# An analytic BPHZ theorem for Regularity Structures

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## Abstract

We prove a general theorem on the stochastic convergence of appropriately renormalized models arising from nonlinear stochastic PDEs. The theory of regularity structures gives a fairly automated framework for studying these problems but previous works had to expend significant effort to obtain these stochastic estimates in an ad-hoc manner.

In contrast, the main result of this article operates as a black box which automatically produces these estimates for nearly all of the equations that fit within the scope of the theory of regularity structures. Our approach leverages multi-scale analysis strongly reminiscent to that used in constructive field theory, but with several significant twists. These come in particular from the presence of "positive renormalizations" caused by the recentering procedure proper to the theory of regularity structure, from the difference in the action of the group of possible renormalization operations, as well as from the fact that we allow for non-Gaussian driving fields.

One rather surprising fact is that although the "canonical lift" is not continuous on any Hölder-type space containing the noise (which is why renormalization is required), we show that the "BPHZ lift" where the renormalization constants are computed using the formula given in [BHZ16], *is* continuous in law when restricted to a class of stationary random fields with sufficiently many moments.

## Contents

## 1 Introduction

The theory of regularity structures [Hai14] presents a systematic and robust methodology for interpreting / solving a wide class of parabolic stochastic partial differential equations (SPDEs). One natural subclass of such SPDEs are equations of the type

$$(\partial_t - \mathcal{L})\varphi = F(\varphi, \zeta) , \tag{1.1}$$

where $\mathcal{L}$ is an elliptic differential operator (think of the standard Laplacian on $d$-dimensional space $\mathbf{R}^d$ or the $d$-dimensional torus $\mathbf{T}^d$) and $F(\cdot, \cdot)$ is a local non-linearity that depends on less derivatives of $\varphi$ than $\mathcal{L}$. The solution $\varphi(t, x)$ can be formally thought of as a random space-time field $\varphi : \mathbf{R}^{d+1} \to \mathbf{R}$, and $\zeta$ is a random space time field, the prototypical example being given by space-time white noise, the centred Gaussian generalised random field satisfying $\mathbf{E}[\zeta(t, x)\zeta(s, y)] = \delta(t - s)\delta(x - y)$.

Equations like (1.1) can formally be obtained as limits of the dynamics of classical models of statistical mechanics in suitable regimes. Two intensely studied examples are the KPZ equation appearing from the evolution of random interfaces,

where $F(\varphi, \zeta) = |\nabla\varphi|^2 + \zeta$, and the $\Phi_d^4$ equation appearing from limits of dynamical Ising models, where $F(\varphi, \zeta) = -\varphi^3 + \zeta$.

An important restriction on the class of equations that [Hai14] is able to treat is *local subcriticality* – a precise definition of this condition is fairly technical[1] but heuristically it states that all non-linear terms appearing on the right hand side of (1.1) behave like perturbations of the noise term as one rescales the equation towards smaller scales and that these perturbations can be written as multi-linear functionals of the driving noise $\zeta$.

An immediate obstacle to rigorously formulating these SPDEs is that the roughness of the driving noise $\zeta$ forces the solution $\varphi$ to live in a space of functions / distributions with insufficient regularity for $F(\varphi, \zeta)$ to have a canonical meaning. As with any other integration theory, the program of giving a rigorous solution theory to such SPDEs entails understanding how to define these solutions as limits of well-defined approximate solutions.

To obtain such well-defined approximations we can convolve the driving noise with some smooth approximate identity $\varrho_\varepsilon$ with $\varepsilon > 0$ and $\lim_{\varepsilon\downarrow 0} \varrho_\varepsilon(\cdot) = \delta(\cdot)$. It is then typically easy to obtain, for every realization of $\zeta$, a local existence theorem for

$$(\partial_t - \mathcal{L})\varphi_\varepsilon = F(\varphi_\varepsilon, \zeta_\varepsilon) , \tag{1.2}$$

where $\zeta_\varepsilon \overset{\text{def}}{=} \zeta * \varrho_\varepsilon$. Naively one would hope to define the solution of (1.1) via the limit $\varphi \overset{\text{def}}{=} \lim_{\varepsilon\downarrow 0} \varphi_\varepsilon$ – unfortunately it is generic for this limit to fail to exist and this signals the need to *renormalize*. The process of renormalization entails prescribing a scheme of modifying the nonlinearity $F$ at each value of $\varepsilon$, giving rise to a family of non-linearities $F_\varepsilon(\cdot, \cdot)$, such that if one considers the modified equations

$$(\partial_t - \mathcal{L})\hat{\varphi}_\varepsilon = F_\varepsilon(\hat{\varphi}_\varepsilon, \zeta_\varepsilon) , \tag{1.3}$$

then the space-time functions $\hat{\varphi}_\varepsilon$ will converge in probability, as $\varepsilon \downarrow 0$, to a non-trivial limiting space-time distribution $\varphi$. The need for renormalization can be seen very explicitly when one uses Picard iteration to replace the non-linearity $F$ with a formal expansion in terms of multilinear functionals of the linear solution.

We now give a detailed presentation of this for the specific case where $F(\varphi, \zeta) = \varphi^3 + \zeta$ and we work on the 3-dimensional torus. We write $G$ for the space-time Green's function of the differential operator $(\partial_t - \mathcal{L})$, recentered so that it averages to 0. Then, using Picard iteration, one obtains a formal expansion of the RHS of (1.2)

$$(\partial_t - \mathcal{L})\varphi_\varepsilon = \zeta_\varepsilon + (G * \zeta_\varepsilon)^3 + 3[G * (G * \zeta_\varepsilon)^3](G * \zeta_\varepsilon)^2$$
$$+ 3[G * (G * \zeta_\varepsilon)^3]^2(G * \zeta_\varepsilon) + [G * (G * \zeta_\varepsilon)^3]^3 + \cdots$$

where $*$ denotes space-time convolution. Observe that $G * \zeta_\varepsilon$ is nothing but the solution to the linear equation underlying (1.2). Some simple graphical notation

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[1] See [BHZ16, Sec. 5] for a formal definition with a wider scope (it also incorporates systems of equations) than the somewhat informal definition given in [Hai14].

borrowed from [Hai14][2] clarifies this expansion – using this for the first three terms of the RHS gives

$$(\partial_t - \mathcal{L})\varphi_\varepsilon = \bullet_\varepsilon + \vee_\varepsilon + 3 \vee_\varepsilon + \cdots . \tag{1.4}$$

As $\varepsilon \downarrow 0$ the second two objects on the RHS (and infinitely many others hiding in the "$\cdots$") will not converge for generic realizations of $\zeta$. However, one can switch to a stochastic point of view and see that the $\varepsilon \downarrow 0$ limits of the space-time functions

$$\widehat{\vee}_\varepsilon \overset{\text{def}}{=} \vee_\varepsilon - 3\, C_{\vee}^\varepsilon \, |_\varepsilon \tag{1.5}$$
$$\widehat{\vee}_\varepsilon \overset{\text{def}}{=} \vee_\varepsilon + 3\, (C_{\vee}^\varepsilon)^2 \, \vee_\varepsilon - C_{\vee}^\varepsilon \, \vee_\varepsilon - 3 C_{\vee}^\varepsilon \, \vee_\varepsilon - 3 C_{\vee}^\varepsilon \, |_\varepsilon$$

exist for *almost every* realization of $\zeta$ in suitable spaces of space-time distributions, provided that we set

$$C_{\vee}^\varepsilon \overset{\text{def}}{=} \mathbf{E}[\vee_\varepsilon(0)] \quad \text{and} \quad C_{\vee}^\varepsilon \overset{\text{def}}{=} \mathbf{E}[\vee_\varepsilon(0)] .$$

We also extended our graphical notation by setting

$$\vee_\varepsilon \overset{\text{def}}{=} (G * \zeta_\varepsilon)^2 , \quad \vee_\varepsilon \overset{\text{def}}{=} G * (G * \zeta_\varepsilon) , \quad \vee_\varepsilon \overset{\text{def}}{=} [G * (G * \zeta_\varepsilon)](G * \zeta_\varepsilon)^2 ,$$

etc. and used the notation $\tau_\varepsilon(0)$ to indicate we are evaluating the space-time function $\tau_\varepsilon$ at $z = 0$.

This $\varepsilon$-dependent subtraction of "counterterms" on individual terms of our formal expansion (1.4) to make each term convergent as $\varepsilon \downarrow 0$ is reminiscent of the *perturbative renormalization* developed for Quantum Field Theory (QFT)[3].

Making these individual terms convergent is only part of the procedure of perturbative renormalization, the other is rewriting the nonlinearity $F$ so that the needed counterterms are automatically generated in the Picard iteration. If, in (1.3), we set $F_\varepsilon(\varphi_\varepsilon, \zeta_\varepsilon) \overset{\text{def}}{=} \varphi_\varepsilon^3 - c_\varepsilon \varphi_\varepsilon + \zeta_\varepsilon$, where $c_\varepsilon \overset{\text{def}}{=} 3 C_{\vee}^\varepsilon + 9 C_{\vee}^\varepsilon$, then, generating an expansion like (1.4) for $\hat{\varphi}_\varepsilon$ and collecting terms appropriately, one will see

$$(\partial_t - \mathcal{L})\hat{\varphi}_\varepsilon = \bullet_\varepsilon + \widehat{\vee}_\varepsilon + 3 \widehat{\vee}_\varepsilon + \cdots , \tag{1.6}$$

where all the divergent terms we did not explicitly write down are also organized with counterterms so that (1.6) is an infinite expansion where each individual term, for almost every realization of $\zeta$, is convergent as $\varepsilon \downarrow 0$.

This analysis of formal expansions is an old story – a systematic procedure for choosing these counterterms is the BPHZ formalism [BP57, Hep69, Zim69] which can be adapted from perturbative QFT to "perturbative SPDE". Unfortunately, this analysis at the level of formal series falls far short of showing that once suitably renormalized, the actual solutions $\hat{\varphi}_\varepsilon$ converge to a limit $\hat{\varphi}$. This proved to be a

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[2]This graphical notation is more than a minor convenience – developing a formalism for seeing all these multilinear functionals of $\zeta$ as combinatorial objects allows us to describe various complicated operations with full precision in great generality.

[3]One important difference is that the perturbative renormalization in QFT is carried out on expectation values, or "amplitudes", not stochastic objects.

difficult problem until the publication of [Hai13, GIP15, Hai14] which was followed
by a substantial amount of progress in the field.

We now summarize the approach of the theory of regularity structures. To
motivate this approach we first observe that the $\varepsilon \downarrow 0$ limits of the left hand sides of
(1.5), which we denote by $\widehat{\psi}$ and $\widehat{\Psi}$, are probabilistic limits which are measurable
functions of $\zeta$. Adopting a deterministic approach, they are not continuous functions
of $\zeta$ in any reasonable topology. Following this thread, if we denote by $\zeta \mapsto \varphi(\zeta)$
the "solution map" that takes a realization of the driving noise to the corresponding
solution to the SPDE (1.1) then it seems natural to expect that $\varphi(\cdot)$ will not be
continuous either – this is discouraging since we would want to define $\varphi(\zeta)$ for
typical realizations of $\zeta$ by setting $\varphi(\zeta) \overset{\text{def}}{=} \lim_{\varepsilon \downarrow 0} \varphi(\zeta_\varepsilon)$.

The theory of regularity structures does not view $\varphi$ as the primary solution
map and rather works with a map defined on a much richer space. It replaces the
(linear) space of realizations of $\zeta$ with a non-linear metric space $\mathscr{M}_0$ of objects
called *models* which encode, loosely speaking, the realization of $\zeta$ *and* sufficiently
many multilinear functionals of $G * \zeta$ so that one can approximate $F$ sufficiently
well.

In order to describe the elements of the space $\mathscr{M}_0$, one must go further then
just specifying a realization of each such multi-linear functional. Additionally, one
must also specify, for every space-time point $z$, a suitable method of recentering
this functional around $z$ so that the recentered functional satisfies certain analytic
bounds close to that point $z$. This recentering is performed via the subtraction of
$z$-dependent counterterms[4] and can be interpreted as a type of "positive renormal-
ization", see [BHZ16]. Moreover, the definition of the space $\mathscr{M}_0$ enforces analytic
and highly non-trivial algebraic constraints on how these counterterms depend on $z$
– the latter constraints are responsible for the non-linearity of the space $\mathscr{M}_0$. The
reward for this increase in complexity is that the map from a model $Z \in \mathscr{M}_0$ to the
solution $\Phi(Z)$ of the equation "driven" by $Z$ is continuous.

At first sight, it is not clear why this would be of any help. Indeed, there is a
natural "naive" way of defining a family of models $(Z_{\text{can}}^{\zeta_\varepsilon})_{\varepsilon,\zeta}$ so that the solution
$\varphi(\zeta_\varepsilon)$ to (1.2) is given by $\Phi(Z_{\text{can}}^{\zeta_\varepsilon})$. Viewing $\zeta$ as random, $(Z_{\text{can}}^{\zeta_\varepsilon})_{\varepsilon,\zeta}$ is then a sequence
of random models but, for essentially the same reasons as above, this sequence
typically fails to converge in $\mathscr{M}_0$ as $\varepsilon \downarrow 0$.

The reason why this perspective is still useful is that there is more than one
way of associating a model to a given realisation of the driving noise. In particular,
it turns out that the renormalisation procedure which was required then in order
to make the terms in the formal series expansion converge can be encoded into a
suitable definition of what we mean by "the model $Z_\varepsilon$ associated to $\zeta_\varepsilon$". Indeed,
it was shown in many different special cases [Hai14, GIP15, HS16, ZZ15, Hos16]
that there is a probabilistically convergent family of models $(Z_{\text{BPHZ}}^{\zeta_\varepsilon})_{\varepsilon,\zeta}$ such that, for
each fixed $\zeta$, the solution $\hat{\varphi}_\varepsilon(\zeta_\varepsilon)$ to (1.3), with a suitable choice of $F_\varepsilon$, is given by
$\Phi(Z_{\text{BPHZ}}^{\zeta_\varepsilon})$. Thanks to the deterministic continuity of the map $Z \mapsto \Phi(Z)$, it then

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[4]These counterterms have no obvious counterpart in perturbative renormalization for QFT.

follows that the probabilistic convergence of the models $Z_{\mathrm{BPHZ}}^{\zeta_\varepsilon}$ is inherited by $\hat{\varphi}_\varepsilon(\zeta_\varepsilon)$. Writing $Z_{\mathrm{BPHZ}}^{\zeta} \stackrel{\mathrm{def}}{=} \lim_{\varepsilon\downarrow 0} Z_{\mathrm{BPHZ}}^{\zeta_\varepsilon}$, one can view $\Phi(Z_{\mathrm{BPHZ}}^{\zeta})$ as "a solution" to the SPDE at hand[5]

While the machinery of [Hai14] was formulated systematically and covered a wide class of SPDEs, two tasks in this procedure were left to be dealt with in a somewhat ad-hoc way:

1. Create a concrete space of models $\mathscr{M}_0$ tailor-made so that the type of SPDEs at hand can be realized as one driven by a model in $\mathscr{M}_0$. This space should have enough flexibility to allow for the renormalization subtractions needed to yield probabilistically convergent families of models.
2. Find a probabilistically convergent family of "renormalised" random models $Z_\varepsilon^\zeta$.

A systematic approach for the first task has recently been presented in [BHZ16]. In particular, given a random, smooth, driving noise $\zeta$ the space of models described by [BHZ16] is rich enough to include a specific random model $Z_{\mathrm{BPHZ}}^{\zeta}$ henceforth refereed to as the "BPHZ lift of $\zeta$" defined in [BHZ16, Eq. 6.22], see also [Hai16, Eq. 3.1].

The goal of the present article is to prove the following implication: if a family of random smooth driving noises $(\zeta_n)_{n\in\mathbf{N}}$ is uniformly bounded in an appropriate sense and converges, as $n \to \infty$, in probability to a random driving noise $\xi$, then $\xi$ admits a BPHZ lift $Z_{\mathrm{BPHZ}}^{\xi}$. Furthermore, the random model $Z_{\mathrm{BPHZ}}^{\xi}$ is obtained as the limit in probability of the random models $(Z_{\mathrm{BPHZ}}^{\zeta_n})_{n\in\mathbf{N}}$. In particular, the BPHZ lift is canonical and stable, i.e. independent of any specific choice of approximation procedure. Our Theorem 2.15 states this result in the Gaussian case and covers in particular all examples of locally sub-critical SPDEs driven by space-time white noise. We state our main result in full generality in Theorem 2.34. The trio of papers [Hai14], [BHZ16], and the present article then give a completely "automatic" and self-contained black box for obtaining local existence theorems for a wide class of SPDEs.

We now describe the approach and outline of this paper. As already mentioned, there are two types of renormalization that appear here – negative renormalizations which compensate for divergences and base-point dependent positive renormalizations which appear in our recentering procedure. The definition of the BPHZ model found in [BHZ16, Hai16] invokes two maps called the positive and negative twisted antipodes. Each of these maps is defined through a recursive formula. A very similar (in spirit) definition is given in Section 3 which is later to shown to coincides with the definition given in [BHZ16, Hai16].

Our first step is to obtain a more explicit integral formula for the BPHZ renor-

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[5]We do not say "the solution" because there will actually be an infinite family of solutions. This degeneracy can be seen in our example – one would expect $\hat{\varphi}_\varepsilon$ to converge for any choice of the form $F_\varepsilon(\varphi,\zeta) = \varphi^3 - (c_\varepsilon + c_1)\varphi + c_2 + \zeta$ where $c_1$ and $c_2$ are finite constants. The full space of solutions to a given locally subcritical SPDE will always be parameterized by a finite number of real parameters.

malized model in terms of a sum over forests and cuts, and this is the content of Section 4. Here the forests come from negative renormalization and are really the same as those of Zimmerman's forest formula [Zim69] – they are families of divergent structures where each pair of structures in each family must be either nested or disjoint. The cuts correspond to locations of positive renormalizations.

The most fundamental obstacle that appears when estimating this explicit integral formula is that renormalizations can obstruct each other. This can happen in two ways: (i) two negative renormalizations can obstruct each other if the two divergent structures are neither nested nor disjoint (ii) two renormalizations of opposite sign can obstruct each other if they overlap, that is a positive renormalization might occur within a divergent structure. As a result, one cannot in general expect to be able to "harvest" all cancellations created by the two renormalization procedures.

The first problem is the famous problem of overlapping divergences of perturbative quantum field theory. Its solution comes from using scale decompositions so that, for any given scale assignment, one can identify which one of a pair of overlapping divergent structures to renormalize. The original approach to this problem was to represent the explicit formula in terms of Fourier variables and splitting up the Fourier integration domain into "Hepp sectors". The resulting estimates can be summarised by the mantra that "overlapping divergences don't overlap in phase-space".

The use of momentum space techniques is too limiting for our purposes. However an alternative approach in position space was described in [FMRS85], where one expands the kernels associated to the lines of a Feynman diagram into an infinite sum of slices, each of which is localized at a particular length scale. A full scale assignment is an assignment of scale to every single line of a Feynman diagram. For each full scale assignment on a Feynman diagram, the authors of [FMRS85] define an operator on the set of Zimmerman forests for that diagram called the "projection onto safe forests". This allows for the sum over Zimmerman forests to be re-organized, as a function of the scale assignment, into sums over subsets of forests we call "intervals". With this reorganization one is guaranteed to harvest the right negative renormalizations for each given scale assignment.

The problem of overlaps between positive and negative renormalizations does not appear in perturbative QFT, but it turns out that it also does not cause a problem. One can show that if one chooses appropriately which cuts to harvest, as a function of scale assignments, then the larger sum over forests and cuts can also be reorganized appropriately for each scale assignment. The mantra here is that "positive and negative renormalizations don't overlap in phase-space".

The set-theoretic aspects and notation of forest projections and intervals, how they can be used to reorganize sums over forests, and sufficient conditions (which we call "compatibility") for the sum over forests and sum over cuts to interact well are all described in Section 5. This is all done abstractly, then in Section 6 we present our multiscale expansion procedure. In Section 7 we make the notions of Section 5 concrete: we present, as a function of each scale assignment, the [FMRS85] forest projection, our algorithm for choosing which cuts to harvest, and

the proof that these two notions are compatible.

In Section 8 we describe how one inductively controls the nested renormalization cancellations that might appear and we obtain full control over one copy of the BPHZ renormalized model. When estimating these nested renormalizations we use seminorms similar to ones introduced in [FMRS85] but simplified. In Section 9 we state and verify conditions which take as input the control over a single copy of the model that we obtained in Section 8 and then gives as output estimates on higher moments of that model.

One key generalization in this work is that we do not assume Gaussianity of the driving noise distribution. This was already done earlier in special cases in [HS15, CS16, SX16], but the novelty here is that we are able to also accommodate very weakly mixing space-time random fields and genuinely non-Gaussian situations[6].

We are also able to deal with driving noises that are more singular than space-time white noise, but we are still restricted to regimes where the multilinear functionals of the noise appearing in our analysis have "regularity" (in some power-counting sense to be made precise below) strictly better than that of white noise. Instances of this limitation were already observed in the works [CQ02, FV10, Hos16, HHL+15]. The reason for this is that the objects indexed by trees that are being renormalised in the theory of regularity structures should really be thought of as "half Feynman diagrams", with the full diagram arising in the moment estimates by gluing together several trees by their leaves, see for example [Hai13]. The analogue of the BPHZ renormalisation procedure arising in our context however takes place at the level of the trees, not at the level of the resulting Feynman diagrams, so that divergent structures straddling more than one of the constituent trees cannot be taken into account, thus leading to divergencies. This is not a mere technical problem but points to a genuine change in behaviour of the corresponding stochastic PDEs, as already seen in [CQ02] in the context of rough paths, where SDEs driven by fractional Brownian motion can only be made sense of for Hurst parameters $H > \frac{1}{4}$, even though the theory of rough paths should in principle "work" all the way down to $H > 0$. The works [MU11, MU12] suggest that even if there is a "canonical" rough path corresponding to fractional Brownian motion with $H \leq \frac{1}{4}$, one would expect it to live over a larger probability space, so that the Lévy area is no longer a measurable function of the underlying path. In the genuinely non-Gaussian case, our analysis points to a similar problem already arising at $H = \frac{1}{3}$ in the case of non-vanishing third cumulants.

### Acknowledgements

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[6]As an example, we can accommodate a driving noise which is given by a Wick power of the linear solution to the stochastic heat equation.

## 2    Setting, preliminary definitions, and the main theorem

The most primitive object of the theory of [Hai14] is the *regularity structure*. In [BHZ16] a very general formulation for the construction of regularity structures is given by defining various algebraic spaces of trees and forests and taking progressive quotients/projections. In this section, we introduce some of the objects and notations from [BHZ16].

### 2.1    Space-time scalings, types, and homogeneities

For the rest of the paper we fix a dimension $d \geq 1$ of space-time. For a multi-index $k = (k_i)_{i=1}^d \in \mathbf{N}^d$ we define the degree of $k$ to given by

$$|k| \stackrel{\text{def}}{=} \sum_{i=1}^d |k_i| \; . \tag{2.1}$$

Most interesting SPDE falling under our framework have underlying linear equations which scale anistropically with respect to the various components of space-time – for example the stochastic heat equation underlying (1.1) scales *parabolically*.

A $d$-dimensional space-time "scaling" is a multi-index $\mathfrak{s} = (\mathfrak{s}_i)_{i=1}^d \in \mathbf{N}^d$ with strictly positive entries. From here on we treat the space-time scaling $\mathfrak{s}$ as fixed.

For $z = (z_1, \ldots, z_d) \in \mathbf{R}^d$ we do not define $|z|$ to be given by the standard Euclidean norm but instead set

$$|z| \stackrel{\text{def}}{=} \sum_{i=1}^d |z_i|^{\frac{1}{\mathfrak{s}_i}} \; . \tag{2.2}$$

There appears to be a bad overload of notation between (2.2) and (2.1) but it will always be clear from context whether the argument of $| \cdot |$ is a multi-index or a space-time vector.

While $| \cdot |$ on $\mathbf{R}^d$ is not a norm it certainly defines a metric. With this metric the space $\mathbf{R}^d$ has Hausdorff dimension $|\mathfrak{s}|$ – in particular the function $|z|^\alpha$ on $\mathbf{R}^d$ is locally integrable at $z = 0$ if and only if $\alpha > -|\mathfrak{s}|$.

We introduce a notion of $\mathfrak{s}$-degree for multi-indices $k = (k_i)_{i=1}^d \in \mathbf{Z}^d$ by setting

$$|k|_{\mathfrak{s}} \stackrel{\text{def}}{=} \sum_{i=1}^d k_i \mathfrak{s}_i \; .$$

This induces a notion of $\mathfrak{s}$-degree for polynomials on $\mathbf{R}^d$: for $k \in \mathbf{N}^d$ the $\mathfrak{s}$-degree of a monomial $\mathbf{R}^d \ni z \mapsto z^k$ is $|k|_{\mathfrak{s}}$ and the $\mathfrak{s}$-degree of a polynomial is the maximum among the $\mathfrak{s}$-degrees of its monomials (with the $\mathfrak{s}$-degree of the 0 polynomial set to $-\infty$).

### 2.1.1   Types and homogeneities

The combinatorial trees we introduce should be thought of as arising, in spirit, from Picard iteration of the mild formulation system of SPDEs one is trying to solve. As we saw in the introduction, for (1.1) one obtains trees where the edges correspond to convolution with the kernel $G$ and the leaves correspond to the driving noise.

To accommodate systems of equations we work with a collection of different convolution kernels and driving noises. To formalize this we fix, for the rest of the article, a finite set of *types* $\mathfrak{L}$ which comes with a partition $\mathfrak{L} = \mathfrak{L}_+ \sqcup \mathfrak{L}_-$. The elements of $\mathfrak{L}_+$ and $\mathfrak{L}_-$ are respectively called kernel types and noise types.

**Definition 2.1**  A homogeneity assignment is a map $|\cdot|_{\mathfrak{s}} : \mathfrak{L} \to \mathbf{R}$ such that

$$\mathfrak{L}_+ = \{\mathfrak{t} \in \mathfrak{L} : |\mathfrak{t}|_{\mathfrak{s}} > 0\} \ \text{and} \ \mathfrak{L}_- = \{\mathfrak{t} \in \mathfrak{L} : |\mathfrak{t}|_{\mathfrak{s}} < 0\} \ .$$

A choice of homogeneity assignment should be considered fixed except in a few specific arguments where we explicitly state this is not case.

For each $\mathfrak{t} \in \mathfrak{L}_+$, $|\mathfrak{t}|_{\mathfrak{s}}$ quantifies the regularizing effect of the corresponding kernel. On the other hand, for $\mathfrak{t} \in \mathfrak{L}_-$, $|\mathfrak{t}|_{\mathfrak{s}}$ encodes the regularity (or rather lack thereof) of the driving noise corresponding to $\mathfrak{t}$. For any set $A$ and map $\mathfrak{t} : A \to \mathfrak{L}$, we write $|\mathfrak{t}(A)|_{\mathfrak{s}} \stackrel{\text{def}}{=} \sum_{a \in A} |\mathfrak{t}(a)|_{\mathfrak{s}}$. Given a homogeneity assignment $|\cdot|_{\mathfrak{s}}$ we extend it to $\mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$ by setting

$$a = k + \sum_{\mathfrak{t} \in \mathfrak{L}} c_{\mathfrak{t}} \mathfrak{t} \in \mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L}) \qquad \Rightarrow \qquad |a|_{\mathfrak{s}} \stackrel{\text{def}}{=} |k|_{\mathfrak{s}} + \sum_{\mathfrak{t} \in \mathfrak{L}} c_{\mathfrak{t}} |\mathfrak{t}|_{\mathfrak{s}} \ .$$

## 2.2   Trees, forests, and decorations

### 2.2.1   Trees

A rooted tree $T$ is an acyclic finite connected simple graph with a distinguished vertex $\varrho_T$ which we call *the root*. This graph can be presented as a set of nodes $N_T$ and edges $E_T \subset N_T^2$. Since we force our trees to have roots it follows that $\varrho_T \in N_T \neq \emptyset$. On the other hand we do allow $E_T = \emptyset$ which forces $|N_T| = 1$; such a tree will be called *trivial* and is also denoted by $\bullet$.

We view the node set $N_T$ of a rooted tree $T$ as being equipped with a partial order $\leq$ where $u \leq v$ if and only if $u$ lies on the unique path connecting $v$ to the root. We then view $T$ as being directed, with every edge being of the form $(u, v) \in E_T$ with $u < v$. With this convention, the root of $T$ is the unique vertex with no incoming edges. We define maps $e_{\mathrm{c}}, e_{\mathrm{p}} : E_T \to N_T$ with the property that for any $e \in E_T$ one has $e = (e_{\mathrm{p}}(e), e_{\mathrm{c}}(e))$ – here c stands for "child" and p stands for parent. When $e$ has been fixed for an expression we may write $e_{\mathrm{p}}$ and $e_{\mathrm{c}}$, suppressing the argument. Note that we inherit a poset structure on edges by setting $e \leq \tilde{e} \Leftrightarrow e_{\mathrm{p}} \leq \tilde{e}_{\mathrm{p}}$.

An morphism of rooted trees $T$ and $\bar{T}$ is a map $f : N_T \sqcup E_T \to N_{\bar{T}} \sqcup E_{\bar{T}}$ with the property that for any $e = (x, y) \in E_T$ one has $f(e) = (f(x), f(y)) \in E_{\bar{T}}$.

A typed rooted tree is a pair $(T, \mathfrak{t})$ where $T$ is a rooted tree and $\mathfrak{t} : E_T \to \mathfrak{L}$ with the properties that (i) if for some $e \in E_T$ one has $\mathfrak{t}(e) \in \mathfrak{L}_-$ then $e$ is maximal in the partial order on $E_T$ and (ii) For distinct $e$, $e' \in E_T$ with $\mathfrak{t}(e), \mathfrak{t}(e') \in \mathfrak{L}_-$ one has $e_\mathfrak{p} \neq e'_\mathfrak{p}$.

A morphism of typed rooted trees is morphism of rooted trees which preserves edge types. The type map $\mathfrak{t}$ is often suppressed from from notation. Henceforth the term "tree" denotes a typed rooted tree.

### 2.2.2   Forests

A typed rooted forest $F$ is a typed directed graph (consisting of a node set $N_F$, an edge set $E_F \subset N_F^2$, and a type map $\mathfrak{t} : E_F \to \mathfrak{L}$) with the property that every connected component of $F$ is a typed rooted tree (we often call each of these components a "tree" of $F$). Henceforth the term "forest" will always refer to a typed rooted forest.

Clearly any typed rooted tree is also a typed rooted forest. We do allow for $N_F = \emptyset$ in the definition of a forest and this empty forest will be denoted by **1**. Note that **1** is *not* a tree. We also write $\varrho(F) \subset N_F$ for the collection of all nodes of $F$ having no incoming edges. Additionally, we view $N_F$ and $E_F$ as posets by inheriting the poset structures from their connected components of $F$ and postulating that pairs of edges / vertices belonging to different connected components are not comparable. We say $f \colon N_F \sqcup E_F \to N_{F'} \sqcup E_{F'}$ is a typed rooted forest morphism from $F$ to $F'$ if it restricts to a morphism of rooted trees on each connected component.

One has the decomposition $E_F = K(F) \sqcup \mathbb{L}(F)$ where

$$\mathbb{L}(F) \stackrel{\text{def}}{=} \{e \in E_F : \mathfrak{t}(e) \in \mathfrak{L}_-\} \ \text{ and } \ K(F) \stackrel{\text{def}}{=} \{e \in E_F : \mathfrak{t}(e) \in \mathfrak{L}_+\} \, .$$

We call the elements of these two sets *noise edges* and *kernel edges*, respectively. While viewing noises as edges was the point of view taken in [BHZ16], where it leads to a cleaner description of the algebraic properties of these objects, it will be much more natural in the present setting to view each noise as a node. We will therefore always restrict ourselves to situations in which, for any given node $v \in N(F)$, there can be at most one kernel edge $e \in \mathbb{L}(F)$ with $e_\mathfrak{p} = v$. As a consequence, we will never work with $\mathbb{L}(F)$, but instead with the set $L(F) \stackrel{\text{def}}{=} e_\mathfrak{p}[\mathbb{L}(F)]$ Observe that, thanks to the above-mentioned restriction, $L(F)$ naturally inherits a type map $\mathfrak{t} : L(F) \to \mathfrak{L}_-$. We also define a subset of nodes $N(F) \stackrel{\text{def}}{=} N_F \setminus e_\mathfrak{c}(\mathbb{L}(F))$ which should be thought of as "true" nodes (they are the ones that will later on correspond to integration variables) and consider the elements of $e_\mathfrak{c}(\mathbb{L}(F))$ as "fictitious" nodes that will not play any role in the sequel. We also use the shorthand $\tilde{N}(F) \stackrel{\text{def}}{=} N(F) \setminus \varrho(F)$. The reason why this shorthand will often be used is that, when renormalizing a subtree $S$ of a larger tree $T$, one should think of "contracting" this subtree to a single node, see [Hai14, BHZ16]. We will then always identify the integration variable corresponding to the contraction of $S$ with

the root of $S$ in $T$, so that $\tilde{N}(S)$ precisely corresponds to the variables that get integrated out by the renormalization procedure.

We often include figures representing trees and forests. To keep things simple our pictures will only distinguish between kernel type and noise type: edges of kernel type will be depicted by straight arrows and edges of noise type will be depicted by a zigzag line, as in the following figure.
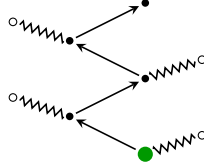


The arrow for an edge $e$ always travels from $e_{\mathrm{p}}$ to $e_{\mathrm{c}}$. Above we have marked the true vertices with solid circles and the fictitious nodes with open circles, with the root being drawn as a larger green dot.

We define the *forest product* $F_1 \cdot F_2$ between two forests $F_1$ and $F_2$ as their disjoint union. Since there is no ordering on the set of components of a forest, the forest product is commutative and associative with the empty forest $\mathbf{1}$ serving as unit. Given a forest $F$ we say that a forest $A$ is a subforest of $F$ if $N_A \subset N_F$, $E_A \subset E_F$ and the natural inclusion map $\iota : N_A \sqcup E_A \to N_F \sqcup E_F$ is a forest morphism. In particular, the empty forest $\mathbf{1}$ is a subforest of every forest. We define $\mathrm{Conn}(F)$ to be the set of connected components of $F$, each viewed as a subtree of $F$ (thus $\mathrm{Conn}(F)$ may contain distinct elements which are isomorphic as trees).

Two subforests $A$ and $B$ of a forest $F$ are said to be disjoint if $N_A \cap N_B = \emptyset$. We also define the union $A \cup B$ and intersection $A \cap B$ of two subforests $A, B$ of $F$ to be the subforests with node and edge sets $N_A \cup N_B$, $E_A \cup E_B$, and $N_A \cap N_B$, $E_A \cap E_B$, respectively. If $B$ is a subforest of $A$ such that $B \cap A$ consists of a union of connected components of $A$ then $A \setminus B$ is also a subforest.

**Remark 2.2** Caution: one may have $G$ a subforest of a forest $F$, $u \in L(F) \cap N(G)$, but $u \notin L(G)$ - this would happen if the unique $e \in \mathbb{L}(F)$ with $e_p = u$ did not belong to $\mathbb{L}(G)$.

A subforest that happens to be a tree is called a subtree. Both the intersection and the union of a non-disjoint pair of subtrees is again a subtree.

### 2.2.3 Decorations

A semi-decorated tree consists of a tree $T$ and two maps: a node label $\mathfrak{n} : N(T) \to \mathbf{N}^d$ and an edge label $\mathfrak{e} : E_T \to \mathbf{N}^d$. Given a forest $T$, node decoration $\mathfrak{n}$ and edge decoration $\mathfrak{e}$ on $T$ we denote the corresponding semi-decorated tree by $T_{\mathfrak{e}}^{\mathfrak{n}}$.

A choice of homogeneity assignment $|\cdot|_s$ induces a notion of a homogeneity for semi-decorated trees by setting

$$|T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} \overset{\text{def}}{=} \sum_{e \in E_T} \left( |\mathfrak{t}(e)|_{\mathfrak{s}} - |\mathfrak{e}(e)|_{\mathfrak{s}} \right) + \sum_{u \in N(T)} |\mathfrak{n}(u)|_{\mathfrak{s}} . \tag{2.3}$$

## 2.3   Regularity structures of semi-decorated trees

### 2.3.1   The reduced regularity structure

We first recall the definition of a regularity structure from [Hai14, Def. 2.1].

**Definition 2.3** A *regularity structure* $\mathscr{T} = (A, \mathscr{T}, G)$ consists of

- An index set $A \subset \mathbf{R}$ which is locally finite and bounded from below.
- A model space $T$, which is a graded vector space $\mathscr{T} = \bigoplus_{\alpha \in A} \mathscr{T}_\alpha$, with each $\mathscr{T}_\alpha$ a Banach space.
- A *structure group* $G$ of linear operators acting on $\mathscr{T}$ such that, for every $\Gamma \in G$, every $\alpha \in A$, and every $\tau \in \mathscr{T}_\alpha$, one has

$$\Gamma\tau - \tau \in \bigoplus_{\beta < \alpha} \mathscr{T}_\beta \,.$$

When formulating our main theorem in the context of the theory of regularity structures the relevant regularity structure will be the *reduced regularity structure* $\mathscr{T}$ described by the procedure of [BHZ16, Sec. 6.4][7]. This procedure takes as input a choice of space-time dimension $d \geq 1$, a $d$-dimensional space-time scaling $\mathfrak{s}$, sets of leaf and kernel types $\mathfrak{L}_+$ and $\mathfrak{L}_-$, a homogeneity assignment $|\cdot|_{\mathfrak{s}}$ on these types, and finally a *rule $R$*. We have introduced all these notions except the last one.

The vector space $\mathscr{T}$ of the reduced regularity structure is the free vector space generated by a collection $B_\circ$ of semi-decorated trees prescribed by the rule $R$. Then one has $A \overset{\text{def}}{=} \{|T^{\mathfrak{n}}_{\mathfrak{e}}|_{\mathfrak{s}} : T^{\mathfrak{n}}_{\mathfrak{e}} \in B_\circ\}$ and for each $\alpha \in A$ one sets $\mathscr{T}_\alpha$ to be the free vector space generated by all semi-decorated trees $T^{\mathfrak{n}}_{\mathfrak{e}} \in B_\circ$ with $|T^{\mathfrak{n}}_{\mathfrak{e}}|_{\mathfrak{s}} = \alpha$.

**Remark 2.4** The collection $B_\circ$ introduced in [BHZ16] is slightly different than the one introduced here since it furthermore allows a tree to be endowed with an additional colored subforest, as well as an additional label $\mathfrak{o}$ which keeps track of some information coming from contracted subtrees. The collection $B_\circ$ we are using here consists of those elements for which $\mathfrak{o} = 0$ and the colored subforest is empty. This is because we will only ever consider the *reduced regularity structure* introduced in [BHZ16, Sec. 6.4].

This collection $B_\circ$ has to be rich enough to describe the solution to the system of SPDEs in question and be closed under positive and negative renormalization operations. A natural method to build $B_\circ$ so that this is the case is to build it up inductively by starting from some primitive objects and then applying operations like integration and point-wise products. We describe this now.

For every $\mathfrak{t} \in \mathfrak{L}_+$ and $k \in \mathbf{N}^d$ we define an operator $\mathscr{I}^k_{\mathfrak{t}}[\cdot]$ on semi-decorated trees as follow: for an arbitrary semi-decorated tree $\tau$ one obtains $\mathscr{I}^k_{\mathfrak{t}}[\tau]$ by adding a new vertex to the set of vertices of $\tau$ which becomes the new root and has a vanishing node label – we then attach the new root to the old one with an edge of type $\mathfrak{t}$ and edge decoration $k$.

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[7]This is a generalisation of the more informal prescription of [Hai14, Sec. 8].

Additionally, for any $\mathfrak{l} \in \mathfrak{L}_-$ we write $\Xi_{\mathfrak{l}}$ for the semi-decorated tree which consists of a single edge of type $\mathfrak{l}$ with vanishing edge and node labels. We also introduce a commutative and associative binary operation on semi-decorated trees called the tree product: for two semi-decorated trees $\tau$ and $\bar{\tau}$, the tree product $\tau \star \bar{\tau}$ is obtained by identifying the roots of $\tau$ and $\bar{\tau}$, which becomes the root of $\tau \star \bar{\tau}$, and then setting the new node label of this root to be the sum of the node label of the two previous roots.

The set of primitive semi-decorated trees we start with is $\{\bullet^n\}_{n \in \mathbf{N}^d} \sqcup \{\Xi_{\mathfrak{l}}\}_{\mathfrak{l} \in \mathfrak{L}_-}$ – the first set is called the set of "abstract polynomials" and the second is called the set of "abstract noises". Then, one builds up $B_\circ$ by applying tree products and applying the operators $\mathcal{I}_t^k$. However, this produces a collection $B_\circ$ which is too large – certainly $A$ will fail to be bounded below. A *rule* is then an algorithm which explicitly specifies when one is allowed to take tree products and apply the operators $\mathcal{I}_t^k$ as one builds $B_\circ$. For a precise definition of what a rule is we point the reader to [BHZ16, Sec. 5.2]. For the present article, this precise definition is irrelevant, one should think of a rule as a way of formalising what one means by a "class of semilinear stochastic PDEs" and $B_\circ$ as the set of trees that are required when trying to formulate elements from that class within the theory of regularity structures. A rule is then "subcritical" if all elements in the corresponding class of SPDEs are subcritical in the sense of [Hai14]. It is "complete" if the class is sufficiently large so that it is closed under the natural renormalisation operations described in [BHZ16]. We now fix a for the rest of the paper a complete subcritical rule $R$ and a corresponding reduced regularity structure $\mathcal{T}$.

**Remark 2.5** For example, if we wish to consider the SPDE formally given by

$$\partial_t \Phi = \Delta \Phi - \Phi^3 + \xi \,,$$

in dimension $d < 4$ and for $\xi$ space-time white noise, then the smallest complete rule allowing to describe this equation contains all equations of the type

$$\partial_t \Phi = \Delta \Phi - P(\Phi) + \xi \,,$$

where $P$ is a polynomial of degree 3.

### 2.3.2   Admissible models

Next we recall the definition of a smooth model on a regularity structure $\mathcal{T}$. In what follows we denote by $\mathscr{C}(\mathbf{R}^d)$ the space of all smooth real valued functions on $\mathbf{R}^d$.

**Definition 2.6** Given a regularity structure $\mathcal{T} = (A, \mathcal{T}, G)$, a smooth model $Z = (\Pi, \Gamma)$ on $\mathcal{T}$ on $\mathbf{R}^d$ with scaling $\mathfrak{s}$ consists of

- A map $\Gamma : \mathbf{R}^d \times \mathbf{R}^d \to G$ such that $\Gamma_{xx} = \mathrm{Id}$, and such $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ for all $x, y, z \in \mathbf{R}^d$.

- A collection $\Pi = (\Pi_x)_{x\in\mathbf{R}^d}$ of continuous linear maps $\Pi_x : \mathscr{T} \to \mathscr{C}^\infty(\mathbf{R}^d)$ such that $\Pi_y = \Pi_x\Gamma_{xy}$ for all $x, y \in \mathbf{R}^d$.

Furthermore, for every $\alpha \in A$ and every compact set $\mathfrak{K} \subset \mathbf{R}^d$, we assume the existence of a finite constant $C_{\alpha,\mathfrak{K}}$ such that the bounds

$$|(\Pi_x\tau)(y)| \leq C_{\alpha,\mathfrak{K}}\|\tau\|_\alpha|x-y|^\alpha \text{ and } \|\Gamma_{xy}\tau\|_\beta \leq C_{\alpha,\mathfrak{K}}\|\tau\|_\alpha|x-y|^{\beta-\alpha} \quad (2.4)$$

hold uniformly in $x, y \in \mathfrak{K}$, $\beta \in A$ with $\beta < \alpha$, and $\tau \in \mathscr{T}_\alpha$. Here for any $\beta \in A$, $\|\cdot\|_\beta$ denotes the norm on the Banach space $\mathscr{T}_\alpha$.

Above, there is no constraint how a model interacts with the various kernel types, noise types, and edge labels – to do this one introduces an "admissibility" condition. This requires associating to each abstract kernel type a concrete convolution kernel.

First, following [Hai14, Sec. 10.3], for any $\zeta \in [0, |\mathfrak{s}|)$ and $m \in \mathbf{N}$ we introduce norms $\|\cdot\|_{\zeta,m}$ on smooth functions $K : \mathbf{R}^d \setminus \{0\} \to \mathbf{R}$ by setting

$$\|K\|_{\zeta,m} \stackrel{\text{def}}{=} \max_{\substack{k\in\mathbf{N}^d \\ |k|_\mathfrak{s}\leq m}} \sup_{x\in\mathbf{R}^d\setminus\{0\}} |x|^{|k|_\mathfrak{s}+|\mathfrak{s}|-\zeta}|D^kK(x)| . \quad (2.5)$$

**Definition 2.7** A *kernel type map* is a tuple $K = (K_\mathfrak{t})_{\mathfrak{t}\in\mathfrak{L}_+}$ where for every $\mathfrak{t} \in \mathfrak{L}_+$

- $K_\mathfrak{t} : \mathbf{R}^d \setminus \{0\} \to \mathbf{R}$ is smooth function.
- $K_\mathfrak{t}(x)$ vanishes if $|x| > 1$.
- For every $m \in \mathbf{N}$ one has $\|K_\mathfrak{t}\|_{|\mathfrak{t}|_\mathfrak{s},m} < \infty$.

**Remark 2.8** Within the scope of regularity structures it is common to also require that for some value of $r > 0$ one has $\int_{\mathbf{R}^d} K_\mathfrak{t}(x)P(x)\, dx = 0$ for any polynomial $P$ of $\mathfrak{s}$-degree strictly less than $r$. Since there is no need for this in any of the theorems stated in this paper we do not enforce any such requirement.

For the rest of the paper we take our kernel type map $K$ as fixed. With all these notions in hand we can now give a definition of an admissible smooth model. This definition only makes sense in the context of the regularity structure built from a set of types and a rule $R$ as described above, so we assume that we are in this setting from now on.

**Definition 2.9** A smooth model $Z = (\Pi, \Gamma)$ is said to be admissible it satisfies all of the following properties.

- For every $z, \bar{z} \in \mathbf{R}^d$, $\Pi_z[\bullet^0](\bar{z}) = 1$.
- For any $\mathfrak{t} \in \mathfrak{L}_+$, $k \in \mathbf{N}^d$ and $\tau \in B_\circ$ such that $\mathscr{I}_\mathfrak{t}^k(\tau) \in B_\circ$ one has

$$(\Pi_z\mathscr{I}_\mathfrak{t}^k(\tau))(\bar{z}) = (D^kK_\mathfrak{t}*\Pi_z\tau)(\bar{z}) - \sum_{|j|_\mathfrak{s}<|\mathscr{I}_\mathfrak{t}^k(\tau)|_\mathfrak{s}} \frac{(\bar{z}-z)^j}{j!}(D^{k+j}K_\mathfrak{t}*\Pi_z\tau)(z) ,$$

for all $z, \bar{z} \in \mathbf{R}^d$. Here $*$ denotes convolution in $\mathbf{R}^d$ and $j$ denotes a multiindex in $\mathbf{N}^d$.

- For any $n \in \mathbf{N}^d$, $z, \bar{z} \in \mathbf{R}^d$, and $\tau \in B_\circ$ such that $\bullet^n \star \tau \in B_\circ$, one has

$$(\Pi_z[\bullet^n \star \tau])(\bar{z}) = (\bar{z} - z)^n (\Pi_z \tau)(\bar{z}) .$$

We denote by $\mathscr{M}_\infty(\mathscr{T})$ the collection of all smooth admissible models on $\mathscr{T}$. The notion of a model is formulated at a deterministic level. When formulating a system of SPDEs as a system "driven" by a model, each realization of the noise yields a different model.

If we turn to solving an SPDE like (1.2) then there is a canonical way to lift each smoothened realization of the driving noise to a corresponding model, which we describe now.

**Definition 2.10** A *smooth noise* is a tuple $\xi = (\xi_\mathfrak{t})_{\mathfrak{t} \in \mathfrak{L}_-}$ with $\xi_\mathfrak{t} \in \mathcal{C}^\infty(\mathbf{R}^d)$ for every $\mathfrak{t} \in \mathfrak{L}_-$. We denote by $\Omega_\infty$ the set of all noises for our fixed set of types $\mathfrak{L}_-$.

Given a smooth noise, there is a canonical way of associating a model to it.

**Definition 2.11** Given $\xi \in \Omega_\infty$, the *canonical lift* $Z_{\mathrm{can}}^\xi = (\Pi, \Gamma) \in \mathscr{M}_\infty(\mathscr{T})$ of $\xi$ is the unique admissible model such that

- for every $\mathfrak{l} \in \mathfrak{L}_-$ and $z \in \mathbf{R}^d$ one has $(\Pi_z \Xi_\mathfrak{l})(\cdot) = \xi_\mathfrak{l}(\cdot)$,
- for every $z, \bar{z} \in \mathbf{R}^d$ and $\tau, \bar{\tau} \in B_\circ$ with $\tau \star \bar{\tau} \in B_\circ$,

$$(\Pi_z[\tau \star \bar{\tau}])(\bar{z}) = (\Pi_z \tau)(\bar{z}) \cdot (\Pi_z \bar{\tau})(\bar{z}) .$$

Reiterating what was said in the introduction, for a fixed realization of $\zeta_\varepsilon$, the equation (1.2) will be viewed as driven by the model $Z_{\mathrm{can}}^{\zeta_\varepsilon}$ and the solution $\varphi(\zeta_\varepsilon)$ can be written in terms of $\Phi(Z_{\mathrm{can}}^{\zeta_\varepsilon})$ where $Z \to \Phi(Z)$ is defined on $\mathscr{M}_\infty(\mathscr{T})$.

Next, we describe the topology that makes $\Phi$ *continuous*. First, we introduce a family of test functions.

**Definition 2.12** Given a homogeneity assignment $|\cdot|_\mathfrak{s}$ we define the following notation. For any smooth function $\psi$ on $\mathbf{R}^d$ we define

$$\|\psi\|_\mathfrak{s} \stackrel{\text{def}}{=} \max \left\{ \sup_{x \in \mathbf{R}^d} |D^k \psi(x)| : \; k \in \mathbf{N}^d, |k|_\mathfrak{s} \leq \left\lceil - \min_{\mathfrak{t} \in \mathfrak{L}_-} |\mathfrak{t}|_\mathfrak{s} \right\rceil \right\} .$$

We define $\mathscr{B}_\mathfrak{s}$ to be the set of all smooth functions $\psi$ on $\mathbf{R}^d$ with $\|\psi\|_\mathfrak{s} \leq 1$ and $\psi(x) = 0$ for all $x \in \mathbf{R}^d$ with $|x| \geq 1$.

Next we define a family of pseudometrics on $\mathscr{M}_\infty(\mathscr{T})$ as follows. For $(\Pi, \Gamma)$, $(\bar{\Pi}, \bar{\Gamma}) \in \mathscr{M}_\infty(\mathscr{T})$, $\alpha \in A$, and compact $\mathfrak{K} \subset \mathbf{R}^d$, we define

$$\|(\Pi, \Gamma); (\bar{\Pi}, \bar{\Gamma})\|_{\alpha, \mathfrak{K}} \stackrel{\text{def}}{=} \|\Pi - \bar{\Pi}\|_{\alpha; \mathfrak{K}} + \|\Gamma - \bar{\Gamma}\|_{\alpha; \mathfrak{K}} \tag{2.6}$$

where

$$\|\Pi - \bar{\Pi}\|_{\alpha; \mathfrak{K}} \stackrel{\text{def}}{=} \sup \left\{ \frac{|\langle (\Pi_x - \bar{\Pi}_x)\tau, \varphi_x^\lambda \rangle|}{\|\tau\|_\alpha \lambda^\alpha} : \begin{array}{l} x \in \mathfrak{K}, \; \tau \in \mathscr{T}_\alpha \\ \lambda \in (0, 1], \; \varphi \in \mathscr{B}_\mathfrak{s}. \end{array} \right\} ,$$

$$\|\Gamma - \bar{\Gamma}\|_{\alpha;\mathfrak{K}} \overset{\text{def}}{=} \sup \left\{ \frac{\|(\Gamma_{xy} - \bar{\Gamma}_{xy})\tau, \varphi_x^\lambda\|_\beta}{\|\tau\|_\alpha \cdot |x - y|^{\alpha - \beta}} : \begin{array}{l} x, y \in \mathfrak{K}, \ x \neq y, \ \varphi \in \mathscr{B}_\mathfrak{s} \\ \tau \in \mathscr{T}_\alpha, \beta \in A, \beta < \alpha. \end{array} \right\}.$$

Endowing $\mathscr{M}_\infty(\mathscr{T})$ with the system of pseudometrics $(\|\bullet; \bullet\|_{\alpha,\mathfrak{K}})_{\alpha,\mathfrak{K}}$, we can view $\mathscr{M}_\infty(\mathscr{T})$ as a metric space. The space $\mathscr{M}_\infty$ is certainly not complete – we denote by $\mathscr{M}_0(\mathscr{T})$ the completion of $\mathscr{M}_\infty(\mathscr{T})$. Elements of $\mathscr{M}_0(\mathscr{T})$ are called admissible models – they include models which are *distributional* in character.

Viewing the map $\Phi(\cdot)$'s target space as an appropriate Hölder-Besov space of distributions one can prove estimates that show $\Phi$ admits a unique extension to $\mathscr{M}_0(\mathscr{T})$. However at this point we have reached the limit of what one can do with a purely pathwise analysis. As we stated in the introduction, if one fixes a realization of $\zeta$ then the deterministic convergence of $\zeta_\varepsilon$ to $\zeta$ as $\varepsilon \downarrow 0$ cannot be turned into a similar statement about convergence of the canonical lifts $Z_{\text{can}}^{\zeta_\varepsilon}$ – the latter will in general not converge in $\mathscr{M}_0(\mathscr{T})$.

## 2.4   Probabilistic assumptions and the main theorem

Switching to a probabilistic approach, even when one views $\zeta$ as a random noise the model-valued random variables $Z_{\text{can}}^{\zeta_\varepsilon}$ will often not converge in a probabilistic sense – one may still need to renormalize.

Before stating our main theorem guaranteeing that this can be done we formalize the stochastic setting we are working in.

**Definition 2.13** We denote by $\mathcal{M}(\Omega_\infty)$ the collection of all $\Omega_\infty$-valued random variables $\xi$ which satisfy all of the following properties.

1. For every $z \in \mathbf{R}^d$, $\mathfrak{t} \in \mathfrak{L}_-$ and $k \in \mathbf{N}^d$, the random variable $D^k \xi_\mathfrak{t}(z)$ has finite moments of all orders with its first moment vanishing.
2. The law of $\xi$ is invariant under the action of $\mathbf{R}^d$ on $\Omega_\infty$ given by translations.

These random variables are assumed to be defined on some underlying probability space $\Omega$ and, when needed, we write $\omega$ for an element of $\Omega$. We will however mostly follow usual custom in suppressing the dependence on $\omega$ from our notations.

Given $\xi \in \mathcal{M}(\Omega_\infty)$ we will describe below in (3.5) and (4.27) how to construct what we call the "BPHZ renormalized lift" of a realization $\xi(\omega)$, henceforth denoted by $Z_{\text{BPHZ}}^{\xi(\omega)}$.

Note that this is a slight abuse of notations since this object actually depends both on a specific realisation $\xi(\omega)$ and on the distribution of $\xi$. We often write $Z_{\text{BPHZ}}^\xi$ for the corresponding model-valued random variable. We will also see in Lemma 4.23 below that this construction is equivalent to that given in [Hai16, Eq. 3.1] and [BHZ16, Thm 6.17]. Additionally we define the space of stationary random models the BPHZ lift takes values in.

**Definition 2.14** Given a regularity structure $\mathscr{T} = (\mathscr{T}, G)$, we denote by $\mathscr{M}_{\text{rand}}(\mathscr{T})$ the space of all $\mathscr{M}_0(\mathscr{T})$-valued random variables $(\Pi, \Gamma)$ which are stationary in the

sense that there exists an action $\tau$ of $\mathbf{R}^d$ onto the underlying probability space by measure preserving maps such that, for every $x, y, h \in \mathbf{R}^d$, the identities

$$\Pi_{x+h}(\omega) = \Pi_x(\tau_h \omega) , \qquad \Gamma_{x+h,y+h}(\omega) = \Gamma_{x,y}(\tau_h \omega) ,$$

hold almost surely.

Then, after fixing a regularity structure $\mathscr{T}$, we can view the BPHZ lift as a map $Z^\bullet_{\text{BPHZ}} : \mathcal{M}(\Omega_\infty) \to \mathscr{M}_{\text{rand}}(\mathscr{T})$ corresponding to $\xi \mapsto Z^\xi_{\text{BPHZ}}$.

Note here that $Z_{\text{BPHZ}}$ maps a space of random variables to another space of random variables defined over the same probability space. Before continuing we state a less technical and more specialized version of our main theorem. Henceforth, given $d \geq 1$, we call a random element of $\Omega_0 = \bigoplus_{\mathfrak{t} \in \mathfrak{L}_-} \mathcal{D}'(\mathbf{R}^d)$ simply a *noise*. Given furthermore a scaling $\mathfrak{s}$, a finite set of types $\mathfrak{L}_-$, and a homogeneity assignment $|\cdot|_\mathfrak{s}$, we say that a Gaussian noise $\xi$ is *compatible* with the homogeneity assignment if, for every $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{L}_-$ there exists a distribution $C_{\mathfrak{t}_1,\mathfrak{t}_2} \in \mathcal{D}'(\mathbf{R}^d)$ with singular support contained in $\{0\}$ such that, for every $f \in \mathcal{D}'(\mathbf{R}^d)$ one has

$$\mathbf{E}[\xi_{\mathfrak{t}_1}(f)\xi_{\mathfrak{t}_2}(f)] = C_{\mathfrak{t}_1,\mathfrak{t}_2}\left( \int_{\mathbf{R}^d} dy f(y - \cdot) f(y) \right) .$$

Denoting by $x \mapsto C_{\mathfrak{t}_1,\mathfrak{t}_2}(x)$ the smooth function representing $C_{\mathfrak{t}_1,\mathfrak{t}_2}$ away from $0$, we also require that for any $g \in \mathcal{D}'(\mathbf{R}^d)$ with $D^k g(0) = 0$ for every $k \in \mathbf{N}^d$ with $|k|_\mathfrak{s} < -|\mathfrak{t}_1|_\mathfrak{s} - |\mathfrak{t}_2|_\mathfrak{s} - |\mathfrak{s}|$ one has

$$C_{\mathfrak{t}_1,\mathfrak{t}_2}(g) = \int_{\mathbf{R}^d} dx\, C_{\mathfrak{t}_1,\mathfrak{t}_2}(x)g(x) .$$

Finally, we impose that there exists $\kappa > 0$ such that for every $k \in \mathbf{N}^d$ with $|k|_\mathfrak{s} \leq 6|\mathfrak{s}|$ and $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{L}_-$ one has

$$\sup_{x \in \mathbf{R}^d \setminus \{0\}} |D^k C_{\mathfrak{t}_1,\mathfrak{t}_2}(x)| \cdot |x|^{-|\mathfrak{t}_1|_\mathfrak{s} - |\mathfrak{t}_2|_\mathfrak{s} + \delta + |k|_\mathfrak{s}} < \infty . \tag{2.7}$$

Furthermore, we say that a family $\xi^{(n)}$ of compatible Gaussian noises is *uniformly compatible* if the quantity (2.7) is bounded for the corresponding covariance functions $C^{(n)}_{\mathfrak{t}_1,\mathfrak{t}_2}$, uniformly over $n$.

**Theorem 2.15** *Fix $d \geq 1$, a scaling $\mathfrak{s}$, finite sets of types $\mathfrak{L}_-$ and $\mathfrak{L}_+$, a homogeneity assignment $|\cdot|_\mathfrak{s}$, a rule $R$ which is complete and subcritical with respect to $|\cdot|_\mathfrak{s}$, and write $\mathscr{T} = (\mathcal{T}, G)$ for the corresponding regularity structure. Assume that for every basis vector $T^\mathfrak{n}_\mathfrak{e} \in \mathcal{T}$ and every subtree $S$ of $T$ with $|N(S)| \geq 2$, the following holds:*

- *For any non-empty leaf typed set $A$ with $\mathfrak{t}(A) \subset \mathfrak{t}(L(T))$ (in the sense of sets, not multi-sets) and $|A| + |L(S)|$ even, one has $|S^0_\mathfrak{e}|_\mathfrak{s} + |\mathfrak{t}(A)|_\mathfrak{s} + |A| \cdot |\mathfrak{s}| > 0$.*
- *For every $u \in L(S)$ one has $|S^0_\mathfrak{e}|_\mathfrak{s} - |\mathfrak{t}(u)|_\mathfrak{s} > 0$.*

- $|S^0_{\mathfrak{e}}|_{\mathfrak{s}} > -\frac{|\mathfrak{s}|}{2}$.

*Then, for every compatible Gaussian noise $\xi$, $Z^{\bullet}_{\text{BPHZ}}$ extends continuously to $\xi$ in the following sense. There exists a unique element $Z^{\xi}_{\text{BPHZ}} \in \mathscr{M}_{\text{rand}}(\mathscr{T})$ such that, for any uniformly compatible sequence of Gaussian noises $(\xi_n)_{n \in \mathbf{N}}$ with $\lim_{n \to \infty} \xi_n = \xi$ in probability in $\Omega_0$, one has $\lim_{n \to \infty} Z^{\xi_n}_{\text{BPHZ}} = Z^{\xi}_{\text{BPHZ}}$ in probability in $\mathscr{M}_0(\mathscr{T})$.*

*Proof.* This is a straightforward consequence of Theorem 2.34 below.    □

The outline of this section is as follows. Mirroring our discussion of models, we will view $\mathcal{M}(\Omega_\infty)$ as a subset of a larger space of random distributions $\mathcal{M}(\Omega_0)$. We will then introduce some quantitative tools for looking at these random distributions which allow us to show that a notion of probabilistic "convergence" of a sequence $(\xi_n)_{n \in \mathbf{N}} \subset \mathcal{M}(\Omega_\infty)$ yields stochastic $L^p$ convergence of the corresponding BPHZ models $(Z^{\xi_n}_{\text{BPHZ}})_{n \in \mathbf{N}}$.

### 2.4.1    Quantitative estimates on cumulants

Our quantitative assumptions on the driving noise all involve their *cumulants*. Throughout the paper, for any (multi)set $B$ of real-valued random variables with finite moments of all orders defined on some underlying probability space $\Omega$, we write $\mathbf{E}^c[B]$ for the joint cumulant of the elements of $B$. Recall that, writing $\mathbf{E}[B]$ as a shorthand for the expectation of the product of all elements of $B$, the cumulant is defined recursively by the identities $\mathbf{E}^c[\emptyset] = 1$ and then $\mathbf{E}[B] = \sum_{\pi \in \mathcal{P}(B)} \prod_{A \in \pi} \mathbf{E}^c[A]$, where $\mathcal{P}(B)$ denotes the set of all partitions of $B$ into disjoint subsets, see for example [GJ87, Sec. 13.5].

We introduce the "large diagonal" of the set $\mathbf{R}^{dN}$ of configurations of $N$ points in $\mathbf{R}^d$ by

$$\text{Diag}_N \stackrel{\text{def}}{=} \left\{ x \in \mathbf{R}^{dN} \,:\, \exists 1 \leq i < j \leq d \text{ with } x_i = x_j \right\}.$$

While we allow for our random driving noises to be quite singular we impose some strong, but natural, regularity assumptions on their cumulants – this is encoded via the following notion.

**Definition 2.16** Suppose that, for $N \geq 2$, we are given a collection[8] $\{\zeta_n\}_{n=1}^N$ of $\mathcal{D}'(\mathbf{R}^d)$-valued random variables that each have moments of all orders, invariant in joint law under the natural action of $\mathbf{R}^d$ on this collection given by simultaneous translation, and are all defined on the same probability space.

We say that such a collection *admits pointwise cumulants* if there exists a distribution $F \in \mathcal{D}'(\mathbf{R}^{dN})$ with the singular support of $F$ contained in $\text{Diag}_N$ (we denote by $F(x)$ the corresponding function, which is smooth away from $\text{Diag}_N$) and satisfying, for every collection of test functions $f_1, \ldots, f_N \in \mathcal{D}(\mathbf{R}^d)$ the identity

$$\mathbf{E}^c[\{\zeta_n(f_n)\}_{n=1}^N] = F(f_1 \otimes \cdots \otimes f_N) \tag{2.8}$$

---

[8]This collection may be a multiset in that the same random variable may appear more than once.

where $(f_1 \otimes \cdots \otimes f_N)(x) \stackrel{\text{def}}{=} \prod_{n=1}^{N} f_n(x_n) \in \mathcal{D}(\mathbf{R}^{dN})$.

As part of this definition, we also enforce two regularity conditions, one on second cumulants and the other on higher cumulants. When $N = 2$, translation invariance guarantees that there exists a unique $\mathscr{R}[F] \in \mathcal{D}'(\mathbf{R}^d)$ such that for every $f, g \in \mathcal{D}'(\mathbf{R}^d)$ one has

$$F(f \otimes g) = \mathscr{R}[F](f * \text{Ref}(g)) \tag{2.9}$$

where $\text{Ref}(g)(x) \stackrel{\text{def}}{=} g(-x)$ and $*$ denotes convolution. Note that the singular support of $\mathscr{R}(F)$ is given by $\{0\}$. Given a distribution $G \in \mathcal{D}'(\mathbf{R}^d)$ with singular support contained in $\{0\}$ and $r \in \mathbf{R}$ we say $G$ is of order $r$ at $\{0\}$ if, for $f \in \mathcal{D}(\mathbf{R}^d)$ with $D^k f(0) = 0$ for every $k \in \mathbf{N}^d$ with $|k|_{\mathfrak{s}} < r$, one has

$$G(f) = \int_{\mathbf{R}^d} f(y)G(y)\, dy \,,$$

When $N \geq 3$, we enforce that $F(x)$ be a locally integrable function on $\mathbf{R}^{dN}$ and for the distribution $F$ to be given by integration against $F(x)$.

For a collection $\{\zeta_n\}_{n=1}^{N}$ admitting pointwise cumulants $F$ as above, we will write $\mathbf{E}^c[\{\zeta_n(x_n)\}_{n=1}^{N}]$ for the value $F(x_1, \ldots, x_N)$ (provided that $x \notin \text{Diag}_N$) and $\mathbf{E}^c[\{\zeta_n\}_{n=1}^{N}]$ for the distribution $F$.

**Definition 2.17**  We define $\mathcal{M}(\Omega_0)$ to be the set of all $\bigoplus_{\mathfrak{t} \in \mathfrak{L}_-} \mathcal{D}'(\mathbf{R}^d)$-valued random variables $\xi$ that satisfy the following properties.

- For every $\mathfrak{t} \in \mathfrak{L}_-$ and $f \in \mathcal{D}(\mathbf{R}^d)$, $\xi_{\mathfrak{t}}(f)$ has moments of all orders.
- The law of $\xi$ is stationary, i.e. invariant under the action of $\mathbf{R}^d$ on $\bigoplus_{\mathfrak{t} \in \mathfrak{L}_-} \mathcal{D}'(\mathbf{R}^d)$ by translation.
- For every $N \geq 2$ and any map $\mathfrak{t} : [N] \to \mathfrak{L}_-$ the collection of generalised random fields $\{\xi_{\mathfrak{t}(n)}\}_{n=1}^{N}$ admits pointwise cumulants.

Clearly, one has $\mathcal{M}(\Omega_\infty) \subset \mathcal{M}(\Omega_0)$. Also, for $\xi, \bar{\xi} \in \mathcal{M}(\Omega_0)$ defined on the same probability space, we say that $\xi$ and $\bar{\xi}$ jointly admit pointwise cumulants if, for every $N \geq 2$, $1 \leq M \leq N$, and $\mathfrak{t} : [N] \to \mathfrak{L}_-$ the multiset $\{\xi_{\mathfrak{t}(m)}\}_{m=1}^{M} \sqcup \{\bar{\xi}_{\mathfrak{t}(n)}\}_{n=M+1}^{N}$ admits pointwise cumulants.

In order to prove theorems that give near optimal results it is natural to take advantage of the fact that for certain random noise-types many cumulants vanish: the most obvious case is that of Gaussian noise where all cumulants of order greater than 2 vanish, but another interesting case is that of noise belonging to a Wiener chaos of fixed order. We define a set that indexed all possible cumulants by

$$\mathfrak{L}_{\text{Cum}}^{\text{all}} \stackrel{\text{def}}{=} \{(\mathfrak{t}, [M]) : M \in \mathbf{N}, \text{ and } \mathfrak{t} : [M] \to \mathfrak{L}_-\} \,.$$

We will then consider noises such that all cumulants outside of some subset $\mathfrak{L}_{\text{Cum}} \subset \mathfrak{L}_{\text{Cum}}^{\text{all}}$ vanish. We make the following natural assumptions on this subset.

**Assumption 2.18** *We assume that we have fixed a set $\mathfrak{L}_{\mathrm{Cum}} \subset \mathfrak{L}_{\mathrm{Cum}}^{\mathrm{all}}$ of allowable cumulants with the following properties:*

1. *For every $(\mathfrak{t}, [M]) \in \mathfrak{L}_{\mathrm{Cum}}$ one has $M \geq 2$.*
2. *$\mathfrak{L}_{\mathrm{Cum}}$ is closed under permutations - that is if $(\mathfrak{t}, [M]) \in \mathfrak{L}_{\mathrm{Cum}}$ then for every permutation $\sigma : [M] \to [M]$ one has $(\mathfrak{t} \circ \sigma, [M]) \in \mathfrak{L}_{\mathrm{Cum}}$. It is natural to think of $\mathfrak{L}_{\mathrm{Cum}}$ as a collection of "leaf typed multisets", thus it makes sense to ask, given a finite set $B$ and a type map $\mathfrak{t} : B \to \mathfrak{L}_-$, whether $(\mathfrak{t}, B) \in \mathfrak{L}_{\mathrm{Cum}}$.*
3. *If $(\mathfrak{t}, B) \in \mathfrak{L}_{\mathrm{Cum}}$ then for any $A \subset B$ with $|A| \geq 2$ one has $(\mathfrak{t}, A) \in \mathfrak{L}_{\mathrm{Cum}}$.*
4. *For every $\bar{\mathfrak{t}} \in \mathfrak{L}_-$ one has $(\mathfrak{t}, [2]) \in \mathfrak{L}_{\mathrm{Cum}}$ where $\mathfrak{t}(1) = \mathfrak{t}(2) = \bar{\mathfrak{t}}$.*

**Remark 2.19** Our main theorem will enforce an assumption of "super-regularity" on the semi-decorated trees. This notion of super-regularity will depend on our choice of $\mathfrak{L}_{\mathrm{Cum}}$, becoming more stringent as this set becomes richer. The second item in the above assumption on $\mathfrak{L}_{\mathrm{Cum}}$ enforces a type of "completeness" that guarantees that the constraint of super-regularity is suitably strong.

**Remark 2.20** Given $(\mathfrak{t}, [N]) \in \mathfrak{L}_{\mathrm{Cum}}^{\mathrm{all}}$ with $\mathfrak{t}(i) = \bar{\mathfrak{t}}$ for $i \in [N]$ we sometimes write $(\bar{\mathfrak{t}}, [N])$ instead of $(\mathfrak{t}, [N])$.

We also define $\mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}})$ and $\mathcal{M}(\Omega_\infty, \mathfrak{L}_{\mathrm{Cum}})$ to be the subsets of $\mathcal{M}(\Omega_0)$ and $\mathcal{M}(\Omega_\infty)$, respectively, consisting of those elements $\xi$ with $\mathbf{E}^c[\{\xi_{\mathfrak{t}(n)}\}_{n=1}^N] = 0$ for every $(\mathfrak{t}, [M]) \in \mathfrak{L}_{\mathrm{Cum}}^{\mathrm{all}} \setminus \mathfrak{L}_{\mathrm{Cum}}$. We now define the following family of "norms" on $\mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}})$. First, we fix, for the entire paper, a smooth non-negative function $u : \mathbf{R}^d \to u$ satisfying $|u(x)| \leq 1$ for all $x$, $u(x) = 1$ for $|x| \leq 1$, and $u(x) = 0$ for $|x| \geq 2$. We then set, for each $k \in \mathbf{N}^d$, $p_k(x) = u(x)x^k$.

**Definition 2.21** For $\xi \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}})$ and $N \geq 2$, we set $\|\xi\|_{N, |\cdot|_{\mathfrak{s}}}$ to be equal to

$$
\mathrm{Diag}(\xi) \vee \max_{\substack{(\mathfrak{t}, [M]) \in \mathfrak{L}_{\mathrm{Cum}} \\ 2 \leq M \leq N}} \sup_{\substack{x \in \mathbf{R}^{dM} \\ x \notin \mathrm{Diag}_M}} \left| \mathbf{E}^c[\{\xi_{\mathfrak{t}(m)}(x_m)\}_{m=1}^M] \right| \left( \sup_{1 \leq i < j \leq M} |x_i - x_j| \right)^{-|\mathfrak{t}([M])|_{\mathfrak{s}}},
$$

where we use the notation $[M] = \{1, \ldots, M\}$ and $|\mathfrak{t}([M])|_{\mathfrak{s}}$ is as in Definition 2.1.

The quantity $\mathrm{Diag}(\xi)$ is defined as follows. If there exists any $(\mathfrak{t}, [2]) \in \mathfrak{L}_{\mathrm{Cum}}$ such that $\mathscr{R}[\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)}, \xi_{\mathfrak{t}(2)}\}]]$ is not of order $(-|\mathfrak{t}([2])|_{\mathfrak{s}} - |\mathfrak{s}|) \vee 0$ at $0$ then we set $\mathrm{Diag}(\xi) \overset{\mathrm{def}}{=} \infty$. Otherwise we set

$$
\mathrm{Diag}(\xi) \overset{\mathrm{def}}{=} \max_{(\mathfrak{t}, [2]) \in \mathfrak{L}_{\mathrm{Cum}}} \max_{\substack{k \in \mathbf{N}^d, \\ |k|_{\mathfrak{s}} < -|\mathfrak{t}([2])|_{\mathfrak{s}} - |\mathfrak{s}|}} \left| \mathscr{R}[\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)}, \xi_{\mathfrak{t}(2)}\}]](p_k) \right| . \tag{2.10}
$$

where $\mathscr{R}[\cdot]$ is as in (2.9).

**Remark 2.22** The point of the definition of $\mathrm{Diag}(\xi)$ is that in general we do not impose that the function $\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)}, \xi_{\mathfrak{t}(2)}\}]$ is locally integrable near the diagonal. In general, there are then several inequivalent ways of extending it to a distribution, and these are precisely parametrised by the values $\tilde{F}(p_k)$.

It turns out that this "norm" also naturally defines a "metric" on elements of $\mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}})$ which are defined on the same probability space. (We intentionally write "metric" in quotation marks since it does not satisfy the triangle inequality.)

**Definition 2.23**  For $\xi, \bar{\xi} \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}})$ defined on the same probability space and jointly admitting pointwise cumulants we set $\|\xi; \bar{\xi}\|_{N, |\cdot|_\mathfrak{s}}$ to be given by

$$\mathrm{Diag}(\xi; \bar{\xi}) \vee \max_{\substack{(\mathfrak{t}, [M]) \in \mathfrak{L}_{\mathrm{Cum}} \\ 2 \leq \hat{M} \leq M \leq N}} \sup_{\substack{x \in \mathbf{R}^{dM} \\ x \notin \mathrm{Diag}_M}} \left| \mathbf{E}^c \left[ \{ \hat{\bar{\xi}}_{\mathfrak{t}(i), i}(x_i) \}_{i=1}^M \right] \right| \left( \sup_{1 \leq i < j \leq M} |x_i - x_j| \right)^{-|\mathfrak{t}([M])|_\mathfrak{s}}$$

where, for any $\hat{M} \leq M$, the generalized random fields $\{ \hat{\bar{\xi}}_{\mathfrak{t}, i} \}_{i=1}^M$ are given by

$$\hat{\bar{\xi}}_{\mathfrak{t}, i} \overset{\text{def}}{=} \xi_\mathfrak{t} \mathbb{1}\{ i \leq \hat{M} \} - \bar{\xi}_\mathfrak{t} \mathbb{1}\{ i \geq \hat{M} \} . \tag{2.11}$$

The quantity $\mathrm{Diag}(\xi; \bar{\xi})$ is defined as follows. If there exists any $(\mathfrak{t}, [2]) \in \mathfrak{L}_{\mathrm{Cum}}$ and $\hat{M} \in \{1, 2\}$ such that $\mathscr{R}[\mathbf{E}^c[\{ \hat{\bar{\xi}}_{\mathfrak{t}(1)}, \hat{\bar{\xi}}_{\mathfrak{t}(2)} \}]]$ is not of order $|\mathfrak{t}([2])|_\mathfrak{s} - |\mathfrak{s}|$ at 0 then we set $\mathrm{Diag}(\xi; \bar{\xi}) \overset{\text{def}}{=} \infty$. Otherwise we set

$$\mathrm{Diag}(\xi; \bar{\xi}) \overset{\text{def}}{=} \max_{\substack{(\mathfrak{t}, [2]) \in \mathfrak{L}_{\mathrm{Cum}} \\ 1 \leq \hat{M} \leq 2}} \max_{\substack{k \in \mathbf{N}^d, \\ |k|_\mathfrak{s} < -|\mathfrak{t}([2])|_\mathfrak{s} - |\mathfrak{s}|}} |\mathscr{R}[\mathbf{E}^c[\{ \hat{\bar{\xi}}_{\mathfrak{t}(1)}, \hat{\bar{\xi}}_{\mathfrak{t}(2)} \}]](p_k)| . \tag{2.12}$$

If $\xi$ and $\bar{\xi}$ do not admit joint pointwise cumulants we set $\|\xi; \bar{\xi}\|_N = \infty$ for every $N$.

### 2.4.2   Super-regular semi-decorated trees

The following assumption on the set of types $\mathfrak{L}$ and on the set $\mathfrak{L}_{\mathrm{Cum}}$ from Assumption 2.18 will be in place throughout the entire paper.

**Assumption 2.24**  *For every* $\mathfrak{t}' \in \mathfrak{L}_-$ *one has* $|\mathfrak{t}'|_\mathfrak{s} > -|\mathfrak{s}|$ *and for every* $(\mathfrak{t}, [M]) \in \mathfrak{L}_{\mathrm{Cum}}$ *with* $M \geq 3$ *one has* $|\mathfrak{t}([M])|_\mathfrak{s} > (1 - M)|\mathfrak{s}|$.

**Remark 2.25**  The above assumption guarantees that only cumulants which need to be renormalized are second cumulants. There is no fundamental obstruction in renormalizing higher cumulants in our approach but we refrain from doing so since this would make the presentation harder to follow.

Next we state a definition and an assumption on semi-decorated trees we will need in order to prove our stochastic bounds.

**Definition 2.26**  Given finite leaf typed sets $A$ and $B$ with $\mathfrak{t}(B) \subset \mathfrak{t}(A)$ (in the sense of sets, not multi-sets) we define $j_A(B)$ to be the minimum value of $|\mathfrak{t}(C)| + |C| \cdot |\mathfrak{s}|$ where $C$ is a non-empty finite leaf typed set with $\mathfrak{t}(C) \subset \mathfrak{t}(A)$ (in the sense of sets, not multi-sets) and with the property that there exists a partition $\pi$ of $C \sqcup B$ with $(\mathfrak{t}, \tilde{B}) \in \mathfrak{L}_{\mathrm{Cum}}$ and $\tilde{B} \cap B \neq \emptyset$ for every $\tilde{B} \in \pi$.

**Remark 2.27** The quantity $j_A(B)$ represents the worse case homogeneity change one sees when the noises of $B$ form cumulants with some other noises drawn from multiple copies of $A$.

**Definition 2.28** Given a homogeneity assignment $|\cdot|_{\mathfrak{s}}$ and a set of cumulants $\mathfrak{L}_{\mathrm{Cum}}$, a semi-decorated tree $T_{\mathfrak{e}}^{\mathfrak{n}}$ is said to be $(|\cdot|_{\mathfrak{s}}, \mathfrak{L}_{\mathrm{Cum}})$-*super-regular* if for every subtree $S$ of $T$ with $|N(S)| > 1$ and $L(S) \neq \emptyset$ and for any $u \in L(S)$ one has

$$|S_{\mathfrak{e}}^0|_{\mathfrak{s}} + \frac{|\mathfrak{s}|}{2} \wedge (-|\mathfrak{t}(u)|_{\mathfrak{s}}) \wedge j_{L(T)}(L(S)) > 0 . \qquad (2.13)$$

### 2.4.3  Main results

We make use of translated and rescaled test functions, namely we write

$$\psi_z^\lambda(x_1, \ldots, x_d) \stackrel{\text{def}}{=} \lambda^{-|\mathfrak{s}|} \psi\big(\lambda^{-\mathfrak{s}_1}(x_1 - z_1), \ldots, \lambda^{-\mathfrak{s}_d}(x_d - z_d)\big) .$$

**Remark 2.29** In what follows we often shrink the co-domain and view $Z_{\text{\tiny BPHZ}}^{\bullet}$ as a map $Z_{\text{\tiny BPHZ}}^{\bullet} : \mathcal{M}(\Omega_\infty, \mathfrak{L}_{\mathrm{Cum}}) \to \mathcal{M}_{\mathrm{rand}}(\mathscr{T})$ given by $\xi \mapsto Z_{\text{\tiny BPHZ}}^{\xi}$. All of our main theorems concern bounds or continuity properties of this map and will take as assumptions conditions that depend on $\mathfrak{L}_{\mathrm{Cum}}$.

**Remark 2.30** In the statement of the main theorem which immediately follows and subsequent proofs, we will use generalisations $\|\bullet\|_{N,\mathfrak{c}}$ and $\|\bullet; \bullet\|_{N,\mathfrak{c}}$ of Definitions 2.21 and 2.23 as well as a notion of "$(\mathfrak{c}, |\cdot|_{\mathfrak{s}}, \mathfrak{L}_{\mathrm{Cum}})$-super-regularity" which generalizes Definition 2.28. Here $\mathfrak{c}$ is a parameter which will be called a *cumulant homogeneity* for $\mathfrak{L}_{\mathrm{Cum}}$ consistent with $|\cdot|_{\mathfrak{s}}$ – this is explained in Appendix A.2. As this notion is a bit technical we prefer to state the main theorem without introducing it. For now, the reader who doesn't wish to delve into these details could just keep in mind that there is an acceptable choice of the parameter $\mathfrak{c}$ for which $(\mathfrak{c}, |\cdot|_{\mathfrak{s}}, \mathfrak{L}_{\mathrm{Cum}})$-super-regularity reduces to the $|\cdot|_{\mathfrak{s}}$ super-regularity of Definition 2.28 and for which the quantities $\|\bullet\|_{N,\mathfrak{c}}$ and $\|\bullet; \bullet\|_{N,\mathfrak{c}}$ reduce to $\|\bullet\|_{N,|\cdot|_{\mathfrak{s}}}$ and $\|\bullet; \bullet\|_{N,|\cdot|_{\mathfrak{s}}}$, respectively, as noted in Remark A.28. In particular, these notions agree in the Gaussian case.

**Theorem 2.31** *Fix $d \geq 1$, a scaling $\mathfrak{s}$, finite sets of types $\mathfrak{L}_-$ and $\mathfrak{L}_+$, a homogeneity assignment $|\cdot|_{\mathfrak{s}}$, a set of cumulants $\mathfrak{L}_{\mathrm{Cum}}$ satisfying Assumption 2.24, and a cumulant homogeneity $\mathfrak{c}$ for $\mathfrak{L}_{\mathrm{Cum}}$ consistent with $|\cdot|_{\mathfrak{s}}$. Let $R$ be a complete subcritical rule and $\mathscr{T}$ be the corresponding reduced regularity structure.*

*Then the map $Z_{\text{\tiny BPHZ}}^{\bullet} : \mathcal{M}(\Omega_\infty, \mathfrak{L}_{\mathrm{Cum}}) \to \mathcal{M}_{\mathrm{rand}}(\mathscr{T})$ has the following properties.*

*For every compact $\mathfrak{K} \subset \mathbf{R}^d$, $\xi \in \mathcal{M}(\Omega_\infty, \mathfrak{L}_{\mathrm{Cum}})$, $p \in \mathbf{N}$, and $\tau \stackrel{\text{def}}{=} T_{\mathfrak{e}}^{\mathfrak{n}} \in B_\circ$ which is $(\mathfrak{c}, |\cdot|_{\mathfrak{s}}, \mathfrak{L}_{\mathrm{Cum}})$-super regular, there exists $C_{\tau,p}$ such that, writing $Z_{\text{\tiny BPHZ}}^{\xi} = (\hat{\Pi}^\xi, \hat{\Gamma}^\xi)$, one has*

$$\mathbf{E}\Big(\hat{\Pi}_z^\xi[\tau](\psi_z^\lambda)\Big)^{2p} \leq C_{\tau,p} \Big( \prod_{e \in K(T)} \|K_{\mathfrak{t}(e)}\|_{|\mathfrak{t}(e)|_{\mathfrak{s}}, m} \Big)^{2p} \|\xi\|_{N,\mathfrak{c}} \lambda^{2p|\tau|_{\mathfrak{s}}} , \qquad (2.14)$$

*where $m \overset{def}{=} 2|\mathfrak{s}| \cdot |N(T)|$ and $N \overset{def}{=} 2p|L(T)|$, uniformly over $\psi \in \mathscr{B}_{\mathfrak{s}}$, $\lambda \in (0,1]$ and $z \in \mathfrak{K}$. Moreover, with the same notation and conventions as above, for any $\bar{\xi} \in \mathcal{M}(\Omega_\infty)$ defined on the same probability space one has*

$$\mathbf{E}\left[\left(\hat{\Pi}_z^\xi - \hat{\Pi}_z^{\bar{\xi}}\right)[\tau](\psi_z^\lambda)\right]^{2p} \tag{2.15}$$
$$\leq C_{\tau,p}\left(\prod_{e \in K(T)} \|K_{\mathfrak{t}(e)}\|_{|\mathfrak{t}(e)|_{\mathfrak{s}},m}\right)^{2p}(\|\xi\|_{N,\mathfrak{c}} \vee \|\bar{\xi}\|_{N,\mathfrak{c}})\|\xi;\bar{\xi}\|_{N,\mathfrak{c}}\lambda^{2p|\tau|_{\mathfrak{s}}} .$$

**Definition 2.32** Given homogeneity assignments $|\cdot|_{\mathfrak{s}}$ and $[\![\cdot]\!]_{\mathfrak{s}}$ and a rule $R$ which is complete and subcritical with respect to $[\![\cdot]\!]_{\mathfrak{s}}$, we say $|\cdot|_{\mathfrak{s}}$ is $R$-consistent with $[\![\cdot]\!]_{\mathfrak{s}}$ if all of the following conditions hold:

- $[\![\cdot]\!]_{\mathfrak{s}}$ satisfies Assumptions 2.24.
- $R$ is also a complete and subcritical rule with respect to $[\![\cdot]\!]_{\mathfrak{s}}$.
- The collection $B_-$ of semi-decorated trees generated by the rule $R$ under either $|\cdot|_{\mathfrak{s}}$ and $[\![\cdot]\!]_{\mathfrak{s}}$ coincide.
- For any semi-decorated tree $T_{\mathfrak{e}}^{\mathfrak{n}} \in B_-$, one has $\lfloor |T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} \rfloor = \lfloor [\![T_{\mathfrak{e}}^{\mathfrak{n}}]\!]_{\mathfrak{s}} \rfloor$.

Combining Theorem 2.31 with the proof of [Hai14, Thm 10.7] immediately gives the following theorem.

**Theorem 2.33** *Fix $d \geq 1$, a scaling $\mathfrak{s}$, finite sets of types $\mathfrak{L}_-$ and $\mathfrak{L}_+$, a homogeneity assignment $[\![\cdot]\!]_{\mathfrak{s}}$ and a set of cumulants $\mathfrak{L}_{\mathrm{Cum}}$ satisfying Assumption 2.24, and a rule $R$ which is complete and subcritical with respect to $[\![\cdot]\!]_{\mathfrak{s}}$. Denote by $\mathscr{T} = (A, \mathscr{T}, G)$ be the corresponding reduced regularity structure with $[\![\cdot]\!]_{\mathfrak{s}}$ serving as the homogeneity assignment.*

*Fix $\kappa > 0$, a second homogeneity assignment $|\cdot|_{\mathfrak{s}}$, and a cumulant homogeneity $\mathfrak{c}$ on $\mathfrak{L}_{\mathrm{Cum}}$ consistent with $|\cdot|_{\mathfrak{s}}$ with the following properties:*

- *$|\cdot|_{\mathfrak{s}}$ is $R$-consistent with $[\![\cdot]\!]_{\mathfrak{s}}$.*
- *For any $\alpha \in A$ with $\alpha < 0$, every semi-decorated tree $T_{\mathfrak{e}}^{\mathfrak{n}} \in \mathscr{T}_\alpha$ is $(\mathfrak{c}, |\cdot|_{\mathfrak{s}}, \mathfrak{L}_{\mathrm{Cum}})$-super regular and*
$$|T_{\mathfrak{e}}^{\mathfrak{n}}|_{\mathfrak{s}} \geq [\![T_{\mathfrak{e}}^{\mathfrak{n}}]\!]_{\mathfrak{s}} + \kappa.$$

*Then the map $Z_{\mathrm{BPHZ}}^\bullet : \mathcal{M}(\Omega_\infty, \mathfrak{L}_{\mathrm{Cum}}) \to \mathcal{M}_{\mathrm{rand}}(\mathscr{T})$ has the following property.*

*For every compact set $\mathfrak{K} \subset \mathbf{R}^d$, $\alpha \in A$, and $p \in \mathbf{N}$ there exists $C$ such that for any $\xi, \bar{\xi} \in \mathcal{M}(\Omega_\infty, \mathfrak{L}_{\mathrm{Cum}})$ one has*

$$\mathbf{E}\|Z_{\mathrm{BPHZ}}^\xi; Z_{\mathrm{BPHZ}}^{\bar{\xi}}\|_{\alpha,\mathfrak{K}}^{2p} \leq C(\|\xi\|_{N,\mathfrak{c}} \vee \|\bar{\xi}\|_{N,\mathfrak{c}})(\|\xi;\bar{\xi}\|_{N,\mathfrak{c}}) , \text{ where}$$
$$N \overset{def}{=} 2\left(p \vee \left\lceil \frac{|\mathfrak{s}|}{\kappa} \right\rceil\right) \cdot \max\{|L(T)| : T_{\mathfrak{e}}^{\mathfrak{n}} \in \mathscr{T}_\alpha \text{ for } \alpha \in A, \ \alpha < 0\} .$$

The following theorem states that if we impose some mild regularity conditions then the map $\xi \mapsto Z_{\mathrm{BPHZ}}^\xi$ admits some nice continuity and regularity properties.

**Theorem 2.34** *Fix $d \geq 1$, a scaling $\mathfrak{s}$, finite sets of types $\mathfrak{L}_-$ and $\mathfrak{L}_+$, a homogeneity assignment $\|\cdot\|_\mathfrak{s}$ and a set of cumulants $\mathfrak{L}_{\mathrm{Cum}}$ satisfying Assumption 2.24, and a cumulant homogeneity $\mathfrak{c}$ on $\mathfrak{L}_{\mathrm{Cum}}$ consistent with $\|\cdot\|_\mathfrak{s}$. Let $R$ be a rule which is complete and subcritical with respect to $\|\cdot\|_\mathfrak{s}$.*

*Fix $\kappa$ satisfying*

$$0 < \kappa < \frac{1}{2} \min_{(\mathfrak{t},[2]) \in \mathfrak{L}_{\mathrm{Cum}}} \left( \lceil -\|\mathfrak{t}([2])\|_\mathfrak{s} \rceil + \|\mathfrak{t}([2])\|_\mathfrak{s} \right),$$

*and small enough such that the homogeneity assignment*

$$[\![\mathfrak{t}]\!]_\mathfrak{s} \stackrel{def}{=} \|\mathfrak{t}\|_\mathfrak{s} \cdot (1 + \kappa \mathbb{1}\{\mathfrak{t} \in \mathfrak{L}_-\}) \text{ for every } \mathfrak{t} \in \mathfrak{L} .$$

*is $R$-consistent with $\|\cdot\|$. Denote by $\mathscr{T} = (A, \mathcal{T}, G)$ the corresponding reduced regularity structure generated by $R$ with $[\![\cdot]\!]_\mathfrak{s}$ as its homogeneity assignment.*

*Assume that for every $\alpha \in A$ with $\alpha < 0$ and every $T_\mathfrak{e}^\mathfrak{n} \in \mathscr{T}_\alpha$, the decorated tree $T_\mathfrak{e}^\mathfrak{n}$ is $(\tilde{\mathfrak{c}}, [\![\cdot]\!]_\mathfrak{s}, \mathfrak{L}_{\mathrm{Cum}})$-super-regular where $\tilde{\mathfrak{c}}$ is the $\kappa$-penalization[9] of $\mathfrak{c}$. Define*

$$\mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}}, \mathfrak{c}) \stackrel{def}{=} \{\xi \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}}) : \|\xi\|_{N,\mathfrak{c},2} < \infty\} ,$$

*where*

$$N \stackrel{def}{=} 2 \cdot \left( 2 \vee \left\lceil \frac{2|\mathfrak{s}|}{\kappa} \right\rceil \right) \max\{|L(T)| : T_\mathfrak{e}^\mathfrak{n} \in \mathscr{T}_\alpha \text{ for } \alpha \in A, \ \alpha < 0\} .$$

*Then there is a unique extension of $Z_{\mathrm{BPHZ}}^\bullet : \mathcal{M}(\Omega_\infty, \mathfrak{L}_{\mathrm{Cum}}) \to \mathscr{M}_{\mathrm{rand}}(\mathscr{T})$ to all of $\mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}}, \mathfrak{c})$ which satisfies the following continuity property: for any $(\xi_n)_{n \in \mathbf{N}} \subset \mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}}, \mathfrak{c})$ and $\xi \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}}, N, \mathfrak{c})$ such that*

*(i) $\sup_{n \geq 0} \|\xi_n\|_{N,\mathfrak{c},1} < \infty$ and*

*(ii) $\xi_n \to \xi$ in probability on $\bigoplus_{\mathfrak{t} \in \mathfrak{L}_-} \mathcal{D}'(\mathbf{R}^d)$*

*$Z_{\mathrm{BPHZ}}^{\xi_n}$ converges to $Z_{\mathrm{BPHZ}}^\xi$ in probability on $\mathscr{M}_0(\mathscr{T})$.*

**Remark 2.35** Of course, this extended map $Z_{\mathrm{BPHZ}}^\bullet$ also satisfies a weaker version of the above property – if (ii) above is replaced by the assumption that $\xi_n \to \xi$ in law on $\bigoplus_{\mathfrak{t} \in \mathfrak{L}_-} \mathcal{D}'(\mathbf{R}^d)$ then $Z_{\mathrm{BPHZ}}^{\xi_n}$ converges to $Z_{\mathrm{BPHZ}}^\xi$ in law on $\mathscr{M}_0(\mathscr{T})$.

**Remark 2.36** We define $Z_{\mathrm{BPHZ}}^\xi$, for any $\xi \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}}, N, \mathfrak{c})$, as the $L^2$ limit $\lim_{n \to \infty} Z_{\mathrm{BPHZ}}^{\xi_n}$ where we set $\xi_n = \xi * \eta_n$ where $(\eta_n)_{n \in \mathbf{N}}$ is a sequence of approximate identities converging in an appropriate sense to a Dirac delta function as $n \to \infty$. Thus we can see this extended $Z_{\mathrm{BPHZ}}^\bullet$ as a map from $\mathcal{M}(\Omega_0, \mathfrak{L}_{\mathrm{Cum}}, N, \mathfrak{c})$ into the space of all measurable maps from $\Omega_0$ to $\mathscr{M}_0(\mathscr{T})$.

**Remark 2.37** Henceforth all of our estimates are claimed to be uniform in $\lambda \in (0, 1]$, even if this is not explicitly stated.

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[9] see Definition B.2.

For the rest of the paper we fix a cumulant homogeneity $\mathfrak{c}$ on $\mathfrak{L}_{\text{Cum}}$ consistent with our fixed homogeneity assignment $|\cdot|_{\mathfrak{s}}$ and a $(\mathfrak{c}, |\cdot|_{\mathfrak{s}}, \mathfrak{L}_{\text{Cum}})$-super regular semi-decorated tree $\overline{T}_{\overline{\mathfrak{c}}}^{\overline{\mathfrak{n}}} \in B_{\circ}$ for which we will prove Theorem 2.31. We gather all the steps to prove Theorem 2.31 and Theorem 2.34 in Section 10.

## 3   Renormalization of combinatorial trees and random fields

### 3.1   Colorings, a new decoration, more homogeneities, and identified forests

#### 3.1.1   Colorings

**Definition 3.1** A colored forest is a pair $(F, \hat{F})$ where $F$ is a typed rooted forest and the "coloring" $\hat{F}$ is a pair $(\hat{F}_1, \hat{F}_2)$ of disjoint subforests of $F$ with the property that $\varrho(\hat{F}_2) \subset \varrho(F)$.

Note that $\hat{F}$ induces a map[10] $\hat{F} : N_F \sqcup E_F \to \{0, 1, 2\}$ by setting $\hat{F}(u) = i$ if $u \in N_{\hat{F}_i} \sqcup E_{\hat{F}_i}$ and $\hat{F}(u) = 0$ otherwise. Clearly, one can recover the tuple $\hat{F}$ from the map $\hat{F}(\cdot)$.

There is a useful alternate notation for specifying colorings of a forest. Given $F$ and two disjoint subforests $A$ and $B$ of $F$ with $\varrho(B) \subset \varrho(F)$ we write $[A]_1 \sqcup [B]_2$ for the coloring $\hat{F} = (A, B)$. If one of the two forests is empty we may drop it, i.e. we may write $(F, [A]_1)$ instead of $(F, [A]_1 \sqcup [\mathbf{1}]_2)$. Additionally, for $i \in \{0, 1, 2\}$ we write $(F, i)$ for the colored forest with the constant coloring $\hat{F}(\cdot) = i$.

#### 3.1.2   Decorated colored forests

A decorated colored forest consists of a colored forest $(F, \hat{F})$ and three maps: $\mathfrak{n} : N(F) \to \mathbf{N}^d$, $\mathfrak{o} : N(\hat{F}_1) \to \mathbf{Z}^d \oplus \mathbf{Z}(\mathfrak{L})$, and $\mathfrak{e} : E_F \to \mathbf{N}^d$, where $\mathbf{Z}(\mathfrak{L})$ denotes the free $\mathbf{Z}$-module generated by $\mathfrak{L}$. Given a colored forest $(F, \hat{F})$ and decorations $\mathfrak{n}$, $\mathfrak{o}$, and $\mathfrak{e}$ on $F$ we denote the corresponding decorated colored forest by $(F, \hat{F})_{\mathfrak{e}}^{\mathfrak{n}, \mathfrak{o}}$.

For the trivial tree we just write $(\bullet, i)^{\mathfrak{n}, \mathfrak{o}}$ and also observe that the empty forest $\mathbf{1}$ is automatically a decorated colored forest. When working with a forest $(F, \hat{F})$ where $\hat{F}_1 = \mathbf{1}$ we abuse notation and write $\mathfrak{o} = 0$ instead of writing $\mathfrak{o} = \emptyset$. Moreover, if the $\mathfrak{o}$-label vanishes or is given by $\emptyset$ we often drop it from the notation.

Observe that given a decorated colored forest $(F, \hat{F})_{\mathfrak{e}}^{\mathfrak{n}, \mathfrak{o}}$ and a subforest $A$ of $F$ there is, by taking restrictions, a corresponding decorated colored forest $(A, \hat{F})_{\mathfrak{e}}^{\mathfrak{n}, \mathfrak{o}}$. Here we abuse notation by not making this restriction on our decorations or colorings explicit in our notation.

Clearly any semi-decorated tree $T_{\mathfrak{e}}^{\mathfrak{n}}$ naturally corresponds to a decorated colored tree $(T, 0)_{\mathfrak{e}}^{\mathfrak{n}}$. Finally, we also remark that the forest product extends naturally to a product on decorated colored forests.

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[10]Here $\hat{F}$ is an overloaded notation but whether it is being treated as a subforest or as a map should always be clear from context.

### 3.1.3   More homogeneities

With the introduction of coloring and $|\cdot|_s$ we define define two homogeneities $|\cdot|_+$ and $|\cdot|_-$ on decorated colored forests as follows: given $\tau = (F, \hat{F})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}$, we set

$$|\tau|_- \overset{\text{def}}{=} \sum_{\substack{e \in E_F \\ \hat{F}(e)=0}} (|\mathfrak{t}(e)|_{\mathfrak{s}} - |\mathfrak{e}(e)|_{\mathfrak{s}}) + \sum_{u \in N(F)} |\mathfrak{n}(u)|_{\mathfrak{s}} \,,$$

$$|\tau|_+ \overset{\text{def}}{=} \sum_{\substack{e \in E_F \\ \hat{F}(e)\neq 2}} (|\mathfrak{t}(e)|_{\mathfrak{s}} - |\mathfrak{e}(e)|_{\mathfrak{s}}) + \sum_{\substack{u \in N(F) \\ \hat{F}(u)\neq 2}} (|\mathfrak{n}(u)|_{\mathfrak{s}} + |\mathfrak{o}(u)|_{\mathfrak{s}}) \,.$$

### 3.1.4   Identified forests

An *identified forest* (or i-forest) is a pair $\sigma = (\sigma, \iota_\sigma)$ where $\sigma$ is a colored forest and $\iota_\sigma$ is a forest morphism from $\sigma$ into $\overline{T}$ which restricts to a tree monomorphism on *each connected component* of $\sigma$. Note that $\iota_\sigma$ is *not* required to be globally injective and in most cases it will not be. Two i-forests $\sigma_1$, $\sigma_2$ are isomorphic if there exists a forest isomorphism $\iota : \sigma_1 \to \sigma_2$ such that $\iota_{\sigma_2} \circ \iota = \iota_{\sigma_1}$.

Sometimes when talking about an i-forest $\sigma$ we may write

$$\sigma = (F^{(1)}, \widehat{F^{(1)}}) \cdots (F^{(k)}, \widehat{F^{(k)}}). \tag{3.1}$$

In this case it should be understood that for $1 \leq j \leq k$ the $F^{(j)}$ are explicit subforests of $\overline{T}$ and there is no need to mention $\iota_\sigma$. Note that there is some subtlety here since the $F^{(j)}$ may not be *disjoint* subforests of $\overline{T}$.

We make decorations explicit in our notation, so writing $\sigma$ indicates the constituent $\sigma$ is undecorated while in the decorated case we write $\sigma_{\mathfrak{e}}^{\mathfrak{n}}$. Note that the decorations of an i-forest are not necessarily inherited from $\overline{T}_{\mathfrak{e}}^{\overline{\mathfrak{n}}}$.

Recall that when we say $S$ is a subtree of $\overline{T}$ we are being more explicit with how $S$ is identified as a sub-object of $\overline{T}$, that is $N_S \subset N_{\overline{T}}$, $N(S) \subset N(\overline{T})$, and $E_S \subset E_{\overline{T}}$ as concrete sets with the type map on $E_S$ being the restriction of the one on $E_{\overline{T}}$. For an i-tree $\sigma$ we usually write $\sigma = (S, 0)$ where $S$ is a subtree of $\overline{T}$.

The set of all decorated i-forests is denoted by $\mathfrak{F}_2$. The subset of $\mathfrak{F}_2$ of decorated i-trees is denoted by $\mathfrak{T}_2$. For $i \in \{0, 1\}$ we define $\mathfrak{T}_i$ as the collection of all decorated i-trees $(T, \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in \mathfrak{T}_2$ with $\hat{T}(\cdot) \leq i$ and define $\mathfrak{F}_i$ analogously. Given any collection of decorated i-forests $\mathfrak{G}$ we denote by $\mathring{\mathfrak{G}}$ the corresponding set of undecorated objects.

We denote by $\langle \mathfrak{F}_2 \rangle$ the unital algebra obtained by endowing free vector space generated by $\mathfrak{F}_2$ with the (linear extension of the) forest product. For any subset $A \subset \mathfrak{F}_2$ we denote by $\langle A \rangle_{\text{for}}$ and $\langle A \rangle$ the subalgebra of $\langle \mathfrak{F}_2 \rangle$ generated by $A$ and the vector subspace of $\langle \mathfrak{F}_2 \rangle$ generated by $A$, respectively.

### 3.2   Co-actions and twisted antipodes

Our algebraic description of positive and negative renormalizations both involve two steps. In the first step one uses a "co-action" extracts object(s) which will be

later assigned a counterterm and in the second step one builds the counter-term by using a "twisted antipode" to perform recursive renormalization procedure *within* the extracted object(s).

Each coaction produces a linear combination of tensor products – in each product one factor consists of the extracted object(s) to be assigned counter-terms while the other factor consists of the part of $\overline{T}$ which is left over. On the other hand each twisted antipode will generate a forest product.

### 3.2.1   Negative renormalization

First, we define a set of decorated i-trees which, due to their power-counting, should be assigned counterterms for negative renormalization. We set

$$\mathfrak{X}_- \stackrel{\text{def}}{=} \left\{ (T,0)_{\mathfrak{e}}^{\mathfrak{n}} \in \underline{\mathfrak{T}_0} : \ \mathfrak{n}(\varrho_T) = 0 \text{ and } |(T,0)_{\mathfrak{e}}^{\mathfrak{n}}|_- < 0 \right\} ,$$

and we define $\mathfrak{p}_- : \langle \underline{\mathfrak{T}_0} \rangle_{\text{for}} \to \langle \mathfrak{X}_- \rangle_{\text{for}}$ to be the algebra homorphism given by projection onto $\langle \mathfrak{X}_- \rangle_{\text{for}}$. The co-action describing the negative renormalization is a linear map

$$\hat{\Delta}_- : \langle \underline{\mathfrak{T}_0} \rangle \to \langle \mathfrak{X}_- \rangle_{\text{for}} \otimes \langle \underline{\mathfrak{T}_1} \rangle .$$

We now specify, for each given i-tree, what sorts of subforests we will try to extract when applying $\hat{\Delta}_-$.

**Definition 3.2** For $(F,0) \in \mathring{\mathfrak{F}}_0$ we define $\mathfrak{A}_1(F,0)$ to be the collection of all subforests $G$ of $F$ with the property that $\text{Conn}(G)$ contains no trivial trees.

Note that one always has $\mathbf{1} \in \mathfrak{A}_1(F,0)$. Now, for any $(T,0)_{\mathfrak{e}}^{\mathfrak{n}} \in \underline{\mathfrak{T}_0}$, we set

$$\hat{\Delta}_-(T,0)_{\mathfrak{e}}^{\mathfrak{n}} \stackrel{\text{def}}{=} \sum_{G \in \mathfrak{A}_1(T,0)} \sum_{\mathfrak{n}_G, \mathfrak{e}_G} \frac{1}{\mathfrak{e}_G!} \binom{\mathfrak{n}}{\mathfrak{n}_G} \mathfrak{p}_- \left[ (G,0)_{\mathfrak{e}}^{\mathfrak{n}_G + \chi \mathfrak{e}_G} \right] \otimes (T, [G]_1)_{\mathfrak{e}_G + \mathfrak{e}}^{\mathfrak{n} - \mathfrak{n}_G, \mathfrak{n}_G + \chi \mathfrak{e}_G} .$$

$$(3.2)$$

Here in the second sum above we are summing over (i) all $\mathfrak{n}_G : N(T) \to \mathbf{N}^d$ supported on $N(G)$ and (ii) all $\mathfrak{e}_G : E_T \to \mathbf{N}^d$ supported on

$$\partial(G,T) \stackrel{\text{def}}{=} \{ (e_{\text{p}}, e_{\text{c}}) \in E_T \setminus E_G : \ e_{\text{p}} \in N_G \} .$$

Also, for forest $T$ and any decoration $\tilde{\mathfrak{e}} : E_T \to \mathbf{N}^d$ we define $\chi \tilde{\mathfrak{e}} : N(T) \to \mathbf{N}^d$ as $\chi \tilde{\mathfrak{e}}(u) \stackrel{\text{def}}{=} \sum_{e \, : \, e_{\text{p}} = u} \tilde{\mathfrak{e}}(e)$.

**Remark 3.3** The appearance of formulas like (3.2) will be quite common – we adopt these conventions on the meaning of $\sum_{\mathfrak{n}_G, \mathfrak{e}_G}$ throughout this paper. Note that they make sense even if $T$ is replaced by a forest and $G$ is a subforest of $F$.

Recursive negative renormalization is described by an algebra homomorphism

$$\hat{\mathscr{A}}_- : \langle \mathfrak{X}_- \rangle_{\text{for}} \to \langle \underline{\mathfrak{T}_1} \rangle_{\text{for}} .$$

One can imagine $\hat{\mathscr{A}}_-$ as given by iteration of a "reduced" analogue of $\hat{\Delta}_-$. By reduced, we mean it is more constrained in what objects it extracts.

**Definition 3.4** For $(F, 0) \in \mathring{\mathfrak{F}}_0$ we define $\overline{\mathfrak{A}}_1(F, 0)$ to be the collection of all $G \in \mathfrak{A}_1(F, 0)$ with the property that for every $T \in \mathrm{Conn}(F)$ one has $T \notin \mathrm{Conn}(G)$.

We first define the map $\hat{\mathscr{A}}_-$ for any $(F, 0)^{\mathfrak{n}}_{\mathfrak{e}} \in \mathfrak{F}_0$ which can be written as a forest product of elements of $\mathfrak{X}_-$.

This definition will be inductive in $|E_F|$ with the base case given by setting $\hat{\mathscr{A}}_- \mathbf{1} = \mathbf{1}$. The induction is given by setting

$$\hat{\mathscr{A}}_-(F, 0)^{\mathfrak{n}}_{\mathfrak{e}} \overset{\mathrm{def}}{=} (-1)^{|\mathrm{Conn}(F)|} \sum_{\substack{G \in \overline{\mathfrak{A}}_1(F, 0) \\ \mathfrak{n}_G, \mathfrak{e}_G}} \frac{1}{\mathfrak{e}_G!} \binom{\mathfrak{n}}{\mathfrak{n}_G} \left[ \hat{\mathscr{A}}_- \mathfrak{p}_-(G, 0)^{\mathfrak{n}_G + \chi \mathfrak{e}_G}_{\mathfrak{e}} \right] \cdot (F, [G]_1)^{\mathfrak{n} - \mathfrak{n}_G}_{\mathfrak{e}_G + \mathfrak{e}} \ .$$

### 3.2.2 Positive renormalization

In order to define $\mathfrak{X}_+$ we first need some new notions. Given a subtree $T$ of $\overline{T}$ and an edge $e \in E_T$ we define $T_{\geq}(e)$ to be the subtree of $T$ formed by all edges $e' \in E_T$ with $e' \geq e$, the corresponding node set being given by the collection of all $u \in N_T$ with $u \geq e_{\mathrm{p}}$. Note that if $e \in K(T)$ then $e_p \notin L(T_{\geq}(e))$ (see Remark 2.2).

Since it will be useful later we also define $T_{\not\geq}(e)$ to be the subtree of $T$ determined by all edges $e' \in E_{\overline{T}}$ with $e' \not\geq e$, the corresponding node set is given by the collection of all $u \in N_{\overline{T}}$ with $u \not\geq e_{\mathrm{p}}$.

We draw a picture to make these definitions clear. On the leftmost picture we draw some subtree $T$ of $\overline{T}$ and specify an edge $e \in E_T$ by drawing a small bisector in the middle of it. In the middle picture we have shaded in the subtree $T_{\geq}(e)$ while on the rightmost picture we have shaded in the subtree $T_{\not\geq}(e)$.
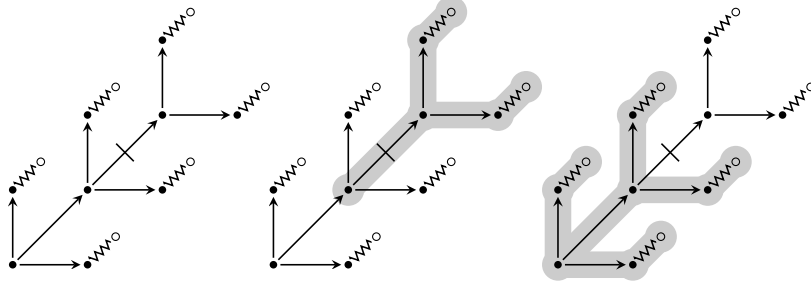


**Definition 3.5** Given a subtree $S$ of $\overline{T}$ and a subtree $S'$ of $S$ we define $\mathfrak{T}(S, S')$ to be the collection of subtrees of $S$ defined as follows

$$\mathfrak{T}(S, S') \overset{\mathrm{def}}{=} \left\{ S_{\geq}(e) : \ e \in K(S), \ e_{\mathrm{p}} \in N_{S'}, \ e_{\mathrm{c}} \notin N_{S'} \right\} .$$

We also define $\widehat{\underline{\mathfrak{T}}_2} \overset{\mathrm{def}}{=} \{(T, \hat{T})^{\mathfrak{n}, \mathfrak{o}}_{\mathfrak{e}} \in \underline{\mathfrak{T}}_2 : \ \hat{T}_2 \neq \mathbf{1}\}$. Finally we set

$$\mathfrak{X}_+ \overset{\mathrm{def}}{=} \left\{ (T, \hat{T})^{\mathfrak{n}, \mathfrak{o}}_{\mathfrak{e}} \in \widehat{\underline{\mathfrak{T}}_2} : \ \forall S \in \mathfrak{T}(T, \hat{T}_2), \ |\tilde{P}(S, \hat{T})^{\mathfrak{n}, \mathfrak{o}}_{\mathfrak{e}}|_+ > 0 \right\} ,$$

where the map $\tilde{P} : \mathfrak{F}_2 \to \mathfrak{F}_2$ acts on decorated colored forests by setting their root $\mathfrak{n}$ label to vanish, that is

$$\tilde{P}(F, \hat{F})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \stackrel{\text{def}}{=} (F, \hat{F})_{\mathfrak{e}}^{\tilde{\mathfrak{n}},\mathfrak{o}} , \qquad \tilde{\mathfrak{n}}(u) = \mathfrak{n}(u)\mathbb{1}\{u \notin \varrho(F)\} .$$

We define $\mathfrak{p}_+ : \langle \widehat{\mathfrak{T}}_2 \rangle_{\text{for}} \to \langle \mathfrak{X}_+ \rangle_{\text{for}}$ to be the algebra homomorphism given by projection onto $\langle \mathfrak{X}_+ \rangle_{\text{for}}$. The co-action for positive renormalization is a linear map

$$\hat{\Delta}_+ : \langle \underline{\mathfrak{T}}_1 \rangle \to \langle \underline{\mathfrak{T}}_1 \rangle \otimes \langle \mathfrak{X}_+ \rangle .$$

**Definition 3.6** Given $(T, \hat{T}) \in \overset{\circ}{\underline{\mathfrak{T}}}_2$ we define $\mathfrak{A}_2(T, \hat{T})$ to be the collection of all subtrees $S$ of $T$ with the following properties that $\varrho_S = \varrho_T$ and that for every $T' \in \text{Conn}(\hat{T}_1)$ one has $T'$ is either a subtree of $S$ or disjoint with $S$.

We then define, for any $(T, \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in \underline{\mathfrak{T}}_1$,

$$\hat{\Delta}_+(T, \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \stackrel{\text{def}}{=} \sum_{\substack{S \in \mathfrak{A}_2(T,\hat{T}) \\ \mathfrak{n}_S, \mathfrak{e}_S}} \frac{1}{\mathfrak{e}_S!} \binom{\mathfrak{n}}{\mathfrak{n}_S} (S, \hat{T})_{\mathfrak{e}}^{\mathfrak{n}_S + \chi \mathfrak{e}_S} \otimes \mathfrak{p}_+ \left( T, [\hat{T}_1 \setminus S]_1 \sqcup [S]_2 \right)_{\mathfrak{e}_S + \mathfrak{e}}^{\mathfrak{n} - \mathfrak{n}_S, \mathfrak{o}} .$$

$$(3.3)$$

Recursive positive renormalization is similarly described by a linear map

$$\hat{\mathscr{A}}_+ : \langle \mathfrak{X}_+ \rangle \to \langle \widehat{\mathfrak{T}}_2 \rangle_{\text{for}} .$$

It suffices to define the map $\hat{\mathscr{A}}_+$ for any $(T, \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in \mathfrak{X}_+$ and then extend linearly. This definition on $\mathfrak{X}_+$ will be inductive with respect to $|E_T \setminus E_{\hat{T}_2}|$. The base case, when $|E_T \setminus E_{\hat{T}_2}| = 0$, is given by setting

$$\hat{\mathscr{A}}_+(T, 2)_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \stackrel{\text{def}}{=} (-1)^{\sum_{u \in N(T)} |\mathfrak{n}(u)|} (T, 2)_{\mathfrak{e}}^{\mathfrak{n},0}.$$

Before stating the inductive definition we need some more notation.

**Definition 3.7** Given $(T, \hat{T}) \in \overset{\circ}{\underline{\mathfrak{T}}}_2$ we define $\overline{\mathfrak{A}}_2(T, \hat{T})$ to be the collection of all $S \in \mathfrak{A}_2(T, \hat{T})$ such that $\hat{T}_2$ is a subtree of $S$ and for every $T' \in \mathfrak{T}(T, \hat{T}_2)$ one has $E_{T'} \cap E_S \neq \emptyset$.

**Definition 3.8** Given $(T, \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in \widehat{\underline{\mathfrak{T}}}_2$ we define $\mathfrak{E}[(T, \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}]$ to be the set of all edge decorations $\mathfrak{e}_T$ on $K(\overline{T})$ supported on $\partial(\hat{T}_2, T)$ such that $(T, \hat{T})_{\mathfrak{e}+\mathfrak{e}_T}^{\mathfrak{n},\mathfrak{o}} \in \mathfrak{X}_+$.

We then set, for any $(T, \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in \mathfrak{X}_+$ with $\hat{T} \neq 2$,

$$\hat{\mathscr{A}}_+(T, \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \stackrel{\text{def}}{=} (-1)^{|\mathfrak{T}(T,\hat{T}_2)|} \sum_{S \in \overline{\mathfrak{A}}_2(T,\hat{T})} \sum_{\mathfrak{n}_S, \mathfrak{e}_S, \mathfrak{f}} \binom{\mathfrak{n} - \hat{\mathfrak{n}}}{\mathfrak{n}_S} \frac{(-1)^{\sum_{u \in N(T)} (|\hat{\mathfrak{n}}(u)| + |\chi \mathfrak{f}(u)|)}}{(\mathfrak{f} + \mathfrak{e}_S)!}$$

$$\cdot (S, \hat{T})^{\mathfrak{n}_S + \hat{\mathfrak{n}} + \chi(\mathfrak{e}_S + \mathfrak{f}),0}_{\mathfrak{e} + \mathfrak{f}} \cdot \mathscr{A}_+ \mathfrak{p}_+ \left( T, [\hat{T}_1 \setminus S]_1 \sqcup [S]_2 \right)^{\mathfrak{n} - \mathfrak{n}_S - \hat{\mathfrak{n}}, \mathfrak{o}}_{\mathfrak{e}_S + \mathfrak{e}},$$

where $\hat{\mathfrak{n}}(u) \overset{\text{def}}{=} \mathfrak{n}(u) \mathbb{1}\{u \in N(\hat{T}_2)\}$ and the sum over $\mathfrak{f}$ is a sum over all edge decorations $\mathfrak{f} \in \mathfrak{E}[(T, \hat{T})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}]$.

Our objective in this section is to describe two ways to map the decorated i-forests of the previous section to space-time functions and random fields.

### 3.3    From combinatorial trees to space-time functions

In this subsection we describe how, for each noise $\xi \in \Omega_\infty$, to map i-forests to functions of the space-time variables $(x_v)_{v \in N^*}$ where we set $N^* \overset{\text{def}}{=} N(\overline{T}) \sqcup \{\circledast\}$. Here $\circledast$ is a new node element and the corresponding $x_\circledast$ is the variable that encodes the base-point (the $z$ of $\Pi_z$). For any finite set $A$ we write $\mathscr{C}_A$ for the algebra of smooth functions $\mathscr{C}^\infty((\mathbf{R}^d)^A)$ and we also use the shorthand $\mathscr{C}_* \overset{\text{def}}{=} \mathscr{C}_{N^*}$.

Throughout this article, given some finite set $A$, some $x \in (\mathbf{R}^d)^A$, and some $B \subset A$, we write $x_B$ for the element of $(\mathbf{R}^d)^B$ with $(x_B)_i = x_i$ for $i \in B$. Similarly, we will always use the shorthands $\int dy_B$ or $\int_B dy$ instead of writing $\int_{(\mathbf{R}^d)^B} dy$. Finally, for $B \subset A$, we will always view $\mathscr{C}_B$ as a subspace of $\mathscr{C}_A$ via the canonical injection $\iota$ given by $(\iota f)(x) = f(x_B)$. For two disjoint sets $A$ and $\bar{A}$ and elements $x \in (\mathbf{R}^d)^A$ and $\bar{x} \in (\mathbf{R}^d)^{\bar{A}}$, we also write $x \sqcup \bar{x}$ for the element $y \in (\mathbf{R}^d)^{A \sqcup \bar{A}}$ given by $y_i = x_i$ for $i \in A$ and $y_i = \bar{x}_i$ for $i \in \bar{A}$. Sometimes, we will make the abuse of notation of writing $x_u$ instead of $x_{\{u\}}$ in expressions like "$x_u \sqcup y_A$".

First, for any i-tree $(T, \hat{T})$ we make the following definitions. We define set $K(T, \hat{T}) \overset{\text{def}}{=} K(T) \cap \hat{T}^{-1}(0)$ and $L(T, \hat{T}) \overset{\text{def}}{=} L(T) \cap \hat{T}^{-1}(0)$. We set

$$N(T, \hat{T}) \overset{\text{def}}{=} N(T) \cap (\hat{T}^{-1}(0) \sqcup \varrho(\hat{T}_1)).$$

We also define an associated map $\text{gv} : N(T) \to N(T, \hat{T}) \sqcup \{\circledast\}$ by setting, for each $u \in N(T)$,

$$\text{gv}(u) \overset{\text{def}}{=} \begin{cases} u & \text{if } u \in N(T, \hat{T}), \\ \varrho_S & \text{if } u \in N(S) \text{ for } S \in \text{Conn}(\hat{T}_1), \\ \circledast & \text{if } u \in N(\hat{T}_2). \end{cases}$$

For any $\xi \in \Omega_\infty$ and decorated i-tree $(T, \hat{T})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} \in \underline{\mathfrak{T}}_2$ we define $\mathring{\Upsilon}^\xi[(T, \hat{T})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}] \in \mathscr{C}_*$ by

$$\mathring{\Upsilon}^\xi[(T, \hat{T})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}](x) \overset{\text{def}}{=} \left( \prod_{u \in N(T)} (x_{\text{gv}(u)})^{\mathfrak{n}(u)} \right) \left( \prod_{e \in K(T, \hat{T})} D^{\mathfrak{e}(e)} K_{\mathfrak{t}(e)}(x_{\text{gv}(e_{\text{p}})} - x_{\text{gv}(e_{\text{c}})}) \right)$$

$$\cdot \left( \prod_{v \in L(T, \hat{T})} \xi_{\mathfrak{t}(v)}(x_v) \right).$$

We then define an algebra homomorphism $\Upsilon^\xi : \langle \underline{\mathfrak{F}}_2 \rangle \to \mathscr{C}_*$ by setting, for $(T, \hat{T})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} \in \underline{\mathfrak{T}}_2$,

$$\Upsilon^\xi[(T, \hat{T})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}](x) \overset{\text{def}}{=} \int_{N(T, \hat{T})} dy \, \mathring{\Upsilon}^\xi[(T, \hat{T})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}](y \sqcup x_{\varrho_T} \sqcup x_\circledast),$$

and then extending multipliciatively and linearly to all of $\Upsilon^\xi$ (in particular, $\Upsilon^\xi[\mathbf{1}] = 1$). We also make the important observation that $\Upsilon^\xi[(T,\hat{T})_\mathfrak{e}^{\mathrm{n},\mathrm{o}}](x)$ depends only on *either* $x_{\varrho_T}$ *or* $x_\otimes$ the former case holds when $\hat{T}_2 = \mathbf{1}$ and the later when $\hat{T}_2 \neq \mathbf{1}$.

### 3.4    The BPHZ renormalized tree

We now make our setting stochastic by fixing for what follows an arbitrary random noise $\xi \in \mathcal{M}(\Omega_\infty)$.

We define an algebra homomorphism $\bar{\Upsilon}^\xi : \langle \mathfrak{F}_1 \rangle_{\mathrm{for}} \to \mathbf{R}$ by setting, for any $\sigma_\mathfrak{e}^{\mathrm{n},\mathrm{o}} \in \underline{\mathfrak{T}_1}$,

$$\bar{\Upsilon}^\xi[\sigma_\mathfrak{e}^{\mathrm{n},\mathrm{o}}] \overset{\mathrm{def}}{=} \mathbf{E}(\Upsilon^\xi \sigma_\mathfrak{e}^{\mathrm{n},\mathrm{o}})(0) , \tag{3.4}$$

and extending this definition multiplicatively and linearly.

**Remark 3.9** It is important to note here that $\bar{\Upsilon}^\xi$ is deterministic and depends solely on the *law* of the random variable $\xi$, not on any random sample drawn from it.

Finally, we define, for each realization $\xi(\omega) \in \Omega_\infty$ of $\xi$, the smooth function $\hat{\Upsilon}^\xi[\overline{T}_\mathfrak{e}^{\overline{\mathrm{n}}}] \in \mathscr{C}_*$, again depending only on $x_{\varrho_{\overline{T}}}$ and $x_\otimes$, by setting

$$\hat{\Upsilon}^{\xi(\omega)}[\overline{T}_\mathfrak{e}^{\overline{\mathrm{n}}}] \overset{\mathrm{def}}{=} \left( \bar{\Upsilon}^\xi[\hat{\mathscr{A}}_- \cdot] \otimes \Upsilon^{\xi(\omega)}[\cdot] \otimes \Upsilon^{\xi(\omega)}[\hat{\mathscr{A}}_+ \cdot] \right)(\mathrm{Id} \otimes \hat{\Delta}_+)\hat{\Delta}_-(\overline{T},0)_\mathfrak{e}^{\overline{\mathrm{n}},0} . \tag{3.5}$$

Above we are committing an abuse of notation, the RHS is technically a sum of triple tensor products each consisting of a scalar and two elements of $\mathscr{C}_*$, however by simply multiplying them together we view the whole sum as an element of $\mathscr{C}_*$.

We emphasize again that in this expression $\bar{\Upsilon}^\xi$ is *deterministic* and depends on the law of the random variable $\xi$, while $\Upsilon^{\xi(\omega)}$ is *random* and depends on the specific sample $\xi(\omega)$. Later on, we will however typically suppress the chance element $\omega$ in our notations and we will simply write $\hat{\Upsilon}^\xi[\overline{T}_\mathfrak{e}^{\overline{\mathrm{n}}}](x)$ for the above expression. We will also view this as a family of random distributions in $\mathcal{D}'(\mathbf{R}^d)$, using the shorthand, for each $z \in \mathbf{R}^d$ and $\varphi \in \mathcal{D}(\mathbf{R}^d)$,

$$\hat{\Upsilon}_z^\xi[\overline{T}_\mathfrak{e}^{\overline{\mathrm{n}}}](\varphi) \overset{\mathrm{def}}{=} \int_{\{\varrho_*,\otimes\}} dx \, \hat{\Upsilon}^\xi[\overline{T}_\mathfrak{e}^{\overline{\mathrm{n}}}](x) \, \varphi(x_{\varrho_*})\delta(x_\otimes - z) ,$$

where here, and in what follows, we simply write $\varrho_*$ instead of $\varrho_{\overline{T}}$ for the root of $\overline{T}$. We refer to the above random distribution as the BPHZ renormalized tree corresponding to $\overline{T}_\mathfrak{e}^{\overline{\mathrm{n}}}$ with basepoint $z$.

## 4    A forest and cut expansion for the BPHZ renormalized tree

The goal of this section is to rewrite the opaque formula for $\hat{\Upsilon}^{\xi(\omega)}[\overline{T}_\mathfrak{e}^{\overline{\mathrm{n}}}]$ given in the previous section in a more explicit form amenable to direct analysis. The fundamental result of this section is Lemma 4.22.

### 4.1 A forest formula for negative renormalizations

For most of this subsection we work with a set of subtrees of $\overline{T}$. We see this set as carrying the inclusion partial order (for which throughout this article we use the convention $\subset \longleftrightarrow \leq$). In general, for any poset $A$ and subset $\mathcal{A} \subset A$ we write $\mathrm{Max}(\mathcal{A})$ or $\overline{\mathcal{A}}$ for the set of maximal elements of $\mathcal{A}$ and $\mathrm{Min}(\mathcal{A})$ or $\underline{\mathcal{A}}$ for the set of minimal elements of $\mathcal{A}$. As an exception to this notation, $\overline{T}$ always refers to our fixed tree.

We write Div for the set of all subtrees $S$ of $\overline{T}$ with $\omega(S) \stackrel{\mathrm{def}}{=} -|(S_{\mathfrak{e}}^0, 0)|_{\mathfrak{s}} > 0$. Note that $S \in \mathrm{Div}$ forces $S$ to be non-trivial.

**Remark 4.1** Note that as a consequence of Definition 2.28, if one has $S \in \mathrm{Div}$ and $e \in \mathbb{L}(\overline{T})$ with $e_{\mathrm{p}} \in N(S)$ then it must be the case that $e \in \mathbb{L}(S)$.

**Definition 4.2** We say that $\mathcal{F}$ is a forest of subtrees if it is a collection of subtrees of $\overline{T}$ with the property that for any pair $S, S' \in \mathcal{F}$ one has either $N_S \subset N_{S'}$, or $N_{S'} \subset N_S$, or $N_S \cap N_{S'} = \emptyset$.

As an example, suppose that $\overline{T}$ is given by
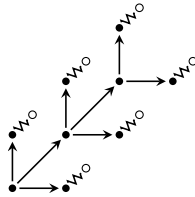


with the root being at the bottom. Indicating subtrees of a tree by shading them, we consider the set $\{S_1, \ldots, S_6\}$ of subtrees of $\overline{T}$ given by



$$(4.1)$$
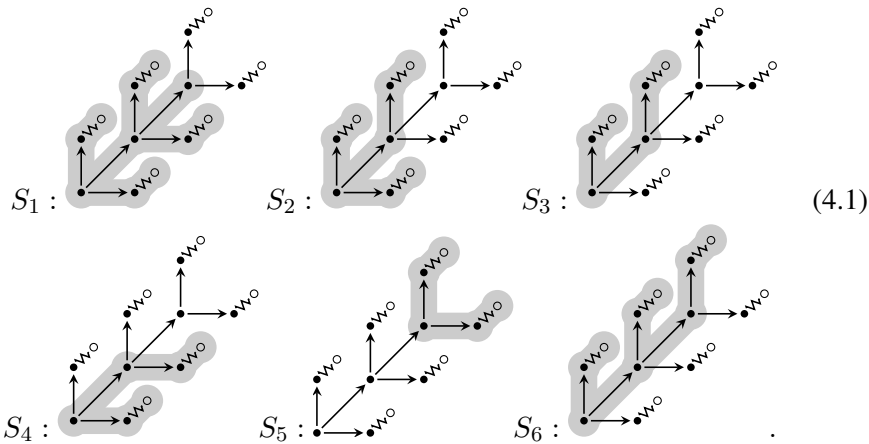
Note here that while $S_3$ and $S_4$ are isomorphic as labelled trees, they represent different subtrees of $\overline{T}$ and are therefore not isomorphic as i-trees. The list of all subsets of $\{S_1, \ldots, S_6\}$ which are forests of subtrees is given by:

$$\emptyset, \ \{S_1\}, \{S_2\}, \{S_3\}, \{S_4\}, \{S_5\}, \{S_6\},$$

$$\{S_3, S_6\}, \{S_2, S_3\}, \{S_1, S_2\}, \{S_1, S_3\}, \{S_1, S_4\}, \{S_2, S_5\},$$
$$\{S_3, S_5\}, \{S_4, S_5\}, \{S_3, S_6\}, \{S_1, S_2, S_3\}, \text{ and } \{S_2, S_3, S_5\} \ .$$

**Remark 4.3** We will use two notions of forest in what follows, one is that of an i-forest and the other is the notion of a forest of subtrees. Clearly any collection $\mathcal{A}$ of subtrees of $\overline{T}$ canonically determines an i-forest where, instead of considering $\mathcal{A}$ as a set with trees as elements, we see it as the i-forest obtained by the disjoint union of these trees.

In the above example $\{S_2, S_3, S_4\}$ can be viewed as an i-forest (and in this case we would write it as $S_2 \cdot S_3 \cdot S_4$), but it is not a forest of subtrees since $S_4$ is not related to $S_2$ (or $S_3$ for that matter) by inclusion. This illustrates that, with this identification, being a forest of subtrees is a strictly stronger property than being an i-forest.

We often use a calligraphic font to indicate forests of subtrees versus standard Roman letters for i-forests. Finally, we observe that in the special case when $\mathcal{F}$ is a forest of subtrees of $\overline{T}$ of depth 1 or 0, the node set $N_F$ of the canonical i-forest $F$ associated to it can be identified as a subset of $N_{\overline{T}}$.

**Definition 4.4** We say that a subtree $S$ is *compatible* with a forest of subtrees $\mathcal{F}$ if $\{S\} \cup \mathcal{F}$ is again a forest of subtrees (note that if $S \in \mathcal{F}$, then this is automatically the case).

We write $\mathbb{F}$ for the collection of all forests of subtrees $\mathcal{F}$ such that each tree in $\mathcal{F}$ belongs to Div. Given a forest of subtrees $\mathcal{F}$ and a subtree $S$ compatible with $\mathcal{F}$ we define the forest of subtrees which are the (immediate) children of $S$ in $\mathcal{F}$ as

$$C_{\mathcal{F}}(S) \stackrel{\text{def}}{=} \text{Max}\{T \in \mathcal{F} : \ T < S\} \ .$$

and the branch of $S$ in $\mathcal{F}$ by $\mathcal{F}_S \stackrel{\text{def}}{=} \{T \in \mathcal{F} : \ T \leq S\}$.

For non-empty $\mathcal{F} \in \mathbb{F}$ and $k \in \mathbf{N}$ we define subsets $D_k(\mathcal{F})$ as follows. We set $D_0(\mathcal{F}) \stackrel{\text{def}}{=} \emptyset$, $D_1(\mathcal{F}) \stackrel{\text{def}}{=} \overline{\mathcal{F}}$, and for $k \geq 1$ we set

$$D_{k+1}(\mathcal{F}) = \bigsqcup_{S \in D_k(\mathcal{F})} C_{\mathcal{F}}(S).$$

We define the depth of $\mathcal{F}$, denoted by $\text{depth}(\mathcal{F})$, to be given by

$$\text{depth}(\mathcal{F}) \stackrel{\text{def}}{=} \inf \left\{ k \in \mathbf{N} : \ \bigcup_{j=0}^{k} D_j(\mathcal{F}) = \mathcal{F} \right\} = \inf\{k \geq 0 : \ D_{k+1}(\mathcal{F}) = \emptyset\} \ .$$

Recalling our previous pictorial example of forests, if we set $\mathcal{F} = \{S_1, S_2, S_3\}$ we have $D_1(\mathcal{F}) = \{S_1\}$, $D_2(\mathcal{F}) = \{S_2\}$, and $D_3(\mathcal{F}) = \{S_3\}$. On the other hand, if we set $\mathcal{F} = \{S_2, S_3, S_5\}$ then $D_1(\mathcal{F}) = \{S_2, S_5\}$ and $D_2(\mathcal{F}) = \{S_3\}$.

We write $\mathbb{F}^{\leq 1}$ for the collection of elements of $\mathbb{F}$ of depth $1$ or $0$. We also define, for any $\mathcal{F} \in \mathbb{F}$,

$$\mathbb{F}[\mathcal{F}] \stackrel{\text{def}}{=} \{\mathcal{G} \in \mathbb{F} : \overline{\mathcal{G}} = \mathcal{F}\},$$ (4.2)

$$\mathbb{F}_<[\mathcal{F}] \stackrel{\text{def}}{=} \{\mathcal{G} \in \mathbb{F}^{\leq 1} : \forall S \in \mathcal{G}, \, \exists T \in \mathcal{F} \text{ with } S < T\}, \text{ and}$$

$$\mathbb{F}_\leq[\mathcal{F}] \stackrel{\text{def}}{=} \{\mathcal{G} \in \mathbb{F}^{\leq 1} : \forall S \in \mathcal{G}, \, \exists T \in \mathcal{F} \text{ with } S \leq T\}.$$

Note that $\mathbb{F}[\mathcal{F}]$ is empty unless $\mathcal{F} \in \mathbb{F}^{\leq 1}$.

Finally, in some of our statements and proofs it is useful to sum over the decorations of an i-forest while keeping the specific i-forest fixed. We do this by introducing the following family of projection maps.

**Definition 4.5** for any *undecorated* i-forest $\sigma \in \mathring{\underline{\mathfrak{F}}}_2$ we write $\mathbf{P}_\sigma$ for the projection onto the subspace of $\langle \underline{\mathfrak{F}}_2 \rangle$ spanned by all decorated i-forests of the form $\sigma_\mathfrak{e}^{\mathfrak{n},\mathfrak{o}}$.

We now describe the class of i-forests produced by the action of $\hat{\mathscr{A}}_-$.

**Definition 4.6** To each $\mathcal{F} \in \mathbb{F}$ we associate an i-forest $\sigma_\mathcal{F} \in \mathring{\underline{\mathfrak{F}}}_2$ as follows. If $\mathcal{F} = \emptyset$ then we set $\sigma_\mathcal{F} \stackrel{\text{def}}{=} \mathbf{1}$. For $\mathcal{F}$ of depth $j > 0$ we set

$$\sigma_\mathcal{F} \stackrel{\text{def}}{=} (F^{(j)}, \widehat{F^{(j)}}) \cdots (F^{(1)}, \widehat{F^{(1)}})$$

where for $1 \leq k \leq j$ we set $F^{(k)} \stackrel{\text{def}}{=} D_k(\mathcal{F})$ and $\widehat{F^{(k)}} = [D_{k+1}(\mathcal{F})]_1$. In particular, $\widehat{F^{(j)}} = 0$.

Below we pictorially present $\sigma_\mathcal{F}$ in the context of our previous example for the case $\mathcal{F} = \{S_1, S_2, S_3\}$. The color blue corresponds to the color 1. The forest product of trees is represented by placing the trees next to each other but note that the order does not matter.
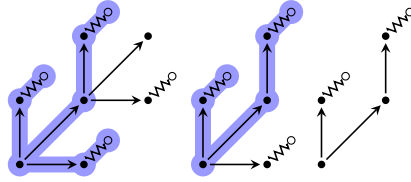


Given a forest of subtrees $\mathcal{F}$ we define $E_\mathcal{F} \subset E_{\overline{T}}$ via $E_\mathcal{F} = \bigsqcup_{S \in \overline{\mathcal{F}}} E_S$. The following lemma states how the action of $\hat{\mathscr{A}}_-$ admits an expansion into forests.

**Lemma 4.7** *Let $\mathcal{F} \in \mathbb{F}^{\leq 1}$ and $F$ be the i-forest corresponding to $\mathcal{F}$. Then for any node labeling $\mathfrak{n}$ on $N(F)$ one has*

$$\left( \sum_{\mathcal{G} \in \mathbb{F}[\mathcal{F}]} \mathbf{P}_{\sigma_\mathcal{G}} \right) \hat{\mathscr{A}}_-(F, 0)_\mathfrak{e}^\mathfrak{n} = \hat{\mathscr{A}}_-(F, 0)_\mathfrak{e}^\mathfrak{n}.$$ (4.3)

*Proof.* We prove the statement via induction in $|E_F|$. The base case, when $\mathcal{F} = \emptyset$ and $F = \mathbf{1}$, is trivial (the sum on the LHS of (4.3) just gives $\mathbf{P}_\emptyset$).

Now suppose $F$ has $k > 0$ edges and the claim has been proven for all $\mathcal{F}' \in \mathbb{F}^{\leq 1}$ with $|E_{\mathcal{F}'}| < k$. Then $(-1)^{|\mathrm{Conn}(F)|}\hat{\mathscr{A}}_-(F, 0)^{\mathfrak{n}}_{\mathfrak{e}}$ is given by

$$\sum_{\mathcal{G} \in \mathbb{F}_<[\mathcal{F}]} \sum_{\mathfrak{n}_G, \mathfrak{e}_G} \frac{1}{\mathfrak{e}_G!} \binom{\mathfrak{n}}{\mathfrak{n}_G} \left( \hat{\mathscr{A}}_- \mathfrak{p}_-(G, 0)^{\mathfrak{n}_G + \chi \mathfrak{e}_G}_{\mathfrak{e}} \right) \cdot (F, [G]_1)^{\mathfrak{n} - \mathfrak{n}_G}_{\mathfrak{e} + \mathfrak{e}_G}$$

$$= \sum_{\mathcal{G} \in \mathbb{F}_<[\mathcal{F}]} \sum_{\mathfrak{n}_G, \mathfrak{e}_G} \frac{1}{\mathfrak{e}_G!} \binom{\mathfrak{n}}{\mathfrak{n}_G} \left( \sum_{\mathcal{G}' \in \mathbb{F}[\mathcal{G}]} \mathbf{P}_{\sigma_{\mathcal{G}'}} \hat{\mathscr{A}}_- \mathfrak{p}_-(G, 0)^{\mathfrak{n}_G + \chi \mathfrak{e}_G}_{\mathfrak{e}} \right) \cdot (F, [G]_1)^{\mathfrak{n} - \mathfrak{n}_G}_{\mathfrak{e} + \mathfrak{e}_G}$$

$$= \sum_{\substack{\mathcal{G} \in \mathbb{F}_<[\mathcal{F}] \\ \mathcal{G}' \in \mathbb{F}[\mathcal{G}]}} \mathbf{P}_{\sigma_{\mathcal{G}' \sqcup \mathcal{F}}} \sum_{\mathfrak{n}_G, \mathfrak{e}_G} \frac{1}{\mathfrak{e}_G!} \binom{\mathfrak{n}}{\mathfrak{n}_G} \left( \hat{\mathscr{A}}_- \mathfrak{p}_-(G, 0)^{\mathfrak{n}_G + \chi \mathfrak{e}_G}_{\mathfrak{e}} \right) \cdot (F, [G]_1)^{\mathfrak{n} - \mathfrak{n}_G}_{\mathfrak{e} + \mathfrak{e}_G}$$

$$= \sum_{\substack{\mathcal{G} \in \mathbb{F}_<[\mathcal{F}] \\ \mathcal{G}' \in \mathbb{F}[\mathcal{G}]}} \mathbf{P}_{\sigma_{\mathcal{G}' \sqcup \mathcal{F}}} \left[ \sum_{\substack{H \in \overline{\mathfrak{A}}_1(F, 0) \\ \mathfrak{n}_H, \mathfrak{e}_H}} \frac{1}{\mathfrak{e}_H!} \binom{\mathfrak{n}}{\mathfrak{n}_H} \left( \hat{\mathscr{A}}_- \mathfrak{p}_-(H, 0)^{\mathfrak{n}_H + \chi \mathfrak{e}_H}_{\mathfrak{e}} \right) \cdot (F, [H]_1)^{\mathfrak{n} - \mathfrak{n}_H}_{\mathfrak{e} + \mathfrak{e}_H} \right]$$

$$= (-1)^{|\mathrm{Conn}(F)|} \left( \sum_{\substack{\mathcal{G} \in \mathbb{F}_<[\mathcal{F}] \\ \mathcal{G}' \in \mathbb{F}[\mathcal{G}]}} \mathbf{P}_{\sigma_{\mathcal{G}' \sqcup \mathcal{F}}} \right) \hat{\mathscr{A}}_-(F, 0)^{\mathfrak{n}}_{\mathfrak{e}}$$

where for each $\mathcal{G} \in \mathbb{F}_<[\mathcal{F}]$ we are writing $G$ for the i-forest corresponding to $\mathcal{G}$ (which is certainly an i-subforest of $F$). Noting that $\bigsqcup_{\mathcal{G} \in \mathbb{F}_<[\mathcal{F}]} \bigsqcup_{\mathcal{G}' \in \mathbb{F}[\mathcal{G}]} (\mathcal{G}' \sqcup \mathcal{F}) = \mathbb{F}[\mathcal{F}]$ completes the proof.  $\square$

We now mix our expansion into forests with a cumulant expansion. For any finite set $L$ we denote by $\bar{\mathcal{P}}(L)$ the set of all partitions $\pi$ of $L$. We also define $\mathcal{P}(L) \stackrel{\text{def}}{=} \bigcup_{A \subset L} \bar{\mathcal{P}}(A)$. We remind the reader that $\bar{\mathcal{P}}(\emptyset) = \{\emptyset\}$.

**Definition 4.8** Given $\pi \in \mathcal{P}[L(\overline{T})]$ and a forest of subtrees $\mathcal{F}$, we say $\mathcal{F}$ and $\pi$ are *compatible* if, for each $S \in \mathcal{F}$, $L(S)$ can be written as union of blocks of $\pi$.

In sums over partitions or forests we sometimes write, as a subscript, "$\pi$ comp. $\mathcal{F}$" to restrict the sum to compatible partitions. We also write $\mathbb{1}_{\mathrm{comp}}(\mathcal{F}, \pi)$ for the indicator function of the condition that $\mathcal{F}$ and $\pi$ are compatible. Given a partition $\pi$, we write $\mathbb{F}_\pi$ for the collection of elements of $\mathbb{F}$ which are compatible with $\pi$.

We now introduce some notions to allow us to explicitly write down the negative renormalization procedure. For any subtree $S$, we set

$$K^{\downarrow}(S) \stackrel{\text{def}}{=} \{e \in K(\bar{T}) : e_{\mathrm{p}} \in N(S), e_{\mathrm{c}} \notin N(S)\},$$
$$\bar{K}^{\downarrow}(S) \stackrel{\text{def}}{=} K^{\downarrow}(S) \sqcup K(S), \text{ and } N^{\downarrow}(S) \stackrel{\text{def}}{=} e_{\mathrm{c}}(K^{\downarrow}(S)) .$$

We include a picture to make some of this notions clear. Below we shaded in a subtree $S$ as before, but we additionally coloured $\varrho_S$ in dark blue, the elements of $\tilde{N}(S)$ in light blue (recall that $\tilde{N}(S) = N(S) \setminus \{\varrho_S\}$), the elements of $N^{\downarrow}(S)$ in red,

and the edges of $K^{\downarrow}(S)$ in light green.
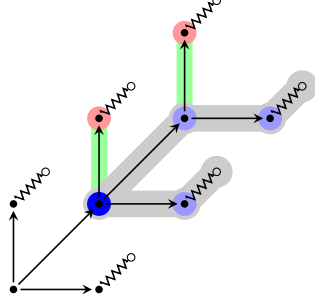
Note that the zigzag at the top right does not belong to $\bar{K}^{\downarrow}(S)$ because $K(\overline{T})$ only contains "kernel edges" by definition.

For a pair of subtrees $S, T$ with $T \leq S$ we define $K_T^{\partial}(S) \overset{\text{def}}{=} K^{\downarrow}(T) \cap K(S)$.

Given a forest of subtrees $\mathcal{F}$ and a subtree $S$ we define

$$\tilde{N}_{\mathcal{F}}(S) \overset{\text{def}}{=} \tilde{N}(S) \setminus \Big( \bigsqcup_{T \in C_{\mathcal{F}}(S)} \tilde{N}(T) \Big), \quad N_{\mathcal{F}}(S) \overset{\text{def}}{=} \tilde{N}_{\mathcal{F}}(S) \sqcup \{\varrho_S\} \ .$$

We also use kernel edge decomposition

$$K(S) = \Big( \bigsqcup_{T \in C_{\mathcal{F}}(S)} K(T) \Big) \sqcup \mathring{K}_{\mathcal{F}}(S) \sqcup K_{\mathcal{F}}^{\partial}(S)$$

where

$$\mathring{K}_{\mathcal{F}}(S) \overset{\text{def}}{=} K(S) \setminus \Big( \bigsqcup_{T \in C_{\mathcal{F}}(S)} \bar{K}^{\downarrow}(T) \Big), \text{ and } K_{\mathcal{F}}^{\partial}(S) \overset{\text{def}}{=} \bigsqcup_{T \in C_{\mathcal{F}}(S)} K_T^{\partial}(S).$$

To keep track of leaf edges we set

$$L_{\mathcal{F}}(S) \overset{\text{def}}{=} L(S) \setminus \Big( \bigsqcup_{T \in C_{\mathcal{F}}(S)} L(T) \Big).$$

To make some of these definitions more concrete we look at the example of $\mathcal{F} = \{S_3, S_6\}$, referring to (4.1). Below we have shaded in $S_6$ in light gray and, on top of this, $S_3$ in dark gray. We also shaded the nodes of $\tilde{N}_{\mathcal{F}}(S_6)$ in light blue, the edge of $\mathring{K}_{\mathcal{F}}(S_6)$ in light green, and the edge of $K_{\mathcal{F}}^{\partial}(S_6) = K_{S_3}^{\partial}(S_6)$ in red. $L_{\mathcal{F}}(S_6)$ consists of just one edge which is the left uppermost zigzag.
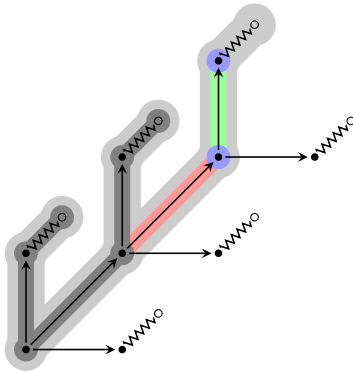
**Remark 4.9** For the rest of the paper we adopt the convention that any notation which takes a subtree of $\overline{T}$ as an argument can also take a collection of subtrees of $\overline{T}$ (not necessarily a forest) as an argument by taking unions over elements of that collection. For example, for a collection $\mathcal{A}$ of subtrees we define

$$\tilde{N}(\mathcal{A}) \stackrel{\text{def}}{=} \bigcup_{S \in \mathcal{A}} \tilde{N}(S)\,, \quad L(\mathcal{A}) \stackrel{\text{def}}{=} \bigcup_{S \in \mathcal{A}} L(S)\,, \quad K^{\downarrow}(\mathcal{A}) \stackrel{\text{def}}{=} \bigcup_{S \in \mathcal{A}} K^{\downarrow}(S)\,,$$

and so on.

We now introduce shorthand for the functions appearing in our integrands. For any set of kernel edges $E \subset K(\overline{T})$ and edge decoration $\mathfrak{e} : K(\overline{T}) \to \mathbf{N}^d$, we define the function $\mathrm{Ker}_{\mathfrak{e}}^E \in \mathscr{C}_A$ with $A = e_{\mathrm{p}}(E) \cup e_{\mathrm{c}}(E)$, by

$$\mathrm{Ker}_{\mathfrak{e}}^E(x) \stackrel{\text{def}}{=} \prod_{e \in E} D^{\bar{\mathfrak{e}}(e)+\mathfrak{e}(e)} K_{\mathfrak{t}(e)}(x_{e_{\mathrm{p}}} - x_{e_{\mathrm{c}}})\,.$$

Given $u \in N^*$, we also define $\mathrm{Ker}_{\mathfrak{e}}^{E,u} \in \mathscr{C}_A$ with $A = \{u\} \cup e_{\mathrm{c}}(E)$, by

$$\mathrm{Ker}_{\mathfrak{e}}^{E,u}(x) \stackrel{\text{def}}{=} \prod_{e \in E} D^{\bar{\mathfrak{e}}(e)+\mathfrak{e}(e)} K_{\mathfrak{t}(e)}(x_u - x_{e_{\mathrm{c}}})\,.$$

For any $L \subset L(\overline{T})$ and $\pi \in \mathcal{P}[L(\overline{T})]$ we define the function $\mathrm{Cu}_{\pi}^L \in \mathscr{C}_L$ by

$$\mathrm{Cu}_{\pi}^L(x) \stackrel{\text{def}}{=} \prod_{\substack{B \in \pi \\ B \subset L}} \mathbf{E}^c[\{\xi_{\mathfrak{t}(u)}(x_u)\}_{u \in B}]\,,$$

where $\mathbf{E}^c$ denotes joint cumulants as before.

Finally, for any $N \subset N(\overline{T})$, $\mathfrak{n} : N \to \mathbf{N}^d$, and distinct $v, v' \in N^*$ we define the functions $\mathrm{X}_{\mathfrak{n}}^N$, $\mathrm{X}_{\mathfrak{n},v}^N$, $\mathrm{X}_{\mathfrak{n}}^{N,v}$, and $\mathrm{X}_{\mathfrak{n},v}^{N,v'} \in \mathscr{C}_*$ via

$$\mathrm{X}_{\mathfrak{n}}^N(x) \stackrel{\text{def}}{=} \prod_{u \in N} x_u^{\mathfrak{n}(u)}, \qquad \mathrm{X}_{\mathfrak{n},v}^N(x) \stackrel{\text{def}}{=} \prod_{u \in N} (x_u - x_v)^{\mathfrak{n}(u)}, \qquad (4.4)$$

$$\mathrm{X}_{\mathfrak{n}}^{N,v}(x) \stackrel{\text{def}}{=} (x_v)^{\sum_{u \in N} \mathfrak{n}(u)}, \quad \mathrm{X}_{\mathfrak{n},v}^{N,v'}(x) = (x_{v'} - x_v)^{\sum_{u \in N} \mathfrak{n}(u)}\,.$$

We also set $\overline{\mathrm{X}}_{\mathfrak{n}}^{N,v}(x) \stackrel{\text{def}}{=} (-x_v)^{\sum_{u \in N} \mathfrak{n}(u)}$.

### 4.1.1   An inductive definition of negative renormalization counterterms

We will begin using multivariable multi-index notation more frequently: we take as a universe of multi-indices $(\mathbf{N}^d)^{N^* \sqcup K(\overline{T})}$. We define $|\cdot|_{\mathfrak{s}}$ and $|\cdot|$ for tuples of multi-indices by summation over entries.

**Definition 4.10** Given a subtree $S$ and any subset $A \subset N^*$ with $\tilde{N}(S) \subset A$, we define the "collapsing map" $\mathrm{Coll}_S$ onto the root of $S$ as the map from $(\mathbf{R}^d)^A$ to itself given by

$$\mathrm{Coll}_S(x)_u \stackrel{\text{def}}{=} \begin{cases} x_{\varrho(S)} & \text{if } u \in \tilde{N}(S), \\ x_u & \text{otherwise.} \end{cases}$$

This allows us to give the following definition.

**Definition 4.11** For any subtree $S \in \mathrm{Div}$ we define $\mathrm{Der}(S)$ to be the set of all multi-indices $k$ supported on $\tilde{N}(S)$ with $|k|_{\mathfrak{s}} < \omega(S)$. We also define an operator $\mathscr{Y}_S : \mathscr{C}_* \mapsto \mathscr{C}_*$ via

$$[\mathscr{Y}_S \varphi](x) = \sum_{k \in \mathrm{Der}(S)} \frac{(x - \mathrm{Coll}_S(x))^k}{k!} (D^k \varphi)(\mathrm{Coll}_S(x)) .$$

Here and below, given any set $M \subset N^*$, we write $M^c$ for its complement in $N^*$.

We now inductively define, for each fixed $\pi \in \mathcal{P}[L(\overline{T})]$ and $\mathcal{F} \in \mathbb{F}_\pi$, operators $H_{\pi,\mathcal{F},S} : \mathscr{C}_* \to \mathscr{C}_*$ for each $S \in \mathcal{F}$. This induction will be with respect to depth of the forest of subtrees $\{T \in \mathcal{F} : T < S\}$. Moreover, for any $M \subset N^*$, the map $H_{\pi,\mathcal{F},S}$ maps $\mathscr{C}_M$ to $\mathscr{C}_{M \setminus \tilde{N}(S)}$.

The base case for our definition occurs when $C_{\mathcal{F}}(S) = \emptyset$ and, in that case, we set for any $\varphi \in \mathscr{C}_*$

$$[H_{\pi,\mathcal{F},S}(\varphi)](x) \stackrel{\text{def}}{=} \int_{\tilde{N}(S)} dy \, \mathrm{Cu}_\pi^{L(S)}(y \sqcup x_{\varrho S}) \, \mathrm{Ker}_0^{K(S)}(y \sqcup x_{\varrho S})(-\mathscr{Y}_S \varphi)(x_{\tilde{N}(S)^c} \sqcup y). \tag{4.5}$$

**Remark 4.12** It is not hard to see that the integral above is absolutely convergent for fixed $x$. For each $e \in K(S)$ we have a uniform bound of the type

$$|D^{\bar{e}} K_{\mathfrak{t}(e)}(y_{e_{\mathfrak{p}}} - y_{e_c})| \lesssim \mathbb{1}\{|y_{e_{\mathfrak{p}}} - y_{e_c}| \leq 1\} \, |y_{e_{\mathfrak{p}}} - y_{e_c}|^{-|\mathfrak{s}|+|\mathfrak{t}(e)|_{\mathfrak{s}}-|\bar{\mathfrak{e}}(e)|_{\mathfrak{s}}}$$

where $|\mathfrak{t}(e)|_{\mathfrak{s}} - |\bar{\mathfrak{e}}(e)|_{\mathfrak{s}} > 0$ and any occurrence of $y_{\varrho S}$ should be replaced by $x_{\varrho S}$. The edges of $K(S)$ form a tree on the vertices of $\tilde{N}(S)$ so the support of $\mathrm{Ker}_0^{K(S)}(y \sqcup x_{\varrho S})$, when seen as a function of $y_{\tilde{N}(S)}$ for fixed $x_{\varrho S}$, is compact. In particular, if the other parts of the integrand of (2.21) can be bounded uniformly in $y$ by some constant, then the entire integral can be bounded by this constant times the the product of $L_1(\mathbf{R})$ norms of the functions $D^{\bar{e}} K_{\mathfrak{t}(e)}(\cdot)$. This is clearly the case as both $\mathrm{Cu}_\pi^L(y)$ and $(-\mathscr{Y}_S \varphi)$ are continuous, and therefore bounded on compact sets.

**Remark 4.13** A second question the reader may ask is what the definition (4.5) is trying to accomplish. The quantity defined from this formula is a renormalization counterterm. One interpretation of the need for renormalization is the following – while working on a problem one may see the appearance of a function $F$ which

fails to belong to $L^1_{\mathrm{loc}}$ in some region (say at the origin for a distribution on $\mathbf{R}^d$ or on the diagonal for a translation invariant distribution on $\mathbf{R}^d \times \mathbf{R}^d$), so that it cannot be canonically identified with a distribution.

The insertion of a regularization parameter $\varepsilon > 0$ which disappears in the $\varepsilon \downarrow 0$ limit may allow one to define a family of bonafide distributions $F_\varepsilon$ but these may fail to converge in any meaningful sense as one takes $\varepsilon \downarrow 0$.

In many cases it is then possible to obtain a convergent family of distributions $\tilde{F}_\varepsilon$ by subtracting from each $F_\varepsilon$ a well-chosen, $\varepsilon$-dependent linear combination of $\delta$ functions and their derivatives situated at these singular regions of $F$. A simple example is encountered when one tries to make sense of $F(t) = |t|^{-\alpha}$ for $\alpha > 1$ with $\alpha \notin \mathbf{N}$ as a distribution on $\mathbf{R}$. A convergent family of distributions can be obtained by defining

$$\tilde{F}_\varepsilon(t) \stackrel{\text{def}}{=} (|t| \vee \varepsilon)^{-\alpha} - \sum_{j=0}^{\lfloor \alpha \rfloor - 1} \left( \int_{\mathbf{R}} ds \, \frac{s^j}{j!} (|s| \vee \varepsilon)^{-\alpha} \right) \delta^{(j)}(t) \,,$$

where $\delta^{(j)}$ denotes the $j$-th weak derivative of the delta function $\delta$ on $\mathbf{R}$. To derive good (uniform in $\varepsilon$) bounds on $\tilde{F}_\varepsilon(f)$ for a test function $f$ one exploits a Taylor remainder estimate for $f$.

For an example closer to our setting, suppose that we have a single divergent subtree $S$ which contains no divergent subtree. Then $S$ can be associated to the distribution on $(\mathbf{R}^d)^{N(S)}$ given by integration against the function $F_\varepsilon$ defined as

$$F_\varepsilon(x) \stackrel{\text{def}}{=} \mathrm{Cu}^L_{\varepsilon,\pi}(x) \mathrm{Ker}^{K(S)}_0(x).$$

We have made the regularization parameter explicit on the RHS. In practice all of our constructions will be applied to probability measures $\mathbb{P}_\varepsilon$ corresponding to some $\xi_\varepsilon \in \mathcal{M}(\Omega_\infty)$ which converge to a limiting measure $\mathbb{P}$ with cumulants that are singular at coinciding points – in such a limit $F_\varepsilon$ fails to be in $L^1_{\mathrm{loc}}[(\mathbf{R}^d)^{N^{\mathrm{ext}}(S)}]$ as $\varepsilon \downarrow 0$. The singular region occurs on the "small" diagonal – those $(x_v)_{v \in N(S)}$ with $x_v = x_{\varrho_S}$ for all $v$. The solution is to subtract an appropriate linear combination of delta functions and their derivatives on this diagonal, i.e. $F_\varepsilon(x) - \tilde{F}_\varepsilon(x)$ is given by

$$\sum_{k \in \mathrm{Der}(S)} \left( \int_{\tilde{N}(S)} dy \, \frac{(y - \mathrm{Coll}_S(x))^k}{k!} F_\varepsilon(x_{\varrho_S} \sqcup y) \right) \prod_{u \in \tilde{N}(S)} \delta^{(k_u)}(x_u - x_{\varrho_S}).$$

In general, $H_{\pi,\mathcal{F},S}$ is given recursively by

$$[H_{\pi,\mathcal{F},S}(\varphi)](x) \stackrel{\text{def}}{=} \int_{\tilde{N}_\mathcal{F}(S)} dy \, \mathrm{Cu}^{L_\mathcal{F}(S)}_\pi(y) \mathrm{Ker}^{\mathring{K}_\mathcal{F}(S)}_0(y \sqcup x_{\varrho_S})$$
$$\cdot \, H_{\pi,\mathcal{F},C_\mathcal{F}(S)} \left[ \mathrm{Ker}^{K^{\partial}_\mathcal{F}(S)}_0 \cdot (-\mathscr{Y}_S \varphi) \right] (x_{\tilde{N}(S)^c} \sqcup y) \,,$$

where $H_{\pi,\mathcal{F},C_\mathcal{F}(S)}$ denotes the composition of the operators $H_{\pi,\mathcal{F},T}$ for all $T \in C_\mathcal{F}(S)$. No order needs to be prescribed for this composition since, for any $T_1, T_2 \in \mathcal{F}$ which are disjoint, the operators $H_{\pi,\mathcal{F},T_1}$ and $H_{\pi,\mathcal{F},T_2}$ commute.

Let us sketch the argument showing that this is indeed the case. We first observe that for any $T \in \mathcal{F}$ and any $\psi \in \mathscr{C}_*$ which depends only on $x_v$ with $v \notin \tilde{N}(T)$ one has

$$H_{\pi,\mathcal{F},T}[\varphi\psi] = \psi H_{\pi,\mathcal{F},T}[\varphi]. \tag{4.6}$$

Then one can unravel the inductive definition of $H_{\pi,\mathcal{F},T}$ to arrive at an expansion of the form

$$H_{\pi,\mathcal{F},T}[\varphi](x) \stackrel{\text{def}}{=} \sum_{k \in \widetilde{\text{Der}}(T)} (D^k\varphi)(\text{Coll}_T(x)) \int_{\tilde{N}(T)} dy\, F_{T,k}(x_{N^\downarrow(T)} \sqcup y)\,, \tag{4.7}$$

where $\widetilde{\text{Der}}(T)$ is some set of multi-indices supported on $\tilde{N}(T)$ and, for each $k$, $F_{T,k}$ is a product of combinatorial factors, derivatives of $\{D^{\bar{\mathfrak{e}}(e)}K_e\}_{e \in K(T)}$ evaluated on differences of $(x_v)_{v \in N(T)}$, and polynomials of those same differences. Using the combination of the observation (4.6), the symmetry of partial derivatives, and the fact that for distinct $i, j \in \{1, 2\}$ one has $N^\downarrow(T_i) \cap \tilde{N}(T_j) = \tilde{N}(T_i) \cap \tilde{N}(T_j) = \emptyset$ it is straightforward to use the expansion (4.7) to prove the claimed commutation rule.

More generally, given $\mathcal{F} \in \mathbb{F}_\pi$ and a subforest $\mathcal{G} \subset \mathcal{F}$ with $\text{depth}(G) \le 1$ we set $H_{\pi,\mathcal{F},\mathcal{G}} \stackrel{\text{def}}{=} \circ_{T \in G} H_{\pi,\mathcal{F},T}$ where on the RHS we are denoting a composition. We will also use this convention for variants of these $H$ operators we introduce later.

### 4.1.2 Deriving an explicit formula for negative renormalizations

In what follows, for any multi-index $\mathfrak{m}$ and real number $r$ we write

$$\mathbb{1}_{<r}(\mathfrak{m}) \stackrel{\text{def}}{=} \mathbb{1}\{|\mathfrak{m}|_\mathfrak{s} < r\}\,.$$

We also define, for any three multi-indices $\mathfrak{n}, \mathfrak{m}, \mathfrak{e}$ and real number $r$,

$$\begin{pmatrix} \mathfrak{n}\ r \\ \mathfrak{m}\ \mathfrak{e} \end{pmatrix} \stackrel{\text{def}}{=} \frac{1}{\mathfrak{e}!}\begin{pmatrix} \mathfrak{n} \\ \mathfrak{m} \end{pmatrix} \mathbb{1}_{<r}(\mathfrak{m} + \mathfrak{e})$$

For any $\mathcal{F} \in \mathbb{F}$, any node decoration $\mathfrak{n}$ on $\overline{T}$, and any $\pi \in \mathcal{P}[L(\overline{T})]$ compatible with $\mathcal{F}$, we define a function $\mathring{\varpi}_\pi^\mathfrak{n}[\mathcal{F}] \in \mathscr{C}_*$ as follows. We set $\mathring{\varpi}_\pi^\mathfrak{n}[\emptyset] = 1$ and, for $\mathcal{F}$ non-empty, we set

$$\mathring{\varpi}_\pi^\mathfrak{n}[\mathcal{F}] \stackrel{\text{def}}{=} \left[ \prod_{S \in \overline{\mathcal{F}}} \text{Cu}_\pi^{L_\mathcal{F}(S)} \text{Ker}_0^{\mathring{K}_\mathcal{F}(S)} H_{\pi,\mathcal{F},C_\mathcal{F}(S)} \left[ X_\mathfrak{n}^{\tilde{N}(S)} \text{Ker}_0^{K_\mathcal{F}^\partial(S)} \right] \right].$$

**Lemma 4.14** *Let $\mathcal{F} \in \mathbb{F}$ and let $F$ be the i-forest corresponding to $\overline{\mathcal{F}}$. Let $\mathfrak{n}$ be a node decoration on $F$ vanishing on $\varrho(F)$. Then one has*

$$\bar{\Upsilon}^\mathbb{P}\left[ \mathbf{P}_{\sigma_\mathcal{F}} \hat{\mathscr{A}}_-(F, 0)_\mathfrak{e}^\mathfrak{n} \right] = (-1)^{|\overline{\mathcal{F}}|} \sum_{\substack{\pi \in \overline{\mathcal{P}}[L(\mathcal{F})] \\ \pi \text{ comp. } \mathcal{F}}} \int_{N_\mathcal{F}} dy\, \delta(y_{\varrho(\overline{\mathcal{F}})}) \mathring{\varpi}_\pi^\mathfrak{n}[\mathcal{F}](y). \tag{4.8}$$

*Additionally, if $\mathfrak{n}$ does not vanish on $\varrho(F)$ then the RHS of (4.8) vanishes.*

*Proof.* The case where $\mathcal{F} = \emptyset$ holds trivially and we ignore this case below.

We work with the RHS of (4.8) and try to transform it to the LHS, working inductively in the depth of $\mathcal{F}$.

We first observe that (4.8) is forest multiplicative, that is we can write

$$\mathbf{P}_{\sigma_{\mathcal{F}}}\hat{\mathscr{A}}_{-}(F,0)^{\mathfrak{n}}_{\mathfrak{e}} = \prod_{S \in \overline{\mathcal{F}}} \mathbf{P}_{\sigma_{\mathcal{F}_S}}\hat{\mathscr{A}}_{-}(S,0)^{\mathfrak{n}}_{\mathfrak{e}} \ .$$

Both the integral and the sum over compatible partitions appearing in the RHS of (4.8) factorize analogously. It follows that for both the base step and the inductive step it suffices to treat the situation where there is a single maximal tree in $\mathcal{F}$, so $F = S$ for some subtree $S$.

We first treat the base case when $\mathcal{F}$ has depth 1. We then have

$$\mathbf{P}_{\sigma_{\{S\}}}\hat{\mathscr{A}}_{-}(S,0)^{\mathfrak{n}}_{\mathfrak{e}} = -(S,0)^{\mathfrak{n}}_{\mathfrak{e}}$$

and it is straightforward to see that

$$\bar{\Upsilon}^{\mathbb{P}}[(S,0)^{\mathfrak{n}}_{\mathfrak{e}}] = - \sum_{\pi \in \bar{\mathcal{P}}[L(S)]} \int_{N(S)} dy \, \delta(y_{\varrho S})\mathring{\varpi}^{\mathfrak{n}}_{\pi}[\{S\}](y).$$

We now treat the inductive case, we assume (4.8) is true for all forests of depth less than $j$ and prove it for a forest $\mathcal{F}$ of depth $j$ with $\overline{\mathcal{F}} = \{S\}$.

We write $\mathcal{G} \stackrel{\text{def}}{=} \mathcal{F} \setminus \{S\}$ and $G$ for the i-forest corresponding to $\overline{\mathcal{G}}$. For $T \in \overline{\mathcal{G}}$ we also write $\mathcal{G}_T = \{U \in \mathcal{F} : U \leq T\}$, observe that $\mathcal{G}_T$ is a forest of subtrees of depth less than $j$.

With this notation in hand we have that $\bar{\Upsilon}^{\mathbb{P}}[\mathbf{P}_{\sigma_{\mathcal{F}}}\hat{\mathscr{A}}_{-}(S,0)^{\mathfrak{n}}_{\mathfrak{e}}]$ is given by

$$- \sum_{\mathfrak{n}_G, \, \mathfrak{e}_G} \bar{\Upsilon}^{\mathbb{P}}[(S,[G]_1)^{\mathfrak{n}-\mathfrak{n}_G}_{\mathfrak{e}+\mathfrak{e}_G}] \cdot \Big[ \prod_{T \in \overline{\mathcal{G}}} \binom{\mathfrak{n} \ \omega(T)}{\mathfrak{n}_T \ \mathfrak{e}_T} \bar{\Upsilon}^{\mathbb{P}}[\mathbf{P}_{\sigma_{\mathcal{G}_T}}\hat{\mathscr{A}}_{-}(T,0)^{\mathfrak{n}_T+\chi\mathfrak{e}_T}_{\mathfrak{e}}] \Big],$$

where above the sum is over node decorations $\mathfrak{n}_G$ on $N(G) \setminus \varrho(G)$ and edge decorations $\mathfrak{e}_G$ on $K^{\partial}_{\mathcal{F}}(S)$ – the edge decorations $\mathfrak{e}_T$ and node decorations $\mathfrak{n}_T$ are given by the restrictions of $\mathfrak{e}_G$ to $K^{\partial}_T(S)$ and $\mathfrak{n}_G$ to $\tilde{N}(T)$, respectively.

Expanding now the RHS of (4.8) and using the fact that every partition $\pi$ of $L(S)$ compatible with $\mathcal{F}$ consists of a partition $\bar{\pi}$ of $L_{\mathcal{F}}(S)$ and, for every $T \in \bar{\mathcal{G}}$, a partition $\pi_T$ compatible with $\mathcal{G}_T$, we obtain

$$- \sum_{\substack{\pi \in \bar{\mathcal{P}}[L(S)] \\ \pi \text{ comp. } \mathcal{F}}} \int_{N_{\mathcal{F}}(S)} dy \, \delta(y_{\varrho S})\mathring{\varpi}^{\mathfrak{n}}_{\pi}[\mathcal{F}](y) \tag{4.9}$$

$$= - \sum_{\bar{\pi} \in \bar{\mathcal{P}}[L_{\mathcal{F}}(S)]} \int_{N_{\mathcal{F}}(S)} dy \, \delta(y_{\varrho S})\mathrm{Cu}^{L_{\mathcal{F}}(S)}_{\bar{\pi}}(y)\mathrm{Ker}^{\mathring{K}_{\mathcal{F}}(S)}_{0}(y)\mathrm{X}^{N_{\mathcal{F}}(S)}_{\mathfrak{n}}(y)$$

$$\cdot \prod_{T \in \bar{\mathcal{G}}} \Big[ \sum_{\substack{\pi_T \in \bar{\mathcal{P}}[L(T)] \\ \pi_T \text{ comp. } \mathcal{G}_T}} H_{\pi_T, \mathcal{G}_T, T}\Big[\mathrm{X}^{\tilde{N}(T)}_{\mathfrak{n}}\mathrm{Ker}^{K^{\partial}_T(S)}_{0}\Big](y) \Big] \ .$$

We want to expand the action of the $H_{\pi_T, \mathcal{G}_T, T}$ above. For any $T \in \overline{\mathcal{G}}$ one has, by the rules of of differential calculus and some manipulation of binomial coefficients,

$$
\mathscr{Y}_T\left[X_\mathfrak{n}^{\tilde{N}(T)} \mathrm{Ker}_0^{K_T^\partial(S)}\right] = \sum_{\substack{j \in \mathrm{Der}(T) \\ \mathfrak{e}_T}} \frac{X_{j,\varrho_T}^{\tilde{N}(T)}}{j!} \binom{j}{\chi \mathfrak{e}_T}\binom{\mathfrak{n}}{j - \chi \mathfrak{e}_T} X_{\mathfrak{n}-j+\chi\mathfrak{e}_T}^{\tilde{N}(T),\varrho_T} \mathrm{Ker}_{\mathfrak{e}_T}^{K_T^\partial(S),\varrho_T}
$$

$$
= \sum_{\mathfrak{n}_T, \mathfrak{e}_T} \binom{\mathfrak{n} \ \omega(T)}{\mathfrak{n}_T \ \mathfrak{e}_T} X_{\mathfrak{n}_T,\varrho_T}^{\tilde{N}(T)} X_{\mathfrak{n}-\mathfrak{n}_T}^{\tilde{N}(T),\varrho_T} \mathrm{Ker}_{\mathfrak{e}_T}^{K_T^\partial(S),\varrho_T}
$$

where we view $j \in \mathrm{Der}(T)$ as node decorations on $\tilde{N}(T)$. We can then write

$$
\sum_{\substack{\pi_T \in \bar{\mathcal{P}}[L(T)] \\ \pi_T \text{ comp. } \mathcal{G}_T}} H_{\pi_T, \mathcal{G}_T, T}\left[X_\mathfrak{n}^{\tilde{N}(T)} \mathrm{Ker}_0^{K_T^\partial(S)}\right](y)
$$

$$
= -\sum_{\substack{\pi_T \in \bar{\mathcal{P}}[L(T)] \\ \pi_T \text{ comp. } \mathcal{G}_T}} \sum_{\mathfrak{n}_T, \mathfrak{e}_T} \binom{\mathfrak{n} \ \omega(T)}{\mathfrak{n}_T \ \mathfrak{e}_T} X_{\mathfrak{n}-\mathfrak{n}_T}^{\tilde{N}(T),\varrho_T}(y) \mathrm{Ker}_{\mathfrak{e}_T}^{K_T^\partial(S),\varrho_T}(y)
$$

$$
\cdot \int_{\tilde{N}_{\mathcal{G}_T}(T)} dz \, \mathrm{Cu}_{\pi_T}^{L_{\mathcal{G}_T}(T)}(z) \mathrm{Ker}_0^{\mathring{K}_{\mathcal{G}_T}(T)}(z \sqcup y_{\varrho_T})
$$

$$
\cdot H_{\pi_T, \mathcal{G}_T, C_{\mathcal{G}_T}(T)}\left[\mathrm{Ker}_0^{K_{\mathcal{G}_T}^\partial(T)} X_{\mathfrak{n}_T,\varrho_T}^{\tilde{N}(T)}\right](y_{N_T^\partial(S)} \sqcup y_{\varrho_T} \sqcup z).
$$

By translation invariance, the final integral does not depend on the value of $y_{\varrho_T}$. We can therefore replace $y_{\varrho_T}$ with 0, which makes $\mathring{\varpi}_{\pi_T}^{\mathfrak{n}_T}$ appear. By our induction hypothesis, the above quantity is thus equal to

$$
- \sum_{\mathfrak{n}_T, \mathfrak{e}_T} \binom{\mathfrak{n} \ \omega(T)}{\mathfrak{n}_T \ \mathfrak{e}_T} X_{\mathfrak{n}-\mathfrak{n}_T}^{\tilde{N}(T),\varrho_T}(y) \, \mathrm{Ker}_{\mathfrak{e}_T}^{K_T^\partial(S),\varrho_T}(y) \, \bar{\Upsilon}^\mathbb{P}\left[\mathbf{P}_{\sigma_{\mathcal{G}_T}} \hat{\mathscr{A}}_-(T, 0)_{\bar{\mathfrak{e}}}^{\mathfrak{n}_T}\right].
$$

Inserting this into the last line of (4.9) for each $T \in \overline{\mathcal{G}}$ gives

$$
\sum_{\bar{\pi} \in \bar{\mathcal{P}}[L_{\mathcal{F}}(S)]} \int_{N_{\mathcal{F}}(S)} dy \, \delta(y_{\varrho_S}) \, \mathrm{Cu}_{\bar{\pi}}^{L_{\mathcal{F}}(S)}(y) \, \mathrm{Ker}_0^{\mathring{K}_{\mathcal{F}}(S)}(y) \, X_\mathfrak{n}^{N_{\mathcal{F}}(S)}(y)
$$

$$
\cdot \left[\prod_{T \in \overline{\mathcal{G}}} \sum_{\mathfrak{n}_T, \mathfrak{e}_T} \binom{\mathfrak{n} \ \omega(T)}{\mathfrak{n}_T \ \mathfrak{e}_T} X_{\mathfrak{n}-\mathfrak{n}_T}^{\tilde{N}(T),\varrho_T}(y) \, \mathrm{Ker}_{\mathfrak{e}_T}^{K_T^\partial(S),\varrho_T}(y) \, \bar{\Upsilon}^\mathbb{P}\left[\mathbf{P}_{\sigma_{\mathcal{G}_T}} \hat{\mathscr{A}}_-(T, 0)_{\bar{\mathfrak{e}}}^{\mathfrak{n}_T+\chi\mathfrak{e}_T}\right]\right]
$$

$$
= -\sum_{\mathfrak{n}_G, \, \mathfrak{e}_G} \bar{\Upsilon}^\mathbb{P}\left[(S, [G]_1)_{\bar{\mathfrak{e}}+\mathfrak{e}_G}^{\mathfrak{n}-\mathfrak{n}_G}\right]\left(\prod_{T \in \overline{\mathcal{G}}} \binom{\mathfrak{n} \ \omega(T)}{\mathfrak{n}_T \ \mathfrak{e}_T} \bar{\Upsilon}^\mathbb{P}\left[\mathbf{P}_{\sigma_{\mathcal{G}_T}} \hat{\mathscr{A}}_-(T, 0)_{\bar{\mathfrak{e}}}^{\mathfrak{n}_T+\chi\mathfrak{e}_T}\right]\right).
$$

$\square$

## 4.2    A cutting formula for positive renormalizations

In the previous subsection we saw that the relevant substructures of $\overline{T}$ with regards to negative renormalizations were the subtrees of $\overline{T}$ of negative homogeneity. The analogous substructures of $\overline{T}$ for positive renormalizations will be edges in $K(\overline{T})$ which are "trunks" of subtrees of $\overline{T}$ of positive homogeneity.

We frequently reference our chosen poset structure on $K(\overline{T})$, namely $e \leq \bar{e}$ if and only if the unique path of edges from $\bar{e}_c$ to the root $\varrho_*$ of $\overline{T}$ contains $e$.

For $\mathscr{C} \subset K(\overline{T})$, $e \in K(\overline{T})$, the set of immediate children of $e$ in $\mathscr{C}$ is given by

$$C_{\mathscr{C}}(e) \stackrel{\text{def}}{=} \text{Min}(\{\bar{e} \in \mathscr{C} : \ \bar{e} > e\}).$$

For a subset $\mathscr{C} \subset K(\overline{T})$ we define the subtrees and sub-forests

$$\overline{T}_{\not\geq}[\mathscr{C}] \stackrel{\text{def}}{=} \bigcap_{e \in \mathscr{C}} \overline{T}_{\not\geq}(e) \text{ and } \mathfrak{T}_{\geq}[\mathscr{C}] \stackrel{\text{def}}{=} \mathfrak{T}(\overline{T}, \overline{T}_{\not\geq}(\mathscr{C})) \, ,$$

where we adopt the convention $\overline{T}_{\not\geq}[\emptyset] \stackrel{\text{def}}{=} \overline{T}$ and $\mathfrak{T}[\cdot, \cdot]$ is as in Definition 3.5. Note that while $\mathfrak{T}_{\geq}[\mathscr{C}]$ is a collection of subtrees of $\overline{T}$ it will in general *not* be a forest of subtrees since distinct elements of $\mathfrak{T}_{\geq}[\mathscr{C}]$ may share roots.

Observe that both $\overline{T}_{\not\geq}[\mathscr{C}]$ and $\mathfrak{T}_{\geq}[\mathscr{C}]$ depend on $\mathscr{C}$ only through $\text{Min}(\mathscr{C})$ and that $\overline{T}_{\not\geq}[\mathscr{C}]$ always contains as a subtree the trivial tree consisting only of $\varrho_*$.

**Remark 4.15**  In what follows we will begin writing co-action extraction / colorings as sums over sets of edges. It then makes sense to label the resulting sums over decorations with sets of edges rather than the subtrees they correspond to. Namely, if $\mathscr{C} \subset K(\overline{T})$ then a sum over $\mathfrak{n}_{\mathscr{C}}$ corresponds to a sum over node decorations which are supported on $N(T_{\not\geq}[\mathscr{C}])$ while a sum over $\mathfrak{e}_{\mathscr{C}}$ corresponds a sum over edge-derivative decorations which are supported on $\underline{\mathscr{C}}$.

**Definition 4.16**  We say $e \in K(\overline{T})$ is a *positive cut* if

$$\left| \tilde{P}(\overline{T}_{\geq}(e), 0)_{\mathfrak{e}}^{\bar{\mathfrak{n}}} \right|_{+} > 0 \, ,$$

where $\tilde{P}$ is the map mentioned before which zeroes node labels at roots (this map naturally induces a map on i-forests).

We denote by $\mathfrak{C} \subset K(\overline{T})$ the collection of all positive cuts and define a function $\gamma : \mathfrak{C} \to \mathbf{N}$ via

$$\gamma(e) \stackrel{\text{def}}{=} \left\lceil \left| \tilde{P}(\overline{T}_{\geq}(e), 0)_{\mathfrak{e}}^{\bar{\mathfrak{n}}} \right|_{+} \right\rceil.$$

The key role of the extended node-label $\mathfrak{o}$ is to store information so that the negative renormalizations from $\hat{\Delta}_{-}$ do not cause $\hat{\Delta}_{+}$ or $\hat{\mathscr{A}}_{+}$ to make *additional* positive renormalizations.

We now take advantage of this to help us unravel the action of $\hat{\mathscr{A}}_{+}$. First recall that any i-tree can be realized as a subtree of $\overline{T}$. We define $\mathfrak{T}_{+}$ to be the collection

of all decorated i-trees of the form $(\overline{T}, \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in \underline{\mathfrak{T}_2}$ with the properties that (i) $\mathfrak{e} \geq \overline{\mathfrak{e}}$ and (ii) for every $e \in K(\overline{T})$ with $\hat{T}(e_\mathsf{p}) \neq 2$ and $\hat{T}(e) = 0$ one has

$$\left| \tilde{P}(\overline{T}_{\geq}(e), 0)_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}} \right|_+ = \left| \tilde{P}(\overline{T}_{\geq}(e), \hat{T})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \right|_+ + |\mathfrak{e}(e) - \overline{\mathfrak{e}}(e)|_{\mathfrak{s}} \ .$$

We then have the following lemma.

**Lemma 4.17** *One has*

$$\hat{\Delta}_1(\overline{T}, 0)_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}} \in \langle \mathfrak{F}_0 \rangle \otimes \langle \underline{\mathfrak{T}_1} \cap \mathfrak{T}_+ \rangle$$
$$\text{and } (\mathrm{Id} \otimes \hat{\Delta}_2)\hat{\Delta}_1(\overline{T}, 0)_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}} \in \langle \mathfrak{F}_0 \rangle \otimes \langle \underline{\mathfrak{T}_1} \rangle \otimes \langle \mathfrak{T}_+ \cap \hat{\underline{\mathfrak{T}_2}} \rangle .$$

Given $\mathscr{C} \subset \mathfrak{C}$ and $k \in \mathbf{N}$ we define subsets $D_k(\mathscr{C})$ of $\mathscr{C}$ as follows. We set $D_0(\mathscr{C}) \stackrel{\text{def}}{=} \emptyset$, $D_1(\mathscr{C}) \stackrel{\text{def}}{=} \underline{\mathscr{C}}$, and for $k > 1$ we set

$$D_k(\mathscr{C}) \stackrel{\text{def}}{=} \bigsqcup_{e \in D_{k-1}(\mathscr{C})} C_{\mathscr{C}}(e).$$

We define the depth of $\mathscr{C} \subset \mathfrak{C}$ via

$$\mathrm{depth}(\mathscr{C}) \stackrel{\text{def}}{=} \inf \left\{ k \in \mathbf{N} : \bigsqcup_{j=1}^{k} D_j(\mathscr{C}) = \mathscr{C} \right\} = \inf\{ k \geq 0 : \ D_{k+1}(\mathscr{C}) = \emptyset \} \ .$$

Below we draw a picture to clarify these notions. We have drawn a tree $\overline{T}$, but have neglected to draw any leaf edges or any of the fictitious nodes attached to their ends. The set $\mathscr{C} \subset K(\overline{T})$ consists of all edges which have a non-zero number of tick marks. For $k \geq 1$ the set $D_k(\mathscr{C})$ then consists of those edges with $k$ tick marks. We see that the set $\mathscr{C}$ has depth 3.
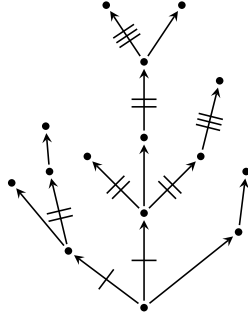


For any $\mathscr{C} \subset \mathfrak{C}$ we define $\mathrm{Div}_{\mathscr{C}}$ to be the collection of those $T \in \mathrm{Div}$ such that $\mathscr{C} \cap K(T) = \emptyset$. We define $\mathbb{F}_{\mathscr{C}}$ to be the set of all forests $\mathcal{F} \in \mathbb{F}$ with $\mathcal{F} \subset \mathrm{Div}_{\mathscr{C}}$.

Conversely, for any $\mathcal{F} \in \mathbb{F}$ we define $\mathfrak{C}_{\mathcal{F}} \stackrel{\text{def}}{=} \mathfrak{C} \setminus K(\mathcal{F})$. so that $\mathcal{F} \in \mathbb{F}_{\mathscr{C}}$ if and only if $\mathscr{C} \subset \mathfrak{C}_{\mathcal{F}}$.

For any $\mathscr{C} \subset \mathfrak{C}$ and $\mathcal{F} \in \mathrm{Div}_{\mathscr{C}}$ we also define

$$\mathcal{F}[\mathscr{C}] \stackrel{\text{def}}{=} \{ T \in \mathcal{F} : T \leq S \text{ for some } S \in \mathfrak{T}_{\geq}[\mathscr{C}] \} \ .$$

Additionally, for $\mathcal{F} \in \mathbb{F}$ and $\mathscr{C}, \mathscr{D} \subset \mathfrak{C}_{\mathcal{F}}$ we define

$$\mathcal{F}[\mathscr{C}, \mathscr{D}] \stackrel{\text{def}}{=} \{T \in \mathcal{F}[\mathscr{C}] : T \leq \overline{T}_{\not\geq}[\mathscr{D}]\} .$$

**Definition 4.18** Given $\mathscr{C} \subset \mathfrak{C}$ and $\mathcal{F} \in \text{Div}_{\mathscr{C}}$ we define $\sigma_{\mathscr{C}, \mathcal{F}} \in \overset{\circ}{\mathfrak{T}}$ as follows.

If $\mathscr{C} = \emptyset$ then we set $\sigma_{\mathscr{C}, \mathcal{F}} \stackrel{\text{def}}{=} (\overline{T}, 2)$. Otherwise $\mathscr{C}$ must be of depth $k \geq 1$ in which case we set

$$\sigma_{\mathscr{C}, \mathcal{F}} \stackrel{\text{def}}{=} (T^{(1)}, \widehat{T^{(1)}}) \cdots (T^{(k)}, \widehat{T^{(k)}}), \tag{4.10}$$

where each of the trees $T^{(j)}$ are subtrees of $\overline{T}$ and for $1 \leq j \leq k$ one has

- $T^{(j)} = \overline{T}_{\not\geq}[D_{j+1}(\mathscr{C})]$,
- $\widehat{T^{(j)}} = [\overline{T}_{\not\geq}[D_j(\mathscr{C})]]_2 \sqcup \left[\overline{\mathcal{F}[D_j(\mathscr{C}), D_{j+1}(\mathscr{C})]}\right]_1$.

We present a pictorial example to clarify the above definitions. We let $\overline{T}$ and $\mathscr{C}$ be as in the immediately previous pictorial example and set $\mathcal{F} = \emptyset$. The tree $\sigma_{\mathscr{C}, \emptyset}$ is then given by the forest product (written in the same order as in (4.10))
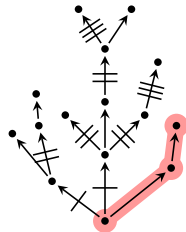


$$\tag{4.11}$$
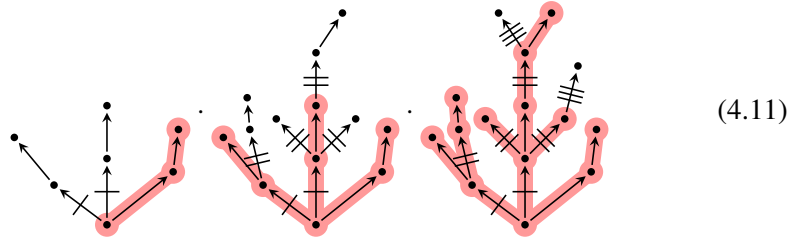
We now want to state the analogues of Lemmas 4.7 and 4.14 for positive renormalizations.

We denote by $\mathfrak{C}^{\leq 1}$ the collection of all subsets of $\mathfrak{C}$ with depth $0$ or $1$ and also set $\mathfrak{C}^{\leq 1}_{\mathcal{F}} \stackrel{\text{def}}{=} \mathfrak{C}^{\leq 1} \cap 2^{\mathfrak{C}_{\mathcal{F}}}$. For any $\mathscr{C} \subset \mathfrak{C}$ and $\mathcal{F} \in \mathbb{F}_{\mathscr{C}}$ we define a coloring on $\overline{T}$ given by

$$\hat{\overline{T}}[\mathscr{C}, \mathcal{F}] \stackrel{\text{def}}{=} \left[\overline{T}_{\not\geq}[\mathscr{C}]\right]_2 \sqcup \left[\overline{\mathcal{F}[\mathscr{C}]}\right]_1 .$$
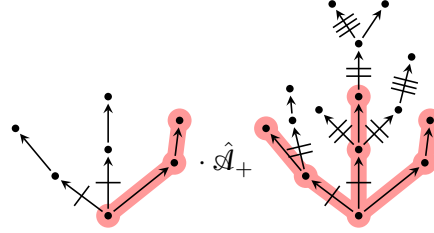
and use the shorthand $\overline{T}[\mathscr{C}, \mathcal{F}] \stackrel{\text{def}}{=} (\overline{T}, \hat{\overline{T}}[\mathscr{C}, \mathcal{F}])$.

For our previous example with $\overline{T}$ and $\mathscr{C}$ one has $(\overline{T}, \hat{\overline{T}}[\mathscr{C}, \emptyset])$ given by



Such i-trees will be produced by the action of $\hat{\Delta}_+$, the action of $\hat{\mathscr{A}}_+$ corresponds to iteratively pulling out more structures, always acting to the right.

Heuristically the tree $\sigma_{\mathscr{C},\emptyset}$ is one of the terms appearing in the expansion of $\hat{\mathscr{A}}_+(\overline{T},\hat{\overline{T}}[\mathscr{C},\emptyset])$. Working through the action of $\hat{\mathscr{A}}_+$ on $(\overline{T},\hat{\overline{T}}[\mathscr{C},\emptyset])$ as specified through its recursive definition, after one step of the recursion many terms will be generated but the term giving rise to $\sigma_{\mathscr{C},\emptyset}$ is



$$\tag{4.12}$$

The term $\sigma_{\mathscr{C},\emptyset}$ then appears from the above term when one goes one step further in the recursive definition of $\hat{\mathscr{A}}_+$, namely by pulling out the middle tree of (4.11) from the second tree of (4.12). We define the collection of cut sets

$$\mathfrak{C}[\mathscr{C},\mathcal{F}] \stackrel{\text{def}}{=} \{\mathscr{D} \subset \mathfrak{C}_{\mathcal{F}} : \; \underline{\mathscr{D}} = \mathscr{C}\}, \tag{4.13}$$

$$\mathfrak{C}_<[\mathscr{C},\mathcal{F}] \stackrel{\text{def}}{=} \Big\{\mathscr{D} \in \mathfrak{C}_{\mathcal{F}}^{\leq 1} : \; \forall e \in \mathscr{D}, \; \exists e' \in \mathscr{C} \text{ with } e' < e \Big\}.$$

Also, for any $\mathscr{C} \subset \mathfrak{C}$, $\mathcal{F} \in \mathbb{F}_{\mathscr{C}}$, and $\mathscr{D} \in \mathfrak{C}_<[\mathscr{C},\mathcal{F}]$ we set

$$\overline{T}[\mathscr{C},\mathscr{D},\mathcal{F}] \stackrel{\text{def}}{=} \left(\overline{T}_{\not\geq}[\mathscr{D}], [\overline{\mathcal{F}[\mathscr{C},\mathscr{D}]}]_1 \sqcup [\overline{T}_{\not\geq}[\mathscr{C}]]_2\right).$$

With all this notation in hand we have the following lemma.

**Lemma 4.19** *Let $\mathcal{F} \in \mathbb{F}$ and $\mathscr{C} \in \mathfrak{C}_{\mathcal{F}}^{\leq 1}$. Then for any node decoration $\mathfrak{n}$, edge decoration $\mathfrak{e}$, and extended node label $\mathfrak{o}$ on $\overline{T}$ such that $\overline{T}[\mathscr{C},\mathcal{F}]_{\bar{\mathfrak{e}}+\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in \mathfrak{T}_+$ one has*

$$\left(\sum_{\mathscr{D} \in \mathfrak{C}[\mathscr{C},\mathcal{F}]} \mathbf{P}_{\sigma_{\mathscr{D},\mathcal{F}}}\right) \hat{\mathscr{A}}_+ \overline{T}[\mathscr{C},\mathcal{F}]_{\bar{\mathfrak{e}}+\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} = \hat{\mathscr{A}}_+ \overline{T}[\mathscr{C},\mathcal{F}]_{\bar{\mathfrak{e}}+\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}.$$

*Proof.* Our proof operates inductively in the quantity

$$|E_{\overline{T}} \setminus E_{\overline{T}_{\not\geq}[\mathscr{C}]}|, \tag{4.14}$$

with $\mathcal{F}$ being fixed for the entire proof. The base case of this induction, which occurs when the quantity (4.14) is 0, is immediate since we then have $\overline{T}[\mathscr{C},\mathcal{F}] = (\overline{T},2)$ – we then have $\hat{\mathscr{A}}_+(\overline{T},2)_{\bar{\mathfrak{e}}+\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} = (-1)^{|\mathfrak{n}|}(\overline{T},2)_{\bar{\mathfrak{e}}+\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}$.

We now turn to the proving the inductive step. Fix $j \in \mathbf{N}$, and suppose the claim has been proven for any $\mathscr{C}$ with $\mathscr{C} \in \mathfrak{C}_{\mathcal{F}}^{\leq 1}$ and (4.14) less than $j$.

Then suppose we are given $\mathscr{C} \in \mathfrak{C}_{\mathcal{F}}^{\leq 1}$ with (4.14) equal to $j+1$. Then we have $(-1)^{|\mathscr{C}|+|\hat{\mathfrak{n}}|} \hat{\mathscr{A}}_+ \overline{T}[\mathscr{C},\mathcal{F}]_{\bar{\mathfrak{e}}+\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}$ is given by

$$\sum_{\mathscr{D} \in \mathfrak{C}_<[\mathscr{C},\mathcal{F}]} \sum_{\mathfrak{e}_{\mathscr{D}},\mathfrak{n}_{\mathscr{D}},\mathfrak{f}_{\mathscr{C}}} \frac{(-1)^{|\mathfrak{f}_{\mathscr{C}}|}}{(\mathfrak{f}_{\mathscr{C}}+\mathfrak{e}_{\mathscr{D}})!} \binom{\mathfrak{n}-\hat{\mathfrak{n}}}{\mathfrak{n}_{\mathscr{D}}} \overline{T}[\mathscr{C},\mathscr{D},\mathcal{F}]_{\bar{\mathfrak{e}}+\mathfrak{e}+\mathfrak{f}_{\mathscr{C}}}^{\hat{\mathfrak{n}}+\mathfrak{n}_{\mathscr{D}}+\chi(\mathfrak{e}_{\mathscr{D}}+\mathfrak{f}_{\mathscr{C}})}$$

$$\cdot \hat{\mathcal{A}}_+\mathfrak{p}_+\overline{T}[\mathscr{D},\mathcal{F}]^{\mathfrak{n}-\mathfrak{n}_{\mathscr{D}}-\hat{\mathfrak{n}},\mathfrak{o}}_{\bar{\mathfrak{e}}+\mathfrak{e}+\mathfrak{e}_{\mathscr{D}}}$$

$$= \sum_{\mathscr{D}\in\mathfrak{C}_<[\mathscr{C},\mathcal{F}]} \sum_{\mathfrak{e}_{\mathscr{D}},\mathfrak{n}_{\mathscr{D}},\mathfrak{f}_\mathscr{C}} \frac{(-1)^{|\mathfrak{f}_\mathscr{C}|}}{(\mathfrak{e}_{\mathscr{D}}+\mathfrak{f}_\mathscr{C})!} \binom{\mathfrak{n}-\hat{\mathfrak{n}}}{\mathfrak{n}_{\mathscr{D}}} \overline{T}[\mathscr{C},\mathscr{D},\mathcal{F}]^{\mathfrak{n}_{\mathscr{D}}+\hat{\mathfrak{n}}+\chi(\mathfrak{e}_{\mathscr{D}}+\mathfrak{f}_\mathscr{C})}_{\bar{\mathfrak{e}}+\mathfrak{e}+\mathfrak{f}_\mathscr{C}}$$

$$\cdot \sum_{\mathscr{E}\in\mathfrak{C}[\mathscr{D},\mathcal{F}]} \mathbf{P}_{\sigma_{\mathscr{E}},\mathcal{F}} \overline{T}[\mathscr{D},\mathcal{F}]^{\mathfrak{n}-\mathfrak{n}_{\mathscr{D}}-\hat{\mathfrak{n}},\mathfrak{o}}_{\bar{\mathfrak{e}}+\mathfrak{e}+\mathfrak{e}_{\mathscr{D}}}$$

$$= \sum_{\substack{\mathscr{D}\in\mathfrak{C}_<[\mathscr{C},\mathcal{F}]\\\mathscr{E}\in\mathfrak{C}[\mathscr{D},\mathcal{F}]}} \mathbf{P}_{\sigma_{\mathscr{E}\sqcup\mathscr{C}},\mathcal{F}} \sum_{\mathfrak{e}_{\mathscr{D}},\mathfrak{n}_{\mathscr{D}},\mathfrak{f}_\mathscr{C}} \frac{(-1)^{|\mathfrak{f}_\mathscr{C}|}}{(\mathfrak{e}_{\mathscr{D}}+\mathfrak{f}_\mathscr{C})!} \binom{\mathfrak{n}}{\mathfrak{n}_{\mathscr{D}}} \overline{T}[\mathscr{C},\mathscr{D},\mathcal{F}]^{\hat{\mathfrak{n}}+\mathfrak{n}_{\mathscr{D}}+\chi(\mathfrak{e}_{\mathscr{D}}+\mathfrak{f}_\mathscr{C})}_{\bar{\mathfrak{e}}+\mathfrak{e}+\mathfrak{f}_\mathscr{C}}$$

$$\cdot \hat{\mathcal{A}}_+\mathfrak{p}_+\overline{T}[\mathscr{D},\mathcal{F}]^{\mathfrak{n}-\mathfrak{n}_{\mathscr{D}}-\hat{\mathfrak{n}},\mathfrak{o}}_{\bar{\mathfrak{e}}+\mathfrak{e}+\mathfrak{e}_{\mathscr{D}}}$$

$$=(-1)^{|\mathscr{C}|+|\hat{\mathfrak{n}}|}\bigg(\sum_{\substack{\mathscr{D}\in\mathfrak{C}_<[\mathscr{C},\mathcal{F}]\\\mathscr{E}\in\mathfrak{C}[\mathscr{D},\mathcal{F}]}} \mathbf{P}_{\sigma_{\mathscr{E}\sqcup\mathscr{C}},\mathcal{F}}\bigg)\hat{\mathcal{A}}_+\overline{T}[\mathscr{C},\mathcal{F}]^{\mathfrak{n},\mathfrak{o}}_{\bar{\mathfrak{e}}+\mathfrak{e}},$$

where $\hat{\mathfrak{n}} = \mathfrak{n}(N(T_{\not\geq}[\mathscr{C}]))$ and the sum over $\mathfrak{f}_\mathscr{C}$ is a sum over edge decorations $\mathfrak{f}_\mathscr{C} \in \mathfrak{E}[\overline{T}[\mathscr{C},\mathcal{F}]^{\mathfrak{n},\mathfrak{o}}_{\bar{\mathfrak{e}}+\mathfrak{e}}]$. Since

$$\bigsqcup_{\mathscr{D}\in\mathfrak{C}_<[\mathscr{C},\mathcal{F}]} \bigsqcup_{\mathscr{E}\in\mathfrak{C}[\mathscr{D},\mathcal{F}]} (\mathscr{C}\sqcup\mathscr{E}) = \mathfrak{C}[\mathscr{C},\mathcal{F}]\,,$$

the result follows.    □

**Remark 4.20** Clearly, the statement of Lemma 4.19 holds if we relax the condition $\mathscr{C} \in \mathfrak{C}^{\leq 1}_{\mathcal{F}}$ to just that $\mathscr{C} \in \mathfrak{C}_{\mathcal{F}}$ provided one modifies what's written on the RHS and sums over $\mathscr{D} \in \mathfrak{C}[\underline{\mathscr{C}},\mathcal{F}]$.

We now introduce some more notation to facilitate writing explicit formulas for the integrands corresponding to the output of the positive twisted antipode.

For any $\mathscr{C} \subset \mathfrak{C}$, $\mathcal{F} \in \mathbb{F}_\mathscr{C}$, and $\mathscr{D} \in \mathfrak{C}_<[\mathscr{C},\mathcal{F}]$ we define, using the convention of Remark 4.9,

$$\tilde{N}[\mathscr{C},\mathcal{F}] \stackrel{\text{def}}{=} \tilde{N}(\mathfrak{T}_\geq[\mathscr{C}]) \setminus \tilde{N}(\mathcal{F}), K[\mathscr{C},\mathcal{F}] \stackrel{\text{def}}{=} K(\mathfrak{T}_\geq[\mathscr{C}]) \setminus (K(\mathcal{F})\sqcup\mathscr{C}),$$

$$L_{\mathscr{C},\mathcal{F}} \stackrel{\text{def}}{=} L(\mathfrak{T}_\geq[\mathscr{C}]) \setminus L(\mathcal{F})\,.$$

For any $\mathscr{D} \subset \mathfrak{C}$ and edge decoration $\mathfrak{e}$ we define $\mathrm{RKer}^{\mathscr{D}}_{\mathfrak{e}} \in \mathscr{C}_*$, depending on $x_v$ with $v \in e_{\mathsf{c}}(\mathscr{D}) \cup e_{\mathsf{p}}(\mathscr{D}) \cup \{\circledast\}$, via

$$\mathrm{RKer}^{\mathscr{D}}_{\mathfrak{e}}(x) \stackrel{\text{def}}{=} (-1)^{|\mathscr{D}|} \prod_{e\in\mathscr{D}} \sum_{|k|_{\mathfrak{s}}<\gamma(e)-|\mathfrak{e}(e)|_{\mathfrak{s}}} \frac{(x_{e_{\mathsf{p}}}-x_\circledast)^k}{k!} D^{k+\bar{\mathfrak{e}}(e)+\mathfrak{e}(e)} K_{\mathfrak{t}(e)}(x_\circledast - x_{e_{\mathsf{c}}}).$$

For any $\mathscr{C} \subset \mathfrak{C}$ and edge decoration $\mathfrak{e}$ on $K(\overline{T})$ we define $\mathrm{Der}(\mathscr{C},\mathfrak{e})$ to be the set of all edge decorations $\mathfrak{f}_\mathscr{C}$ supported on $\underline{\mathscr{C}}$ with the property that for every $e \in \underline{\mathscr{C}}$ one has $|\mathfrak{f}_\mathscr{C}(e)|_{\mathfrak{s}} \leq \gamma(e) - |\mathfrak{e}(e)|_{\mathfrak{s}}$.

For any node decoration $\mathfrak{n}$ on $N(T)$ with $\mathfrak{n} \leq \overline{\mathfrak{n}}$ and edge decoration $\mathfrak{e}$ on $K(\overline{T})$ we define $\ddot{\varpi}_{\mathfrak{e}}^{\mathfrak{n}}[\mathscr{C}, \mathcal{F}](x) \in \mathscr{C}_*$, depending on $x_v$ with $v \in \tilde{N}[\mathscr{C}, \mathcal{F}] \sqcup \{\circledast\}$, by setting

$$
\ddot{\varpi}_{\mathfrak{e}}^{\mathfrak{n}}[\mathscr{C}, \mathcal{F}] \stackrel{\text{def}}{=} \mathrm{RKer}_{\mathfrak{e}}^{\mathscr{C} \backslash \underline{\mathscr{C}}} \cdot \mathrm{Ker}_{\mathfrak{e}}^{K[\mathscr{C}, \mathcal{F}]} \cdot \Big( \sum_{\mathfrak{f}_{\mathscr{C}} \in \mathrm{Der}(\mathscr{C}, \mathfrak{e})} \frac{\mathrm{Ker}_{\mathfrak{e}+\mathfrak{f}_{\mathscr{C}}}^{\mathscr{C}, \otimes}}{\mathfrak{f}_{\mathscr{C}}!} \overline{\mathrm{X}}_{\chi \mathfrak{f}_{\mathscr{C}}}^{e_{\mathrm{p}}(\underline{\mathscr{C}}), \otimes} \Big)
$$
$$
\cdot \Big( \prod_{T \in \overline{\mathcal{F}[\mathscr{C}]}} \mathrm{X}_{\mathfrak{n}, \otimes}^{\tilde{N}(T), \varrho_T} \Big) \cdot \mathrm{X}_{\mathfrak{n}, \otimes}^{\tilde{N}[\mathscr{C}, \mathcal{F}]} \cdot \overline{\mathrm{X}}_{\mathfrak{n}}^{N(\overline{T}_{\not\geq}[\mathscr{C}]), \otimes} .
$$

Finally, for an edge decoration $\mathfrak{e}$ and $\mathscr{C} \subset \mathfrak{C}$ we define an indicator function

$$
\mathbb{1}_{<\gamma}^{\mathscr{C}}(\mathfrak{e}) \stackrel{\text{def}}{=} \prod_{e \in \mathscr{C}} \mathbb{1}\{|\mathfrak{e}(e)|_{\mathfrak{s}} < \gamma(e)\}
$$

**Lemma 4.21** *Let $\mathscr{C} \subset \mathfrak{C}$ be non-empty and $\mathcal{F} \in \mathbb{F}_{\mathscr{C}}$. Then for any node decoration $\mathfrak{n}$ on $N(T)$ with $\mathfrak{n} \leq \overline{\mathfrak{n}}$, edge decoration $\mathfrak{e}$ on $K(\overline{T})$, and extended node label $\mathfrak{o}$ on $\overline{T}$ such that $\overline{T}[\mathscr{C}, \mathcal{F}]_{\overline{\mathfrak{e}}+\mathfrak{e}}^{\mathfrak{n}, \mathfrak{o}} \in \mathfrak{T}_+$ one has*

$$
\Upsilon^{\xi} \Big[ \mathbf{P}_{\sigma_{\mathscr{C}, \mathcal{F}}} \hat{\mathscr{A}}_+ \overline{T}[\mathscr{C}, \mathcal{F}]_{\overline{\mathfrak{e}}+\mathfrak{e}}^{\mathfrak{n}, \mathfrak{o}} \Big](x_{\circledast}) \tag{4.15}
$$
$$
= (-1)^{|\mathscr{C}|} \int_{\tilde{N}[\mathscr{C}, \mathcal{F}]} dy \, \ddot{\varpi}_{\mathfrak{e}}^{\mathfrak{n}}[\mathscr{C}, \mathcal{F}](y \sqcup x_{\circledast}) \Big( \prod_{u \in L_{\mathscr{C}, \mathcal{F}}} \xi_{\mathfrak{t}(u)}(y_u) \Big)
$$

*Proof.* We prove the statement by induction in $\mathrm{depth}(\mathscr{C})$ for fixed $\mathcal{F}$. The base case, when $\mathscr{C} = \emptyset$, is a straightforward computation.

We move to the inductive step. Fix $j \in \mathbf{N}$ and suppose the claim has been proven for any $\mathscr{C} \subset \mathfrak{C}_{\mathcal{F}}$ with $\mathrm{depth}(\mathscr{C}) \leq j$. Now suppose we are given $\mathscr{C} \subset \mathfrak{C}_{\mathcal{F}}$ with $\mathrm{depth}(\mathscr{C}) = j + 1$. Writing $\mathscr{D} = \mathscr{C} \backslash \underline{\mathscr{C}}$ and $\hat{\mathfrak{n}} = \overline{\mathfrak{n}}(T_{\not\geq}[\mathscr{C}])$, we then have

$$
\mathbf{P}_{\sigma_{\mathscr{C}, \mathcal{F}}} \hat{\mathscr{A}}_+ \hat{T}[\mathscr{C}, \mathcal{F}]_{\overline{\mathfrak{e}}+\mathfrak{e}}^{\mathfrak{n}, \mathfrak{o}} \tag{4.16}
$$
$$
= (-1)^{|\mathscr{C}|+|\hat{\mathfrak{n}}|} \sum_{\mathfrak{e}_{\mathscr{D}}, \mathfrak{n}_{\mathscr{D}}, \mathfrak{f}_{\mathscr{C}}} \frac{(-1)^{|\mathfrak{f}_{\mathscr{C}}|}}{(\mathfrak{f}_{\mathscr{C}} + \mathfrak{e}_{\mathscr{D}})!} \binom{\mathfrak{n} - \hat{\mathfrak{n}}}{\mathfrak{n}_{\mathscr{D}}} \overline{T}[\mathscr{C}, \mathscr{D}, \mathcal{F}]_{\overline{\mathfrak{e}}+\mathfrak{e}+\mathfrak{f}_{\mathscr{C}}}^{\mathfrak{n}_{\mathscr{D}}+\chi(\mathfrak{e}_{\mathscr{D}}+\mathfrak{f}_{\mathscr{C}})+\hat{\mathfrak{n}}}
$$
$$
\cdot \hat{\mathscr{A}}_+ \mathfrak{p}_+ \overline{T}[\mathscr{D}, \mathcal{F}]_{\overline{\mathfrak{e}}+\mathfrak{e}+\mathfrak{e}_{\mathscr{D}}}^{\mathfrak{n}-\hat{\mathfrak{n}}-\mathfrak{n}_{\mathscr{D}}, \mathfrak{o}} \cdot \mathbb{1}_{<\gamma}^{\mathscr{D}}(\mathfrak{e}_{\mathscr{D}} + \mathfrak{e}) \cdot \mathbb{1}_{<\gamma}^{\mathscr{C}}(\mathfrak{f}_{\mathscr{C}} + \mathfrak{e}).
$$

We rewrite the corresponding RHS of (4.15). By the binomial identity one has

$$
\overline{\mathrm{X}}_{\hat{\mathfrak{n}}}^{N(\overline{T}), \otimes} \mathrm{X}_{\mathfrak{n}-\hat{\mathfrak{n}}, \otimes}^{\tilde{N}[\mathscr{C}, \mathcal{F}]} \Big( \prod_{T \in \overline{\mathcal{F}[\mathscr{C}]}} \mathrm{X}_{\mathfrak{n}-\hat{\mathfrak{n}}, \otimes}^{\tilde{N}(T), \varrho_T} \Big)
$$
$$
= \sum_{\mathfrak{n}_{\mathscr{D}}} \binom{\mathfrak{n} - \hat{\mathfrak{n}}}{\mathfrak{n}_{\mathscr{D}}} \mathrm{X}_{\mathfrak{n}_{\mathscr{D}}}^{\tilde{N}[\mathscr{C}, \mathcal{F}] \cap N(\overline{T}_{\not\geq}[\mathscr{D}])} \Big( \prod_{T \in \overline{\mathcal{F}[\mathscr{C}, \mathscr{D}]}} \mathrm{X}_{\mathfrak{n}_{\mathscr{D}}}^{\tilde{N}(T), \varrho_T} \Big) \overline{\mathrm{X}}_{\hat{\mathfrak{n}}}^{N(\overline{T}), \otimes} \tag{4.17}
$$
$$
\cdot \overline{\mathrm{X}}_{\mathfrak{n}-\hat{\mathfrak{n}}-\mathfrak{n}_{\mathscr{D}}}^{N(\overline{T}_{\not\geq}[\mathscr{D}]), \otimes} \cdot \mathrm{X}_{\mathfrak{n}-\hat{\mathfrak{n}}-\mathfrak{n}_{\mathscr{D}}, \otimes}^{\tilde{N}[\mathscr{C}, \mathcal{F}]} \cdot \Big( \prod_{T \in \overline{\mathcal{F}[\mathscr{D}]}} \mathrm{X}_{\mathfrak{n}-\hat{\mathfrak{n}}-\mathfrak{n}_{\mathscr{D}}, \otimes}^{\tilde{N}(T), \varrho_T} \Big),
$$

where in keeping with our convention, the sum over $\mathfrak{n}_\mathscr{D}$ is a sum over node decorations supported on $N(\overline{T}_{\npreceq}[\mathscr{D}])$ – however that the binomial coefficient forces it to be supported on $N(\overline{T}_{\npreceq}[\mathscr{D}]) \setminus N(\overline{T}_{\npreceq}[\mathscr{C}])$.

We also have

$$
\begin{aligned}
\mathrm{RKer}_{\mathfrak{e}}^{\mathscr{D}} &= (-1)^{|\mathscr{D}|} \sum_{\mathfrak{e}_\mathscr{D}} \frac{1}{\mathfrak{e}_\mathscr{D}!} X_{\chi\mathfrak{e}_\mathscr{D},\otimes}^{e_{\mathrm{p}}(\mathscr{D})} \cdot \mathrm{Ker}_{\mathfrak{e}+\mathfrak{e}_\mathscr{D}}^{\mathscr{D},\otimes} \mathbb{1}_{<\gamma}^{\mathscr{D}}(\mathfrak{e}_\mathscr{D}+\mathfrak{e}) \\
&= (-1)^{|\mathscr{D}|} \sum_{\mathfrak{f}_\mathscr{D},\mathfrak{e}_\mathscr{D}} \frac{1}{\mathfrak{e}_\mathscr{D}!} \binom{\mathfrak{e}_\mathscr{D}}{\mathfrak{f}_\mathscr{D}} X_{\chi(\mathfrak{e}_\mathscr{D}-\mathfrak{f}_\mathscr{D}),\otimes}^{e_{\mathrm{p}}(\mathscr{D})} \overline{X}_{\chi\mathfrak{f}_\mathscr{D}}^{e_{\mathrm{p}}(\mathscr{D}),\otimes} \mathrm{Ker}_{\mathfrak{e}+\mathfrak{e}_\mathscr{D}}^{\mathscr{D},\otimes} \mathbb{1}_{<\gamma}^{\mathscr{D}}(\mathfrak{e}_\mathscr{D}+\mathfrak{e}) \\
&= (-1)^{|\mathscr{D}|} \sum_{\mathfrak{f}_\mathscr{D},\mathfrak{e}_\mathscr{D}} \frac{1}{\mathfrak{e}_\mathscr{D}!} \frac{1}{\mathfrak{f}_\mathscr{D}!} X_{\chi\mathfrak{e}_\mathscr{D},\otimes}^{e_{\mathrm{p}}(\mathscr{D})} \overline{X}_{\chi\mathfrak{f}_\mathscr{D}}^{e_{\mathrm{p}}(\mathscr{D}),\otimes} \mathrm{Ker}_{\mathfrak{e}+\mathfrak{e}_\mathscr{D}+\mathfrak{f}_\mathscr{D}}^{\mathscr{D},\otimes} \quad (4.18)\\
&\qquad\qquad \cdot \mathbb{1}_{<\gamma}^{\mathscr{D}}(\mathfrak{e}_\mathscr{D}+\mathfrak{f}_\mathscr{D}+\mathfrak{e}) \\
&= (-1)^{|\mathscr{D}|} \sum_{\mathfrak{e}_\mathscr{D}} \frac{1}{\mathfrak{e}_\mathscr{D}!} X_{\chi\mathfrak{e}_\mathscr{D},\otimes}^{e_{\mathrm{p}}(\mathscr{D})} \bigg( \sum_{\mathfrak{f}_\mathscr{D}\in\mathrm{Der}(\mathscr{D},\mathfrak{e}+\mathfrak{e}_\mathscr{D})} \frac{\mathrm{Ker}_{\mathfrak{e}+\mathfrak{e}_\mathscr{D}+\mathfrak{f}_\mathscr{D}}^{\mathscr{D},\otimes}}{\mathfrak{f}_\mathscr{D}!} \overline{X}_{\chi\mathfrak{f}_\mathscr{D}}^{e_{\mathrm{p}}(\mathscr{D}),\otimes} \bigg)
\end{aligned}
$$

where in going to the second line we used the binomial identity and in going to the third line we performed a change of summation $\mathfrak{e}_\mathscr{D}' \overset{\mathrm{def}}{=} \mathfrak{e}_\mathscr{D} - \mathfrak{f}_\mathscr{D}$. Note that there is an indicator function $\mathbb{1}_{<\gamma}^{\mathscr{D}}(\mathfrak{e}_\mathscr{D}+\mathfrak{e})$ implicit in the last line, when this vanishes $\mathrm{Der}(\mathscr{D},\mathfrak{e}+\mathfrak{e}_\mathscr{D})$ is empty.

Using (4.18) we have that $\mathrm{Ker}_{\mathfrak{e}}^{K[\mathscr{C},\mathcal{F}]}\mathrm{RKer}_{\mathfrak{e}}^{\mathscr{C}\setminus\mathscr{C}}$ is equal to

$$
\sum_{\mathfrak{e}_\mathscr{D}} \mathbb{1}_{<\gamma}^{\mathscr{D}}(\mathfrak{e}_\mathscr{D}+\mathfrak{e}) \frac{1}{\mathfrak{e}_\mathscr{D}!} \mathrm{Ker}_{\mathfrak{e}}^{K[\mathscr{D},\mathcal{F}]}\mathrm{RKer}_{\mathfrak{e}}^{\mathscr{D}\setminus\mathscr{D}} \quad (4.19)
$$

$$
\cdot \mathrm{Ker}_{\mathfrak{e}}^{K[\mathscr{C},\mathcal{F}]\cap K(T_{\npreceq}[\mathscr{D}])}(-1)^{|\mathscr{D}|} X_{\chi\mathfrak{e}_\mathscr{D},\otimes}^{e_{\mathrm{p}}(\mathscr{D})} \bigg( \sum_{\mathfrak{f}_\mathscr{D}\in\mathrm{Der}(\mathscr{D},\mathfrak{e}+\mathfrak{e}_\mathscr{D})} \frac{\mathrm{Ker}_{\mathfrak{e}+\mathfrak{e}_\mathscr{D}+\mathfrak{f}_\mathscr{D}}^{\mathscr{D},\otimes}}{\mathfrak{f}_\mathscr{D}!} \overline{X}_{\chi\mathfrak{f}_\mathscr{D}}^{e_{\mathrm{p}}(\mathscr{D}),\otimes} \bigg)
$$

We then put this all together to see that the RHS of (4.15) is equal to

$$
(-1)^{|\mathscr{C}|} \sum_{\mathfrak{e}_\mathscr{D},\mathfrak{n}_\mathscr{D},\mathfrak{f}_\mathscr{C}} \frac{1}{(\mathfrak{e}_\mathscr{D}+\mathfrak{f}_\mathscr{C})!} \binom{\mathfrak{n}}{\mathfrak{n}_\mathscr{D}} \mathbb{1}_{<\gamma}^{\mathscr{D}}(\mathfrak{e}_\mathscr{D}+\mathfrak{e}) \cdot \mathbb{1}_{<\gamma}^{\mathscr{C}}(\mathfrak{f}_\mathscr{C}(e)+\mathfrak{e}(e))
$$

$$
\cdot \int dy_{\tilde{N}[\mathscr{C},\mathcal{F}]\setminus\tilde{N}[\mathscr{D},\mathcal{F}]} X_{\chi\mathfrak{e}_\mathscr{D},\otimes}^{e_{\mathrm{p}}(\mathscr{D})}(y\sqcup x_\otimes)\mathrm{Ker}_{\mathfrak{e}}^{\mathscr{D},\otimes}(y\sqcup x_\otimes)\mathrm{Ker}_{\mathfrak{e}}^{K[\mathscr{C},\mathcal{F}]\cap K(T_{\npreceq}[\mathscr{D}])}(y)
$$

$$
\cdot X_{\mathfrak{n}_\mathscr{D}}^{\tilde{N}[\mathscr{C},\mathcal{F}]\cap N(T_{\npreceq}[\mathscr{D}])}(y) \bigg( \prod_{T\in\overline{\mathcal{F}[\mathscr{C},\mathscr{D}]}} X_{\mathfrak{n}_\mathscr{D}}^{\tilde{N}(T),\varrho_T}(y) \bigg) \overline{X}_{\hat{\mathfrak{n}}}^{N(\overline{T}),\otimes}
$$

$$
\cdot \mathrm{Ker}_{\mathfrak{e}+\mathfrak{f}_\mathscr{C}}^{\mathscr{C},\otimes} \overline{X}_{\chi\mathfrak{f}_\mathscr{C}}^{e_{\mathrm{p}}(\mathscr{C}),\otimes} \bigg( \prod_{u\in L_{\mathscr{C},\mathcal{F}}\setminus L_{\mathscr{D},\mathcal{F}}} \xi_{\mathfrak{t}(u)}(y_u) \bigg)
$$

$$
\cdot (-1)^{|\mathscr{D}|} \int dy_{\tilde{N}[\mathscr{D},\mathcal{F}]} \dddot{\varpi}_{\mathfrak{e}+\mathfrak{e}_\mathscr{D}}^{\mathfrak{n}-\hat{\mathfrak{n}}-\mathfrak{n}_\mathscr{D}}[\mathscr{D},\mathcal{F}](y\sqcup x_\otimes).
$$

Here $\dddot{\varpi}_{\mathfrak{e}+\mathfrak{e}_\mathscr{D}}^{\mathfrak{n}-\hat{\mathfrak{n}}-\mathfrak{n}_\mathscr{D}}[\mathscr{D},\mathcal{F}]$ is built using the second lines of (4.17) and (4.19).

Using our inductive hypothesis for the very last line we see that the expression above is equal to

$$(-1)^{|\mathscr{C}|} \sum_{\mathfrak{e}_{\mathscr{D}}, \mathfrak{n}_{\mathscr{D}}, \mathfrak{f}_{\mathscr{C}}} \frac{1}{(\mathfrak{f}_{\mathscr{C}} + \mathfrak{e}_{\mathscr{D}})!} \binom{\mathfrak{n}}{\mathfrak{n}_{\mathscr{D}}} \mathbb{1}_{<\gamma}^{\mathscr{D}}(\mathfrak{e}_{\mathscr{D}}\mathfrak{e})$$

$$\cdot \Upsilon^{\xi} \Big[ (-1)^{|\mathfrak{f}_{\mathscr{C}}| + |\hat{\mathfrak{n}}|} \overline{T}[\mathscr{C}, \mathscr{D}, \mathcal{F}]_{\bar{\mathfrak{e}}+\mathfrak{e}+\mathfrak{f}}^{\mathfrak{n}_{\mathscr{D}}+\hat{\mathfrak{n}}+\chi(\mathfrak{e}_{\mathscr{D}}+\mathfrak{f}_{\mathscr{C}})} \Big] (x_{\otimes})$$

$$\cdot \Upsilon^{\xi} \Big[ \mathbf{P}_{\sigma_{\mathscr{D}}, \mathcal{F}} \hat{\mathscr{A}}_{+} \overline{T}[\mathscr{D}, \mathcal{F}]_{\bar{\mathfrak{e}}+\mathfrak{e}_{\mathscr{D}}}^{\mathfrak{n}-\hat{\mathfrak{n}}-\mathfrak{n}_{\mathscr{D}}, \mathfrak{o}} \Big] (x_{\otimes}).$$

Comparing this with (4.16) we arrive at the desired result. $\qquad\square$

### 4.3 An explicit formula for the BPHZ renormalized tree

In this subsection we put together the results of the previous two subsections and prove a proposition on the explicit form of chaos kernels for the BPHZ formula. First, for any forest of subtrees $\mathcal{F}$ we define

$$\tilde{N}(\mathcal{F}, \overline{T}) \overset{\text{def}}{=} \tilde{N}(\overline{T}) \setminus \tilde{N}(\mathcal{F}), \ N(\mathcal{F}, \overline{T}) \overset{\text{def}}{=} \tilde{N}(\overline{T}) \setminus \tilde{N}(\mathcal{F}),$$

$$K(\mathcal{F}, \overline{T}) \overset{\text{def}}{=} K(\overline{T}) \setminus \bar{K}^{\downarrow}(\overline{\mathcal{F}}), \ \text{and} \ L(\mathcal{F}, \overline{T}) \overset{\text{def}}{=} L(\overline{T}) \setminus L(\mathcal{F}) \ .$$

Then, for any $\tilde{L} \subset L(\overline{T})$, $\pi \in \mathcal{P}[L(\overline{T}) \setminus \tilde{L}]$, $\mathscr{C} \subset \mathfrak{C}$, and $\mathcal{F} \in \mathbb{F}_{\mathscr{C}}$, we define $\mathcal{W}_{\tilde{L}}^{\pi}[\mathcal{F}, \mathscr{C}] \in \mathscr{C}_{\tilde{L} \cup \{\varrho_*, \otimes\}}$ via the following integral formula where we write $y \in (\mathbf{R}^d)^{\tilde{N}(\mathcal{F}, \overline{T}) \setminus \tilde{L}}$, $x \in (\mathbf{R}^d)^{\tilde{L} \cup \{\varrho_*, \otimes\}}$, and $z = x \sqcup y$.

$$\mathcal{W}_{\tilde{L}}^{\pi}[\mathcal{F}, \mathscr{C}](x_{\tilde{L} \cup \{\varrho_*, \otimes\}})$$
$$\overset{\text{def}}{=} \int_{\tilde{N}(\mathcal{F}, \overline{T}) \setminus \tilde{L}} dy \, \mathrm{Cu}_{\pi}^{L(\mathcal{F}, \overline{T})}(z) \cdot \mathrm{Ker}_0^{K(\mathcal{F}, \overline{T}) \setminus \mathscr{C}}(z) \cdot \mathrm{RKer}_0^{K(\mathcal{F}, \overline{T}) \cap \mathscr{C}}(z) \cdot \mathrm{X}_{\bar{\mathfrak{n}}, \otimes}^{N(\mathcal{F}, \overline{T})}(z)$$
$$\cdot \Big( \prod_{S \in \overline{\mathcal{F}}} H_{\pi, \mathcal{F}, S} \Big[ \mathrm{RKer}_0^{K^{\downarrow}(S) \cap \mathscr{C}} \cdot \mathrm{Ker}_0^{K^{\downarrow}(S) \setminus \mathscr{C}} \mathrm{X}_{\bar{\mathfrak{n}}, \otimes}^{\tilde{N}(S)} \Big] (z_{N^{\downarrow}(S)} \sqcup z_{\varrho_S}) \Big) \ .$$

**Proposition 4.22** *For any $\xi \in \mathcal{M}(\Omega_{\infty})$ and $x \in (\mathbf{R}^d)^{\{\otimes, \varrho_*\}}$,*

$$\hat{\Upsilon}^{\xi}[\overline{T}_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}](x) = \sum_{\substack{\tilde{L} \subset L(\overline{T}) \\ \pi \in \mathcal{P}[L(\overline{T}) \setminus \tilde{L}] \\ \mathcal{G} \in \mathbb{F}_{\pi}, \, \mathscr{D} \subset \mathfrak{C}_{\mathcal{G}}}} \int_{\tilde{L} \setminus \{\varrho_*\}} dy \, \mathcal{W}_{\tilde{L}}^{\pi}[\mathcal{G}, \mathscr{D}](z) \cdot \mathrm{Wick}\big(\{\xi_{\mathfrak{t}(u)}(z_u)\}_{u \in \tilde{L}}\big) \quad (4.20)$$

*where $z = x \sqcup y$ and, for any multiset $A$ of random variables, $\mathrm{Wick}(A)$ denotes the Wick product of the elements of $A$ (see [HS15, Def. 4.5] or [AT06, Appendix B]).*

*Proof.* Starting from the LHS, we have

$$(\mathrm{id} \otimes \hat{\Delta}_{+})\hat{\Delta}_{-}(\overline{T}, 0)_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}} = \sum_{\substack{\mathcal{F} \in \mathbb{F}^{\leq 1} \\ \mathscr{C} \in \mathfrak{C}_{\overline{\mathcal{F}}}^{\leq 1}}} \sum_{\substack{\mathfrak{n}_F, \mathfrak{e}_F \\ \mathfrak{n}_{\mathscr{C}}, \mathfrak{e}_{\mathscr{C}}}} A(\mathcal{F}, \mathscr{C}, \mathfrak{n}_F, \mathfrak{e}_F, \mathfrak{n}_{\mathscr{C}}, \mathfrak{e}_{\mathscr{C}}) \ , \quad (4.21)$$

where $A(\mathcal{F}, \mathscr{C}, \mathfrak{n}_F, \mathfrak{e}_F, \mathfrak{n}_{\mathscr{C}}, \mathfrak{e}_{\mathscr{C}})$ is given by

$$
\Big( \prod_{S \in \mathcal{F}} \mathbb{1}_{<\omega(S)}(\mathfrak{n}_S + \mathfrak{e}_S) \Big) \binom{\overline{\mathbf{n}}}{\mathfrak{n}_F} \frac{1}{\mathfrak{e}_F!} (F, 0)_{\overline{\mathfrak{e}}}^{\mathfrak{n}_F + \chi \mathfrak{e}_F, 0} \tag{4.22}
$$

$$
\otimes \binom{\overline{\mathbf{n}} - \mathfrak{n}_F}{\mathfrak{n}_{\mathscr{C}}} \frac{1}{\mathfrak{e}_{\mathscr{C}}!} \mathbb{1}^{\mathscr{C}}_{<\gamma}(\mathfrak{e}_{\mathscr{C}} + \mathfrak{e}_F) (\overline{T}_{\not\succeq}[\mathscr{C}], [F]_1)_{\overline{\mathfrak{e}} + \mathfrak{e}_F}^{\mathfrak{n}_{\mathscr{C}} + \chi \mathfrak{e}_{\mathscr{C}}, 0}
$$

$$
\otimes \overline{T}[\mathscr{C}, \mathcal{F}]_{\overline{\mathfrak{e}} + \mathfrak{e}_F + \mathfrak{e}_{\mathscr{C}}}^{\mathfrak{n} - \mathfrak{n}_F - \mathfrak{n}_{\mathscr{C}}, \mathfrak{o}_F} \, ,
$$

where $\mathfrak{o}_F \overset{\text{def}}{=} \mathfrak{n}_F + \chi \mathfrak{e}_F$. Note that we used Lemma 4.17 here. Above, $F$ is the i-forest corresponding to $\mathcal{F} \in \mathbb{F}^{\leq 1}$, the sum over decorations in (4.21) is a sum over node decorations $\mathfrak{n}_F$ on $N(F) \setminus \varrho(F)$, $\mathfrak{n}_{\mathscr{C}}$ on $\overline{T}_{\not\succeq}[\mathscr{C}]$, edge decorations $\mathfrak{e}_F$ on $K^{\downarrow}(\mathcal{F})$ and $\mathfrak{e}_{\mathscr{C}}$ on $\mathscr{C}$. The edge decorations $\mathfrak{e}_S$ and node decorations $\mathfrak{n}_S$ appearing in (4.22) are given by the restrictions of $\mathfrak{e}_F$ to $K_S^{\partial}(\overline{T})$ and $\mathfrak{n}_F$ to $\tilde{N}(S)$, respectively.

Observe that we have

$$
\bigsqcup_{\tilde{L} \subset L(\overline{T})} \bigsqcup_{\pi \in \bar{\mathcal{P}}[L(\overline{T}) \setminus \tilde{L}]} \bigsqcup_{\substack{\mathcal{F} \in \mathbb{F}_\pi \\ \mathscr{C} \subset \mathfrak{C}_{\mathcal{F}}}} \{(\mathcal{F}, \mathscr{C}, \pi)\} = \bigsqcup_{\substack{\mathcal{F} \in \mathbb{F} \\ \mathscr{C} \subset \mathfrak{C}_{\mathcal{F}}}} \bigsqcup_{\substack{\pi \in \bar{\mathcal{P}}[L(\mathcal{F})] \\ \pi \text{ comp. } \mathcal{F}}} \bigsqcup_{\bar{\pi} \in \mathcal{P}[L(\mathcal{F}, \overline{T}) \setminus \tilde{L}]} \{(\mathcal{F}, \mathscr{C}, \pi \sqcup \bar{\pi})\},
$$

$$
\sum_{\bar{\pi} \in \mathcal{P}[L(\mathcal{F}, \overline{T})]} \mathrm{Cu}_{\pi}^{L(\mathcal{F}, \overline{T})}(y) \mathrm{Wick}\Big( \{\xi_{\mathfrak{t}(u)}(y_u)\}_{u \in L(\mathcal{F}, \overline{T}) \setminus \mathrm{supp}(\pi)} \Big) = \prod_{u \in L(\mathcal{F}, \overline{T})} \xi_{\mathfrak{t}(u)}(y_u),
$$

where the second inequality holds for any $\mathcal{F} \in \mathbb{F}$. We can use the two identities above to interchange summations and eliminate the Wick monomials on the RHS of (4.20) to see that, again writing $z = y_{\tilde{N}(\mathcal{F}, \overline{T})} \sqcup x_{\{\varrho_*, \otimes\}}$, it is equal to

$$
\sum_{\substack{\mathcal{F} \in \mathbb{F}^{\leq 1} \\ \mathscr{C} \in \mathfrak{C}_{\mathcal{F}}^{\leq 1}}} \sum_{\substack{\mathcal{G} \in \mathbb{F}[\mathcal{F}] \\ \mathscr{D} \in \mathfrak{C}[\mathscr{C}, \mathcal{F}]}} \sum_{\substack{\pi \in \bar{\mathcal{P}}[L(\mathcal{G})] \\ \pi \text{ comp. } \mathcal{G}}} \int_{\tilde{N}(\mathcal{F}, \overline{T})} dy \, \Big( \prod_{u \in L(\mathcal{F}, \overline{T})} \xi_{\mathfrak{t}(u)}(z) \Big) \mathrm{X}_{\overline{\mathfrak{n}}, \otimes}^{N(\mathcal{F}, \overline{T})}(z) \tag{4.23}
$$

$$
\cdot \mathrm{Ker}_0^{K(\mathcal{F}, \overline{T}) \setminus \mathscr{D}}(z) \cdot \mathrm{RKer}_0^{K(\mathcal{F}, \overline{T}) \cap \mathscr{D}}(z)
$$

$$
\cdot \Big( \prod_{S \in \mathcal{F}} H_{\pi, \mathcal{G}, S} \Big[ \mathrm{RKer}_0^{K^{\downarrow}(S) \cap \mathscr{D}} \cdot \mathrm{Ker}_0^{K^{\downarrow}(S) \setminus \mathscr{D}} \mathrm{X}_{\overline{\mathfrak{n}}, \otimes}^{\tilde{N}(S)}(z) \Big] (z_{N^{\downarrow}(S)} \sqcup z_{\varrho_S}) \Big) \, .
$$

For the time being we fix $\mathcal{F} \in \mathbb{F}^{\leq 1}$, $\mathscr{C} \in \mathfrak{C}_{\mathcal{F}}^{\leq 1}$, and $\mathscr{D} \in \mathfrak{C}[\mathscr{C}, \mathcal{F}]$. Then by straightforward computation,

$$
\sum_{\substack{\mathcal{G} \in \mathbb{F}[\mathcal{F}] \\ \pi \in \bar{\mathcal{P}}[L(\mathcal{G})] \\ \pi \text{ comp. } \mathcal{G}}} \Big( \prod_{S \in \mathcal{F}} H_{\pi, \mathcal{G}, S} \Big[ \mathrm{RKer}_0^{K^{\downarrow}(S) \cap \mathscr{D}} \cdot \mathrm{Ker}_0^{K^{\downarrow}(S) \setminus \mathscr{D}} \mathrm{X}_{\overline{\mathfrak{n}}, \otimes}^{\tilde{N}(S)} \Big] (z) \Big) \tag{4.24}
$$

$$
= \sum_{\substack{\mathcal{G} \in \mathbb{F}[\mathcal{F}] \\ \pi \in \bar{\mathcal{P}}[L(\mathcal{G})] \\ \pi \text{ comp. } \mathcal{G},}} \sum_{\tilde{\mathfrak{n}}_F} \binom{\overline{\mathbf{n}}}{\tilde{\mathfrak{n}}_F} \mathrm{RKer}_0^{K(\mathcal{F}, \overline{T}) \cap \mathscr{D}}(z)
$$

$$
\cdot \Big( \prod_{S \in \mathcal{F}} \mathrm{X}_{\tilde{\mathfrak{n}}_F, \otimes}^{\tilde{N}(S), \varrho_S}(z) H_{\pi, \mathcal{G}, S} \Big[ \mathrm{RKer}_0^{K^{\downarrow}(S) \cap \mathscr{D}} \mathrm{Ker}_0^{K^{\downarrow}(S) \setminus \mathscr{D}} \mathrm{X}_{\overline{\mathfrak{n}} - \tilde{\mathfrak{n}}_F}^{\tilde{N}(S)} \Big] (z) \Big)
$$

$$
= (-1)^{|\mathcal{F}|} \sum_{\substack{\tilde{\mathfrak{n}}_F, \mathfrak{n}_F, \\ \mathfrak{e}_F}} \binom{\overline{\mathfrak{n}}}{\tilde{\mathfrak{n}}_F} \Big( \prod_{S \in \mathcal{F}} X_{\tilde{\mathfrak{n}}_F, \otimes}^{\tilde{N}(S), \varrho_S}(z) X_{\mathfrak{n} - \tilde{\mathfrak{n}}_F - \mathfrak{n}_F, \otimes}^{\tilde{N}(S), \varrho_S}(z) \binom{\overline{\mathfrak{n}} - \tilde{\mathfrak{n}}_F \ \omega(S)}{\mathfrak{n}_S \ \mathfrak{e}_S} \Big)
$$

$$
\cdot \mathrm{RKer}_{\mathfrak{e}_F}^{K^{\downarrow}(\mathcal{F}) \cap \mathscr{D}}(z) \mathrm{Ker}_{\mathfrak{e}_F}^{K^{\downarrow}(\mathcal{F}) \backslash \mathscr{D}}(z) \sum_{\substack{\mathcal{G} \in \mathbb{F}[\mathcal{F}] \\ \pi \in \tilde{\mathcal{P}}[L(\mathcal{G})] \\ \pi \text{ comp. } \mathcal{G}}} \int_{N_{\mathcal{G}}(\overline{\mathcal{F}})} dw \delta(w_{\varrho(\mathcal{F})}) \mathring{\varpi}_{\pi}^{\mathfrak{n}_F + \chi \mathfrak{e}_F}[\mathcal{G}](w)
$$

$$
= \sum_{\substack{\tilde{\mathfrak{n}}_F, \mathfrak{n}_F, \\ \mathfrak{e}_F}} \binom{\overline{\mathfrak{n}}}{\tilde{\mathfrak{n}}_F} \binom{\overline{\mathfrak{n}} - \tilde{\mathfrak{n}}_F}{\mathfrak{n}_F} \frac{1}{\mathfrak{e}_F!} \Big( \prod_{S \in \mathcal{F}} X_{\overline{\mathfrak{n}} - \mathfrak{n}_F, \otimes}^{\tilde{N}(S), \varrho_S}(z) \mathbb{1}_{<\omega(S)}(\mathfrak{n}_S + \mathfrak{e}_S) \Big)
$$

$$
\cdot \mathrm{RKer}_{\mathfrak{e}_F}^{K^{\downarrow}(\mathcal{F}) \cap \mathscr{D}}(z) \mathrm{Ker}_{\mathfrak{e}_F}^{K^{\downarrow}(\mathcal{F}) \backslash \mathscr{D}}(z) \cdot \bar{\Upsilon}^{\mathbb{P}} \Big[ \hat{\mathscr{A}}_{-}(F, 0)_{\mathfrak{e}}^{\mathfrak{n}_F + \chi \mathfrak{e}_F} \Big].
$$

In going to the second line we used the binomial identity and the fact that we can replace an $X_{\overline{\mathfrak{n}} - \mathfrak{n}_F, \varrho_S}^{\tilde{N}(S)}$ with $X_{\overline{\mathfrak{n}} - \tilde{\mathfrak{n}}_F}^{\tilde{N}(S)}$ in the argument of $H_{\pi, \mathcal{G}, S}$ thanks to translation invariance. In going to the last line we used Lemmas 4.7 and 4.14. By the the Chu-Vandermonde identity we have that for each fixed $\mathfrak{n}_F$,

$$
\sum_{\tilde{\mathfrak{n}}_F} \binom{\overline{\mathfrak{n}}}{\tilde{\mathfrak{n}}_F} \binom{\overline{\mathfrak{n}} - \tilde{\mathfrak{n}}_F}{\mathfrak{n}_F} = \binom{\overline{\mathfrak{n}}}{\mathfrak{n}_F}.
$$

It follows that (4.23) is equal to

$$
\sum_{\substack{\mathcal{F} \in \mathbb{F}^{\leq 1} \\ \mathscr{C} \in \mathfrak{C}_{\overline{\mathcal{F}}}^{\leq 1} \\ \mathscr{D} \in \mathfrak{C}[\mathscr{C}, \mathcal{F}]}} \sum_{\mathfrak{n}_F, \mathfrak{e}_F} \binom{\overline{\mathfrak{n}}}{\mathfrak{n}_F} \frac{1}{\mathfrak{e}_F!} \int_{\tilde{N}(\mathcal{F}, \overline{T})} dy \, \mathrm{Ker}_0^{K(\mathcal{F}, \overline{T}) \backslash \mathscr{D}}(z) \cdot \mathrm{RKer}_0^{K(\mathcal{F}, \overline{T}) \cap \mathscr{D}}(z)
$$

$$
\cdot X_{\overline{\mathfrak{n}}, \otimes}^{N(\mathcal{F}, \overline{T})}(z) \cdot \Big( \prod_{S \in \mathcal{F}} X_{\mathfrak{n} - \mathfrak{n}_F, \otimes}^{\tilde{N}(S), \varrho_S}(z) \mathbb{1}_{<\omega(S)}(\mathfrak{n}_S + \mathfrak{e}_S) \Big) \cdot \Big( \prod_{u \in L(\mathcal{F}, \overline{T})} \xi_{\mathfrak{t}(u)}(z_u) \Big)
$$

$$
\cdot \mathrm{RKer}_{\mathfrak{e}_F}^{K^{\downarrow}(\mathcal{F}) \cap \mathscr{D}}(z) \mathrm{Ker}_{\mathfrak{e}_F}^{K^{\downarrow}(\mathcal{F}) \backslash \mathscr{D}}(z) \cdot \bar{\Upsilon}^{\mathbb{P}} \Big[ \hat{\mathscr{A}}_{-}(F, 0)_{\mathfrak{e}}^{\mathfrak{n}_F + \chi \mathfrak{e}_F} \Big].
$$

$$
(4.25)
$$

Next we observe that for fixed $\mathcal{F} \in \mathbb{F}^{\leq 1}$, $\mathscr{C} \in \mathfrak{C}_{\overline{\mathcal{F}}}^{\leq 1}$, $\mathfrak{n}_F$, and $\mathfrak{e}_F$ we have

$$
\mathrm{RKer}_0^{K(\mathcal{F}, \overline{T}) \cap \mathscr{C}} \cdot \mathrm{RKer}_{\mathfrak{e}_F}^{K^{\downarrow}(\mathcal{F}) \cap \mathscr{C}}
$$

$$
= (-1)^{|\mathscr{C}|} \sum_{\mathfrak{e}_{\mathscr{C}}} \frac{1}{\mathfrak{e}_{\mathscr{C}}!} X_{\mathfrak{e}_{\mathscr{C}}, \otimes}^{e_p(\mathscr{C})} \Big( \sum_{\mathfrak{f}_{\mathscr{C}} \in \mathrm{Der}(\mathscr{C}, \mathfrak{e} + \mathfrak{e}_F)} \frac{\mathrm{Ker}_{\mathfrak{e} + \mathfrak{e}_{\mathscr{C}} + \mathfrak{f}_{\mathscr{C}}}^{\otimes, \mathscr{C}}}{\mathfrak{f}_{\mathscr{C}}!} \overline{X}_{\chi \mathfrak{f}_{\mathscr{C}}}^{e_p(\mathscr{C}), \otimes} \Big)
$$

and we also have

$$
X_{\overline{\mathfrak{n}}, \otimes}^{N(\mathcal{F}, \overline{T})} \Big( \prod_{S \in \mathcal{F}} X_{\mathfrak{n} - \mathfrak{n}_F, \otimes}^{\tilde{N}(S), \varrho_S} \Big) = \sum_{\mathfrak{n}_{\mathscr{C}}} \binom{\mathfrak{n} - \mathfrak{n}_F}{\mathfrak{n}_{\mathscr{C}}} X_{\mathfrak{n}_{\mathscr{C}}}^{N_{\mathcal{F}}(\overline{T} \succeq [\mathscr{C}])} \Big( \prod_{\substack{S \in \mathcal{F} \\ S \leq \overline{T} \not\succeq [\mathscr{C}]}} X_{\mathfrak{n}_{\mathscr{C}}}^{\tilde{N}(S), \varrho_S} \Big)
$$

$$\cdot \overline{\mathrm{X}}^{N(\overline{T}_{\not\geq}[\mathscr{C}]),\otimes}_{\overline{\mathfrak{n}}-\mathfrak{n}_F-\mathfrak{n}_{\mathscr{C}}} \mathrm{X}^{\tilde{N}[\mathscr{C},\mathcal{F}]}_{\overline{\mathfrak{n}},\otimes} \Big( \prod_{S\in\mathcal{F}[\mathscr{C}]} \mathrm{X}^{\tilde{N}(S),\varrho_S}_{\overline{\mathfrak{n}}-\mathfrak{n}_F-\mathfrak{n}_{\mathscr{C}},\otimes} \Big) \ .$$

With these facts in hand one can carefully factorize the integral of (4.25), and then use Lemmas 4.19 and 4.21 to write (4.25) as

$$\sum_{\substack{\mathcal{F}\in\mathbb{F}^{\leq 1} \\ \mathscr{C}\in\mathfrak{C}^{\leq 1}_{\overline{\mathcal{F}}}}} \sum_{\substack{\mathfrak{n}_F,\mathfrak{e}_F, \\ \mathfrak{e}_{\mathscr{C}},\mathfrak{n}_{\mathscr{C}}}} \binom{\overline{\mathfrak{n}}}{\mathfrak{n}_F} \binom{\overline{\mathfrak{n}}-\mathfrak{n}_F}{\mathfrak{n}_{\mathscr{C}}} \frac{1}{\mathfrak{e}_F!} \frac{1}{\mathfrak{e}_{\mathscr{C}}!} \Big( \prod_{S\in\mathcal{F}} \mathbb{1}_{<\omega(S)}(\mathfrak{n}_S+\mathfrak{e}_S) \Big)$$

$$\cdot \mathbb{1}^{\mathscr{C}}_{<\gamma}(\mathfrak{e}_{\mathscr{C}}+\mathfrak{e}_F) \Upsilon^{\zeta} \Big[ (\overline{T}_{\not\geq}[\mathscr{C}],[F]_1)^{\mathfrak{n}_{\mathscr{C}}+\chi\mathfrak{e}_{\mathscr{C}},0}_{\overline{\mathfrak{e}}+\mathfrak{e}_F} \Big]$$

$$\cdot \Upsilon^{\zeta} \Big[ \hat{\mathscr{A}}_+ \overline{T}[\mathscr{C},\mathcal{F}]^{\overline{\mathfrak{n}}-\mathfrak{n}_F-\mathfrak{n}_{\mathscr{C}},\mathfrak{o}_F}_{\overline{\mathfrak{e}}+\mathfrak{e}_F+\mathfrak{e}_{\mathscr{C}}} \Big] \cdot \overline{\Upsilon}^{\mathbb{P}} \Big[ \hat{\mathscr{A}}_-(F,0)^{\mathfrak{n}_F+\chi\mathfrak{e}_F,0}_0 \Big].$$

The proof is finished upon observing that the above expression is what one obtains by applying to (4.21) the operator $\overline{\Upsilon}^{\xi}[\cdot] \otimes \Upsilon^{\xi(\omega)}[\cdot] \otimes \Upsilon^{\xi(\omega)}[\cdot]$ .     $\square$

## 4.4   From the BPHZ renormalized tree to the BPHZ model

We close this section by checking that our BPHZ renormalized tree agrees with the BPHZ renormalized model, using freely the notations and terminology of [BHZ16].

**Lemma 4.23** *Fix a realization $\xi(\omega) \in \Omega_\infty$. Let $Z^{\xi(\omega)}_{\mathrm{BPHZ}} = (\Pi^{\xi(\omega)}, \Gamma^{\xi(\omega)})$ be the "BPHZ model" which is defined as the restriction to the reduced regularity structure of the model obtained by applying the BPHZ renormalization procedure of [BHZ16, Thm 6.17] to the random model corresponding to the canonical lift of $\xi(\omega)$ on the extended regularity structure. Then, for every $z \in \mathbf{R}^d$, $(\Pi_z \overline{T}^{\overline{\mathfrak{n}}}_{\overline{\mathfrak{e}}})(\cdot)$ and $\hat{\Upsilon}^{\xi}_z[\overline{T}^{\overline{\mathfrak{n}}}_{\overline{\mathfrak{e}}}](\cdot)$ as defined in* (3.5) *coincide as random distributions.*

*Proof.* There is little to prove here other than to describe why the minor differences between our framework and that of [BHZ16] lead to absolutely no difference in the resulting analytic expressions.

The process of going from elements of our "underlined" spaces of identified objects to their appropriate representatives in the corresponding "un-identified" and non-underlined spaces imported from [BHZ16] entails forgetting the identification map, performing a contraction, and possibly replacing a forest product with a tree product. This "forgetting" is denoted by an algebra homomorphism $\mathscr{U} : \langle \underline{\mathfrak{F}_2} \rangle \to \langle \mathfrak{F}_2 \rangle$ given by dropping the identification data from any element in $\underline{\mathfrak{F}_2}$ and then extending by linearity.

As in [BHZ16] we use a variety of contraction maps. We start with a map $\mathscr{k} : \langle \mathfrak{F}_2 \rangle \to \langle \mathfrak{F}_2 \rangle$ defined exactly as $\mathscr{K}$ is in [BHZ16, Def. 3.14] but with a minor difference in how decorations are treated – in the definition of $\mathscr{k}$ the new $[\mathfrak{o}]$ label should be replaced by $[\mathfrak{o}]'$ given by

$$[\mathfrak{o}]'(x) \stackrel{\text{def}}{=} \sum_{y\in x} \mathfrak{o}(y) + \sum_{e\in E_F\cap\hat{F}^{-1}(2)\cap x} (\mathfrak{t}(e)-\mathfrak{e}(e)).$$

We then define, for $i \in \{1, 2\}$, maps $\mathring{k}_i \stackrel{\text{def}}{=} \mathring{k} \circ \Phi_i$ and $\widehat{\mathring{k}}_i \stackrel{\text{def}}{=} \mathring{k} \circ \widehat{\Phi}_i$ where $\Phi_i, \widehat{\Phi}_i : \langle \mathfrak{F}_2 \rangle \to \langle \mathfrak{F}_2 \rangle$ are defined by [BHZ16, Eq. 3.22] along with the immediately preceding and following paragraphs.

Finally, we define $\mathbb{O} : \langle \mathfrak{F}_2 \rangle \to \langle \mathfrak{F}_2 \rangle$ to be the map that sets all extended node labels $\mathfrak{o}$ that appear to zero. Then it is straightforward to check that

$$\left( \mathscr{U} \otimes \mathring{k}\mathscr{U} \otimes \widehat{\mathring{k}}_2 \mathscr{U} \right) (\mathrm{Id} \otimes \hat{\Delta}_+) \hat{\Delta}_- (\overline{T}, 0)_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}} = (\mathrm{Id} \otimes \mathbb{O} \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta_2)\Delta_1 \overline{T}_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}},$$

where we are implicitly using that the RHS can be identified as an element of $\mathscr{T}_-^{\mathrm{ex}} \otimes \mathscr{T}^{\mathrm{ex}} \otimes \mathscr{T}_+^{\mathrm{ex}}$.

For the negative twisted antipodes $\tilde{\mathscr{A}}_- : \mathscr{T}_-^{\mathrm{ex}} \to \widehat{\mathscr{T}}_-^{\mathrm{ex}}$ it is again straightforward to see that, for every $(F, 0)_{\mathfrak{e}}^{\mathfrak{n}} \in \widetilde{\mathfrak{F}}_0$ which, $(i)$ can be written as a forest product of elements of $\mathfrak{X}_-$ and $(ii)$ satisfies $\mathring{k}_1 \mathscr{U}(F, 0)_{\mathfrak{e}}^{\mathfrak{n}} \in \mathscr{T}_-^{\mathrm{ex}}$, one has the identity

$$\mathring{k}_1 \mathscr{U} \hat{\mathscr{A}}_- (F, 0)_{\mathfrak{e}}^{\mathfrak{n}} = \mathbb{O}\tilde{\mathscr{A}}_- \mathscr{U}(F, 0)_{\mathfrak{e}}^{\mathfrak{n}} .$$

This claim can be verified inductively in $|E_F|$.

We turn to positive twisted antipode – for any i-tree of the form $(\overline{T}, \hat{T})_{\mathfrak{e}}^{\mathfrak{n}, \mathfrak{o}}$ with $\widehat{\mathring{k}}_2 \mathscr{U}(\overline{T}, \hat{T})_{\mathfrak{e}}^{\mathfrak{n}, \mathfrak{o}} \in \mathscr{T}_+^{\mathrm{ex}}$ one has

$$\mathscr{J} \widehat{\mathring{k}}_2 \mathscr{U} \hat{\mathscr{A}}_+ (\overline{T}, \hat{T})_{\mathfrak{e}}^{\mathfrak{n}, \mathfrak{o}} = \mathbb{O}\tilde{\mathscr{A}} \widehat{\mathring{k}}_2 \mathscr{U}(\overline{T}, \hat{T})_{\mathfrak{e}}^{\mathfrak{n}, \mathfrak{o}},$$

where $\mathscr{J} : \langle \widehat{\mathfrak{T}}_2 \rangle_{\mathrm{for}} \to \langle \widehat{\mathfrak{T}}_2 \rangle$ is the operation of joining roots as given in [BHZ16, Def. 4.6] and the equality above is in the sense of elements of $\widehat{\mathscr{T}}_+^{\mathrm{ex}}$. This claim can be verified inductively in $|E_{\overline{T}} \setminus E_{\overline{T} \not\succeq [\mathscr{C}]}|$.

It follows that one has equality

$$\left( \mathring{k}_1 \mathscr{U} \hat{\mathscr{A}}_- \otimes \mathring{k}\mathscr{U} \otimes \mathscr{J} \widehat{\mathring{k}}_2 \mathscr{U} \hat{\mathscr{A}}_+ \right)(\mathrm{Id} \otimes \hat{\Delta}_+) \hat{\Delta}_- (\overline{T}, 0)_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}} \qquad (4.26)$$

$$= (\mathbb{O}\tilde{\mathscr{A}}_- \otimes \mathbb{O} \otimes \mathbb{O}\tilde{\mathscr{A}}_+)(\mathrm{Id} \otimes \Delta_2)\Delta_1 \overline{T}_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}}$$

as elements of $\widehat{\mathscr{T}}_-^{\mathrm{ex}} \otimes \mathscr{T}^{\mathrm{ex}} \otimes \widehat{\mathscr{T}}_+^{\mathrm{ex}}$.

If we denote by $\mathbf{\Pi}^{\xi(\cdot)} : \mathscr{T}^{\mathrm{ex}} \to \mathscr{C}^\infty$ the unique *multiplicative* admissible random map with $\mathbf{\Pi}^{\xi(\omega)}[\Xi_{\mathfrak{t}}] \stackrel{\text{def}}{=} \xi_{\mathfrak{t}}(\omega)$ for all $\mathfrak{t} \in \mathfrak{L}_-$ and again write $Z_{\mathrm{BPHZ}}^{\xi(\omega)} = (\Pi^{\xi(\omega)}, \Gamma^{\xi(\omega)})$ then for any $x_\oplus, x_{\varrho_*} \in \mathbf{R}^d$ one has

$$(\Pi_{x_\oplus}^{\xi(\omega)} \overline{T}_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}})(x_{\varrho_*}) \qquad (4.27)$$

$$= (g^-(\mathbf{\Pi}^{\xi(\cdot)}) \otimes \mathbf{\Pi}^{\xi(\omega)}[\cdot](x_{\varrho_*}) \otimes g_{x_\oplus}^+(\mathbf{\Pi}^{\xi(\omega)}))(\tilde{\mathscr{A}}_- \otimes \mathrm{Id} \otimes \tilde{\mathscr{A}}_+)(\mathrm{Id} \otimes \Delta_2)\Delta_1 \overline{T}_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}}$$

$$= (g^-(\mathbf{\Pi}^{\xi(\cdot)}) \otimes \mathbf{\Pi}^{\xi(\omega)}[\cdot](x_{\varrho_*}) \otimes g_{x_\oplus}^+(\mathbf{\Pi}^{\xi(\omega)}))(\mathbb{O}\tilde{\mathscr{A}}_- \otimes \mathbb{O} \otimes \mathbb{O}\tilde{\mathscr{A}}_+)(\mathrm{Id} \otimes \Delta_2)\Delta_1 \overline{T}_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}} .$$

The first equality is by definition, and the second holds because the maps $g^-(\mathbf{\Pi}^{\xi(\cdot)})$, $\mathbf{\Pi}^{\xi(\omega)}[\cdot](x_{\varrho_*})$, and $g_{x_\oplus}^+(\mathbf{\Pi}^{\xi(\omega)})$ all ignore the extended node label.

We then observe the following.

- For any $(\overline{T}, \hat{T})_\mathfrak{e}^\mathfrak{n} \in \underline{\mathfrak{T}_1}$ with $\Bbbk \mathscr{U}(\overline{T}, \hat{T})_\mathfrak{e}^\mathfrak{n}](x_{\varrho*}) \in \mathscr{T}^{\mathrm{ex}}$

$$\boldsymbol{\Pi}^{\xi(\omega)}[\Bbbk \mathscr{U}(\overline{T}, \hat{T})_\mathfrak{e}^\mathfrak{n}](x_{\varrho*}) = \Upsilon^{\xi(\omega)}[(\overline{T}, \hat{T})_\mathfrak{e}^\mathfrak{n}](x_{\varrho*}).$$

- For any $(F, \hat{F})_\mathfrak{e}^\mathfrak{n} \in \langle \underline{\widehat{\mathfrak{T}_2}} \rangle_{\mathrm{for}}$ with $\mathscr{J} \widehat{\Bbbk}_2 \mathscr{U}(F, \hat{F})_\mathfrak{e}^\mathfrak{n} \in \widehat{\mathscr{T}}_+^{\mathrm{ex}}$

$$g_{x_\otimes}^+(\boldsymbol{\Pi}^{\xi(\omega)}) \mathscr{J} \widehat{\Bbbk}_2 \mathscr{U}(F, \hat{F})_\mathfrak{e}^\mathfrak{n} = \Upsilon^{\xi(\omega)}[(F, \hat{F})_\mathfrak{e}^\mathfrak{n}](x_\otimes).$$

- For any $(F, \hat{F})_\mathfrak{e}^\mathfrak{n} \in \underline{\mathfrak{F}_1}$ with $\Bbbk_1 \mathscr{U}(F, \hat{F})_\mathfrak{e}^\mathfrak{n} \in \widehat{\mathscr{T}}_-^{\mathrm{ex}}$

$$g^-(\boldsymbol{\Pi}^{\xi(\cdot)}) \Bbbk_1 \mathscr{U}(F, \hat{F})_\mathfrak{e}^\mathfrak{n} = \bar{\Upsilon}^\xi[(F, \hat{F})_\mathfrak{e}^\mathfrak{n}].$$

Using these identities, all that is left is comparing (3.5) to (4.27).  □

### 4.5 An example

We have verified that in our setting we have recovered the BPHZ model of [BHZ16]. In fact, the BPHZ model has already appeared in previous works using the framework of regularity structures, but not under that name. In this subsection we compare, for a single symbol, our formulas for the BPHZ model with those of the renormalized model appearing in [HQ15]. The symbol we choose to look at is $\vee$.

The context here is that of the KPZ equation, we we are working on $\mathbf{R}^2$ with space-time scaling $\mathfrak{s} = (2, 1)$. We have a single kernel type $\mathfrak{t}$ which corresponds to the spatial derivative of the space-time heat kernel on $\mathbf{R}^2$ and a single noise type $\mathfrak{l}$ which corresponds to the driving noise. We specify a homogeneity assignment $|\cdot|_\mathfrak{s}$ by setting $|\mathfrak{t}|_\mathfrak{s} = 1$ and $|\mathfrak{l}|_\mathfrak{s} = -3/2 - \kappa$ where $\kappa \in (0, 1/10)$. We also fixed a random, smooth noise time map given by setting $\xi_\varepsilon = \xi * \varrho_\varepsilon$.

Using the symbolic notation for kernels and integrals of [HQ15] one has



$$(4.28)$$

We have modified the notations of [HQ15] slightly, the new blue vertex corresponds to the base point $\otimes$ of the model and the green dot now represents the variable for the root of the tree. However all the kernel notation remains exactly the same.

Switching back to our conventions the combinatorial tree $T_0^0$ corresponding to $\vee$ is given by
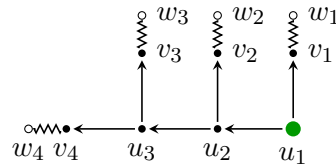
One has $N(T) = \{u_i\}_{i=1}^3 \sqcup \{v_i\}_{i=1}^4$ with $\varrho_T = u_1$ and $L(T) = \{v_i\}_{i=1}^4$.

We now describe what the RHS of (4.20) looks like in this example. Assuming that we are considering centred Gaussian approximations to space-time white noise, only second order cumulants are non-zero, so that any summand corresponding to a pair $(\tilde{L}, \pi)$ not appearing in the list below does vanish. (In the general case considered in [HS15], one has to keep a few more terms.)

1. $\tilde{L} = \{v_1, v_2, v_3, v_4\}, \pi = \emptyset$ .
2. $\tilde{L} = \{v_1, v_2\}, \pi = \{\{v_3, v_4\}\}$ .
3. $\tilde{L} = \{v_3, v_4\}, \pi = \{\{v_1, v_2\}\}$ .
4. $\tilde{L} = \{v_1, v_4\}, \pi = \{\{v_2, v_3\}\} + v_3 \leftrightarrow v_4$ .
5. $\tilde{L} = \{v_2, v_3\}, \pi = \{\{v_1, v_4\}\} + v_3 \leftrightarrow v_4$ .
6. $\tilde{L} = \emptyset, \pi = \{\{v_1, v_4\}, \{v_2, v_3\}\} + v_3 \leftrightarrow v_4$ .
7. $\tilde{L} = \emptyset, \pi = \{\{v_1, v_2\}, \{v_3, v_4\}\}$ .

Here, we write $v_3 \leftrightarrow v_4$ to denote the same term with $v_3$ and $v_4$ exchanged.

Concerning positive renormalizations, the set $\mathfrak{C}$ contains only a single edge: $\mathfrak{C} = \{(u_1, u_2)\}$. Using the same numbering as above (and ignoring the permuted situations we didn't explicitly write out), we have the following possibilities for $\mathbb{F}_\pi$

1. $\{\emptyset\}$ .
2. $\{\emptyset, \{S_1\}\}$ .
3. $\{\emptyset, \{S_2\}\}$ .
4. $\{\emptyset, \{S_3\}\}$.
5. $\{\emptyset\}$ .
6. $\{\emptyset, \{S_3\}, \{T\}, \{S_3, T\}\}$
7. $\{\emptyset, \{S_1\}, \{S_2\}, \{S_1, S_2\}, \{T, S_1, S_2\}, \{S_1, T\}, \{S_2, T\}, \{T\}\}$.

Here the trees $S_1$, $S_2$, and $S_3$ are respectively given by
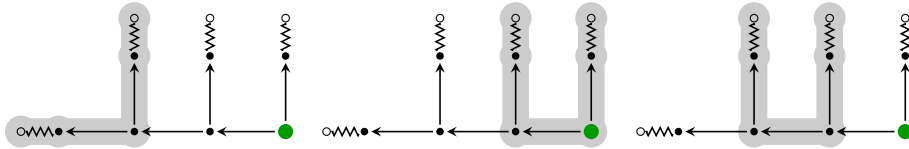


The sum corresponding to scenario 1 gives the first term in (4.28).

The sum for scenario 2 vanishes for either of two reasons: one is that the renormalization of $S_1$ is a "Wick"-type renormalization (so the term with $S_1$ renormalized precisely kills the one without) and a second reason is that the kernel $K'$ in [HQ15] is chosen to annihilates constants.

The sum for scenario 3 gives the second term and third term in (4.28), the third term comes when one chooses $\mathscr{C} = \{(u_1, u_2)\}$ which obstructs the renormalization.

The sums for scenarios 4 and 5 give, respectively, the fourth and fifth terms in (4.28) with the factors 2 coming from the permutation we mentioned.

For the sum in scenario 6 we first note that one has

$$\sum_{\mathcal{F} \in \mathbb{F}_\pi} \mathcal{W}_\emptyset^\pi[\mathcal{F}, \emptyset] = \sum_{\substack{\mathcal{F} \in \mathbb{F}_\pi \\ \mathcal{F} \not\ni T}} (\mathcal{W}_\emptyset^\pi[\mathcal{F}, \emptyset] + \mathcal{W}_\emptyset^\pi[\mathcal{F} \cup \{T\}, \emptyset])$$

$$= \sum_{\substack{\mathcal{F}\in\mathbb{F}_\pi \\ \mathcal{F}\not\ni T}} (\mathcal{W}^\pi_\emptyset[\mathcal{F},\emptyset] - \mathcal{W}^\pi_\emptyset[\mathcal{F},\emptyset]) = 0\ .$$

This sort of cancellation is quite common when $\tilde{L}=\emptyset$. The rest of the sum in scenario 6 (given by $\mathcal{W}^\pi_\emptyset[\emptyset,\{(u_1,u_2)\}] + \mathcal{W}^\pi_\emptyset[\{S_3\},\{(u_1,u_2)\}]$) gives the sixth term in (4.28).

Finally, the same arguments used for scenario 2 also lead to the vanishing of the sum in scenario 7.

## 5   Reorganizing sums

Our main goal moving forward is to obtain, for $\xi\in\mathcal{M}(\Omega_\infty,\mathfrak{L}_{\mathrm{Cum}})$, uniform in $\lambda\in(0,1]$, $L^{2p}$ estimates for the random variable $\hat{\Upsilon}^\xi[\overline{T}^{\overline{\mathfrak{n}}}_{\overline{\mathfrak{e}}}](\psi^\lambda_z)$ for some continuous function $\psi$ as in Theorem 2.31 and $z\in\mathbf{R}^d$. Our main theorem promises estimates uniform in $\psi$ but this comes automatically[11] by choosing $\psi$ to be the generator of a wavelet basis, so from now on we treat $\psi$ as fixed.   The theorem also gives uniformity in $z$ but this will be automatic from stationarity.

If we define, for $\tilde{L}\subset L(\overline{T})$ and $\pi\in\bar{\mathcal{P}}[L(T)\setminus\tilde{L}]$, the kernels $\mathcal{W}^\pi_{\tilde{L},\lambda}\in\mathscr{C}_{\tilde{L}\cup\{\otimes,\varrho_*\}}$ by setting, for each $x\in(\mathbf{R}^d)^{\tilde{L}\cup\{\otimes,\varrho_*\}}$,

$$\mathcal{W}^\pi_{\tilde{L},\lambda}(x) \stackrel{\mathrm{def}}{=} \sum_{\substack{\mathcal{F}\in\mathbb{F}_\pi \\ \mathscr{C}\subset\mathfrak{C}_\mathcal{F}}} \mathcal{W}^\pi_{\tilde{L}}[\mathcal{F},\mathscr{C}](x)\cdot\psi^\lambda_{x_\otimes}(x_{\varrho_*}) \tag{5.1}$$

then the functions $\{\sum_{\pi\in\bar{\mathcal{P}}[L(\overline{T})\setminus\tilde{L}]} \mathcal{W}^\pi_{\tilde{L},\lambda}\}_{\tilde{L}\subset L_{\overline{T}}}$ play the role of chaos kernels in a non-Gaussian analogue to the Wiener chaos decomposition for the random variable[12] $\hat{\Upsilon}^\xi_{\bar{z}}[\overline{T}^{\overline{\mathfrak{n}}}_{\overline{\mathfrak{e}}}](\psi^\lambda_{\bar{z}})$. In order to obtain the mentioned $L^{2p}$ estimates we try to get good control over the behavior of the kernels $\mathcal{W}^\pi_{\tilde{L}}$ for fixed[13] $\tilde{L}\subset L(\overline{T})$ and $\pi\in\bar{\mathcal{P}}[L(T)\setminus \tilde{L}]$ and throughout Sections 5 to 9 we treat both of these parameters as fixed.

In what follows we enforce the condition $\xi\in\mathcal{M}(\Omega_\infty,\mathfrak{L}_{\mathrm{Cum}})$, we then have that $\mathcal{W}^\pi_{\tilde{L}}$ vanishes unless $\pi$ is a partition on $L(T)\setminus\tilde{L}$ which satisfies the property that for every $B\in\pi$ one has $(\mathfrak{t},B)\in\mathfrak{L}_{\mathrm{Cum}}$. Therefore we assume this of $\pi$, note that this prevents $\pi$ from containing any singletons.

We also write $\mathcal{W}_\lambda[\mathcal{F},\mathscr{C}]\in\mathscr{C}_{\tilde{L}\cup\{\varrho_*,\otimes\}}$ for summand of (4.20) for fixed $\mathcal{F}\in\mathbb{F}_\pi$ and $\mathscr{C}\subset\mathfrak{C}_\mathcal{F}$.

One does not expect to get good control over the LHS of (5.1) by inserting absolute values and controlling each summand on the right-hand side separately – doing so would prevent us from harvesting any of the numerous cancellations created

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[11]See [Hai14, Remark 10.8 and Sec. 3.1].

[12]Being pedantic, this is true upon fixing $x_\otimes=\bar{z}$ and then integrating $x_{\varrho_*}$ if $\varrho_*\notin\tilde{L}$.

[13]Applying the triangle inequality to deal with the sum over $\tilde{L}$ and $\pi$ introduces combinatorial factors that are unimportant in our context.

by our renormalization procedures. Exploiting a renormalization cancellation for an edge $e \in \mathfrak{C}$ or divergent subtree $S \in \mathrm{Div}$ requires us to estimate together

$$\mathcal{W}_\lambda[\mathcal{F}, \mathscr{C}] \text{ and } \mathcal{W}_\lambda[\mathcal{F}, \mathscr{C} \sqcup \{e\}]$$
$$\text{or } \mathcal{W}_\lambda[\mathcal{F}, \mathscr{C}] \text{ and } \mathcal{W}_\lambda[\mathcal{F} \sqcup \{S\}, \mathscr{C}].$$

In many examples it is impossible to harvest all the renormalizations of $\mathfrak{C}$ and $\mathrm{Div}$ simultaneously, this is, in our context, the problem of "overlapping divergences". The solution to this problem is the observation that "overlapping divergences don't overlap in phase space". More concretely we perform a multiscale expansion of the quantities we wish to control and then design an algorithm which tells us which renormalizations to harvest for a fixed *scale assignment* **n** as in [HQ15]. This algorithm requires some basic manipulations with partially ordered sets (posets). To make notions clear, we introduce these concepts prematurely, that is before we introduce the multiscale expansion itself. This means that many of the notions of this section will be used in an **n**-dependent way in the sequel.

All posets have a natural notion of "intervals", that is subsets of the poset which are either empty or contain unique minimal and maximal elements along with all elements of the poset that fall between these two extremal elements.

**Definition 5.1** Given a poset $\mathbb{A}$, $\mathbb{M} \subset \mathbb{A}$ is said to be an interval of $\mathbb{A}$ if $\mathbb{M}$ is either empty or there exist elements $s(\mathbb{M})$ and $b(\mathbb{M})$ such that

$$\mathbb{M} = \{a \in \mathbb{A} \ : \ s(\mathbb{M}) \le a \le b(\mathbb{M})\} \ .$$

For a non-empty interval $\mathbb{M}$ we also use the notation $\mathbb{M} = [s(\mathbb{M}), b(\mathbb{M})]$. If elements of $\mathbb{A}$ happen to be sets themselves, equipped with the partial order given by inclusion (which will always be the case for the intervals we consider), we write $\delta(\mathbb{M}) \overset{\text{def}}{=} b(\mathbb{M}) \setminus s(\mathbb{M})$. As a mnemonic one should think of $s(\cdot)$ and $b(\cdot)$ as standing for "small" and "big", respectively. In what follows we view $\mathbb{F}_\pi$ as a poset by equipping it with the inclusion partial order.

**Remark 5.2** If $\mathbb{M}$ is an interval of $\mathbb{F}_\pi$ then for any $\mathscr{C} \subset \mathfrak{C}$ one has that $\mathbb{M} \cap \mathbb{F}_{\mathscr{C},\pi}$ is an interval of $\mathbb{F}_\pi$.

This is immediate when $\mathbb{M} = \emptyset$ or $\mathbb{M}$ is non-empty and $s(\mathbb{M}) \notin \mathbb{F}_{\mathscr{C},\pi}$, in both situations $\mathbb{M} \cap \mathbb{F}_{\mathscr{C},\pi} = \emptyset$. If $\mathbb{M}$ is non-empty with $s(\mathbb{M}) \in \mathbb{F}_{\mathscr{C},\pi}$, then it is straightforward to check that $\mathbb{M} \cap \mathbb{F}_{\mathscr{C},\pi} = [s(\mathbb{M}), b(\mathbb{M}) \cap \mathrm{Div}_\mathscr{C}]$.

For the rest of the paper an *interval of forests* refers to an interval of $\mathbb{F}_\pi$. The intervals of forests of interest to us are specified as pullbacks of certain maps from $\mathbb{F}_\pi$ to itself called forest projections.

**Definition 5.3** A map $P : \mathbb{F}_\pi \to \mathbb{F}_\pi$ is said to be a forest projection if for any $\mathcal{S} \in \mathbb{F}_\pi$, $P^{-1}[\mathcal{S}]$ is either empty or an interval of forests $\mathbb{M}$ with $s(\mathbb{M}) = \mathcal{S}$.

Given $\mathscr{C} \subset \mathfrak{C}$ and a forest projection $P$, we sometimes use the shorthand $P_{\mathscr{C}}^{-1}[\cdot] = P^{-1}[\cdot] \cap \mathbb{F}_{\mathscr{C},\pi}$. A simple observation that we use frequently is that

$$\mathscr{C}_1 \subset \mathscr{C}_2 \subset \mathfrak{C} \quad \Rightarrow \quad P_{\mathscr{C}_2}^{-1}[\mathcal{S}] \subset P_{\mathscr{C}_1}^{-1}[\mathcal{S}] \quad \forall \mathcal{S} \in \mathbb{F}_\pi . \tag{5.2}$$

For the rest of this subsection assume that we have fixed a forest projection $P$. Given any forest projection $P$ and $\mathscr{C} \subset \mathfrak{C}$ we define $\mathfrak{M}^P(\mathscr{C}) \subset 2^{\mathbb{F}_\pi}$ via setting

$$\mathfrak{M}^P(\mathscr{C}) \stackrel{\text{def}}{=} \left\{ \mathbb{M} \neq \emptyset \,:\, \exists \mathcal{F} \in \mathbb{F}_\pi \text{ with } P_{\mathscr{C}}^{-1}[\mathcal{F}] = \mathbb{M} \right\} .$$

Conversely, for any forest projection $P$ and any $\mathbb{M} \subset \mathbb{F}_\pi$ we define

$$\mathfrak{C}^P(\mathbb{M}) \stackrel{\text{def}}{=} \left\{ \mathscr{C} \subset \mathfrak{C} \,:\, P_{\mathscr{C}}^{-1}[s(\mathbb{M})] = \mathbb{M} \right\} .$$

Clearly $\mathfrak{C}^P(\mathbb{M})$ is empty unless $\mathbb{M}$ is an interval of forests. Then, for any forest projection $P$ one can rewrite the set $\{(\mathcal{F},\mathscr{C}) \,:\, \mathscr{C} \subset \mathfrak{C} \;\&\; \mathcal{F} \in \mathbb{F}_{\mathscr{C},\pi}\}$ as

$$\bigsqcup_{\mathscr{C} \subset \mathfrak{C}} \bigsqcup_{\mathbb{M} \in \mathfrak{M}^P(\mathscr{C})} \{(\mathcal{F},\mathscr{C}) \,:\, \mathcal{F} \in \mathbb{M}\} \;=\; \bigsqcup_{\mathbb{M} \subset \mathbb{F}_\pi} \bigsqcup_{\mathscr{C} \in \mathfrak{C}^P(\mathbb{M})} \{(\mathcal{F},\mathscr{C}) \,:\, \mathcal{F} \in \mathbb{M}\} . \tag{5.3}$$

We use the shorthand $\mathfrak{M}^P \stackrel{\text{def}}{=} \bigcup_{\mathscr{C} \subset \mathfrak{C}} \mathfrak{M}^P(\mathscr{C})$. Equipping $2^{\mathfrak{C}}$ with the inclusion partial order, we use the term *interval of cuttings* to refer to an interval of $2^{\mathfrak{C}}$.

**Remark 5.4** In this paper the use of the term "interval" will always refer to an interval of the poset $\mathbb{F}_\pi$ or $2^{\mathfrak{C}}$. However, the reader should be cautious since these two posets are themselves collections of subsets of other posets – Div and $\mathfrak{C}$, respectively.

Our maximum and minimum operations are always applied to *collections of elements* of a poset. For example, given an interval of forests $\mathbb{M}$, $\overline{s(\mathbb{M})}$ denotes the collection of maximal subtrees of the forest $s(\mathbb{M})$ – the selection of maximal elements is taking place with respect to the poset structure of Div since $s(\mathbb{M}) \subset$ Div.

Before stating our next level we define, for any interval of forests $\mathbb{M}$,

$$\mathscr{C}_{\mathbb{M}} \stackrel{\text{def}}{=} \mathfrak{C} \setminus \Big( \bigcup_{S \in \overline{b(\mathbb{M})}} K(S) \Big).$$

**Lemma 5.5** *For any forest projection $P$ and $\mathbb{M} \in \mathfrak{M}^P$ one has $\overline{\mathfrak{C}^P(\mathbb{M})} = \{\mathscr{C}_{\mathbb{M}}\}$.*

*Proof.* First we observe that for any $\mathscr{C} \in \mathfrak{C}^P(\mathbb{M})$ one must have $\mathscr{C} \subset \mathscr{C}_{\mathbb{M}}$ since $\mathbb{M} \subset \mathbb{F}_{\mathscr{C},\pi}$. Next we show $\mathscr{C}_{\mathbb{M}} \in \mathfrak{C}^P(\mathbb{M})$. Fixing some $\mathscr{C} \in \mathfrak{C}^P(\mathbb{M})$ (the latter set is non-empty by assumption) we have

$$P_{\mathscr{C}}^{-1}[s(\mathbb{M})] = \mathbb{M} \subset P_{\mathscr{C}_{\mathbb{M}}}^{-1}[s(\mathbb{M})].$$

The subset relation is a consequence of the facts that $\mathbb{M} \subset P^{-1}[s(\mathbb{M})]$ and $\mathbb{M} \subset \mathbb{F}_{\mathscr{C}_{\mathbb{M}},\pi}$. We also have $P_{\mathscr{C}_{\mathbb{M}}}^{-1}[s(\mathbb{M})] \subset P_{\mathscr{C}}^{-1}[s(\mathbb{M})] = \mathbb{M}$ by virtue of (5.2). $\qquad\square$

**Definition 5.6** A cut rule is a map $\mathscr{G} : \mathbb{F}_\pi \to \mathfrak{C}$ such that for each $\mathcal{F} \in \mathbb{F}_\pi$, $\mathscr{G}(\mathcal{F}) \subset \mathfrak{C}_\mathcal{F}$.

When we perform a multiscale expansion, each scale assignment $\mathbf{n}$ will determine a forest projection $P^{\mathbf{n}}$ which will be used to organize the sum over forests of divergent subtrees and a cut rule $\mathscr{G}^{\mathbf{n}}$ for organizing the sum over cut sets.

We now formulate a criterion which will guarantee that conflicts do not arise between negative renormalizations and positive cuts we wish to harvest, that is $P^{\mathbf{n}}$ and $\mathscr{G}^{\mathbf{n}}$ do not get in each other's way.

**Definition 5.7** Given a forest projection $P$ and a cut rule $\mathscr{G}$ we say that $\mathscr{G}$ and $P$ are *compatible* if, for all $\mathbb{M} \in \mathfrak{M}^P$, $e \in \mathscr{G}(b(\mathbb{M}))$, and $\mathscr{C} \subset \mathfrak{C}$, one has

$$\mathscr{C} \cup \{e\} \in \mathfrak{C}^P(\mathbb{M}) \quad \Longleftrightarrow \quad \mathscr{C} \setminus \{e\} \in \mathfrak{C}^P(\mathbb{M}).$$

For any compatible $\mathscr{G}$ and $P$ we define, for each $\mathbb{M} \in \mathfrak{M}^P$, a collection of intervals of cut sets

$$\mathfrak{G}_\mathscr{G}^P(\mathbb{M}) \stackrel{\text{def}}{=} \left\{ [\mathscr{C}, \mathscr{C} \sqcup \mathscr{G}(b(\mathbb{M})] : \mathscr{C} \in \mathfrak{C}^P(\mathbb{M}), \ \mathscr{C} \cap \mathscr{G}(b(\mathbb{M}) = \varnothing \right\}.$$

Note that, for each $\mathbb{M} \in \mathfrak{M}^P$, by compatibility one has that $\mathfrak{G}_\mathscr{G}^P(\mathbb{M})$ is a partition of $\mathfrak{C}^P(\mathbb{M})$. The following proposition is essentially immediate.

**Proposition 5.8** *Given a forest projection $P$ and a compatible $\mathscr{G} \subset \mathfrak{C}$, one has*

$$\bigsqcup_{\mathscr{C} \subset \mathfrak{C}} \bigsqcup_{\mathcal{F} \in \mathbb{F}_{\mathscr{C},\pi}} \{(\mathcal{F}, \mathscr{C})\} = \bigsqcup_{\mathbb{M} \in \mathfrak{M}^P} \bigsqcup_{\mathbb{G} \in \mathfrak{G}_\mathscr{G}^P(\mathbb{M})} \bigsqcup_{\substack{\mathcal{F} \in \mathbb{M} \\ \mathscr{C} \in \mathbb{G}}} \{(\mathcal{F}, \mathscr{C})\}.$$

*Proof.* It follows from (5.3) that the left hand side is equal to

$$\bigsqcup_{\mathbb{M} \in \mathfrak{M}^P} \bigsqcup_{\mathscr{C} \in \mathfrak{C}^P(\mathbb{M})} \bigsqcup_{\mathcal{F} \in \mathbb{M}} \{(\mathcal{F}, \mathscr{C})\},$$

since $\mathfrak{C}^P(\mathbb{M})$ is empty unless $\mathbb{M} \in \mathfrak{M}^P$. The claim then follows from the fact that all subsets in $\mathfrak{G}_\mathscr{G}^P(\mathbb{M})$ are disjoint by definition and every $\mathscr{C} \in \mathfrak{C}^P(\mathbb{M})$ has a unique decomposition $\mathscr{C} = \mathscr{D} \sqcup \mathscr{A}$ with $\mathscr{D} \in \mathfrak{C}^P(\mathbb{M})$ and $\mathscr{D} \cap \mathscr{G}(b(\mathbb{M})) = \varnothing$, as well as $\mathscr{A} \subset \mathscr{G}(b(\mathbb{M}))$. Indeed, simply set $\mathscr{D} = \mathscr{C} \setminus \mathscr{G}(b(\mathbb{M}))$ and $\mathscr{A} = \mathscr{C} \cap \mathscr{G}(b(\mathbb{M}))$: one then has $\mathscr{D} \in \mathfrak{C}^P(\mathbb{M})$ by the compatibility of $\mathscr{G}$ and $P$. $\qquad\square$

# 6 Multiscale expansion

Given any set $A$ we denote by $A^{(2)}$ the collection of all two element subsets of $A$.

We also define

$$\mathcal{E}_\pi \stackrel{\text{def}}{=} \bigsqcup_{B \in \pi} B^{(2)}.$$

When viewed as a set of edges, $\mathcal{E}_\pi$ forms a union of complete graphs with vertex sets given by the true nodes of the blocks of $\pi$.

We also define $\mathcal{E}_\otimes \overset{\text{def}}{=} \{\{\otimes, u\} : u \in N(\overline{T})\}$ and

$$\mathcal{E} \overset{\text{def}}{=} K(\overline{T}) \sqcup \mathcal{E}_\pi \sqcup \mathcal{E}_\otimes, \tag{6.1}$$

For any subset $A \subset N(\overline{T})$ we write $\mathcal{E}_\otimes(A) \overset{\text{def}}{=} \{\{\otimes, u\} : u \in A\} \subset \mathcal{E}_\otimes$.

**Remark 6.1** Note that we often view the elements of $K(\overline{T})$ as sets of two element subsets of $N(\overline{T})$ in (6.1). When we are doing this there will be pairs $\{a, b\}$ that may appear "twice", once as an element of $K(\overline{T})$ and another as an element of $\mathcal{E}_\pi$, however we see these occurrences as distinct and distinguishable. This should also be kept in mind when we present some subsets of $\mathcal{E}$ as disjoint unions of subsets of $K(\overline{T})$, $\mathcal{E}_\pi$, and $\mathcal{E}_\otimes$.

We define a *global scale assigment* for $\overline{T}$ to be a tuple

$$\mathbf{n} = (n_e)_{e \in \mathcal{E}} \in \mathbf{N}^\mathcal{E} \ .$$

We fix $\psi : \mathbf{R} \to [0, 1]$ to be a smooth function supported on $[3/8, 1]$ and with the property that $\sum_{n \in \mathbf{Z}} \psi(2^n x) = 1$ for $x \neq 0$. We then define another family of functions $\{\Psi^{(k)}\}_{k \in \mathbf{N}}$, $\Psi^{(k)} : \mathbf{R}^d \to [0, 1]$ by setting $\Psi^{(k)}(0) = \delta_{k,0}$ and, for $x \neq 0$,

$$\Psi^{(k)}(x) \overset{\text{def}}{=} \begin{cases} \sum_{n \leq 0} \psi(2^n |x|) & \text{if } k = 0 \\ \psi(2^k |x|) & \text{if } k \neq 0. \end{cases}$$

Given $e \in \mathcal{E}$ with $e = \{a, b\}$ and a global scale assignment $\mathbf{n}$ we define $\Psi_\mathbf{n}^e \in \mathscr{C}_*$ via $\Psi_\mathbf{n}^e(x_a - x_b)$. Given $E \subset \mathcal{E}$ we define $\Psi_\mathbf{n}^E \in \mathscr{C}_*$ via

$$\Psi_\mathbf{n}^E \overset{\text{def}}{=} \prod_{e \in E} \Psi_\mathbf{n}^e \ .$$

We now define single scale analogues of some of the functions we introduced earlier, as well as for the functions

$$\widehat{\mathrm{RKer}}_{\mathfrak{e}}^{\mathscr{D}} \overset{\text{def}}{=} \prod_{e \in \mathscr{D}} \left( \mathrm{Ker}_{\mathfrak{e}}^{\{e\}} + \mathrm{RKer}_{\mathfrak{e}}^{\{e\}} \right). \tag{6.2}$$

For any $\mathbf{n} \in \mathbf{N}^\mathcal{E}$ we set

$$\mathrm{Ker}_{\mathfrak{e},\mathbf{n}}^E \overset{\text{def}}{=} \mathrm{Ker}_{\mathfrak{e}}^E \cdot \Psi_\mathbf{n}^E, \qquad \widehat{\mathrm{RKer}}_{\mathfrak{e},\mathbf{n}}^{\mathscr{D}} \overset{\text{def}}{=} \widehat{\mathrm{RKer}}_{\mathfrak{e}}^{\mathscr{D}} \cdot \Psi_\mathbf{n}^{\mathscr{D}},$$

$$\mathrm{RKer}_{\mathfrak{e},\mathbf{n}}^{\mathscr{D}} \overset{\text{def}}{=} \mathrm{RKer}_{\mathfrak{e}}^{\mathscr{D}} \cdot \Psi_\mathbf{n}^{\mathscr{D}}, \qquad \mathrm{X}_{\mathbf{n},v,\mathbf{n}}^N \overset{\text{def}}{=} \mathrm{X}_{\mathbf{n},v}^N \cdot \prod_{u \in N} \Psi_\mathbf{n}^{\{\{\otimes, u\}\}},$$

as well as $\psi_\mathbf{n}^\lambda \in \mathscr{C}_{\{\otimes, \varrho_*\}}$ via $\psi_\mathbf{n}^\lambda(x) \overset{\text{def}}{=} \psi_{x_\otimes}^\lambda(x_{\varrho_*}) \cdot \Psi_\mathbf{n}^{\{\otimes, \varrho_*\}}(x)$.

We now describe our multiscale expansion for cumulants. Our expansion will be more involved for second cumulants. Since we allow $\mathfrak{t} \in \mathfrak{L}_-$ to have $|\mathfrak{t}|_\mathfrak{s} \in (-\frac{2}{3}|\mathfrak{s}|, -|\mathfrak{s}|)$ this means that if $\|\xi\|_{N,\mathfrak{c}}$ only contained information about cumulants away from the diagonal then the quantity $\|\xi\|_{N,\mathfrak{c}}$ could not be used to control, for arbitrary $\xi \in \mathcal{M}(\Omega_\infty)$, $\delta > 0$, and $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{L}_-$ with $|\mathfrak{t}|_\mathfrak{s} + |\mathfrak{t}'|_\mathfrak{s} \le -|\mathfrak{s}|$, the quantity

$$\fint_{\substack{y \in \mathbf{R}^d \\ |y|_\mathfrak{s} < \delta}} dx\, \mathbf{E}^c[\{\xi_\mathfrak{t}(0), \xi_{\mathfrak{t}'}(x)\}].$$

In particular, if we use a brutal multiscale bounds on the above quantity and only use the information on $\|\xi\|_{N,\mathfrak{c}}$ given away from the diagonal then by power-counting the above integral will look divergent. However our norm $\|\xi\|_{N,\mathfrak{c}}$ includes data on the behavior of the diagonal through (2.10) so there is no real problem here. In order to overcome this in our power-counting analysis, we perform a built-in "fictitious"-renormalization of second cumulants which uses the data from (2.10). For each $B \subset L(\overline{T})$ with $B = \{u, v\}$ and $|\mathfrak{t}(B)|_\mathfrak{s} \le -|\mathfrak{s}|$ we define, by applying the same trick as used in [HQ15, Lem. A.4], a family of functions $\{\widetilde{\mathrm{Cu}}_{B,j}(x_u, x_v)\}_{j=0}^\infty$ with the property that each $\widetilde{\mathrm{Cu}}_{B,j}(x_u, x_v)$ is translation invariant (i.e., expressable as a function of $x_u - x_v$) and furthermore

1. The identity

$$\sum_{j=0}^\infty \widetilde{\mathrm{Cu}}_{B,j} = \mathbf{E}^c[\{\xi_{\mathfrak{t}(u)}, \xi_{\mathfrak{t}(v)}\}]$$

  holds in the sense of distributions on $(\mathbf{R}^d)^B$.
2. $\widetilde{\mathrm{Cu}}_{B,j}$ is supported on $(x_u, x_v)$ with $2^{-n-2} \le |x_u - x_v| \le 2^{-n}$.
3. One has, uniform in $j \ge 0$ and $x_B \in \mathbf{R}^{2d}$,

$$|\widetilde{\mathrm{Cu}}_{B,j}(x_B)| \lesssim \|\xi\|_{2,\mathfrak{c}} \cdot |x_u - x_v|^{|\mathfrak{t}(B)|_\mathfrak{s}} .$$

4. For any polynomial $Q$ on $\mathbf{R}^d$ of $\mathfrak{s}$-degree strictly less than $-|\mathfrak{s}| - |\mathfrak{t}(B)|_\mathfrak{s}$ and every $j > 0$, one has

$$\int dx\, \widetilde{\mathrm{Cu}}_{B,j}(0, x) Q(x) = 0. \tag{6.3}$$

**Remark 6.2** Note that we do *not* impose (6.3) for $j = 0$. Indeed, this would in general be in contradiction with the first item.

For any $B \subset L(\overline{T})$ and any global scale assignment $\mathbf{n}$ we set

$$\mathrm{Cu}_{B,\mathbf{n}}(x) \overset{\text{def}}{=} \begin{cases} \widetilde{\mathrm{Cu}}_{B,n_B}(x) & \text{if } |B| = 2 \text{ and } |\mathfrak{t}(B)|_\mathfrak{s} \le -|\mathfrak{s}| \\ \Psi_\mathbf{n}^{B^{(2)}}(x) \cdot \mathbf{E}^c[\{\xi_{\mathfrak{t}(u)}(x_u)\}_{u \in B}] & \text{otherwise} \end{cases}$$

Finally, for any $L \subset L(\overline{T})$ and global scale assignment $\mathbf{n}$ we set

$$\mathrm{Cu}_{\pi,\mathbf{n}}^L \overset{\text{def}}{=} \prod_{\substack{B \in \pi, \\ B \subset L}} \mathrm{Cu}_{B,\mathbf{n}}.$$

We now define operators $H^{\mathbf{n}}_{\pi,\mathbb{M},S} : \mathscr{C}_* \to \mathscr{C}_{\tilde{N}(S)^c}$, where $\mathbb{M}$ is an interval of forests, $S \in b(\mathbb{M})$, and $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$. This definition is again recursive and for $\varphi \in \mathscr{C}_*$ we set, for any $x \in (\mathbf{R}^d)^{\tilde{N}(S)^c}$,

$$[H^{\mathbf{n}}_{\pi,\mathbb{M},S}\varphi](x) \stackrel{\text{def}}{=} \int_{\tilde{N}_{b(\mathbb{M})}(S)} dy \, \mathrm{Cu}^{L_{b(\mathbb{M})}(S)}_{\pi,\mathbf{n}}(y) \cdot \mathrm{Ker}^{\mathring{K}_{b(\mathbb{M})}(S)}_{0,\mathbf{n}}(x_{\varrho_S} \sqcup y)$$
$$\cdot H^{\mathbf{n}}_{\pi,\mathbb{M},C_{b(\mathbb{M})}(S)}\left[\mathrm{Ker}^{K^{\partial}_{b(\mathbb{M})}(S)}_{0,\mathbf{n}} \cdot [\mathscr{Y}^{\#}_{S,\mathbb{M}}\varphi]\right](x \sqcup y)$$

where

$$\mathscr{Y}^{\#}_{S,\mathbb{M}}\varphi = \begin{cases} -\mathscr{Y}_S\varphi & \text{if } S \in s(\mathbb{M}) \\ (\mathrm{Id} - \mathscr{Y}_S)\varphi & \text{if } S \in \delta(\mathbb{M}). \end{cases}$$

Recall that the operators $\mathscr{Y}_S$ were introduced in Definition 4.11. The notation $H^{\mathbf{n}}_{\pi,\mathbb{M},C_{b(\mathbb{M})}(S)}$ denotes the composition of the operators $\{H^{\mathbf{n}}_{\pi,\mathbb{M},T}\}_{T \in C_{b(\mathbb{M})}(S)}$. Again, no order needs to be prescribed for this composition because these operators commute – the justification of this claim is essentially the same as that given for the commutation proved in Remark 4.13. The difference here is that some instances of $\mathscr{Y}_S$ are replaced by $(\mathrm{Id} - \mathscr{Y}_S)$ and all the cumulants and kernels are replaced by single slices of their multiscale expansions but these changes make no real difference when checking that these operators commute.

We can now define single slice, partially resummed chaos kernels. For any $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$, any interval of forests $\mathbb{M}$, and any interval of cuttings $\mathbb{G}$ with $b(\mathbb{G}) \subset \mathfrak{C}_{b(\mathbb{M})}$, we define $\mathcal{W}^{\mathbf{n}}_{\lambda}[\mathbb{M},\mathbb{G}] \in \mathscr{C}_{\tilde{L} \cup \{\varrho_*,\otimes\}}$ via setting, for each $x \in (\mathbf{R}^d)^{\tilde{L} \cup \{\otimes,\varrho_*\}}$,

$$\mathcal{W}^{\mathbf{n}}_{\lambda}[\mathbb{M},\mathbb{G}](x) \stackrel{\text{def}}{=} \int_{\tilde{N}(b(\mathbb{M}),\overline{T})\backslash(\tilde{L}\cup\{\otimes\})} dy \, \psi^{\lambda}_{\mathbf{n}}(x \sqcup y) \cdot \mathrm{X}^{\tilde{N}(b(\mathbb{M}),\overline{T})}_{\overline{\mathrm{n}},\otimes,\mathbf{n}}(x \sqcup y) \qquad (6.4)$$
$$\cdot \mathrm{Cu}^{L(b(\mathbb{M}),\overline{T})}_{\pi,\mathbf{n}}(x \sqcup y) \, \mathrm{Ker}^{K(b(\mathbb{M}),\overline{T})\backslash b(\mathbb{G})}_{0,\mathbf{n}}(x \sqcup y)$$
$$\cdot \mathrm{RKer}^{s(\mathbb{G})\backslash K^{\downarrow}(\overline{b(\mathbb{M})})}_{0,\mathbf{n}}(x \sqcup y) \, \widehat{\mathrm{RKer}}^{\delta(\mathbb{G})\backslash K^{\downarrow}(\overline{b(\mathbb{M})})}_{0,\mathbf{n}}(x \sqcup y)$$
$$\cdot \left( \prod_{S \in \overline{b(\mathbb{M})}} H^{\mathbf{n}}_{\pi,\mathbb{M},S}\left[\mathrm{RKer}^{s(\mathbb{G})\cap K^{\downarrow}(S)}_{0,\mathbf{n}} \, \widehat{\mathrm{RKer}}^{\delta(\mathbb{G})\cap K^{\downarrow}(S)}_{0,\mathbf{n}} \, \mathrm{Ker}^{K^{\downarrow}(S)\backslash b(\mathbb{G})}_{0,\mathbf{n}} \mathrm{X}^{\tilde{N}(S)}_{\overline{\mathrm{n}},\otimes,\mathbf{n}}\right](x \sqcup y) \right).$$

With our definitions the following lemma is straightforward.

**Lemma 6.3** *For any $\mathscr{C} \subset \mathfrak{C}$ and $\mathcal{F} \in \mathbb{F}_{\mathscr{C},\pi}$ one has*

$$\sum_{\mathbf{n} \in \mathbf{N}^{\mathcal{E}}} \mathcal{W}^{\mathbf{n}}_{\lambda}[\{\mathcal{F}\},\{\mathscr{C}\}] = \mathcal{W}_{\lambda}[\mathcal{F},\mathscr{C}], \qquad (6.5)$$

*where for the sum on the the LHS we have absolute convergence pointwise in $x_{\tilde{L}}$.*

*On the other hand, for fixed $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$, interval of forests $\mathbb{M}$, and any interval of cut sets $\mathbb{G}$ with $b(\mathbb{G}) \subset \mathfrak{C}_{b(\mathbb{M})}$, one has*

$$\sum_{\substack{\mathcal{F} \in \mathbb{M} \\ \mathscr{C} \in \mathbb{G}}} \mathcal{W}^{\mathbf{n}}_{\lambda}[\{\mathcal{F}\},\{\mathscr{C}\}] = \mathcal{W}^{\mathbf{n}}_{\lambda}[\mathbb{M},\mathbb{G}]. \qquad (6.6)$$

*Proof.* Both statements are straightforward but we sketch how to show (6.6) since the notation there is somewhat heavy. Note that the fact that we stated this identity for a "single scale" slice (i.e. with an **n**) plays no role (a similar identity holds before the multiscale expansion) but we have stated things this way for later use.

The heuristic for (6.6) is as follows: for any interval $\mathbb{A}$ of sets where the ordering is given by inclusion one has

$$\sum_{\mathcal{A} \in \mathbb{A}} \prod_{a \in \mathcal{A}} (-y_a) = \Big( \prod_{a \in s(\mathbb{A})} (-y_a) \Big) \Big( \prod_{a \in \delta(\mathbb{A})} (1 - y_a) \Big)$$

where the $\{y_a\}_{a \in b(\mathbb{A})}$ are indeterminates which need not be commuting as long as one interprets the products on either side consistently.

Fix $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$. We first show that for any interval of forests $\mathbb{M}$,

$$\sum_{\mathcal{F} \in \mathbb{M}} \mathcal{W}_\lambda^{\mathbf{n}}[\{\mathcal{F}\}, \varnothing] = \mathcal{W}_\lambda^{\mathbf{n}}[\mathbb{M}, \varnothing] . \tag{6.7}$$

We prove the above identity via induction on $|\delta(\mathbb{M})|$. The base case, which occurs when $|\mathbb{M}| = 1$ and $|\delta(\mathbb{M})| = 0$, is immediate. For the inductive step fix $l > 0$, assume the claim has been proven whenever $|\delta(\mathbb{M})| < l$, and fix $\mathbb{M}$ with $|\delta(\mathbb{M})| = l$.

Fix $T \in \delta(\mathbb{M})$, we prove the claim in the case where there exists $\tilde{T} \in b(\mathbb{M})$ with $T \in C_{b(\mathbb{M})}(\tilde{T})$. The case when there is no such $\tilde{T}$ is easier.

Let $\mathbb{M}_1 \stackrel{\text{def}}{=} \{\mathcal{F} \in \mathbb{M} : \mathcal{F} \not\ni T\}$ and $\mathbb{M}_2 \stackrel{\text{def}}{=} \{\mathcal{F} \sqcup \{T\} : \mathcal{F} \in \mathbb{M}_1\}$. Note that $\mathbb{M}_1$ and $\mathbb{M}_2$ are intervals that partition $\mathbb{M}$. Therefore, by our inductive hypothesis,

$$\mathcal{W}_\lambda^{\mathbf{n}}[\mathbb{M}, \varnothing] = \mathcal{W}_\lambda^{\mathbf{n}}[\mathbb{M}_1, \varnothing] + \mathcal{W}_\lambda^{\mathbf{n}}[\mathbb{M}_2, \varnothing] . \tag{6.8}$$

Clearly one has, for all $S \in b(\mathbb{M})$ with $T \not\leq S$,

$$H_{\pi, \mathbb{M}_1, S}^{\mathbf{n}} = H_{\pi, \mathbb{M}_2, S}^{\mathbf{n}} . \tag{6.9}$$

Next we claim that for every $S \in b(\mathbb{M})$ with $S > T$ one has

$$H_{\pi, \mathbb{M}_1, S}^{\mathbf{n}} + H_{\pi, \mathbb{M}_2, S}^{\mathbf{n}} = H_{\pi, \mathbb{M}, S}^{\mathbf{n}} . \tag{6.10}$$

Proving this claim finishes our proof since the combination of (6.9) and (6.10) yields (6.7). We prove the claim using an auxilliary induction in

$$\text{depth}(\{S' \in b(\mathbb{M}) : S > S' > T\}) .$$

The inductive step for this induction is immediate upon writing out both sides of (6.10) and remembering that $\mathscr{Y}_{S, \mathbb{M}_1}^{\#} = \mathscr{Y}_{S, \mathbb{M}_2}^{\#} = \mathscr{Y}_{S, \mathbb{M}}^{\#}$. What remains is to check base case of this induction which occurs when $S = \tilde{T}$.

To obtain (6.10) when $S = \tilde{T}$ we first observe that $C_{b(\mathbb{M})}(\tilde{T}) = C_{b(\mathbb{M}_2)}(\tilde{T}) = C_{b(\mathbb{M}_1)}(\tilde{T}) \sqcup \{T\}$ and then rewrite, for $i = 1, 2$, $H_{\pi, \mathbb{M}_i, \tilde{T}}^{\mathbf{n}}[\varphi]$ as

$$\int_{\tilde{N}_{b(\mathbb{M})}(\tilde{T})} dy \, \text{Cu}_{\pi, \mathbf{n}}^{L_{b(\mathbb{M})}(\tilde{T})}(y) \cdot \text{Ker}_{0, \mathbf{n}}^{\mathring{K}_{b(\mathbb{M})}(\tilde{T})}(x_{\varrho_{\tilde{T}}} \sqcup y) \int_{\tilde{N}_{b(\mathbb{M})}(T)} dz \, \text{Cu}_{\pi, \mathbf{n}}^{L_{b(\mathbb{M})}(T)}( \tag{6.11}$$

$$\cdot \operatorname{Ker}_{0,\mathbf{n}}^{\mathring{K}_{b(\mathbb{M})}(\tilde{T})}(w)(\mathscr{Y}^{(i)} \circ H_{\pi,C_{b(\mathbb{M})}(\tilde{T})\backslash\{T\}}^{\mathbf{n}}[\operatorname{Ker}_{0,\mathbf{n}}^{K_{b(\mathbb{M})}^{\partial}(S)} \mathscr{Y}_{\tilde{T},\mathbb{M}}^{\#}\varphi])(w),$$

where $w = x_{\varrho_{\tilde{T}}} \sqcup y \sqcup z$, $\mathscr{Y}^{(1)} = \operatorname{Id}$, and $\mathscr{Y}^{(2)} = (-\mathscr{Y}_T)$. In obtaining (6.11) for $i = 1$ we used (4.6).

The corresponding identity for summing over $\mathscr{C} \in \mathbb{G}$ is easier to check, one just expands, for each $e \in \delta(\mathbb{G})$, the $\widehat{\operatorname{RKer}}_{0,\mathbf{n}}^{\{e\}}$ appearing in the RHS of (6.4) as $\operatorname{Ker}_{0,\mathbf{n}}^{\{e\}} + \operatorname{RKer}_{0,\mathbf{n}}^{\{e\}}$. $\qquad\square$

**Remark 6.4** In what follows we will perform various interchanges between integrals and infinite sums over scale assignments with an implicit assumption of equality. Once we fix a final method of organizing these sums and integrals we will establish their absolute convergence which will a posteriori justify such interchanges.

## 7    The projection onto safe forests and choosing positive cuts

We now introduce the family of forest projections $\{P^{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{N}}\varepsilon$ used to organize the sum over forests, these are the "projections onto safe forests" of [FMRS85]. For any subtree $S$ of $\overline{T}$, we define the edge sets

$$\mathcal{E}_{\varnothing}^{\text{int}}(S) \stackrel{\text{def}}{=} K(S) \sqcup \{e \in \mathcal{E}_{\pi} : e \subset N(S)\}$$
$$\mathcal{E}_{\varnothing}^{\text{ext}}(S) \stackrel{\text{def}}{=} \{e \in \mathcal{E} \setminus \mathcal{E}^{\text{int}}(S) : e \cap N(S) \neq \varnothing\}.$$

Heuristically, one determines whether the divergent subtree $S$ needs to be renormalized under slice $\mathbf{n}$ based on the relative values of the quantities

$$\operatorname{int}_{\varnothing}^{\mathbf{n}}(S) \stackrel{\text{def}}{=} \min\{n_e : e \in \mathcal{E}_{\varnothing}^{\text{int}}(S)\} \text{ and } \operatorname{ext}_{\varnothing}^{\mathbf{n}}(S) \stackrel{\text{def}}{=} \max\{n_e : e \in \mathcal{E}_{\varnothing}^{\text{ext}}(S)\} . \tag{7.1}$$

We describe the general idea in the scenario where $S$ is the only divergent structure. In this scenario one would sum over the scales assignments in the following way: we freeze the value of the scales external to $\operatorname{ext}_{\varnothing}^{\mathbf{n}}(S)$ and perform a sum on scales internal to $S$, after this is done we sum over the scales external to $S$.

It is the sum over the scales internal to $S$ where our bounds blow up due to the divergence of $S$.

In the parlance of [FMRS85], $S$ is *dangerous* for the scale assignment $\mathbf{n}$ if

$$\operatorname{int}_{\varnothing}^{\mathbf{n}}(S) > \operatorname{ext}_{\varnothing}^{\mathbf{n}}(S). \tag{7.2}$$

The contribution of $S$ for a given scale will be of the order $2^{\omega(S)\operatorname{int}_{\varnothing}^{\mathbf{n}}(S)}$, which means that the our estimates on the sum over internal scales, subject to the constraint (7.2), will diverge. These are the scale assignments for which we need to exploit the cancellation that occurs between the original integral and the counter-term we subtract for $S$ – we need to harvest this negative renormalization.

On the other hand, the sum over internal scales subject to the constraint

$$\text{int}_{\emptyset}^{\mathbf{n}}(S) \leq \text{ext}_{\emptyset}^{\mathbf{n}}(S) \tag{7.3}$$

is not a problem, since we have an upper bound on our sum. When (7.3) holds [FMRS85] calls $S$ *safe* for the scale assignment $\mathbf{n}$. We get a bound of order $2^{\omega(S)\text{ext}_{\emptyset}^{\mathbf{n}}(S)}$ after performing the sum over scales internal to $S$. If $S$ is the only divergence, then our estimate on the entire integral, written as a product of exponential factors, will have the scale $\text{ext}_{\emptyset}^{\mathbf{n}}(S)$ multiplied by a negative constant in the exponent – this comes from the fact that any structure larger than $S$ would be power-counting convergent in this scenario. Therefore the sum over $\text{ext}_{\emptyset}^{\mathbf{n}}(S)$ also creates no problem.

Loosely speaking, the issue of overlapping negative renormalizations is clarified by working in this multiscale approach since, for a given scale assignment $\mathbf{n}$, the set of divergent structures which are dangerous for $\mathbf{n}$ form a forest.

We now generalize the discussion above, as well as make it more concrete. When working with forests $\mathcal{F}$ of divergences, the quantities appearing in (7.1) should be replaced by analogues which are computed "mod $\mathcal{F}$", that is they take the renormalizations of $\mathcal{F}$ into consideration with regards to the notions of internal and external. The forest projections $P^{\mathbf{n}}$ we use will map a forest $\mathcal{F}$ to the subset $\mathcal{S} \subset \mathcal{F}$ which consists of all the elements of $\mathcal{F}$ which are safe for $\mathbf{n}$ when the notions of internal and external are taken "mod $\mathcal{F}$". Defining the projection $P^{\mathbf{n}}$ requires us to introduce some notions.

Fix a subtree $S$ and let $\mathcal{F} \in \mathbb{F}_{\pi}$. We now describe some useful notation. We define the immediate ancestor of $S$ in $\mathcal{F}$ by

$$A_{\mathcal{F}}(S) \stackrel{\text{def}}{=} \begin{cases} T & \text{if } \text{Min}(\{\tilde{T} \in \mathcal{F} : \tilde{T} > S\}) = \{T\}, \\ \overline{T}^* & \text{if } \{\tilde{T} \in \mathcal{F} : \tilde{T} > S\} = \emptyset \end{cases}$$

where we view $\overline{T}^*$ as an undirected multigraph with node set $N^*$ and edge set $\mathcal{E}^{\text{int}}(\overline{T}^*) \stackrel{\text{def}}{=} \mathcal{E}$.

We also define the edge sets

$$\mathcal{E}^{\text{int}}(C_{\mathcal{F}}(S)) \stackrel{\text{def}}{=} \bigsqcup_{T \in C_{\mathcal{F}}(S)} \mathcal{E}^{\text{int}}(T),$$

$$\mathcal{E}_{\mathcal{F}}^{\text{int}} S \stackrel{\text{def}}{=} \mathcal{E}^{\text{int}}(S) \setminus \mathcal{E}^{\text{int}}(C_{\mathcal{F}}(S)),$$

$$\text{and } \mathcal{E}_{\mathcal{F}}^{\text{ext}}(S) \stackrel{\text{def}}{=} \mathcal{E}^{\text{ext}}(S) \cap \mathcal{E}^{\text{int}}(A_{\mathcal{F}}(S)).$$

The definitions generalizing (7.1) are then

$$\begin{aligned} \text{int}_{\mathcal{F}}^{\mathbf{n}}(S) &\stackrel{\text{def}}{=} \min\{n_e : e \in \mathcal{E}_{\mathcal{F}}^{\text{int}} S\}, \\ \text{ext}_{\mathcal{F}}^{\mathbf{n}}(S) &\stackrel{\text{def}}{=} \max\{n_e : e \in \mathcal{E}_{\mathcal{F}}^{\text{ext}}(S)\}. \end{aligned} \tag{7.4}$$

We now introduce the forest projections we will use. For each $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$, we define $P^{\mathbf{n}} : \mathbb{F}_{\pi} \to \mathbb{F}_{\pi}$ by

$$P^{\mathbf{n}}[\mathcal{F}] \stackrel{\text{def}}{=} \{S \in \mathcal{F} : \text{int}_{\mathcal{F}}^{\mathbf{n}}(S) \leq \text{ext}_{\mathcal{F}}^{\mathbf{n}}(S)\}.$$

We now present the main arguments (Lemma 7.1 and Proposition 7.3) showing that $P^{\mathbf{n}}$ is a forest projection – here we are adapting [FMRS85, Lem. 2.2, 2.3].

**Lemma 7.1** *For any* $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$, $\mathcal{F} \in \mathbb{F}_\pi$ *and* $S \in \mathcal{F}$, *one has*

$$\mathrm{int}_{\mathcal{F}}^{\mathbf{n}}(S) = \mathrm{int}_{P^{\mathbf{n}}[\mathcal{F}]}^{\mathbf{n}}(S), \qquad \mathrm{ext}_{\mathcal{F}}^{\mathbf{n}}(S) = \mathrm{ext}_{P^{\mathbf{n}}[\mathcal{F}]}^{\mathbf{n}}(S).$$

*Proof.* Since $P^{\mathbf{n}}[\mathcal{F}] \subset \mathcal{F}$ and since both $\mathcal{E}_{\mathcal{F}}^{\mathrm{int}}$ and $\mathcal{E}_{\mathcal{F}}^{\mathrm{ext}}$ are decreasing in $\mathcal{F}$, it is immediate that one has

$$\mathrm{int}_{\mathcal{F}}^{\mathbf{n}}(S) \geq \mathrm{int}_{P^{\mathbf{n}}[\mathcal{F}]}^{\mathbf{n}}(S), \qquad \mathrm{ext}_{\mathcal{F}}^{\mathbf{n}}(S) \leq \mathrm{ext}_{P^{\mathbf{n}}[\mathcal{F}]}^{\mathbf{n}}(S). \tag{7.5}$$

We start by turning the first inequality of (7.5) into an equality. Fix $S$ and $\mathcal{F}$ and define

$$K \stackrel{\mathrm{def}}{=} \mathcal{E}^{\mathrm{int}}(C_{P^{\mathbf{n}}[\mathcal{F}]}(S)), \qquad \tilde{K} \stackrel{\mathrm{def}}{=} \mathcal{E}^{\mathrm{int}}(C_{\mathcal{F}}(S)).$$

The claim is non-trivial only if $\tilde{K} \neq K$. It then suffices to show that for any $e \in \tilde{K} \setminus K$ there exists $\tilde{e} \in \mathcal{E}^{\mathrm{int}}(S) \setminus \tilde{K}$ with $n_{\tilde{e}} \leq n_e$.

Fix such an $e$. Observe that if $T \in \mathcal{F}$ with $T < S$ and $\mathcal{E}^{\mathrm{int}}(T) \ni e$ then it must be the case that $T \in \mathcal{F} \setminus P^{\mathbf{n}}[\mathcal{F}]$, moreover there is at least one such tree $T$. In particular, there is a (unique) sequence of trees $T_1, \dots, T_{k+1} \in \mathcal{F} \setminus P^{\mathbf{n}}[\mathcal{F}]$, with $k \geq 0$, such that

- $T_1$ is the minimal element of $\mathcal{F}$ with $e \in \mathcal{E}^{\mathrm{int}}(T_1)$,
- for $1 \leq j \leq k$ one has $T_{j+1} = A_{\mathcal{F}}(T_j)$, and $T_{k+1} = S$.

By our assumptions, for $1 \leq j \leq k$ we have

$$\mathrm{int}_{\mathcal{F}}^{\mathbf{n}}(T_j) > \mathrm{ext}_{\mathcal{F}}^{\mathbf{n}}(T_j). \tag{7.6}$$

For each such $j$, fix a choice of $e_j \in \mathcal{E}^{\mathrm{ext}}(T_j) \cap K(T_{j+1})$. Then our desired claim follows by setting $\tilde{e} = e_k$ since this edge belongs to $\mathcal{E}^{\mathrm{int}}(S) \setminus \tilde{K}$ and (7.6) implies

$$n_e > n_{e_1} > n_{e_2} > \cdots > n_{e_k}.$$

We now turn the second inequality of (7.5) into an equality. For this, we start by introducing the shorthand

$$T \stackrel{\mathrm{def}}{=} A_{\mathcal{F}}(S) \quad \text{and} \quad \widetilde{T} \stackrel{\mathrm{def}}{=} A_{P^{\mathbf{n}}[\mathcal{F}]}(S).$$

With this notation, the claim is trivial if $\mathcal{E}^{\mathrm{ext}}(S) \cap (\mathcal{E}^{\mathrm{int}}(\tilde{T}) \setminus \mathcal{E}^{\mathrm{int}}(T))$ is empty so we suppose this is not the case. (In particular, this means that $A_{\mathcal{F}}(S) \neq \overline{T}^*$.)

In a way analogous to above, it remains to show that for any $e \in \mathcal{E}^{\mathrm{ext}}(S) \cap (\mathcal{E}^{\mathrm{int}}(\tilde{T}) \setminus \mathcal{E}^{\mathrm{int}}(T))$, one can find $\tilde{e} \in \mathcal{E}^{\mathrm{ext}}(S) \cap \mathcal{E}^{\mathrm{int}}(T)$ with $n_{\tilde{e}} \geq n_e$. Fixing such an $e$, we define similarly to before the unique sequence of subtrees $T_1, \dots, T_k$ such that

- $T_1 = S$ and, for $1 \leq j \leq k$, $T_{j+1} = A_{\mathcal{F}}(T_j)$.
- $T_k$ is the minimum element of $\mathcal{F}$ with $S \leq T_k$ and $e \in \mathcal{E}^{\mathrm{ext}}(T_k) \cap \mathcal{E}^{\mathrm{int}}(A_{\mathcal{F}}(T_k))$.

One then has $T_j \in \mathcal{F} \setminus P^{\mathbf{n}}[\mathcal{F}]$ for $j = 1, \ldots, k$ so that, in particular, (7.6) holds. We then set $e_k = e$ and, for each for $1 \leq j < k$, we pick some $e_j \in \mathcal{E}^{\text{ext}}(T_j) \cap \mathcal{E}^{\text{int}}(T_{j+1})$ arbitrarily. It then follows again by (7.6) that $n_{e_j} > n_{e_{j+1}}$ so that, by setting $\tilde{e} = e_1$ our claim is proved. $\qquad\square$

**Corollary 7.2** *For any* $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$, $P^{\mathbf{n}} \circ P^{\mathbf{n}} = P^{\mathbf{n}}$.

We can now show that $P^{\mathbf{n}}$ is a forest projection.

**Proposition 7.3** *For any* $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$, $P^{\mathbf{n}}$ *is a forest projection.*

*Proof.* Fix $\mathbf{n}$. By the previous corollary it follows that if for some $\mathcal{S} \in \mathbb{F}_\pi$ the set $(P^{\mathbf{n}})^{-1}[\mathcal{S}]$ is non-empty then $P^{\mathbf{n}}[\mathcal{S}] = \mathcal{S}$. Fix such an $\mathcal{S}$.

Since $P^{\mathbf{n}}[\mathcal{F}] \subset \mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}_\pi$ it follows that $\mathcal{S}$ is the unique minimal element of $(P^{\mathbf{n}})^{-1}[\mathcal{S}]$, so all that is left to show is that this pullback is an interval. Define

$$\mathcal{G} \stackrel{\text{def}}{=} \{T \in \text{Div} : \{T\} \cup \mathcal{S} \in \mathbb{F}_\pi \text{ and } \text{int}^{\mathbf{n}}_{\mathcal{S}}(T) > \text{ext}^{\mathbf{n}}_{\mathcal{S}}(T)\} . \qquad (7.7)$$

Since $P^{\mathbf{n}}[\mathcal{S}] = \mathcal{S}$, it follows from the definitions of $P^{\mathbf{n}}$ and $\mathcal{G}$ that $\mathcal{G} \cap \mathcal{S} = \emptyset$. We claim that one also had $\mathcal{G} \sqcup \mathcal{S} \in \mathbb{F}_\pi$. For this, it suffices to show that for any $S_1, S_2 \in \mathcal{G}$ one has $N(S_1)$ and $N(S_2)$ either nested or disjoint.

Suppose on the contrary that these two sets are neither disjoint nor nested, it follows that $N(S_1) \setminus N(S_2)$, $N(S_2) \setminus N(S_1)$, and $N(S_1) \cap N(S_2)$ are all non-empty. In particular, there must be two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ with $e_1 \in \mathcal{E}^{\text{int}}(S_1) \cap \mathcal{E}^{\text{ext}}(S_2)$ and $e_2 \in \mathcal{E}^{\text{int}}(S_2) \cap \mathcal{E}^{\text{ext}}(S_1)$.

Moreover, the condition that $S_1, S_2$ be compatible with the forest $\mathcal{S}$ implies that $A_{\mathcal{S}}(S_1) = A_{\mathcal{S}}(S_2)$ contains both $S_1$ and $S_2$ and that the node set given by the trees in $C_{\mathcal{S}}(S_1)$ does not intersect the node set of $S_2$, and vice-versa. As a consequence, for $i, j \in \{1, 2\}$, $i \neq j$, one has $e_i \in \mathcal{E}^{\text{int}}(A_{\mathcal{S}}(S_j)) \setminus \mathcal{E}^{\text{int}}(C_{\mathcal{S}}(S_i))$.

Since $\text{ext}^{\mathbf{n}}_{\mathcal{S}}(S_1) < \text{int}^{\mathbf{n}}_{\mathcal{S}}(S_1)$ we must have $n_{e_2} < n_{e_1}$, on the other hand $\text{ext}^{\mathbf{n}}_{\mathcal{S}}(S_2) < \text{int}^{\mathbf{n}}_{\mathcal{S}}(S_2)$ forces the reverse inequality which gives us a contradiction, thus proving our claim that $\mathcal{G} \sqcup \mathcal{S} \in \mathbb{F}_\pi$

We complete the proof by showing that $(P^{\mathbf{n}})^{-1}[\mathcal{S}] = [\mathcal{S}, \mathcal{G} \sqcup \mathcal{S}]$. For this, we first show that for $\mathcal{F}$ such that $\mathcal{S} \subset \mathcal{F} \subset (\mathcal{G} \sqcup \mathcal{S})$, one has $P^{\mathbf{n}}[\mathcal{F}] = \mathcal{S}$. Observe that $P^{\mathbf{n}}[\mathcal{F}] \subset \mathcal{S}$ since for any $T \in (\mathcal{F} \setminus \mathcal{S}) \subset \mathcal{G}$ one has

$$\text{ext}^{\mathbf{n}}_{\mathcal{F}}(T) \leq \text{ext}^{\mathbf{n}}_{\mathcal{S}}(T) < \text{int}^{\mathbf{n}}_{\mathcal{S}}(T) \leq \text{int}^{\mathbf{n}}_{\mathcal{F}}(T).$$

The middle inequality is a consequence of $T \in \mathcal{G}$ and the outer inequalities are a consequence of $\mathcal{S} \subset \mathcal{F}$.

The fact that one also has $\mathcal{S} \subset P^{\mathbf{n}}[\mathcal{F}]$ follows from the chain of inequalities

$$\text{int}^{\mathbf{n}}_{\mathcal{F}}(T) = \text{int}^{\mathbf{n}}_{P^{\mathbf{n}}[\mathcal{F}]}[T] \leq \text{int}^{\mathbf{n}}_{\mathcal{S}}[T] \leq \text{ext}^{\mathbf{n}}_{\mathcal{S}}[T] \leq \text{ext}^{\mathbf{n}}_{P^{\mathbf{n}}[\mathcal{F}]}[T] = \text{ext}^{\mathbf{n}}_{\mathcal{F}}[T]$$

which hold for arbitrary $T \in \mathcal{S}$. Lemma 7.1 gives the outermost equalities. Working inward, the next two inequalities follow from $P^{\mathbf{n}}[\mathcal{F}] \subset \mathcal{S}$ and the center inequality is a consequence of $P^{\mathbf{n}}[\mathcal{S}] = \mathcal{S}$.

All that is left is proving that $\mathcal{G} \sqcup \mathcal{S}$ is the unique maximal element of $(P^{\mathbf{n}})^{-1}[\mathcal{S}]$, namely that if $P^{\mathbf{n}}[\mathcal{F}] = \mathcal{S}$ then $(\mathcal{F} \setminus \mathcal{S}) \subset \mathcal{G}$. For any $T \in \mathcal{F} \setminus \mathcal{S}$ we have indeed

$$\mathrm{ext}_{\mathcal{S}}^{\mathbf{n}}[T] = \mathrm{ext}_{\mathcal{F}}^{\mathbf{n}}[T] < \mathrm{int}_{\mathcal{F}}^{\mathbf{n}}[T] = \mathrm{int}_{\mathcal{S}}^{\mathbf{n}}[T]$$

where the outer equalities are given by Lemma 7.1 and the middle inequality follows from assuming $T \in \mathcal{F} \setminus P^{\mathbf{n}}[\mathcal{F}]$. □

We say a subset $\tilde{\mathcal{E}} \subset \mathcal{E}$ connects $u, v \in N^*$ if one can find a sequence of $e_1, \ldots, e_k \in \mathcal{E}$ with $u \in e_1$, $v \in e_k$ and $e_j \cap e_{j+1} \neq \emptyset$ for $1 \leq j \leq k-1$.

Then given $u, v \in N^*$, $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$, and $\mathcal{F} \in \mathbb{F}_\pi$, we define

$$\mathbf{n}_{\mathcal{F}}(u,v) \stackrel{\mathrm{def}}{=} \max \left\{ \min\{n_e : e \in \mathcal{E}' \setminus \mathcal{E}^{\mathrm{int}}(\mathcal{F})\} : \mathcal{E}' \subset \mathcal{E} \text{ connects } u, v \right\}. \quad (7.8)$$

We now define, for each $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$, by setting, for each $\mathcal{F} \in \mathbb{F}_\pi$,

$$\mathscr{G}^{\mathbf{n}}(\mathcal{F}) \stackrel{\mathrm{def}}{=} \{e \in \mathfrak{C} : \mathbf{n}_{\mathcal{F}}(\circledast, e_{\mathrm{p}}) > \mathbf{n}_{\mathcal{F}}(e_{\mathrm{p}}, e_{\mathrm{c}})\}. \quad (7.9)$$

These represent precisely those edges for which there is a cancellation between the term $\mathrm{Ker}_{\mathfrak{e}}^{\{e\}}$ and its Taylor expansion $\mathrm{RKer}_{\mathfrak{e}}^{\{e\}}$ appearing in (6.2). We also introduce the shorthand $\mathfrak{M}^{\mathbf{n}} \stackrel{\mathrm{def}}{=} \mathfrak{M}^{P^{\mathbf{n}}}$, $\mathfrak{G}^{\mathbf{n}}(\cdot) \stackrel{\mathrm{def}}{=} \mathfrak{G}_{\mathscr{G}^{\mathbf{n}}}^{P^{\mathbf{n}}}(\cdot)$, and $\mathfrak{C}^{\mathbf{n}}(\cdot) \stackrel{\mathrm{def}}{=} \mathfrak{C}^{P^{\mathbf{n}}}(\cdot)$. We can now state our final combinatorial result.

**Proposition 7.4** *For any $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$ the forest projection $P^{\mathbf{n}}$ is compatible with the cut rule $\mathscr{G}^{\mathbf{n}}$ in the sense of Definition 5.7.*

*Proof.* Fix $\mathbf{n} \in \mathbf{N}^{\mathcal{E}}$ and $\mathbb{M} \in \mathfrak{M}^{\mathbf{n}}$. To keep notation light, for arbitrary $\mathscr{D} \subset \mathfrak{C}$ we write similarly to before $P_{\mathscr{D}}^{-1}[\cdot]$ to denote $(P_{\mathscr{D}}^{\mathbf{n}})^{-1}[\cdot] \cap \mathbb{F}_{\mathscr{D},\pi}$.

First we prove that for any $e \in \mathscr{C}_{\mathbb{M}}$ and any $\mathscr{C} \subset \mathfrak{C}$

$$\mathscr{C} \setminus \{e\} \in \mathfrak{C}^{\mathbf{n}}(\mathbb{M}) \quad \Rightarrow \quad \mathscr{C} \cup \{e\} \in \mathfrak{C}^{\mathbf{n}}(\mathbb{M}).$$

Fix $e \in \mathscr{C}_{\mathbb{M}}$ and let $\mathscr{C} \in \mathfrak{C}^{\mathbf{n}}(\mathbb{M})$ with $\mathscr{C} \not\ni e$. The fact that $\mathscr{C} \sqcup \{e\} \in \mathfrak{C}^{\mathbf{n}}(\mathbb{M})$ follows from the observation that, since $\mathscr{C} \subset \mathscr{C}_{\mathbb{M}}$, one has the inclusions

$$P_{\mathscr{C}_{\mathbb{M}}}^{-1}[s(\mathbb{M})] \subset P_{\mathscr{C} \sqcup \{e\}}^{-1}[s(\mathbb{M})] \subset P_{\mathscr{C}}^{-1}[s(\mathbb{M})],$$

where the outermost sets are both equal to $\mathbb{M}$.

We now turn to the proof of the converse statement. Fix $e \in \mathscr{C}_{\mathbb{M}}$ and suppose that $\widetilde{\mathscr{C}} \subset \mathscr{C}_{\mathbb{M}}$, $\widetilde{\mathscr{C}} \ni e$, and $\widetilde{\mathscr{C}} \in \mathfrak{C}^{\mathbf{n}}(\mathbb{M})$. We now additionally assume that $\widetilde{\mathscr{C}} \setminus \{e\} \notin \mathfrak{C}^{\mathbf{n}}(\mathbb{M})$ and show that this forces $e \notin \mathscr{G}^{\mathbf{n}}(b(\mathbb{M}))$, thus establishing the claim. Our assumptions imply

$$b(P_{\widetilde{\mathscr{C}} \setminus \{e\}}^{-1}[s(\mathbb{M})]) \supsetneq b(P_{\widetilde{\mathscr{C}}}^{-1}[s(\mathbb{M})]).$$

For any element $T$ of the left hand side which is not an element of the right hand side, it must be the case that $e \in \mathcal{E}^{\mathrm{int}}(T)$ and $T \not\leq S$ for any $S \in b(\mathbb{M})$ (in particular, $T \notin s(\mathbb{M})$). It follows that

$$\mathbf{n}_{b(\mathbb{M})}(\circledast, e_{\mathrm{p}}) \leq \mathrm{ext}_{s(\mathbb{M})}^{\mathbf{n}}(T) < \mathrm{int}_{s(\mathbb{M})}^{\mathbf{n}}(T) \leq \mathbf{n}_{b(\mathbb{M})}(e_{\mathrm{c}}, e_{\mathrm{p}}),$$

as a consequence of the fact that $T \notin P^{\mathbf{n}}[s(\mathbb{M})]$ and $e \notin \mathcal{E}^{\mathrm{int}}(S)$ for all $S \in s(\mathbb{M})$. It follows that indeed $e \notin \mathscr{G}^{\mathbf{n}}(b(\mathbb{M}))$ as announced. □

## 8 Summing over scales

### 8.1 Interchanging the sum over scales and intervals

In Lemma 6.3 we showed, in an abstract sense, how a sum over forests and cuts could be reorganized into a smaller sum over intervals of forests $\mathbb{M}$ and intervals of cuts $\mathbb{G}$. In the previous section this was made more concrete, we specified an algorithm which, given a scale assignment $\mathbf{n}$, specifies a particular forest projection $P^{\mathbf{n}}$ and compatible cut rule $\mathscr{G}^{\mathbf{n}}(\cdot)$ for which we can apply Lemma 6.3.

We now reorganize our sums again, first summing over pairs $(\mathbb{M}, \mathbb{G})$ of intervals of forests and cuttings and then summing over the scale assignments $\mathbf{n}$ which allow a given pair of intervals to arise from from $P^{\mathbf{n}}$ and $\mathscr{G}^{\mathbf{n}}$. To this end we define

$$\mathfrak{R} \stackrel{\text{def}}{=} \left\{ (\mathbb{M}, \mathbb{G}) \in 2^{\mathbb{F}_\pi} \times 2^{2^{\mathfrak{C}}} \; : \; \exists \mathbf{n} \in \mathbf{N}^{\mathcal{E}} \text{ such that } \mathbb{M} \in \mathfrak{M}^{\mathbf{n}} \text{ and } \mathbb{G} \in \mathfrak{G}^{\mathbf{n}}(\mathbb{M}) \right\}$$

and, for any $(\mathbb{M}, \mathbb{G}) \in \mathfrak{R}$ and $\lambda \in (0, 1]$, we set

$$\mathcal{N}_{\mathbb{M}, \mathbb{G}, \lambda} \stackrel{\text{def}}{=} \left\{ \mathbf{n} \in \mathbf{N}^{\mathcal{E}} \; : \; \mathbb{M} \in \mathfrak{M}^{\mathbf{n}}, \; \mathbb{G} \in \mathfrak{G}^{\mathbf{n}}(\mathbb{M}), \; \text{ and } n_{\{\oplus, \varrho*\}} \geq \lfloor -\log_2(\lambda) \rfloor \right\} .$$

We then have the following lemma.

**Lemma 8.1** *For any $\lambda \in (0, 1]$, one has*

$$\mathcal{W}_{\bar{L}}^{\pi} = \sum_{(\mathbb{M}, \mathbb{G}) \in \mathfrak{R}} \sum_{\mathbf{n} \in \mathcal{N}_{\mathbb{M}, \mathbb{G}, \lambda}} \mathcal{W}_{\lambda}^{\mathbf{n}}[\mathbb{M}, \mathbb{G}]$$

*where the LHS is defined as in* (5.1).

*Proof.* By Lemma 6.3 and Proposition 5.8, and furthermore freely interchanging finite sums with infinite ones,

$$\mathcal{W}_{\bar{L}}^{\pi} = \sum_{\mathbf{n} \in \mathbf{N}^{\mathcal{E}}} \sum_{\substack{\mathcal{F} \in \mathbb{F}_\pi \\ \mathscr{C} \subset \mathfrak{C}_{\mathcal{F}}}} \mathcal{W}_{\lambda}^{\mathbf{n}}[\{\mathcal{F}\}, \{\mathscr{C}\}] = \sum_{\mathbf{n} \in \mathbf{N}^{\mathcal{E}}} \sum_{\substack{\mathbb{M} \in \mathfrak{M}^{\mathbf{n}} \\ \mathbb{G} \in \mathfrak{G}^{\mathbf{n}}(\mathbb{M})}} \mathcal{W}_{\lambda}^{\mathbf{n}}[\mathbb{M}, \mathbb{G}]$$

$$= \sum_{\mathbf{n} \in \mathbf{N}^{\mathcal{E}}} \sum_{(\mathbb{M}, \mathbb{G}) \in \mathfrak{R}} \mathbb{1}\left\{ \begin{matrix} \mathbb{M} \in \mathfrak{M}^{\mathbf{n}}, \\ \mathbb{G} \in \mathfrak{G}^{\mathbf{n}}(\mathbb{M}) \end{matrix} \right\} \mathcal{W}_{\lambda}^{\mathbf{n}}[\mathbb{M}, \mathbb{G}]$$

$$= \sum_{(\mathbb{M}, \mathbb{G}) \in \mathfrak{R}} \sum_{\mathbf{n} \in \mathbf{N}^{\mathcal{E}}} \mathbb{1}\left\{ \begin{matrix} \mathbb{M} \in \mathfrak{M}^{\mathbf{n}}, \\ \mathbb{G} \in \mathfrak{G}^{\mathbf{n}}(\mathbb{M}) \end{matrix} \right\} \mathcal{W}_{\lambda}^{\mathbf{n}}[\mathbb{M}, \mathbb{G}] = \sum_{(\mathbb{M}, \mathbb{G}) \in \mathfrak{R}} \sum_{\mathbf{n} \in \mathcal{N}_{\mathbb{M}, \mathbb{G}, \lambda}} \mathcal{W}_{\lambda}^{\mathbf{n}}[\mathbb{M}, \mathbb{G}].$$

Note that the constraint that $n_{\{\oplus, \varrho*\}} \geq \lfloor -\log_2(\lambda) \rfloor$ comes for free since the presence of $\psi^{\lambda}$ makes $\mathcal{W}_{\lambda}^{\mathbf{n}}[\cdot, \cdot]$ vanishes if this is not the case. $\qquad\square$

The goal of the subsequent sections is to prove the following theorem.

**Proposition 8.2** *For any* $(\mathbb{M}, \mathbb{G}) \in \mathfrak{R}$, *and any* $p \in \mathbf{N}$ *there exists* $C_p$ *such that, uniform in* $\xi \in \mathcal{M}(\Omega_\infty)$ *and* $x_\oplus \in \mathbf{R}^d$, *one has*

$$\mathbf{E}\Big[\Big(\sum_{\mathbf{n} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda}} \int_{\tilde{L} \cup \{\varrho_*\}} dx \, \mathcal{W}_\lambda^{\mathbf{n}}[\mathbb{M}, \mathbb{G}](x) \operatorname{Wick}\big(\{\xi_{\mathfrak{t}(u)}(x_u)\}_{u \in \tilde{L}}\big)\Big)^{2p}\Big] \quad (8.1)$$

$$\leq C_p \Big(\prod_{e \in K(\overline{T})} \|K_{\mathfrak{t}(e)}\|_{|\mathfrak{t}(e)|_{\mathfrak{s}}, m}\Big)^{2p} \|\xi\|_{j,\mathfrak{c}} \lambda^{2p\alpha}$$

*uniform in* $\lambda \in (0,1]$. *Here* $m \overset{\text{def}}{=} 2|\mathfrak{s}| \cdot |N(\overline{T})|$, $j \overset{\text{def}}{=} 2p|L(\overline{T})|$, *and* $\alpha = |\overline{T}_{\overline{\mathfrak{c}}}^{\overline{\mathbf{n}}}|_{\mathfrak{s}}$.

We pick an arbitrary $(\mathbb{M}, \mathbb{G}) \in \mathfrak{R}$ and treat it as fixed for the remainder of the paper, writing $\mathcal{W}_\lambda^{\mathbf{n}}$ instead of $\mathcal{W}_\lambda^{\mathbf{n}}[\mathbb{M}, \mathbb{G}]$. We also fix the shorthands $\mathcal{S} \overset{\text{def}}{=} s(\mathbb{M})$, $\mathcal{B} \overset{\text{def}}{=} b(\mathbb{M})$, $\mathcal{D} \overset{\text{def}}{=} \delta(\mathbb{M})$, $\mathscr{S} \overset{\text{def}}{=} s(\mathbb{G})$, $\mathscr{B} \overset{\text{def}}{=} b(\mathbb{G})$, *and* $\mathscr{D} \overset{\text{def}}{=} \delta(\mathbb{G})$.

## 8.2   Summing over scales inductively

In order to efficiently prove a theorem that applies for any value of depth$(\mathcal{B})$ it is natural to formulate our analytic scheme in a way that operates inductively with respect to this quantity.

To that end we will decompose the single sum over global scale assignments $\mathcal{N}_{\mathbb{M},\mathbb{G},\lambda}$ into a family of sums which facilitate summing the scales internal to single $T \in \mathcal{B}$ conditioned on the values of relevant external scales (these two sets of quantities being dependent through the requirement that $T \in \mathcal{S}$ or $T \in \mathcal{D}$). In what follows, for any $\tilde{\mathcal{E}} \subset \mathcal{E}$ and $\mathbf{j} \in \mathbf{N}^{\mathcal{E}}$ we write $\mathbf{j}{\restriction}\mathcal{E}' \in \mathbf{N}^{\mathcal{E}'}$ for the restriction of $\mathbf{j}$ to the edges in $\mathcal{E}'$.

The most external set of edges is given by

$$\mathcal{E}_{\mathcal{B}}^{\text{int}}(\overline{T}^*) \overset{\text{def}}{=} (K^{\downarrow}(\overline{\mathcal{B}}) \cup K(\mathcal{B}, \overline{T})) \sqcup \{e \in \mathcal{E}_\pi : e \subset L(\mathcal{B}, \overline{T})\} \sqcup \mathcal{E}_\oplus. \quad (8.2)$$

We now introduce some sets of "partial" scale assignments, writing

$$\partial \mathcal{N}_{\mathcal{B},\lambda} \overset{\text{def}}{=} \{\mathbf{k} \in \mathbf{N}^{\mathcal{E}_{\mathcal{B}}^{\text{int}}(\overline{T}^*)} : \exists \mathbf{j} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda} \text{ with } \mathbf{j}{\restriction}\mathcal{E}_{\mathcal{B}}^{\text{int}}(\overline{T}^*) = \mathbf{k}\}$$

and, for any $S \in \mathcal{B}$, $\mathcal{E}' \subset \mathcal{E}$ with $\mathcal{E}' \supset \mathcal{E}_{\mathcal{B}}^{\text{ext}}(S)$, and $\mathbf{j} \in \mathbf{N}^{\mathcal{E}'}$, setting

$$\mathring{\mathcal{N}}_S(\mathbf{j}) \overset{\text{def}}{=} \Big\{\mathbf{k} \in \mathbf{N}^{\mathcal{E}_{\mathcal{B}}^{\text{int}}(S)} : \exists \tilde{\mathbf{j}} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda} \text{ with } \tilde{\mathbf{j}}{\restriction}\mathcal{E}' = \mathbf{j} \text{ and } \tilde{\mathbf{j}}{\restriction}\mathcal{E}_{\mathcal{B}}^{\text{int}}(S) = \mathbf{k}\Big\}. \quad (8.3)$$

Note that for every $\mathbf{k} \in \mathring{\mathcal{N}}_S(\mathbf{j})$ one then has

$$\operatorname{int}_{\mathcal{B}}^{\mathbf{k}}(S) \leq \operatorname{ext}_{\mathcal{B}}^{\mathbf{j}}(S) \qquad \text{if } S \in \mathcal{S},$$
$$\operatorname{int}_{\mathcal{B}}^{\mathbf{k}}(S) > \operatorname{ext}_{\mathcal{B}}^{\mathbf{j}}(S) \qquad \text{if } S \in \mathcal{D}.$$

The following lemma verifies that the set of scale assignments $\mathring{\mathcal{N}}_S(\mathbf{j})$ really only depends on $\mathbf{j}$'s values on the edges of of $\mathcal{E}_{\mathcal{B}}^{\text{ext}}(S)$.

**Lemma 8.3** *For any $S \in \mathcal{B}$ and $\mathbf{j} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda}$ one has*

$$\mathring{\mathcal{N}}_S(\mathbf{j}{\restriction}\mathcal{E}_{\mathcal{B}}^{\mathrm{ext}}(S)) = \mathring{\mathcal{N}}_S(\mathbf{j}). \qquad (8.4)$$

*Proof.* Clearly the RHS of (8.4) is contained on the LHS, we now prove the reverse inclusion. Fix $\mathbf{k} \in \mathring{\mathcal{N}}_S(\mathbf{j}{\restriction}\mathcal{E}_{\mathcal{B}}^{\mathrm{ext}}(S))$. We are guaranteed the existence of corresponding $\tilde{\mathbf{j}} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda}$ with $\tilde{\mathbf{j}}{\restriction}\mathcal{E}_{\mathcal{B}}^{\mathrm{ext}}(S) = \mathbf{j}{\restriction}\mathcal{E}_{\mathcal{B}}^{\mathrm{ext}}(S)$ and $\tilde{\mathbf{j}}{\restriction}\mathcal{E}_{\mathcal{B}}^{\mathrm{int}}(S) = \mathbf{k}$. We then define $\hat{\mathbf{j}} \in \mathbf{N}^{\mathcal{E}}$ by setting $\hat{\mathbf{j}} = (\tilde{\mathbf{j}}{\restriction}\mathcal{E}^{\mathrm{int}}(S)) \sqcup (\mathbf{j}{\restriction}\mathcal{E} \setminus \mathcal{E}^{\mathrm{int}}(S))$. To finish our proof we need to show that $\hat{\mathbf{j}} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda}$.

To prove that $\mathbb{M} \in \mathfrak{M}^{\hat{\mathbf{j}}}$ it suffices to show that $P^{\hat{\mathbf{j}}}[\mathcal{B}] = \mathcal{S}$ and that for any $\mathscr{C} \in \mathbb{G}$ and any $T \in \mathrm{Div}_\pi \setminus \mathcal{B}$, which is both compatible with $\mathcal{B}$ and does not contain any edge of $\mathscr{C}$, one has $\mathrm{int}_{\mathcal{B}}^{\hat{\mathbf{j}}}(T) \leq \mathrm{ext}_{\mathcal{B}}^{\hat{\mathbf{j}}}(T)$. These two statements can together be rewritten as the claim that for every $T \in \mathrm{Div}_\pi$ compatible with $\mathcal{B}$ and disjoint from $\mathscr{C}$, one has

$$\begin{cases} \mathrm{int}_{\mathcal{B}}^{\hat{\mathbf{j}}}(T) \leq \mathrm{ext}_{\mathcal{B}}^{\hat{\mathbf{j}}}(T) & \text{if } T \notin \mathcal{D} \\ \mathrm{int}_{\mathcal{B}}^{\hat{\mathbf{j}}}(T) > \mathrm{ext}_{\mathcal{B}}^{\hat{\mathbf{j}}}(T) & \text{if } T \in \mathcal{D}. \end{cases}$$

This claim can be checked in the following four cases: (i) $T$ is disjoint from all elements of $\mathcal{F}$, (ii) $T$ properly contains at least one element of $\mathcal{F}$, (iii) $T$ is properly contained in an element of $\mathcal{F}$, or (iv) $T$ is an element of $\mathcal{F}$.

In the first two cases our claim follows from the fact that both $\mathcal{E}_{\mathcal{B}}^{\mathrm{int}}(T)$ and $\mathcal{E}_{\mathcal{B}}^{\mathrm{ext}}(T)$ consist of edges where $\hat{\mathbf{j}}$ is determined by $\mathbf{j} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda}$, while for the last two cases it is because $\hat{\mathbf{j}}$ is determined by $\tilde{\mathbf{j}} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda}$ on these edges.

Clearly all edges whose scale assignments are involved determining $\mathscr{G}^{\hat{\mathbf{j}}}(\overline{\mathcal{B}})$ are edges where $\hat{\mathbf{j}}$ is determined by $\mathbf{j}$, so it follows that $\mathbb{G} \in \mathfrak{G}^{\hat{\mathbf{j}}}(\mathbb{M})$. Finally, we also have $\hat{j}_{\{\otimes,\varrho_*\}} = j_{\{\otimes,\varrho_*\}} \geq \lfloor -\log_2(\lambda) \rfloor$. This shows that $\hat{\mathbf{j}} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda}$ as desired. $\qquad \square$

We immediately have the following corollary.

**Corollary 8.4** *For any $F : \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda} :\to \mathbf{R}$ one has*

$$\sum_{\mathbf{n} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda}} F(\mathbf{n}) = \sum_{\mathbf{n}^{(0)} \in A_0} \sum_{\mathbf{n}^{(1)} \in A_1(\mathbf{n}^{(0)})} \cdots \sum_{\mathbf{n}^{(j)} \in A_j(\mathbf{n}^{(j-1)})} F(\mathbf{n}^{(0)} \sqcup \mathbf{n}^{(1)} \sqcup \cdots \sqcup \mathbf{n}^{(j)})$$

*where $j \overset{def}{=} \mathrm{depth}(\mathcal{B})$, $A_0 \overset{def}{=} \partial\mathcal{N}_{\mathcal{B},\lambda}$, and for each $1 \leq i \leq j$ we inductively set, for each $\mathbf{n}^{(i-1)} \in A_{i-1}$,*

$$A_i \overset{def}{=} \underset{S \in D_i(\mathcal{B})}{\bigtimes} \mathcal{N}_S(\mathbf{n}^{(i-1)}).$$

Here, we implicitly make the identifications $(\mathbf{k}_1, \ldots, \mathbf{k}_j) \sim \mathbf{k}_1 \sqcup \ldots \sqcup \mathbf{k}_j$ for scale assignments $\mathbf{k}_i$ that involve disjoint sets of edges.

We now inductively define a family of operators $\hat{H}_S^{\mathbf{j}} : \mathscr{C}_* \to \mathscr{C}_*$, where $S \in \mathcal{B}$ and $\mathbf{j} \in \mathbf{N}^{\mathcal{E}'}$ with $\mathcal{E}' \supset \mathcal{E}_{\mathcal{B}}^{\mathrm{ext}}(S)$ by setting

$$
[\hat{H}_S^{\mathbf{j}}\varphi](x) \stackrel{\mathrm{def}}{=} \sum_{\mathbf{k} \in \mathring{\mathcal{N}}_S(\mathbf{j})} \int_{\tilde{N}_{\mathcal{B}}(S)} dy \, \mathrm{Cu}_{\pi,\mathbf{k}}^{L_{\mathcal{B}}(S)}(y) \, \mathrm{Ker}_{0,\mathbf{k}}^{\mathring{K}_{\mathcal{B}}(S)}(y \sqcup x_{\varrho_S})
$$
$$
\times \hat{H}_{C_{\mathcal{B}}(S)}^{\mathbf{k}}\Big[\mathrm{Ker}_{0,\mathbf{k}}^{K_{\mathcal{B}}^{\partial}(S)}[\mathscr{Y}_{S,\mathbb{M}}^{\#}\varphi]\Big](x_{\tilde{N}(S)^c} \sqcup y) ,
\tag{8.5}
$$

with the base case of the induction given by setting $\hat{H}_{\emptyset}^{\mathbf{j}}$ to be the identity operator. The operators $\hat{H}_S^{\mathbf{j}}$ are partially summed analogues of the operators $H_{\mathbb{M},S}^{\cdot}$ which perform the summation of scales inside $S$ but outside $C_{\mathcal{B}}(S)$ (note that this operator contains in its definition all expressions within $\mathcal{W}_\lambda^{\bullet}$ which depend on these scales).

We define, for each $\mathbf{j} \in \partial\mathcal{N}_{\mathcal{B},\lambda}$, a function $\mathring{\mathcal{W}}_\lambda^{\mathbf{j}} \in \mathscr{C}_{\tilde{N}(\overline{\mathcal{B}})^c}$ via

$$
\mathring{\mathcal{W}}_\lambda^{\mathbf{j}} \stackrel{\mathrm{def}}{=} \psi^\lambda \cdot \mathrm{Ker}_{0,\mathbf{j}}^{K(\mathcal{B},\overline{T})\backslash\mathcal{B}} \mathrm{Cu}_{\pi,\mathbf{j}}^{L(\mathcal{B},\overline{T})} \widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\mathcal{B}\backslash K^{\downarrow}(\overline{\mathcal{B}})} X_{\overline{\mathbf{n}},\otimes,\mathbf{j}}^{\tilde{N}(\mathcal{B},\overline{T})}
$$
$$
\cdot \prod_{S \in \overline{\mathcal{B}}} \hat{H}_S^{\mathbf{j}}\Big[\widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\mathcal{B}\cap K^{\downarrow}(S)} \mathrm{Ker}_{0,\mathbf{j}}^{K^{\downarrow}(S)\backslash\mathcal{B}} X_{\overline{\mathbf{n}},\otimes,\mathbf{j}}^{\tilde{N}(S)}\Big] ,
\tag{8.6}
$$

where for any $\mathscr{C} \subset \mathfrak{C}$ and $\mathbf{j} \in \partial\mathcal{N}_{\mathcal{B},\lambda}$ we set

$$
\widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\mathscr{C}} \stackrel{\mathrm{def}}{=} \mathrm{RKer}_{0,\mathbf{j}}^{\mathscr{C}\cap\mathscr{S}} \cdot \widehat{\mathrm{RKer}}_{0,\mathbf{j}}^{\mathscr{C}\cap\mathscr{D}} .
$$

Lemma 8.3 and Corollary 8.4 together give the following lemma.

**Lemma 8.5**

$$
\sum_{\mathbf{j} \in \partial\mathcal{N}_{\mathcal{B},\lambda}} \int_{\tilde{N}(\mathcal{B},\overline{T})\backslash\tilde{L}} dy \, \mathring{\mathcal{W}}_\lambda^{\mathbf{j}}(x \sqcup y) = \sum_{\mathbf{n} \in \mathcal{N}_{\mathbb{M},\mathbb{G},\lambda}} \mathcal{W}_\lambda^{\mathbf{n}}(x) .
\tag{8.7}
$$

### 8.3 Estimates on renormalization

#### 8.3.1 More notational preliminaries

In what follows we will frequently use the generalized Taylor remainder estimate of [Hai14, Prop. A.1]. In view of this, it is natural to define, for any set of multi-indices $A$ and subset $N \subset N^*$, the set of multi-indices

$$
\partial_N A \stackrel{\mathrm{def}}{=} \Big\{ k \in (\mathbf{N}^d)^{N^*} \setminus A : \exists j \in (\mathbf{N}^d)^N \text{ with } \|j\| = 1, \, k - j \in A \Big\} .
$$

Note that if $N = \{u\}$ we sometimes write $\partial_u A$ instead of $\partial_{\{u\}} A$.

We now introduce some notation for the renormalization of second cumulants. First we set $R(\pi) \stackrel{\mathrm{def}}{=} \{B \in \pi : \ell(B) > 0\}$ where $\ell(\cdot)$ is defined in (A.20).

The second cumulants corresponding to $B \in R(\pi)$ will be renormalized in a manner similar to the divergent subtrees – in particular one needs to choose a distinguished vertex for each such $B$ which serves the same rule that $\varrho_S$ does for

$S \in \mathrm{Div}$. To that end for each $B \in R(\pi)$ we label the elements of $B$ with either a $+$ or $-$, so for each such $B$ we can write $B = \{(+, B), (-, B)\}$ – this labeling is arbitrary except for one constraint: we require that for every $S \in \mathcal{B}$ and every $B \in L_{\mathcal{B}}(S)$ one has $(+, B) \neq \varrho_S$. One should think of $(-, B)$ as serving as the "root" of $B$ for renormalization.

Moving forward we take these labelling as fixed. We then define, for every $B \in R(\pi)$, an operator $\mathscr{Y}_B : \mathscr{C}_* \to \mathscr{C}_*$ via setting, for each $\varphi \in \mathscr{C}_*$,

$$[\mathscr{Y}_B \varphi](z) \stackrel{\text{def}}{=} \sum_{b \in \mathrm{Der}(B)} (D^b \varphi)(\mathrm{Coll}_B(z))(z_{(+,B)} - z_{(-,B)})^b , \qquad (8.8)$$

where

$$\mathrm{Coll}_B(z)_u \stackrel{\text{def}}{=} \begin{cases} z_{(-,B)} & \text{if } u \in B, \\ z_u & \text{otherwise,} \end{cases} \qquad (8.9)$$

and $\mathrm{Der}(B)$ is the set of all multi-indices $b$ supported on $(+, B)$ and satisfying $|b|_{\mathfrak{s}} < \mathfrak{f}(B)$. We also set $\overline{\mathrm{Der}}(B) \stackrel{\text{def}}{=} \mathrm{Der}(B) \sqcup \partial_{(+,B)}\mathrm{Der}(B)$ and for any subset $A \subset R(\pi)$ we set

$$\overline{\mathrm{Der}}(A) \stackrel{\text{def}}{=} \left\{ \sum_{B \in A} m_B : m_B \in \overline{\mathrm{Der}}(B) \right\} . \qquad (8.10)$$

Finally, for any $S \in \mathcal{B}$ we define

$$\overline{\mathrm{Der}}(\pi, S) \stackrel{\text{def}}{=} \left\{ \sum_{\substack{B \in R(\pi) \\ B \subset L_{\mathcal{B}}(S)}} k_B : k_B \in \overline{\mathrm{Der}}(B) \right\} . $$

This finishes our additional notation for renormalizing second cumulants, we now introduce other useful shorthands. We define a map $\mathfrak{h} : K(\overline{T}) \to \mathbf{R}$ by setting

$$\mathfrak{h}(e) \stackrel{\text{def}}{=} |\mathfrak{s}| - |\mathfrak{t}(e)|_{\mathfrak{s}} + |\overline{\mathfrak{e}}(e)|_{\mathfrak{s}} . \qquad (8.11)$$

The mnemonic here is that $\mathfrak{h}$ stands for homogeneity, the kernel $\mathrm{Ker}_0^{\{e\}}$ blows up like $|x_{e_{\mathrm{p}}} - x_{e_{\mathrm{c}}}|^{-\mathfrak{h}(e)}$ as $|x_{e_{\mathrm{p}}} - x_{e_{\mathrm{c}}}| \to 0$.

We also introduce notation for various domain constraints. For $z, w \in \mathbf{R}^d$ and $t \in \mathbf{R}$ we write $z \stackrel{t}{\longleftrightarrow} w$ for the condition

$$C^{-1} 2^{-t} \leq |z - w| \leq C 2^{-t} \qquad (8.12)$$

write $z \stackrel{t}{\longleftrightarrow\!\!\!\!\rightsquigarrow} w$ for the condition

$$|z - w| \leq C 2^{-t} . \qquad (8.13)$$

In both (8.12) and (8.13) one chooses a fixed value $C > 0$ (not dependent on $t$).

**Remark 8.6** Note that from line to line the constant $C$ implicit in the notations (8.12) and (8.13) may change but remains suppressed from the notation.

In particular, when the notations (8.12) and/or (8.13) appear in the assumptions of a lemma or proposition one is allowed to choose any value(s) of $C$. Any proportionality constant hidden in the notation $\lesssim$ appearing in the conclusion may then depend on the choices of constants $C$ made in the assumption. Moreover, if there is another use of the notations (8.12) and/or (8.13) in the conclusion of the lemma or proposition, then implicit constants for these conditions cannot be chosen arbitrarily, but have to be taken sufficiently large in a way that may depend on the constants chosen for the analogous expression in the assumptions.

In the end, all of these constants influence the overall factor $C_{\tau,p}$ in (2.31).

### 8.3.2    Inductive estimates for negative renormalizations

After renormalizing everything in $\mathcal{F} \subset \mathcal{B}$, the sets $\tilde{N}(\mathcal{F})^c$ and $\tilde{N}(\mathcal{F})$ are the sets of free and integrated out variables of our integrand, respectively. Since the forest $\mathcal{B}$ is now fixed once and for all, we also write $\mathrm{int}^\bullet$ and $\mathrm{ext}^\bullet$ instead of $\mathrm{int}_{\mathcal{B}}^\bullet$ and $\mathrm{ext}_{\mathcal{B}}^\bullet$ and, for any $T \in \mathrm{Div}$, we define $\bar{\omega}(T) \stackrel{\text{def}}{=} \lfloor \omega(T) \rfloor < \omega(T)$.

The goal in this subsection is to estimate, for any forest $\mathcal{F} \subset \mathcal{B}$ of depth 1 and any scale assignment $\mathbf{j} \in \mathbf{N}^{\mathcal{E}'}$ with $\mathcal{E}' \supset \mathcal{E}_{\mathcal{B}}^{\mathrm{ext}}(\mathcal{F})$, quantities of the form $\hat{H}_{\mathcal{F}}^{\mathbf{j}}(\varphi)$. In order to facilitate this, we define a family of seminorms $\| \cdot \|_{\mathcal{F},\mathbf{j}}$ on the functions of $\mathscr{C}_*$. These seminorms control the derivatives of $\varphi$ that are generated when one renormalizes the trees $\mathcal{F}$, as well as the derivatives generated by the renormalization of the sub-divergences within these trees. To that end, for any forest $\mathcal{F} \subset \mathcal{B}$ with $\mathrm{depth}(\mathcal{F}) \leq 1$, we define

$$\widetilde{\mathrm{Der}}(\mathcal{F}) \stackrel{\text{def}}{=} \left\{ \sum_{S \in \mathcal{G}} k_S + \sum_{\substack{B \in R(\pi),\ B \subset L(\mathcal{F}) \\ B \not\subset L(\mathcal{G})}} k_B : \begin{array}{l} \mathcal{G} \subset \mathcal{B} \text{ with } \mathcal{G} \in \mathbb{F}_{\leq}[\mathcal{F}] \\ k_S \in \overline{\mathrm{Der}}(S),\ k_B \in \overline{\mathrm{Der}}(B) \end{array} \right\},$$

(8.14)

where $\mathbb{F}_{\leq}[\mathcal{F}]$ is defined as in (4.2) and for any tree in $T \in \mathrm{Div}$, we set

$$\overline{\mathrm{Der}}(T) \stackrel{\text{def}}{=} \mathrm{Der}(T) \cup \partial_{\tilde{N}(T)} \mathrm{Der}(T)$$

where the notation $\mathrm{Der}(T)$ was introduced in Definition 4.11.

The control over derivatives will be modulated by a certain scaling. For $\mathcal{F}$ and $\mathbf{j}$ as before and $b \in \widetilde{\mathrm{Der}}(\mathcal{F})$ we define a differential operator with constant coefficients on $\mathscr{C}_*$ by writing

$$\mathbb{D}_{\mathcal{F}}^{b,\mathbf{j}} \stackrel{\text{def}}{=} \prod_{S \in \mathcal{F}} \frac{D^{b_S}}{2^{\mathrm{ext}^{\mathbf{j}}(S)|b_S|_\mathfrak{s}}},$$

where we use the fact that each such $b$ admits a unique decomposition $b = \sum_{S \in \mathcal{F}} b_S$ with $b_S$ supported on $\tilde{N}(S)$. Additionally, we build a seminorm which controls the result of acting on a test function $\varphi$ with the renormalisation operator $\hat{H}_{\mathcal{F}}^{\mathbf{j}}$. This is achieved by taking suitable suprema of $\varphi$ and some of its partial derivatives over

the variables to be integrated. Recall that $\varrho(\mathcal{F}) \subset \tilde{N}(\mathcal{F})^c$ denotes the set of roots of $\mathcal{F}$. Then, for any $V \supset \varrho(\mathcal{F})$ fixed $x \in (\mathbf{R}^d)^V$, $\mathcal{E}' \supset \mathcal{E}_{\mathcal{B}}^{\mathrm{ext}}(\mathcal{F})$, and $\mathbf{j} \in \mathbf{N}^{\mathcal{E}'}$, we define $\mathrm{Dom}(\mathcal{F}, \mathbf{j}, x)$ to be the set of all $y \in (\mathbf{R}^d)^{\tilde{N}(\mathcal{F})}$ such that the following conditions hold

- For each $S \in \mathcal{F} \cap \mathcal{S}$, one has $y_v = x_{\varrho_S}$ for all nodes $v \in \tilde{N}(S)$.
- For each $S \in \mathcal{F} \cap \mathcal{D}$, one has $y_v \overset{\mathrm{ext}^{\mathbf{j}}(S)}{\rightsquigarrow} x_{\varrho_S}$ for all nodes $v \in \tilde{N}(S)$.

In other words, we restrict ourselves to coordinates $y$ for which "safe" trees are collapsed to a point, while "unsafe" trees are restricted to be of diameter of order $2^{-\mathrm{ext}^{\mathbf{j}}(S)}$. We can now define the previously mentioned seminorms.

**Definition 8.7** Let $\mathcal{F} \subset \mathcal{B}$ with $\mathrm{depth}(\mathcal{F}) \leq 1$. Let $\mathbf{j} = \mathbf{N}^{\mathcal{E}'}$ with $\mathcal{E}' \supset \mathcal{E}_{\mathcal{B}}(\mathcal{F})$. For $x \in (\mathbf{R}^d)^{\tilde{N}(\mathcal{F})^c}$, we define a family of seminorms $\| \cdot \|_{\mathcal{F},\mathbf{j}}(x)$ on $\mathscr{C}_*$ by

$$\|\varphi\|_{\mathcal{F},\mathbf{j}}(x) \overset{\mathrm{def}}{=} \sup \left\{ \left| \left( \mathbb{D}_{\mathcal{F}}^{b,\mathbf{j}} \varphi \right)(x \sqcup y) \right| : \begin{array}{l} b \in \widetilde{\mathrm{Der}}(\mathcal{F}) \text{ and} \\ y \in \mathrm{Dom}(\mathcal{F}, \mathbf{j}, x) \end{array} \right\} .$$

In the particular case when $\mathcal{F} = \{S\}$ (i.e. it only consists of a single tree), we also write $\|\varphi\|_{S,\mathbf{j}}(x)$ instead of $\|\varphi\|_{\{S\},\mathbf{j}}(x)$.

**Remark 8.8** We make a few simple observations about the seminorms defined in Definition 8.7. The first is that $\| \cdot \|_{\mathcal{F},\mathbf{j}}(x)$ clearly only depends on $\mathbf{j}$'s values on $\mathcal{E}_{\mathcal{B}}(\mathcal{F})$. The second is that the case $\mathcal{F} = \emptyset$ is somewhat degenerate, here the "seminorm" is really just a point-wise absolute value.

We then have the following lemmas.

**Lemma 8.9** *Let $\mathcal{F} \subset \mathcal{B}$ with $\mathrm{depth}(\mathcal{F}) \leq 1$. Then uniform in $\mathbf{j} \in \mathbf{N}^{\mathcal{E}'}$ with $\mathcal{E}' \supset \mathcal{E}_{\mathcal{B}}(\mathcal{F})$, $x \in (\mathbf{R}^d)^{\tilde{N}(\mathcal{F})^c}$, and $f, g \in \mathscr{C}_*$, one has*

$$\|fg\|_{\mathcal{F},\mathbf{j}}(x) \lesssim \|f\|_{\mathcal{F},\mathbf{j}}(x) \, \|g\|_{\mathcal{F},\mathbf{j}}(x)$$

*Furthermore, if $g \in \mathscr{C}_{\tilde{N}(\mathcal{F})^c}$ then $\|fg\|_{\mathcal{F},\mathbf{j}}(x) = \|f\|_{\mathcal{F},\mathbf{j}}(x) \, g(x)$.*

*Proof.* The first claim follows from Leibniz's Rule and the second is immediate from the definitions. $\qquad \square$

We use a shorthand for the kernel norms of (2.5) by writing, for any subset of edges $E \subset K(\overline{T})$,

$$\|E\|_{\mathfrak{t}} \overset{\mathrm{def}}{=} \prod_{e \in E} \|K_{\mathfrak{t}(e)}\|_{|\mathfrak{t}(e)|_{\mathfrak{s}}, 2|\mathfrak{s}| \times |N(\overline{T})|} .$$

**Lemma 8.10** *Let $S \in \mathcal{B}$ and $\mathcal{F} \stackrel{def}{=} C_{\mathcal{B}}(S)$. Then uniform in $\mathbf{k} \in \mathbf{N}^{\mathcal{E}'}$ with $\mathcal{E}' \supset \mathcal{E}_{\mathcal{B}}^{\mathrm{int}}(S)$, $x \in (\mathbf{R}^d)^{\tilde{N}(\mathcal{F})^c}$, and any multi-index $p$ supported on $e_{\mathrm{c}}[K_{\mathcal{B}}^{\partial}(S)]$ with $|p(u)|_{\mathfrak{s}} \leq |\mathfrak{s}|$ for every $u$ in that node set, one has*

$$\left\| D^p \mathrm{Ker}_{0,\mathbf{k}}^{K_{\mathcal{B}}^{\partial}(S)} \right\|_{\mathcal{F},\mathbf{k}}(x) \lesssim \|K_{\mathcal{B}}^{\partial}(S)\|_{\mathfrak{t}} \cdot \prod_{e \in K_{\mathcal{B}}^{\partial}(S)} 2^{(\mathfrak{h}(e)+|p(e_{\mathrm{c}})|_{\mathfrak{s}})k_e} . \qquad (8.15)$$

*Furthermore, the left hand side vanishes unless, for all $T \in \mathcal{F}$ and $e \in K_T^{\partial}(S)$, one has $x_{e_{\mathrm{c}}} \stackrel{k_e}{\longleftrightarrow} x_{\varrho_T}$.*

*Proof.* When $\mathcal{F} = \emptyset$ the statement is an immediate consequence of the definition of (2.5) so we turn to the case of $\mathcal{F} \neq \emptyset$ Using Lemma 8.9 it suffices to show that for each $T \in \mathcal{F}$, $e \in K_T^{\partial}(S)$

$$\|D^{p'} \mathrm{Ker}_{\mathfrak{e},\mathbf{k}}^{\{e\}}\|_{T,\mathbf{k}}(x) \lesssim \mathbb{1}\{x_{e_{\mathrm{c}}} \stackrel{k_e}{\longleftrightarrow} x_{\varrho_T}\} \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)} 2^{(\mathfrak{h}(e)+|p'|_{\mathfrak{s}}+|j(e_{\mathrm{c}})|_{\mathfrak{s}})k_e} , \quad (8.16)$$

where the multi-index $p'$ is supported on $e_{\mathrm{c}}$ and is bounded as in the assumption. By definition, we have

$$\|D^{p'} \mathrm{Ker}_{0,\mathbf{k}}^{\{e\}}\|_{T,\mathbf{k}}(x) = \sup_{\substack{b \in \widetilde{\mathrm{Der}}(\{T\}) \\ y \in \mathrm{Dom}(\{T\},\mathbf{k},x)}} 2^{-|b|_{\mathfrak{s}}\mathrm{ext}^{\mathbf{k}}(T)} |(D^{b+p'} \mathrm{Ker}_{0,\mathbf{k}}^{\{e\}})(x_{e_{\mathrm{c}}} \sqcup y_{e_{\mathrm{p}}})| .$$

$$(8.17)$$

Note now that the bound $\mathrm{ext}^{\mathbf{k}}(T) \geq k_e$ holds by the definition of $\mathrm{ext}^{\mathbf{k}}(T)$ and the fact that $e \in K_T^{\partial}(S) \subset K_{\mathcal{B}}^{\mathrm{ext}}(T)$, so that

$$|D^{b+p'} \mathrm{Ker}_{0,\mathbf{k}}^{\{e\}}(\cdot)| \lesssim 2^{(|b|_{\mathfrak{s}}+|p'|_{\mathfrak{s}}+\mathfrak{h}(e))k_e} \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)}$$

$$\leq 2^{(|p'|_{\mathfrak{s}}+\mathfrak{h}(e))k_e+|b|_{\mathfrak{s}}\mathrm{ext}^{\mathbf{k}}(T)} \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)}$$

yielding

$$\|\mathrm{Ker}_{0,\mathbf{k}}^{\{e\}}\|_{T,\mathbf{k}}(x) \lesssim 2^{(\mathfrak{h}(e)+|p'|_{\mathfrak{s}})k_e} \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)} .$$

It remains to show that all points $x$ in the support of the left hand side of (8.15) satisfy $x_{e_{\mathrm{c}}} \stackrel{k_e}{\longleftrightarrow} x_{\varrho_T}$. The support property of $\mathrm{Ker}_{0,\mathbf{k}}^{\{e\}}$ enforces $x_{e_{\mathrm{c}}} \stackrel{k_e}{\longleftrightarrow} y_{e_{\mathrm{p}}}$ for all $y$'s over which the supremum in our seminorm is taken. On the other hand the condition $y \in \mathrm{Dom}(\{T\}, \mathbf{k}, x)$ forces $y_{e_{\mathrm{p}}} \stackrel{\mathrm{ext}^{\mathbf{k}}(T)}{\longleftrightarrow} x_{\varrho_T}$, so that the required relation follows from the triangle inequality, thus completing the proof. $\qquad \square$

**Lemma 8.11** *Let $S \in \mathcal{B}$ and $\mathcal{F} \stackrel{def}{=} C_{\mathcal{B}}(S)$. Then, uniform in $\mathbf{j} \in \mathbf{N}^{\mathcal{E}'}$ with $\mathcal{E}' \supset \mathcal{E}_{\mathcal{B}}^{\mathrm{ext}}(S)$, $\mathbf{k} \in \mathring{\mathcal{N}}_S(\mathbf{j})$, $m \in \mathrm{Der}(\pi, S)$, and $x \in (\mathbf{R}^d)^{\tilde{N}(\mathcal{F})^c}$ satisfying the constraints*

$$x_{e_{\mathrm{p}}} \stackrel{k_e}{\leftrightsquigarrow} x_{e_{\mathrm{c}}} \text{ for all } e \in \mathring{K}_{\mathcal{B}}(S), \qquad (8.18)$$

$$x_{e_{\mathrm{c}}} \stackrel{k_e}{\leftrightsquigarrow} x_{\varrho_T} \text{ for all } T \in \mathcal{F} \text{ and } e \in K_T^{\partial}(S),$$

*one has the bound*

$$\|D^m \mathscr{Y}_{\mathbb{M},S}^{\#}\varphi\|_{\mathcal{F},\mathbf{k}}(x) \lesssim \|\varphi\|_{S,\mathbf{j}}(x_{\tilde{N}(S)^c}) \cdot 2^{\bar{\omega}^{\#}(S)[\mathrm{ext}^{\mathbf{j}}(S)-\mathrm{int}^{\mathbf{k}}(S)]+|m|_{\mathfrak{s}}\mathrm{int}^{\mathbf{k}}(S)} , \quad (8.19)$$

*where*

$$\bar{\omega}^{\#}(S) \overset{\text{def}}{=} \begin{cases} \bar{\omega}(S) & \text{if } S \in \mathcal{S}, \\ \bar{\omega}(S) + 1 & \text{if } S \in \mathcal{D} . \end{cases} \tag{8.20}$$

*Proof.* We first treat the case where $\mathcal{F} \neq \emptyset$. We fix $\mathbf{j}$ as above and $\mathbf{k} \in \mathring{\mathcal{N}}_S(\mathbf{j})$.

Throughout the proof we implicitly assume that $x$ satisfies the constraints (8.18).

We first establish the desired bound when $S \in \mathcal{S}$. A term by term estimate gives

$$\|D^m \mathscr{Y}_S \varphi\|_{\mathcal{F},\mathbf{k}}(x) \lesssim \max_{p+m \in \text{Der}(S)} \left[ \|\mathbf{X}_{p,\varrho_S}^{\tilde{N}(S)}\|_{\mathcal{F},\mathbf{k}}(x) \cdot |(D^{p+m}\varphi)(\text{Coll}_S(x))| \right] . \tag{8.21}$$

Here, the factor $(D^{p+m}\varphi)(\text{Coll}_S(x))$ could be pulled out of the seminorm because it does not depend on any of the variables in $\tilde{N}(\mathcal{F})$.

It is easy to see that we have the bound

$$\|\mathbf{X}_{p,\varrho_S}^{\tilde{N}(S)}\|_{\mathcal{F},\mathbf{k}}(x) = \sup \left\{ |\mathbb{D}_{\mathcal{F}}^{b,\mathbf{k}} \mathbf{X}_{p,\varrho_S}^{\tilde{N}(S)}|(x \sqcup y) : \begin{array}{l} b \in \widetilde{\text{Der}}(\mathcal{F}), \; b \leq p \\ y \in \text{Dom}(\mathcal{F}, \mathbf{k}, x) \end{array} \right\}$$
$$\lesssim 2^{-|p|_{\mathfrak{s}} \text{int}^{\mathbf{k}}(S)} .$$

Here we used that our condition on $x$ forces the distances between coordinates in $N_{\mathcal{F}}(S)$ to be at most of order $2^{-\text{int}^{\mathbf{k}}(S)}$, while the constraint $y \in \text{Dom}(\mathcal{F}, \mathbf{k}, x)$ forces the distances between coordinates in $N_T$ for each $T \in \mathcal{F}$ to be of order at most $2^{-\text{ext}^{\mathbf{k}}(T)}$. Furthermore, for any $T \in \mathcal{F}$ we have

$$\text{ext}^{\mathbf{k}}(T) \geq \text{int}^{\mathbf{k}}(S). \tag{8.22}$$

Inserting this into (8.21) we have, for $x \in (\mathbf{R}^d)^{\tilde{N}(S)^c}$,

$$\|D^m \mathscr{Y}_S \varphi\|_{\mathcal{F},\mathbf{k}}(x) \lesssim \max \left\{ 2^{-|p|_{\mathfrak{s}} \text{int}^{\mathbf{k}}(S)} |(D^{p+m}\varphi)(\text{Coll}_S(x))| : \; p+m \in \text{Der}(S) \right\}$$
$$= \max \left\{ 2^{|p|_{\mathfrak{s}}(\text{ext}^{\mathbf{j}}(S) - \text{int}^{\mathbf{k}}(S)) + |m|_{\mathfrak{s}} \text{ext}^{\mathbf{j}}(S)} : \; p+m \in \text{Der}(S) \right\} \cdot \|\varphi\|_{S,\mathbf{j}}(x)$$
$$= 2^{(\bar{\omega}(S) - |m|_{\mathfrak{s}})(\text{ext}^{\mathbf{j}}(S) - \text{int}^{\mathbf{k}}(S)) + |m|_{\mathfrak{s}} \text{ext}^{\mathbf{j}}(S)} \cdot \|\varphi\|_{S,\mathbf{j}}(x) ,$$

where in the last line we used the fact that $S \in \mathcal{S}$ implies that $\text{ext}^{\mathbf{j}}(S) \geq \text{int}^{\mathbf{k}}(S)$.

We turn to case $S \in \mathcal{D}$ and start with the estimate

$$\|D^m (1 - \mathscr{Y}_S)\varphi\|_{\mathcal{F},\mathbf{k}}(x)$$
$$\leq \sup \left\{ 2^{-|b|_{\mathfrak{s}} \text{int}^{\mathbf{k}}(S)} \left| D^{b+m}[(1 - \mathscr{Y}_S)\varphi](x \sqcup y) \right| : \begin{array}{l} b \in \widetilde{\text{Der}}(\mathcal{F}) \\ y \in \text{Dom}(\mathcal{F}, \mathbf{k}, x) \end{array} \right\} ,$$

where we have an inequality because we used (8.22). We treat the cases $|b+m|_{\mathfrak{s}} > \bar{\omega}(S)$ and $|b+m|_{\mathfrak{s}} \leq \bar{\omega}(S)$ separately. In the former case one has $D^{b+m}\mathscr{Y}_S\varphi = 0$ and we arrive at the bound

$$\sup \left\{ 2^{-|b|_{\mathfrak{s}} \text{int}^{\mathbf{k}}(S)} |(D^{b+m}\varphi)(y \sqcup x)| : \begin{array}{l} b \in \widetilde{\text{Der}}(\mathcal{F}), \; |b+m|_{\mathfrak{s}} > \bar{\omega}(S) \\ y \in \text{Dom}(\mathcal{F}, \mathbf{k}, x) \end{array} \right\}$$

$$\leq \max\left\{2^{|b|_\mathfrak{s}[\mathrm{ext}^\mathbf{j}(S)-\mathrm{int}^\mathbf{k}(S)]+|m|_\mathfrak{s}\mathrm{ext}^\mathbf{j}(S)} : \begin{array}{c} b \in \widetilde{\mathrm{Der}}(\mathcal{F})\,, \\ |b+m|_\mathfrak{s} > \bar\omega(S) \end{array}\right\} \cdot \|\varphi\|_{S,\mathbf{j}}(x)$$

$$\leq 2^{(\bar\omega(S)+1-|m|_\mathfrak{s})[\mathrm{ext}^\mathbf{j}(S)-\mathrm{int}^\mathbf{k}(S)]+|m|_\mathfrak{s}\mathrm{ext}^\mathbf{j}(S)} \cdot \|\varphi\|_{S,\mathbf{j}}(x)\,. \tag{8.23}$$

To obtain the first inequality of (8.23) observe that the condition $y \in \mathrm{Dom}(\mathcal{F},\mathbf{k},x)$ implies that one also has

$$x_{\tilde{N}_\mathcal{F}(S)} \sqcup y \in \mathrm{Dom}(\{S\},\mathbf{j},x). \tag{8.24}$$

For the second equality we used $\mathrm{ext}^\mathbf{j}(S) < \mathrm{int}^\mathbf{k}(S)$.

We now treat the case of $b \in \widetilde{\mathrm{Der}}(\mathcal{F})$ with $|b+m|_\mathfrak{s} \leq \bar\omega(S)$ and write $D^{b+m}(1-\mathscr{Y}_S)\varphi(x)$ as

$$D^{b+m}\varphi(x) - \sum_{\substack{p, \\ p+m\in\mathrm{Der}(S)}} \frac{(x-\mathrm{Coll}_S(x))^p}{p!}(D^{b+p+m}\varphi)(\mathrm{Coll}_S(x))\,. \tag{8.25}$$

Viewing (8.25) as a Taylor remainder of order $\bar\omega(S) - |b|_\mathfrak{s} - |m|_\mathfrak{s}$ for $D^{b+m}\varphi$, we can apply [Hai14, Prop. A.1] followed by (8.24) to get the estimate

$$\sup\left\{2^{-|b|_\mathfrak{s}\mathrm{int}^\mathbf{k}(S)}\left|D^b[(1-\mathscr{Y}_S)\varphi](y \sqcup x)\right| : \begin{array}{c} b \in \widetilde{\mathrm{Der}}(\mathcal{F}),\ |b+m|_\mathfrak{s} \leq \bar\omega(S) \\ y \in \mathrm{Dom}(\mathcal{F},\mathbf{k},x) \end{array}\right\}$$

$$\leq 2^{(\bar\omega(S)+1-|m|_\mathfrak{s})[\mathrm{ext}^\mathbf{j}(S)-\mathrm{int}^\mathbf{k}(S)]+|m|_\mathfrak{s}\mathrm{ext}^\mathbf{j}(S)} \cdot \|\varphi\|_{S,\mathbf{j}}(x)\,.$$

Combining this with (8.23) yields the required bound in the case $S \in \mathcal{D}$ and thus concludes the proof for when $C_\mathcal{B}(S) \neq \emptyset$.

The case $C_\mathcal{B}(S) = \emptyset$ follows by the same argument (a Taylor remainder estimate when $S \in \mathcal{D}$ and a term by term estimate when $S \in \mathcal{S}$) but is strictly easier. □

We now state and prove the advertised bound on the operators $\hat{H}_\mathcal{F}^\mathbf{j}$ which was the motivation for the introduction of these seminorms.

**Lemma 8.12** *Let $\mathcal{F} \subset \mathcal{B}$ with $\mathrm{depth}(\mathcal{F}) \leq 1$. Then, uniform in $x \in (\mathbf{R}^d)^{\tilde{N}(\mathcal{F})^c}$, $\mathbf{j} \in \mathbf{N}^{\mathcal{E}'}$ with $\mathcal{E}' \supset \mathcal{E}_\mathcal{B}^{\mathrm{ext}}(\mathcal{F})$, and $\varphi \in \mathscr{C}_*$, one has the bound*

$$\left|\hat{H}_\mathcal{F}^\mathbf{j}[\varphi](x)\right| \lesssim \left(\prod_{S\in\mathcal{F}} 2^{\omega(S)\mathrm{ext}^\mathbf{j}(S)}\|K(S)\|_\mathfrak{t} \cdot \|\xi\|_{|L(S)|,\mathfrak{c}}\right)\|\varphi\|_{\mathcal{F},\mathbf{j}}(x)\,. \tag{8.26}$$

*Proof.* Our proof uses two nested inductions. The outer one, which is also the less trivial one, is an induction in the quantity

$$\mathrm{depth}_\mathcal{B}(\mathcal{F}) \overset{\mathrm{def}}{=} \mathrm{depth}[\mathcal{F} \sqcup \{T \in \mathcal{B} : \exists S \in \mathcal{F} \text{ with } T < S\}]\,. \tag{8.27}$$

The second, simpler, induction step is then in the cardinality of $\mathcal{F}$ for a fixed value of $\mathrm{depth}_\mathcal{B}(\mathcal{F})$. For both inductions we focus on the inductive steps, the base case being strictly easier to verify.

Fix $m \geq 1$ and assume that (8.26) has already been proven for all forests $\mathcal{G}$ of depth 1 with $\mathrm{depth}_{\mathcal{B}}(\mathcal{G}) \leq m$. Our aim is to then prove (8.26) for any $\mathcal{F}$ of cardinality one, i.e. for $\mathcal{F} = \{S\} \subset \mathcal{B}$ with $\mathrm{depth}_{\mathcal{B}}(\{S\}) = m + 1$.

Fix $\mathbf{j} \stackrel{\text{def}}{=} \mathbf{j}_S \in \mathbf{N}^{\mathcal{E}_{\mathcal{B}}^{\mathrm{ext}}(S)}$, our concern to control the corresponding sum over $\mathbf{k} \in \mathring{\mathcal{N}}_S(\mathbf{j})$ appearing in the definition of $\hat{H}_S^{\mathbf{j}}$ (see (8.5)). We will use Lemmas 8.9, 8.10, 8.11 and then appeal to Theorem A.11 to control the integral over $\tilde{N}_{\mathcal{B}}(S)$, the application of this theorem will occur with $x \in (\mathbf{R}^d)^{\tilde{N}(S)^c}$ fixed but our our estimates will be uniform in $x$ (however dependence on $x$ may sometimes be suppressed from the notation).

The multigraph underlying our application of Theorem A.11 is given by a quotient of the multigraph $\mathcal{E}_{\mathcal{B}}^{\mathrm{int}}(S)$ where, for each $T \in C_{\mathcal{B}}(S)$, one performs a contraction and identifies the collection of vertices $N(T)$ as a single equivalence class of points which we identify with $\varrho_T$.

Then our set of vertices is given by $\mathcal{V} \stackrel{\text{def}}{=} N_{\mathcal{B}}(S)$ with $\mathcal{V}_0 \stackrel{\text{def}}{=} \tilde{N}_{\mathcal{B}}(S)$ (so $\varrho_S$ serves the role of the pinned vertex) and our multigraph $\mathbf{G}$ is given by

$$\mathbf{G} \stackrel{\text{def}}{=} \mathring{K}_{\mathcal{B}}(S) \sqcup \left\{ \{u, \varrho_T\} : T \in C_{\mathcal{B}}(S), \ u \in e_{\mathrm{c}}[K_T^\partial(S)] \right\} \sqcup \mathbf{C} \,, \qquad (8.28)$$

where we selectively view $\mathring{K}_{\mathcal{B}}(S) \subset \mathcal{V}^{(2)}$ as a set of undirected edges and $\mathbf{C}$ is the set of contractions given by $\mathbf{C} \stackrel{\text{def}}{=} \{e \in \mathcal{E}_\pi : \ e \subset L_{\mathcal{B}}(S)\}$. We define a bijection $\mathfrak{q}_S : \mathcal{E}_{\mathcal{B}}^{\mathrm{int}}(S) \to \mathbf{G}$ in the natural way: by asking that $\mathfrak{q}_S$ maps $\mathcal{E}_\pi \cap \mathcal{E}_{\mathcal{B}}^{\mathrm{int}}(S)$ onto $\mathbf{C}$, $\mathfrak{q}_{\mathbf{S}}$ maps $\mathring{K}_{\mathcal{B}}(S)$ onto $\mathring{K}_{\mathcal{B}}(S)$, and $\mathfrak{q}_S$ maps $K_{\mathcal{B}}^\partial(S)$ onto the middle set of (8.28) as follows: for $(e_{\mathrm{p}}, e_{\mathrm{c}}) \in K_T^\partial(S)$ for $T \in \mathcal{C}_{\mathcal{B}}(S)$ one sets $\mathfrak{q}_S(\{e_{\mathrm{c}}, e_{\mathrm{p}}\}) = \{e_{\mathrm{c}}, \varrho_T\}$. The map $\mathfrak{q}_S$ induces a corresponding bijection between $\mathbf{N}^{\mathcal{E}_{\mathcal{B}}^{\mathrm{int}}(S)}$ and $\mathbf{N}^{\mathbf{G}}$. In what follows, we abuse notation and treat this bijection as an identification. As a start we define $\mathcal{N}_{\mathbf{G}} \subset \mathbf{N}^{\mathbf{G}}$ by setting $\mathcal{N}_{\mathbf{G}} \stackrel{\text{def}}{=} \mathring{\mathcal{N}}_S(\mathbf{j})$.

For each $\mathbf{k} \in \mathcal{N}_{\mathbf{G}}$ we define a function $F^{\mathbf{k}} \in \mathscr{C}_{\mathcal{V}}$ by setting

$$F^{\mathbf{k}}(y) \stackrel{\text{def}}{=} \mathrm{Cu}_{\pi,\mathbf{k}}^{L_{\mathcal{B}}(S)}(y) \mathrm{Ker}_{0,\mathbf{k}}^{\mathring{K}_{\mathcal{B}}(S)}(y) \hat{H}_{C_{\mathcal{B}}(S)}^{\mathbf{k}} \left[ \mathrm{Ker}_{0,\mathbf{k}}^{K_{\mathcal{B}}^\partial(S)} [\mathscr{Y}_{S,\mathbb{M}}^{\#} \varphi] \right] (x \sqcup y) \,.$$

So we then have

$$\hat{H}_S^{\mathbf{j}}[\varphi](x \sqcup y_{\varrho_S}) = \sum_{\mathbf{k} \in \mathring{\mathcal{N}}_S(\mathbf{j})} \int_{\tilde{N}_{\mathcal{B}}(S)} dy \, F^{\mathbf{k}}(y) \,.$$

Define $\hat{\mathfrak{q}}_S : N(S) \to \mathcal{V}$ via setting $\hat{\mathfrak{q}}(S)(u) \stackrel{\text{def}}{=} \varrho_T$ if there exists $T \in C_{\mathcal{B}}(S)$ with $u \in N(T)$, we set $\hat{\mathfrak{q}}_S(u) \stackrel{\text{def}}{=} u$ otherwise. We now define a total homogeneity $\varsigma$ on the trees of $\widehat{\mathcal{U}}_{\mathcal{V}}$ (see Definitions A.1 and A.4 below) by setting

$$\varsigma \stackrel{\text{def}}{=} -\bar{\omega}^{\#}(S)\delta^{\uparrow}[\mathcal{V}] + \sum_{e \in K_{\mathcal{B}}(S)} \mathfrak{h}(e)\delta^{\uparrow}[\{e_{\mathrm{c}}, \hat{\mathfrak{q}}_S(e_{\mathrm{p}})\}] + \varsigma^{\mathbf{C}} + \varsigma^R + \sum_{T \in C_{\mathcal{B}}(S)} \omega(T)\delta^{\uparrow}[\varrho_T] \,, \tag{8.29}$$

where the total homogeneity $\varsigma^{\mathbf{C}}$ is given by

$$\varsigma^{\mathbf{C}} \overset{\text{def}}{=} \sum_{\substack{B \in \pi \\ B \subset L_{\mathcal{B}}(S)}} \mathfrak{c}^{B} \, . \tag{8.30}$$

The total homogeneity $\mathfrak{c}^{B}$ on the coalescence trees of $\widehat{\mathcal{U}}_{\mathcal{V}}$ is defined in Definition A.30. The total homogeneity $\varsigma^{R}$ is given by setting, for each $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$,

$$\varsigma^{R}_{\mathbf{T}} \overset{\text{def}}{=} \sum_{B \in R(\mathbf{T})} \ell(B)\Big( \delta^{\Uparrow}_{\mathbf{T}}[B] - \delta^{\uparrow}_{\mathbf{T}}[B] \Big), \text{ where}$$

$$R(\mathbf{T}) \overset{\text{def}}{=} \{B \in \pi : \ B \subset L_{\mathcal{B}}(S), \ \mathbf{L}_{B^{\uparrow},} = B, \text{ and } \ell(B) > 0\} \, .$$

The notation $\ell(B)$ was defined in (A.20). and, for $a \in \mathring{\mathbf{T}}$ we write $\mathbf{L}_a$ for the set of leaves of $\mathbf{T}$ which are descendants of $a$. In Lemma 8.13 we establish that $\varsigma$ is subdivergence free on $\mathcal{V}$ for the set of scales $\mathcal{N}_{\mathbf{G}}$. Taking this for granted for the moment, we check the other conditions of Theorem A.11. Observe that the total homogeneity $\varsigma$ is of order $\omega(S) - \bar{\omega}(S)^{\#}$ which is negative when $S \in \mathcal{D}$ and positive when $S \in \mathcal{S}$. Additionally, one has

$$\mathcal{N}_{\mathbf{G}} \overset{\text{def}}{=} \begin{cases} \mathcal{N}_{\mathbf{G}, > \text{ext}^{\mathbf{j}}(S)} & \text{if } S \in \mathcal{D} \, , \\ \mathcal{N}_{\mathbf{G}, \leq \text{ext}^{\mathbf{j}}(S)} & \text{if } S \in \mathcal{S} \, . \end{cases}$$

We are then done if we can exhibit a modification of $\tilde{F} = (\tilde{F}^{\mathbf{k}})_{\mathbf{k} \in \mathcal{N}_{\mathbf{G}}} \in \text{Mod}(F)$ such that $\tilde{F}$ is is bounded by $\varsigma$ in the sense of Definition A.5 with

$$\|\tilde{F}\|_{\varsigma, \mathcal{N}_{\mathbf{G}}} \lesssim \|\varphi\|_{S, \mathbf{j}}(x) \cdot 2^{\bar{\omega}^{\#}(S)\text{ext}^{\mathbf{j}}(S)} \|K(S)\|_{\mathfrak{t}} \cdot \|\xi\|_{|L(S)|, \mathfrak{c}} \, . \tag{8.31}$$

For any $\mathbf{k} \in \mathcal{N}_{\mathbf{G}}$ we will set

$$\tilde{F}^{\mathbf{k}} \overset{\text{def}}{=} \begin{cases} F^{\mathbf{k}} & \text{if } R(\mathcal{T}(\mathbf{k})) = \emptyset \, , \\ \mathring{F}^{\mathbf{k}} & \text{if } R(\mathcal{T}(\mathbf{k})) \neq \emptyset \, . \end{cases}$$

We will define $\mathring{F}^{\mathbf{k}}$ on a tree by tree basis later. It suffices to check the domain condition and desired supremum bound in each of the two cases separately.

We treat the first case. The domain condition (A.4) is easy to check so we turn to the desired supremum bounds needed for (8.31).

Fix $\mathbf{T} \in \mathcal{U}_{\mathcal{V}}$ with $R(\mathbf{T}) = \emptyset$. We now obtain the desired uniform in $\mathbf{s} \in \text{Lab}_{\mathbf{T}}$ and $\mathbf{k} \in \mathcal{N}_{\mathbf{G}, \text{tr}}(\mathbf{T}, \mathbf{s})$ estimates. Fix appropriate $\mathbf{s}$ and $\mathbf{k}$, we start by applying the inductive hypothesis to $\hat{H}^{\bullet}_{C_{\mathcal{B}}(S)}$ along with Lemmas 8.9, 8.10, and 8.11 which yields

$$|F^{\mathbf{k}}(y)| \lesssim \Big( \prod_{T \in C_{\mathcal{B}}(S)} 2^{\omega(T)\text{ext}^{\mathbf{k}}(T)} \|K(T)\|_{\mathfrak{t}} \cdot \|\xi\|_{|L(T)|, \mathfrak{c}} \Big) \cdot \Big\| \mathscr{Y}^{\#}_{S} \varphi \Big\|_{C_{\mathcal{B}}(S), \mathbf{k}}(x \sqcup y)$$

$$\cdot \Big| \text{Ker}^{\mathring{K}_{\mathcal{B}}(S)}_{0, \mathbf{k}}(y \sqcup x_{\varrho S}) \text{Cu}^{L_{\mathcal{B}}(S)}_{\pi, \mathbf{k}}(y \sqcup x_{\varrho S}) \Big| \cdot \Big\| \text{Ker}^{K^{\partial}_{\mathcal{B}}(S)}_{0, \mathbf{k}} \Big\|_{C_{\mathcal{B}}(S), \mathbf{k}}(x \sqcup y)$$

$$\lesssim \|\varphi\|_{S,\mathbf{j}}(x) \cdot 2^{\bar{\omega}^{\#}(S)\mathrm{ext}^{\mathbf{j}}(S)} \tag{8.32}$$

$$\cdot\, 2^{-\bar{\omega}^{\#}(S)\mathrm{int}^{\mathbf{k}}(S)} \Big( \prod_{T\in C_{\mathcal{B}}(S)} 2^{\omega(T)\mathrm{ext}^{\mathbf{k}}(T)} \|K(T)\|_{\mathfrak{t}} \cdot \|\xi\|_{|L(T)|,\mathfrak{c}} \Big)$$

$$\cdot\, \Big|\mathrm{Ker}_{0,\mathbf{k}}^{\mathring{K}_{\mathcal{B}}(S)}(y \sqcup x_{\varrho S})\mathrm{Cu}_{\pi,\mathbf{k}}^{L_{\mathcal{B}}(S)}(y \sqcup x_{\varrho S})\Big| \cdot \Big\|\mathrm{Ker}_{0,\mathbf{k}}^{K_{\mathcal{B}}^{\partial}(S)}\Big\|_{C_{\mathcal{B}}(S),\mathbf{k}}(x \sqcup y)\,.$$

The desired bound follows using the supremum estimates

$$\Big|\mathrm{Ker}_{0,\mathbf{k}}^{\mathring{K}_{\mathcal{B}}(S)}(y \sqcup x_{\varrho S})\mathrm{Cu}_{\pi,\mathbf{k}}^{L_{\mathcal{B}}(S)}(y \sqcup x_{\varrho S})\Big| \lesssim \|\mathring{K}_{\mathcal{B}}(S)\|_{\mathfrak{t}}\|\xi\|_{L_{\mathcal{B}}(S),\mathfrak{c}} \cdot 2^{\langle \varsigma_{\mathbf{T}}^{C} + \varsigma_{\mathbf{T}}[\mathring{K}_{\mathcal{B}}(S)],\mathbf{s}\rangle}$$

$$\text{and } \Big\|\mathrm{Ker}_{0,\mathbf{k}}^{K_{\mathcal{B}}^{\partial}(S)}\Big\|_{C_{\mathcal{B}}(S),\mathbf{k}}(x \sqcup y) \lesssim \|K_{\mathcal{B}}^{\partial}(S)\|_{\mathfrak{t}} \cdot 2^{\langle \varsigma_{\mathbf{T}}[\mathring{K}_{\mathcal{B}}(S)],\mathbf{s}\rangle}\,.$$

Now fix $\mathbf{T} \in \mathcal{U}_{\mathcal{V}}$ with $R(\mathbf{T}) \neq \emptyset$. For any $\mathbf{k} \in \mathcal{N}_{\mathbf{G}}$ with $\mathcal{T}(\mathbf{k}) = \mathbf{T}$ we set, for $y \in (\mathbf{R}^{d})^{\mathcal{V}_0}$,

$$\mathring{F}^{\mathbf{k}}(y) \stackrel{\mathrm{def}}{=} \mathrm{Cu}_{\pi,\mathbf{k}}^{L_{\mathcal{B}}(S)}(y \sqcup x_{\varrho S}) \cdot \Big( \prod_{B\in R(\mathbf{T})} (1 - \mathscr{Y}_B) \Big)[G^{\mathbf{k}}](x \sqcup y), \text{ where}$$

$$G^{\mathbf{k}}(x \sqcup y) \stackrel{\mathrm{def}}{=} \mathrm{Ker}_{0,\mathbf{k}}^{\mathring{K}_{\mathcal{B}}(S)}(y \sqcup x_{\varrho S})\hat{H}_{C_{\mathcal{B}}(S)}^{\mathbf{k}}\Big[\mathrm{Ker}_{0,\mathbf{k}}^{K_{\mathcal{B}}^{\partial}(S)}(\mathscr{Y}_S^{\#}\varphi)\Big](x \sqcup y)\,,$$

and the notation $\mathscr{Y}_B$ was defined in (8.8). Clearly

$$F^{\mathbf{k}}(y) - \mathring{F}^{\mathbf{k}}(y) \stackrel{\mathrm{def}}{=} \mathrm{Cu}_{\pi,\mathbf{k}}^{L_{\mathcal{B}}(S)}(y \sqcup x_{\varrho S}) \cdot \sum_{\substack{\mathcal{A}\subset R(\mathbf{T}) \\ \mathcal{A}\neq\emptyset}} \Big( \prod_{B\in\mathcal{A}} (-\mathscr{Y}_B) \Big)[G^{\mathbf{k}}](x \sqcup y)\,.$$

We claim that for each fixed $x$ and for each $\mathcal{A}$ in the above sum the integral over $y_{\mathcal{V}_0}$ of the corresponding summand vanishes. Fixing $\mathcal{A}$ and expanding the summand completely yields a linear combination of terms – in each individual term all the dependence on $y_{(+,B)}$ can extracted as a factor of the form

$$(y_{(+,B)} - z_{(-,B)})^n \widetilde{\mathrm{Cu}}_{B,k_B}(z_B)\,,$$

where $z = y \sqcup x_{\varrho*}$ and $n \in \mathbf{N}^d$ with $|n|_{\mathfrak{s}} \leq \ell(B)$. Since $B \in R(\mathbf{T})$ on must have $k_B > 0$ and therefore the above quantity vanishes when integrating $y_{(+,B)}$ by (6.3).

The required domain constraint for $\tilde{F}^{\mathbf{k}}$ is straightforward to check and for the supremum bound we observe that by the estimate of [Hai14, Prop. A.1] one has, uniform in $\mathbf{s}$, $\mathbf{k} \in \mathcal{N}_{\mathrm{tr}}(\mathbf{T},\mathbf{s})$, and in $y$ with $y_{(+,B)} \stackrel{k_B}{\rightsquigarrow} y_{(-,B)}$ for every $B \in R(\mathbf{T})$ – which is required to be in the support of $\mathrm{Cu}_{\pi,\mathbf{k}}^{L_{\mathcal{B}}(S)}$ – the bound

$$\sup_y \Big|\Big( \prod_{B\in R(\mathbf{T})} (1 - \mathscr{Y}_B) \Big)[G^{\mathbf{k}}](x \sqcup y)\Big|$$

$$\lesssim \sup_y \max_{\substack{f\in\overline{\mathrm{Der}}(R(\mathbf{T})) \\ a+b+c=f}} \Big( \prod_{B\in R(\mathbf{T})} 2^{|f_{(+,B)}|_{\mathfrak{s}}\mathbf{s}(B^{\uparrow})} \Big) |D^a\mathrm{Ker}_{0,\mathbf{k}}^{\mathring{K}_{\mathcal{B}}(S)}(y)| \tag{8.33}$$

$$\cdot\, |\hat{H}_{C_{\mathcal{B}}(S)}^{\mathbf{k}}\Big[(D^b\mathrm{Ker}_{0,\mathbf{k}}^{K_{\mathcal{B}}^{\partial}(S)})[D^c\mathscr{Y}_{S,\mathbb{M}}^{\#}\varphi]\Big](x \sqcup y)|\,,$$

where $\overline{\mathrm{Der}}(R(\mathbf{T}))$ was defined in (8.10) and for the multi-indices $m = a, b, c, f$ and any $B \in \mathbf{R}(T)$ we use the notation $m_{(+,B)}$ for the multi-index which is given by $m$ on the site $(+, B)$ but vanishes everywhere else. We proceed as earlier and use our inductive hypothesis along with Lemmas 8.10 and 8.11 to get

$$
|\hat{H}^{\mathbf{k}}_{C_{\mathcal{B}}(S)}\Big[(D^b \mathrm{Ker}^{K^{\partial}_{\mathcal{B}}(S)}_{0,\mathbf{k}})[D^c \mathscr{Y}^{\#}_{S,\mathbb{M}}\varphi]\Big](x \sqcup y)| \tag{8.34}
$$
$$
\lesssim \Big( \prod_{T \in C_{\mathcal{B}}(S)} 2^{\omega(T)\mathrm{ext}^{\mathbf{k}}(T)} \|K(T)\|_{\mathfrak{t}} \cdot \|\xi\|_{|L(T)|,\mathfrak{c}} \Big) \cdot \Big\| D^c \mathscr{Y}^{\#}_{S} \varphi \Big\|_{C_{\mathcal{B}}(S),\mathbf{k}} (x \sqcup y)
$$
$$
\cdot \Big\| D^b \mathrm{Ker}^{K^{\partial}_{\mathcal{B}}(S)}_{0,\mathbf{k}} \Big\|_{C_{\mathcal{B}}(S),\mathbf{k}} (y)
$$
$$
\lesssim \Big( \prod_{T \in C_{\mathcal{B}}(S)} 2^{\omega(T)\mathbf{s}(\varrho^{\uparrow}_T)} \|K(T)\|_{\mathfrak{t}} \cdot \|\xi\|_{|L(T)|,\mathfrak{c}} \Big) \cdot 2^{\bar{\omega}^{\#}(S)\mathrm{ext}^{\mathbf{j}}(S)} 2^{-\bar{\omega}^{\#}(S)\mathbf{s}(\varrho_{\mathbf{T}})} \|\varphi\|_{S,\mathbf{j}}(x)
$$
$$
\cdot \Big\| K^{\partial}_{\mathcal{B}}(S) \Big\|_{\mathfrak{t}} \cdot 2^{\langle \varsigma_{\mathbf{T}}[K^{\partial}_{\mathcal{B}}(S)],\mathbf{s} \rangle} \Big( \prod_{B \in \mathbf{R}(\mathbf{T})} 2^{|b_{(+,B)}+c_{(+,B)}|_{\mathfrak{s}}\mathbf{s}(B^{\Uparrow})} \Big) \;.
$$

In the last inequality above the key observation is that for each $B \in R(\mathbf{T})$, the scale $\mathbf{s}(B^{\Uparrow})$ dominates the scale $\mathbf{s}(\cdot)$ at which we would pay the cost of $|b_{(+,B)}|_{\mathfrak{s}}$ or $|c_{(+,B)}|_{\mathfrak{s}}$ with regards to Lemmas 8.10 and 8.11. Similarly, one has the bound

$$
|D^a \mathrm{Ker}^{\mathring{K}_{\mathcal{B}}(S)}_{0,\mathbf{k}}(y)| \lesssim \|\mathring{K}_{\mathcal{B}}(S)\|_{\mathfrak{t}} 2^{\langle \varsigma_{\mathbf{T}}[\mathring{K}_{\mathcal{B}}(S)],\mathbf{s} \rangle} \prod_{B \in \mathbf{R}(\mathbf{T})} 2^{|a_{(+,B)}|_{\mathfrak{s}}\mathbf{s}(B^{\Uparrow})} \Big) \;. \tag{8.35}
$$

Inserting (8.34) and (8.35) into (8.33) we get

$$
\sup_y \Big| \Big( \prod_{B \in \mathbf{R}(\mathbf{T})} (1 - \mathscr{Y}_B) \Big)[\tilde{G}^{\mathbf{k}}](x \sqcup y) \Big| \tag{8.36}
$$
$$
\lesssim \|K_{\mathcal{B}}(S)\|_{\mathfrak{t}} 2^{\langle \varsigma_{\mathbf{T}}[K_{\mathcal{B}}(S)],\mathbf{s} \rangle} \Big( \prod_{T \in C_{\mathcal{B}}(S)} 2^{\omega(T)\mathbf{s}(\varrho^{\uparrow}_T)} \|\xi\|_{|L(T)|,\mathfrak{c}} \Big) \cdot 2^{\bar{\omega}^{\#}(S)\mathrm{ext}^{\mathbf{j}}(S)}
$$
$$
\cdot 2^{-\bar{\omega}^{\#}(S)\mathbf{s}(\varrho_{\mathbf{T}})} \|\varphi\|_{S,\mathbf{j}}(x) \cdot \max_{\substack{f \in \overline{\mathrm{Der}}(R(\mathbf{T})) \\ a+b+c=f}} \Big( \prod_{B \in \mathbf{R}(\mathbf{T})} 2^{|f_{(+,B)}|_{\mathfrak{s}}[\mathbf{s}(B^{\Uparrow})-\mathbf{s}(B^{\uparrow})]} \Big)
$$

The desired supremum bound is then obtained by noticing that the maximum of the last factor is obtained when $|f_{(+,B)}|_{\mathfrak{s}} = \hbar(B)$ for each $B \in R(\mathbf{T})$ and by combining this with the earlier used supremum bound on $\mathrm{Cu}^{L_{\mathcal{B}}(S)}_{\pi,\mathbf{k}}$. This concludes the proof of (8.31).

We now prove the inductive step for the easier induction: let $n, m \geq 1$ and suppose that (8.26) has been proven for every forest $\mathcal{G} \subset \mathcal{B}$ of depth 1 with either $\mathrm{depth}_{\mathcal{B}}(\mathcal{G}) \leq n$ or $\mathrm{depth}_{\mathcal{B}}(\mathcal{G}) = n + 1$ and $|\mathcal{G}| \leq m$.

Now suppose $\mathcal{F} \subset \mathcal{B}$ is of depth 1, $\mathrm{depth}_{\mathcal{B}}(\mathcal{F}) = n + 1$, and $|\mathcal{F}| = m + 1$. We prove (8.26) in this case. Fix $S \in \mathcal{F}$ and write $\mathcal{G} \overset{\text{def}}{=} \mathcal{F} \setminus \{S\}$, then for $x \in \mathbf{R}^{\tilde{N}(\mathcal{F})^c}$,

$$
\hat{H}^{\mathbf{j}}_{\mathcal{F}}[\varphi](x) = \hat{H}^{\mathbf{j}}_{\mathcal{G}}\Big[\hat{H}^{\mathbf{j}}_{S}[\varphi]\Big](x) \;.
$$

By applying the inductive hypothesis for $\mathcal{G}$ we have the bound

$$\left|\hat{H}^{\mathbf{j}}_{\mathcal{F}}[\varphi](x)\right| \lesssim \Big( \prod_{T \in \mathcal{G}} 2^{\omega(T)\mathrm{ext}^{\mathbf{j}}(T)} \|K(T)\|_{\mathfrak{t}} \cdot \|\xi\|_{|L(T)|,\mathfrak{c}} \Big) \left\|\hat{H}^{\mathbf{j}}_S[\varphi]\right\|_{\mathcal{G},\mathbf{j}}(x_{\tilde{N}(\mathcal{F})^c}) \;.$$

It suffices to prove

$$\left\|\hat{H}^{\mathbf{j}}_S[\varphi]\right\|_{\mathcal{G},\mathbf{j}}(x) \lesssim 2^{\omega(S)\mathrm{ext}^{\mathbf{j}}(S)}\|K(S)\|_{\mathfrak{t}} \cdot \|\xi\|_{|L(S)|,\mathfrak{c}} \cdot \|\varphi\|_{\mathcal{F},\mathbf{j}}(x) \;. \tag{8.37}$$

This is not hard to see since

$$\begin{aligned}
&\|\hat{H}^{\mathbf{j}}_S[\varphi]\|_{\mathcal{G},\mathbf{j}}(x) \\
&= \sup \left\{ \left|\left(\mathbb{D}^{b,\mathbf{j}}_{\mathcal{G}}\hat{H}^{\mathbf{j}}_S[\varphi]\right)(x,y)\right| : \begin{array}{l} b \in \mathrm{Der}(\mathcal{G}) \text{ and} \\ y \in \mathrm{Dom}(\mathcal{G},\mathbf{j},x) \end{array} \right\} \\
&= \sup \left\{ \left|\hat{H}^{\mathbf{j}}_S[\mathbb{D}^{b,\mathbf{j}}_{\mathcal{G}}\varphi](x,y)\right| : \begin{array}{l} b \in \mathrm{Der}(\mathcal{G}) \text{ and} \\ y \in \mathrm{Dom}(\mathcal{G},\mathbf{j},x) \end{array} \right\} \\
&\lesssim \sup \left\{ \left|\left(\mathbb{D}^{\bar{b},\mathbf{j}}_{\{S\}}\mathbb{D}^{b,\mathbf{j}}_{\mathcal{G}}\varphi\right)(x,y,z)\right| : \begin{array}{l} \bar{b} \in \mathrm{Der}(\{S\}),\ b \in \mathrm{Der}(\mathcal{G}), \\ y_{\tilde{N}_{\mathcal{G}}} \in \mathrm{Dom}(\mathcal{G},\mathbf{j},x), \\ \text{and } z_{\tilde{N}(S)} \in \mathrm{Dom}(\{S\},\mathbf{j},x) \end{array} \right\} \\
&\quad \cdot 2^{\omega(S)\mathrm{ext}^{\mathbf{j}}(S)}\|K(S)\|_{\mathfrak{t}} \cdot \|\xi\|_{|L(S)|,\mathfrak{c}} \;,
\end{aligned}$$

where we used the inductive hypothesis for $\{S\}$ in the final inequality. $\qquad\square$

In the following lemma and later sections we use the following notation: for any $K(\overline{T})$ connected subset $A \subset N(\overline{T})$ we write $T(A)$ for the maximal subtree of $\overline{T}$ with true node set $A$ and also write $\varrho_A \overset{\text{def}}{=} \varrho_{T(A)}$. Observe that if $e \in L(\overline{T})$ with $e_{\mathrm{p}} \in A$ then $e \in L(T(A))$.

**Lemma 8.13** *In the context of Lemma 8.12, the total homogeneity $\varsigma$ given by (8.29) is subdivergence free in $N_{\mathcal{B}}(S)$ for the set of scales $\mathcal{N}_{\mathbf{G}}$.*

*Proof.* Fix an arbitrary $\mathbf{T} \in \mathcal{U}_{\mathcal{V}}$. For any $a \in \mathring{\mathbf{T}}$ we define

$$\mathbf{N}(a) \overset{\text{def}}{=} \mathbf{L}_a \sqcup \Big( \bigsqcup_{\substack{T \in C_{\mathcal{B}}(S) \\ \varrho_T \in \mathbf{L}_a}} \tilde{N}(T) \Big) \;. \tag{8.38}$$

We also set

$$\mathcal{Q} \overset{\text{def}}{=} \Big\{ \mathbf{N}(a) : \ a \in \mathring{\mathbf{T}} \setminus \{\varrho_{\mathbf{T}}\} \Big\} \;. \tag{8.39}$$

Observe that $N(T) \notin \mathcal{Q}$ for any $T \in C_{\mathcal{B}}(S)$ and that any node-set $M \in \mathcal{Q}$ is edge connected by $\mathcal{E}^{\mathrm{int}}(S)$.

We define a map $\tilde{\varsigma} : 2^{N(S)} \setminus \{\varnothing\} \to \mathbf{R}$ as follows: for $M \subset N(S)$ we set

$$\tilde{\varsigma}(M) \overset{\text{def}}{=} \sum_{\substack{e \in K(S) \\ e \subset M}} \mathfrak{h}(e) - \Big( \sum_{\substack{B \in \pi, \\ B \not\subset M}} |\mathfrak{t}(B \cap M)|_{\mathfrak{s},\mathfrak{c},L(\overline{T})} \Big) \tag{8.40}$$

$$- (|M| - 1)|\mathfrak{s}| - \sum_{\substack{B \in \pi \\ B \subset M}} \left[ |\mathfrak{t}(B)|_\mathfrak{s} + \ell(B)\mathbb{1}\{M = B\} \right] .$$

Observe that if $M \subset N(S)$ is $K(S)$-connected then $\tilde{\varsigma}(M) \leq -|T(M)^0_{\mathfrak{e}}|_\mathfrak{s}$. Furthermore, it is straightforward to check that for any $a \in \mathring{\mathbf{T}} \setminus \{\varrho_\mathbf{T}\}$,

$$\sum_{b \in \mathbf{T}_{\geq a}} \varsigma_\mathbf{T}(b) - (|\mathbf{L}_a| - 1)|\mathfrak{s}| = \tilde{\varsigma}(\mathbf{N}(a)) . \tag{8.41}$$

Our desired result will follow if we show $\tilde{\varsigma}(M) < 0$ for all $M \in \mathcal{Q}$. We do this by splitting into various cases. While we are not always explicit about it, at each point in this proof we are always restricting ourselves to the complement of all the cases treated previously.

First we assume $M \in \mathcal{Q}$ contains no $K(S)$-connected components of cardinality more than 1. In this subcase the first sum on the RHS of (8.41) vanishes and by $\mathcal{E}^{\text{int}}(S)$-connectivity this forces $M \subset L(S)$. Moreover, there must be a unique block $\bar{B} \in \pi$ with $M \subset \bar{B}$. Therefore $(\mathfrak{t}, M) \in \mathfrak{L}^{\text{all}}_{\text{Cum}}$ and $\tilde{\varsigma}(M)$ is bounded above by the RHS of (A.21) which is negative by Lemma A.26.

Next we treat the case where $M$ is not $K(S)$ connected but has at least one component of cardinality at least 2. Then, for some $k \geq 1$,

$$M = \left( \bigsqcup_{j=1}^{k} M_j \right) \sqcup \tilde{M} , \tag{8.42}$$

where the $M_j$ are the $K(S)$-components of $M$ of cardinality at least 2 and $\tilde{M} \subset L(S)$ is given by the union of the $K(S)$-components of $M$ of cardinality 1.

Now we treat the subcase where $k = 1$ and $\tilde{M}$ is non-empty. By $\mathcal{E}^{\text{int}}(S)$-connectivity there exist $B_1, \dots, B_n \in \pi$ such that $\cup_{m=1}^n B_m \supset [\tilde{M} \sqcup L(T(M_1))]$ and, for every $m \in [n]$, one has $B_m \cap L(T(M_1)) \neq \emptyset$. Now if it is the case that for every $m$ one has $B_m \cap M \in \mathfrak{L}_{\text{Cum}}$ then we are done since one has

$$\tilde{\varsigma}(M) \leq -|T(M_1)^0_{\mathfrak{e}}|_\mathfrak{s} - |\mathfrak{t}(\tilde{M})|_\mathfrak{s} - |\mathfrak{s}| \cdot |\tilde{M}| < 0$$

where the second inequality comes from (2.13). On the other hand, if there exists $m \in [n]$ for which $B_m \cap M \notin \mathfrak{L}_{\text{Cum}}$ then it follows that $B_m \cap M = B_m \cap L(T(M_1)) = \{u\}$ so $|\mathfrak{t}(B_m \cap M)|_{\mathfrak{s},\mathfrak{e},L(\overline{T})} = 0$. Thus by adding and subtracting $|\mathfrak{t}(u)|_\mathfrak{s}$ we get

$$\tilde{\varsigma}(M) \leq -\left[ |T(M_1)^0_{\mathfrak{e}}|_\mathfrak{s} - |\mathfrak{t}(\mathfrak{u})|_\mathfrak{s} \right] - |\mathfrak{t}(\tilde{M})|_\mathfrak{s} - |\mathfrak{s}| \cdot |\tilde{M}| < 0 ,$$

where in the second inequality we used (2.13) to see that the bracketed quantity is strictly positive. This finishes the subcase where $k = 1$ and $|\tilde{M}| \neq 0$. We next treat the subcase where $k \geq 2$ and $|\tilde{M}|$ is arbitrary. Then one has

$$\tilde{\varsigma}(M) \leq -(k-1)|\mathfrak{s}| - \sum_{j=1}^{k} |T(M_j)^0_{\mathfrak{e}}|_\mathfrak{s} - |\mathfrak{t}(\tilde{M})|_\mathfrak{s} - |\mathfrak{s}| \cdot |\tilde{M}| \tag{8.43}$$

$$\leq -(k-1)|\mathfrak{s}| - \sum_{j=1}^{k} |T(M_j)_{\bar{\mathfrak{c}}}^0|_{\mathfrak{s}} < -(k-1)|\mathfrak{s}| + \frac{|\mathfrak{s}|}{2}k \leq 0 \,,$$

where in last inequality we used (2.13).

We now treat the case where $M \in \mathcal{Q}$, is $K(S)$-connected, and $T(M)$ is not compatible with $\pi$. It follows that there exists some $B \in \pi$ with $B \not\subset L(T(M))$ and $B \cap L(T(M)) \neq \emptyset$. Then one has

$$\tilde{\varsigma}(M) = -|T(M)_{\bar{\mathfrak{c}}}^0|_{\mathfrak{s}} + (|\mathfrak{t}(B \cap L(T(M)))|_{\mathfrak{s}} - |\mathfrak{t}(B \cap L(T(M)))|_{\mathfrak{s},\mathfrak{c},L(\overline{T})}) < 0 \,,$$

where the final inequality comes Definition A.27.

We are now left with the case that $M \in \mathcal{Q}$, $M$ is $K(S)$-connected, and $T(M)$ is compatible with $\pi$. In this case $\tilde{\varsigma}(M) = -|T(M)_{\bar{\mathfrak{c}}}^0|_{\mathfrak{s}}$. We claim that indeed $|T(M)_{\bar{\mathfrak{c}}}^0|_{\mathfrak{s}} > 0$ as required since one would otherwise have $T(M) \in \text{Div}$ and, combining this with the fact that $T(M)$ is compatible with $\mathcal{B}$ and $\pi$, this would force $T \notin \mathcal{Q}$ by the definition of the set of scale assignments $\mathring{\mathcal{N}}_S$. $\qquad\square$

### 8.3.3 Estimates with positive renormalizations

To make formulas more readable we introduce the notation $\exp_2[t] \stackrel{\text{def}}{=} 2^t$ for $t \in \mathbf{R}$. Also, for $u, v \in \tilde{N}(\mathcal{B})^c$ we define $\mathbf{j}_{\mathcal{B}}(u,v)$ as in (7.8).

**Lemma 8.14** *Let $e \in \mathscr{B}$ and $c > 0$. Uniform in $\mathbf{j} \in \partial\mathcal{N}_{\mathcal{B},\lambda}$ satisfying*

$$|\mathbf{j}_{\mathcal{B}}(e_{\mathrm{c}}, e_{\mathrm{p}}) - j_e|, \ |\mathbf{j}_{\mathcal{B}}(e_{\mathrm{p}}, \circledast) - j_{\{e_p, \circledast\}}| < c \,, \tag{8.44}$$

*any multi-index $k$ supported on $\{e_{\mathrm{c}}, e_{\mathrm{p}}\}$, and $x = (x_{\circledast}, x_{e_{\mathrm{c}}}, x_{\varepsilon_{\mathrm{p}}})$ such that $x_{e_{\mathrm{c}}} \stackrel{j_{\{\circledast, e_{\mathrm{c}}\}}}{\longleftrightarrow} x_{\circledast}$ and $x_{e_{\mathrm{p}}} \stackrel{j_{\{\circledast, e_{\mathrm{p}}\}}}{\longleftrightarrow} x_{\circledast}$, one has the estimate*

$$\left| D^k \widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\{e\}}(x) \right| \lesssim \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)} 2^{\eta(j_e, j_{\{\circledast, e_{\mathrm{p}}\}}, j_{\{\circledast, e_{\mathrm{c}}\}}, k_{e_{\mathrm{c}}}, k_{e_{\mathrm{p}}}, e)}$$

*where $\eta(j_e, j_{\{\circledast, e_{\mathrm{p}}\}}, j_{\{\circledast, e_{\mathrm{c}}\}}, k_{e_{\mathrm{c}}}, k_{e_{\mathrm{p}}}, e)$ is given by*

$$\begin{cases} \gamma(e)(j_e - j_{\{\circledast, e_{\mathrm{p}}\}}) + (\mathfrak{h}(e) + |k_{e_{\mathrm{c}}}|_{\mathfrak{s}})j_e + |k_{e_{\mathrm{p}}}|_{\mathfrak{s}}j_{\{\circledast, e_{\mathrm{p}}\}} & \text{if } e \in \mathscr{D}, \\ -(\gamma(e) - 1 + |k_{e_{\mathrm{p}}}|_{\mathfrak{s}})j_{\{\circledast, e_{\mathrm{p}}\}} + (|k_{e_{\mathrm{c}}}|_{\mathfrak{s}} + \mathfrak{h}(e) + \gamma(e) - 1)j_{\{\circledast, e_{\mathrm{c}}\}} & \text{if } e \in \mathscr{S} \,. \end{cases}$$

*Furthermore, the LHS vanishes unless the condition $x_{e_{\mathrm{c}}} \stackrel{j_e}{\longleftrightarrow} x_{e_{\mathrm{p}}}$ holds.*

*Proof.* We prove the lemma in the case when $k = 0$, upon inspection of how each type of derivative modifies a Taylor expansion (or Taylor remainder) then the estimate for $k \neq 0$ follows easily.

We first treat the case when $e \in \mathscr{D}$ in which case $\widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\{e\}}(x)$ is given by

$$\left( \mathrm{Ker}_0^{\{e\}}(x_{e_{\mathrm{p}}}, x_{e_{\mathrm{c}}}) - \sum_{m \in \mathrm{Der}(e)} \frac{(x_{e_{\mathrm{p}}} - x_{\circledast})^m}{m!} (D^m \mathrm{Ker}_0^{\{e\}})(x_{\circledast}, x_{e_{\mathrm{c}}}) \right) \Psi^{j_e}(x_{e_{\mathrm{p}}} - x_{e_{\mathrm{c}}}),$$

$$\tag{8.45}$$

where $\mathrm{Der}(e)$ denotes the set of multi-indices supported on $e_{\mathrm{p}}$ with $|m|_{\mathfrak{s}} < \gamma(e)$. By using [Hai14, Prop. A.1] it follows that, uniform in $x$ satisfying $x_{e_{\mathrm{p}}} \overset{je}{\longleftrightarrow} x_{e_{\mathrm{c}}}$ (enforced by the second factor of (8.45)) and $x_{e_{\mathrm{p}}} \overset{j_{\{\otimes, e_{\mathrm{p}}\}}}{\longleftrightarrow} x_{\otimes}$ one has a bound

$$
\left| \mathrm{Ker}_0^{\{e\}}(x_{e_{\mathrm{p}}}, x_{e_{\mathrm{c}}}) - \sum_{m \in \mathrm{Der}(e)} \frac{(x_{e_{\mathrm{p}}} - x_{\otimes})^m}{m!} (D^m \mathrm{Ker}_0^{\{e\}})(x_{\otimes}, x_{e_{\mathrm{c}}}) \right|
$$

$$
\lesssim \sup_{n \in \partial_{e_{\mathrm{p}}} \mathrm{Der}(e)} |x_{e_{\mathrm{p}}} - x_{\otimes}|^{|n|_s} \cdot \sup_{\substack{y, y' \in \mathbf{R}^d \\ y \overset{je}{\longleftrightarrow} y'}} |D^n \mathrm{Ker}_0^{\{e\}}(y, y')|
$$

$$
\lesssim \max_{n \in \partial_{e_{\mathrm{p}}} \mathrm{Der}(e)} 2^{-j_{\{\otimes, e_{\mathrm{p}}\}}|n|_s} \cdot \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)} \cdot 2^{(\mathfrak{h}(e) + |n|_{\mathfrak{s}})j_e} \ .
$$

Here the use of the scale $j_e$ vs $j_{\{\otimes, e_{\mathrm{c}}\}}$ is interchangeable – the combinations of the conditions (8.44), the fact that $\mathbf{j}(e_{\mathrm{p}}, \otimes) > \mathbf{j}(e_{\mathrm{c}}, e_p)$, $x_{e_{\mathrm{p}}} \overset{je}{\longleftrightarrow} x_{e_{\mathrm{c}}}$, $x_{e_{\mathrm{p}}} \overset{j_{\{\otimes, e_{\mathrm{p}}\}}}{\longleftrightarrow} x_{\otimes}$, and $x_{e_{\mathrm{c}}} \overset{j_{\{\otimes, e_{\mathrm{c}}\}}}{\longleftrightarrow} x_{\otimes}$ and the triangle inequality mean that there exists some combinatorial constant $C > 0$, depending on $c$, such that if $|j_{\{\otimes, e_{\mathrm{c}}\}} - j_e| \le C$ does not hold then our domain in $x$ is empty.

The desired bound now follows upon using the constraint $x_{e_{\mathrm{p}}} \overset{j_{\{\otimes, e_{\mathrm{p}}\}}}{\longleftrightarrow} x_{\otimes}$, the fact that $j_{\{\otimes, e_{\mathrm{p}}\}} + 2c > j_e$, and that $\min_{n \in \partial_{e_{\mathrm{p}}} \mathrm{Der}(e)} |n|_{\mathfrak{s}} = \gamma(e)$. Now we turn into the case where $e \in \mathscr{S}$. By a similar triangle inequality argument we may assume that there exists some combinatorial constant $C > 0$, depending on $c$, such that

$$
j_{\{\otimes, e_{\mathrm{c}}\}} + C \ge j_e \ge j_{\{\otimes, e_{\mathrm{p}}\}} \ . \tag{8.46}
$$

Under this assumption we do a term by term estimate which gives

$$
\left| \sum_{m \in \mathrm{Der}(e)} \frac{(x_{e_{\mathrm{p}}} - x_{\otimes})^m}{m!} D^m \mathrm{Ker}_0^{\{e\}}(x_{\otimes}, x_{e_{\mathrm{c}}}) \right|
$$

$$
\lesssim \max_{m \in \mathrm{Der}(e)} |(x_{e_{\mathrm{p}}} - x_{\otimes})^m| \cdot \sup_{\substack{y, y' \in \mathbf{R}^d \\ y \overset{j_{\{\otimes, e_{\mathrm{c}}\}}}{\longleftrightarrow} y'}} |D^m \mathrm{Ker}_0^{\{e\}})(y, y')|
$$

$$
\lesssim \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)} \cdot \max_{m \in \mathrm{Der}(e)} 2^{(\mathfrak{h}(e) + |m|_{\mathfrak{s}})j_{\{\otimes, e_{\mathrm{c}}\}} - |m|_{\mathfrak{s}} j_{\{\otimes, e_{\mathrm{p}}\}}}
$$

$$
\lesssim \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)} \cdot 2^{(\mathfrak{h}(e) + \gamma(e) - 1)j_{\{\otimes, e_{\mathrm{c}}\}} - (\gamma(e) - 1)j_{\{\otimes, e_{\mathrm{p}}\}}} \ ,
$$

where in the final inequality we used (8.46) and that $\max_{m \in \mathrm{Der}(e)} |m|_{\mathfrak{s}} = \gamma(e) - 1$. $\qquad \square$

**Lemma 8.15** *Let $S \in \overline{\mathcal{B}}$, $c > 0$, and $k \in (\mathbf{N}^d)^{N^*}$ be supported on $N^{\downarrow}(S) \sqcup \{\varrho_S\}$. One has, uniform in $\mathbf{j} \in \mathcal{E}^{\mathrm{ext}}(S)$ satisfying (8.44) for every $e \in \mathfrak{C}_{\mathcal{B}}$, and $x \in \mathbf{R}^{\tilde{N}(S)^c}$ satisfying*

$$
x_{\varrho_S} \overset{j_{\{\otimes, \varrho_S\}}}{\longleftrightarrow} x_{\otimes} \ , \quad x_{e_{\mathrm{c}}} \overset{j_{\{\otimes, e_{\mathrm{c}}\}}}{\longleftrightarrow} x_{\otimes} \quad \forall e \in K^{\downarrow}(S) \ ,
$$

*the bound,*

$$\left| D^k \hat{H}_S^{\mathbf{j}} \left[ \widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\mathscr{B} \cap K^{\downarrow}(S)} \mathrm{Ker}_{0,\mathbf{j}}^{K^{\downarrow}(S) \setminus \mathscr{B}} X_{\overline{\mathfrak{n}},\mathbf{j},\otimes}^{\tilde{N}(S)} \right] (x) \right|$$

$$\lesssim \exp_2 \left[ \omega(S) \mathrm{ext}^{\mathbf{j}}(S) + j_{(\varrho_S,\otimes)} |\overline{\mathfrak{n}}(\tilde{N}(S))|_{\mathfrak{s}} + \sum_{e \in K^{\downarrow}(S) \setminus \mathscr{B}} (\mathfrak{h}(e) + |k_{e_{\mathrm{c}}}|_{\mathfrak{s}}) j_e \right.$$

$$+ \left( \sum_{e \in K^{\downarrow}(S) \cap \mathscr{S}} -(\gamma(e) - 1) j_{\{\otimes,\varrho_S\}} + (\mathfrak{h}(e) + |k_{e_{\mathrm{c}}}|_{\mathfrak{s}} + \gamma(e) - 1) j_{\{\otimes,e_{\mathrm{c}}\}} \right)$$

$$+ \left( \sum_{e \in K^{\downarrow}(S) \cap \mathscr{D}} \gamma(e)(j_e - j_{(\otimes,\varrho_S)}) + (\mathfrak{h}(e) + |k_{e_{\mathrm{c}}}|_{\mathfrak{s}}) j_e \right) \right]$$

$$\cdot \|\overline{K}^{\downarrow}(S)\|_{\mathfrak{t}} \cdot \|\xi\|_{|L(S)|} \ .$$

$$(8.47)$$

*Furthermore, there exists a combinatorial constant $C > 0$ such that the LHS vanishes unless $x_{\varrho_S} \xleftrightarrow{j_e} x_{e_{\mathrm{c}}}$ for all $e \in K^{\downarrow}(S)$, and $|j_{\{\otimes,u\}} - j_{(\otimes,\varrho_S)}| \leq C$ for each $u \in \tilde{N}(S)$.*

*Proof.* First observe that the differential operator $D^k$ commutes with $\hat{H}_S^{\mathbf{j}}$. We use Lemma 8.26, the desired bound is obtained by estimating $\|D^k\Theta\|_{S,\mathbf{j}}(x_{\tilde{N}(S)^c})$ where $\Theta$ is given by either: $X_{\overline{\mathfrak{n}},\mathbf{j},\otimes}^{\{u\}}$ for $u \in \tilde{N}(S)$, $\mathrm{Ker}_{0,\mathbf{j}}^{\{e\}}$ for $e \in K^{\downarrow}(S)$, or $\widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\{e\}}$ for $e \in K^{\downarrow}(S) \cap \mathscr{B}$.

The proof of the first two scenarios mirror that of Lemma 8.10 and one obtains

$$\|D^k\Theta\|_{S,\mathbf{j}}(x_{\tilde{N}(S)^c}) \lesssim \begin{cases} 2^{-|\overline{\mathfrak{n}}(u)|_{\mathfrak{s}} j_{\{\otimes,u\}}} \mathbb{1}\{x_{\varrho_S} \xleftrightarrow{j_{\{\otimes,u\}}} x_{\otimes}\} & \text{if } \Theta = X_{\overline{\mathfrak{n}},\mathbf{j},\otimes}^{\{u\}} \ , \\ 2^{(\mathfrak{h}(e) + |k_{e_{\mathrm{c}}}|_{\mathfrak{s}}) j_e} \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)} \mathbb{1}\{x_{\varrho_S} \xleftrightarrow{j_e} x_{e_{\mathrm{c}}}\} & \text{if } \Theta = \mathrm{Ker}_{0,\mathbf{j}}^{\{e\}} \ . \end{cases}$$

Our bound for the factor $X_{\overline{\mathfrak{n}},\otimes,\mathbf{j}}^{\tilde{N}(S)}$ then gives us the constraint that $x_{\varrho_S} \xleftrightarrow{j_{\{\otimes,u\}}} x_{\otimes}$ for every $u \in \tilde{N}(S)$. Combining this with the inherited $x_{\varrho_S} \xleftrightarrow{j_{\{\otimes,\varrho_S\}}} x_{\otimes}$ constraint it follows that there exists a combinatorial constant $C > 0$ such that the LHS of (8.47) vanishes unless

$$|j_{\{\otimes,u\}} - j_{(\otimes,\varrho_S)}| \leq C \text{ for every } u \in \tilde{N}(S) \ . \tag{8.48}$$

We now turn to the case where $\Theta = \widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\{e\}}$ for $e \in K^{\downarrow}(S) \cap \mathscr{B}$ which further splits into two subcases depending on whether $e \in \mathscr{D}$ or $e \in \mathscr{S}$. We claim that one has

$$\|D^k \widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\{e\}}\|_{S,\mathbf{j}}(x_{\tilde{N}(S)^c}) \lesssim \mathbb{1}\{x_{\varrho_S} \xleftrightarrow{j_e} x_{e_{\mathrm{c}}}\} \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)} 2^{\eta(j_e, j_{\{\otimes,\varrho_S\}}, j_{\{\otimes,e_{\mathrm{c}}\}}, k_{e_{\mathrm{c}}}, 0, e)} \ ,$$

where $\eta(\cdot)$ is as in the statement of Lemma 8.14. We only sketch the proofs here since the details are very similar to those of Lemma 8.14.

Writing $\widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\{e\}}(z) = \widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\{e\}}(z_{e_{\mathrm{p}}}, z_{e_{\mathrm{c}}}, z_{\otimes})$ we claim that for any $b \in \widetilde{\mathrm{Der}}(\{S\})$ we have, using a Taylor remainder bound or a term-by-term Taylor series bound, the following estimate, uniform $\mathbf{j} \in \partial \mathcal{N}_S$ satisfying our assumptions, (8.48) and in $x_{\tilde{N}(S)^c}$ satisfying both $x_{\varrho_S} \overset{j_{\{\otimes, \varrho_S\}}}{\longleftrightarrow} x_{\otimes}$ and $x_{e_{\mathrm{c}}} \overset{j_{\{\otimes, e_{\mathrm{c}}\}}}{\longleftrightarrow} x_{\otimes}$,

$$
\sup_{y_{\tilde{N}(S)} \in \mathrm{Dom}(\{S\}, \mathbf{j}, x)} \left| D^{b+k} \widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\{e\}}(y_{e_{\mathrm{p}}}, x_{e_{\mathrm{c}}}, x_{\otimes}) \right| \tag{8.49}
$$

$$
\lesssim \sup_{\substack{y_{\tilde{N}(S)} \in \mathrm{Dom}(\{S\}, \mathbf{j}, x_{\varrho_S}) \\ n \in A_e}} |y_{u(e)} - x_{\otimes}|^{|n|_{\mathfrak{s}} - |b_{e_{\mathrm{p}}}|_{\mathfrak{s}}} \cdot \sup_{\substack{w, w' \in \mathbf{R}^d \\ w \overset{je}{\longleftrightarrow} w'}} |D^{n+k_{e_{\mathrm{c}}}} \mathrm{Ker}_0^{\{e\}}(w - w')|
$$

$$
\lesssim \|K_{\mathfrak{t}(e)}\|_{\mathfrak{t}(e)} \cdot 2^{(\mathfrak{h}(e) + |k_{e_{\mathrm{c}}}|_{\mathfrak{s}})\theta_2(\mathbf{j}, e) + |b_{e_{\mathrm{p}}}|_{\mathfrak{s}} j_{\{\otimes, \varrho_S\}}} \max_{n \in A_e} 2^{|n|_{\mathfrak{s}} \theta_1(\mathbf{j}, e)}.
$$

Here, if $e \in \mathscr{D}$ one has $u(e) \overset{\mathrm{def}}{=} e_{\mathrm{p}}$, $A_e \overset{\mathrm{def}}{=} \partial_{e_{\mathrm{p}}} \mathrm{Der}(e)$, $\theta_1(\mathbf{j}, e) \overset{\mathrm{def}}{=} (j_e - j_{\{\otimes, \varrho_S\}})$, and $\theta_2(\mathbf{j}, e) \overset{\mathrm{def}}{=} j_e$. On the other hand, if $e \in \mathscr{S}$ then $A_e \overset{\mathrm{def}}{=} \mathrm{Der}(e)$, $\theta_1(\mathbf{j}, e) \overset{\mathrm{def}}{=} (j_{\{\otimes, e_{\mathrm{c}}\}} - j_{\{\otimes, \varrho_S\}})$, and $\theta_2(\mathbf{j}, e) \overset{\mathrm{def}}{=} j_{\{\otimes, e_{\mathrm{p}}\}}$.

We also used $y_{e_{\mathrm{p}}} \overset{j_{\{\otimes, \varrho_S\}}}{\longleftrightarrow} x_{\otimes}$ which comes from the inherited constraint $x_{\varrho_S} \overset{j_{\{\otimes, \varrho_S\}}}{\longleftrightarrow} x_{\otimes}$ along with the domain of the supremum for $y_{e_{\mathrm{p}}}$. We also used the fact that by (8.44) and (8.48) there is a combinatorial constant $C' > 0$, depending on $c$, such that $j_{\{\otimes, e_{\mathrm{p}}\}} + C' > j_e$ if $e \in \mathscr{D}$ and $j_{\{\otimes, e_{\mathrm{p}}\}} \le j_e + C'$ if $e \in \mathscr{S}$ which is good enough, the discrepancies $C'$ only contributes to the constant of proportionality in our bound.

The desired bound on $\|\widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\{e\}}\|_{S,\mathbf{j}}(x_{\tilde{N}(S)^c})$ follows from (8.49) by observing from multiplying the last line of (8.49) by $2^{-\mathrm{ext}^{\mathbf{j}}(S)|b|_{e_{\mathrm{p}}}}$, $\mathrm{ext}^{\mathbf{j}}(S) \ge j_{\{\otimes, \varrho_S\}}$, and using the observation of the above paragraph when taking the maximum over $n$. For the indicator function on the RHS one observes that the LHS of (8.49) vanishes unless $x_{\varrho_S} \overset{je}{\longleftrightarrow} x_{e_{\mathrm{c}}}$, this follows by similar reasoning as in Lemma 8.10. $\qquad \square$

## 8.4  Full control over a single tree

We start by taking a quotient of the multigraph $\mathcal{E}^{\mathrm{int}}(\overline{T}^*)$ (defined in (8.2)) by contracting every $T \in \mathcal{B}$ and identifying it with $\varrho_T$, doing this we obtain a multigraph $\mathbf{H}$ on the vertex set $\mathcal{X} \overset{\mathrm{def}}{=} \tilde{N}(\overline{\mathcal{B}})^c$. In particular,

$$
\mathbf{H} \overset{\mathrm{def}}{=} K(\mathcal{B}, \overline{T}) \sqcup \left\{ \{e_{\mathrm{c}}, \varrho_T\} : \begin{array}{l} T \in \overline{\mathcal{B}}, \\ e \in K^{\downarrow}(T) \end{array} \right\} \sqcup \{e \in \mathcal{E}_\pi : e \subset L(\mathcal{B}, T)\} . \tag{8.50}
$$

We write $\mathfrak{q} : \mathbf{N}^{\mathcal{E}^{\mathrm{int}}(\overline{T}^*)} \to \mathbf{N}^{\mathbf{H}}$ for the natural bijection between these two sets and then set $\mathcal{N}_{\mathbf{H}, \lambda} \overset{\mathrm{def}}{=} \mathfrak{q}(\partial \mathcal{N}_{\mathcal{B}, \lambda})$ and $\mathcal{N}_{\mathbf{H}} \overset{\mathrm{def}}{=} \cup_{\lambda \in (0,1]} \mathcal{N}_{\mathbf{H}, \lambda}$.

From now on we switch viewpoints replacing, for every $\lambda \in (0, 1]$, the family $\mathring{\mathcal{W}}_\lambda = (\mathring{\mathcal{W}}_\lambda^{\mathbf{j}})_{\mathbf{j} \in \partial \mathcal{N}_{\mathcal{B}, \lambda}}$ with $\mathring{\mathcal{W}}_\lambda = (\mathring{\mathcal{W}}_\lambda^{\mathbf{n}})_{\mathbf{n} \in \mathcal{N}_{\mathbf{H}}}$ where we set, for $\mathbf{n} \in \mathcal{N}_{\mathbf{H}}$, $\mathring{\mathcal{W}}_\lambda^{\mathbf{n}} = \mathring{\mathcal{W}}_\lambda^{\mathfrak{q}^{-1}(\mathbf{n})}$ if $\mathbf{n} \in \mathcal{N}_{\mathbf{H}, \lambda}$ and $\mathring{\mathcal{W}}_\lambda^{\mathbf{n}} = 0$ otherwise.

We also define $\hat{\mathfrak{q}} : N^* \to \mathcal{X}$ via setting $\hat{\mathfrak{q}}(u) \overset{\mathrm{def}}{=} \varrho_T$ if there exists $T \in \overline{\mathcal{B}}$ with $u \in N(T)$, we set $\hat{\mathfrak{q}}(u) \overset{\mathrm{def}}{=} u$ otherwise.

We now define a total homogeneity $\varsigma$ on the trees of $\widehat{\mathcal{U}}_{\mathcal{X}}$ by setting

$$\varsigma \overset{\text{def}}{=} - \sum_{\substack{u \in N(\overline{T})}} |\overline{\mathfrak{n}}(u)|_{\mathfrak{s}} \delta^{\uparrow}[\{\hat{\mathfrak{q}}(u), \circledast\}] + \sum_{\substack{B \in \pi \\ B \subset L(\mathcal{B}, \overline{T})}} \mathfrak{c}^B + \sum_{T \in \overline{\mathcal{B}}} \omega(T) \delta^{\uparrow}[\varrho_T] + \varsigma^R$$

$$+ \sum_{\substack{e \in K(\mathcal{B}, \overline{T}) \sqcup K^{\downarrow}(\overline{\mathcal{B}}) \\ e \notin \mathscr{S}}} \mathfrak{h}(e) \delta^{\uparrow}[\{e_{\mathrm{c}}, \hat{\mathfrak{q}}(e_{\mathrm{p}})\}] + \sum_{e \in \mathscr{D}} \gamma(e) \Big( \delta^{\uparrow}[\{e_{\mathrm{c}}, \hat{\mathfrak{q}}(e_{\mathrm{p}})\}] - \delta^{\uparrow}[\{\circledast, \hat{\mathfrak{q}}(e_{\mathrm{p}})\}] \Big)$$

$$+ \Big[ \sum_{e \in \mathscr{S}} (\gamma(e) - 1) \Big( \delta^{\uparrow}[\{e_{\mathrm{c}}, \circledast\}] - \delta^{\uparrow}[\{\circledast, \hat{\mathfrak{q}}(e_{\mathrm{p}})\}] \Big) + \mathfrak{h}(e) \delta^{\uparrow}[\{e_{\mathrm{c}}, \circledast\}] \Big],$$

$$(8.51)$$

where the the total homogeneity $\varsigma^R$ is defined by setting, for each $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{X}}$,

$$\varsigma_{\mathbf{T}}^R \overset{\text{def}}{=} \sum_{B \in R(\mathbf{T})} \mathcal{f}(B) \Big( \delta^{\Uparrow}[B] - \delta^{\uparrow}[B] \Big)$$

and $R(\mathbf{T}) \overset{\text{def}}{=} \big\{ B \in \pi : \ B \subset L(\mathcal{B}, \overline{T}), \ \mathcal{f}(B) > 0, \ \text{and } \mathbf{L}_{B\uparrow} = B \big\}$. We can then state the final lemma of this section.

**Lemma 8.16** *If one defines $\varsigma$ as in* (8.51)*, then uniform in $\lambda \in (0, 1]$, there exists a $F_\lambda \in \mathrm{Mod}_\ell(\mathring{\mathcal{W}}_\lambda)$ such that*

$$\|F_\lambda\|_{\varsigma, \mathcal{N}_{\mathbf{H}}, \tilde{L}} \lesssim \lambda^{-|\mathfrak{s}|} \|K(\overline{T})\|_{\mathfrak{t}} \cdot \|\xi\|_{|L(\overline{T}) \setminus \tilde{L}|} \,,$$

*where we are using the notation of* (A.13).

When we say this condition holds uniform in $\lambda$ we are including uniformity in the combinatorial constant appearing in (A.7).

*Proof.* We set, for each $\mathbf{n} \in \mathcal{N}_{\mathbf{H}}$,

$$F_\lambda^{\mathbf{n}} \overset{\text{def}}{=} \begin{cases} \mathring{\mathcal{W}}_\lambda^{\mathbf{n}} & \text{if } \mathbf{n} \in \mathcal{N}_{\mathbf{H}, \lambda} \text{ and } R(\mathcal{T}(\mathbf{n})) = \varnothing \,, \\ \mathring{F}_\lambda^{\mathbf{n}} & \text{if } \mathbf{n} \in \mathcal{N}_{\mathbf{H}, \lambda} \text{ and } R(\mathcal{T}(\mathbf{n})) \neq \varnothing \,, \\ 0 & \text{otherwise.} \end{cases}$$

We will define $\mathring{F}_\lambda^{\mathbf{n}}$ on a tree by tree basis later. It suffices to check the domain condition and desired supremum bounds in each of the three cases separately. For the first case one this is done by combining all the lemmas of Section 8.3, and all the previous lemmas of this section, note that by the condition (A.1) there is come combinatorial constant $c$ such that one has the condition (8.44) for every $e \in \mathcal{B}$ and every $\mathbf{n}$ appearing in our supremum .

For the second case we fix $\mathbf{T} \in \mathcal{U}_{\mathcal{X}}$ with $R(\mathbf{T}) \neq \varnothing$ and $\mathbf{n} \in \mathcal{N}_{\mathbf{H}, \lambda}$ with $\mathcal{T}(\mathbf{n}) = \mathbf{T}$. We then set

$$\mathring{F}_\lambda^{\mathbf{n}} \overset{\text{def}}{=} \mathrm{Cu}_{\pi, \mathbf{n}}^{\hat{R}(\mathbf{T})} \cdot \Big( \prod_{B \in R(\mathbf{T})} (1 - \mathscr{Y}_B) \Big) [\hat{G}_\lambda^{\mathbf{j}}] \,,$$

where $\hat{R}(\mathbf{T}) \overset{\text{def}}{=} \bigcup_{B \in R(\mathbf{T})} B$, the operators $\mathscr{Y}_B$ were defined in (8.8), and

$$\hat{G}_\lambda^{\mathbf{j}} \overset{\text{def}}{=} \psi_{\mathbf{j}}^\lambda \cdot \mathrm{Ker}_{0,\mathbf{j}}^{K(\mathcal{B},\overline{T}) \setminus \mathcal{B}} \mathrm{Cu}_{\pi,\mathbf{n}}^{L(\mathcal{B},\overline{T}) \setminus \hat{R}(\mathbf{T})} \widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\mathcal{B} \setminus K^{\downarrow}(\overline{\mathcal{B}})} \mathrm{X}_{\overline{\mathbf{n}},\otimes,\mathbf{j}}^{\tilde{N}(\mathcal{B},\overline{T})}$$
$$\cdot \prod_{S \in \overline{\mathcal{B}}} \hat{H}_S^{\mathbf{j}} \left[ \widetilde{\mathrm{RKer}}_{0,\mathbf{j}}^{\mathcal{B} \cap K^{\downarrow}(S)} \mathrm{Ker}_{0,\mathbf{j}}^{K^{\downarrow}(S) \setminus \mathcal{B}} \mathrm{X}_{\overline{\mathbf{n}},\mathbf{j},\otimes}^{\tilde{N}(S)} \right].$$

An important observation is that the sets $\tilde{L}$ and $\bigcup_{B \in R(\mathbf{T})} B$ must be disjoint. Therefore one can use the same argument as in Lemma 8.26 to show that for any $x \in (\mathbf{R}^d)^{\tilde{L}}$ one has

$$\int_{\mathcal{X}_0 \setminus \tilde{L}} dy \; \mathring{F}_\lambda^{\mathbf{n}}(x \sqcup y) = \int_{\mathcal{X}_0 \setminus \tilde{L}} dy \; \tilde{\mathcal{W}}_\lambda^{\mathbf{n}}(x \sqcup y).$$

The domain condition for $\mathring{F}_\lambda^{\mathbf{n}}$ is also straightforward to check. The desired supremum bound for $\mathring{F}_\lambda^{\mathbf{n}}$ follows by the same sort of Taylor remainder argument as in Lemma 8.26. For multi-indices $m$ supported on $\tilde{L}$ one gets the desired supremum bounds for on the corresponding derivative by following the same argument – note that the operator $D^m$ commutes with the $\mathscr{Y}_B$ and only acts on the $\hat{G}_\lambda^{\mathbf{j}}$. $\qquad\square$

## 9   Moments of a single tree

We now state the propositions which combined with Theorem A.32 establish the main theorem.

**Proposition 9.1** *The total homogeneity $\varsigma$ given by (8.51) satisfies the conditions of Theorem A.32 with $\mathcal{X}$, $\mathcal{X}_0$, $\mathbf{H}$ and $\mathcal{N}_{\mathbf{H}}$ as given, $\otimes$ as the pinned vertex, $\mathbf{H}_* \overset{\text{def}}{=} \{\otimes, \varrho_*\}$, $\alpha \overset{\text{def}}{=} -|\overline{T}_{\tilde{\mathfrak{e}}}^{\overline{\mathbf{n}},0}| - |\mathfrak{t}(\tilde{L})|_{\mathfrak{s}} - |\mathfrak{s}|$, and $\ell \overset{\text{def}}{=} \tilde{L}$ with its type map $\mathfrak{t}$.*

*Proof.* It is straightforward to check $\varsigma$ is of order $-|\overline{T}_{\tilde{\mathfrak{e}}}^{\overline{\mathbf{n}},0}| - |\mathfrak{t}(\tilde{L})|_{\mathfrak{s}} - |\mathfrak{s}|$. We split the verification of (A.23) and (A.24) into Lemma 9.2 and Lemma 9.3. $\qquad\square$

Before stating and proving Lemma 9.2 and Lemma 9.3 we give introduce more notation and state an observation.

For any $u \in N(\overline{T})$ we write $\overline{T}_{\geq}(u)$ for the maximal subtree of $\overline{T}$ with true node set $\{v \in N(\overline{T}) : v \geq u\}$. Also, for any $K(\overline{T})$ connected $M \subset N(\overline{T})$ we write

$$M^{\mathrm{cu}} \overset{\text{def}}{=} (L(\overline{T}) \setminus \tilde{L}) \cap M, \; M^{\mathrm{ext}} \overset{\text{def}}{=} \tilde{L} \cap M,$$
$$K(M) \overset{\text{def}}{=} K(T(M)), \; \text{ and } \; K^{\downarrow}(M) \overset{\text{def}}{=} K^{\downarrow}(T(M)).$$

Finally, an important observation we will use is that for any $K(\overline{T})$-connected subset $A \subset N(\overline{T})$ one has

$$|T(A)_{\tilde{\mathfrak{e}}}^{\overline{\mathbf{n}}}|_{\mathfrak{s}} + \sum_{e \in K^{\downarrow}(A)} |\tilde{P}[\overline{T}_{\geq}(e)_{\tilde{\mathfrak{e}}}^{\overline{\mathbf{n}}}]|_{\mathfrak{s}} = |\overline{T}_{\geq}(\varrho_A)_{\tilde{\mathfrak{e}}}^{\overline{\mathbf{n}}}|. \tag{9.1}$$

**Lemma 9.2** *The total homogeneity $\varsigma$, as defined by* (8.51)*, satisfies condition* (A.23) *of Theorem A.32.*

*Proof.* We define a map $\tilde{\varsigma} : 2^{N^*} \to \mathbf{R}$ as follows, for $M \subset N^*$ we set

$$
\tilde{\varsigma}(M) \stackrel{\text{def}}{=} -(|M|-1)|\mathfrak{s}| + \sum_{\substack{e \in K(\overline{T}) \setminus \mathscr{S} \\ e \subset M}} \mathfrak{h}(e) \tag{9.2}
$$

$$
- \mathbb{1}\{\otimes \in M\} \left[ |\overline{\mathfrak{n}}(M \setminus \{\otimes\})|_{\mathfrak{s}} + \sum_{e \in \mathscr{D}} \gamma(e) \mathbb{1} \left\{ \begin{array}{c} e_{\text{c}} \notin M \\ e_{\text{p}} \in M \end{array} \right\} \right]
$$

$$
+ \mathbb{1}\{\otimes \in M\} \sum_{e \in \mathscr{S}} \left[ \mathfrak{h}(e) + (\gamma(e)-1)\mathbb{1}\{e_{\text{p}} \notin M\} \right] \mathbb{1}\{e_{\text{c}} \in M\}
$$

$$
- |\mathfrak{t}(M^{\text{ext}})|_{\mathfrak{s}} - \sum_{\substack{B \in \pi \\ B \subset M^{\text{cu}}}} |\mathfrak{t}(B)|_{\mathfrak{s}} - \sum_{\substack{B \in \pi \\ B \not\subset M^{\text{cu}}}} |\mathfrak{t}(B \cap M^{\text{cu}})|_{\mathfrak{s}, \mathfrak{c}, L(\overline{T})} - \sum_{\substack{B \in \pi \\ M=B}} \ell(B) \,.
$$

We fix, for the remainder of this proof, $\mathbf{T} \in \mathcal{U}_{\mathcal{X}}$. We define

$$
\mathbf{N}(a) \stackrel{\text{def}}{=} \mathbf{L}_a \sqcup \left( \bigsqcup_{\substack{T \in C_{\mathcal{B}}(\overline{T}) \\ \varrho_T \in \mathbf{L}_a}} \tilde{N}(T) \right), \tag{9.3}
$$

and also set $\mathcal{Q} \stackrel{\text{def}}{=} \left\{ \mathbf{N}(a) : \ a \in \mathring{\mathbf{T}} \setminus \{\varrho_{\mathbf{T}}\} \right\}$.

We claim that for any $a \in \mathring{\mathbf{T}} \setminus \{\varrho_{\mathbf{T}}\}$,

$$
\left( \sum_{b \in \mathbf{T}_{\geq a}} \varsigma(b) \right) - (|\mathbf{L}_a|-1)|\mathfrak{s}| \leq \tilde{\varsigma}(\mathbf{N}(a)) \,. \tag{9.4}
$$

To obtain (9.4) note that our choice of cut rule means that for each $M \in \mathcal{Q}$,

1. $e \in \mathscr{S}$ and $e_{\text{p}}, \otimes \in M$ together imply $e_{\text{c}} \in M$.
2. $e \in \mathscr{D}$ and $e_{\text{c}}, e_{\text{p}} \in M$ together imply $\otimes \in M$.

We have an inequality because of the third term on the last line of (9.2).

To prove the lemma it suffices to show that for all $M \in \mathcal{Q}$ one has

$$
\tilde{\varsigma}(M) < \mathbb{1}\{M \not\ni \otimes\} \left[ \hbar_{\mathfrak{c},\ell}(M^{\text{ext}}) \wedge \jmath_\ell(M^{\text{ext}}) \wedge \frac{|\mathfrak{s}|}{2} \right], \tag{9.5}
$$

where $\hbar_{\mathfrak{c},\ell}(\cdot)$ is defined as in (A.19). When $M$ satisfies both $M \not\ni \otimes$ and $M^{\text{ext}} = \emptyset$ one can show $\tilde{\varsigma}(M) < 0$ by copying the argument of Lemma 8.13 nearly verbatim so this case is done.

The proof of the lemma will be completed by proving (9.5) in the two remaining cases, either: (i) $M \in \mathcal{Q}$, $M \not\ni \otimes$, and $M^{\text{ext}} \neq \emptyset$, or (ii) $M \in \mathcal{Q}$ and $M \ni \otimes$.

We first treat case (i), again by splitting into subcases. Note that in this case the RHS of (9.5) is non-negative so for some subcases we just show $\tilde{\varsigma}(M) < 0$.

Note that in case (i) it is impossible for $M$ to consist of $K(\overline{T})$-components of cardinality 1. By **H**-connectivity this would forces $M = M^{\mathrm{cu}}$ so $M^{\mathrm{ext}} = \emptyset$.

Now suppose we are in a subcase of case $(i)$ where $M$ is not $K(\overline{T})$-connected and has at least one $K(\overline{T})$-component of cardinality greater than or equal to 2. We then have a decomposition of $M$ as in (8.42) and same arguments used in Lemma 8.13 can be copied verbatim to show that one always has $\tilde{\varsigma}(M) < 0$ in this subcase.

What is left in case (i) is to establish (9.5) for any $K(\overline{T})$-connected $M \in \mathcal{Q}$ with $M^{\mathrm{ext}} \neq \emptyset$, and $M \not\ni \otimes$ – this is then an immediate consequence of the inequality $\tilde{\varsigma}(M) \leq -|T(M)^0_{\mathfrak{e}}|_{\mathfrak{s}}$ and Definition A.27.

Now we turn to case (ii) where it is convenient to prove a stronger bound than (9.5). Define another function $\hat{\varsigma} : 2^{N^*} \to \mathbf{R}$ by using the same definition for $\tilde{\varsigma}$ as in (9.2) but replacing the entire last line of (9.2) with $-|\mathfrak{t}(M^{\mathrm{cu}} \sqcup M^{\mathrm{ext}})|_{\mathfrak{s}}$. In case $(ii)$ it suffices to show $\hat{\varsigma}(M) < 0$. Let $M \in \mathcal{Q}$ with $M \ni \otimes$, writing $\{M_j\}_{j=1}^n$ for the $K(\overline{T})$ connected components of $M \setminus \{\otimes\}$ it is straightforward to see that

$$\hat{\varsigma}(M) = \sum_{j=1}^n \hat{\varsigma}(M_j \sqcup \{\otimes\})$$

so it suffices to prove $\hat{\varsigma}(\widehat{M} \sqcup \{\otimes\}) < 0$ for all $\widehat{M} \in \widehat{\mathcal{Q}}$ where we have defined

$$\widehat{\mathcal{Q}} \stackrel{\mathrm{def}}{=} \left\{ \widehat{M} \subset N(\overline{T}) : \begin{array}{c} \widehat{M} \text{ is } K(\overline{T}) - \text{connected and } \exists\, M \in \mathcal{Q},\ u \in \mathcal{X}_0 \\ \text{such that } M \ni \otimes \text{ and } \widehat{M} = \overline{T}_{\geq}(u) \cap M \end{array} \right\}.$$

Observe that for any $\widehat{M} \in \widehat{\mathcal{Q}}$ and $S \in \mathcal{B}$ one has $N(S) \subset \widehat{M}$ or $N(S) \cap \widehat{M} = \emptyset$. A second important observation is that for any $\widehat{M} \in \widehat{\mathcal{Q}}$ one has

$$\sum_{e \in K^{\downarrow}(\widehat{M})} |\tilde{P}[\overline{T}_{\geq}(e)_{\mathfrak{e}}^{\bar{\mathfrak{n}}}]|_{\mathfrak{s}} \leq \sum_{e \in \mathscr{D} \cap K^{\downarrow}(\widehat{M})} \gamma(e) \tag{9.6}$$

because (i) the conditions $e \in K^{\downarrow}(\widehat{M})$ and $|\tilde{P}[\overline{T}_{\geq}(e)_{\mathfrak{e}}^{\bar{\mathfrak{n}}}]|_{\mathfrak{s}} > 0$ together force $e \in \mathscr{D}$ and (ii) the definition of $\gamma(e)$.

The claim we will prove, which will finish the proof of this lemma, is that for every $\widehat{M} \in \widehat{\mathcal{Q}}$

$$\hat{\varsigma}(\widehat{M} \sqcup \{\otimes\}) \leq \begin{cases} -|\mathfrak{s}| - |\overline{T}_{\geq}(\varrho_{\widehat{M}})_{\mathfrak{e}}^{\bar{\mathfrak{n}}}|_{\mathfrak{s}} & \text{if } \varrho_{\widehat{M}} \notin e_{\mathrm{c}}(\mathscr{S}) \\ -\mathrm{frac}(|\tilde{P}[\overline{T}_{\geq}(\tilde{e})_{\mathfrak{e}}^{\bar{\mathfrak{n}}}]|_{\mathfrak{s}}) & \text{if } \varrho_{\widehat{M}} = \tilde{e}_{\mathrm{c}} \text{ for } \tilde{e} \in \mathscr{S}, \end{cases} \tag{9.7}$$

where for $t \geq 0$ we define $\mathrm{frac}(t) = t - \lfloor t \rfloor$. Note that in the second case of the RHS such an edge $\tilde{e}$ is necessarily unique, this makes the formula well-defined. We prove (9.7) inductively with respect to the cardinality of $\widehat{M} \cap \mathcal{X}_0$.

We start with the base case where $|\widehat{M} \cap \mathcal{X}_0| = 1$ – here there are no edges $e \in \mathscr{S}$ with $e \subset \widehat{M}$ and the LHS of (9.7) is equal to

$$
\begin{cases}
-|\mathfrak{s}| - |T(\widehat{M})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}|_{\mathfrak{s}} - \displaystyle\sum_{e \in \mathscr{D} \cap K^{\downarrow}(\widehat{M})} \gamma(e) & \text{if } \varrho_{\widehat{M}} \notin e_{\mathrm{c}}^{-1}(\mathscr{S}), \\[2em]
-|T(\widehat{M})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}|_{\mathfrak{s}} + (\mathfrak{h}(\tilde{e}) - |\mathfrak{s}|) + \gamma(\tilde{e}) - 1 - \displaystyle\sum_{e \in \mathscr{D} \cap K^{\downarrow}(\widehat{M})} \gamma(e) & \text{if } \tilde{e} \in \mathscr{S}, \ \varrho_{\widehat{M}} = \tilde{e}_{\mathrm{c}} \, .
\end{cases}
$$
$$(9.8)$$

Now let us verify that (9.8) implies (9.7). When $\varrho_{\widehat{M}} \notin e_{\mathrm{c}}^{-1}(\mathscr{S})$ this follows from (9.6), when $\varrho_{\widehat{M}} = \tilde{e}_{\mathrm{c}}$ ones uses (9.1) and (9.6) to obtain

$$
\begin{aligned}
\gamma(\tilde{e}) - 1 &= |\tilde{P}[\overline{T}_{\geq}(\tilde{e})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}]|_{\mathfrak{s}} - \mathrm{frac}(|\tilde{P}[\overline{T}_{\geq}(\tilde{e})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}]|_{\mathfrak{s}}) \\
&= (|\mathfrak{s}| - \mathfrak{h}(\tilde{e})) + |\overline{T}_{\geq}(\varrho_{\widehat{M}})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}|_{\mathfrak{s}} - \mathrm{frac}(|\tilde{P}[\overline{T}_{\geq}(\tilde{e})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}]|_{\mathfrak{s}}) \\
&= (|\mathfrak{s}| - \mathfrak{h}(\tilde{e})) + |T(\widehat{M})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}|_{\mathfrak{s}} + \sum_{e \in K^{\downarrow}(\widehat{M})} |\tilde{P}[\overline{T}_{\geq}(e)^{\bar{\mathfrak{n}}}_{\mathfrak{c}}]|_{\mathfrak{s}} - \mathrm{frac}(|\tilde{P}[\overline{T}_{\geq}(\tilde{e})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}]|) \\
&\leq (|\mathfrak{s}| - \mathfrak{h}(\tilde{e})) + |T(\widehat{M})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}|_{\mathfrak{s}} + \sum_{e \in \mathscr{D} \cap K^{\downarrow}(\widehat{M})} \gamma(e) - \mathrm{frac}(|\tilde{P}[\overline{T}_{\geq}(\tilde{e})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}]|) \, .
\end{aligned}
$$
$$(9.9)$$

We now turn to the inductive step. We fix $q \geq 2$ and assume that (9.7) holds for all elements of $\widehat{\mathcal{Q}}$ whose intersection with $\mathcal{X}_0$ is of cardinality strictly less than $q$. Suppose that we are given $\widehat{M} \in \widehat{\mathcal{Q}}$ with $|\widehat{M} \cap \mathcal{X}_0| = q$.

We first define

$$
\overline{M} \overset{\text{def}}{=} \begin{cases} N(S) & \text{if } \varrho_{\widehat{M}} = \varrho_S, \text{ for some } S \in \overline{\mathcal{B}} \\ \{\varrho_{\widehat{M}}\} & \text{otherwise,} \end{cases}
$$

$$
K^{\downarrow}_{\varrho,\mathrm{int}}(\widehat{M}) \overset{\text{def}}{=} K^{\downarrow}(\overline{M}) \cap K(\widehat{M}), \ \text{ and } \ K^{\downarrow}_{\varrho,\mathrm{ext}}(\widehat{M}) \overset{\text{def}}{=} K^{\downarrow}(\overline{M}) \setminus K(\widehat{M}) \, .
$$

Next, we decompose $\widehat{M} \setminus \overline{M}$ into $K(\overline{T})$-connected components $\{\widehat{M}_j\}_{j=1}^n$, then the LHS of (9.7) is then equal to

$$
\begin{aligned}
&- |T(\overline{M})^{\bar{\mathfrak{n}}}_{\mathfrak{c}}|_{\mathfrak{s}} - |\mathfrak{s}| + \sum_{j=1}^n \left[ \hat{\varsigma}(\widehat{M}_j \sqcup \{\circledast\}) - |\mathfrak{t}(\widehat{M}^{\mathrm{ext}}_j)|_{\mathfrak{s}} \right] \\
&+ \sum_{e \in K^{\downarrow}_{\varrho,\mathrm{int}}(\widehat{M})} \left[ \mathbb{1}\{e \notin \mathscr{S}\}\mathfrak{h}(e) - \mathbb{1}\{e \in \mathscr{S}\}(\gamma(e) - 1) \right] \\
&- \sum_{e \in K^{\downarrow}_{\varrho,\mathrm{ext}}(\widehat{M})} \mathbb{1}\{e \in \mathscr{D}\}\gamma(e) + \mathbb{1}\left\{ \begin{array}{c} \varrho_{\widehat{M}} = \tilde{e}_{\mathrm{c}} \\ \text{for } \tilde{e} \in \mathscr{S} \end{array} \right\}[\mathfrak{h}(\tilde{e}) + (\gamma(\tilde{e}) - 1)]
\end{aligned}
$$
$$(9.10)$$

Note that the two summation sets appearing in the first two sets are in bijection where one assigns each $\widehat{M}_j$ to the unique $e$ with $e_{\mathrm{c}} = \varrho_{\widehat{M}_j}$.

For each $j$, $\widehat{M}_j \in \widehat{\mathcal{Q}}$ and $|\widehat{M}_j \cap \mathcal{X}_0| < q$ – thus using our inductive hypothesis combined with (9.9) gives

$$
\sum_{j=1}^{n} \hat{\varsigma}(\widehat{M}_j \sqcup \{\circledast\}) + \sum_{e \in K_{\varrho,\mathrm{int}}^{\downarrow}(\widehat{M})} \left[ \mathbb{1}\{e \notin \mathscr{S}\}\mathfrak{h}(e) - \mathbb{1}\{e \in \mathscr{S}\}(\gamma(e) - 1) \right] \quad (9.11)
$$

$$
\leq - \sum_{e \in K_{\varrho,\mathrm{int}}^{\downarrow}(\widehat{M})} \mathbb{1}\{e \notin \mathscr{S}\}\left( |\bar{T}_{\geq}(e_{\mathrm{c}})_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}|_{\mathfrak{s}} + |\mathfrak{s}| - \mathfrak{h}(e) \right) + \mathbb{1}\{e \in \mathscr{S}\}|\tilde{P}[\bar{T}_{\geq}(e)_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}]|_{\mathfrak{s}}
$$

$$
= \sum_{e \in K_{\varrho,\mathrm{int}}^{\downarrow}(\widehat{M})} |\tilde{P}[\bar{T}_{\geq}(e)_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}]|_{\mathfrak{s}} .
$$

On the other hand,

$$
- \sum_{e \in K_{\varrho,\mathrm{ext}}^{\downarrow}(\widehat{M})} \mathbb{1}\{e \in \mathscr{D}\}\gamma(e) \leq - \sum_{e \in K_{\varrho,\mathrm{ext}}^{\downarrow}(\widehat{M})} |\tilde{P}[\bar{T}_{\geq}(e)_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}]|_{\mathfrak{s}} . \quad (9.12)
$$

By using (9.11) and (9.12) in (9.10) we see $\hat{\varsigma}(\widehat{M} \sqcup \{\circledast\})$ is bounded above by

$$
- |T(\overline{M})_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}|_{\mathfrak{s}} - |\mathfrak{s}| + \mathbb{1}\left\{ \begin{array}{c} \varrho_{\widehat{M}} = \tilde{e}_{\mathrm{c}} \\ \text{for } \tilde{e} \in \mathscr{S} \end{array} \right\}[\mathfrak{h}(\tilde{e}) + \gamma(\tilde{e}) - 1] - \sum_{e \in K^{\downarrow}(\overline{M})} |\tilde{P}[\bar{T}_{\geq}(e)_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}]|_{\mathfrak{s}}
$$

$$
= - |\bar{T}_{\geq}(\varrho_{\widehat{M}})_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}|_{\mathfrak{s}} - |\mathfrak{s}| + \mathbb{1}\left\{ \begin{array}{c} \varrho_{\widehat{M}} = \tilde{e}_{\mathrm{c}} \\ \text{for } \tilde{e} \in \mathscr{S} \end{array} \right\}[\mathfrak{h}(\tilde{e}) + \gamma(\tilde{e}) - 1].
$$

This is the desired bound when $\varrho_{\widehat{M}} \notin e_{\mathrm{c}}(\mathscr{S})$. It is also the desired bound when the indicator function is non-vanishing since in this case

$$
- |\overline{T}_{\geq}(\varrho_{\widehat{M}})_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}|_{\mathfrak{s}} - |\mathfrak{s}| + \mathfrak{h}(\tilde{e}) = |\tilde{P}[\overline{T}_{\geq}(\tilde{e})_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}]|_{\mathfrak{s}}
$$
$$
\text{and} \quad \gamma(\tilde{e}) - 1 = |\tilde{P}[\overline{T}_{\geq}(\tilde{e})_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}]|_{\mathfrak{s}} - \mathrm{frac}\{|\tilde{P}[\overline{T}_{\geq}(\tilde{e})_{\bar{\mathfrak{e}}}^{\bar{\mathfrak{n}}}]|_{\mathfrak{s}}\} .
$$

$\square$

**Lemma 9.3** *The total homogeneity $\varsigma$, as defined by* (8.51)*, satisfies condition* (A.24) *of Theorem A.32.*

*Proof.* We start by defining a map $\tilde{\varsigma} : 2^{N^*} \to \mathbf{R}$ as follows, for $M \subset N^*$ we set

$$
\tilde{\varsigma}(M) \overset{\mathrm{def}}{=} \sum_{\substack{e \in K(D_{2p}) \setminus \mathscr{S} \\ e \cap M \neq \emptyset}} \mathfrak{h}(e) - |\mathfrak{t}(M^{\mathrm{cu}})|_{\mathfrak{s}} - |\mathfrak{t}(M^{\mathrm{ext}})|_{\mathfrak{s}} + \sum_{e \in \mathscr{D}} \mathbb{1}\left\{ \begin{array}{c} e_{\mathrm{c}} \in M \\ e_{\mathrm{p}} \notin M \end{array} \right\}\gamma(e) \quad (9.13)
$$

$$
+ \left[ \sum_{e \in \mathscr{S}} \mathfrak{h}(e)\mathbb{1}\{e_{\mathrm{c}} \in M\} - (\gamma(e) - 1)\mathbb{1}\left\{ \begin{array}{c} e_{\mathrm{p}} \in M \\ e_{\mathrm{c}} \notin M \end{array} \right\} \right]
$$

$$
- |\bar{\mathfrak{n}}(M)|_{\mathfrak{s}} - |M| \, |\mathfrak{s}|.
$$

Fix for the remainder of the proof $\mathbf{T} \in \mathcal{U}_{\mathcal{X}}$.

We claim that for $a \in \mathring{\mathbf{T}}$ with $a \leq \{\otimes, \varrho_*\}^{\uparrow}$ one has, writing $M \stackrel{\text{def}}{=} N^* \setminus \mathbf{N}(a)$ (where $\mathbf{N}(a)$ is defined as in (9.3)),

$$\sum_{b \in \mathbf{T}_{\not\geq a}} \varsigma_{\mathbf{T}}(b) - |\mathcal{X} \setminus \mathbf{L}_a| \, |\mathfrak{s}| - |\mathfrak{t}(M^{\text{ext}})|_{\mathfrak{s}} \geq \tilde{\varsigma}(M)$$

The claim is easily justified upon remembering that $e \in \mathfrak{C}$, $e_{\text{c}} \in N^* \setminus \mathbf{N}(a)$ and $e_{\text{p}} \notin N^* \setminus \mathbf{N}(a)$ together imply $e \in \mathscr{D}$ and that one has, using item 3 of Definition A.14, the inequality

$$\sum_{b \in \mathbf{T}_{\not\geq a}} \sum_{B \in \pi} \mathfrak{c}^B(b) \geq -|\mathfrak{t}(M^{\text{cu}})|_{\mathfrak{s}} \, .$$

We now define a collection of node sets $\mathcal{Z} \subset 2^{\tilde{N}(\overline{T})}$ by setting

$$\mathcal{Z} \stackrel{\text{def}}{=} \left\{ M \subset \tilde{N}(\overline{T}) : \quad \begin{array}{c} \exists a \in \mathring{\mathbf{T}} \text{ with } a \leq \otimes \text{ and } a \leq \varrho_* \text{ such that} \\ M \text{ is an } K(\overline{T})\text{-connected component of } N^* \setminus \mathbf{N}(a) \end{array} \right\} \, .$$

The lemma will be proved if we show that, for every $M \in \mathcal{Z}$, $\tilde{\varsigma}(M) > 0$. Here we used that if $M \subset \tilde{N}(\overline{T})$ decomposes into $K(\overline{T})$-components $\{M_j\}_{j=1}^n$ then $\tilde{\varsigma}(M) = \sum_{j=1}^n \tilde{\varsigma}(M_j)$.

We now fix $M \in \mathcal{Z}$ and write $\tilde{e}$ for the unique edge in $e \in K(\overline{T})$ with $\tilde{e}_{\text{c}} = \varrho_M$. We start by observing that $\tilde{e} \in \mathfrak{C} \Rightarrow \tilde{e} \in \mathscr{D}$, and $\tilde{e}$ is the only edge that could possibly contribute to the sum occurring in the last term of the first line of (9.13).

The proof of the lemma is finished upon proving the claim

$$\tilde{\varsigma}(M) \geq -|\tilde{P}[\overline{T}_{\geq}(\tilde{e})]^{\overline{\mathfrak{n}}}_{\tilde{\mathfrak{c}}}|_{\mathfrak{s}} + \gamma(\tilde{e}) \cdot \mathbb{1}\{\tilde{e} \in \mathfrak{C}\} > 0. \tag{9.14}$$

The last inequality of (9.14) is a consequence of the definition of $\mathfrak{C}$ and $\gamma(\cdot)$.

We now prove the first inequality of (9.14). First observe that

$$\tilde{\varsigma}(M) = - |T(M)^{\overline{\mathfrak{n}}}_{\tilde{\mathfrak{c}}}|_{\mathfrak{s}} + \mathfrak{h}(\tilde{e}) - |\mathfrak{s}| + \gamma(\tilde{e}) \cdot \mathbb{1}\{\tilde{e} \in \mathfrak{C}\} \tag{9.15}$$
$$+ \sum_{e \in K^{\downarrow}(M) \setminus \mathscr{S}} \mathfrak{h}(e) - \sum_{e \in K^{\downarrow}(M) \cap \mathscr{S}} (\gamma(e) - 1)$$

Conversely, by using (9.1) we have

$$-|\tilde{P}[\overline{T}_{\geq}(\tilde{e})]^{\overline{\mathfrak{n}}}_{\tilde{\mathfrak{c}}}|_{\mathfrak{s}} = - |\overline{T}_{\geq}(\varrho_M)^{\overline{\mathfrak{n}}}_{\tilde{\mathfrak{c}}}|_{\mathfrak{s}} + \mathfrak{h}(\tilde{e}) - |\mathfrak{s}| \tag{9.16}$$
$$= - |T(M)^{\overline{\mathfrak{n}}}_{\tilde{\mathfrak{c}}}|_{\mathfrak{s}} + \mathfrak{h}(\tilde{e}) - |\mathfrak{s}| - \sum_{e \in K^{\downarrow}(M)} |\tilde{P}[\overline{T}_{\geq}(e)]^{\overline{\mathfrak{n}}}_{\tilde{\mathfrak{c}}}|_{\mathfrak{s}} \, .$$

The claim is then proved upon observing that

$$\forall e \in K^{\downarrow}(M) \setminus \mathscr{S}, \; |\tilde{P}[\overline{T}_{\geq}(e)]^{\overline{\mathfrak{n}}}_{\tilde{\mathfrak{c}}}|_{\mathfrak{s}} > -\mathfrak{h}(e) \tag{9.17}$$
$$\text{and } \forall e \in K^{\downarrow}(M) \cap \mathscr{S}, \; |\tilde{P}[\overline{T}_{\geq}(e)]^{\overline{\mathfrak{n}}}_{\tilde{\mathfrak{c}}}|_{\mathfrak{s}} > (\gamma(e) - 1) \, .$$

The second inequality of (9.17) follows from the definition of $\gamma(\cdot)$ (and holds for all $e \in \mathfrak{C}$). The first inequality of (9.17) actually holds for all $e \in K(\overline{T})$ – one writes

$$|\tilde{P}[\overline{T}_{\geq}(e)]_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}}|_{\mathfrak{s}} = \left(|\overline{T}_{\geq}(e_{\mathsf{c}})_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}}|_{\mathfrak{s}} + |\mathfrak{s}|\right) - \mathfrak{h}(e)$$

and then notices that the quantity in parentheses on the RHS must be strictly positive by either by the assumption of super-regularity as in Definition 2.28 or Assumption 2.24. Using (9.17) in (9.16) and then using (9.15) gives the first inequality of (9.14) as required. □

## 10   Proofs of the main theorems

We now describe how everything fits together to prove Theorem 2.31.

*Proof Proposition 8.2.* Here we combine the outputs of Lemma 8.16 with Proposition 9.1 as input to Theorem A.32 – the desired bound is obtained from (A.25) and setting $r \overset{\text{def}}{=} -\lfloor \log_2(\lambda) \rfloor$. □

*Proof Theorem 2.31.* Note that the sums over $\tilde{L} \subset L(\overline{T})$, $\pi \in \overline{\mathcal{P}}[L(\overline{T}) \setminus \tilde{L}]$, and $(\mathbb{M}, \mathbb{G}) \in \mathfrak{R}$ are all finite and therefore contribute only combinatorial factors – it follows that the estimate of Proposition 8.2 combined with the identifications of Lemma 4.23 and Proposition 4.22 gives one (2.14) of Theorem 2.31.

We now describe how a simple modification of our analysis gives (2.15). By stationarity, it suffices to prove the bound for $z = 0$. Observe that $X^\xi \overset{\text{def}}{=} \hat{\Upsilon}_0^\xi[\overline{T}_{\overline{\mathfrak{e}}}^{\overline{\mathfrak{n}}}](\psi_0^\lambda)$, seen as a function of $\xi \in \mathcal{M}(\Omega_\infty)$ is in fact a $|L(\overline{T})|$-linear function of $\xi$.

Now we write, for $\xi, \bar{\xi} \in \mathcal{M}(\Omega_\infty)$,

$$\mathbf{E}[(X^\xi - X^{\bar{\xi}})^{2p}] = \sum_{j=0}^{2p} (-1)^{2p-j} \binom{2p}{j} \mathbf{E}[(X^\xi)^j (X^{\bar{\xi}})^{2p-j}]$$

$$= \sum_{j=0}^{2p} (-1)^{2p-j} \binom{2p}{j} \mathbf{E}[(X^\xi)^j (X^{\bar{\xi}})^{2p-j} - (X^\xi)^{2p}].$$

We will estimate the above sum term by term. Fixing $j$, note the we are estimating can be written as the difference of a single $2p|L(\overline{T})|$-linear functional on $\mathcal{M}(\Omega_\infty)$ evaluated at two different sets of arguments. Denoting this functional by $\mathfrak{X}$ and choosing a convenient ordering for its arguments we then write

$$(X^\xi)^j (X^{\bar{\xi}})^{2p-j} - (X^\xi)^{2p}$$
$$= \mathfrak{X}(\xi^{\otimes j|L(\overline{T})|} \otimes \bar{\xi}^{\otimes(2p-j)|L(\overline{T})|}) - \mathfrak{X}(\xi^{\otimes 2p|L(\overline{T})|})$$
$$= \sum_{k=0}^{(2p-j)|L(\overline{T})|-1} \mathfrak{X}(\xi^{\otimes j|L(\overline{T})|+k} \otimes (\bar{\xi} - \xi) \otimes \bar{\xi}^{\otimes(2p-j)|L(\overline{T})|-k-1}).$$

Finally, by performing the same analysis as used for the uniform bound, one obtains, for each fixed $k$ in the above sum,

$$\left| \mathbf{E}[\mathfrak{X}(\xi^{\otimes j|L(\overline{T})|+k} \otimes (\xi - \bar{\xi}) \otimes \bar{\xi}^{\otimes(2p-j)|L(\overline{T})|-k-1})] \right|$$

$$\lesssim \left( \prod_{e \in K(T)} \|K_{\mathfrak{t}(e)}\|_{|\mathfrak{t}(e)|_{\mathfrak{s}},m} \right)^{2p} (\|\xi\|_{N,\mathfrak{c}} \vee \|\bar{\xi}\|_{N,\mathfrak{c}}) \|\xi; \bar{\xi}\|_{j,\mathfrak{c}} \lambda^{2p|\tau|_{\mathfrak{s}}} .$$

The factor of $\|\xi; \bar{\xi}\|_{N,\mathfrak{c}}$ is obtained because at least one of the cumulants appearing in our estimates will involve the difference $\xi - \bar{\xi}$.                    □

Next we prove Theorem 2.34.

*Proof Theorem 2.34.* The main ingredients of this proof are Theorem 2.31, Proposition B.3, and a simple diagonalization argument.

First we construct the mentioned extension of $Z_{\text{BPHZ}}^{\bullet}$. We define a homogeneity assignment $|\cdot|_{\mathfrak{s}}$ on $\mathfrak{L}$ by setting, for each $\mathfrak{t} \in \mathfrak{L}_-$, $|\mathfrak{t}|_{\mathfrak{s}} \overset{\text{def}}{=} \|\mathfrak{t}\|_{\mathfrak{s}} \cdot (1 + \frac{\kappa}{2} \mathbb{1}\{\mathfrak{t} \in \mathfrak{L}_-\})$. Note that $|\cdot|_{\mathfrak{s}}$ is $R$-consistent with $[\![\cdot]\!]_{\mathfrak{s}}$. We claim that for any $\xi \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\text{Cum}}, \mathfrak{c})$ the map $\eta \mapsto Z_{\text{BPHZ}}^{\xi*\eta}$ from $(\text{Moll}, \|\cdot\|_{\kappa/2,0})$ into the $L^2$ space of random models on $\mathscr{M}_{\infty}(\mathscr{T})$ has some uniform continuity properties.

To show this we first define $\tilde{\mathfrak{c}}$ to be the $\kappa/2$-penalization of $\mathfrak{c}$. Applying Theorem 2.31 followed by Proposition B.3, we conclude that for every compact set $\mathfrak{K} \subset \mathbf{R}^d$ and $\alpha \in A$ one has the bound

$$\mathbf{E}\|Z_{\text{BPHZ}}^{\xi*\eta_1}; Z_{\text{BPHZ}}^{\xi*\eta_2}\|_{\alpha,\mathfrak{K}}^2 \lesssim (\|\xi * \eta_1\|_{N,\tilde{\mathfrak{c}}} \vee \|\xi * \eta_2\|_{N,\tilde{\mathfrak{c}}}) \|\xi * \eta_1; \xi * \eta_2\|_{N,\tilde{\mathfrak{c}}} \quad (10.1)$$

$$\lesssim \|\xi\|_{N,\mathfrak{c},1}^3 \cdot \|\eta_1 - \eta_2\|_{\kappa/2,0} ,$$

uniformly over $\eta_1, \eta_2 \in \text{Moll}$ and $\xi \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\text{Cum}}, \mathfrak{c})$. In what follows we denote by $(\eta_n)_{n \in \mathbf{N}} \subset \text{Moll}$ an arbitrary sequence satisfying $\lim_{n \to \infty} \|\eta_n - \delta\|_{\frac{\kappa}{2},0} = 0$. We see that the map $\eta \mapsto Z_{\text{BPHZ}}^{\xi*\eta}$ extends uniquely to $(\overline{\text{Moll}}, \|\cdot\|_{\kappa/2})$ by setting

$$Z_{\text{BPHZ}}^{\xi*\delta} \overset{\text{def}}{=} \lim_{n \to \infty} Z_{\text{BPHZ}}^{\xi*\eta_n} ,$$

where the limit is in $L^2$.

We then define the extension by setting, for each $\xi \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\text{Cum}}, N, \mathfrak{c}) \setminus \mathcal{M}(\Omega_{\infty}, \mathfrak{L}_{\text{Cum}})$, $Z_{\text{BPHZ}}^{\xi} \overset{\text{def}}{=} Z_{\text{BPHZ}}^{\xi*\delta}$.

Observe that for $\xi \in \mathcal{M}(\Omega_{\infty}, \mathfrak{L}_{\text{Cum}})$ one has the equality $Z_{\text{BPHZ}}^{\xi*\delta} = Z_{\text{BPHZ}}^{\xi}$ since one has the convergence in probability of $\xi * \eta_n \to \xi$ with respect to very strong topologies on $\Omega_{\infty}$ along with a convergence of cumulants (with also sit in regular spaces with uniform bounds), the map $\mathbf{Z}_{\text{BPHZ}}^{\bullet}$ is a continuous function of the data consisting of (i) realizations in $\Omega_{\infty}$, equipped with a suitably strong topology, and (ii) and corresponding cumulants. The uniqueness of this extension is immediate since by Proposition B.3 we see that $\|\xi\|_{N,\mathfrak{c},2} \leq \infty$ implies $\sup_{n \geq 0} \|\xi * \eta_n\|_{N,\mathfrak{c},1} < \infty$.

We now prove the desired continuity property. Suppose that we are given random noises $(\xi_n)_{n \geq 0}$ with $\xi_n \to \xi$ in probability and $\sup_{n \geq 0} \|\xi_n\|_{N,\mathfrak{c},1} < \infty$. We

define, for every $\alpha \in A$, compact $\mathfrak{K} \subset \mathbf{R}^d$, and random models $Z, Z'$ defined on the same probability space as $\xi$,

$$d_{\alpha,\mathfrak{K}}(Z, Z') \stackrel{\text{def}}{=} \mathbf{E}[1 \wedge \|Z; Z'\|_{\alpha,\mathfrak{K}}] \ .$$

The family of pseudometrics $\{d_{\alpha,\mathfrak{K}}(\cdot, \cdot)\}_{\alpha,\mathfrak{K}}$ then generates the topology of convergence in probability on the space of random models.

Fix $\alpha$ and $\mathfrak{K}$ as above. For any $n, m \in \mathbf{N}$ one has

$$d_{\alpha,\mathfrak{K}}(Z_{\text{BPHZ}}^{\xi}, Z_{\text{BPHZ}}^{\xi_n}) \leq d_{\alpha,\mathfrak{K}}(Z_{\text{BPHZ}}^{\xi}, Z_{\text{BPHZ}}^{\xi*\eta_m}) + d_{\alpha,\mathfrak{K}}(Z_{\text{BPHZ}}^{\xi*\eta_m}, Z_{\text{BPHZ}}^{\xi_n*\eta_m})$$
$$+ d_{\alpha,\mathfrak{K}}(Z_{\text{BPHZ}}^{\xi_n*\eta_m}, Z_{\text{BPHZ}}^{\xi_n}) \ . \tag{10.2}$$

Now fix $\varepsilon > 0$. By applying the bound (10.1) it follows that there exists $M > 0$ such that for any $m \geq M$ and any $n \in \mathbf{N}$ both the first and third terms on the RHS of the above inequality are each smaller than $\frac{\varepsilon}{3}$.

We now claim that for any $m \in \mathbf{N}$ one can find $n_m$ such that for any $n > n_m$ one has

$$d_{\alpha,\mathfrak{K}}(Z_{\text{BPHZ}}^{\xi*\eta_m}, Z_{\text{BPHZ}}^{\xi_n*\eta_m}) < \frac{\varepsilon}{3} \ .$$

This follows by observing that the map $Z_{\text{BPHZ}}^{\bullet*\eta_m}$, viewed as a function of a frozen realization in $\bigoplus_{\mathfrak{t} \in \mathfrak{L}_-} \mathcal{D}'(\mathbf{R}^d)$ and cumulants living in a sufficiently regular space, is again a continuous map and so one can push through the convergence of $\xi_n \to \xi$ in probability along with the convergence of cumulants with uniform bounds.

By fixing $m \geq M$ we see that for any $n \geq n_m$ one has $d_{\alpha,\mathfrak{K}}(Z_{\text{BPHZ}}^{\xi}, Z_{\text{BPHZ}}^{\xi_n}) < \varepsilon$ as required, thus concluding the proof.                                                    $\square$

## Appendix A   Multiscale Clustering

### A.1   Multiscale clustering for a single tree

Suppose we are given an undirected connected multigraph $\mathbf{G}$ on a vertex set $\mathcal{V} \stackrel{\text{def}}{=} \{v_0, \ldots, v_p\}$ with $p \geq 1$. Here, when we say that $\mathbf{G}$ is a multigraph one should view $\mathbf{G}$ as some set equipped with some map $\iota_{\mathbf{G}} : \mathbf{G} \to \mathcal{V}^{(2)}$.

The map $\iota_{\mathbf{G}}$ may not be an injection which is why we call $\mathbf{G}$ a multigraph, however the "duplicated" edges are seen as distinguishable amongst themselves. In practice we will often just present the vertex set $\mathcal{V}$ and the multigraph $\mathbf{G}$ and the map $\iota_{\mathbf{G}}$ will be obvious.

We consider $v_0$ to be the "root" of $\mathbf{G}$. We also write $\mathcal{V}_0 = \mathcal{V} \setminus \{v_0\}$.

**Definition A.1**  A *coalescence tree* $\mathbf{T}$ on a vertex set $\mathcal{V}$ is a rooted tree with at least three nodes with the following structures and properties:

- The set of leaves of $\mathbf{T}$ is identified with the set $\mathcal{V}$.
- Writing $\mathring{\mathbf{T}}$ for the set of internal nodes (i.e. nodes that are not leaves) and $\varrho_{\mathbf{T}}$ for the root of $\mathbf{T}$, we require that every $u \in \mathring{\mathbf{T}} \setminus \{\varrho_{\mathbf{T}}\}$ has degree at least 3 and that $\varrho_{\mathbf{T}}$ has degree at least 2.

We then also equip the set of nodes of **T** with the poset structure corresponding to viewing **T** has a Hasse diagram with its root $\varrho_{\mathbf{T}}$ as the unique minimal element.

Note that unlike in [HQ15] these coalescence trees are not necessarily binary, an internal node is allowed to have more than 2 "children". We denote by $\widehat{\mathcal{U}}_{\mathcal{V}}$ the (finite) set of *all* coalescence trees **T** on the vertex set $\mathcal{V}$.

The trees $\widehat{\mathcal{U}}_{\mathcal{V}}$ will be used to organize our multiscale expansions, for each fixed $x_{v_0}$ they slice the domain $(\mathbf{R}^d)^{\mathcal{V}_0}$ into regions characterized by the relative distances of the variables $x_{\mathcal{V}} = (x_v)_{v \in \mathcal{V}}$. These domains can be visualized in the following way. For each fixed $x_{\mathcal{V}}$, define a family of equivalence relations $\{\sim_n\}_{n \in \mathbf{N}}$ on the set $\mathcal{V}$ by taking $\sim_n$ to be the transitive closure of the relation $R_n$ given by

$$v R_n v' \quad \Leftrightarrow \quad \exists e \in \mathbf{G} \text{ with } \iota_{\mathbf{G}}(e) = \{v, v'\} \text{ and } |x_v - x_{v'}| \leq 2^{-n} .$$

The tree **T** should then be thought of as pictorially representing one particular way the equivalence classes of the equivalence relations $\sim_n$ could merge as one takes $n$ from $-\infty$ to $0$ – if one ignores issues of coinciding position variables then at some value of $n$ all the elements of $\mathcal{V}$ are singletons and by $n = 0$ there is only one equivalence class. The domain associated to a tree **T** would then correspond to all values of $x_{\mathcal{V}_0}$ for which the "coalescence history" prescribed by **T** holds.

The description above is only for intuition, concrete definitions are given below. However one can already see that it is natural to label the inner nodes of **T** with numbers that indicate the specific scale at which a coalescence event takes place.

**Definition A.2** Given a vertex set $\mathcal{V}$ and $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$, we define $\mathrm{Lab}_{\mathbf{T}}$ to be the collection of all maps $\mathbf{s} : \mathring{T} \to \mathbf{N}$ with the property that $u < v \Rightarrow \mathbf{s}(u) < \mathbf{s}(v)$. The pair $(\mathbf{T}, \mathbf{s})$ is then called a *labeled coalescence tree*.

We define the set of all possible *scale assignments* to be given by $\mathbf{n} = (n_e)_{e \in \mathbf{G}} \in \mathbf{N}^{\mathbf{G}}$. We usually fix a possibly smaller set of scales $\mathcal{N}_{\mathbf{G}} \subset \mathbf{N}^{\mathbf{G}}$ relevant to the given context when deploying the machinery of this section.

For any fixed $\mathbf{n} \in \mathbf{N}^{\mathbf{G}}$ and $r \in \mathbf{N}$ we define the sub-multigraph $\mathbf{G}_r^{\mathbf{n}} = \{e \in \mathbf{G} : n_e \geq r\}$ and also define $\mathcal{V}_r^{\mathbf{n}} \subset 2^{\mathcal{V}}$ to be the collection of vertex sets of the connected components of $\mathbf{G}_r^{\mathbf{n}}$. We consider singletons as connected components so that, for every $r$, $\mathcal{V}_r^{\mathbf{n}}$ is a partition of $\mathcal{V}$.

The sequence $(\mathcal{V}_r^{\mathbf{n}})_{r \in \mathbf{N}}$ determines a labelled coalescence tree $(\mathbf{T}, \mathbf{s})$ via the following procedure. The set of nodes for **T** is given by $\mathbf{T} = \bigcup_{r=0}^{\infty} \mathcal{V}_r^{\mathbf{n}}$. Since elements of **T** are themselves subsets of $\mathcal{V}$, they are partially ordered by inclusion. Given two distinct nodes $a, b \in \mathring{\mathbf{T}}$ we then connect $a$ and $b$ if $a \subset b$ maximally in **T**. In this way, the set of leaves is indeed given by $\mathcal{V} \subset \mathbf{T}$ since, for $r$ sufficiently large, $\mathcal{V}_r^{\mathbf{n}}$ consists purely of singletons. The root is always given by $\varrho_{\mathbf{T}} = \mathcal{V}$, by considering $r$ sufficiently small. It is easy to verify that the required properties hold for **T** as a consequence of the fact that the children of any node, viewed as subsets of $\mathcal{V}$, form a non-trivial partition of that node. The labeling $\mathbf{s}(\cdot)$ on internal nodes is defined as follows. For each $a \in \mathring{\mathbf{T}}$, we set

$$\mathbf{s}(a) \overset{\mathrm{def}}{=} \max\{r \in \mathbf{N} : a \in \mathcal{V}_r^{\mathbf{n}}\} .$$

This is always finite since elements of $\mathring{\mathbf{T}}$ are not singletons, while there always exists some $r$ such that $\mathcal{V}_r^{\mathbf{n}} = \{\{v\} : v \in \mathcal{V}\}$. This completes our construction of the labeled coalescence tree $(\mathbf{T}, \mathbf{s})$ with the caveat that for purely aesthetic reasons we identify the "singleton" leaves of $\mathbf{T}$ with their lone constituent element. We will henceforth always treat the elements of $\mathring{\mathbf{T}}$ as "abstract" nodes, once we are done constructing the tree we forget how they correspond to non-singleton subsets of $\mathcal{V}$.

The above procedure gives us a map $\hat{\mathcal{T}} : \mathbf{N}^{\mathbf{G}} \to \widehat{\mathcal{U}}_{\mathcal{V}} \ltimes \text{Lab}_\bullet$ taking scale assignments to labeled coalescence trees, we write it $\mathbf{n} \mapsto (\mathcal{T}(\mathbf{n}), \mathbf{s}(\mathbf{n}))$. We then define $\mathcal{U}_{\mathcal{V}} \stackrel{\text{def}}{=} \mathcal{T}(\mathcal{N}_{\mathbf{G}})$.

**Remark A.3** The usage of the notation $\mathcal{U}_{\mathcal{V}}$ instead of $\widehat{\mathcal{U}}_{\mathcal{V}}$ indicates that we are looking at a sub-collection of coalesence trees which have been determined by fixing a multigraph $\mathbf{G}$ on $\mathcal{V}$ and a corresponding set of scales $\mathcal{N}_{\mathbf{G}}$ even though this is suppressed from the notation $\mathcal{U}_{\mathcal{V}}$.

Given $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$ and $f \subset \mathcal{V}$ we write $f^{\uparrow}$ for the maximal internal node which is a *proper* ancestor of all the elements of $f$. When $f = \{a\}$ we may write $a^{\uparrow}$ instead of $\{a\}^{\uparrow}$. We define $f^{\Uparrow}$ to be the maximal internal node which is a proper ancestor of $f^{\uparrow}$ if $f^{\uparrow} \neq \varrho_{\mathbf{T}}$, otherwise we set $f^{\Uparrow} = f^{\uparrow} = \varrho_{\mathbf{T}}$. For $a \in \mathring{\mathbf{T}}$ we write $\mathbf{L}_a$ for the set of leaves of $\mathbf{T}$ which are descendants of $a$.

Let us give a concrete example to clarify these definitions. For our vertex set we fix $\mathcal{V} = \{v_0, \ldots, v_6\}$. Below we fix a graph $\mathbf{G}$ by presenting it pictorially and also specify a scale assignment $\mathbf{n} \in \mathbf{N}^{\mathbf{G}}$ by labeling each of the edges.
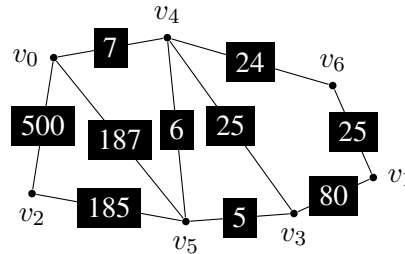


Below we draw the corresponding coalescence tree $\mathbf{T} = \mathcal{T}(\mathbf{n})$, labeling the leaves using the set $\mathcal{V}$ and introducing new labels for the internal nodes of $\mathbf{T}$.
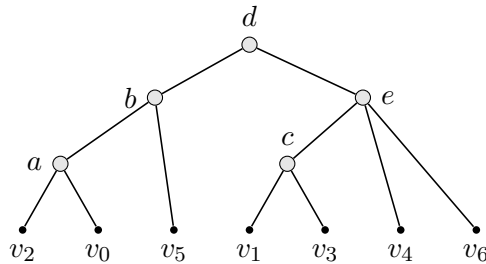


The labeling $\mathbf{s} = \mathbf{s}(\mathbf{n}) \in \text{Lab}_{\mathbf{T}}$ is then given by

| $u \in \mathring{\mathbf{T}}$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $\mathbf{s}(u)$ | 500 | 187 | 80 | 7 | 25 |

In this example, we have

$$\{v_2, v_5\}^{\uparrow} = \{v_5\}^{\uparrow} = b\,, \quad \{v_2, v_5\}^{\Uparrow} = d\,, \quad \{v_1, v_3, v_5\}^{\Uparrow} = \{v_1, v_3, v_5\}^{\uparrow} = d\,.$$

We also define, for any $(\mathbf{T}, \mathbf{s}) \in \widehat{\mathcal{U}}_{\mathcal{V}} \ltimes \mathrm{Lab}_{\bullet}$, $\mathcal{N}_{\mathrm{tri}}(\mathbf{T}, \mathbf{s}) \subset \mathcal{N}_{\mathbf{G}}$ to be the set of all those scale assignments $\mathbf{n}$ with $\hat{\mathcal{T}}(\mathbf{n}) = (\mathbf{T}, \mathbf{s})$ and the property that for every $e \in \mathbf{G}$

$$|n_e - \mathbf{s}(e^{\uparrow})| < 2C \cdot |\mathcal{V}|, \tag{A.1}$$

where $C > 0$ is chosen to be the same as (A.2). Here we are abusing notation and writing $e^{\uparrow}$ instead of $\iota_{\mathbf{G}}(e)^{\uparrow}$, we commit this abuse of notation frequently. Clearly $|\mathcal{N}_{\mathrm{tri}}(\mathbf{T}, \mathbf{s})|$ is finite and bounded uniform in $(\mathbf{T}, \mathbf{s}) \in \widehat{\mathcal{U}}_{\mathcal{V}} \ltimes \mathrm{Lab}_{\bullet}$.

For each $(\mathbf{T}, \mathbf{s}) \in \widehat{\mathcal{U}}_{\mathcal{V}} \ltimes \mathbf{S}_{\bullet}$ we define

$$\mathcal{D}(\mathbf{T}, \mathbf{s}, x_{v_0}) \stackrel{\mathrm{def}}{=} \{x \in (\mathbf{R}^d)^{\mathcal{V}_0} : \ \forall e = \{v_i, v_j\} \in \mathcal{V}^{(2)}, x_{v_i} \stackrel{\mathbf{s}(e^{\uparrow})}{\longleftrightarrow} x_{v_j}\}, \tag{A.2}$$

where we've used the notation (8.12).

**Definition A.4** For $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$ we call a map $\varsigma_{\mathbf{T}} : \mathring{\mathbf{T}} \to \mathbf{R}$ a $\mathbf{T}$-homogeneity. A collection of such maps $\varsigma = (\varsigma_{\mathbf{T}})_{\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}}$ is called a *total homogeneity*. Addition of total homogeneities is defined pointwise.

We introduce two special families of total homogeneities which will play the role of "Kronecker deltas" out of which we will build other total homogenieties. Given any subset $\tilde{\mathcal{V}} \subset \mathcal{V}$, the total homogeneities $\delta^{\uparrow}[\tilde{\mathcal{V}}]$ and $\delta^{\Uparrow}[\tilde{\mathcal{V}}]$ are given by setting, for every $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$ and $a \in \mathring{\mathbf{T}}$,

$$\delta^{\uparrow}_{\mathbf{T}}[\tilde{\mathcal{V}}](a) \stackrel{\mathrm{def}}{=} \mathbb{1}\left\{a = \tilde{\mathcal{V}}^{\uparrow, \mathbf{T}}\right\} \quad \text{and} \quad \delta^{\Uparrow}_{\mathbf{T}}[\tilde{\mathcal{V}}](a) \stackrel{\mathrm{def}}{=} \mathbb{1}\left\{a = \tilde{\mathcal{V}}^{\Uparrow, \mathbf{T}}\right\}, \tag{A.3}$$

where the superscript $\mathbf{T}$ is used to remind readers that these operations are $\mathbf{T}$-dependent. For $u \in \mathcal{V}$ we will write $\delta^{\uparrow}[u]$ or $\delta^{\Uparrow}[u]$ instead of $\delta^{\uparrow}[\{u\}]$ or $\delta^{\Uparrow}[\{u\}]$.

Given $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$, a $\mathbf{T}$-homogeneity $\varsigma_{\mathbf{T}}$, and $\mathbf{s} \in \mathrm{Lab}_{\mathbf{T}}$ we often use the shorthand

$$\langle \varsigma_{\mathbf{T}}, \mathbf{s} \rangle = \sum_{a \in \mathring{\mathbf{T}}} \varsigma_{\mathbf{T}}(a) \mathbf{s}(a)\,.$$

**Definition A.5** Given a set of scale assignments $\mathcal{N}_{\mathbf{G}}$ and a total homogeneity $\varsigma$ we say that a family of continuous compactly supported functions $F = (F^{\mathbf{n}})_{\mathbf{n} \in \mathcal{N}_{\mathbf{G}}}$ on $(\mathbf{R}^d)^{\mathcal{V}_0}$ is bounded by $\varsigma$ if the following conditions hold.

1. There exists $x_{v_0} \in \mathbf{R}^d$ such that for each $(\mathbf{T}, \mathbf{s}) \in \mathcal{U}_{\mathcal{V}} \ltimes \mathrm{Lab}_{\bullet}$, and $\mathbf{n} \in \mathcal{N}_{\mathrm{tri}}(\mathbf{T}, \mathbf{s})$, one has

$$\mathrm{supp}\,(F^{\mathbf{n}}(\cdot)) \subset \mathcal{D}(\mathbf{T}, \mathbf{s}, x_{v_0})\,. \tag{A.4}$$

2. One has the bound

$$\|F\|_{\varsigma,\mathcal{N}_{\mathbf{G}}} \stackrel{\text{def}}{=} \sup_{\substack{\mathbf{T}\in\mathcal{U}_{\mathcal{V}} \\ \mathbf{s}\in\mathrm{Lab}_{\mathbf{T}} \\ \mathbf{n}\in\mathcal{N}_{\mathrm{tri}}(\mathbf{T},\mathbf{s})}} 2^{-\langle\varsigma_{\mathbf{T}},\mathbf{s}\rangle} \sup_{x\in(\mathbf{R}^d)^{\mathcal{V}_0}} |F^{\mathbf{n}}(x)| < \infty. \tag{A.5}$$

(In the particular case $\mathcal{N}_{\mathbf{G}} = \mathbf{N}^{\mathbf{G}}$ we will also just write $\|F\|_{\varsigma}$.)

**Remark A.6** Because of the domain constraint (A.4), it is clear that $F^{\mathbf{n}}$ must vanish unless $\mathbf{n} \in \mathcal{N}_{\mathrm{tri},\mathbf{G}}$, where

$$\mathcal{N}_{\mathrm{tri},\mathbf{G}} \stackrel{\text{def}}{=} \bigsqcup_{(\mathbf{T},\mathbf{s})\in\mathcal{U}_{\mathcal{V}}\ltimes\mathrm{Lab}_{\bullet}} \mathcal{N}_{\mathrm{tri}}(\mathbf{T},\mathbf{s}) .$$

**Remark A.7** The notion of being "bounded" by a total homogeneity $\varsigma$ depends on a invisible "combinatorial" constant $C$ hidden in (A.2) – this affects both the domain constraint (A.4) and the definition of $\|\cdot\|_{\varsigma,\mathcal{N}_{\mathbf{G}}}$. In practice we want to be able to formulate that this constant $C$ can be chosen independently of certain parameters. Thus, if we have a collections of families of functions $F_\theta = (F_\theta^{\mathbf{n}})_{\mathbf{n}\in\mathcal{N}_{\mathbf{G}}}$ where $\theta$ varies as a parameter in some set $\Theta$ we say that a the collection of families $F_\theta$ are bounded uniform in $\theta \in \Theta$ by a total homogeneity $\varsigma$ if one can use the same constant $C$ in (A.2) for all values of $\theta \in \Theta$.

**Definition A.8** Given a set of scale assignments $\mathcal{N}_{\mathbf{G}}$ and a total homogeneity $\varsigma$, we say that $\varsigma$ is subdivergence free in $\bar{\mathcal{V}} \subset \mathcal{V}$ for the set of scales $\mathcal{N}_{\mathbf{G}}$ if for every $\mathbf{T} \in \mathcal{U}_{\mathcal{V}}$ and every $a \in \mathring{\mathbf{T}} \setminus \{\varrho_{\mathbf{T}}\}$ with $\mathbf{L}_a \subset \bar{\mathcal{V}}$ one has

$$\sum_{b\in\mathring{\mathbf{T}}_{\geq a}} \varsigma_{\mathbf{T}}(b) < (|\mathbf{L}_a| - 1)|\mathfrak{s}|. \tag{A.6}$$

**Definition A.9** We say a total homogeneity $\varsigma$ is of order $\alpha \in \mathbf{R}$ if for every $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$ one has $\sum_{a\in\mathring{\mathbf{T}}} \varsigma_{\mathbf{T}}(a) - (|\mathcal{V}| - 1)|\mathfrak{s}| = \alpha$.

The following definition will be useful in getting good bounds on various integrals.

**Definition A.10** Given a family of continuous compactly supported functions $F = (F^{\mathbf{n}})_{\mathbf{n}\in\mathcal{N}_{\mathbf{G}}}$ on $(\mathbf{R}^d)^{\mathcal{V}_0}$ and a subset $A \subset \mathcal{V}_0$ we denote by $\mathrm{Mod}_A(F)$ the collection of all family of continuous compactly supported functions $\tilde{F} = (\tilde{F}^{\mathbf{n}})_{\mathbf{n}\in\mathcal{N}_{\mathbf{G}}}$ such that there exists $x_{v_0} \in \mathbf{R}^d$ with the property that for every $\mathbf{n} \in \mathcal{N}_{\mathbf{G}}$ one has

$$\mathrm{supp}\left(\tilde{F}^{\mathbf{n}}(\cdot)\right) \subset \mathcal{D}(\mathbf{T},\mathbf{s},x_{v_0}) \tag{A.7}$$

and for any $x \in (\mathbf{R}^d)^A$

$$\int_{\mathcal{V}_0\setminus A} dy\, F^{\mathbf{n}}(x \sqcup y) = \int_{\mathcal{V}_0\setminus A} dy\, \tilde{F}^{\mathbf{n}}(x \sqcup y) .$$

We also use the notation $\mathrm{Mod}(F) = \bigcup_{A\subset\mathcal{V}_0} \mathrm{Mod}_A(F)$.

**Theorem A.11** *Suppose we are given a set of scales $\mathcal{N}_{\mathbf{G}}$ and a family of smooth compactly supported functions $F = (F^{\mathbf{n}})_{\mathbf{n} \in \mathcal{N}_{\mathbf{G}}}$ on $(\mathbf{R}^d)^{\mathcal{V}_0}$ and a total homogeneity $\varsigma$ on the trees of $\widehat{\mathcal{U}}_{\mathcal{V}}$ which is of order $\alpha$ and subdivergence free on $\mathcal{V}$ for the set of scales $\mathcal{N}_{\mathbf{G}}$.*

*For $r \in \mathbf{N}$ we define*

$$\mathcal{N}_{\mathbf{G}, > r} \stackrel{\text{def}}{=} \{\mathbf{n} \in \mathcal{N}_{\mathbf{G}} : \min_{e \in \mathbf{G}} n_e > r\} \text{ and } \mathcal{N}_{\mathbf{G}, \leq r} \stackrel{\text{def}}{=} \{\mathbf{n} \in \mathcal{N}_{\mathbf{G}} : \min_{e \in \mathbf{G}} n_e \leq r\} .$$

$$(A.8)$$

*Then, for $\alpha > 0$, one has*

$$\sum_{\mathbf{n} \in \mathcal{N}_{\mathbf{G}, \leq r}} \int_{\mathcal{V}_0} dy \, |F^{\mathbf{n}}(y)| \leq \text{const}(|\mathcal{V}|) 2^{\alpha r} \inf_{\tilde{F} \in \text{Mod}(F)} \|\tilde{F}\|_{\varsigma, \mathcal{N}_{\mathbf{G}, \leq r}}$$

*while for $\alpha < 0$ one has*

$$\sum_{\mathbf{n} \in \mathcal{N}_{\mathbf{G}, > r}} \int_{\mathcal{V}_0} dy \, |F^{\mathbf{n}}(y)| \leq \text{const}(|\mathcal{V}|) 2^{\alpha r} \inf_{\tilde{F} \in \text{Mod}(F)} \|\tilde{F}\|_{\varsigma, \mathcal{N}_{\mathbf{G}, > r}} .$$

*Here, $\text{const}(|\mathcal{V}|)$ is a combinatorial factor depending only on $|\mathcal{V}|$ and not on $r$.*

*Proof.* This is essentially a special case of [HQ15, Lem. A.10] with $\nu_\star$ equal to the root of $\mathbf{T}$. The only difference is that our "subdivergence-free condition" does not include the root itself. In the case $\alpha < 0$, Definition A.9 implies that (A.6) also holds for the root and we can apply [HQ15, Lem. A.10]. In the case $\alpha > 0$, this is not the case, but the proof of [HQ15, Lem. A.10] still applies, the only difference being that the sum appearing in the base case $|\mathring{\mathbf{T}}| = 1$ runs over large scales instead of small scales. □

For the next theorem and what follows, for any $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$ and $a \in \mathring{\mathbf{T}}$ we define

$$\mathbf{T}_{\not\geq a} \stackrel{\text{def}}{=} \{b \in \mathring{\mathbf{T}} : b \not\geq a\} .$$

**Theorem A.12** *Let $\mathcal{N}_{\mathbf{G}}$ be fixed and let $\mathbf{G}_* \subset \mathbf{G}$ be non-empty subset of edges which connects the collection of vertices of $\mathcal{V}$ it is incident with, we denote this collection of vertices by $\mathcal{V}_*$.*

*Suppose that we are given a family of functions $F = (F^{\mathbf{n}})_{\mathbf{n} \in \mathcal{N}_{\mathbf{G}}}$ and a total homogeneity $\varsigma$ which is sub-divergence free on $\mathcal{V}$ for the scales $\mathcal{N}_{\mathbf{G}}$, of order $\alpha < 0$, and satisfies the following large scale integrability condition: for every $\mathbf{T} \in \mathcal{U}_{\mathcal{V}}$ and $u \in \mathring{\mathbf{T}}$ with both $u \leq (\mathcal{V}_*)^{\uparrow}$ and $\mathbf{T}_{\not\geq u} \neq \emptyset$ one has*

$$\sum_{w \in \mathbf{T}_{\not\geq u}} \varsigma_{\mathbf{T}}(w) > |\mathfrak{s}| \, |\mathcal{V} \setminus \mathbf{L}_u| .$$

$$(A.9)$$

*Then if we set, for any $r \in \mathbf{N}$,*

$$\mathcal{N}_{\mathbf{G},>r,\mathbf{G}_*} \stackrel{\text{def}}{=} \left\{ \mathbf{n} \in \mathcal{N}_{\mathbf{G}} : \min_{e \in \mathbf{G}_*} n_e \geq r \right\}$$

*we have the bound, uniform in $r$,*

$$\sum_{\mathbf{n} \in \mathcal{N}_{\mathbf{G},>r,\mathbf{G}_*}} \int dy_{\mathcal{V}_0} \, |F^{\mathbf{n}}(y)| \lesssim 2^{\alpha r} \inf_{\tilde{F} \in \text{Mod}(F)} \|\tilde{F}\|_{\varsigma,\mathcal{N}_{\mathbf{G}}} \ .$$

*Proof.* This is precisely the content of [HQ15, Lem. A.10]. □

We close the section by introducing more notation. Given any total homogeneity $\varsigma$ on the set of coalesence trees $\widehat{\mathcal{U}}_{\mathcal{V}}$, any $A \subset \mathcal{V}$, and any multi-index $(k_v)_{v \in A} \in (\mathbf{N}^d)^A$ we define another total homogeneity $D^k \varsigma$ by setting

$$D^k \varsigma \stackrel{\text{def}}{=} \varsigma + \sum_{v \in A} |k_v|_{\mathfrak{s}} \delta^{\uparrow}[v] \ . \tag{A.10}$$

**Definition A.13** Given a family of compactly supported functions $(F^{\mathbf{n}})_{\mathbf{n} \in \mathcal{N}_{\mathbf{G}}}$ on $(\mathbf{R}^d)^{\mathcal{V}_0}$ and a total homogeneity $\varsigma$ on the trees of $\widehat{\mathcal{U}}_{\mathcal{V}}$ and a subset $\tilde{\mathcal{V}} \subset \mathcal{V}_0$, we define

$$\|F\|_{\varsigma,\mathcal{N}_{\mathbf{G}},\tilde{\mathcal{V}}} \stackrel{\text{def}}{=} \sup_{k \in A} \|F_k\|_{D^k \varsigma,\mathcal{N}_{\mathbf{G}}}$$

where

$$A \stackrel{\text{def}}{=} \left\{ k \in (\mathbf{N}^d)^{N^*} : \ k \text{ supported on } \tilde{\mathcal{V}} \text{ and } \sup_{u \in N^*} |k(u)|_{\mathfrak{s}} \leq |\mathfrak{s}| \right\},$$

and for $k \in A$ we defined $F_k = (F_k^{\mathbf{n}})_{\mathbf{n} \in \mathcal{N}_{\mathbf{G}}}$ via $F_k^{\mathbf{n}} \stackrel{\text{def}}{=} D^k F^{\mathbf{n}}$.

## A.2   Multiscale clustering for cumulants

The types of bounds on cumulants we ask for in Definitions 2.21 and 2.23 place major limitations on the types of driving noises we can accommodate – we are basically restricted to space-time random fields that are either Gaussian, Gaussian-like (the kind of situation one encounters when considering a central-limit type convergence result as in [HS15]), or non-Gaussian with rapidly decreasing cumulants.

In order to work in a more general setting and which as few restrictions as possible we will require additional data: in addition to a set of types $\mathfrak{L} = \mathfrak{L}_+ \sqcup \mathfrak{L}_-$ and a homogeneity assignment $|\cdot|_{\mathfrak{s}}$ for $\mathfrak{L}$, we prescribe (i) a set of non-vanishing cumulants and (ii) a corresponding *cumulant homogeneity*.

To describe the latter, we use the multiclustering notation introduced earlier to describe our cumulant bounds. For $N \in \mathbf{N}$ with $N \geq 2$ we view $[N]$ as a vertex set, we take $\mathbf{G} \stackrel{\text{def}}{=} [N]^{(2)}$ to be the complete graph and we set $1 \in [N]$ to be the pinned vertex. We set $\mathcal{N}_{\mathbf{G}} \stackrel{\text{def}}{=} \mathbf{N}^{\mathbf{G}}$ for our set of scales. One then has $\mathcal{U}_{[N]} = \widehat{\mathcal{U}}_{[N]}$, the set of all coalescence trees on $[N]$.

**Definition A.14** A cumulant homogeneity $\mathfrak{c}$ for $\mathfrak{L}_{\mathrm{Cum}}$ is a collection

$$\mathfrak{c} = \{\mathfrak{c}^{(\mathfrak{t},[M])} : (\mathfrak{t},[M]) \in \mathfrak{L}_{\mathrm{Cum}}\}$$

where each $\mathfrak{c}^{(\mathfrak{t},[M])}$ is a total homogeneity on the trees of $\mathcal{U}_{[M]}$, so $\mathfrak{c}^{(\mathfrak{t},[M])} = (\mathfrak{c}_{\mathbf{T}}^{(\mathfrak{t},[M])})_{\mathbf{T}\in\mathcal{U}_{[M]}}$.

We also impose that $\mathfrak{c}$ is imposed to be invariant under permutations in the following sense. For each $(\mathfrak{t},[M]) \in \mathfrak{L}_{\mathrm{Cum}}$ a permutation $\sigma : [M] \to [M]$ induces bijections $\sigma : 2^{[M]} \to 2^{[M]}$ and $\sigma : \widehat{\mathcal{U}}_{[M]} \to \widehat{\mathcal{U}}_{[M]}$ since there is a canonical association $\mathbf{T} \in \widehat{\mathcal{U}}_{[M]} \leftrightarrow N_{\mathbf{T}} \subset 2^{[M]}$. We then require that for any $A \subset [M]$ with at least 2 elements and for any $\mathbf{T} \in \widehat{\mathcal{U}}_{[M]}$ one has

$$\mathfrak{c}_{\mathbf{T}}^{(\mathfrak{t},[M])}(A^{\uparrow,\mathbf{T}}) = \mathfrak{c}_{\sigma(\mathbf{T})}^{(\mathfrak{t}\circ\sigma,[M])}(\sigma(A)^{\uparrow,\mathbf{T}}) \,.$$

Thus, given a finite set $B$ equipped with a type map $\mathfrak{t} : B \to \mathfrak{L}_-$ we can write $\mathfrak{c}^{(\mathfrak{t},B)}$ without ambiguity, and this notation is compatible with the analogous notation introduced in Assumption 2.18.

**Remark A.15** Note that for any cumulant homogeneity on a set of cumulants $\mathfrak{L}_{\mathrm{Cum}}$ one has that for any $(\mathfrak{t},[2]) \in \mathfrak{L}_{\mathrm{Cum}}$ the total homogeneity $\mathfrak{c}^{(\mathfrak{t},[2])}$ is encoded by just a single scalar value. We sometime abuse notation and treat $\mathfrak{c}^{(\mathfrak{t},[2])}$ as a scalar.

We will need the following notion of consistency.

**Definition A.16** Given a homogeneity assignment $|\cdot|_{\mathfrak{s}}$, we say $\mathfrak{c}$ is *consistent* with $|\cdot|_{\mathfrak{s}}$ if, for every finite set $B$, every $(\mathfrak{t},B) \in \mathfrak{L}_{\mathrm{Cum}}$ and $\mathbf{T} \in \widehat{\mathcal{U}}_B$ the following conditions hold.

1. Total homogeneities are 'correct': $\sum_{u\in\mathring{\mathbf{T}}} \mathfrak{c}_{\mathbf{T}}^{(\mathfrak{t},B)}(u) = -|\mathfrak{t}(B)|_{\mathfrak{s}}$ .
2. For every $A \subset B$,

$$\sum_{\substack{u\in\mathring{\mathbf{T}} \\ u\leq v \text{ for some } v\in A}} \mathfrak{c}_{\mathbf{T}}^{(\mathfrak{t},B)}(u) \geq -|\mathfrak{t}(A)|_{\mathfrak{s}} \,. \tag{A.11}$$

3. For every $a \in \mathring{\mathbf{T}}$,

$$\sum_{u\in\mathbf{T}_{\geq a}} \mathfrak{c}_{\mathbf{T}}^{(\mathfrak{t},B)}(u) \leq -|\mathfrak{t}(\mathbf{L}_a)|_{\mathfrak{s}} \,. \tag{A.12}$$

4. If $M \geq 3$ then for every $a \in \mathring{\mathbf{T}}$ with $|\mathbf{L}_a| \leq 3$ one has

$$\sum_{u\in\mathbf{T}_{\geq a}} \mathfrak{c}_{\mathbf{T}}^{(\mathfrak{t},[M])}(u) < |\mathfrak{s}| \cdot (|\mathbf{L}_a| - 1) \,. \tag{A.13}$$

**Remark A.17** It is possible to see that if item 1 of Definition A.14 is satisfied, then items 2 and 3 are equivalent. One way of interpreting these conditions is that $\mathfrak{c}_{\mathbf{T}}^{(\mathfrak{t},B)}$ is always obtained by distributing the homogeneity $-|\mathfrak{t}(a)|_{\mathfrak{s}}$ of each leaf $a$ among the nodes of $\mathring{\mathbf{T}}$ connecting $a$ to the root.

With these definitions at hand, we are now ready to formulate our "distance" on the space $\mathcal{M}(\Omega_0, \mathfrak{L}_{\text{Cum}})$ of stationary noise processes.

**Definition A.18** Given a set of cumulants $\mathfrak{L}_{\text{Cum}}$ and a cumulant homogeneity $\mathfrak{c}$ for $\mathfrak{L}_{\text{Cum}}$, $N \geq 2$, $r \in \mathbf{N}$, and $\xi \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\text{Cum}})$ we set

$$\|\xi\|_{N,\mathfrak{c},r} \stackrel{\text{def}}{=} \text{Diag}_{\mathfrak{c}}(\xi) \vee \max_{\substack{M \leq N \\ (\mathfrak{t},[M]) \in \mathfrak{L}_{\text{Cum}}}} \max_{k \in D_{M,r}} \|F_{k,(\mathfrak{t},[M])}\|_{D^k \mathfrak{c}^{(\mathfrak{t},[M])}}$$

Here we have set

$$D_{M,r} \stackrel{\text{def}}{=} \{k = (k_i)_{i=1}^M \in (\mathbf{N}^d)^M : \forall i \in [M], \ |k_i|_{\mathfrak{s}} \leq 3r|\mathfrak{s}|\}, \qquad (A.14)$$

We have also defined the family $F_{k,(\mathfrak{t},[M])} = (F^{\mathbf{n}}_{k,(\mathfrak{t},[M])} : \mathbf{n} \in \mathbf{N}^{[M]^2})$ to be given by setting $F^{\mathbf{n}}_{k,(\mathfrak{t},[M])}(x_{[M]\setminus\{1\}})$ equal to

$$\Big( \prod_{1 \leq j < m \leq M} \Psi^{(n_{\{j,m\}})}(x_j - x_m) \Big) D^k \mathbf{E}^c[\{\xi_{\mathfrak{t}(i)}(x_i)\}_{i=1}^M] \Big|_{x_1 = 0} .$$

and we are using the notation of (A.5). We also used the set of derivatives

The term $\text{Diag}_{\mathfrak{c}}(\xi)$ is defined as follows. If there exists any $(\mathfrak{t}, [2]) \in \mathfrak{L}_{\text{Cum}}$ such that $\mathscr{R}[\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)}, \xi_{\mathfrak{t}(2)}\}]]$ is not of order $(\mathfrak{c}_{(\mathfrak{t},[2])} - |\mathfrak{s}|) \vee 0$ at $0$ then we set $\text{Diag}(\xi) \stackrel{\text{def}}{=} \infty$. Otherwise we set

$$\text{Diag}_{\mathfrak{c}}(\xi) \stackrel{\text{def}}{=} \max_{(\mathfrak{t},[2]) \in \mathfrak{L}_{\text{Cum}}} \max_{\substack{k \in \mathbf{N}^d, \\ |k|_{\mathfrak{s}} < \mathfrak{c}^{(\mathfrak{t},[2])} - |\mathfrak{s}|}} |\mathscr{R}[\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)}, \xi_{\mathfrak{t}(2)}\}]](p_k)| . \qquad (A.15)$$

where $\mathscr{R}[\cdot]$ is as in (2.9).

**Definition A.19** Given a set of cumulants $\mathfrak{L}_{\text{Cum}}$, a cumulant homogeneity $\mathfrak{c}$ for $\mathfrak{L}_{\text{Cum}}$, $N \geq 2$, $r \in \mathbf{N}$ and $\xi, \bar{\xi} \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\text{Cum}})$ defined on the same probability space and jointly admitting pointwise cumulants, we define the quantity $\|\xi; \bar{\xi}\|_{N,\mathfrak{c},r}$ by

$$\|\xi; \bar{\xi}\|_{N,\mathfrak{c},r} \stackrel{\text{def}}{=} \text{Diag}_{\mathfrak{c}}(\xi; \bar{\xi}) \vee \max_{\substack{\hat{M} \leq M \leq N \\ (\mathfrak{t},[M]) \in \mathfrak{L}_{\text{Cum}}}} \max_{k \in A_{M,r}} \|\hat{F}_{k,(\mathfrak{t},[M]),\hat{M}}\|_{D^k \mathfrak{c}^{(\mathfrak{t},M)}, \mathbf{N}^{[M]^{(2)}}} .$$

Here $D_{M,r}$ is defined as in (A.14). We have set the family $\hat{F}_{k,(\mathfrak{t},[M]),\hat{M}} = (\hat{F}^{\mathbf{n}}_{(\mathfrak{t},[M]),\hat{M}} : \mathbf{n} \in \mathbf{N}^{[M]^2})$ to be given by setting $\hat{F}^{\mathbf{n}}_{k,(\mathfrak{t},[M]),\hat{M}}(x_{[M]\setminus\{1\}})$ equal to

$$\Big( \prod_{1 \leq j < m \leq M} \Psi^{(n_{\{j,m\}})}(x_j - x_m) \Big) D^k \mathbf{E}^c[\{\hat{\xi}_{\mathfrak{t}(i),i}(x_i)\}_{i=1}^M] \Big|_{x_1 = 0} ,$$

and the random fields $(\hat{\xi}_{\mathfrak{t},i})_{i=1}^M$ above are given by

$$\hat{\xi}_{\mathfrak{t},i}(x) \stackrel{\text{def}}{=} \xi_{\mathfrak{t}}(x) \mathbb{1}\{i \leq \hat{M}\} - \bar{\xi}_{\mathfrak{t}}(x) \mathbb{1}\{i \geq \hat{M}\} .$$

We set The quantity $\mathrm{Diag}_{\mathfrak{c}}(\xi; \bar{\xi})$ is defined as follows. If there exists any $(\mathfrak{t}, [2]) \in \mathfrak{L}_{\mathrm{Cum}}$ and $\hat{M} \in \{1, 2\}$ such that $\mathscr{R}[\mathbf{E}^c[\{\hat{\xi}_{\mathfrak{t}(1)}, \hat{\xi}_{\mathfrak{t}(2)}\}]]$ is not of order $\mathfrak{c}_{(\mathfrak{t},[2])} - |\mathfrak{s}|$ at 0 then we set $\mathrm{Diag}_{\mathfrak{c}}(\xi; \bar{\xi}) \overset{\mathrm{def}}{=} \infty$. Otherwise we set

$$\mathrm{Diag}_{\mathfrak{c}}(\xi; \bar{\xi}) \overset{\mathrm{def}}{=} \max_{\substack{(\mathfrak{t},[2]) \in \mathfrak{L}_{\mathrm{Cum}} \\ 1 \leq \hat{M} \leq 2}} \max_{\substack{k \in \mathbf{N}^d, \\ |k|_{\mathfrak{s}} < \mathfrak{c}^{(\mathfrak{t},[2])} - |\mathfrak{s}|}} |\mathscr{R}[\mathbf{E}^c[\{\hat{\xi}_{\mathfrak{t}(1)}, \hat{\xi}_{\mathfrak{t}(2)}\}]](p_k)| . \tag{A.16}$$

If $\xi$ and $\bar{\xi}$ don't admit joint pointwise cumulants we set $\|\xi; \bar{\xi}\|_{N,\mathfrak{c}} = \infty$ for every $N$.

**Remark A.20** We often write $\|\xi|_{N,\mathfrak{c}}$ or $\|\xi; \bar{\xi}\|_{N,\mathfrak{c}}$ instead of $\|\xi\|_{N,\mathfrak{c},0}$ or $\|\xi; \bar{\xi}\|_{N,\mathfrak{c},0}$.

**Remark A.21** Going back to an earlier mentioned example, suppose $\psi$ is a random, stationary, centered Gaussian, element of the scaled Holder-Besov space $C_{\mathfrak{s}}^\alpha(\mathbf{R}^d)$ where $\alpha \in (-\frac{|\mathfrak{s}|}{4}, 0)$. Then for any sequence of smooth compactly supported approximate identities $(\eta_n)_{n \in \mathbf{N}}$ appropriately converging to a Dirac delta as $n \to \infty$ one expects that $(\psi * \eta_n)^2(\cdot) - \mathbf{E}[(\psi * \eta_n)^2(0)]$ should converge in probability on $C_{\mathfrak{s}}^{2\alpha}(\mathbf{R}^d)$ to a limit we denote $\psi^{\diamond 2}$.

The random distribution $\psi^{\diamond 2}$ is sometimes referred to as the "second Wick power" of $\psi$. Suppose we are looking at a system of equations driven by only $\psi^{\diamond 2}$. Writing $\mathfrak{t}_-$ for the type-label for this noise one could set $|\mathfrak{t}_-|_{\mathfrak{s}} = 2\alpha$ and we now describe a cumulant homogeneity $\mathfrak{c}$ one could use in this situation.

For any $N \geq 2$, $\mathbf{T} \in \widehat{\mathcal{U}}_{[N]}$, and $a \in \mathring{\mathbf{T}}$ one sets,

$$\mathfrak{c}_{\mathbf{T}}^{(\mathfrak{t}_-,[N])}(a) \overset{\mathrm{def}}{=} 2\alpha \cdot (d(a) - 1) + 2\alpha \mathbb{1}\{a = \varrho_{\mathbf{T}}\}$$

where for $a \in N_{\mathbf{T}}$ we write $d(a) \overset{\mathrm{def}}{=} |\mathrm{Min}(\{b \in N_{\mathbf{T}} : b > a\})|$.

**Definition A.22** Assume we are given a cumulant homogeneity $\mathfrak{c}$ on $\mathfrak{L}_{\mathrm{Cum}}$ consistent with a homogeneity assignment $|\cdot|_{\mathfrak{s}}$, and finite set $A$ and $D$ equipped with a type map $\mathfrak{t} : A \sqcup D \to \mathfrak{L}_-$. We then define the quantity $|\mathfrak{t}(A)|_{\mathfrak{s},\mathfrak{c},D} \leq 0$ as follows: if $A \notin \mathfrak{L}_{\mathrm{Cum}}$ we set $|\mathfrak{t}(A)|_{\mathfrak{s},\mathfrak{c},D} \overset{\mathrm{def}}{=} 0$ and otherwise we set $|\mathfrak{t}(A)|_{\mathfrak{s},\mathfrak{c},D}$ equal to

$$\inf \left\{ -\sum_{u \in \mathbf{T}_{\geq a}} \mathfrak{c}^{(\mathfrak{t}', A \sqcup [N])}(u) : \begin{array}{l} N \geq 1, \ (\mathfrak{t}', A \sqcup [N]) \in \mathfrak{L}_{\mathrm{Cum}} \\ \mathfrak{t}' {\restriction} A = \mathfrak{t}, \ \mathfrak{t}'([N]) \subset \mathfrak{t}(D) \\ \mathbf{T} \in \mathcal{U}_{A \sqcup [N]} \text{ with } a \in \mathring{\mathbf{T}} \text{ s.t. } \mathbf{L}_a = A. \end{array} \right\} . \tag{A.17}$$

Above, we are enforcing that $\mathfrak{t}'([N]) \subset \mathfrak{t}(D)$ in the sense of sets, not multi-sets.

**Remark A.23** Observe that for any leaf typed sets $A$ and $D$ we have $|A| \leq 3 \Rightarrow |\mathfrak{t}(A)|_{\mathfrak{s},\mathfrak{c},D} \geq [(|A| - 1) \vee 0] \cdot |\mathfrak{s}|$ thanks to (A.13) and by (A.12) one has

$$|\mathfrak{t}(A)|_{\mathfrak{s},\mathfrak{c},D} \geq |\mathfrak{t}(A)|_{\mathfrak{s}} . \tag{A.18}$$

The above inequality is used many times throughout the paper, often implicitly.

In order to get the desired stochastic estimates when we use a cumulant homogeneity $\mathfrak{c}$ to control cumulants we also need a stronger notion of super-regularity.

**Definition A.24** Given a homogeneity assignment $|\cdot|_{\mathfrak{s}}$, a set of cumulants $\mathfrak{L}_{\mathrm{Cum}}$, and a cumulant homogeneity $\mathfrak{c}$ on $\mathfrak{L}_{\mathrm{Cum}}$ consistent with $|\cdot|_{\mathfrak{s}}$ and a leaf-typed finite set $D$ we define a map $h_{\mathfrak{c},D}$ defined on leaf typed finite sets $(\mathfrak{t}, A)$ as follows. If $A = \varnothing$ we set $\hbar_{\mathfrak{c},D}(\varnothing) \stackrel{\mathrm{def}}{=} 0$ and for $A \neq \varnothing$ we set

$$\hbar_{\mathfrak{c},D}(A) \stackrel{\mathrm{def}}{=} \min_{\varnothing \neq B \subset A} (|\mathfrak{t}(B)|_{\mathfrak{s},\mathfrak{c},D} - |\mathfrak{t}(B)|_{\mathfrak{s}}) \ . \tag{A.19}$$

**Remark A.25** $\hbar_{\mathfrak{c},D}(A)$ gives the minimum homogeneity gain from having at least one element of $A$ participate in a cumulant external to $A$ with noises drawn from $D$.

For the renormalization of second cumulants it is convenient to define, for any leaf typed set $B$,
$$\ell(B) \stackrel{\mathrm{def}}{=} \mathbf{1}\{|B| = 2\}(\lceil -|\mathfrak{t}(B)|_{\mathfrak{s}} - |\mathfrak{s}|\rceil \vee 0) \ . \tag{A.20}$$

The quantity $\ell(B)$ gives the power-counting gain from the renomalization of a second cumulant corresponding to $B$. The next lemma states that we don't need to renormalize any other cumulants.

**Lemma A.26** *Let $M$ and $D$ be a leaf typed sets and $(\mathfrak{t}, M) \in \mathfrak{L}_{\mathrm{Cum}}$. Then*

$$|\mathfrak{t}(M)|_{\mathfrak{s},\mathfrak{c},D} \wedge (\ell(M) + |\mathfrak{t}(M)|_{\mathfrak{s}}) + (|M| - 1) \cdot |\mathfrak{s}| > 0 \tag{A.21}$$

*Proof.* If $|M| = 2$ then this follows from either (A.13) or the definition of $\ell(M)$. If $|M| \geq 3$ this follows from either Remark A.23 or Assumption 2.24. $\qquad\square$

**Definition A.27** A semi-decorated $T_{\mathfrak{e}}^{\mathfrak{n}}$ is said to be $(\mathfrak{c}, |\cdot|_{\mathfrak{s}}, \mathfrak{L}_{\mathrm{Cum}})$-*super-regular* if, for every subtree $S$ of $T$ with $|N(S)| > 1$, one has

$$|S_{\mathfrak{e}}^{0}|_{\mathfrak{s}} + \left(\frac{|\mathfrak{s}|}{2} \wedge \hbar_{\mathfrak{c},L(T)}(L(S)) \wedge \jmath_{L(T)}(L(S))\right) > 0 \ . \tag{A.22}$$

**Remark A.28** Given a set of non-vanishing cumulants $\mathfrak{L}_{\mathrm{Cum}}$ and a homogeneity assignment $|\cdot|_{\mathfrak{s}}$ one choice of cumulant homogeneity $\mathfrak{c}$ for $\mathfrak{L}_{\mathrm{Cum}}$ consistent with $|\cdot|_{\mathfrak{s}}$ is given by setting, for each $(\mathfrak{t}, [M]) \in \mathfrak{L}_{\mathrm{Cum}}$, the total homogeneity $(\mathfrak{c}_{\mathbf{T}}^{(\mathfrak{t},[M])})_{\mathbf{T} \in \mathcal{U}_{[M]}}$ to be given by $\mathfrak{c}_{\mathbf{T}}^{(\mathfrak{t},[M])}(a) \stackrel{\mathrm{def}}{=} |\mathfrak{t}([M])|_{\mathfrak{s}} \cdot \mathbb{1}\{a = \varrho_{\mathbf{T}}\}$ for every $\mathbf{T} \in \mathcal{U}_{[M]}$ and $a \in \overset{\circ}{\mathbf{T}}$. This is precisely the choice of $\mathfrak{c}$ being referred to in Remark 2.30.

It is useful to formalize how the cumulant homogeneity will be used with regards to larger integrands, which motivates the following definitions.
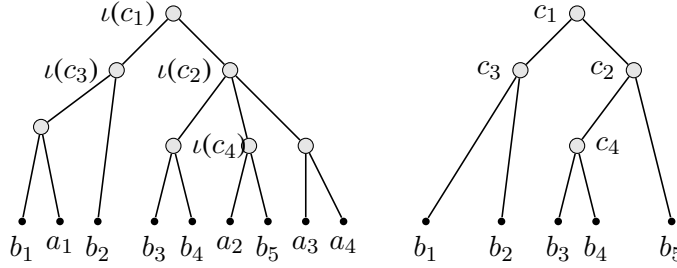
**Definition A.29** Let $A$ be a finite set and let $B \subset A$ be of cardinality at least 2. For each $\mathbf{T} \in \widehat{\mathcal{U}}_A$ there is a natural choice of $\mathbf{S} \in \widehat{\mathcal{U}}_B$ corresponding to the "the restriction of $\mathbf{T}$ to $B$". First recall that $N_{\mathbf{T}}$ can be identified with a subset of $2^A$.

Using this identification, we then define $N_{\mathbf{S}} = \{D \cap B : D \in N_{\mathbf{T}}, \ D \cap B \neq \emptyset\}$. We then equip $N_{\mathbf{S}}$ with the reverse inclusion partial order as in Section A – this finishes the specification of $\mathbf{S} \in \mathcal{U}_B$. For what follows, recall that, except for the leaves, we usually treat $N_{\mathbf{S}}$ as an abstract set (except for the leaves).

Note that there is a natural injection $\iota : \mathring{\mathbf{S}} \to \mathring{\mathbf{T}}$ given by the map $a \mapsto (\mathbf{L}_a)^{\uparrow,\mathbf{T}}$ where $a \in \mathring{\mathbf{S}}$ and the set of descendents of $a$ given by $L_{\mathbf{a}}$ is determined by $\mathbf{S}$.

If a coalesence tree $\mathbf{S}$ on a vertex set $B$ is obtained in such a way from a a coalesence tree $\mathbf{T}$ on a vertex set $A \supset B$ then we say $\mathbf{S}$ is a *restriction* of $\mathbf{T}$ to $B$ and sometimes write $\mathbf{S} \overset{\text{def}}{=} \mathbf{T}\!\restriction_B$.

We draw a picture to make this clearer. Here $A = \{a_i\}_{i=1}^4 \sqcup \{b_i\}_{i=1}^5$ and $B = \{b_i\}_{i=1}^5$. To the left we have drawn a choice of $\mathbf{T} \in \widehat{\mathcal{U}}_A$ and on the right the corresponding $\mathbf{T}\!\restriction_B \in \mathcal{U}_B$.



**Definition A.30** Let $\mathfrak{c}$ be a cumulant homogeneity, let $A$ be a finite set, and let $B \subset A$ be of cardinality at least 2 and suppose we also have a type map $\mathsf{t} : B \to \mathfrak{L}_-$. We define a total homogeneity $\mathfrak{c}^{(\mathsf{t},B)} = (\mathfrak{c}_{\mathbf{T}}^{(\mathsf{t},B)})_{\mathbf{T} \in \widehat{\mathcal{U}}_A}$ as follows. Fix $\mathbf{T} \in \widehat{\mathcal{U}}_A$, let $\mathbf{S} \overset{\text{def}}{=} \mathbf{T}\!\restriction_B$, and let $\iota$ be the corresponding injection $\iota : \mathring{\mathbf{S}} \to \mathring{\mathbf{T}}$. We then set

$$\mathfrak{c}_{\mathbf{T}}^{(\mathsf{t},B)}(u) \overset{\text{def}}{=} \begin{cases} \mathfrak{c}_{\mathbf{S}}^{(\mathsf{t},B)}(a) & \text{if } u = \iota(a) \text{ for } a \in \mathring{\mathbf{S}} \\ 0 & \text{otherwise.} \end{cases}$$

### A.3   Multiscale clustering for Wick contractions

**Definition A.31** Let $A$ be a finite set and $p$ be a positive integer. We denote by $\{A^{(j)}\}_{j=1}^p$ $p$ distinct copies of the set $A$. A $p$-fold Wick contraction on $A$ is a partition $\pi$ of $\bigsqcup_{j=1}^p A^{(j)}$ which satisfies the condition that for any $B \in \pi$ we have $B \not\subset A^{(j)}$ for any $1 \leq j \leq p$. In other words, each element of the partition is required to straddle at least two copies of $A$.

**Theorem A.32** *Let $\mathcal{X}$ be a finite set of cardinality at least two with a distinguished element $\circledast \in \mathcal{X}$ and a subset $\ell \subset \mathcal{X}_0 \overset{\text{def}}{=} \mathcal{X} \setminus \{\circledast\}$ equipped with a type map $\mathsf{t} : \ell \to \mathfrak{L}_-$.*

*Let $\mathbf{H} \subset \mathcal{X}^{(2)}$ be a connected multigraph on $\mathcal{X}$ and $\mathcal{N}_{\mathbf{H}} \subset \mathbf{N}^{\mathbf{H}}$ be a set of scales. Let $\mathbf{H}_* \subset \mathbf{H}$ be a non-empty connected subset of edges such that $\mathcal{X}_*$, the set of vertices incident to $\mathbf{H}_*$, is disjoint from $\ell$.*

*Let $\varsigma$ be a total homogeneity on $\widehat{\mathcal{U}}_{\mathcal{X}}$ with the following properties:*

1. $\varsigma$ *is of order* $\alpha < |\mathsf{t}(\ell)|_{\mathfrak{s}}$.
2. *Using the notation* $\hbar_{\varsigma,.}(\cdot)$ *of* (A.19), *one has, for every* $\mathbf{T} \in \mathcal{U}_{\mathcal{X}}$ *and* $a \in \mathring{\mathbf{T}}$,

$$\sum_{b \in \mathbf{T}_{\geq a}} \varsigma_{\mathbf{T}}(b) < |\mathfrak{s}| \, (|\mathbf{L}_a| - 1) + |\mathsf{t}(\mathbf{L}_a^{\text{ext}})|_{\mathfrak{s}} \tag{A.23}$$

$$+ \mathbb{1}\{\mathbf{L}_a \not\supseteq \circledast\} \left[ \hbar_{\varsigma,\ell}(\mathbf{L}_a^{\text{ext}}) \wedge j_{\ell}(\mathbf{L}_a^{\text{ext}}) \wedge \frac{|\mathfrak{s}|}{2} \right] \, ,$$

   *where we used the notation* $\mathbf{L}_a^{\text{ext}} \stackrel{\text{def}}{=} \{u \in \ell \, : \, u \in \mathbf{L}_a\}$.
3. *For any* $\mathbf{T} \in \mathcal{U}_{\mathcal{X}}$ *and* $a \in \mathring{\mathbf{T}}$ *with both* $a \leq \mathcal{X}_*^{\uparrow}$ *and* $\mathbf{T}_{\not\geq a} \neq \emptyset$ *one has*

$$\sum_{b \in \mathbf{T}_{\not\geq a}} \varsigma_{\mathbf{T}}(b) > |\mathfrak{s}| \, |\mathcal{X} \setminus \mathbf{L}_a| + |\mathsf{t}(L \setminus \mathbf{L}_a^{\text{ext}})|_{\mathfrak{s}} \, . \tag{A.24}$$

*Then, for any* $p \in \mathbf{N}$, *there exists* $C > 0$ *such that for any family of functions* $G = (G^{\mathbf{n}})_{\mathbf{n} \in \mathcal{N}_{\mathbf{H}}}$ *on* $(\mathbf{R}^d)^{\mathcal{X}_0}$ *bounded by* $\varsigma$ *(for some choice of* $x_{\circledast} \in \mathbf{R}^d$ *where* $\circledast$ *serves as the pinned vertex of* $\mathcal{X}$*), one has the estimate*

$$\mathbf{E}\Big( \int_{\mathcal{X}_0} dy \sum_{\mathbf{n} \in \mathcal{N}_{\mathbf{H}, > r, \mathbf{H}_*}} G^{\mathbf{n}}(y) \cdot \text{Wick}(\{\xi_{\mathsf{t}(u)}(y_u)\})_{u \in \ell}) \Big)^{2p} \tag{A.25}$$

$$\leq C \|\xi\|_{2p|\ell|, \varsigma} \cdot 2^{2p\beta r} \cdot \inf_{\tilde{G} \in \text{Mod}_{\ell}(G)} \|\tilde{G}\|_{\varsigma, \mathcal{N}_{\mathbf{H}}, \ell}^{2p},$$

*where the norm on the RHS was defined in Definition A.13 and* $\beta \stackrel{\text{def}}{=} \alpha - |\mathsf{t}(\ell)|_{\mathfrak{s}}$.

*Proof.* Observe that the LHS of (A.25) does not change if we replace $G$ with $\tilde{G}$ in $\text{Mod}_{\ell}(G)$ – for the rest of the proof we fix such a $\tilde{G}$. Next, recall that

$$\sum_{\mathbf{n} \in \mathcal{N}_{\mathbf{H}, > r, \mathbf{H}_*}} \tilde{G}^{\mathbf{n}} = \sum_{\mathbf{T} \in \mathcal{U}_{\mathcal{X}}} \Big( \sum_{\substack{\mathbf{n} \in \mathcal{N}_{\mathbf{H}, > r, \mathbf{H}_*} \\ \mathcal{T}(\mathbf{n}) = \mathbf{T}}} \tilde{G}^{\mathbf{n}} \Big) \, ,$$

where the number of terms in outer sum on the RHS is bounded by some finite combinatorial constant depending on $|\mathcal{X}|$. Fix some choice of $\mathbf{T} \in \mathcal{U}_{\mathcal{X}}$. By the triangle inequality it suffices to prove (A.25) where on the LHS we replace

$$\sum_{\mathbf{n} \in \mathcal{N}_{\mathbf{H}, > r, \mathbf{H}_*}} G^{\mathbf{n}} \quad \text{with} \quad \hat{G} \stackrel{\text{def}}{=} \sum_{\substack{\mathbf{n} \in \mathcal{N}_{\mathbf{H}, > r, \mathbf{H}_*} \\ \mathcal{T}(\mathbf{n}) = \mathbf{T}}} \tilde{G}^{\mathbf{n}}$$

and on the RHS of (A.25) we drop the infinum over $\tilde{G}$ (since it has been fixed).

We write $\mathcal{X}_0^{(j)}$ for $j = 1, \dots, 2p$ be $2p$ disjoint copies of $\mathcal{X}_0$. We write $\mathcal{V}_0 \stackrel{\text{def}}{=} \bigsqcup_{j=1}^{2p} \mathcal{X}_0^{(j)}$, $\mathcal{V} \stackrel{\text{def}}{=} \mathcal{V}_0 \sqcup \{\circledast\}$, and $\mathcal{X}^{(j)} \stackrel{\text{def}}{=} \mathcal{X}_0^{(j)} \sqcup \{\circledast\}$. We also set $\ell^{\text{all}} \stackrel{\text{def}}{=} \bigsqcup_{j=1}^{2p} \ell^{(j)} \subset \mathcal{V}_0$, so that $2p$-fold Wick contractions are naturally identified with partitions of $\ell^{\text{all}}$. Note that the type map $\mathsf{t}$ naturally yields a type map on $\ell^{\text{all}}$ which we again call $\mathsf{t}$.

Using standard facts about the expansion of moments of Wick powers into sums over Wick contractions (see [HS15, Lem. 4.5]) it suffices to show that for any any fixed $2p$-fold Wick contraction $\pi$ on $\ell$ one has

$$\Big| \int_{\mathcal{V}_0} dy \, \Big( \prod_{j=1}^{2p} \hat{G}(y^{(j)}) \Big) \Big( \prod_{B \in \pi} \mathbf{E}^c[\{\xi_{\mathsf{t}(u)}(y_u)\}_{u \in B}] \Big) \Big| \tag{A.26}$$
$$\lesssim \|\tilde{G}\|_{\varsigma, \mathcal{N}_{\mathbf{H},\lambda,\ell}}^{2p} \|\xi\|_{2p|\ell|,\mathfrak{c}} \lambda^{2p\beta} \ .$$

where we use the shorthand $y^{(j)} \stackrel{\text{def}}{=} y_{\mathcal{X}_0^{(j)}}$.

For $j \in [2p]$ and $\mathbf{T} \in \mathcal{U}_{\mathcal{X}}$, we write $\mathbf{H}^{(j)}$ and $\mathbf{T}^{(j)}$ for the corresponding copies of $\mathbf{H}$ and $\mathbf{T}$ on the vertex set $\mathcal{X}^{(j)}$. To prove (A.26) we will apply Theorem A.12 with $\mathcal{V}$ as the vertex set, and $\circledast$ as the root vertex – we know that $\hat{G}$ satisfies (A.4) for some choice of $x_\circledast \in \mathbf{R}^d$ – this will be fixed throughout our whole application of Theorem A.12 but our bounds don't depend on this variable.

The underlying multigraph is given by $\mathbf{G} \stackrel{\text{def}}{=} \tilde{\mathbf{G}} \sqcup \mathbf{C}$ where

$$\tilde{\mathbf{G}} \stackrel{\text{def}}{=} \bigsqcup_{j=1}^{2p} \mathbf{H}^{(j)} \ \text{ and } \ \mathbf{C} \stackrel{\text{def}}{=} \Big\{ \{v, v'\} \in \mathcal{V}^{(2)} : \exists B \in \pi \text{ with } v, v' \in B \Big\} \ .$$

We define a distinguished set of edges $\mathbf{G}_* = \bigsqcup_{j=1}^{2p} \mathbf{H}_*^{(j)}$ where $\mathbf{H}_*^{(j)}$ denote the copies of $\mathbf{H}_*$. There is a natural identification of $\bigtimes_{j=1}^{2p} \mathbf{N}^{\mathbf{H}^{(j)}}$ with $\mathbf{N}^{\tilde{\mathbf{G}}}$ and using this identification we define our set of scales $\mathcal{N}_{\mathbf{G}} \subset \mathbf{N}^{\tilde{\mathbf{G}}} \times \mathbf{N}^{\mathbf{C}} = \mathbf{N}^{\mathbf{G}}$ by

$$\mathcal{N}_{\mathbf{G}} \stackrel{\text{def}}{=} \Big( \bigtimes_{j=1}^{2p} \big[ \mathcal{N}_{\mathbf{H},>r,\mathbf{H}_*} \cap \mathcal{T}^{-1}(\mathbf{T}^{(j)}) \big] \Big) \times \mathbf{N}^{\mathbf{C}}.$$

Note that then $\mathcal{N}_{\mathbf{G},>r,\mathbf{G}_*} = \mathcal{N}_{\mathbf{G}}$. We write $\mathcal{U}_{\mathcal{V}}$ for the corresponding set of coalescence trees on $\mathcal{V}$. Elements $\mathbf{m} \in \mathcal{N}_{\mathbf{G}}$ are often denoted as pairs $\mathbf{m} = (\mathbf{n}, \mathbf{j}) \in \mathbf{N}^{\tilde{\mathbf{G}}} \times \mathbf{N}^{\mathbf{C}}$.

Let the family of functions $F = (F^{\mathbf{m}})_{\mathbf{m} \in \mathcal{N}_{\mathbf{G}}}$ on $(\mathbf{R}^d)^{\mathcal{V}_0}$ for which we apply Theorem A.12 be given by

$$F_\lambda^{(\mathbf{n},\mathbf{j})}(y) \stackrel{\text{def}}{=} \Big( \prod_{j=1}^{2p} \tilde{G}^{\mathbf{n}_j}(y^{(j)}) \Big) \mathrm{WickCum}^{\mathbf{j}}(y) \tag{A.27}$$

where we used the natural correspondance $(\mathbf{n}_j)_{j=1}^{2p} \leftrightarrow \mathbf{n}$ and set

$$\mathrm{WickCum}^{\mathbf{j}}(y_{\ell \mathrm{all}}) \stackrel{\text{def}}{=} \prod_{B \in \pi} \Big[ \mathbf{E}^c[\{\xi_{\mathsf{t}(u)}(y_u)\}_{u \in B}] \prod_{\{u,v\} \in B^{(2)}} \Psi^{(j_{\{u,v\}})}(y_u - y_v) \Big].$$

We will define a total homogeneity $\tilde{\varsigma} = \{\tilde{\varsigma}_{\mathbf{S}}\}_{\mathbf{S} \in \widehat{\mathcal{U}}_{\mathcal{V}}}$ and $\tilde{F} \in \mathrm{Mod}(F)$ such that $\tilde{F}$ is bounded by $\tilde{\varsigma}$ (the root vertex position for (A.4) will be given by $x_\circledast$) and with the property that

$$\|\tilde{F}\|_{\tilde{\varsigma}, \mathcal{N}_{\mathbf{G}}} \lesssim \|\xi\|_{2p|\ell|,\mathfrak{c}} \cdot \|\tilde{G}\|_{\varsigma, \mathcal{N}_{\mathbf{H}},\ell}^{2p} \ . \tag{A.28}$$

We will then check that $\tilde{\varsigma}$ satisfies the conditions of Theorem A.12.

We start by defining $\tilde{\varsigma}$. For each $j \in [2p]$ and $\mathbf{S} \in \widehat{\mathcal{U}}_{\mathcal{V}}$ we define a map $\theta_{j,\mathbf{S}} : \mathring{\mathbf{T}}^{(j)} \to \mathring{\mathbf{S}}$ by setting, for any $a \in \mathring{\mathbf{T}}^{(j)}$,

$$\theta_{j,\mathbf{S}}(a) \stackrel{\text{def}}{=} (\mathbf{L}_a)^{\uparrow,\mathbf{S}} .$$

For the remainder of the proof, whenever we write $\uparrow$ or $\Uparrow$ without specifying a tree then this operation is taken in $\mathbf{S}$.

For $j \in [2p]$ we define total homogeneities $\varsigma^j = \{\varsigma_{\mathbf{S}}^j\}_{\mathbf{S} \in \mathcal{U}_{\mathcal{V}}}$ by setting, for each $\mathbf{S} \in \mathcal{U}_{\mathcal{V}}$ and $a \in \mathring{\mathbf{S}}$,

$$\varsigma_{\mathbf{S}}^j(a) \stackrel{\text{def}}{=} \sum_{b \in \theta_{j,\mathbf{S}}^{-1}(a)} \varsigma_{\mathbf{T}^{(j)}}(b)$$

where the total homogeneity $\varsigma$ was fixed in the assumption, $\mathbf{T}$ is the previously fixed tree in $\mathcal{U}_{\mathcal{X}}$, and the $\varsigma_{\mathbf{T}^{(j)}}$ are the corresponding copies of $\varsigma_{\mathbf{T}}$.

We introduce more total homogeneities on the trees of $\widehat{\mathcal{U}}_{\mathcal{V}}$, setting $\bar{\varsigma} \stackrel{\text{def}}{=} \sum_{j=1}^{2p} \varsigma^j$, $\varsigma^C \stackrel{\text{def}}{=} \sum_{B \in \pi} \mathfrak{c}^B$ where we use Definition A.30, as well as $\varsigma^R$ by setting, for each $\mathbf{S} \in \mathcal{U}_{\mathcal{V}}$,

$$\varsigma_{\mathbf{S}}^R \stackrel{\text{def}}{=} \sum_{B \in R(\mathbf{S})} \ell(B)\left(\delta_{\mathbf{S}}^{\Uparrow}[B] - \delta_{\mathbf{S}}^{\uparrow}[B]\right), \text{ where}$$

$$R(\mathbf{S}) \stackrel{\text{def}}{=} \{B \in \pi : \ell(B) > 0 \text{ and } \mathbf{L}_{B^{\uparrow}} = B\} ,$$

and $\ell(B)$ was defined in (A.20). Finally, we define $\tilde{\varsigma} \stackrel{\text{def}}{=} \bar{\varsigma} + \varsigma^C + \varsigma^R$.

The family $\tilde{F}$ will be given by setting

$$\tilde{F}^{\mathbf{k}} \stackrel{\text{def}}{=} \begin{cases} F^{\mathbf{k}} & \text{if } R(\mathcal{T}(\mathbf{k})) = \varnothing , \\ \mathring{F}^{\mathbf{k}} & \text{if } R(\mathcal{T}(\mathbf{k})) \neq \varnothing . \end{cases}$$

We will define $\mathring{F}^{\mathbf{k}}$ on a tree by tree basis later. It suffices to check the domain condition and desired supremum bound in each of the two cases separately.

We first treat the former case, that is when $\mathbf{S} \in \mathcal{U}_{\mathcal{V}}$ satisfies $R(\mathbf{S}) = \varnothing$. The domain property (A.4) is straightforward to check and next we show the supremum bound. Uniformly over $j \in [2p]$, $\mathbf{s} \in \text{Lab}_{\mathbf{S}}$, and $(\mathbf{n}, \mathbf{j}) \in \mathcal{N}_{\text{tri}}(\mathbf{S}, \mathbf{s})$, one has

$$\sup_{y \in (\mathbf{R}^d)^{\mathcal{V}_0}} |\tilde{G}^{\mathbf{n}^j}(y^{(j)})| \lesssim 2^{\langle \varsigma_{\mathbf{S}}^j, \mathbf{s} \rangle} \|\tilde{G}\|_{\varsigma,\mathcal{N}_{\mathbf{H}}} ,$$

as well as

$$\sup_{y \in (\mathbf{R}^d)^{\mathcal{V}_0}} |\text{WickCum}^{\mathbf{j}}(y_{\ell^{\text{all}}})| \lesssim 2^{\langle \varsigma_{\mathbf{S}}^C, \mathbf{s} \rangle} \|\xi\|_{2p|\ell|,\mathfrak{c}} .$$

Combining these two estimates and setting $\mathbf{m} = (\mathbf{n}, \mathbf{j})$ as above yields

$$\sup_{y \in (\mathbf{R}^d)^{\mathcal{V}_0}} |F^{\mathbf{m}}(y)| \lesssim 2^{\langle \varsigma_{\mathbf{S}}, \mathbf{s} \rangle} \|\tilde{G}\|_{\varsigma,\mathcal{N}_{\mathbf{H}}}^{2p} \|\xi\|_{2p|\ell|,\mathfrak{c}} .$$

Now we treat the other case for $\tilde{F}$, we fix $\mathbf{S} \in \mathcal{U}_\mathcal{X}$ and suppose that $R(\mathbf{S}) \neq \emptyset$. We work similarly to Lemma 8.26 here. We again label, for each $B \in R(\mathbf{S})$, each of the elements of $B$ with a $+$ or $-$, define sets of multi-indices $\mathrm{Der}(B)$, and operators $\mathscr{Y}_B$. Then for any $\mathbf{m} = (\mathbf{n}, \mathbf{j}) \in \mathcal{N}_G$ with $\mathcal{T}(\mathbf{m}) = \mathbf{S}$ we define $\mathring{F}_\lambda^{\mathbf{m}}$ via

$$\mathring{F}^{(\mathbf{n},\mathbf{j})}(y) \overset{\text{def}}{=} \Big( \prod_{B \in R(\mathbf{S})} (1 - \mathscr{Y}_B) \Big) \Big[ \bigotimes_{j=1}^{2p} \tilde{G}^{\mathbf{n}^j} \Big](y) \cdot \mathrm{WickCum}^{\mathbf{j}}(y_{\ell \mathrm{all}}) \ .$$

To see that both $\mathring{F}_\lambda^{(\mathbf{n},\mathbf{j})}$ and $F_\lambda^{(\mathbf{n},\mathbf{j})}$ integrate to the same value, one proceeds as in Lemma 8.26. The desired domain constraint for $\mathring{F}_\lambda^{(\mathbf{n},\mathbf{j})}$ is also straightforward to check – all that is left is the supremum bound.

Using [Hai14, Prop. A.1] we get, uniform in $\mathbf{s} \in \mathrm{Lab}_\mathbf{S}$ and $(\mathbf{n}, \mathbf{j}) \in \mathcal{N}_{\mathrm{tri}}(\mathbf{S}, \mathbf{s})$, the supremum bound

$$\sup_{y \in (\mathbf{R}^d)^{\mathcal{V}_0}} \Big| \mathring{F}_\lambda^{(\mathbf{n},\mathbf{j})}(y) \Big| \lesssim \max_{k \in \mathrm{Der}(R(\mathbf{S}))} \Big( \prod_{B \in R(\mathbf{S})} 2^{-|k_{(+,B)}|_\mathbf{s} j_B} \Big) \qquad (\text{A.29})$$

$$\cdot \sup_{y \in (\mathbf{R}^d)^{\mathcal{V}_0}} \Big| \mathrm{WickCum}^{\mathbf{j}}(y_{\ell \mathrm{all}}) \prod_{j=1}^{2p} D^{k_j} \tilde{G}^{\mathbf{n}^j}(y^{(j)}) \Big|$$

where $\mathrm{Der}(R(\mathbf{S})) \overset{\text{def}}{=} \sum_{B \in R(\mathbf{S})} \partial_{(+,B)} \mathrm{Der}(B)$ and, for any $k \in (\mathbf{N}^d)^\mathcal{V}$ and $j \in [2p]$, we set $k_j \overset{\text{def}}{=} \sum_{l \in \ell^{(j)}} k_l$ . Now for any $k$ appearing in (A.29) we have, uniform in $\mathbf{s} \in \mathrm{Lab}_\mathbf{S}$ and $(\mathbf{n}, \mathbf{j}) \in \mathcal{N}_{\mathrm{tri}}(\mathbf{S}, \mathbf{s})$,

$$\sup_{y \in (\mathbf{R}^d)^{\mathcal{V}_0}} |D^{k_j} \hat{G}^{\mathbf{n}^j}(y^{(j)})|$$

$$\leq \exp_2 \Big[ \Big\langle \varsigma_\mathbf{S}^j + \sum_{\substack{B \in R(\mathbf{S}) \\ (+,B) \in \ell^{(j)}}} |k_{(+,B)}|_\mathbf{s} \mathbb{1}\Big\{ \bullet = \theta_{j,\mathbf{S}}[(+, B)^{\uparrow, \mathbf{T}^{(j)}}] \Big\}, \mathbf{s} \Big\rangle \Big] \|\tilde{G}\|_{\varsigma, \mathcal{N}_\mathbf{H}, \ell}^{2p}$$

$$\leq \exp_2 \Big[ \Big\langle \varsigma_\mathbf{S}^j + \sum_{\substack{B \in R(\mathbf{S}) \\ (+,B) \in \ell^{(j)}}} |k_{(+,B)}|_\mathbf{s} \delta_\mathbf{S}^{\Uparrow}[B], \mathbf{s} \Big\rangle \Big] \|\tilde{G}\|_{\varsigma, \mathcal{N}_\mathbf{H}, \ell}^{2p} \ .$$

In going to the third line above we used the fact that in the partial order on the nodes of $\mathbf{S}$ one has $B^{\Uparrow} \geq \theta_{j,\mathbf{S}}[(+, B)^{\uparrow, \mathbf{T}^{(j)}}]$ which can be justified as follows: the inner node on the left corresponds to the first coalescence event in $\mathbf{S}$ where the component containing $(+, B)$ contains a distinct element from the same copy of $\mathcal{X}_0$ that $(+, B)$ is from, however this must happen strictly after $(+, B)$ combines with $(-, B)$ to form a component of cardinality 2.

It follows that uniform in $\mathbf{s} \in \mathrm{Lab}_\mathbf{S}$ and $(\mathbf{n}, \mathbf{j}) \in \mathcal{N}_{\mathrm{tri}}(\mathbf{S}, \mathbf{s})$ we have the bound

$$\sup_{y \in (\mathbf{R}^d)^{\mathcal{V}_0}} \Big| \mathring{F}_\lambda^{(\mathbf{n},\mathbf{j})}(y) \Big| \lesssim 2^{\langle \bar{\varsigma}\mathbf{s}, \mathbf{s} \rangle} \|\xi\|_{2p|\ell|, \mathfrak{c}} \|\tilde{G}\|_{\varsigma, \mathcal{N}_\mathbf{H}, \ell}^{2p}$$

$$\cdot \max_{k \in \mathrm{Der}(\mathbf{S})} \exp_2 \Big[ \Big\langle \sum_{B \in R(\mathbf{S})} |k_{(+,B)}|_\mathbf{s} (\delta_\mathbf{S}^{\Uparrow}[B] - \delta_\mathbf{S}^{\uparrow}[B]), \mathbf{s} \Big\rangle \Big] \ .$$

To obtain the desired result observe that the maximum over $k$ on the RHS is acheived when $|k_{(+,B)}|_{\mathfrak{s}} = \mathcal{f}(B)$ for each $B \in R(\mathbf{S})$. This finishes the proof of (A.28).

We now check that $\tilde{\zeta}$ satisfies the assumptions of Theorem A.12. It is easy to see that the total homogeneity $\tilde{\zeta}$ is of order $\beta \overset{\text{def}}{=} 2p(\alpha - |t(\ell)|_{\mathfrak{s}}) < 0$.

Next we prove that $\tilde{\zeta}$ is subdivergence free on $\mathcal{V}$ for the set of scales $\mathcal{N}_{\mathbf{G}}$. In what follows we make frequent and implicit use of Assumption 2.24 and the following trivial bound – for any $L' \subset \ell^{\text{all}}$

$$- \sum_{\substack{B \in \pi \\ B \subset L'}} |t(B)|_{\mathfrak{s}} - \sum_{\substack{B \in \pi \\ B \not\subset L'}} |t(B)|_{\mathfrak{s},\mathfrak{c},\ell} \leq -|t(L')|_{\mathfrak{s}} . \tag{A.30}$$

Fix $\mathbf{S} \in \mathcal{U}_{\mathcal{V}}$ and $a \in \mathring{\mathbf{S}}$. For each $j \in [2p]$ and $A \subset [2p]$ we write

$$\mathbf{L}_a^j \overset{\text{def}}{=} \mathbf{L}_a \cap \mathcal{X}^{(j)}, \;\; \mathbf{L}_a^{j,\text{ext}} \overset{\text{def}}{=} \mathbf{L}_a^j \cap \ell^j, \;\; \mathbf{L}_a^A \overset{\text{def}}{=} \bigsqcup_{i \in A} \mathbf{L}_a^i, \;\; \text{and} \;\; \mathbf{L}_a^{A,\text{ext}} = \bigsqcup_{i \in A} \mathbf{L}_a^{i,\text{ext}} .$$

We also define three sets of indices

$$J_1 \overset{\text{def}}{=} \{j \in [2p] : |\mathbf{L}_a^{(j)}| = 1\}, \;\; J_2 \overset{\text{def}}{=} \{j \in [2p] : |\mathbf{L}_a^{(j)}| \geq 2\}, \;\; \text{and} \;\; J \overset{\text{def}}{=} J_1 \sqcup J_2 . \tag{A.31}$$

Given $a \in \mathring{\mathbf{S}}$ and $j \in [2p]$, we also write $a^j$ as a shorthand for

$$a^j \overset{\text{def}}{=} (\mathbf{L}_a^j)^{\uparrow,\mathbf{T}^{(j)}} .$$

We first treat the case when $\mathbf{L}_a \not\ni \circledast$; this is done by checking a variety of subcases. Restricting to this case, for every $j \in J_1$ we have, by $\tilde{\mathbf{G}}$-connectivity, $\mathbf{L}_a^{(j)} \subset \ell^{(j)}$. In the subcase where $J_2 = \emptyset$ we then have $\mathbf{L}_a^{J,\text{ext}} = \mathbf{L}_a$ and

$$\sum_{b \in \mathbf{S}_{\geq a}} \tilde{\varsigma}_{\mathbf{S}}(b) \leq - \sum_{\substack{B \in \pi \\ B \subset \mathbf{L}_a}} (\mathcal{f}(B)\mathbb{1}\{\mathbf{L}_a = B\} + |t(B)|_{\mathfrak{s}}) - \sum_{\substack{B \in \pi \\ B \not\subset \mathbf{L}_a}} |t(B \cap \mathbf{L}_a)|_{\mathfrak{s},\mathfrak{c},\ell}.$$

Again, by $\tilde{\mathbf{G}}$-connectivity, there is a unique $\bar{B} \in \pi$ with $\mathbf{L}_a \subset \bar{B}$ – it follows that the RHS above is bounded by the LHS of (A.21) with $M = \mathbf{L}_a$. The desired bound then follows from Lemma A.26 and this finishes the subcase $J_2 = \emptyset$.

For the subcase $J_1 = \emptyset$ and $J_2 = \{j\}$, we write $B^{a,j} \overset{\text{def}}{=} B \cap \mathbf{L}_a^{j,\text{ext}}$ and set $\pi^{a,j} \overset{\text{def}}{=} \{B \in \pi : B^{a,j} \neq \emptyset\}$. With this notation, we have

$$\sum_{b \in \mathbf{S}_{\geq a}} \tilde{\varsigma}_{\mathbf{S}}(b) \leq \sum_{c \in \mathbf{T}_{\geq a^j}^{(j)}} \varsigma_{\mathbf{T}^{(j)}}(c) - \sum_{B \in \pi^{a,j}} |t(B^{a,j})|_{\mathfrak{s},\mathfrak{c},\ell} . \tag{A.32}$$

To see that the RHS is bounded by $(|\mathbf{L}_a| - 1)|\mathfrak{s}| = (|\mathbf{L}_a^j| - 1)|\mathfrak{s}|$ first note that it is immediate by (A.23) when the second sum on the RHS of (A.32) is empty. When this sum is not empty we then use $|t(\bullet)|_{\mathfrak{s}} \leq |t(\bullet)|_{\mathfrak{s},\mathfrak{c},\ell}$ for all but one block $B$ contributing to the sum to get the inequality

$$- \sum_{B \in \pi^{a,j}} |t(B^{a,j})|_{\mathfrak{s},\mathfrak{c},\ell} \leq -|t(\mathbf{L}_a^{j,\text{ext}})|_{\mathfrak{s}} + \min_{B \in \pi^{a,j}} (|t(B^{a,j})|_{\mathfrak{s},\mathfrak{c},\ell} - |t(B^{a,j})|_{\mathfrak{s}}) . \tag{A.33}$$

The last term on the RHS of (A.33) is bounded above by $\hbar_{\mathfrak{c},\ell}(\mathbf{L}_a^{j,\mathrm{ext}})$ and therefore by (A.23) the RHS of (A.32) must be strictly bounded above by $(|\mathbf{L}_a^j| - 1)|\mathfrak{s}|$.

Now we treat the subcase where $J_2 = \{j\}$ and $J_1 \neq \emptyset$. Since $\mathbf{L}_a$ is connected by $\mathbf{G}$ and $\mathbf{L}_a \not\ni \otimes$ it follows that there exist $B_1, \ldots, B_n \in \pi$ such that $\cup_{m=1}^n B_m \supset \mathbf{L}_a^{J,\mathrm{ext}}$ and, for every $m \in [n]$, one has $B_m \cap \mathbf{L}_a^{j,\mathrm{ext}} \neq \emptyset$. Now if it is the case that for every $m$ one has $B_m \cap \mathbf{L}_a^{J,\mathrm{ext}} \in \mathfrak{L}_{\mathrm{Cum}}$ then,

$$-|\mathsf{t}(\mathbf{L}_a^{J_1})|_\mathfrak{s} - |\mathfrak{s}| \cdot |\mathbf{L}_a^{J_1}| \leq -\jmath_\ell(\mathbf{L}_a^{j,\mathrm{ext}}),$$

and also using $\sum_{b \in \mathbf{S}_{\geq a}} \varsigma_{\mathbf{S}}^C(b) \leq -|\mathsf{t}(\mathbf{L}_a^{j,\mathrm{ext}})|_\mathfrak{s} - |\mathsf{t}(\mathbf{L}_a^{J_1})|_\mathfrak{s}$ we have

$$\sum_{b \in \mathbf{S}_{\geq a}} \tilde{\varsigma}_{\mathbf{S}}(b) \leq \sum_{c \in \mathbf{T}_{\geq a(j)}^{(j)}} \varsigma_{\mathbf{T}^{(j)}}(c) - |\mathsf{t}(\mathbf{L}_a^{j,\mathrm{ext}})|_\mathfrak{s} - |\mathsf{t}(\mathbf{L}_a^{J_1})|_\mathfrak{s} \tag{A.34}$$

$$\leq \sum_{c \in \mathbf{T}_{\geq a(j)}^{(j)}} \varsigma_{\mathbf{T}^{(j)}}(c) - |\mathsf{t}(\mathbf{L}_a^{j,\mathrm{ext}})|_\mathfrak{s} - \jmath_\ell(\mathbf{L}_a^{j,\mathrm{ext}}) + |\mathfrak{s}| \cdot |\mathbf{L}_a^{J_1}|$$

and we then get the desired bound by applying (A.23). On the other hand, if there exists $m \in [n]$ such that $B_m \cap \mathbf{L}_a^{J,\mathrm{ext}} \notin \mathfrak{L}_{\mathrm{Cum}}$ then it follows that $|B_m \cap \mathbf{L}_a^{J,\mathrm{ext}}| = 1$ and $|\mathsf{t}(B_m \cap \mathbf{L}_a^{\mathrm{ext}})|_{\mathfrak{s},\mathfrak{c},\ell} = 0$, so writing $B_m \cap \mathbf{L}_a^{j,\mathrm{ext}} = \{u\}$ we have we have

$$\sum_{b \in \mathbf{S}_{\geq a}} \tilde{\varsigma}_{\mathbf{S}}(b) \leq \sum_{c \in \mathbf{T}_{\geq a(j)}^{(j)}} \varsigma_{\mathbf{T}^{(j)}}(c) - |\mathsf{t}(\mathbf{L}_a^{j,\mathrm{ext}} \setminus \{u\})|_\mathfrak{s} - |\mathsf{t}(\mathbf{L}_a^{J_1})|_\mathfrak{s}$$

$$\leq \sum_{c \in \mathbf{T}_{\geq a(j)}^{(j)}} \varsigma_{\mathbf{T}^{(j)}}(c) - |\mathsf{t}(\mathbf{L}_a^{j,\mathrm{ext}})|_\mathfrak{s} + [|\mathsf{t}(\{u\})|_\mathfrak{s} - |\mathsf{t}(\{u\})|_{\mathfrak{s},\mathfrak{c},\ell}] + |\mathfrak{s}| \cdot |J_1|$$

The bracketed quantity on the last line is bounded below by $-\hbar_{\mathfrak{c},\ell}(\mathbf{L}^{j,\mathrm{ext}})$ and so by (A.23) the last line without the last term is bounded above by $(|\mathbf{L}_a^j| - 1) \cdot |\mathfrak{s}|$ which finishes this subcase.

Finally, suppose $|J_2| \geq 2$. We then have

$$\sum_{b \in \mathbf{S}_{\geq a}} \tilde{\varsigma}_{\mathbf{S}}(b) \leq \Big[\sum_{j \in J_2} \Big(\sum_{c \in \mathbf{T}_{\geq a^j}^{(j)}} \varsigma_{\mathbf{T}^{(j)}}(c) - |\mathsf{t}(\mathbf{L}_a^{j,\mathrm{ext}})|_\mathfrak{s}\Big)\Big] - \sum_{j \in J_1} |\mathsf{t}(\mathbf{L}_a^{j,\mathrm{ext}})|_\mathfrak{s}$$

$$< \sum_{j \in J_2} |\mathfrak{s}|\Big[|\mathbf{L}_a^j| - \frac{1}{2}\Big] - \sum_{j \in J_1} |\mathsf{t}(\mathbf{L}_a^{j,\mathrm{ext}})|_\mathfrak{s}. \tag{A.35}$$

In going to the second line of (A.35) we used, for each $j \in J_2$, (A.23). Now by using $\sum_{j \in J_2} |\mathbf{L}_a^j| = |\mathbf{L}_a| - |J_1|$ and Assumption 2.24 we see that the RHS of (A.35) is strictly bounded above by $|\mathfrak{s}|(|\mathbf{L}_a| - \frac{1}{2}|J_2|)$ so we are done.

We now treat the case $\mathbf{L}_a \ni \otimes$. Here $\mathbf{L}_a^j \ni \otimes$ for each $j \in [2p]$ and (A.23) gives

$$\sum_{b \in \mathbf{S}_{\geq a}} \tilde{\varsigma}_{\mathbf{S}}(b) \leq \sum_{j \in J_2} \Big(\sum_{c \in \mathbf{T}_{\geq a^j}^{(j)}} \varsigma_{\mathbf{T}^{(j)}}(c) - |\mathsf{t}(\mathbf{L}_a^{j,\mathrm{ext}})|_\mathfrak{s}\Big)$$

$$< |\mathfrak{s}| \sum_{j \in J_2} (|\mathbf{L}_a^j| - 1) = |\mathfrak{s}|(|\mathbf{L}_a| - 1). \tag{A.36}$$

This completes the proof that the total homogeneity $\tilde\varsigma$ is subdivergence free on $\mathcal{V}$ for the set of scales $\mathcal{N}_{\mathbf{G}}$.

We now turn to showing (A.9) in our context. Suppose we are given $u \in \mathring{\mathbf{S}}$ satisfying both $u \le (\mathcal{V}_*)^\uparrow$ and $\mathbf{S}_{\not\ge u} \ne \emptyset$. We define, for each $j \in [2p]$,

$$\mathbf{Q}_u^j \stackrel{\text{def}}{=} \mathcal{X}^{(j)} \setminus \mathbf{L}_u \ \text{ and } \ D \stackrel{\text{def}}{=} \{j \in [2p] : \mathbf{Q}_u^j \ne \emptyset\} .$$

Clearly $D$ is non-empty. We then have

$$\sum_{w \in \mathbf{S}_{\not\ge u}} \tilde\varsigma_{\mathbf{S}}(w) - |\mathcal{V} \setminus \mathbf{L}_u| \, |\mathfrak{s}|$$

$$\ge - \sum_{w \in \mathbf{S}_{\not\ge u}} \sum_{B \in \pi} \mathfrak{c}^{(\mathfrak{t},B)}(w) + \sum_{j \in D} \Big[ \sum_{w \in \mathbf{T}_{\not\ge u^j}^{(j)}} \varsigma_{\mathbf{T}^{(j)}}(w) - |\mathcal{X}^{(j)} \setminus \mathbf{L}_{u^j}| \, |\mathfrak{s}| \Big]$$

$$\ge \sum_{j \in D} \Big[ \sum_{w \in \mathbf{T}_{\not\ge u^j}^{(j)}} \varsigma_{\mathbf{T}^{(j)}}(w) - |\mathfrak{t}(\ell^{(j)} \setminus \mathbf{L}_u^j)|_{\mathfrak{s}} - |\mathcal{X}^{(j)} \setminus \mathbf{L}_{u^j}| \, |\mathfrak{s}| \Big] > 0 .$$

In the second inequality we used (A.24) and in the first we used

$$\sum_{w \in \mathbf{S}_{\not\ge u}} \sum_{B \in \pi} \mathfrak{c}^{(\mathfrak{t},B)}(w) = - |\mathfrak{t}(\ell^{\text{all}})|_{\mathfrak{s}} - \sum_{w \in \mathbf{S}_{\ge u}} \mathfrak{c}^{(\mathfrak{t},B)}(w)$$

$$\le - |\mathfrak{t}(\ell^{\text{all}})|_{\mathfrak{s}} + |\mathfrak{t}(\mathbf{L}_u^{\text{ext}})|_{\mathfrak{s}} = - |\mathfrak{t}(\ell^{\text{all}} \setminus \mathbf{L}_u^{\text{ext}})|_{\mathfrak{s}} .$$

$\square$

## Appendix B    Convergence of mollified approximations

Throughout this section we take the dimension $d$ and space-time scaling $\mathfrak{s}$ as fixed along with a set of noise types $\mathfrak{L}_-$ and a set of non-vanishing cumulants $\mathfrak{L}_{\text{Cum}}$.

The primary question of this section is as follows: given $\xi \in \mathcal{M}(\Omega_0)$ and a sequence of smooth, compactly supported approximate identities $\{\eta_\varepsilon\}_{\varepsilon \in (0,1]}$ on $\mathbf{R}^d$ which converge to the Dirac delta function $\delta$, does $\|\xi; \xi_\varepsilon\|_{\mathfrak{c},N} \to 0$ as $\varepsilon \downarrow 0$ where we use the notation of Definition A.19 and we set $\xi_\varepsilon = \{\xi_{\mathfrak{t}} * \eta_\varepsilon\}_{\mathfrak{t} \in \mathfrak{L}_-}$.

We denote by Moll the collection of all smooth functions $\eta \in \mathscr{C}(\mathbf{R}^d)$ which are supported on the closed ball $\{x \in \mathbf{R}^d : |x| \le 1\}$ and satisfy $\int dx\, \eta(x) = 1$. The following lemma is immediate.

**Lemma B.1** *For any $\eta \in$ Moll and $\zeta \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\text{Cum}})$ the noise $\zeta * \eta = (\zeta_{\mathfrak{t}} * \eta)_{\mathfrak{t} \in \mathfrak{L}_-}$ belongs to $\mathcal{M}(\Omega_\infty, \mathfrak{L}_{\text{Cum}})$.*

Throughout this section $\delta$ will always denote the Dirac delta distribution on $\mathbf{R}^d$. We also write $\overline{\text{Moll}} \stackrel{\text{def}}{=} \text{Moll} \sqcup \{\delta\}$, and also quantitative notions of the convergence in $\overline{\text{Moll}}$ by setting, for each $\kappa \in [0,1)$, $r \in \mathbf{N}$, and any distribution $\eta \in \mathcal{D}'(\mathbf{R}^d)$ determined by a smooth function away from the origin,

$$\|\eta\|_{\kappa,r} \stackrel{\text{def}}{=} \max_{\substack{k \in \mathbf{N}^d \\ |k|_{\mathfrak{s}} < 4r|\mathfrak{s}|}} \Big[ |\eta(p_k)| \vee \big( \sup_{y \in \mathbf{R}^d \setminus \{0\}} |D^k \eta(y)| \cdot |y|^{|\mathfrak{s}|+|k|_{\mathfrak{s}}+\kappa} \big) \Big].$$

As an example, observe that if one fixes $\eta \in \text{Moll}$ and defines, for each $\varepsilon > 0$,

$$\eta^\varepsilon(x_1, \ldots, x_d) \overset{\text{def}}{=} \varepsilon^{-|\mathfrak{s}|}\eta\left(\varepsilon^{-\mathfrak{s}_1}x_1, \ldots, \varepsilon^{-\mathfrak{s}_d}x_d\right), \tag{B.1}$$

then for any $\kappa \in [0,1)$ one has $\|\eta^\varepsilon - \delta\|_{\kappa,r} \sim \varepsilon^\kappa$ as $\varepsilon \downarrow 0$ for fixed $r$. In what follows we use the notation

$$\widetilde{\text{Moll}} \overset{\text{def}}{=} \text{Moll} \sqcup \{\eta_1 - \eta_2 : \eta_1, \eta_2 \in \text{Moll}\}$$

Before stating the main proposition of this section we need additional notation.

**Definition B.2** Given a cumulant homogeneity $\mathfrak{c}$ on $\mathfrak{L}_{\text{Cum}}$ we define the the $\kappa$-penalization of $\mathfrak{c}$ to be the cumulant homogeneity $\tilde{\mathfrak{c}}$ on $\mathfrak{L}_{\text{Cum}}$ defined via setting, for each $(\mathfrak{t}, [N]) \in \mathfrak{L}_{\text{Cum}}$,

$$\tilde{\mathfrak{c}}^{(\mathfrak{t},[N])} \overset{\text{def}}{=} \mathfrak{c}^{(\mathfrak{t},[N])} + \kappa \sum_{v \in [N]} \delta^\uparrow[v],$$

where we are using the notation of (A.3).

**Proposition B.3** *For any $\xi \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\text{Cum}})$ and $\eta \in \text{Moll}$ one has the random noise type valued variable $\xi * \eta \overset{\text{def}}{=} (\xi_\mathfrak{t} * \eta)_{\mathfrak{t} \in \mathfrak{L}_-}$ is an element of $\mathcal{M}(\Omega_\infty, \mathfrak{L}_{\text{Cum}})$.*

*Additionally, for any $N \geq 2$, and cumulant homogeneity $\mathfrak{c}$ on $\mathfrak{L}_{\text{Cum}}$ and $r \in \mathbf{N}$, one has, uniform in $\xi \in \mathcal{M}(\Omega_0, \mathfrak{L}_{\text{Cum}})$, and $\kappa$ satisfying*

$$0 < \kappa < \frac{1}{2} \min_{(\mathfrak{t},[2]) \in \mathfrak{L}_{\text{Cum}}} \left( \lceil \mathfrak{c}^{(\mathfrak{t},[2])} \rceil - \mathfrak{c}^{(\mathfrak{t},[2])} \right),$$

*and $\eta_1, \eta_2 \in \text{Moll}_r \sqcup \{0\}$, the bounds*

$$\|\xi * \eta_1; \xi * \eta_2\|_{N,\mathfrak{c}',r} \lesssim \|\eta_1 - \eta_2\|_\kappa \cdot (\|\eta_1\|_\kappa \vee \|\eta_2\|_\kappa)^{N-1} \cdot \|\xi\|_{N,\mathfrak{c},r+1}, \tag{B.2}$$

*where $\mathfrak{c}'$ is the $\kappa$-penalization of $\mathfrak{c}$.*

**Remark B.4** Note that by setting $\eta_2 \overset{\text{def}}{=} 0$ in the above proposition we see that (B.2) actually implies
$$\|\xi * \eta_1\|_{N,\mathfrak{c}',r} \lesssim \|\eta_1\|_\kappa^N \cdot \|\xi\|_{N,\mathfrak{c},r+1}.$$

*Proof.* It is immediate that $\xi * \eta_1$ and $\xi * \eta_2$ jointly admit pointwise cumulants. We first establish control over second cumulants.

Fix $(\mathfrak{t}, [2]) \in \mathfrak{L}_{\text{Cum}}$ and the shorthand $\alpha \overset{\text{def}}{=} \mathfrak{c}^{(\mathfrak{t},[2])} - |\mathfrak{s}|$. For any $\beta \in \mathbf{R}$, $q \in \mathbf{N}$, and $H \in \mathcal{S}'(\mathbf{R}^d)$ with singular support contained in $\{0\}$, we define

$$[\![H]\!]_{\beta,q} \overset{\text{def}}{=} \left( \max_{\substack{k \in \mathbf{N}^d \\ |k|_\mathfrak{s} < \alpha}} H(p_k) \right) + \max_{k \in A_q} \sup_{x \in \mathbf{R}^d} |D^k H(x)| \cdot |x|^{\beta + |k|_\mathfrak{s}}.$$

Here $A_q \overset{\text{def}}{=} \{k \in \mathbf{N}^d : |k|_{\mathfrak{s}} \leq 3q|\mathfrak{s}|\}$ Clearly $[\![\mathscr{R}\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)}, \xi_{\mathfrak{t}(2)}\}]]\!]_{\alpha, 2r+2} \leq \|\xi\|_{\mathfrak{c}, 2, r+1}$ and conversely $[\![\mathscr{R}\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)}, \xi_{\mathfrak{t}(2)}\}]]\!]_{\alpha, 2r}$ controls the contribution of second cumulants to $\|\xi\|_{\mathfrak{c}, 2, r}$.

For any $H$ as above, $q \leq 3$, and $\varrho \in \widetilde{\text{Moll}}$ we claim that for $\theta \in \{\alpha, \alpha + \kappa\}$,

$$[\![H * \varrho]\!]_{\theta + \kappa, q} \lesssim \|\varrho\|_\kappa \cdot [\![H]\!]_{\theta, q+1} . \tag{B.3}$$

The desired bound on second cumulants follows from the claim since by applying it twice we get

$$[\![\mathscr{R}(\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)} * \eta_1, \xi_{\mathfrak{t}(2)} * (\eta_1 - \eta_2)\}])]\!]_{\alpha + 2\kappa, 2r}$$
$$= [\![\mathscr{R}(\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)}, \xi_{\mathfrak{t}(2)}\}]) * \eta_1 * \text{Ref}[\eta_1 - \eta_2]]\!]_{\alpha + 2\kappa, 0}$$
$$\lesssim [\![\mathscr{R}(\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)}, \xi_{\mathfrak{t}(2)}\}]) * \eta_1]\!]_{\alpha + \kappa, 2r+1} \cdot \|\eta_1 - \eta_2\|_\kappa$$
$$\lesssim [\![\mathscr{R}(\mathbf{E}^c[\{\xi_{\mathfrak{t}(1)}, \xi_{\mathfrak{t}(2)}\}])]\!]_{\alpha, 2r+2} \cdot \|\eta_1\|_\kappa \cdot \|\eta_1 - \eta_2\|_\kappa,$$

where $\text{Ref}(g)(x) = g(-x)$ for any distribution $g$ on $\mathbf{R}^d$.

We now prove (B.3), assuming without loss of generality the RHS side of the bound is finite. Observe that for $m \in A_q$, $D^m(H * \varrho)$ has empty singular support and is given everywhere by integration against the smooth function

$$D^m(H * \varrho)(x) \overset{\text{def}}{=} \sum_{|k|_{\mathfrak{s}} < \alpha} \frac{H(p_k)}{k!}(D^{k+m}\varrho)(x) + D^m I(x)$$

$$\text{where } I(x) \overset{\text{def}}{=} \int_{\mathbf{R}^d} dz \left( \varrho(x - z) - \sum_{|k|_{\mathfrak{s}} < \alpha} \frac{(-z)^k}{k!}(D^k \varrho)(x) \right) H(z) .$$

Since $m \in A_q$ we get the bound, uniform in $x \in \mathbf{R}^d$,

$$\left| \sum_{|k|_{\mathfrak{s}} < \alpha} \frac{H(p_k)}{k!}(D^{k+m}\varrho)(x) \right| \lesssim \|\varrho\|_\kappa \cdot [\![H]\!]_{\theta, 0} \cdot |x|^{-|m|_{\mathfrak{s}} - |\mathfrak{s}| - \kappa} . \tag{B.4}$$

For the second term we perform a multiscale expansion where we again use coalescence trees to organize our bounds. Here we will be leaving more variables frozen instead of integrating all but one of them so we will perform more of the analysis by hand instead of just referencing the earlier multiclustering lemmas.

As in [HQ15, Lem. A.4], we can construct a families of smooth functions $\{\varrho_j\}_{j=0}^\infty$ and $\{H_j\}_{j=0}^\infty$ such that, for $F = H$ or $\varrho$,

1. The identity $\sum_{j=0}^\infty F_j = F$ holds in the sense of distributions on $\mathbf{R}^d$.
2. $F_j(x)$ is supported on $x$ with $2^{-j-2} \leq |x| \leq 2^{-j}$.
3. For any $k \in \mathbf{N}^d$ with $|k|_{\mathfrak{s}} < 10|\mathfrak{s}|$ one has, uniform in $j \geq 0$ and $x \in \mathbf{R}^d$,

$$|D^k \varrho_j(x)| \lesssim \|\varrho\|_\kappa \cdot |x|^{-|\mathfrak{s}| - |k|_{\mathfrak{s}} - \kappa} .$$

4. For each $k \in \mathbf{N}^d$ one has, uniform in $j \geq 0$, and $x \in \mathbf{R}^d$ and with $k \in A_2$ one has

$$|D^k H_j(x)| \lesssim [\![H]\!]_{\theta, ||k||} \cdot |x|^{-\theta - |k|_{\mathfrak{s}}}$$

5. For any $j > 0$, $\int dx\, \varrho_j(x) = 0$.
6. For any $j > 0$ and any $k \in \mathbf{N}^d$ with $|k|_{\mathfrak{s}} \leq \alpha$, $\int dx\, x^k \cdot H_j(x) = 0$

The integrand determining $I$ as a function of $x$, $z$ which we view as the positions of the vertices $a$ and $b$, respectively. We also introduce a root vertex $c$ pinned at 0. Thus we set $\mathcal{V} \stackrel{\text{def}}{=} \{a, b, c\}$, $\mathcal{V}_0 \stackrel{\text{def}}{=} \{a, b\}$, and we fix the multigraph $\mathbf{G}$ to be the complete graph on $\mathcal{V}$. We also set $\mathcal{N}_{\mathbf{G}} \stackrel{\text{def}}{=} \mathbf{N}^{\mathbf{G}}$.

Fix $\bar{n} \geq 0$. Then for any $x \in \mathbf{R}^d$ with $2^{-\bar{n}-2} \leq |x| < 2^{-\bar{n}}$ one has, by exploiting triangle inequality constraints on scale indices,

$$D^m I(x) = \sum_{\substack{\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}} \\ \mathbf{s} \in \text{Lab}_{\mathbf{T}}(\bar{n}) \\ \mathbf{n} \in \mathcal{N}_{\text{tri}}(\mathbf{T}, \mathbf{s})}} \int dz\, \mathring{I}_m^{\mathbf{n}}(x, z) + \int_{\mathbf{R}^d} dz \sum_{|k|_{\mathfrak{s}} < \alpha} (D^{k+m}\varrho)(x) \frac{(-z)^k}{k!} H_0(z) \quad \text{(B.5)}$$

where $\mathring{I}_m^{\mathbf{n}}(x, z) \stackrel{\text{def}}{=} D^m \varrho_{n_{\{a,b\}}}(x - z) H_{n_{\{b,c\}}}(z) \Psi^{(n_{\{a,c\}})}(x)$, for $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$,

$$\text{Lab}_{\mathbf{T}}(\bar{n}) \stackrel{\text{def}}{=} \left\{ \mathbf{s} \in \text{Lab}_{\mathbf{T}} : \, \mathbf{s}(\{a, c\}^{\uparrow}) - \bar{n} | \leq 12 \right\},$$

and we choose some constant $C \geq 4$ for the constraint defining $\mathcal{N}_{\text{tri}}(\mathbf{T}, \mathbf{s})$. The second integral on the RHS of (B.5) satisfies a bound like (B.4) so we are left with estimating the contribution from the first integral.

Since there are only four coalesence trees in $\widehat{\mathcal{U}}_{\mathcal{V}}$ we present $\varsigma$ by showing these four trees below (which we will denote, going from left to right, $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4$) and then labeling the internal nodes with the weight that $\varsigma$ gives them:
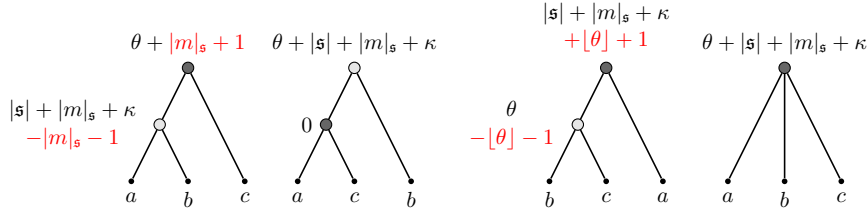


Writing $\mathring{I}_m \stackrel{\text{def}}{=} \{\mathring{I}_m^{\mathbf{n}}\}_{\mathbf{n} \in \mathcal{N}_{\mathbf{G}}}$, we claim that there is a modification $\tilde{I}_m \in \text{Mod}_{\{b\}}(\mathring{I}_m)$ which is bounded by a total homogeneity $\varsigma$ with

$$\|\tilde{I}_m\|_{\varsigma, \mathcal{N}_{\mathbf{G}}} \lesssim \llbracket H \rrbracket_{\theta, j+1} \cdot \|\varrho\|_{\kappa}.$$

We have darkened the internal node $\{a, c\}^{\uparrow}$ whose scale $\mathbf{s}(\{a, c\}^{\uparrow})$ will be frozen to be close to $\bar{n}$ and we have colored in red the contributions coming from renormalization.

For any $\mathbf{n} \in \mathcal{N}_{\mathbf{G}}$ with $\mathcal{T}(\mathbf{n}) \in \{\mathbf{T}_2, \mathbf{T}_4\}$ we set $\tilde{I}_m^{\mathbf{n}} \stackrel{\text{def}}{=} \mathring{I}_m^{\mathbf{n}}$ and the desired estimate comes from the bound

$$|\mathring{I}_m^{\mathbf{n}}| \lesssim (2^{n_{\{a,b\}}(|m|_{\mathfrak{s}} + |\mathfrak{s}| + \theta)} \|\varrho\|_{\theta})(2^{n_{\{b,c\}}\alpha} \llbracket H \rrbracket_{\alpha, 0}).$$

For any $\mathbf{n} \in \mathcal{N}_{\mathbf{G}}$ with $\mathcal{T}(\mathbf{n}) = \mathbf{T}_1$ we set

$$\tilde{I}^{\mathbf{n}}_m(y_a, y_b) \overset{\text{def}}{=} D^m \varrho_{n_{\{a,b\}}}(y_a - y_b) \cdot \Psi^{(n_{\{a,c\}})}(y_a)$$

$$\cdot \left( H_{n_{\{b,c\}}}(y_b) - \sum_{\substack{k \in \mathbf{N}^d \\ |k|_{\mathfrak{s}} \leq |m|_{\mathfrak{s}}}} \frac{(y_b - y_a)^k}{k!} H_{n_{\{b,c\}}}(y_a) \right) .$$

To see that this is compatible with $\tilde{I}^m \in \mathrm{Mod}_{\{b\}}(\mathring{I}^m)$ observe that for $\mathbf{n}$ as above we have $n_{\{a,b\}} \neq 0$ and so $\int dy_b\, \tilde{I}^{\mathbf{n}}_m(y_a, y_b) = \int dy_b\, \mathring{I}^{\mathbf{n}}_m(y_a, y_b)$ - the support constraint is also easy to check. Writing $\mathbf{s} = \mathbf{s}(\mathbf{n})$ and $\mathring{\mathbf{T}}_1 = \{\varrho_{\mathbf{T}_1}, u\}$, a Taylor remainder estimate then gives

$$|\tilde{I}^{\mathbf{n}}_m(y_a, y_b)| \lesssim 2^{\mathbf{s}(u)(|m|_{\mathfrak{s}} + |\mathfrak{s}| + \kappa)} \cdot \|\varrho\|_{\kappa} \cdot 2^{\mathbf{s}(\varrho_{\mathbf{T}_1})\theta + (\mathbf{s}(\varrho_{\mathbf{T}_1}) - \mathbf{s}(u))(|m|_{\mathfrak{s}} + 1)} [\![ H ]\!]_{\theta, j+1} .$$

Finally, for any $\mathbf{n} \in \mathcal{N}_{\mathbf{G}}$ with $\mathcal{T}(\mathbf{n}) = \mathbf{T}_3$ we set

$$\tilde{I}^{\mathbf{n}}_m(y_a, y_b) \overset{\text{def}}{=} H_{n_{\{b,c\}}}(y_b) \cdot \Psi^{(n_{\{a,c\}})}(y_a)$$

$$\left( D^m \varrho_{n_{\{a,b\}}}(y_a - y_b) - \sum_{|k|_{\mathfrak{s}} < \alpha} D^{k+m} \varrho_{n_{\{a,b\}}}(y_a) \frac{(-y_b)^k}{k!} \right)$$

As before, writing $\mathbf{s} \overset{\text{def}}{=} \mathbf{s}(\mathbf{n})$ and $\mathring{\mathbf{T}}_3 = \{\varrho_{\mathbf{T}_3}, v\}$, a Taylor remainder estimate gives

$$\|\tilde{I}^{\mathbf{n}}_m(y_a, y_b)\| \lesssim \exp_2 \left[ \mathbf{s}(\varrho_{\mathbf{T}_3})(|m|_{\mathfrak{s}} + |\mathfrak{s}| + \kappa) + (\mathbf{s}(\varrho_{\mathbf{T}_3}) - \mathbf{s}(v))(\lfloor \theta \rfloor + 1) \right] \cdot \|\varrho\|_{\kappa}$$

$$\cdot \exp_2 \left[ \mathbf{s}(v)\theta \right] [\![ H ]\!]_{\theta, j}$$

The proof for our bounds for second cumulants is finished upon observing that for $1 \leq i \leq 4$,

$$\sum_{\mathbf{s} \in \mathrm{Lab}_{\mathbf{T}_i}(\bar{n})} \sum_{\mathbf{n} \in \mathcal{N}_{\mathrm{tri}}(\mathbf{T}, \mathbf{s})} \int dy_b\, |\tilde{I}^{\mathbf{n}}_m(y_a, y_b)| \lesssim \sum_{\mathbf{s} \in \mathrm{Lab}_{\mathbf{T}_i}(\bar{n})} 2^{\langle \varsigma_{\mathbf{T}_i}, \mathbf{s} \rangle} 2^{-|\mathfrak{s}| \mathbf{s}(b^{\uparrow})} \|\tilde{I}_m\|_{\varsigma, \mathcal{N}_{\mathbf{G}}}$$

$$\lesssim 2^{\bar{n}(\theta + |m|_{\mathfrak{s}} + \kappa)} \|\tilde{I}_m\|_{\varsigma, \mathcal{N}_{\mathbf{G}}} .$$

The factor $2^{-|\mathfrak{s}| \mathbf{s}(b^{\uparrow})}$ in going to the RHS of the first line comes from the integration of $y_b$. In going to the last line we summed over the labelings $\mathbf{s} \in \mathrm{Lab}_{\mathbf{T}_i}$ - they key point is that $\mathbf{s}(\{a, c\}^{\uparrow})$ can be treated as fixed to the value $\bar{n}$. For $\mathbf{T}_4$ the bound is then immediate since there is no other scale labeling to sum over. For $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ one has an infinite sum over the scale labeling for the other internal node - this scale will be constrainted to be greater than $\bar{n}$ for $\mathbf{T}_1, \mathbf{T}_3$ and less than $\bar{n}$ for $\mathbf{T}_2$.

We now obtain the desired bounds for higher cumulants so fix $M \geq 3$. Let $\mathbf{G}$ be the complete graph on $[M]$. We will also commit an abuse of notation - for any

multi-index $k \in \mathbf{N}^{[M]}$ and function $H$ on $(\mathbf{R}^d)^{[M]}$ we will also denote by $H_k$ the family of functions $H_k \stackrel{\mathrm{def}}{=} (H_k^{\mathbf{n}})_{\mathbf{n} \in \mathbf{N}^{\mathbf{G}}}$ given by

$$H_k^{\mathbf{n}}(x_1, \ldots, x_M) \stackrel{\mathrm{def}}{=} \left( D^k H(x_1, \ldots, x_M) \right) \cdot \prod_{\{i,j\} \in \mathbf{G}} \Psi^{(n_{\{i,j\}})}(x_i - x_j).$$

Now given a function $H$ smooth on $(\mathbf{R}^d)^{[M]} \setminus \mathrm{Diag}_M$, invariant under $M$-fold simultaneous $\mathbf{R}^d$ translations [14], a total homogeneity $\varsigma$ on the trees of $\widehat{\mathcal{U}}_{[M]}$, a subset $J \subset [M]$, and $p \in \mathbf{N}$ we define

$$\llbracket H \rrbracket_{\varsigma, J, p} \stackrel{\mathrm{def}}{=} \max_{k \in A_{J,p}} \| H_k \|_{D^k \varsigma, \mathbf{N}^{\mathbf{G}}}$$

where $A_{J,p} \stackrel{\mathrm{def}}{=} \left\{ (k_i)_{i=1}^M \in (\mathbf{N}^d)^{[M]} : \forall 1 \le j \le M, \; \|k_j\| \le (p + \mathbb{1}\{j \in J\}) |\mathfrak{s}| \right\}.$

On the right hand side of the first line we are using the notation of (A.5), choosing arbitrarily a particular element of $[M]$ as a root and a value to fix it as (due to translation invariance these choices do not matter). For any $j \in [J]$, $\varrho \in \widetilde{\mathrm{Moll}}$, and $H$ as above we define

$$(H *_j \varrho)(x_1, \ldots, x_M) \stackrel{\mathrm{def}}{=} \int_{\mathbf{R}^d} dz_j \, H(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_M) \varrho(x_j - z_j) \,.$$

The claim we wish to prove for higher cumulants is that for any $j \in J \subset [M]$, $p \le 4$, and $\varrho \in \widetilde{\mathrm{Moll}}$ along with a total homogeneity $\varsigma$ and function $H$ as above,

$$\llbracket H *_j \varrho \rrbracket_{\varsigma + \kappa \delta^{\uparrow}[j], J \setminus \{j\}, p} \lesssim \llbracket H \rrbracket_{\varsigma, J, p} \cdot \|\varrho\|_\kappa. \tag{B.6}$$

Applying claim $M$ times will give us the desired bound. To see this first note that for any $(\mathfrak{t}, [M]) \in \mathfrak{L}_{\mathrm{Cum}}$ one has $\llbracket \mathbf{E}^c[\{\xi_{\mathfrak{t}(i)}\}_{i=1}^M] \rrbracket_{\mathfrak{c}, [M], r} \le \|\xi\|_{M, \mathfrak{c}, r+1}$ and that $\llbracket \mathbf{E}^c[\{\xi_{\mathfrak{t}(i)}\}_{i=1}^M] \rrbracket_{\mathfrak{c}, \varnothing, r}$ controls the contribution of $\mathbf{E}^c[\{\xi_{\mathfrak{t}(i)}\}_{i=1}^M]$ to $\|\xi\|_{M, \mathfrak{c}, r}$.

We now turn to proving, by using a multiscale expansion, the claim (B.6). Fix appropriate $j$, $J$, $\theta$, $p$, and $\varrho$. The underlying vertex set is given by $\mathcal{V} \stackrel{\mathrm{def}}{=} [M] \sqcup \{o\}$ where the additional vertex $o$ corresponds to the integration variable $z_j$ in our formula for $H *_j \varrho$. We choose $j \in \mathcal{V}$ as a root vertex (so $\mathcal{V}_0 \stackrel{\mathrm{def}}{=} \mathcal{V} \setminus \{j\}$) and arbitrarily pin its position $y_j$ to some value in $\mathbf{R}^d$. We also fix $\mathbf{G}' \stackrel{\mathrm{def}}{=} \mathbf{G} \sqcup \{\{o, j\}\}$ and set $\mathcal{N}_{\mathbf{G}'} \stackrel{\mathrm{def}}{=} \mathbf{N}^{\mathbf{G}'}$.

For any fixed $\bar{\mathbf{n}} \in \mathbf{N}^{\mathbf{G}}$ and $n_{\{o,j\}} \in \mathbf{N}$ observe that if one sets $\mathbf{S} \stackrel{\mathrm{def}}{=} \mathcal{T}(\bar{\mathbf{n}}) \in \widehat{\mathcal{U}}_{[M]}$ and $\mathbf{T} \stackrel{\mathrm{def}}{=} \mathcal{T}(\bar{\mathbf{n}} \sqcup n_{\{o,j\}}) \in \mathcal{U}_{\mathcal{V}}$ then $\mathbf{S} = \mathbf{T} \upharpoonright_{[M]}$ (see Definition A.29).

Now fix $k \in A_{J \setminus \{j\}, p}$. For any $\mathbf{n} \in \mathcal{N}_{\mathbf{G}}$ we define a family of functions $\mathring{I}_k = \{\mathring{I}_k^{\mathbf{n}}\}_{\mathbf{n} \in \mathcal{N}_{\mathbf{G}'}}$ via setting

$$\mathring{I}_k^{\mathbf{n}}(y_{\mathcal{V}_0}) \stackrel{\mathrm{def}}{=} \varrho_{n_{\{o,j\}}}(y_j - y_o) \tilde{H}_k^{\mathbf{n}}(y_{\mathcal{V}_0}) \,, \text{ where}$$

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[14]that is, $H(x_1, \ldots, x_M) = H(x_1 + h, \ldots, x_M + h)$ for any $(x_i)_{i=1}^M \in (\mathbf{R}^d)^{[M]}$ and $h \in \mathbf{R}^d$

$$\tilde{H}_k^{\mathbf{n}}(y_{\mathcal{V}_0}) \overset{\text{def}}{=} \Big( D^k H(y_1, \ldots, y_{j-1}, y_o, y_{j+1}, y_M) \Big) \prod_{\substack{\{i,i'\}\in\mathbf{G}' \\ \{i,i'\}\neq\{o,j\}}} \Psi^{(n_{\{i,j\}})}(y_i - y_j) \,.$$

With these definitions we have

$$D^k(H *_j \varrho)(y_{\mathcal{V}_0\setminus\{o\}}) = \sum_{\mathbf{T}\in\widehat{\mathcal{U}}_{\mathcal{V}}} \sum_{\mathbf{s}\in\mathrm{Lab}_{\mathbf{T}}} \sum_{\mathbf{n}\in\mathcal{N}_{\mathrm{tr}}(\mathbf{T},\mathbf{s})} \int dy_o\, \mathring{I}_k^{\mathbf{n}}(y_{\mathcal{V}_0}) \,.$$

To prove (B.6) it suffices to find a modification $\tilde{I}_k \overset{\text{def}}{=} \{\tilde{I}_k^{\mathbf{n}}\}_{\mathbf{n}\in\mathcal{N}_{\mathbf{G}'}} \in \mathrm{Mod}_{\{o\}}(\mathring{I}_k)$ such that, uniform in $(\mathbf{S},\bar{\mathbf{s}}) \in \widehat{\mathcal{U}}_{[M]} \ltimes \mathrm{Lab}_\bullet$ and $\bar{\mathbf{n}} \in \mathcal{N}_{\mathrm{tr}}(\mathbf{S},\bar{\mathbf{s}})$, one has the bound

$$\sum_{\substack{\mathbf{T}\in\mathcal{U}_{\mathcal{V}} \\ \mathbf{T}\restriction_{[M]}=\mathbf{S}}} \sum_{\substack{\mathbf{s}\in\mathrm{Lab}_{\mathbf{T}} \\ \mathbf{s}\restriction_{\mathring{\mathbf{S}}}=\bar{\mathbf{s}}}} \sum_{\substack{\mathbf{n}\in\mathcal{N}_{\mathrm{tr}}(\mathbf{T},\mathbf{s}) \\ \mathbf{n}\restriction_{\mathbf{G}}=\bar{\mathbf{n}}}} \Big| \int_{\mathbf{R}^d} dy_o\, \tilde{I}^{\mathbf{n}}(y_{\mathcal{V}_0}) \Big| \lesssim \|\varrho\|_\kappa \, \llbracket H \rrbracket_{\varsigma,J,p} \cdot 2^{\langle D^k\varsigma_{\mathbf{S}}+\kappa\delta_{\mathbf{S}}^{\uparrow}[j],\bar{\mathbf{s}}\rangle} \,.$$

(B.7)

Above we are committing an abuse of notation - for each $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$ with $\mathbf{T}\restriction_{[M]} = \mathbf{S}$ we use the inclusion map[15] $\iota : \mathring{\mathbf{S}} \to \mathring{\mathbf{T}}$ to identify $\mathring{\mathbf{S}}$ as a subset of $\mathring{\mathbf{T}}$. This conventiion is used when we write the condition $\mathbf{s}\restriction_{\mathring{\mathbf{S}}} = \bar{\mathbf{s}}$ and is used throughout the rest of the proof. Any use of the notation $\mathbf{L}_\bullet$ in what follows refer to using the relevant tree of $\widehat{\mathcal{U}}_{\mathcal{V}}$ to determine descendent leaf vertices.

We now obtain (B.7). We start with defining a total homogeneity $\hat{\varsigma}$ on the trees of $\widehat{\mathcal{U}}_{\mathcal{V}}$ by setting, for $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$ and $a \in \mathring{\mathbf{T}}$,

$$\hat{\varsigma}_{\mathbf{T}}(a) \overset{\text{def}}{=} (|\mathbf{s}| + \kappa) \cdot \delta_{\mathbf{T}}^{\uparrow}[\{o,j\}] + \mathbb{1}\Big\{ \mathbf{L}_{\{o,j\}^{\uparrow},\mathbf{T}} = \{o,j\} \Big\} \Big( \delta_{\mathbf{T}}^{\Uparrow}[\{o,j\}] - \delta_{\mathbf{T}}^{\uparrow}[\{o,j\}] \Big)$$
$$+ \begin{cases} \varsigma_{\mathbf{S}}(a) & \text{if } a \in \mathring{\mathbf{S}} \text{ for } \mathbf{S} \overset{\text{def}}{=} \mathbf{T}\restriction_{[M]} \\ 0 & \text{otherwise.} \end{cases}$$

We now specify $\tilde{I}_k \in \mathrm{Mod}_{\{o\}}(\mathring{I}_k)$ with

$$\|\tilde{I}_k\|_{\hat{\varsigma},\mathcal{N}_{\mathbf{G}'}} \lesssim \|\varrho\|_\kappa \cdot \llbracket H \rrbracket_{\varsigma,J,p} \,.$$

(B.8)

For each $\mathbf{T} \in \mathcal{U}_{\mathcal{V}}$ and $\mathbf{n} \in \mathcal{N}_{\mathbf{G}'}$ with $\mathcal{T}(\mathbf{n}) = \mathbf{T}$ we set $\tilde{I}_k^{\mathbf{n}} \overset{\text{def}}{=} \mathring{I}_k^{\mathbf{n}}$ if $\mathbf{L}_{\{o,j\}^{\uparrow},\mathbf{T}} \supsetneq \{o,j\}$ and if $\mathbf{L}_{\{o,j\}^{\uparrow},\mathbf{T}} = \{o,j\}$ we set

$$\tilde{I}_k^{\mathbf{n}}(y_{\mathcal{V}_0}) \overset{\text{def}}{=} \Big( \tilde{H}_k^{\mathbf{n}}(y_{\mathcal{V}_0\setminus\{o\}}, y_o) - \tilde{H}_k^{\mathbf{n}}(y_{\mathcal{V}_0\setminus\{o\}}, y_j) \Big) \cdot \varrho_{n_{\{o,j\}}}(y_j - y_o) \,.$$

The necessary integration property, support property, and the estimate (B.8) are all straightforward to check. Now fix $(\mathbf{S},\bar{\mathbf{s}}) \in \widehat{\mathcal{U}}_{[M]} \ltimes \mathrm{Lab}_\bullet$, $\bar{\mathbf{n}} \in \mathcal{N}_{\mathrm{tr}}(\mathbf{S},\bar{\mathbf{s}})$, and $\mathbf{T} \in \widehat{\mathcal{U}}_{\mathcal{V}}$ with $\mathbf{T}\restriction_{[M]} = \mathbf{S}$. By using the bound (B.8) we get

$$\sum_{\substack{\mathbf{s}\in\mathrm{Lab}_{\mathbf{T}} \\ \mathbf{s}\restriction_{\mathring{\mathbf{S}}}=\bar{\mathbf{s}}}} \sum_{\substack{\mathbf{n}\in\mathcal{N}_{\mathrm{tr}}(\mathbf{T},\mathbf{s}) \\ \mathbf{n}\restriction_{\mathbf{G}}=\bar{\mathbf{n}}}} \Big| \int_{\mathbf{R}^d} dy_o\, \tilde{I}_k^{\mathbf{n}}(y_{\mathcal{V}_0}) \Big| \lesssim \|\varrho\|_\kappa \cdot \llbracket H \rrbracket_{\varsigma,J,p} \cdot \sum_{\substack{\mathbf{s}\in\mathrm{Lab}_{\mathbf{T}} \\ \mathbf{s}\restriction_{\mathring{\mathbf{S}}}=\bar{\mathbf{s}}}} 2^{\langle\hat{\varsigma}_{\mathbf{T}},\mathbf{s}\rangle-\mathbf{s}(o^{\uparrow})|\mathbf{s}|} \,.$$

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[15] See Definition A.29.

Again, the factor $2^{-\mathbf{s}(o^\uparrow)|\mathfrak{s}|}$ comes from integration. What remains is showing that

$$\sum_{\substack{\mathbf{s}\in\mathrm{Lab}_\mathbf{T} \\ \mathbf{s}\restriction_{\mathring{\mathbf{S}}}=\bar{\mathbf{s}}}} 2^{\langle\hat{\varsigma}_\mathbf{T},\mathbf{s}\rangle-\mathbf{s}(o^\uparrow)|\mathfrak{s}|} \lesssim 2^{\langle D^k\varsigma_\mathbf{S}+\theta\delta_\mathbf{s}^\uparrow[j],\bar{\mathbf{s}}\rangle} \ . \tag{B.9}$$

To verify the above inequality one can split into three different cases. Remember that it must always be the case that $\mathbf{L}_{o^\uparrow,\mathbf{T}} \ni j$ because of our choice of $\mathbf{G}'$. We draw an example with $M = 3$, the first tree represents $\mathbf{S}$ and the the three other trees show examples of our three cases.
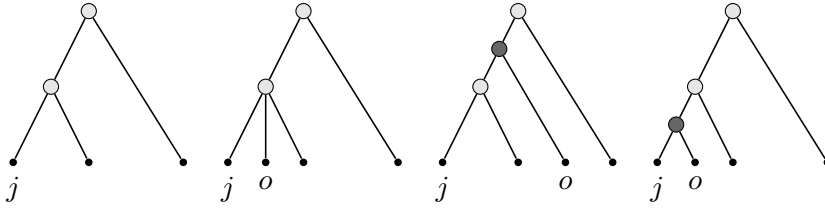


The first case occurs when $\mathring{\mathbf{T}} = \mathring{\mathbf{S}}$, here there is nothing to prove since the sum of the LHS only has one term (where $\mathbf{s} = \bar{\mathbf{s}}$) so both sides of the inequality are equal.

In the second and third cases one has $\mathring{\mathbf{T}} = \mathring{\mathbf{S}} \sqcup \{b\}$ where $b = \{o, j\}^\uparrow$ is represented with a darker node in our pictures. The sum over $\mathbf{s}$ with $\mathbf{s}\restriction_{\mathring{\mathbf{S}}} = \bar{\mathbf{s}}$ is just a sum over the value of $\mathbf{s}(b)$ constrained to satisfy an upper bound in the second case or lower bound in the third case.

For the second case we assume that $\mathbf{L}_{\{o,j\}^\uparrow,\mathbf{T}} \supsetneq \{o, j\}$. It follows that $b$ is not maximal in the partial order of $\mathring{\mathbf{T}}$ and in fact $j^{\uparrow,\mathbf{T}} = j^{\uparrow,\mathbf{S}} > b$. We then have

$$\sum_{\substack{\mathbf{s}\in\mathrm{Lab}_\mathbf{T} \\ \mathbf{s}\restriction_{\mathring{\mathbf{S}}}=\bar{\mathbf{s}}}} 2^{\langle\hat{\varsigma}_\mathbf{T},\mathbf{s}\rangle-\mathbf{s}(o^\uparrow)|\mathfrak{s}|} \lesssim \sum_{\mathbf{s}(b)<\bar{\mathbf{s}}(j^\uparrow)} 2^{(|\mathfrak{s}|+\kappa-|\mathfrak{s}|)\mathbf{s}(b)} \prod_{a\in\mathring{\mathbf{S}}} 2^{\hat{\varsigma}_\mathbf{T}(a)\bar{\mathbf{s}}(a)}$$

$$\lesssim 2^{\kappa\bar{\mathbf{s}}(j^\uparrow)} \prod_{a\in\mathring{\mathbf{S}}} 2^{\hat{\varsigma}_\mathbf{T}(a)\bar{\mathbf{s}}(a)} = 2^{\langle D^k\varsigma_\mathbf{S}+\kappa\delta_\mathbf{s}^\uparrow[j],\bar{\mathbf{s}}\rangle} \ .$$

In the third case one has $\mathbf{L}_{\{o,j\}^\uparrow,\mathbf{T}} = \{o, j\}$. It follows that $b$ is maximal in $\mathring{\mathbf{T}}$ and $b^\uparrow = j^{\uparrow,\mathbf{S}}$, thus we have

$$\sum_{\substack{\mathbf{s}\in\mathrm{Lab}_\mathbf{T} \\ \mathbf{s}\restriction_{\mathring{\mathbf{S}}}=\bar{\mathbf{s}}}} 2^{\langle\hat{\varsigma}_\mathbf{T},\mathbf{s}\rangle-\mathbf{s}(o^\uparrow)|\mathfrak{s}|} \lesssim \sum_{\mathbf{s}(b)>\bar{\mathbf{s}}(j^{\uparrow,\mathbf{S}})} 2^{(|\mathfrak{s}|+\kappa-|\mathfrak{s}|-1)\mathbf{s}(b)+\bar{\mathbf{s}}(j^{\uparrow,\mathbf{S}})} \prod_{a\in\mathring{\mathbf{S}}} 2^{\hat{\varsigma}_\mathbf{T}(a)\bar{\mathbf{s}}(a)}$$

$$\lesssim 2^{\kappa\bar{\mathbf{s}}(j^{\uparrow,\mathbf{S}})} \prod_{a\in\mathring{\mathbf{S}}} 2^{\hat{\varsigma}_\mathbf{T}(a)\bar{\mathbf{s}}(a)} = 2^{\langle D^k\varsigma_\mathbf{S}+\kappa\delta_\mathbf{s}^\uparrow[j],\bar{\mathbf{s}}\rangle} \ .$$

$\square$

## Appendix C   Symbolic index

In this appendix, we collect the most used symbols of the article, together with their meaning and the page where they were first introduced.

| Symbol | Meaning | Page |
|---|---|---|
| $\bullet$ | The trivial tree consisting only of a root. | 10 |
| $\mathbf{1}$ | The empty forest. Not considered a tree. | 11 |
| $\mathfrak{L}$ | Finite set of "types", used to distinguish each of the kernels and noises appearing in the system SPDE. | 10 |
| $\mathfrak{L}_+$ | Set of $\mathfrak{t} \in \mathfrak{L}$ which are kernel types, satisfy $|\mathfrak{t}|_\mathfrak{s} > 0$. | 10 |
| $\mathfrak{L}_-$ | Set of $\mathfrak{t} \in \mathfrak{L}$ which are leaf types, satisfy $|\mathfrak{t}|_\mathfrak{s} < 0$. | 10 |
| $\mathscr{B}_\mathfrak{s}$ | Set of test functions used to define seminorms on space of models. | 16 |
| $\mathfrak{L}_{\mathrm{Cum}}^{\mathrm{all}}$ | Prescribed set of non-vanishing cumulants. | 21 |
| $p_k$ | For $k \in \mathbf{N}^d$, test function given by spatial truncation of $x^k$ | 21 |
| $\jmath_A(B)$ | Minimum homogeneity gain from noises in $B$ contracting with noises with types from $A$. | 22 |
| $\overline{\mathcal{A}}$ | Set of maximal elements of the poset $\mathcal{A}$. | 33 |
| $\mathrm{Max}\,\mathcal{A}$ | Set of maximal elements of the poset $\mathcal{A}$. | 33 |
| $\underline{\mathcal{A}}$ | Set of minimal elements of the poset $\mathcal{A}$. | 33 |
| $\mathrm{Min}\,\mathcal{A}$ | Set of minimal elements of the poset $\mathcal{A}$. | 33 |
| Div | All superficially divergent subtrees | 33 |
| $\mathfrak{C}$ | All potentially useful cuts (edges). | 44 |
| $\mathscr{C}_\mathbb{M}$ | Collection of edges in $\mathfrak{C}$ which are not in any element of $\mathbb{M}$. | 60 |
| $\mathfrak{C}^P(\mathbb{M})$ | Collection of cut sets $\mathscr{C}$ generating $\mathbb{M}$: $P_\mathscr{C}^{-1}[s(\mathbb{M})] = \mathbb{M}$. | 60 |
| $\mathfrak{M}^P(\mathscr{C})$ | All forest intervals $\mathbb{M}$ generated by $\mathscr{C}$. | 60 |
| $P^\mathbf{n}$ | Discards "dangerous" trees with respect to $\mathbf{n}$. | 67 |
| $\mathscr{G}^\mathbf{n}(\mathcal{F})$ | Kernel edges benefiting from cancellation when $\mathcal{F}$ is contracted. | 70 |
| $\mathcal{F}, \mathcal{G}$ | Generic forests of subtrees. | 33 |
| $\mathbb{F}$ | All forests of superficially divergent subtrees. | 34 |
| $\mathbb{F}^{\leq k}$ | Forests in $\mathbb{F}$ of depth at most $k$. | 35 |
| $\mathbb{F}[\mathcal{F}]$ | All forests $\mathcal{G}$ with $\mathrm{Max}\,\mathcal{G} = \mathcal{F}$. | 35 |
| $\mathbb{F}_<[\mathcal{F}]$ | All $\mathcal{G} \in \mathbb{F}^{\leq 1}$ with trees strictly contained in trees of $\mathcal{F}$. | 35 |
| $\mathbb{F}_\leq[\mathcal{F}]$ | All $\mathcal{G} \in \mathbb{F}^{\leq 1}$ with trees contained in trees of $\mathcal{F}$. | 35 |
| $N_F$ | All nodes of the forest $F$. | 11 |
| $E_F$ | All edges of the forest $F$. | 11 |
| $\mathbb{L}(F)$ | Edges of noise type. | 11 |

| Symbol | Meaning | Page |
|---|---|---|
| $K(F)$ | Edges of kernel type. | 11 |
| $L(F)$ | True nodes corresponding to $\mathbb{L}(F)$, inherits types. | 11 |
| $\overline{T}_{\not\geq}[\mathscr{C}]$ | Intersection of all subtrees $\{\overline{T}_{\not\geq}(e) : e \in \mathrm{Min}(\mathscr{C})\}$ | 44 |
| $N(F)$ | True nodes: $N(F) = N_F \setminus e_c(L(F))$. | 11 |
| $N^*$ | True nodes of $\bar{T}$, plus $\otimes$. | 31 |
| $\otimes$ | Node representing basepoint of the model. | 31 |
| $\tilde{N}(F)$ | Defined as $\tilde{N}(F) = N(F) \setminus \varrho(F)$. | 11 |
| $T_{\geq}(e)$ | Subtree of $T$ above the edge $e$ (including $e$). | 29 |
| $T_{\not\geq}(e)$ | Subtree of $T$ formed by edges below or not comparable to the edge $e$. | 29 |
| $\overline{T}_{\not\geq}[\mathscr{C}]$ | Intersection of all subtrees $\{\overline{T}_{\not\geq}(e) : e \in \mathrm{Min}(\mathscr{C})\}$ | 44 |
| $\mathfrak{T}_{\geq}[\mathscr{C}]$ | Collection of maximal subtrees of $\overline{T}$ whose roots are of degree 1 and are edge disjoint $\overline{T}_{\not\geq}[\mathscr{C}]$. | 44 |
| $K^{\downarrow}(S)$ | Kernel edges incoming to $S$ (i.e. $e_p \in N_S$, $e_c \notin N_S$) in $\bar{T}$. | 36 |
| $\bar{K}^{\downarrow}(S)$ | Defined as $K^{\downarrow}(S) \cup K(S)$. | 36 |
| $K^{\partial}_T(S)$ | Defined for $T \leq S$ as $K^{\downarrow}(T) \cap K(S)$. | 37 |
| $N^{\downarrow}(S)$ | Defined as $e_c(K^{\downarrow}(S))$. | 36 |
| $C_{\mathcal{F}}(S)$ | Maximal elements of $\mathcal{F}$ restricted to subtrees of $S$. | 34 |
| $\tilde{N}_{\mathcal{F}}(S)$ | Nodes in $\tilde{N}(S)$, but not in $\tilde{N}(T)$ for $T \in C_{\mathcal{F}}(S)$. | 37 |
| $N_{\mathcal{F}}(S)$ | Defined as $\tilde{N}_{\mathcal{F}}(S) \cup \{\varrho_S\}$. | 37 |
| $\mathring{K}_{\mathcal{F}}(S)$ | Kernel edges in $S$ that neither belong to nor are adjacent to any subtree of $C_{\mathcal{F}}(S)$. | 37 |
| $K^{\partial}_{\mathcal{F}}(S)$ | Union of $K^{\partial}_T(S)$ over $T \in C_{\mathcal{F}}(S)$. | 37 |
| $\mathit{f}(B)$ | Gives power-counting gain for renormalization of $\mathbf{E}^c_{\mathfrak{t}(B)}$. | 110 |
| $\mathfrak{h}_{\mathfrak{c},D}(B)$ | Worst-case homogeneity gain from noises in $B$ participating in "external" versus "internal" cumulants with external noises drawn from $D$ | 110 |
| $|\mathfrak{t}(B)|_{\mathfrak{s}}$ | Total homogeneity of the typed set $B$ | 10 |
| $|\mathfrak{t}(B)|_{\mathfrak{s},\mathfrak{c},D}$ | Worst-case homogeneity given to noises $B$ when part of an external cumulant with noises from $D$. | 109 |
| $\underline{\mathfrak{F}_2}$ | Collection of all decorated i-forests. | 27 |
| $\underline{\mathfrak{T}_2}$ | Collection of all decorated i-trees. | 27 |
| $\underline{\mathfrak{F}_i},\ \underline{\mathfrak{T}_i}$ | For $i \in \{0,1\}$, set of decorated i-forests or i-trees with color at most $i$. | 27 |
| $\widehat{\underline{\mathfrak{T}_2}}$ | Collection of all decorated i-trees with root colored 2. | 29 |
| $L_{\mathcal{F}}(S)$ | $e \in L(S)$ but not in $L(T)$ for every $T \in C_{\mathcal{F}}(S)$. | 37 |
| $\mathscr{C}(\mathbf{R}^d)$ | Collection of smooth real valued functions on $\mathbf{R}^d$. | 14 |
| $\mathscr{C}_A$ | All smooth real valued functions on $(\mathbf{R}^d)^A$. | 31 |
| $\mathscr{C}_*$ | Collection of all smooth real valued functions on $(\mathbf{R}^d)^{N^*}$. | 31 |
| $\mathrm{X}^{N,v'}_{\mathrm{n},v}$, etc. | Various notations for products of monomials or binomials in $\mathscr{C}_*$. | 38 |
| $H_{\pi,\mathcal{F},S}$ | Renormalized distribution for divergent $S$. | 40 |
| $C_{\mathscr{C}}(e)$ | For $e \in \mathfrak{C}$, $\mathscr{C} \subset \mathbb{C}_+$, given by $\mathrm{Min}(\{\bar{e} \in \mathscr{C} : \bar{e} > e\})$. | 44 |

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