1}$  in the expressions of  $\dot{\bar{x}}_{\rho+1}, \dots, \dot{\bar{x}}_n$ , respectively.  $\triangleright$   
This brings back the time derivatives of  $y, y_1, \dots, y_\rho$ , but with the order reduced by one.
- 8:    $x_{\rho+1} \leftarrow \bar{x}_{\rho+1}, \dots, x_n \leftarrow \bar{x}_n, \dot{x}_{\rho+1} \leftarrow \dot{\bar{x}}_{\rho+1}, \dots, \dot{x}_n \leftarrow \dot{\bar{x}}_n$ .  $\triangleright$  Update the old coordinates before the next iteration.
- 9: **end while**

holds. With this fact, the change of coordinate

$$\begin{aligned} \bar{x}_n &= x_n - \sum_{j=0}^{\rho-1} (-1)^j \left(\frac{b_0}{b_m}\right)^j y^{(\rho-1-j)} \\ &\quad + \sum_{i=1}^{\rho-1} \sum_{j=0}^{\rho-i-1} (-1)^j \left(\frac{b_0}{b_m}\right)^j y_i^{(\rho-i-1-j)} \end{aligned} \quad (59)$$

yields

$$\begin{aligned} \dot{\bar{x}}_n &= -\frac{b_0}{b_m}x_{\rho+1} + y_n + (-1)^\rho \left(\frac{b_0}{b_m}\right)^{(\rho)} y \\ &\quad - \sum_{i=1}^{\rho} (-1)^{\rho-i} \left(\frac{b_0}{b_m}\right)^{(\rho-i)} y_i, \end{aligned} \quad (60)$$

which does not contain time derivatives of  $y$  and  $y_i$ . In the same spirit, applying the change of coordinates specified by Algorithm 1, we are able to remove the terms containing the time derivatives of  $y$  and  $y_i$  in each equation of the inverse dynamics. The resulting inverse dynamics in the new coordinates (we use  $\bar{x}_i$ ,  $i = \rho + 1, \dots, n$  with a slight abuse of notation) are described by the equations

$$\begin{aligned} \dot{\bar{x}} &= A_{\bar{b}}(t)\bar{x} + b_{\bar{x}y}(t)y + \sum_{i=1}^n b_{\bar{x}\phi,0,i}(t)\phi_{0,i}(y) \\ &\quad + \sum_{i=1}^n \sum_{j=1}^q b_{\bar{x}\phi,i,j}(t)\phi_{i,j}(y), \end{aligned} \quad (61)$$

$$\begin{aligned} u_g &= \frac{1}{b_m(t)} \left( -\bar{x}_{\rho+1} + y^{(\rho)} + \sum_{j=0}^{\rho-1} a_{u_g y^{(j)}}(t)y^{(j)} \right. \\ &\quad \left. + \sum_{i=1}^{\rho} \sum_{j=0}^{\rho-i} a_{u_g y_i^{(j)}}(t)y_i^{(j)} \right), \end{aligned} \quad (62)$$

where  $\bar{x}(t) = [\bar{x}_{\rho+1}(t), \dots, \bar{x}_n(t)]^\top \in \mathbb{R}^m$ ,  $A_{\bar{b}} = S - \bar{b}e_1^\top$ ,  $\bar{b}(t) = \frac{1}{b_m(t)}[b_{m-1}(t), \dots, b_0(t)]^\top \in \mathbb{R}^m$ .

*Remark 10:* The time-varying vectors  $b_{\bar{x}y}(t)$ ,  $b_{\bar{x}\phi,0,i}(t)$ ,  $b_{\bar{x}\phi,i,j}(t)$  and the time-varying scalars  $a_{u_g y^{(j)}}(t)$ ,  $a_{u_g y_i^{(j)}}(t)$  are unknown as they depend on the unknown  $\theta(t)$ . However, as a consequence of Assumption 2, they are bounded.  $\diamond$

*Assumption 4 (Strong minimum-phase property):* The time-varying system (41) has a strong minimum-phase property in the sense that the inverse dynamics (61) are input-to-state stable (ISS) with respect to the inputs  $y$ ,  $\phi_{0,i}(y)$ ,  $\phi_{i,j}(y)$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, q$ . Moreover, there exists an ISS Lyapunov function  $\underline{\gamma}_{\bar{x}}|\bar{x}|^2 \leq V_{\bar{x}}(\bar{x}, t) \leq \bar{\gamma}_{\bar{x}}|\bar{x}|^2$ ,  $0 \leq \underline{\gamma}_{\bar{x}} \leq \bar{\gamma}_{\bar{x}}$ , and the time derivative of  $V_{\bar{x}}$  along the trajectories of the inverse dynamics satisfies the inequality

$$\dot{V}_{\bar{x}} \leq -|\bar{x}|^2 + \sigma_{\bar{x}y}y^2 + \sigma_{\bar{x}\phi_0}|\phi_0(y)|^2 + \sigma_{\bar{x}\Phi}|\Phi(y)|_{\mathbb{F}}^2, \quad (63)$$

for some constant  $\sigma_{(\cdot)} > 0$ .  $\diamond$

*Remark 11:* Assumption 4 is verified if  $\bar{x} = 0$  is a globally exponentially stable equilibrium of the zero dynamics described by  $\dot{\bar{x}} = A_{\bar{b}}(t)\bar{x}$ , see e.g. Lemma 4.6 in [33]. Some works (e.g. [11] and [16]) exploit this exponential stability property as a substitute for the classical minimum-phase assumption. Note, finally, that Assumption 4 is not more restrictive than the classical minimum-phase assumption because for time-invariant systems Assumption 4 reduces to minimum-phaseness.  $\diamond$

### C. Filter Design

Consider now the state estimation error dynamics (52) with  $u_g$  given by (62), which yields

$$\begin{aligned} \dot{\varepsilon} &= A_k \varepsilon + \Phi^\top(y)\Delta_a + \begin{bmatrix} 0_{(\rho-1) \times 1} \\ \Delta_b \end{bmatrix} \frac{1}{b_m} \left( -\bar{x}_{\rho+1} + y^{(\rho)} \right. \\ &\quad \left. + \sum_{j=0}^{\rho-1} a_{u_g y^{(j)}}(t)y^{(j)} + \sum_{i=1}^{\rho} \sum_{j=0}^{\rho-i} a_{u_g y_i^{(j)}}(t)y_i^{(j)} \right). \end{aligned} \quad (64)$$

Similarly to what is done in Section III-B, we need to use a change of coordinates to remove the time derivative terms brought by  $u_g$ . Implementing a change of coordinates in the same spirit of Algorithm 1, the state estimation error dynamics in the new coordinates  $\bar{\varepsilon}$  are described by the equations

$$\begin{aligned} \dot{\bar{\varepsilon}} &= A_k \bar{\varepsilon} - \bar{\Delta}_b \bar{x}_{\rho+1} + b_{\bar{\varepsilon}y}(t)y \\ &\quad + \sum_{i=1}^n b_{\bar{\varepsilon}\phi,0,i}(t)\phi_{0,i}(y) + \sum_{i=1}^n \sum_{j=1}^q b_{\bar{\varepsilon}\phi,i,j}(t)\phi_{i,j}(y), \end{aligned} \quad (65)$$

where  $\bar{\Delta}_b = [0_{1 \times (\rho-1)}, \Delta_b^\top]^\top \frac{1}{b_m}$ .

*Remark 12:* The time derivative terms are injected into the  $\varepsilon$ -dynamics via the vector of gains  $\Delta_b(t)$ . Similarly to Remark 10, the time-varying vectors  $\bar{\Delta}_b(t)$ ,  $b_{\bar{\varepsilon}y}(t)$ ,  $b_{\bar{\varepsilon}\phi,0,i}(t)$ ,  $b_{\bar{\varepsilon}\phi,i,j}(t)$  are unknown, yet bounded, due to Assumption 2. We will see that as long as these parameters are bounded they do not affect the controller design. In particular, when  $b(t)$  is constant,  $\Delta_b(t) \equiv 0$ , provided  $\ell_b = b$ , thus  $\bar{\Delta}_b$ ,  $b_{\bar{\varepsilon}y}(t)$ ,  $b_{\bar{\varepsilon}\phi,0,i}(t)$ ,  $b_{\bar{\varepsilon}\phi,i,j}(t)$  are all identically 0 and  $\bar{\varepsilon} = \varepsilon$ , which yields  $\dot{\varepsilon} = A_k \varepsilon + \Phi^\top(y)\Delta_a$ , a simplified case that has been dealt with in [17].  $\diamond$



Similarly to the description of the ISS inverse dynamics, we want the state estimation error dynamics to be ISS, but in this case, rather than assuming it, we can guarantee such a property by designing the  $K$ -filters.

**Proposition 2:** The state estimation error dynamics are ISS with respect to the inputs  $\bar{x}_{\rho+1}$ ,  $y$ ,  $\phi_{0,i}(y)$ ,  $\phi_{i,j}(y)$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, q$ , if the vector of filter gains is given by  $k = \frac{1}{2}X_{\bar{\varepsilon}}e_1$ ,  $X_{\bar{\varepsilon}} = X_{\bar{\varepsilon}}^{\top} \succ 0$ , and  $X_{\bar{\varepsilon}}$  satisfies the Riccati inequality<sup>6</sup>

$$SX_{\bar{\varepsilon}} + X_{\bar{\varepsilon}}S^{\top} - X_{\bar{\varepsilon}}(e_1e_1^{\top} - \gamma_{\bar{\varepsilon}}^{-1}I)X_{\bar{\varepsilon}} + Q_{\bar{\varepsilon}} \preceq 0, \quad (66)$$

where

$$Q_{\bar{\varepsilon}} = \left( \begin{array}{c} \delta_{\bar{\Delta}_b} + \frac{\delta_{b_{\bar{\varepsilon}y}}}{\epsilon_{b_{\bar{\varepsilon}y}}} + \sum_{i=1}^n \frac{\delta_{b_{\bar{\varepsilon}\phi_{0,i}}}}{\epsilon_{b_{\bar{\varepsilon}\phi_{0,i}}}} + \sum_{i=1}^n \sum_{j=1}^q \frac{\delta_{b_{\bar{\varepsilon}\phi_{i,j}}}}{\epsilon_{b_{\bar{\varepsilon}\phi_{i,j}}}} \\ \epsilon_{\bar{\Delta}_b} \end{array} \right) I, \quad (67)$$

and  $\epsilon_{(\cdot)} > 0$ . Moreover, there exists an ISS Lyapunov function  $V_{\bar{\varepsilon}} = \gamma_{\bar{\varepsilon}}|\bar{\varepsilon}|_{P_{\bar{\varepsilon}}}^2$ , with  $P_{\bar{\varepsilon}} = X_{\bar{\varepsilon}}^{-1}$  and the time derivative of  $V_{\bar{\varepsilon}}$  along the trajectories of the state estimation error dynamics satisfies the inequality

$$\begin{aligned} \dot{V}_{\bar{\varepsilon}} \leq & -|\bar{\varepsilon}|^2 + \epsilon_{b_{\bar{\varepsilon}y}}\delta_{b_{\bar{\varepsilon}y}}y^2 + \sum_{i=1}^n \epsilon_{b_{\bar{\varepsilon}\phi_{0,i}}}\delta_{b_{\bar{\varepsilon}\phi_{0,i}}}\phi_{0,i}^2(y) \\ & + \sum_{i=1}^n \sum_{j=1}^q \epsilon_{b_{\bar{\varepsilon}\phi_{i,j}}}\delta_{b_{\bar{\varepsilon}\phi_{i,j}}}\phi_{i,j}^2(y) + \epsilon_{\bar{\Delta}_b}\delta_{\bar{\Delta}_b}\bar{x}_{\rho+1}^2, \end{aligned} \quad (68)$$

where  $\epsilon_{(\cdot)} > 0$ , or in a more compact (yet more conservative) form,

$$\begin{aligned} \dot{V}_{\bar{\varepsilon}} \leq & -|\bar{\varepsilon}|^2 + \sigma_{\bar{\varepsilon}y}y^2 + \sigma_{\bar{\varepsilon}\phi_0}|\phi_0(y)|^2 + \sigma_{\bar{\varepsilon}\Phi}|\Phi(y)|_{\mathbb{F}}^2 \\ & + \sigma_{\bar{\varepsilon}\bar{x}_{\rho+1}}\bar{x}_{\rho+1}^2, \end{aligned} \quad (69)$$

for some constant  $\sigma_{(\cdot)} > 0$ .  $\diamond$

*Proof:* Taking the time derivative of  $V_{\bar{\varepsilon}} = \gamma_{\bar{\varepsilon}}|\bar{\varepsilon}|_{P_{\bar{\varepsilon}}}^2$  along the trajectories of system (65) yields

$$\begin{aligned} \dot{V}_{\bar{\varepsilon}} = & 2\gamma_{\bar{\varepsilon}}\bar{\varepsilon}^{\top}P_{\bar{\varepsilon}}\left(A_k\bar{\varepsilon} + b_{\bar{\varepsilon}y}(t)y + \sum_{i=1}^n b_{\bar{\varepsilon}\phi_{0,i}}(t)\phi_{0,i}(y) \right. \\ & \left. + \sum_{i=1}^n \sum_{j=1}^q b_{\bar{\varepsilon}\phi_{i,j}}(t)\phi_{i,j}(y) - \bar{\Delta}_b\bar{x}_{\rho+1}\right) \\ \leq & \gamma_{\bar{\varepsilon}}\left(\bar{\varepsilon}^{\top}(P_{\bar{\varepsilon}}S + S^{\top}P_{\bar{\varepsilon}} - e_1e_1^{\top} + P_{\bar{\varepsilon}}Q_{\bar{\varepsilon}}P_{\bar{\varepsilon}})\bar{\varepsilon}\right) \\ & + \epsilon_{b_{\bar{\varepsilon}y}}\delta_{b_{\bar{\varepsilon}y}}y^2 + \sum_{i=1}^n \epsilon_{b_{\bar{\varepsilon}\phi_{0,i}}}\delta_{b_{\bar{\varepsilon}\phi_{0,i}}}\phi_{0,i}^2(y) \\ & + \sum_{i=1}^n \sum_{j=1}^q \epsilon_{b_{\bar{\varepsilon}\phi_{i,j}}}\delta_{b_{\bar{\varepsilon}\phi_{i,j}}}\phi_{i,j}^2(y) + \epsilon_{\bar{\Delta}_b}\delta_{\bar{\Delta}_b}\bar{x}_{\rho+1}^2. \end{aligned} \quad (70)$$

Left-multiplying and right-multiplying by  $P_{\bar{\varepsilon}}$  on both sides of (66) yields

$$P_{\bar{\varepsilon}}S + S^{\top}P_{\bar{\varepsilon}} - (e_1e_1^{\top} - \gamma_{\bar{\varepsilon}}^{-1}I) + P_{\bar{\varepsilon}}Q_{\bar{\varepsilon}}P_{\bar{\varepsilon}} \preceq 0 \quad (71)$$

or, equivalently,

$$P_{\bar{\varepsilon}}S + S^{\top}P_{\bar{\varepsilon}} - e_1e_1^{\top} + P_{\bar{\varepsilon}}Q_{\bar{\varepsilon}}P_{\bar{\varepsilon}} \preceq -\gamma_{\bar{\varepsilon}}^{-1}I. \quad (72)$$

<sup>6</sup>The solvability of (66) has been discussed in [17].

Substituting (72) into (70) yields (68). Finally, defining  $\sigma_{\bar{\varepsilon}y} = \epsilon_{b_{\bar{\varepsilon}y}}\delta_{b_{\bar{\varepsilon}y}}$ ,  $\sigma_{\bar{\varepsilon}\phi_0} = \max_i \epsilon_{b_{\bar{\varepsilon}\phi_{0,i}}}\delta_{b_{\bar{\varepsilon}\phi_{0,i}}}$ ,  $\sigma_{\bar{\varepsilon}\phi_{i,j}} = \max_{i,j} \epsilon_{b_{\bar{\varepsilon}\phi_{i,j}}}\delta_{b_{\bar{\varepsilon}\phi_{i,j}}}$ , and  $\sigma_{\bar{\varepsilon}\bar{x}_{\rho+1}} = \epsilon_{\bar{\Delta}_b}\delta_{\bar{\Delta}_b}$  yields (69), which completes the proof.  $\blacksquare$

**Remark 13:** In practice  $Q_{\bar{\varepsilon}}$  is tuned to achieve better filtering performance rather than computed analytically. This is feasible since there exist  $\epsilon_{(\cdot)}$  for any bounded  $\delta_{(\cdot)}$  such that  $Q_{\bar{\varepsilon}}$  can be set to an arbitrary positive multiple of  $I$ , due to (67). Moreover,  $\epsilon_{(\cdot)}$  and  $\delta_{(\cdot)}$  do not affect the controller design, as the  $\sigma(\cdot)$ -related terms in (69) are dominated adaptively as shown in the subsection that follows. In this sense, neither  $\epsilon_{(\cdot)}$  nor  $\delta_{(\cdot)}$  are implemented or need to be known.  $\diamond$

## D. Controller Design

In Sections III-B and III-C we have established the ISS of the inverse dynamics and the state estimation error dynamics. However, before proceeding to design the controller, we have to consider (53) in the new coordinates. Note that  $\varepsilon_2$  can be written as

$$\varepsilon_2 = \bar{\varepsilon}_2 + a_{\varepsilon_2y(1)}(t)\dot{y} + Y_{\varepsilon_2}(y), \quad (73)$$

where  $Y_{\varepsilon_2}(y) = a_{\varepsilon_2y}(t)y + \sum_{i=1}^n a_{\varepsilon_2\phi_{0,i}}(t)\phi_{0,i}(y) + \sum_{i=1}^n \sum_{j=1}^q a_{\varepsilon_2\phi_{i,j}}(t)\phi_{i,j}(y)$  and  $a_{\varepsilon_2y(1)}(t) = \frac{\Delta_{b_m}(t)}{b_m(t)}$ . Two special cases, in which either  $\rho = 1$  or  $\rho \geq 2$  and  $b_m$  is constant, and therefore  $a_{\varepsilon_2\phi_{0,i}}(t) = 0$ , for all  $t \geq 0$ , have been discussed in [19]. In general,  $a_{\varepsilon_2y(1)}(t) \neq 0$  and, as a result,  $\varepsilon_2$  contains  $\dot{y}$ . Substituting (73) into the first equation of (53) yields

$$(1 - a_{\varepsilon_2y(1)})\dot{y} = \omega_0 + \bar{\omega}^{\top}\ell_{\theta} + \ell_{b_m}v_{m,2} + \bar{\varepsilon}_2 + Y_{\varepsilon_2}. \quad (74)$$

Noting that  $\frac{1}{1 - a_{\varepsilon_2y(1)}} = \frac{b_m}{b_m - \Delta_{b_m}} = \frac{b_m}{\ell_{b_m}}$ , we can write the dynamics of  $y$  as

$$\dot{y} = \frac{b_m(t)}{\ell_{b_m}}(\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2) + \bar{\omega}^{\top}\left(\frac{b_m(t)}{\ell_{b_m}}\ell_{\theta}\right) + b_m(t)v_{m,2}. \quad (75)$$

Observe that the effect of the  $a_{\varepsilon_2y(1)}(t)\dot{y}$  term is to bring the time-varying parameters back to the dynamics of  $y$ , which requires the *congelation of variables* method again. To do this, we need first to augment system (53) with the  $\xi$ ,  $\Xi$  and  $v$ -dynamics, which are not needed in the classical constant parameter scenarios but necessary in the current setup. It turns out that the extended system is in the so-called *parametric*

block-strict-feedback form [2], described by the equations

$$\dot{\xi} = A_k \xi + ky + \phi_0(y), \quad (76)$$

$$\dot{\Xi}^\top = A_k \Xi^\top + \Phi^\top(y), \quad (77)$$

$$\begin{aligned} \dot{y} &= \frac{b_m(t)}{\ell_{b_m}} (\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2) + \bar{\omega}^\top \begin{pmatrix} b_m(t) \\ \ell_{b_m} \end{pmatrix} \ell_\theta \\ &\quad + b_m(t) v_{m,2}, \end{aligned} \quad (78)$$

$$\dot{v}_{0,2} = v_{1,2},$$

$\vdots$

$$\dot{v}_{m-1,2} = v_{m,2},$$

$$\dot{v}_{m,2} = -k_1 v_{m,1} + v_{m,3},$$

$\vdots$

$$\dot{v}_{m,\rho-1} = -k_{\rho-1} v_{m,1} + v_{m,\rho}, \quad (79)$$

$$\dot{v}_{m,\rho} = -k_\rho v_{m,1} + v_{m,\rho+1} + g(y)u.$$

In these equations, (76) and (77) describe the state evolution of the filters of the regressors; equations (79) give the integrator-chain structure used for *backstepping*; and equations (78) are the key part of the design that contains the dynamics of the output  $y$ . Recall that  $\omega_0 = \phi_{0,1} + \xi_2$  and  $\bar{\omega} = [0, v_{m-1,2}, \dots, v_{0,2}, (\Phi)_1^\top + (\Xi)_2^\top]^\top$ . The *congelation of variables* method requires an ISS-like property of the state variables coupled with the time-varying parameters. It turns out that we need to first establish ISS properties for (76) and (77), and the zero dynamics of (78) before developing the backstepping design. For the subsystems described by (76) and (77) we have the following result.

*Lemma 1:* Let the filter gain  $k$  be as in Proposition 2. Then system (76) is ISS with respect to the inputs  $y, \phi_{0,i}(y)$  and system (77) is ISS with respect to the inputs  $\phi_{i,j}(y)$ , where  $i = 1, \dots, n, j = 1, \dots, q$ . Moreover, there exist two ISS Lyapunov functions  $V_\xi = |\xi|_{P_\xi}^2, V_\Xi = \text{tr}(\Xi P_\Xi \Xi^\top)$ , with  $P_\xi = P_\Xi = \gamma_\varepsilon P_\varepsilon \succ 0$ , such that the time derivative of  $V_\xi$  along the trajectories of (76) satisfies

$$\dot{V}_\xi \leq -|\xi|^2 + \sigma_{\xi y} y^2 + \sigma_{\xi \phi_0} |\phi_0(y)|^2 \quad (80)$$

and the time derivative of  $V_\Xi$  along the trajectories of (77) satisfies

$$\dot{V}_\Xi \leq -|\Xi|_{\mathbb{F}}^2 + \sigma_{\Xi \Phi} |\Phi(y)|_{\mathbb{F}}^2, \quad (81)$$

for some constant  $\sigma_{(\cdot)} > 0$ .  $\diamond$

*Proof:* Noting (66) and the fact that  $P_\xi = \gamma_\varepsilon P_\varepsilon$  yields

$$A_k^\top P_\xi + P_\xi A_k \preceq -I - \gamma_\varepsilon P_\varepsilon Q_\varepsilon P_\varepsilon. \quad (82)$$

Define  $\bar{Q}_\xi = \gamma_\varepsilon P_\varepsilon Q_\varepsilon P_\varepsilon \succ 0$ , take the time derivative of  $V_\xi = |\xi|_{P_\xi}^2$  along the trajectories of (77), and invoke Young's inequality to obtain

$$\begin{aligned} \dot{V}_\xi &= \xi^\top (A_k^\top P_\xi + P_\xi A_k) \xi + 2\xi^\top P_\xi (ky + \phi_0) \\ &= -\xi^\top (I + \bar{Q}_\xi) \xi + 2\xi^\top P_\xi (ky + \phi_0) \\ &\leq -|\xi|^2 + \sigma_{\xi y} y^2 + \sigma_{\xi \phi_0} |\phi_0(y)|^2. \end{aligned} \quad (83)$$

Similarly, we take the time derivative of  $V_\Xi = \text{tr}(\Xi P_\Xi \Xi^\top)$  along the trajectories of (77), which yields

$$\begin{aligned} \dot{V}_\Xi &= \sum_{i=1}^n ((\Xi^\top)_i)^\top (A_k^\top P_\Xi + P_\Xi A_k) (\Xi^\top)_i + 2(\Xi^\top)_i P_\Xi (\Phi^\top)_i \\ &= \sum_{i=1}^n ((\Xi^\top)_i)^\top (I + \bar{Q}_\Xi) (\Xi^\top)_i + 2(\Xi^\top)_i P_\Xi (\Phi^\top)_i \\ &\leq |\Xi|_{\mathbb{F}}^2 + \sigma_{\Xi \Phi} |\Phi(y)|_{\mathbb{F}}^2, \end{aligned} \quad (84)$$

where  $P_\Xi = P_\xi, \bar{Q}_\Xi = \bar{Q}_\xi$ , and this completes the proof.  $\blacksquare$

The remaining work is to investigate if ISS holds for the inverse dynamics of (78). To do this, first let

$$\begin{aligned} v_{m,2} &= \frac{1}{b_m} \dot{y} - \frac{1}{\ell_{b_m}} (\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2) - \frac{1}{\ell_{b_m}} \bar{\omega}^\top \ell_\theta \\ &= -\frac{\ell_{b_{m-1}}}{\ell_{b_m}} v_{m-1,2} - \dots - \frac{\ell_{b_0}}{\ell_{b_m}} v_{0,2} + \frac{1}{b_m} \dot{y} \\ &\quad - ((\Xi)_2^\top + (\Phi)_1^\top) \ell_a - \frac{1}{\ell_{b_m}} (\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2) \end{aligned} \quad (85)$$

and then define the change of coordinates:  $\bar{v}_{0,2} = v_{0,2}, \dots, \bar{v}_{m-2,2} = v_{m-2,2}, \bar{v}_{m-1,2} = v_{m-1,2} - \frac{1}{b_m} \dot{y}$ . The inverse dynamics of (78) are then described by

$$\dot{\bar{v}} = A_{\ell_b} \bar{v} + g_{\bar{v}}(y, \xi, \Xi, \bar{\varepsilon}_2, t), \quad (86)$$

where  $A_{\ell_b} = S - e_m \ell_b^\top, \ell_b = [\frac{\ell_{b_0}}{\ell_{b_m}}, \dots, \frac{\ell_{b_{m-1}}}{\ell_{b_m}}]^\top$ , and  $g_{\bar{v}}(y, \xi, \Xi, \bar{\varepsilon}_2, t) = [0, \dots, 0, \frac{1}{b_m} \dot{y}, -(\frac{\ell_{b_{m-1}}}{\ell_{b_m}} + (\frac{1}{b_m})^{(1)}) \dot{y} - ((\Xi)_2^\top + (\Phi)_2^\top) \ell_a - \frac{1}{\ell_{b_m}} (\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2)]^\top$ . Exploiting the flexibility of the *congelation of variables* method we can always select  $\ell_b$  to construct a Hurwitz  $A_{\ell_b}$ , and therefore ISS of system (86) can be established as shown in the lemma that follows.

*Lemma 2:* Suppose  $\ell_b = [\ell_{b_m}, \dots, \ell_{b_0}]^\top$  is such that the polynomial  $\ell_{b_m} s^m + \ell_{b_{m-1}} s^{m-1} + \dots + \ell_{b_0}$  is Hurwitz. Then system (86) is ISS with respect to the inputs  $y, \phi_{0,i}(y), \phi_{i,j}(y), \xi_2, (\Xi)_{j2}$  and  $\bar{\varepsilon}_2$ , where  $i = 1, \dots, n, j = 1, \dots, q$ . Moreover, there is an ISS Lyapunov function  $V_{\bar{v}} = |\bar{v}|_{P_{\bar{v}}}^2$ , with  $P_{\bar{v}} = P_{\bar{v}}^\top \succ 0$ , such that the time derivative of  $V_{\bar{v}}$  along the trajectories of (86) satisfies

$$\begin{aligned} \dot{V}_{\bar{v}} &\leq -|\bar{v}|^2 + \sigma_{\bar{v} y} y^2 + \sigma_{\bar{v} \phi_0} |\phi_0(y)|^2 + \sigma_{\bar{v} \Phi} |\Phi(y)|_{\mathbb{F}}^2 \\ &\quad + \sigma_{\bar{v} \xi_2} \xi_2^2 + \sigma_{\bar{v} (\Xi)_2} |(\Xi)_2|^2 + \sigma_{\bar{v} \bar{\varepsilon}_2} \bar{\varepsilon}_2^2, \end{aligned} \quad (87)$$

where  $\sigma_{(\cdot)} > 0$  are constant.  $\diamond$

*Proof:* Since  $\ell_{b_m} s^m + \ell_{b_{m-1}} s^{m-1} + \dots + \ell_{b_0}$  is Hurwitz,  $A_{\ell_b} = S - e_m \ell_b^\top$  is also Hurwitz, and therefore there exist  $P_{\bar{v}} = P_{\bar{v}}^\top \succ 0$  and  $Q_{\bar{v}} = Q_{\bar{v}}^\top \succ 0$  such that  $A_{\ell_b}^\top P_{\bar{v}} + P_{\bar{v}} A_{\ell_b} + Q_{\bar{v}} = 0$ . Without loss of generality we assume that  $Q_{\bar{v}} = I + \bar{Q}_{\bar{v}}$ , where  $\bar{Q}_{\bar{v}} = \bar{Q}_{\bar{v}}^\top \succ 0$ . This condition can always be satisfied by scaling  $P_{\bar{v}}$ . Taking the time derivative of  $V_{\bar{v}}$  along the trajectories of (86) yields

$$\begin{aligned} \dot{V}_{\bar{v}} &= \bar{v}^\top (A_{\ell_b}^\top P_{\bar{v}} + P_{\bar{v}} A_{\ell_b}) \bar{v} + \bar{v}^\top P_{\bar{v}} g_{\bar{v}} + g_{\bar{v}}^\top P_{\bar{v}} \bar{v} \\ &= -\bar{v}^\top (I + \bar{Q}_{\bar{v}}) \bar{v} + 2\bar{v}^\top P_{\bar{v}} g_{\bar{v}}. \end{aligned} \quad (88)$$

Note that in  $g_{\bar{v}}(y, \xi, \Xi, \bar{\varepsilon}_2, t)$  all the coefficients coupled with the inputs  $y, \phi_{0,i}(y), \phi_{i,j}(y), \xi_2, (\Xi)_{j2}, \bar{\varepsilon}_2$  are bounded. Thus, by Young's inequality, the condition

$$-\bar{v}^\top \bar{Q}_{\bar{v}} \bar{v} + 2\bar{v}^\top P_{\bar{v}} g_{\bar{v}} - (\sigma_{\bar{v}y} y^2 + \sigma_{\bar{v}\phi_0} |\phi_0(y)|^2 + \sigma_{\bar{v}\Phi} |\Phi(y)|_{\mathbb{F}}^2 + \sigma_{\bar{v}\xi_2} \xi_2^2 + \sigma_{\bar{v}(\Xi)_2} |(\Xi)_2|^2 + \sigma_{\bar{v}\bar{\varepsilon}_2} \bar{\varepsilon}_2^2) \leq 0 \quad (89)$$

holds for some constant  $\sigma_{(\cdot)} > 0$ . Substituting (89) into (88) yields (87), which completes the proof. ■

Having established the ISS properties of (76), (77) and the zero dynamics of (78), we proceed to the *backstepping* design on the chain of integrators (79). Define the error variables

$$z_1 = y, \quad (90)$$

$$z_i = v_{m,i} - \alpha_{i-1}, \quad i = 2, \dots, \rho, \quad (91)$$

the tuning functions

$$\tau_1 = (\omega - \hat{\rho} \bar{\alpha}_1 e_1) z_1, \quad (92)$$

$$\tau_i = \tau_{i-1} - \frac{\partial \alpha_{i-1}}{\partial y} \omega z_i, \quad i = 2, \dots, \rho, \quad (93)$$

the virtual control laws

$$\alpha_1 = \hat{\rho} \bar{\alpha}_1 = -\hat{\rho} \kappa z_1, \quad (94)$$

$$\alpha_2 = -\hat{b}_m z_1 - (c_2 + \zeta_2) z_2 + \beta_2 + \frac{\partial \alpha_1}{\partial \theta} \Gamma_\theta \tau_2, \quad (95)$$

$$\alpha_i = -z_{i-1} - (c_i + \zeta_i) z_i + \beta_i + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma_\theta \tau_i - \sum_{j=2}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma_\theta \frac{\partial \alpha_{i-1}}{\partial y} \omega z_j, \quad i = 3, \dots, \rho, \quad (96)$$

with

$$\kappa = c_1 + \frac{\epsilon_{\hat{\theta}}}{2} |\hat{\theta}|^2 + \hat{\zeta}_y + \hat{\zeta}_{\phi_0} |\bar{\phi}_0(y)|^2 + \hat{\zeta}_{\Phi} |\bar{\Phi}(y)|_{\mathbb{F}}^2, \quad (97)$$

$$\zeta_2 = \frac{1}{2\epsilon_{\hat{\theta}}} + \frac{\rho \delta_{\Delta_{b_m}}}{2\epsilon_{\Delta_{b_m}}} + \frac{1}{2} \left( \frac{\partial \alpha_1}{\partial y} \right)^2 \times \left( \epsilon_{\Delta_{b_m}} \delta_{\Delta_{b_m}} (\hat{\rho}^2 \kappa^2 + 1) + \epsilon_{\Delta_{\bar{\theta}}} \delta_{\Delta_{\bar{\theta}}} + \epsilon_{Y_{\varepsilon_2}} + \epsilon_{\bar{\varepsilon}_2} \right), \quad (98)$$

$$\zeta_i = \frac{1}{2} \left( \frac{\partial \alpha_{i-1}}{\partial y} \right)^2 \times \left( \epsilon_{\Delta_{b_m}} \delta_{\Delta_{b_m}} (\hat{\rho}^2 \kappa^2 + 1) + \epsilon_{\Delta_{\bar{\theta}}} \delta_{\Delta_{\bar{\theta}}} + \epsilon_{Y_{\varepsilon_2}} + \epsilon_{\bar{\varepsilon}_2} \right), \quad (99)$$

$$\begin{aligned} \beta_i &= \frac{\partial \alpha_{i-1}}{\partial y} (\omega_0 + \omega^\top \hat{\theta}) + \frac{\partial \alpha_{i-1}}{\partial \xi} (A_k \xi + ky + \phi_0) \\ &+ \sum_{j=1}^q \frac{\partial \alpha_{i-1}}{\partial (\Xi^\top)_j} (A_k (\Xi^\top)_j + (\Phi^\top)_j) + k_i v_{m,1} \\ &+ \sum_{j=1}^{m+i-1} \frac{\partial \alpha_{i-1}}{\partial \lambda_j} (-k_j \lambda_1 + \lambda_{j+1}) + \frac{\partial \alpha_{i-1}}{\partial \hat{\rho}} \hat{\rho} \\ &+ \frac{\partial \alpha_{i-1}}{\partial \hat{\zeta}_y} \hat{\zeta}_y + \frac{\partial \alpha_{i-1}}{\partial \hat{\zeta}_{\phi_0}} \hat{\zeta}_{\phi_0} + \frac{\partial \alpha_{i-1}}{\partial \hat{\zeta}_{\Phi}} \hat{\zeta}_{\Phi}, \end{aligned} \quad (100)$$

$$i = 2, \dots, \rho,$$

the control law

$$u = \frac{1}{g(y)} (\alpha_\rho - v_{m,\rho+1}), \quad (101)$$

and the parameter update laws

$$\dot{\hat{\rho}} = \gamma_\rho \text{sgn}(\ell_{b_m}) \kappa z_1^2, \quad (102)$$

$$\dot{\hat{\zeta}}_y = \gamma_{\zeta_y} z_1^2, \quad \dot{\hat{\zeta}}_{\phi_0} = \gamma_{\zeta_{\phi_0}} |\phi_0|^2, \quad \dot{\hat{\zeta}}_{\Phi} = \gamma_{\zeta_{\Phi}} |\Phi|_{\mathbb{F}}^2, \quad (103)$$

$$\dot{\hat{\theta}} = \Gamma_\theta \tau_\rho, \quad (104)$$

where  $c_i > 0, i = 1, \dots, \rho, \epsilon_{(\cdot)} > 0, \gamma_{(\cdot)} > 0, \Gamma_\theta = \Gamma_\theta^\top \succ 0, \bar{\theta}(t) = \frac{b_m(t)}{\ell_{b_m}} \ell_\theta$ , and  $\Delta_{\bar{\theta}} = \bar{\theta}(t) - \ell_\theta$ . In the definition of  $\kappa, \bar{\phi}_0(y), \bar{\Phi}(y)$  are defined such that  $\phi_0(y) = \bar{\phi}_0(y)y, \Phi(y) = \bar{\Phi}(y)y$ , which is feasible due to Remark 9. Moreover, the initial value of the parameter estimates are selected such that  $\hat{\rho}(0) > 0, \hat{\zeta}_{(\cdot)}(0) > 0$ .

*Remark 14:* We use dynamically updated ‘‘estimates’’  $\hat{\zeta}_{(\cdot)}$  as the coefficient of the additional damping terms due to the convention mentioned in Remark 3, since the required damping coefficients are in general hard to compute (this fact is indicated by the proof of Proposition 3 that follows) and not vanishing even when all system parameters are constant. Meanwhile, thanks to these adaptive damping terms, we do not need to know  $\delta_{\Delta_{\bar{\theta}}}$  for a reason similar to what is explained in Remark 13. ◊

*Proposition 3:* Consider the adaptive controller described by equations (90)–(104) for the system described by equations (76)–(79). Suppose Assumptions 2, 3, and 4 hold. Then the closed-loop signals  $z, \bar{x}, \bar{\varepsilon}, \xi, \Xi, \bar{v}, \hat{\theta}, \hat{\rho}$ , and  $\hat{\zeta}_{(\cdot)}$  are bounded. ◊

*Proof:* We first analyze the *backstepping* error variables  $z_i$  step by step.

*Step 1.* Consider the dynamics of  $z_1$ , which are described by

$$\begin{aligned} \dot{z}_1 &= \frac{b_m}{\ell_{b_m}} (\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2) + \bar{\omega}^\top \frac{b_m}{\ell_{b_m}} \ell_\theta + b_m v_{m,2} \\ &= (\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2) + \bar{\omega}^\top \bar{\theta} + \ell_{b_m} v_{m,2} \\ &\quad + \Delta_{\frac{b_m}{\ell_{b_m}}} (\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2) + \bar{\omega}^\top \Delta_{\bar{\theta}} + \Delta_{b_m} v_{m,2} \\ &= (\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2) + \bar{\omega}^\top \hat{\theta} + \bar{\alpha}_1 + b_m z_2 \\ &\quad + \bar{\omega}^\top (\ell_\theta - \hat{\theta}) - \ell_{b_m} \left( \frac{1}{\ell_{b_m}} - \hat{\rho} \right) \bar{\alpha}_1 \\ &\quad + \Delta_{\frac{b_m}{\ell_{b_m}}} (\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2) + \bar{\omega}^\top \Delta_{\bar{\theta}} + \Delta_{b_m} \hat{\rho} \bar{\alpha}_1, \end{aligned} \quad (105)$$

where  $\Delta_{\frac{b_m}{\ell_{b_m}}}(t) = \frac{b_m(t)}{\ell_{b_m}} - 1$  (recall also that  $\bar{\theta}(t) = \frac{b_m(t)}{\ell_{b_m}} \ell_\theta$  and  $\Delta_{\bar{\theta}} = \bar{\theta}(t) - \ell_\theta$ ). Note that  $z_2 = v_{m,2} - \hat{\rho} \bar{\alpha}_1$  and

$$b_m z_2 = \hat{b}_m z_2 + (\ell_{b_m} - \hat{b}_m) z_2 + \Delta_{b_m} z_2, \quad (106)$$

which yields

$$\begin{aligned} &\bar{\omega}^\top (\ell_\theta - \hat{\theta}) + b_m z_2 \\ &= (\omega - \hat{\rho} \bar{\alpha}_1 e_1)^\top (\ell_\theta - \hat{\theta}) + \hat{b}_m z_2 + \Delta_{b_m} z_2. \end{aligned} \quad (107)$$

Considering  $V_{z_1} = \frac{1}{2}z_1^2$  and taking the time derivative of  $V_{z_1}$  along the trajectories of (105) yields

$$\begin{aligned} \dot{V}_{z_1} = & z_1(\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2) + z_1\bar{\omega}^\top \hat{\theta} + \bar{\alpha}_1 z_1 + \hat{b}_m z_1 z_2 \\ & + \Delta_{\frac{b_m}{\bar{b}_m}}(\omega_0 + Y_{\varepsilon_2} + \bar{\varepsilon}_2)z_1 + z_1\bar{\omega}^\top \Delta_{\bar{\theta}} + \Delta_{b_m} z_1 z_2 \\ & + z_1(\omega - \hat{\rho}\bar{\alpha}_1 e_1)^\top (\ell_\theta - \hat{\theta}) - \ell_{b_m}(\ell_{\bar{b}_m}^{-1} - \hat{\rho})\bar{\alpha}_1 z_1 \\ & + \Delta_{b_m} \hat{\rho}\bar{\alpha}_1 z_1. \end{aligned} \quad (108)$$

Invoking Young's inequality several times yields

$$\begin{aligned} \dot{V}_{z_1} \leq & -\kappa z_1^2 + \left(\frac{\epsilon_{\hat{\theta}}}{2}|\hat{\theta}|^2 + \sigma_{z_1 y}\right)z_1^2 + \sigma_{z_1 \phi_0}|\phi_0|^2 \\ & + \sigma_{z_1 \Phi}|\Phi|_F^2 + \sigma_{z_1 \bar{\varepsilon}_2}\bar{\varepsilon}_2^2 + \sigma_{z_1 \xi_2}\xi_2^2 + \sigma_{z_1 (\Xi)_2}|(\Xi)_2|^2 \\ & + \sigma_{z_1 \bar{v}}\bar{v}^2 + \left(\frac{1}{2\epsilon_{\hat{\theta}}} + \frac{\delta_{b_m}}{2\epsilon_{\Delta_{b_m}}}\right)z_2^2 + R_1 - \Delta_{b_m} \hat{\rho}\bar{\alpha}_1 z_1^2, \end{aligned} \quad (109)$$

where  $R_1 = z_1(\omega - \hat{\rho}\bar{\alpha}_1 e_1)^\top (\ell_\theta - \hat{\theta}) - \ell_{b_m}(\ell_{\bar{b}_m}^{-1} - \hat{\rho})\bar{\alpha}_1 z_1$  consists of the remaining terms to be cancelled by the update law/tuning function design. Moreover, using the same argument as in Section II-B and Section II-C, we can show that  $-\Delta_{b_m} \hat{\rho}\bar{\alpha}_1 z_1^2 \leq 0$ , and therefore this term can be dropped hereafter.

*Step 2, ..., ρ.* Consider the sum of the functions  $V_{z_i} = \frac{1}{2}z_i^2$ ,  $i = 1, \dots, \rho$ , and take the time derivative of the sum along the trajectories of the system, which yields<sup>7</sup>

$$\begin{aligned} \sum_{i=1}^{\rho} \dot{V}_{z_i} = & \dot{V}_{z_1} - R_1 - \sum_{i=2}^{\rho} (c_i + \zeta_i)z_i^2 - \sum_{i=2}^{\rho} z_i \frac{\partial \alpha_{i-1}}{\partial y} \times \\ & (\bar{\varepsilon}_2 + Y_{\varepsilon_2} + \bar{\omega}^\top \Delta_{\bar{\theta}} + (\alpha_1 + z_2)\Delta_{b_m}) + R_\rho \\ \leq & \dot{V}_{z_1} - R_1 - \sum_{i=1}^{\rho} (c_i + \zeta_i)z_i^2 + \frac{(\rho-1)\delta_{\Delta_{b_m}}}{2\epsilon_{\Delta_{b_m}}}z_2^2 \\ & + \sum_{i=2}^{\rho} \frac{1}{2}z_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \\ & \left(\epsilon_{\Delta_{b_m}} \delta_{\Delta_{b_m}} (\hat{\rho}^2 \kappa^2 + 1) + \epsilon_{\Delta_{\bar{\theta}}} \delta_{\Delta_{\bar{\theta}}} + \epsilon_{Y_{\varepsilon_2}} + \epsilon_{\bar{\varepsilon}_2}\right) \\ & + \sum_{i=2}^{\rho} \sigma_{z_i y} z_1^2 + \sum_{i=2}^{\rho} \sigma_{z_i \phi_0} |\phi_0|^2 + \sum_{i=2}^{\rho} \sigma_{z_i \Phi} |\Phi|_F^2 \\ & + \sum_{i=2}^{\rho} \sigma_{z_i \bar{\varepsilon}_2} \bar{\varepsilon}_2^2 + \sum_{i=2}^{\rho} \sigma_{z_i \xi_2} \xi_2^2 + \sum_{i=2}^{\rho} \sigma_{z_i (\Xi)_2} |(\Xi)_2|^2 \\ & + \sum_{i=2}^{\rho} \sigma_{z_i \bar{v}} |\bar{v}|^2 + R_\rho \\ = & - \sum_{i=1}^{\rho} c_i z_i^2 - \hat{\zeta}_y z_1^2 - \hat{\zeta}_{\phi_0} |\phi_0|^2 - \hat{\zeta}_{\Phi} |\Phi|_F^2 \\ & + \sigma_{z_1 y} z_1^2 + \sigma_{z_1 \phi_0} |\phi_0|^2 + \sigma_{z_1 \Phi} |\Phi|_F^2 + \sigma_{z_1 \bar{\varepsilon}_2} \bar{\varepsilon}_2^2 \\ & + \sigma_{z_1 \xi_2} \xi_2^2 + \sigma_{z_1 (\Xi)_2} |(\Xi)_2|^2 + \sigma_{z_1 \bar{v}} |\bar{v}|^2 + R_\rho, \end{aligned} \quad (110)$$

<sup>7</sup>Some technical details such as cancellations related to the tuning function design are omitted as they are already well known and not directly related to the modifications for parameter variations. The readers can refer to [2] for these details.

where  $\sigma_{z(\cdot)} = \sum_{i=1}^{\rho} \sigma_{z_i(\cdot)} > 0$  and  $R_\rho = \tau_\rho^\top (\ell_\theta - \hat{\theta}) - \ell_{b_m}(\ell_{\bar{b}_m}^{-1} - \hat{\rho})\bar{\alpha}_1 z_1$  consists of the remaining terms to be cancelled by the update laws. Then considering the function  $V_z = \sum_{i=1}^{\rho} V_{z_i} + \frac{1}{2}|\ell_\theta - \hat{\theta}|_{\Gamma^{-1}}^2 + \frac{|\ell_{b_m}|}{2\gamma_{\hat{\rho}}}|\ell_{\bar{b}_m}^{-1} - \hat{\rho}|^2$  and taking its time derivative along the system trajectories yields

$$\begin{aligned} \dot{V}_z \leq & - \sum_{i=1}^{\rho} c_i z_i^2 - \hat{\zeta}_y z_1^2 - \hat{\zeta}_{\phi_0} |\phi_0|^2 - \hat{\zeta}_{\Phi} |\Phi|_F^2 \\ & + \sigma_{z_1 y} z_1^2 + \sigma_{z_1 \phi_0} |\phi_0|^2 + \sigma_{z_1 \Phi} |\Phi|_F^2 + \sigma_{z_1 \bar{\varepsilon}_2} \bar{\varepsilon}_2^2 \\ & + \sigma_{z_1 \xi_2} \xi_2^2 + \sigma_{z_1 (\Xi)_2} |(\Xi)_2|^2 + \sigma_{z_1 \bar{v}} |\bar{v}|^2. \end{aligned} \quad (111)$$

Consider now the Lyapunov function candidate  $V = V_z + \gamma_{V_{\bar{x}}} V_{\bar{x}} + \gamma_{V_{\bar{\varepsilon}}} V_{\bar{\varepsilon}} + \gamma_{V_{\xi}} V_{\xi} + \gamma_{V_{\Xi}} V_{\Xi} + \gamma_{V_{\bar{v}}} V_{\bar{v}} + \frac{1}{2\gamma_{\zeta_y}} (\zeta_y - \hat{\zeta}_y)^2 + \frac{1}{2\gamma_{\zeta_{\phi_0}}} (\zeta_{\phi_0} - \hat{\zeta}_{\phi_0})^2 + \frac{1}{2\gamma_{\zeta_{\Phi}}} (\zeta_{\Phi} - \hat{\zeta}_{\Phi})^2$ , where  $\gamma_{V_{\bar{x}}} = \sigma_{\bar{x}\rho+1}(\sigma_{z\bar{\varepsilon}_2} + \sigma_{\bar{v}\bar{\varepsilon}_2}\sigma_{z\bar{v}})$ ,  $\gamma_{V_{\bar{\varepsilon}}} = \sigma_{z\bar{\varepsilon}_2} + \sigma_{\bar{v}\bar{\varepsilon}_2}\sigma_{z\bar{v}}$ ,  $\gamma_{V_{\xi}} = \sigma_{z\xi_2} + \sigma_{\bar{v}\xi_2}\sigma_{z\bar{v}}$ ,  $\gamma_{V_{\Xi}} = \sigma_{z(\Xi)_2} + \sigma_{\bar{v}(\Xi)_2}\sigma_{z\bar{v}}$ ,  $\gamma_{V_{\bar{v}}} = \sigma_{z\bar{v}}$  are the scaling coefficients of the corresponding partial Lyapunov function candidate, and  $\zeta_y = \sigma_{zy} + \gamma_{V_{\bar{x}}}\sigma_{\bar{x}y} + \gamma_{V_{\bar{\varepsilon}}}\sigma_{\bar{\varepsilon}y} + \gamma_{V_{\xi}}\sigma_{\xi y} + \gamma_{V_{\bar{v}}}\sigma_{\bar{v}y}$ ,  $\zeta_{\phi_0} = \sigma_{z\phi_0} + \gamma_{V_{\bar{x}}}\sigma_{\bar{x}\phi_0} + \gamma_{V_{\bar{\varepsilon}}}\sigma_{\bar{\varepsilon}\phi_0} + \gamma_{V_{\xi}}\sigma_{\xi\phi_0} + \gamma_{V_{\bar{v}}}\sigma_{\bar{v}\phi_0}$ ,  $\zeta_{\Phi} = \sigma_{z\Phi} + \gamma_{V_{\bar{x}}}\sigma_{\bar{x}\Phi} + \gamma_{V_{\bar{\varepsilon}}}\sigma_{\bar{\varepsilon}\Phi} + \gamma_{V_{\Xi}}\sigma_{\Xi\Phi} + \gamma_{V_{\bar{v}}}\sigma_{\bar{v}\Phi}$  are the required damping coefficients to be compensated by  $\hat{\zeta}_y$ ,  $\hat{\zeta}_{\phi_0}$ , and  $\hat{\zeta}_{\Phi}$ , respectively. Taking the time derivative of  $V$  along the trajectories of the system yields

$$\begin{aligned} \dot{V} \leq & - \sum_{i=1}^{\rho} c_i z_i^2 + (\zeta_y - \hat{\zeta}_y)(z_1^2 - \gamma_{\zeta_y}^{-1} \hat{\zeta}_y) \\ & + (\zeta_{\phi_0} - \hat{\zeta}_{\phi_0})(|\phi_0|^2 - \gamma_{\zeta_{\phi_0}}^{-1} \hat{\zeta}_{\phi_0}) \\ & + (\zeta_{\Phi} - \hat{\zeta}_{\Phi})(|\Phi|_F^2 - \gamma_{\zeta_{\Phi}}^{-1} \hat{\zeta}_{\Phi}) = - \sum_{i=1}^{\rho} c_i z_i^2 \leq 0. \end{aligned} \quad (112)$$

Hence  $z$ ,  $\bar{x}$ ,  $\bar{\varepsilon}$ ,  $\xi$ ,  $\Xi$ ,  $\bar{v}$ ,  $\hat{\theta}$ ,  $\hat{\rho}$ , and  $\hat{\zeta}_{(\cdot)}$  are bounded, which completes the proof. ■

We should not forget that the invariance-like proof of asymptotic output regulation requires the boundedness of  $\varepsilon$ . In Proposition 3 we have proved the boundedness of  $\bar{\varepsilon}$  after the change of coordinates described by Algorithm 1. However, it is not easy to directly imply the boundedness of  $\varepsilon$  since Algorithm 1 involves the time derivatives of  $y$ ,  $\phi_{0,i}(y)$ , and  $\phi_{i,j}(y)$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, q$ , the boundedness of which is difficult to conclude. Recall that these time derivatives are present because  $u$  has to be decomposed at the design stage with the help of the inverse dynamics. Now that we have completed the design, it is more convenient to directly use the boundedness of  $u$  for concluding the boundedness of  $\varepsilon$ , provided that we can first prove the boundedness of  $\lambda$ , as shown in what follows.

*Theorem 1:* Consider the system described by the equations (76)–(79) and the adaptive controller described by the equations (90)–(104). Suppose the same assumptions hold as in Proposition 3. Then, all closed-loop signals are bounded and  $\lim_{t \rightarrow +\infty} y(t) = 0$ , that is, asymptotic output regulation to 0 is achieved. ◊

*Proof:* We can directly conclude the boundedness of  $\xi$ ,  $\Xi$ ,  $\hat{\theta}$ ,  $\hat{\rho}$ , and  $\hat{\zeta}_{(\cdot)}$  from Proposition 3, and therefore, in this

proof we only need to establish the boundedness of  $\lambda$ ,  $\varepsilon$ , and  $x$ . First recall (50), which yields

$$\begin{bmatrix} v_{0,2} \\ \vdots \\ v_{m,2} \\ \vdots \\ v_{m,\rho} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ * & * & 1 & \ddots & 0 \\ * & * & * & \ddots & 0 \\ * & * & * & * & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}, \quad (113)$$

where “\*” represents terms that are not important. Note that  $v_{0,2} = \lambda_2$  is bounded due to the boundedness of  $\bar{v}$  and thus  $\lambda_2$  is also bounded. Note that by Vieta’s formula,

$$k_1 = -\text{tr}(A_k) \quad (114)$$

and since  $A_k$  is Hurwitz,  $\text{tr}(A_k) < 0$ . Hence  $k_1 > 0$ . Consider the dynamics of the first state variable of the input filter (46), that is,  $\dot{\lambda}_1 = -k_1\lambda_1 + \lambda_2$  with a bounded input  $\lambda_2$ . Thus  $\lambda_1$  is also bounded due to the bounded-input bounded-output property. Rewriting (113) yields

$$\begin{bmatrix} v_{0,2} \\ \vdots \\ v_{m,2} \\ \vdots \\ v_{m,\rho} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \ddots & 0 \\ * & * & \ddots & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} + \begin{bmatrix} 0 \\ * \\ * \\ * \end{bmatrix} \lambda_1. \quad (115)$$

Since  $\bar{v}$ ,  $\lambda_1$ , and  $z_1$  (or  $y$ ) are bounded, we can conclude that  $\lambda_3, \dots, \lambda_{m+1}$  are bounded by exploiting the lower-triangular structure of the matrix in (115). Since  $\lambda_1, \dots, \lambda_{m+1}$  are bounded, by Proposition 3,  $\alpha_1$  is bounded. Note that  $z_2$  is bounded, thus  $v_{m,2}$  is bounded, which further guarantees the boundedness of  $\lambda_{m+2}$  due to (115). In the same spirit, the boundedness of  $\lambda_{m+3}, \dots, \lambda_n$  can be established in a recursive way similar to [2], which proves that  $\lambda$  is bounded.

The boundedness of  $\lambda$  and Proposition 3 yields the boundedness of  $\alpha_\rho$  and  $v_{m,\rho+1}$ , and therefore the boundedness of  $u$ , which further proves the boundedness of  $\varepsilon$  along with the boundedness of  $y$  due to (52). Since  $x = \xi + \Omega^\top \ell_\theta + \varepsilon$ , we can conclude that  $x$  is also bounded. Finally, consider (112) and note that  $\dot{z}$  is also bounded due to the boundedness of the parameters and all other closed-loop state variables, hence Barbalat’s lemma yields  $\lim_{t \rightarrow +\infty} z(t) = 0$ , and also  $\lim_{t \rightarrow +\infty} y(t) = 0$ , which completes the proof. ■

*Remark 15:* Using the fact that  $\lim_{t \rightarrow +\infty} z(t) = 0$  we can proceed to prove the convergence of  $\xi$ ,  $\Xi$ ,  $\lambda$ ,  $\varepsilon$  and  $x$  to 0 by exploiting the converging-input converging-output property of the corresponding subsystems or the dependency on converging signals. ◇

*Remark 16:* We do not implement any *projection operation* in the parameter update laws and therefore the proposed method cannot guarantee boundedness of the parameter estimates in the presence of noise or additive disturbances. The reason for this is that we intend to present a plain scheme that precisely and concisely shows the spirit of the *congelation of variables* method, which exploits nonlinear damping and small-gain-like design to cope with time-varying parameters,

instead of exploiting robust modifications to the update laws, as done in some of the aforementioned works. This, however, does not mean that the proposed method is not compatible with such robust modifications. In fact one can replace the classical *adaptive backstepping* procedure with the *adaptive backstepping* with *projection* method [34] to guarantee boundedness of the parameter estimates. This, however, is not pursued in the paper. ◇

## IV. SIMULATIONS

To compare the proposed controller with the classical adaptive controller, consider the nonlinear system described by the equations

$$\begin{aligned} \dot{x}_1 &= a_1(t)x_1^2 + x_2, \\ \dot{x}_2 &= a_2(t)x_1^2 + x_3 + b_1(t)u, \\ \dot{x}_3 &= a_3(t)x_1^2 + b_0(t)u, \\ y &= x_1, \end{aligned} \quad (116)$$

where the time-varying parameters are defined by

$$b_1(t) = 1 + 0.2 \sin(5t), \quad b_0(t) = 6 + \sin(20t), \quad (117)$$

$$a(t) = [1, 1, 1]^\top - 20 \operatorname{sgn}\left(\frac{\partial \alpha_1}{\partial y} z_2\right) \frac{(\bar{\omega})_{3:5}}{|(\bar{\omega})_{3:5}|}, \quad (118)$$

with  $(\bar{\omega})_{3:5} = [(\bar{\omega})_3, (\bar{\omega})_4, (\bar{\omega})_5]^\top$ . Each of these parameters comprise of a constant nominal part and a time-varying ( $a(t)$  is also state-dependent) part designed to destabilize the system. It is not difficult to verify that Assumption 4 is satisfied since  $\frac{b_0}{b_1} \geq \frac{5}{1.2} > 0$ . Consider now two controllers: Controller 1 is the classical *adaptive backstepping* controller, and Controller 2 is the controller proposed in this paper. To compare fairly, set the common controller parameters as  $c_1 = c_2 = 1$ ,  $\Gamma_\theta = I$ ,  $\gamma_\rho = 1$  and the initial conditions  $\hat{\theta}(0) = 0$ ,  $\hat{\rho}(0) = 1$  for both controllers. Each of the controllers uses an identical set of *K-filters* given by (44)–(46). The filter gains are obtained by solving the algebraic Riccati equation (66) with  $Q_{\bar{\varepsilon}} = 10$  and  $\gamma_{\bar{\varepsilon}} = 100$ , and the filter states are initialized to 0. For the parameters solely used in Controller 2, set  $\gamma_{(\cdot)} = 1$ ,  $\epsilon_{(\cdot)} = 1$ ,  $\delta_{\Delta_{b_m}} = 0.2$ ,  $\epsilon_{\Delta_{\bar{\theta}}} \delta_{\Delta_{\bar{\theta}}} = 1$  (note that one does not need to know  $\delta_{\Delta_{\bar{\theta}}}$  as mentioned in Remark 14), and set the initial conditions to  $\hat{\zeta}_y(0) = 2$ ,  $\hat{\zeta}_\Phi(0) = 1$  (non-zero initial conditions provide additional damping from the beginning to counteract the parameter variations). The initial condition for the system state is set to  $x(0) = [1, 0, 0]^\top$ .

Two scenarios are explored: in the first scenario, each controller is applied to a separate yet identical system while the state-dependent time-varying parameters of both systems are generated by the closed-loop system controlled by Controller 1, and the second scenario has the same setting as the first scenario except that the state-dependent time-varying parameters are generated by the closed-loop system controlled by Controller 2. In both scenarios, “Baseline” results are the responses of the closed-loop system with constant nominal parameters controlled by Controller 1, which demonstrate the performance of the classical controller in the case of constant parameters. The responses of the system state variables in each scenario are plotted in Fig. 4 and Fig. 6, respectively,

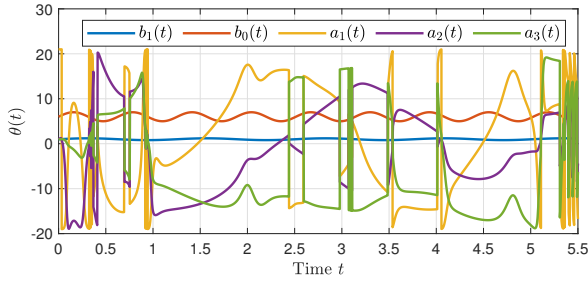


Fig. 3. Scenario 1: time-varying parameters generated by the closed-loop system controlled by Controller 1.

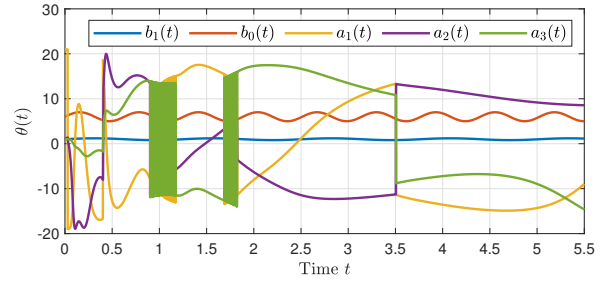


Fig. 5. Scenario 2: time-varying parameters generated by the closed-loop system controlled by Controller 2.

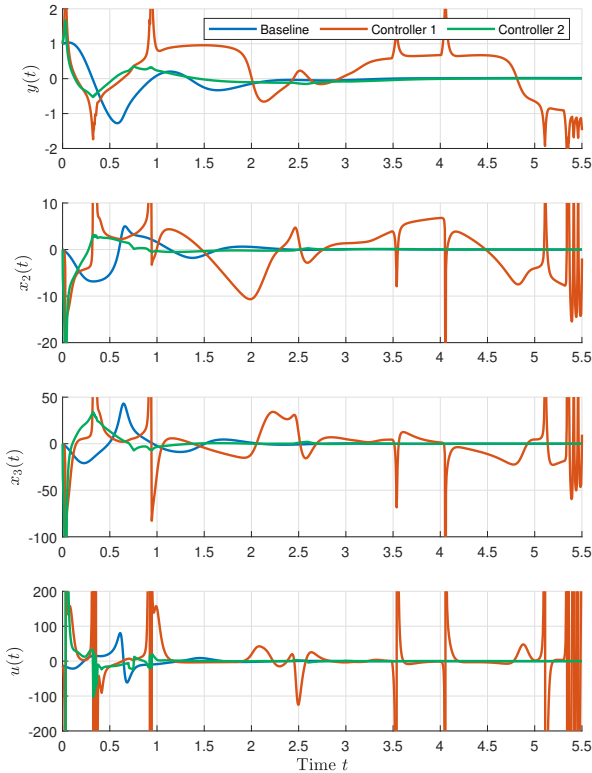


Fig. 4. Scenario 1: time histories of the system state and control effort driven by different controllers and the parameters shown in Fig. 3.

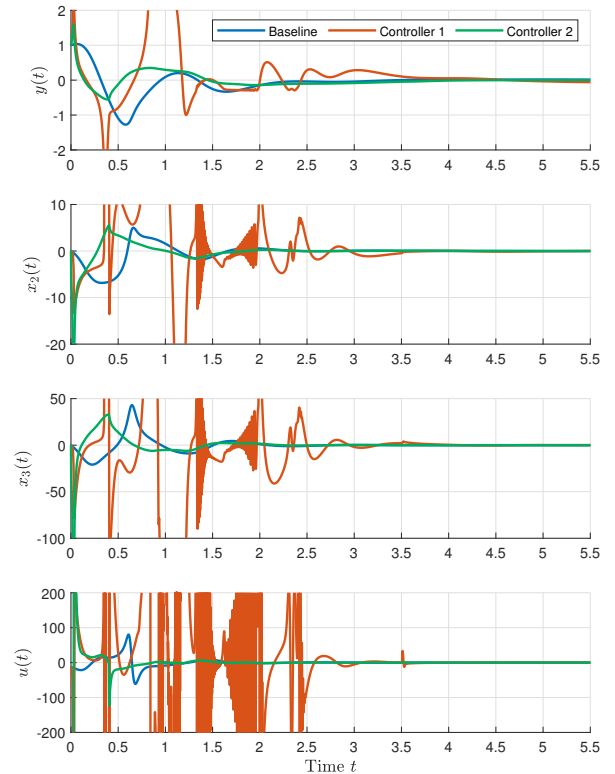


Fig. 6. Scenario 2: time histories of the system state and control effort driven by different controllers and the parameters shown in Fig. 5.

and the parameters used in each scenario are shown in Fig. 3 and Fig. 5, respectively. These results show that the proposed controller (Controller 2) outperforms the classical controller (Controller 1) in the presence of time-varying parameters and effectively prevents the oscillations caused by parameter variations.

## V. CONCLUSIONS AND FUTURE WORK

This paper discusses a new adaptive control scheme to cope with time-varying parameters based on the so-called *congelation of variables* method. Several examples with full-state feedback, including scalar systems with time-varying

parameters in the feedback path and in the input path,  $n$ -dimensional systems with unmatched time-varying parameters, to illustrate the preliminary results, are considered. The output regulation problem for a more general class of nonlinear systems, to which the previous results are not directly applicable due to the coupling between the input and the time-varying perturbation, is then discussed. To solve this problem, ISS of the inverse dynamics, a counterpart of minimum-phasesness in classical adaptive control schemes, is exploited to convert the coupling between the input and the time-varying perturbation into the coupling between the output and the time-varying perturbation. A set of  $K$ -filters that guarantee ISS

state estimation error dynamics are also designed to replace the unmeasured state variables. Finally, a controller with adaptively updated damping terms is designed to guarantee convergence of the output to 0 and boundedness of all closed-loop signals, via a small-gain-like analysis. The simulation results show performance improvement resulting from the use of the proposed controller compared with the classical adaptive controller in the presence of time-varying parameters, even if the time-varying parameters are both unknown and persistently varying.

In future work the knowledge of the internal model of time-varying parameters can be exploited to avoid an over-conservative controller design.

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### REFERENCES

- [1] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. Prentice Hall, 1989.
- [2] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and Adaptive Control Design*, 1st ed. New York, NY, USA: John Wiley & Sons, Inc, 1995.
- [3] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. PTR Prentice-Hall Upper Saddle River, NJ, 1996.
- [4] G. Tao, *Adaptive Control Design and Analysis*. John Wiley & Sons, 2003.
- [5] A. Astolfi, D. Karagiannis, and R. Ortega, *Nonlinear and Adaptive Control with Applications*. Springer Science & Business Media, 2007.
- [6] G. C. Goodwin and E. K. Teoh, "Adaptive control of a class of linear time varying systems," *IFAC Proc. Volumes*, vol. 16, no. 9, pp. 1–6, 1983.
- [7] J. G. Rueda-Escobedo and J. A. Moreno, "Discontinuous gradient algorithm for finite-time estimation of time-varying parameters," *Int. J. of Control*, vol. 89, no. 9, pp. 1838–1848, 2016.
- [8] H. Ríos, D. Efimov, J. A. Moreno, W. Perruquetti, and J. G. Rueda-Escobedo, "Time-varying parameter identification algorithms: Finite and fixed-time convergence," *IEEE Trans. Automat. Contr.*, vol. 62, no. 7, pp. 3671–3678, 2017.
- [9] G. Kreisselmeier, "Adaptive control of a class of slowly time-varying plants," *Syst. Control Lett.*, vol. 8, no. 2, pp. 97–103, 1986.
- [10] R. H. Middleton and G. C. Goodwin, "Adaptive control of time-varying linear systems," *IEEE Trans. Automat. Contr.*, vol. 33, no. 2, pp. 150–155, 1988.
- [11] K. Tsakalis and P. A. Ioannou, "Adaptive control of linear time-varying plants," *Automatica*, vol. 23, no. 4, pp. 459–468, 1987.
- [12] Y. Zhang and P. A. Ioannou, "Adaptive control of linear time varying systems," in *Proc. 35th IEEE Conf. on Decision and Control*, vol. 1. IEEE, 1996, pp. 837–842.
- [13] Y. Zhang, B. Fidan, and P. A. Ioannou, "Backstepping control of linear time-varying systems with known and unknown parameters," *IEEE Trans. Automat. Contr.*, vol. 48, no. 11, pp. 1908–1925, 2003.
- [14] R. Marino and P. Tomei, "Global adaptive output-feedback control of nonlinear systems. I. Linear parameterization," *IEEE Trans. Automat. Contr.*, vol. 38, no. 1, pp. 17–32, 1993.
- [15] —, "An adaptive output feedback control for a class of nonlinear systems with time-varying parameters," *IEEE Trans. Automat. Contr.*, vol. 44, no. 11, pp. 2190–2194, 1999.
- [16] —, "Adaptive control of linear time-varying systems," *Automatica*, vol. 39, no. 4, pp. 651–659, 2003.
- [17] K. Chen and A. Astolfi, "Adaptive control of linear systems with time-varying parameters," in *Proc. 2018 Amer. Control Conf.* IEEE, pp. 80–85.
- [18] —, "I&I adaptive control for systems with varying parameters," in *Proc. 57th IEEE Conf. on Decision and Control*. IEEE, 2018, pp. 2205–2210.
- [19] —, "Output-feedback adaptive control for systems with time-varying parameters," *IFAC-PapersOnLine*, vol. 52, no. 16, pp. 586–591, 2019.
- [20] —, "Output-feedback I&I adaptive control for linear systems with time-varying parameters," in *Proc. 58th IEEE Conf. on Decision and Control*. IEEE, 2019, pp. 1965–1970.
- [21] —, "Adaptive control for nonlinear systems with time-varying parameters and control coefficient," in *Proc. of IFAC World Congr. 2020*. IFAC. [Online]. Available: <https://www.rayseven.com/rt7/runtime/vdi/ifac2020/files/finalpapers/635/attachments/0834.pdf?1593798616>
- [22] G. C. Goodwin and D. Q. Mayne, "A parameter estimation perspective of continuous time model reference adaptive control," *Automatica*, vol. 23, no. 1, pp. 57–70, 1987.
- [23] J.-B. Pomet and L. Praly, "Adaptive nonlinear regulation: Estimation from the Lyapunov equation," *IEEE Trans. Automat. Contr.*, vol. 37, no. 6, pp. 729–740, 1992.
- [24] J. Zhou and C. Wen, *Adaptive Backstepping Control of Uncertain Systems: Nonsmooth Nonlinearities, Interactions or Time-Variations*. Springer, 2008.
- [25] R. Marino and P. Tomei, "Robust adaptive regulation of linear time-varying systems," *IEEE Trans. Automat. Contr.*, vol. 45, no. 7, pp. 1301–1311, 2000.
- [26] A. M. Annaswamy and K. S. Narendra, "Adaptive control of simple time-varying systems," in *Proc. 28th IEEE Conf. on Decision and Control*. IEEE, 1989, pp. 1014–1018.
- [27] W. Lin and C. Qian, "Adaptive control of nonlinearly parameterized systems: the smooth feedback case," *IEEE Trans. Automat. Contr.*, vol. 47, no. 8, pp. 1249–1266, 2002.
- [28] C. Wang and L. Guo, "Adaptive cooperative tracking control for a class of nonlinear time-varying multi-agent systems," *J. of the Franklin Inst.*, vol. 354, no. 15, pp. 6766–6782, 2017.
- [29] Merriam-Webster Staff, *Merriam-Webster's Collegiate Dictionary*. Merriam-Webster, 2004.
- [30] J. Nestruev, *Smooth Manifolds and Observables*. Springer Science & Business Media, 2006.
- [31] Z. Li and M. Krstic, "Optimal-design of adaptive tracking controllers for nonlinear-systems," *Automatica*, vol. 33, no. 8, pp. 1459–1473, 1997.
- [32] G. Kreisselmeier, "Adaptive observers with exponential rate of convergence," *IEEE Trans. Automat. Contr.*, vol. 22, no. 1, pp. 2–8, 1977.
- [33] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, 2002.
- [34] F. Ikhouane and M. Krstic, "Adaptive backstepping with parameter projection: Robustness and asymptotic performance," *Automatica*, vol. 34, no. 4, pp. 429–435, 1998.



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