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Applications of Berkovich spaces

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Declaration of originality

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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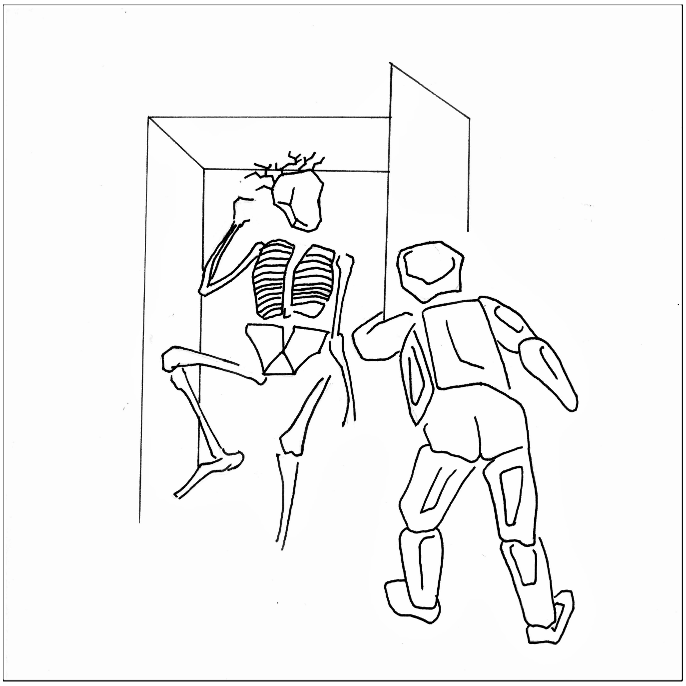
Abstract

This thesis applies the techniques of non-archimedean geometry to the study of degenerations and compactifications of algebraic varieties. The central object we investigate is the so-called essential skeleton, a combinatorial object that lies embedded in non-archimedean spaces and encodes an important part of the geometry of the space. This originates in the work of Kontsevich and Soibelman on mirror symmetry, an important development in algebraic geometry that has its roots in mathematical physics. The interplay of the theory of Berkovich spaces, the ideas of mirror symmetry and the tools of birational geometry gives form and meaning to the study of the essential skeleton.

Chapters 3 and 4 are built on the research paper *The essential skeleton of a product of degenerations*, in collaboration with Morgan Brown [BM19]. We establish the behaviour of the essential skeleton under some natural operations, and we merge the language of logarithmic geometry into the construction of Berkovich skeletons. As main application, we compute the essential skeleton of some degenerations of hyperkähler varieties. We consider Hilbert schemes of a semistable degeneration of K3 surfaces, and generalised Kummer constructions applied to a semistable degeneration of abelian surfaces. In both cases we find that the dual complex of the $2n$-dimensional degeneration is homeomorphic to a point, n-simplex, or $\mathbb{CP}^n$, depending on the type of the degeneration and in accordance with the predictions of mirror symmetry.

Chapters 5 to 7 are based on the joint work *Essential skeletons of pairs and the geometric P=W conjecture* with Mirko Mauri and Matthew Stevenson [MMS18]. We introduce and study an explicit formulation of the weight function, a key tool to define the essential skeleton, in the case of varieties defined over a non-archimedean trivially-valued field. As a result, we employ these techniques to compute the dual boundary complexes of certain character varieties: this provides the first evidence for the geometric P=W conjecture in the compact case, and the first application of Berkovich geometry in non-abelian Hodge theory.

We investigate the essential skeleton, as geometrical figure which is able to establish striking connections among different mathematical worlds. With a pinch of irony, we think of this as an essential skeleton in the closet, hence a secret tool whose discovery and disclosure cause surprise and a new perception of some mathematical challenges.



A non-archimedean illustration of an essential skeleton in the closet: dual complexes and infinite tree of valuations face an imaginative torus fibration. The inspiration of this picture is a caricature from 1792 which depicts the skeleton of Count of Mirabeau in the closet of king Louis XVI of France, regarding a scandal of duplicity at the time of the French Revolution. (Picture by Mirko Mauri)

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Contents

Notation

Throughout we keep largely to the following notation.

(X, Δ)	pair	2.1.1
$\Delta = \sum \alpha_i \Delta_i$	boundary, with $\alpha_i \in (0, 1]$	2.1.1
$\mathrm{Diff}^*_W(\Delta)$	different of (X, Δ) with respect to a stratum W	7.2.6
logCY	log Calabi–Yau	2.1.1
snc	simple normal crossing	2.1.1
lc	log canonical	2.3
dlt	divisorially log terminal	2.3
$\mathcal{K}$	non-archimedean field	2.1.1
$\mathcal{K}^\circ$	valuation ring of $\mathcal{K}$	2.1.1
$\mathcal{K}^{\circ\circ}$	maximal ideal of $\mathcal{K}^\circ$	2.1.1
$\widetilde{\mathcal{K}} = \mathcal{K}^\circ / \mathcal{K}^{\circ\circ}$	residue field of $\mathcal{K}^\circ$	2.1.1
$\mathcal{K}_r$	Gauss extension on $\mathcal{K}$	6.3
K	discretely-valued field	2.1.1
R	valuation ring of K	2.1.1
k	residue field of R	2.1.1
$S = \operatorname{Spec} R$	affine space associated to R	2.1.1
s	closed point of S	2.1.1
X^{an}	Berkovich analytification of X	2.1.1
$X^{\beth}$	$\beth$-analytification of X	2.1.1
X^{bir}	subspace of birational points	2.1.1
X^{mon}	subspace of monomial points	2.1.3
X^{div}	subspace of divisorial points	2.1.3
$x = (\xi_x, \lvert \cdot \rvert_x)$	point of the Berkovich space	2.1.1
$\ker \colon X^{\mathrm{an}} \to X$	kernel map	2.1.1
$\mathscr{X}$	model over S	2.1.2
$\mathscr{X}_K$	generic fibre of the model $\mathscr{X}$	2.1.2
$\mathscr{X}_k$	special fibre of the model $\mathscr{X}$	2.1.2
$\mathcal{D}(\cdot)$	dual complex	2.2
$\mathcal{DMR}(X, \Delta)$	dual complex of a lc pair	7.2.1
$A * B$	join of the topological spaces A and B	2.2

M^{gp}	groupification of a monoid M	2.4.1
$\mathcal{X}^+ = (\mathcal{X}, \mathcal{M}_{\mathcal{X}})$	logarithmic scheme	2.4.1
$\mathcal{C}_{\mathcal{X}} = \mathcal{M}_{\mathcal{X}}/\mathcal{O}_{\mathcal{X}}^{\times}$	characteristic sheaf of $\mathcal{X}^+$	2.4.1
$\mathcal{I}_{\mathcal{X}}$	ideal sheaf in $\mathcal{O}_{\mathcal{X}}$ generated by $\mathcal{M}_{\mathcal{X}} \setminus \mathcal{O}_{\mathcal{X}}^{\times}$	2.4.1
$S^+ = (S, s)$	logarithmic scheme on S induced by the closed point s	2.4.1
$\mathscr{X}^+$	logarithmic scheme on $\mathscr{X}$ induced by special fibre $\mathscr{X}_k$	2.4.1
$F_{\mathcal{X}}$	Kato fan associated to a log-regular log scheme $\mathcal{X}^+$	2.5
$D_{\mathcal{X}}$	divisor associated to a log-regular log scheme $\mathcal{X}^+$	2.5
$\mathrm{Sk}(\mathscr{X}^+)$	skeleton associated to a log-regular log scheme over S^+	3.2
$\rho_{\mathscr{X}}$	retraction onto the skeleton $\mathrm{Sk}(\mathscr{X})$	3.2
$\mathrm{Sk}(X^+)$	skeleton associated to X^+ over a trivially-valued field	5.3
$\mathrm{Sk}_x(X^+)$	face of $\mathrm{Sk}(X^+)$ corresponding to $x \in F_X$	5.3
ρ_{X^+}	retraction onto the skeleton $\mathrm{Sk}(X^+)$	5.3.3.1
wt_{ω}	weight function associated to ω	3.3, 5.2.4
$\mathrm{Sk}(X, \Delta, \omega)$	Kontsevich–Soibelman skeleton of (X, Δ) with respect to ω	3.3, 5.3.7
$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta)$	essential skeleton of the pair (X, Δ)	3.4.1, 5.3.7
$\phi_{\mathrm{triv},L}(\cdot,\cdot)$	trivial metric on line bundle L	5.2.2
$A_{(X,\Delta)}(\cdot)$	log discrepancy function of (X, Δ)	5.2.4
$\lVert \cdot \rVert_{\mathrm{Tem}}$	Temkin's metric	6.2.2
M_G	G-character variety of curve of Riemann surface of genus 1	1.4.3

1

Introduction

The theory of Berkovich spaces was introduced in the late 1980s by Vladimir Berkovich as a new approach to non-archimedean analytic geometry. Refining the notion of a rigid analytic space conceived by Tate in [Tat71], Berkovich spaces are at the same time generalizations and geometrical analogues of complex manifolds. In particular, this theory associates, to each variety defined over a non-archimedean field, a space called the Berkovich analytification, with nice topological properties, which reflect and illuminate the geometry of the underlying variety. This approach has been successfully applied in several domains of mathematics, ranging from moduli spaces, birational geometry and mirror symmetry to Arakelov geometry, p-adic analysis and complex dynamics.

One of the fundamental concepts underlying the importance of Berkovich spaces in algebraic geometry is the notion of skeletons. These arise naturally in the context of degenerations or compactifications of algebraic varieties. In these situations, Berkovich geometry provides a natural framework for studying properties of the original variety in terms of those of the degenerate fibre or boundary divisor.

The focus of this thesis is on the notion of essential skeleton, a canonical and minimal skeleton which captures the fundamental geometric structure of a Berkovich analytification. Inspired by [KS06] and then developed in [MN15; NX16], the geometry of essential skeletons enjoys a deep connection with mirror symmetry and birational geometry. The results presented in the following chapters build on and strengthen this interplay, which we will now elaborate on; for a more exhaustive treatment and additional background we refer to the surveys [Gro13; Nic16].

1.1. Essential skeleton

1.1.1. Mirror symmetry

Mirror symmetry is a fast-moving research area at the boundary between mathematics and theoretical physics. Originated from observations in string theory, it suggests the existence of a duality between Calabi–Yau manifolds, complex manifolds with a nowhere vanishing holomorphic form of maximal degree. It predicts that every Calabi-Yau manifold X should have a mirror partner $\check{X}$, such that the complex geometry of $\check{X}$ is equivalent to the symplectic geometry of X, in some appropriate sense, and vice versa. Explicit constructions and concrete applications of this idea have led to important results. For instance, in [COGP91] the numbers of rational curves of fixed degree on the quintic threefold were predicted by performing certain period calculations on the mirror partner, at a time where these numbers were far beyond the reach of algebraic geometry.

The search for a rigorous definition of a mirror pair, and for methods to construct mirror partners still remains an open challenge, but good progress has been made. Indeed, in the mid-nineties three approaches arose: a *combinatorial explanation* due to Batyrev and later generalised by Batyrev and Borisov [Bat94; BB96a; BB96b] uses toric geometry and a duality for lattice polytopes to construct mirror pairs of complete intersections in toric varieties. A *categorical formulation* introduced by Kontsevich [Kon95] under the name of homological mirror symmetry expresses mirror symmetry as an equivalence of categories, the Fukaya category of X and the derived category of coherent sheaves on $\check{X}$. A concrete *geometric interpretation* proposed by Strominger, Yau and Zaslow in [SYZ96] relates mirror symmetry to a duality between fibres of Lagrangian torus fibrations.

1.1.2. SYZ conjecture

In its current formulation, the Strominger–Yau–Zaslow (SYZ) conjecture concerns Calabi–Yau manifolds in certain degenerating families rather than individual manifolds. More precisely, we consider a projective family $(X_t)_t$ of Calabi–Yau varieties of dimension n over a punctured disk, such that the family is maximally degenerate. The condition of maximal degeneracy, also called large complex structure limit, means that the monodromy operator on the degree n cohomology of X_t has a Jordan block of maximal size, that is $n+1$.

Conjecture 1.1.2.1 (SYZ conjecture). *A general fibre X_t admits a fibration $\rho : X_t \to S$, whose fibres are special Lagrangian tori, away from a locus of codimension 2 in the base S. Moreover, the mirror partner $\check{X}_t$ of X_t is obtained by dualizing the special Lagrangian toric fibres of ρ and by suitably compactifying the resulting space.*

1.1.3. Gromov–Hausdorff limit

In [GW00; KS01] a metric approach is adopted to describe the base S of the conjectural SYZ fibration, and more generally to explain the structure of Calabi–Yau manifolds near the large complex structure limit. If we fix a polarization on $(X_t)_t$, then by Yau's theorem every smooth fibre X_t carries a canonical Ricci-flat metric g_t^{CY}. We rescale the metrics to obtain a family of Ricci-flat manifolds (X_t, g_t) with bounded diameter.

On the set of compact metric spaces we consider the notion of Gromov–Hausdorff distance: roughly speaking, we say that two metric spaces M_1 and M_2 are ε-close if there exists a metric space M containing both M_1 and M_2 such that M_1 belongs to the ε-neighborhood of M_2 and vice versa. From a result by Gromov, see [Pet06, §10, Corollary 31], the space of compact Ricci-flat manifolds with bounded diameter is precompact, hence any sequence of such manifolds admits a converging subsequence with respect to the Gromov–Hausdorff distance. We note that a priori the Gromov–Hausdorff limit of a given sequence is non-unique, as it may depend on the choice of the subsequence, and is just a metric space. However, in the setting of the SYZ fibration such limit is expected to be unique, and in [Oda18, §3] Odaka proves its uniqueness when the manifolds X_t are principally polarized abelian varieties.

Conjecture 1.1.3.1. *[KS01, §3.1, Conjecture 1] The limit (B, g_B) of (X_t, g_t) in the Gromov–Hausdorff metric as $t \to 0$ is such that*

(i) *(B, g_B) is a compact metric space, which contains a smooth oriented Riemannian manifold $(B^{sm}, g_{B^{sm}})$ of dimension n as a dense open metric subspace, and the Hausdorff dimension of $B \setminus B^{sm}$ is less than or equal to $n-2$;*

(ii) *B^{sm} carries an integral affine structure, that is B^{sm} admits an atlas of charts such that the transition functions belong to $GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$;*

(iii) *the metric $g_{B^{sm}}$ has a potential, that is $g_{B^{sm}}$ is locally given in affine coordinates by a symmetric matrix $(g_{ij}) = (\partial^2 F/\partial x_i \partial x_j)$, where F is a smooth function;*

(iv) *the real Monge–Ampère equation is satisfied, that is in affine coordinates the metric volume element $\det(g_{ij}) = \det(\partial^2 F/\partial x_i \partial x_j)$ is constant.*

A metric space satisfying the conditions $(ii) - (iv)$ is called an integral Monge-Ampère manifold, and given such a manifold there exists a well defined and canonical notion of dual integral Monge-Ampère manifold.

Conjecture 1.1.3.2. *[KS06, §5.1, Conjecture 2] Smooth parts of Gromov–Hausdorff limits of dual families of Calabi–Yau manifolds are dual integral Monge–Ampère manifolds.*

In the above conjecture, the duality of mirror symmetry is explained in terms of Gromov–Hausdorff limits, and integral Monge–Ampère manifolds can be viewed as real analogues of Calabi–Yau manifolds. Indeed, given an integral Monge-Ampère manifold one can construct a 1-parameter family of non-compact Calabi–Yau complex manifolds, each of which is a torus bundle over B^{sm}. A further conjecture by Kontsevich and Soibelman [KS01, §3.1, Conjecture 2] predicts that the family of these torus fibrations is metrically close to the family $(X_t, g_t)_t$ outside of a singular set. This implies that the base S of the SYZ fibration in Conjecture 1.1.2.1 is expected to be isometric to the Gromov–Hausdorff limit B.

Finally, we note that Conjecture 1.1.3.1 has been recently established for the case of $K3$ surfaces by Odaka and Oshima [OO18].

1.1.4. Non-archimedean picture

We consider, in particular, the algebro-geometric setting in which X is a Calabi–Yau variety over $K = \mathbb{C}((t))$, that is a smooth projective scheme over K with trivial canonical line bundle. Consistently with the setting of the SYZ conjecture, we assume that X is maximally degenerate, and moreover that it extends to a projective semistable model $\mathscr{X}$ over $R = \mathbb{C}[[t]]$, that is a projective regular scheme with reduced strict normal crossing (snc) degenerate fibre $\mathscr{X}_0$.

A natural question, raised by Kontsevich and Soibelman and concerning such Calabi–Yau varieties, is whether it is possible to define B and calculate the Monge-Ampère manifold $(B^{\mathrm{sm}}, g_{B^{\mathrm{sm}}})$ purely algebraically, without the use of transcendental methods and Calabi–Yau metrics. As first insight on this problem, they suggest to relate the metric limit, the Gromov–Hausdorff limit of the family, to the algebro-geometric limit, the degenerate fibre $\mathscr{X}_0$. Roughly speaking, the limit B should coincide with (a canonical subcomplex of) the dual intersection complex $\mathcal{D}(\mathscr{X}_0)$ of $\mathscr{X}_0$. This is a finite simplicial complex that encodes the combinatorics of the irreducible components of the divisor $\mathscr{X}_0$ and their intersections. Indeed, it has as many vertices v_i as the irreducible components D_i of $\mathscr{X}_0$, and a simplex $< v_{i_0}, \dots, v_{i_r} >$ for each component of $D_{i_0} \cap \dots \cap D_{i_r} \neq \emptyset$.

A different choice for the model $\mathscr{X}$ corresponds to a birational modification, and determines a different dual complex with the same homotopy type of $\mathcal{D}(\mathscr{X}_0)$ [Dan75; Ste06]. The search of a canonical complex associated to X leads to the notion of essential divisors [KS01, §3.3]. An irreducible component D of $\mathscr{X}_0$ is called essential if the order of vanishing at D of a global volume form ω on X is minimal among all components of $\mathscr{X}_0$. More generally, a simplex $< v_{i_0}, \dots, v_{i_r} >$ is essential if all D_{i_j} are essential.

Finally, to determine the smooth part B^{sm} and its integral affine structure from the subcomplex of essential simplices in $\mathcal{D}(\mathscr{X}_0)$, Kontsevich and Soibelman propose to use ideas from Berkovich's theory of non-archimedean analytic spaces; see Conjecture 1.1.7.2.

1.1.5. Essential skeleton

Let X be a smooth proper variety over $K = \mathbb{C}((t))$. The Berkovich analytification of X in the sense of [Ber90; Ber93], denoted X^{an}, is a space of valuations associated to X. More precisely, X^{an} is the topological space consisting of real valuations on the residue fields of

X that extend the t-adic valuation on K. In particular, we denote by X^{bir} the subspace of valuations defined on the function field of X. In recent years, there has been much interest in certain combinatorial subspaces of X^{an} onto which X^{an} admits a strong deformation retraction, called skeletons. One common source of skeletons are the dual complexes of suitable models of X over $R = \mathbb{C}[[t]]$.

Consider a proper regular scheme $\mathscr{X}$ over R whose base change to K is isomorphic to X and whose central fibre is an snc divisor. Given such a model $\mathscr{X}$, the order of vanishing along a component D of the special fibre $\mathscr{X}_0$ defines a valuation, called divisorial, on the function field of X, and hence a point of X^{an}. In this way, the vertices of the dual complex of $\mathscr{X}_0$ are embedded in X^{an}. The results in [Ber99; MN15] show that this extends to an embedding of the entire dual complex $\mathcal{D}(\mathscr{X}_0)$ into X^{an}; the image, denoted $\mathrm{Sk}(\mathscr{X})$, is called the skeleton of $\mathscr{X}$. Moreover, the embedding admits a canonical continuous retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$, which is a strong deformation retraction by [Ber99; Thu07].

The Berkovich skeletons all depend on the auxiliary choice of a model, and they are not intrinsic to the variety X in question. Inspired by the work of Kontsevich and Soibelman, and motivated to construct a canonical skeleton in X^{an}, Mustață and Nicaise introduce in [MN15] the notions of weight functions and of the essential skeleton of X.

The weight functions are real-valued functions associated to pluricanonical forms on X; roughly, given a pluricanonical form ω and an irreducible component D in $\mathscr{X}_0$, the weight function wt_ω measures the order of vanishing at D of the form ω. These functions are piecewise affine on the Berkovich skeleton of any snc model of X and strictly increasing as one moves away from the Berkovich skeleton. Thus, the locus in X^{an} where the weight functions attain their minimal value is contained in any Berkovich skeleton.

Definition 1.1.5.1. The essential skeleton of X is

$$\mathrm{Sk}^{\mathrm{ess}}(X) \coloneqq \bigcup_{\omega} \overline{\{x \in X^{\mathrm{div}} \,|\, \mathrm{wt}_\omega(x) \text{ is minimal } \}} \subseteq X^{\mathrm{bir}}$$

where ω runs through the set of non-zero regular pluricanonical forms of X and X^{div} denotes the set of divisorial valuations.

It follows from the definition that the essential skeleton is a birational invariant of X and is a union of faces in any Berkovich skeleton. From the study of the weight functions on Berkovich curves in [BN16], we deduce that pluricanonical forms, not only canonical forms, are needed in the definition of the essential skeleton, and moreover, that in general different pluricanonical forms have different minimal loci. In order to analyse further properties of the essential skeleton, one can apply the techniques of birational geometry.

1.1.6. Birational geometry

An alternative approach to determine a canonical dual complex associated to a variety over $K = \mathbb{C}((t))$ is by enlarging the class of models of X we can work with. Indeed, if X has dimension 1, one can consider the unique minimal snc model of X, in the sense of the minimal model program (MMP): this determines a distinguished homeomorphism class with the homotopy type of the dual complex of any snc model of X. In higher dimension, minimal snc models do not always exist, so we take into account divisorially log terminal (dlt) models. These can be viewed as snc models with mild singularities, such that the dual complex is still well defined. Even if dlt models of X are not unique, the skeleton does not depend on the choice of a minimal dlt model, hence its homeomorphism type is an invariant of X.

Theorem 1.1.6.1. *[NX16, Theorem 3.3.3] If the canonical line bundle of X is semiample, then the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X)$ coincides with the skeleton $\mathrm{Sk}(\mathscr{X})$ of any minimal dlt model $\mathscr{X}$ of X.*

Therefore, to construct a canonical skeleton in X^{an} one can rely on the techniques of birational geometry (minimal dlt models), or alternatively use tools from non-archimedean geometry (weight functions). These methods express a minimality condition in two different ways, but both determine the same canonical and minimal skeleton of X topologically.

Depending on the purpose, one approach might be more suitable than the other. For instance, by the birational characterization of the essential skeleton and the results in [dFKX17], one proves that the map $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X}) = \mathrm{Sk}^{\mathrm{ess}}(X)$ is a strong deformation retraction, as for any Berkovich skeleton. Moreover, by means of the techniques of birational geometry further interesting properties of the map $\rho_{\mathscr{X}}$ are established; see Theorem 1.1.7.3. Nevertheless, the advantage of the non-archimedean viewpoint is that the dual complex $\mathcal{D}(\mathscr{X}_0)$ for any minimal dlt model $\mathscr{X}$ is embedded in the topological space X^{an} and the faces of $\mathcal{D}(\mathscr{X}_0)$ can be singled out as minima of weight functions without constructing explicitly the model. In the main results of Chapter 4 and Chapter 7 this approach turns out to be especially convenient.

1.1.7. Non-archimedean SYZ fibration

The non-archimedean interpretation of mirror symmetry continues with the following series of ideas, starting from a non-archimedean analogue of a Lagrangian fibration.

Let $\mathbb{G}_m = \mathrm{Spec}(K[z, z^{-1}])$ be the multiplicative group over $K = \mathbb{C}((t))$ and n a positive integer. The tropicalization map of $\mathbb{G}_m^n$

$$\rho_{\mathrm{trop}} : (\mathbb{G}_m^n)^{\mathrm{an}} \to \mathbb{R}^n \, : \, (v_1, \dots, v_n) \mapsto (v_1(z), \dots, v_n(z))$$

is a continuous map whose fibres are K-affinoid tori.

Definition 1.1.7.1. [KS06, §4.1] A continuous map $f : Z \to W$ from a smooth K-analytic space of dimension n into a Hausdorff topological space W is an n-dimensional affinoid torus fibration if for every point $w \in W$ there is a neighborhood U such that the map $f^{-1}(U) \to U$ is isomorphic to $\rho_{\mathrm{trop}}^{-1}(V) \to V$ for some open subset $V \subseteq \mathbb{R}^n$, that is there exists a commutative diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\simeq} & \rho_{\mathrm{trop}}^{-1}(V) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \rho_{\mathrm{trop}}} \\ U & \xrightarrow{\simeq} & V \end{array}$$

such that the top horizontal map is an isomorphism of K-analytic spaces and the lower horizontal map is a homeomorphism.

An affinoid torus fibration $f : Z \to W$ induces an integral affine structure on W [KS06, §4.1, Theorem 1]. Then [KS06] formulates a conjecture which relates the Gromov-Hausdorff limit of a Calabi–Yau variety with non-archimedean geometry, and yields a description of the integral affine structure on B^{sm} in purely algebraic terms.

Conjecture 1.1.7.2. *[KS06, §5.2, Conjecture 3] Let X be a maximally degenerate Calabi–Yau variety over K of dimension n. There exists a map $\rho_{\mathrm{an}} : X^{\mathrm{an}} \to B$ such that*

(i) B^{sm} coincides with the set of points of B on which the map ρ_{an} is an n-dimensional affinoid torus fibration;

(ii) the integral affine structure on B^{sm} induced by the Gromov–Hausdorff limit coincides with the integral affine structure induced by the affinoid torus fibration.

Building on the suggestion that the Gromov–Hausdorff limit B should coincide with the essential skeleton, Nicaise, Xu and Yu describe the origin of an integral affine structure on the essential skeleton which is expected to match the properties of Conjecture 1.1.7.2.

Theorem 1.1.7.3. *[NXY18, Theorem 6.1] Let $\mathscr{X}$ be a minimal dlt model of X. The retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}^{\mathrm{ess}}(X)$ associated with $\mathscr{X}$ is an n-dimensional affinoid torus fibration away from a locus of codimension 2 in the base, which consists of the simplices in $\mathrm{Sk}(\mathscr{X})$ of codimension $\geqslant 2$.*

Although it does not solve the original SYZ conjecture, this outstanding result may be used for the ultimate aim of mirror symmetry, the construction of the mirror partner. In this perspective, the tools of differential and symplectic geometry are replaced by the non-archimedean SYZ fibration, and by non-archimedean enumerative computations and wall-crossing structures [KS06; KY18; Yu16a; Yu16b].

1.2. Questions

The relation of the essential skeleton to the geometry of dual complexes, the structure of minimal models and the conjectures in mirror symmetry make its structure deserving of further investigation. Natural and interesting questions arise regarding the properties of the essential skeletons and the following are particularly relevant to this thesis.

1. Can we explicitly compute the essential skeleton as a topological space? Which geometric features of X are reflected in and constrain $\mathrm{Sk}^{\mathrm{ess}}(X)$? In the setting of mirror symmetry, do such computations confirm the predictions and strengthen the conjectural connection between the Gromov–Hausdorff limit and the essential skeleton?

2. How is $\mathrm{Sk}^{\mathrm{ess}}(X)$ related to other algebro-geometric or metric constructions on X? For instance, does the formation of $\mathrm{Sk}^{\mathrm{ess}}(X)$ commute with natural operations in algebraic geometry?

3. Can we generalize the formation of the essential skeleton to the setting of pairs (X, D) or to varieties over other non-archimedean fields? Inspired by the case of the SYZ conjecture, are there other contexts where a non-archimedean interpretation might provide a new viewpoint on the subject?

Concerning 1, we mention that the essential skeleton has been computed in a few important cases.

- If X is a $K3$ surface over $K = \mathbb{C}((t))$ with semistable reduction, then, by Kulikov classification of degenerations of $K3$ surfaces [Kul77], $\mathrm{Sk}^{\mathrm{ess}}(X)$ is either a point, a closed interval or a sphere $\mathbb{S}^2$, depending only on the type of X, that is the nilpotency index of the logarithm of the monodromy operator on the second cohomology group.

- If X is an abelian variety, then the essential skeleton coincides with Berkovich's canonical skeleton for an abelian variety [HN17]; in particular, if X has semi-abelian reduction, then $\mathrm{Sk}^{\mathrm{ess}}(X)$ is homeomorphic to a real torus $(\mathbb{S}^1)^{t(X)}$, where $t(X)$ denotes the toric rank of the reduction of X.

- If X is a maximally degenerate strict Calabi–Yau variety of dimension n over K, that is X is geometrically simply connected and the Hodge numbers $h^{i,0}(X) = 0$ for $0 < i < n$, then $\mathrm{Sk}^{\mathrm{ess}}(X)$ is a closed pseudomanifold of dimension n with the rational homology of a sphere $\mathbb{S}^n$ [NX16]. In [KX16] it is proved that $\mathrm{Sk}^{\mathrm{ess}}(X)$ is homeomorphic to $\mathbb{S}^n$ when $n \leqslant 3$ and there are partial results when $n = 4$; in general, it is expected to be homeomorphic to $\mathbb{S}^n$.

- The case of hyperkähler manifolds is treated in detail in Section 1.3.

Our research aims for the answer to these questions and this manuscript is concerned with the positive outcomes. In the following sections, we report on the main results and on the techniques developed along the way.

1.3. Essential skeleton of hyperkähler varieties

In [BM19] in collaboration with Brown, we consider certain degenerations of hyperkähler varieties. By the analysis of the weight function and its locus of minimality, we compute the essential skeleton; this proves certain predictions from mirror symmetry and gives a positive answer to Question 1. These results are the subject of Chapter 4.

1.3.1. Degenerations of hyperkähler varieties

Hyperkähler varieties are very important in the birational classification of algebraic varieties. Indeed, by Beauville-Bogomolov's decomposition theorem, any Calabi–Yau manifold can be decomposed, up to a finite cover, into a product of irreducible factors of three types: abelian varieties, strict Calabi-Yau varieties, and hyperkähler varieties [Bea83]. We recall the definition.

Definition 1.3.1.1. A smooth projective variety is said to be hyperkähler if it is simply connected and the space of its global holomorphic 2-forms is spanned by a symplectic form. Such a variety has necessarily trivial canonical line bundle and has even dimension.

Example 1.3.1.2. Up to deformation, few examples are known. For any K3 surface S, the Hilbert scheme $S^{[n]}$ of n points on S is hyperkähler. Another family arises from generalizing the Kummer construction: let A be an abelian surface, then the Hilbert scheme $\mathrm{Hilb}^{n+1}(A)$ admits a multiplication map to A and the fibre $K_n(A)$ over the identity is a hyperkähler variety. Together with two examples due to O'Grady in dimension 6 and 10 [O'G03; O'G99], these four constructions give the only known deformation classes of hyperkähler varieties.

Families of $2n$-dimensional hyperkähler manifolds over a punctured disk Δ^*, and similarly hyperkähler varieties over $K = \mathbb{C}((t))$, are distinguished into three types (I, II and III), according to the nilpotency index of the monodromy operator on the second cohomology group. By [NX16, Theorem 4.1.10] and [KLSV17, Proposition 6.18], a type III degeneration is equivalent to a maximally degenerate family, which we recall means that the monodromy operator on the $2n$-th cohomology group has a Jordan block of size $2n+1$. We expect that the topology of the essential skeleton is determined by the type; a first evidence is provided by Kollár, Laza, Saccà and Voisin.

Theorem 1.3.1.3. *[KLSV17, Theorem 0.10] Let $\mathscr{X}$ be a minimal dlt degeneration of $2n$-dimensional hyperkähler manifolds over the disk Δ. Then the dual complex of the special fibre has dimension $(\nu - 1)n$, where ν denotes the type of the degeneration. Moreover, in the type II case the $\mathbb{Q}$-homology of the dual complex is trivial, and in the type III case the dual complex has the rational homology of $\mathbb{CP}^n$.*

1.3.2. Main result

The following result considers some degenerations of hyperkähler varieties and enhances the rational homological description of the essential skeleton of [KLSV17] into a topological characterization.

Theorem 1.3.2.1 (§ 4.3-4.4). *Let S be a $K3$ surface, and A an abelian surface over $\mathbb{C}((t))$, admitting snc degenerations with reduced special fibre. If X is isomorphic to the Hilbert scheme of n points on S or to the n-th generalised Kummer variety on A, then the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X)$ is homeomophic to a point, to the standard n-simplex or to $\mathbb{CP}^n$ according to the degeneration type of S and A.*

Considering the relation of the essential skeleton to the dual complex of maximally degenerate minimal dlt degenerations, and conjecturally to the base of the SYZ fibration, Theorem 1.3.2.1 provides evidence for two following "mirror" conjectures.

Conjecture 1.3.2.2. *Let $\mathscr{X}$ be a maximally degenerate minimal dlt degeneration of $2n$-dimensional hyperkähler manifolds over $\mathbb{C}((t))$. Then the dual complex of the special fibre of $\mathscr{X}$ is homeomorphic to $\mathbb{CP}^n$.*

Conjecture 1.3.2.3. *[Huy03] Let $X \to B$ be a connected Lagrangian fibration of a projective hyperkähler manifold of dimension 2n such that the base B is normal. Then B is isomorphic to $\mathbb{CP}^n$.*

Conjecture 1.3.2.3 holds when B is smooth by Hwang [Hwa08] and when X is of dimension 4 by Huybrechts–Xu and Bogomolov–Kurnosov [BK18; HX19]. Moreover, it holds for deformations of Hilbert schemes of $K3$ surfaces by Markman [Mar14] in combination with a result of Matsushita [Mat17], for deformations of generalised Kummer varieties by Yoshioka [Yos16] also in combination with [Mat17], and for deformations of O'Grady's example in dimension 6 by Mongardi–Rapagnetta (in progress).

1.4. Essential skeleton and the geometric P=W conjecture

In a joint work with Mauri and Stevenson [MMS18], we determine dual complexes of varieties arising in non-abelian Hodge theory, called character varieties, by means of the construction of an essential skeleton. As a result, this provides new evidence for a conjecture, called the geometric P=W conjecture, and the first application of Berkovich geometry in non-abelian Hodge theory. These results respond positively to Question 3, and we refer to Chapter 7 for an extended discussion.

1.4.1. The geometric P=W conjecture

The cornerstone of non-abelian Hodge theory is the Corlette–Simpson correspondence: for a reductive algebraic group G, this is a homeomorphism between the G-character variety, or Betti moduli space,

$$M_B \coloneqq \mathrm{Hom}(\pi_1(X), G) /\!\!/ G$$

of G-representations of the topological fundamental group of a smooth curve X over $\mathbb{C}$, and Hitchin's moduli space M_{Dol} of semistable principal Higgs G-bundles on X with vanishing Chern classes, also known as the Dolbeault moduli space. See for instance [Sim94] for further details and generalizations.

The spaces M_B and M_{Dol} are non-proper varieties, and the behaviour at infinity of the Corlette–Simpson correspondence is a topic of great interest in the literature [KNPS15]. More precisely, consider:

- a compactification $\overline{M}_B$ of M_B, resp. $\overline{M}_{\mathrm{Dol}}$ of M_{Dol};
- the boundary $\partial M_B \coloneqq \overline{M}_B \setminus M_B$, resp. $\partial M_{\mathrm{Dol}} \coloneqq \overline{M}_{\mathrm{Dol}} \setminus M_{\mathrm{Dol}}$;
- a neighborhood at infinity N_B of M_B (i.e. a tubular neighborhood of ∂M_B), resp. N_{Dol} of M_{Dol};
- a punctured neighborhood at infinity $N_B^* \coloneqq N_B \setminus \partial M_B$, resp. $N_{\mathrm{Dol}}^* \coloneqq N_{\mathrm{Dol}} \setminus \partial M_{\mathrm{Dol}}$.

Note that the Corlette–Simpson correspondence induces a homotopy equivalence $N_{\mathrm{Dol}}^* \sim N_B^*$. Hitchin's moduli space M_{Dol} comes equipped with the Hitchin map

$$H : M_{\mathrm{Dol}} \to \mathbb{C}^N,$$

with $2N = \dim_{\mathbb{C}}(M_{\mathrm{Dol}})$ by [Hit87, Equation 4.4], which induces a map from N_{Dol}^* to a neighborhood at infinity of $\mathbb{C}^N$. Composing with the radial projection to the sphere $\mathbb{S}^{2N-1}$, we obtain a map

$$h : N_{\mathrm{Dol}}^* \xrightarrow{H} \mathbb{C}^N \setminus \{0\} \xrightarrow{\sim} \mathbb{S}^{2N-1}.$$

Now, assume that the dual boundary complex $\mathcal{D}(\partial M_B)$ is well defined. By means of a partition of unity, one can define a map from N_B^* to the dual boundary complex $\mathcal{D}(\partial M_B)$, written

$$\alpha : N_B^* \to \mathcal{D}(\partial M_B).$$

If ∂M_B is an snc divisor, the homotopy type of $\mathcal{D}(\partial M_B)$ is independent of the choice of the snc compactification; see [Dan75] for a reference.

The geometric $P = W$ conjecture proposes a correspondence between the dual boundary complex of M_B and the sphere at infinity of the Hitchin base for M_{Dol}.

Conjecture 1.4.1.1 (Geometric $P = W$ conjecture). *There exists a homotopy equivalence*

$$\mathcal{D}(\partial M_B) \sim \mathbb{S}^{2N-1} \tag{1.4.1.2}$$

such that the following diagram is homotopy commutative

$$\begin{array}{ccc} N_{\mathrm{Dol}}^* & \xrightarrow{\sim} & N_B^* \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathbb{S}^{2N-1} & \xrightarrow{\sim} & \mathcal{D}(\partial M_B). \end{array} \tag{1.4.1.3}$$

The results in [Sim16] provide evidence for the conjecture: when M_B is the SL_2-character variety of local systems on a punctured sphere (such that conjugacy classes of the monodromies around the punctures are fixed), Simpson proves in [Sim16, Theorem 1.1] that the dual boundary complex $\mathcal{D}(\partial M_B)$ has the homotopy type of a sphere; see also [Kom15, Theorem 1.4]. However, there is no known proof of the commutativity of the diagram 1.4.1.3. In the same paper, Simpson suggests to study the case of character varieties associated to compact Riemann surfaces. In the sequel, we explain our contribution in the genus one case.

1.4.2. Singular compactifications of M_B

In order to address the geometric $P = W$ conjecture, one must make sense of the dual boundary complex $\mathcal{D}(\partial M_B)$ of M_B. It is not a priori clear how one can do so, since M_B can be singular, hence it may not admit an snc compactification. Thus, the task is to find mildly singular compactifications to which a dual complex may still be attached. Our solution is to consider dlt compactifications; see Definition 2.3.0.3 for details. Indeed, such compactifications have well defined dual complex, whose homotopy type is independent of the choice of a specific dlt compactification by [dFKX17]. Further, when the group G is either GL_n or SL_n, the existence of dlt compactifications follows from the existence of dlt blow-ups by Hacon ([Fuj11, Theorem 10.4], [KK10, Theorem 3.1]) and the fact that M_B has canonical and $\mathbb{Q}$-factorial singularities, as shown in [BS19, Theorems 1.20 and 1.21].

Among all possible dlt compactifications of M_B, it is convenient to restrict to special ones, more precisely to the dlt log Calabi–Yau (logCY) compactifications. This is an algebraic condition which rigidifies the configuration of divisors at infinity, and in practice it simplifies the description of the dual complex. The dual complex of any dlt logCY compactification identifies a distinguished homeomorphism class in the homotopy equivalence class of the dual complex of any dlt compactification. Moreover, it is expected that M_B actually admits a logCY compactification; see [Sim16]. These observations suggest the following strengthening of the homotopy equivalence in 1.4.1.2.

Conjecture 1.4.2.1. *The Betti moduli space M_B admits a dlt logCY compactification $(\overline{M}_B, \partial M_B)$ and the associated dual complex $\mathcal{D}(\partial M_B)$ is homeomorphic to a sphere.*

1.4.3. Main result

Let G be either GL_n or SL_n, and let M_G be the G-character variety of the fundamental group of a Riemann surface of genus one. In this case, M_G can be realized as the GIT quotient

$$\{(A, B) \in G \times G \colon AB = BA\} /\!\!/ G,$$

where G acts by conjugation on each factor. For example, when $G = \mathrm{GL}_n$, M_{GL_n} is isomorphic to the n-fold symmetric product $(\mathbb{C}^* \times \mathbb{C}^*)^{(n)}$ of the torus $\mathbb{C}^* \times \mathbb{C}^*$.

We reinterpret the dual complex $\mathcal{D}(\partial M_G)$ as the locus of minimal value of a suitable weight function inside a Berkovich space. By way of this non-archimedean description, we prove Conjecture 1.4.2.1 for M_G.

Theorem 1.4.3.1 (§ 7.2-7.3). *The dual boundary complex $\mathcal{D}(\partial M_G)$ of a dlt logCY compactification of M_G has the homeomorphism type of $\mathbb{S}^{2n-1}$ if $G = \mathrm{GL}_n$, and of $\mathbb{S}^{2n-3}$ if*

$G = \mathrm{SL}_n$.

1.5. Techniques

The results in Sections 1.3 and 1.4 display the essential skeleton as central object in connection with the birational geometry of degenerations, the topology of dual complexes of compactification, and the theory of Berkovich spaces. In this section, we report the main directions in which we develop the theory of Berkovich skeletons, weight functions and essential skeletons, in order to prove Theorems 1.3.2.1 and 1.4.3.1. Moreover, the outcomes of this section provide positive answer to Question 2.

Our contributions are treated in detail in Chapters 3, 5 and 6, building on the works by Boucksom, Favre, Jonsson, Mustață, Nicaise, Xu [BFJ16; JM12; MN15; NX16], Temkin [Tem16], Thuiller [Thu07], Gubler, Rabinoff and Werner [GRW16].

1.5.1. Generalization of weight functions and essential skeleton

Our aim is to enlarge the set of non-archimedean tools (Berkovich skeletons, weight functions and essential skeletons) to varieties over $\mathbb{C}$ and to pairs (X, Δ), where X is a normal proper variety and $\Delta = \sum \alpha_i \Delta_i$ is an effective $\mathbb{Q}$-divisor. Indeed, such settings arise frequently in the minimal model program and in the study of compactifications of algebraic varieties.

Let (X, Δ_X) be a pair over a non-archimedean field $\mathcal{K}$ which is trivially-valued, for instance the complex numbers $\mathbb{C}$ equipped with the trivial norm, or discretely-valued with valuation ring R and residue characteristic zero, like $\mathbb{C}((t))$ with the t-adic valuation. In the discretely-valued case, a model $(\mathscr{X}, \Delta_{\mathscr{X}})$ for (X, Δ_X) is the datum of a model $\mathscr{X}$ of X over R such that the closure of any component of Δ_X in $\mathscr{X}$ has non-empty intersection with $\mathscr{X}_0$ and $\Delta_{\mathscr{X}} = \overline{\Delta_X} + \mathscr{X}_0$; in the trivially-valued setting a model of (X, Δ_X) is just the pair (X, Δ_X) itself. If $\mathscr{X}$ is regular and the support of $\Delta_{\mathscr{X}}$ is snc (or more generally if the pair $(\mathscr{X}, \Delta_{\mathscr{X}})$ log-regular, as we will see in Section 1.5.2), one constructs a skeleton $\mathrm{Sk}(\mathscr{X}, \Delta_{\mathscr{X}}) \subseteq X^{\mathrm{an}}$ [BM19; GRW16; Thu07].

For discretely-valued fields, this corresponds geometrically to adding horizontal divisors to the special fibre and yields the addition of some unbounded faces to the dual complex $\mathcal{D}(\mathscr{X}_0)$. In the trivially-valued case, the skeleton $\mathrm{Sk}(X, \Delta_X)$ has the structure of a cone complex, with the vertex corresponding to the trivial valuation; indeed, the valuations given by the order of vanishing along irreducible components of Δ_X can be rescaled by a positive real number. This induces an action of $\mathbb{R}_{\geqslant 0}$ on X^{an}, such that the quotient $\mathrm{Sk}(X, \Delta_X)^*/\mathbb{R}_{\geqslant 0}$, also called link, is homeomorphic to the dual complex $\mathcal{D}(\Delta_X)$; here the punctured skeleton $\mathrm{Sk}(X, \Delta_X)^*$ is obtained by removing the vertex of the cone.

Building on [BFJ16; JM12; MN15], we extend the definition and the properties of the weight functions to the trivially-valued setting and to pluricanonical forms with poles along Δ_X. The weight function at a divisorial valuation is a log discrepancy of the divisor, a standard numerical invariant which arises in birational geometry; see [Kol13, Definition 2.4]. We state below one of the main results on weight functions, we refer to Sections 3.3, 5.2 and 5.3 for further details, and to Examples 3.3.1.7 and 3.3.1.8 for some explicit examples.

Proposition 1.5.1.1. *Let (X, Δ_X) be an snc regular pair and let ω be a non-zero regular section of $\mathcal{O}_X(m(K_X + \Delta_X))$.*

(i) (§ 5.2.3-5.2.4) There is a unique maximal lower-semicontinuous extension $\mathrm{wt}_\omega \colon X^{\mathrm{an}} \to \mathbb{R} \cup \{+\infty\}$ of the weight function $\mathrm{wt}_\omega \colon X^{\mathrm{div}} \to \mathbb{R}$ defined on divisorial valuations.

(ii) (Propositions 3.3.1.6 and 5.3.7.4) For every point x of X^{an} and every snc model $(\mathscr{X}, \Delta_{\mathscr{X}})$ of (X, Δ_X), we have $\mathrm{wt}_\omega(x) \geqslant \mathrm{wt}_\omega(\rho_{\mathscr{X}}(x))$, where $\rho_{\mathscr{X}}$ denotes the retraction of X^{an} to the skeleton $\mathrm{Sk}(\mathscr{X}, \Delta_{\mathscr{X}})$.

Definition 1.5.1.2. The essential skeleton of (X, Δ_X) is

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) \coloneqq \bigcup_{\omega} \overline{\{x \in X^{\mathrm{div}} | \ \mathrm{wt}_\omega(x) \text{ is minimal } \}} \subseteq X^{\mathrm{bir}}$$

where ω runs through the set of non-zero regular sections of $\mathcal{O}_X(m(K_X+\Delta_X))$ and $m \geqslant 0$.

Finally, we establish a compatibility result between the weight functions in the trivially-valued and discretely-valued settings. Given a 1-parameter family of complex varieties, we compare the essential skeleton of the total space as scheme over $\mathbb{C}$, with the essential skeleton of the generic fibre as variety over $\mathbb{C}((t))$. Loosely speaking, in Proposition 5.3.8.9 we prove that the essential skeleton in the former setting is a cone over the essential skeleton in the latter.

1.5.2. Logarithmic version of Berkovich skeletons

The heuristic of the main results mentioned in Sections 1.3 and 1.4 boils down to the characterization and computation of the essential skeleton of products of varieties, and of finite quotients. The first step is showing that the formation of the essential skeleton commutes with the product (Z, Δ_Z) of two pairs (X, Δ_X) and (Y, Δ_Y). As the essential skeleton is contained in any Berkovich skeleton, we want to produce nice models $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ for the product such that the locus of minimality of the weight functions in $\mathrm{Sk}(\mathscr{Z}, \Delta_{\mathscr{Z}})$ can be related to the essential skeletons of the two factors:

$$\begin{array}{ccccc} \mathrm{Sk}(\mathscr{Z}, \Delta_{\mathscr{Z}}) & & \mathrm{Sk}(\mathscr{X}, \Delta_{\mathscr{X}}) & \times & \mathrm{Sk}(\mathscr{Y}, \Delta_{\mathscr{Y}}) \\ \cup\!| & & \cup\!| & & \cup\!| \\ \mathrm{Sk}^{\mathrm{ess}}(Z, \Delta_Z) & \overset{?}{\longleftrightarrow} & \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) & \times & \mathrm{Sk}^{\mathrm{ess}}(Y, \Delta_Y). \end{array}$$

To this purpose, we work in the context of logarithmic geometry. In a nutshell, log geometry is a generalization of the theory of schemes, where each log scheme is a scheme equipped with an additional structure, called log structure. For instance, log structures can be associated to compactifications or degenerations of the scheme, and they provide a suitable framework to incorporate the theories of toroidal embeddings and of differential forms with logarithmic poles [KKMSD73].

We consider log-regular log schemes: these are a generalization of toroidal embeddings and of snc pairs, their log structure is induced by a divisor and they have an associated combinatorial structure called the Kato fan [Kat94]; see Section 2.5 for more details. In Sections 3.2 and 5.3, we define a logarithmic version of the Berkovich skeleton for a log-regular model $\mathscr{X}^+ = (\mathscr{X}, D_{\mathscr{X}})$ of (X, Δ_X), where the log structure $D_{\mathscr{X}}$ is induced by $\overline{\Delta_X} + \mathscr{X}_0$ in the discretely-valued setting, and by Δ_X in the trivially-valued case. This construction gives rise to a polyhedral complex in X^{an} whose faces correspond to the points of the Kato fan of $\mathscr{X}^+$.

Given two log-regular models $\mathscr{X}^+$ and $\mathscr{Y}^+$, their product $\mathscr{Z}^+ = (\mathscr{Z}, D_{\mathscr{Z}})$ (in the category of fine and saturated log schemes) is a log-regular model of $(Z = X \times Y, \Delta_Z = \Delta_X \times Y + X \times \Delta_Y)$, hence $\mathscr{Z}^+$ has an associated skeleton. By functoriality, the projection maps $(\mathrm{pr}_X, \mathrm{pr}_Y)\colon Z \to X \times Y$ induce a map of skeletons

$$(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X}^+)}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y}^+)}) : \mathrm{Sk}(\mathscr{Z}^+) \to \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+).$$

Proposition 1.5.2.1 (Propositions 3.2.5.3 and 5.3.6.6). *The map* $(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X}^+)}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y}^+)})$ *is a homeomorphism, assuming in the discretely-valued field case that the residue field is algebraically closed and that* $\mathscr{X}$ *is semistable, that is the special fibre* $\mathscr{X}_0$ *is reduced.*

Working with log-regular models is convenient since log-regularity is preserved under taking products while snc models are not; see Example 3.2.5.5 for evidence. Moreover, the assumption of reducedness of the special fibre is crucial in Proposition 1.5.2.1 in the discretely-valued case: we illustrate a counterexample in Example 3.2.5.4.

Given the homeomorphism between a skeleton of the product and the product of skeletons, it is straightforward to compute the weight function on $\mathrm{Sk}(\mathscr{Z}^+)$ and relate the essential skeletons, just using the properties of the skeletons and of weight functions; see Theorem 3.3.2.9.

Along the same lines, we treat the case of the finite quotient X/G of a variety of dimension $\geqslant 2$. Indeed, starting with a model $\mathscr{Y}$ for the quotient, we construct suitable models $\mathscr{X}$ for X and compute the weight functions of the fibres in $\mathrm{Sk}(\mathscr{X})$ over $\mathrm{Sk}(\mathscr{Y})$. This allows to express the essential skeleton on X/G in terms of the quotient of the essential skeleton of X by G; see Proposition 4.2.0.9 and Lemma 7.2.4.11.

1.5.3. Comparison with Temkin's metrics

We adopt the formalism of metrics on line bundles of Berkovich spaces in order to give a complementary perspective on the theory of weight functions.

Given a pair (X, Δ_X), each weight function is associated to a pluricanonical form on X with prescribed poles along the divisor Δ_X. One can view the weight functions as metrics on the pluricanonical sheaves of the Berkovich analytification of X: this theory is developed by Temkin in [Tem16], where he proves ([Tem16, Theorem 8.3.3]) that the weight function of [MN15] in the discretely-valued setting coincides with an intrinsic metric on $(\omega_{X/K}^{[m]})^{\mathrm{an}}$, called Temkin's metric. Thus, Temkin's metrics can be used to define the essential skeleton of Berkovich spaces more in general (for quasi-smooth analytic space over any field); this approach is adopted in [HN11, Proposition 4.3.2] and [KY18]. We extend Temkin's comparison theorem to the trivially-valued case.

Theorem 1.5.3.1 (Theorem 6.5.0.5). *If X is a smooth proper variety over a trivially-valued field K of characteristic zero, then Temkin's metric on $(\omega_{X/K}^{[m]})^{\mathrm{an}}$ coincides with the weight metric.*

This result has two benefits. On one hand, it provides evidence for the appropriateness of our construction of the weight function in the trivially-valued case, and unifies the definitions of the weight function in the discretely and trivially-valued settings. On the other, it gives a computable expression for Temkin's metric in the trivially-valued case.

2

Preliminaries

This first chapter recalls some definitions and develops the necessary background for the rest of the thesis. In particular, we introduce the notions of

1. Berkovich analytification (2.1.1.5), monomial and divisorial valuations (2.1.3),
2. dual complex (2.2.0.1),
3. log discrepancy (2.3.0.2) and dlt singularities (2.3.0.3),
4. log scheme (2.4.1.5) and log-regularity (2.4.2.1),
5. Kato fans and their main properties (2.5).

2.1. Reminders on Berkovich spaces

Introduced in the late 1980s, Berkovich's theory of analytic spaces associates to each variety a space of valuations, called the Berkovich analytification, with nice topological properties, which reflect and illuminate the geometry of the underlying algebraic variety. We recall the definition and the main constructions we will need in the sequel.

2.1.1. Varieties and Berkovich analytification

Definition 2.1.1.1. A variety is an integral separated scheme of finite type over a field. A *pair* (resp. a *sub-pair*) (X, Δ) is the datum of a normal variety X and an effective divisor $\Delta = \sum \alpha_i \Delta_i$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and the coefficients $\alpha_i \in (0, 1]$ (resp. in $(-\infty, 1]$). In particular, Δ is called a *boundary* (resp. *sub-boundary*).

(2.1.1.2) The irreducible components of the intersection $\Delta_{i_1} \cap \ldots \cap \Delta_{i_r}$ of r components of Δ are called *strata* of codimension r. We write $\Delta^{=1} \coloneqq \sum_{i\,:\,\alpha_i=1} \Delta_i$.

We say that the pair (X, Δ) is *log Calabi–Yau* (logCY) if $K_X + \Delta$ is $\mathbb{Q}$-linearly equivalent to zero, written $K_X + \Delta \sim_{\mathbb{Q}} 0$.

We say that (X, Δ) is *simple normal crossing* (snc) if X is a regular variety and the support of Δ has simple normal crossings; see [Kol13, Definition 1.7].

Definition 2.1.1.3. A *non-archimedean field* is a field $\mathcal{K}$ equipped with a complete multiplicative norm $|\cdot|\colon \mathcal{K} \to \mathbb{R}_{\geqslant 0}$ that satisfies the ultrametric inequality, i.e. $|a+b| \leqslant \max\{|a|, |b|\}$. Equivalently, we can work with the associated valuation $v = -\log(|\cdot|) : \mathcal{K}^\times \to \mathbb{R}$, where $v(a+b) \geqslant \min\{v(a), v(b)\}$. We denote by $\mathcal{K}^\circ \coloneqq \{|\cdot| \leqslant 1\}$ the valuation ring of $\mathcal{K}$, by $\mathcal{K}^{\circ\circ} \coloneqq \{|\cdot| < 1\}$ its maximal ideal, and by $\widetilde{\mathcal{K}} \coloneqq \mathcal{K}^\circ/\mathcal{K}^{\circ\circ}$ the residue field.

(2.1.1.4) We say that $\mathcal{K}$ is *trivially-valued* if $|\mathcal{K}^\times| = \{1\}$; we denote this norm by $|\cdot|_0$ and equivalently the trivial valuation by v_0. Every field can be equipped with the trivial norm; in particular, this allows to think of the field of complex numbers $\mathbb{C}$ as a non-archimedean field.

We say that $\mathcal{K}$ is *discretely-valued* if there is $r \in (0,1)$ such that $|\mathcal{K}^\times| = r^{\mathbb{Z}}$. In this case, the tuple $(\mathcal{K}, \mathcal{K}^\circ, \mathcal{K}^{\circ\circ}, \widetilde{\mathcal{K}})$ is usually denoted by $(K, R, \mathfrak{m}, k)$. We write $S = \operatorname{Spec} R$ and we denote by s the closed point of S. Let $\mathscr{X}$ be an R-scheme of finite type; we denote by $\mathscr{X}_k$ the special fibre of $\mathscr{X}$ and by $\mathscr{X}_K$ the generic fibre. We will always assume that the residue field k is algebraically closed; for instance, we consider the case where $K = \mathbb{C}((t))$ is endowed with the t-adic norm, so the valuation ring is $\mathbb{C}[[t]]$ and the residue field is $\mathbb{C}$.

Definition 2.1.1.5. Given a variety X over a non-archimedean field $\mathcal{K}$, we define the *Berkovich analytification* X^{an} of X in the sense of [Ber90; Ber93] as follows. As a set, X^{an} consists of pairs $x = (\xi_x, |\cdot|_x)$, where $\xi_x \in X$ is a scheme-theoretic point of X, and $|\cdot|_x$ is an absolute value on the residue field $\mathcal{K}(\xi_x)$ of X at ξ_x that extends $|\cdot|$ on $\mathcal{K}$. Equivalently, we will often work with the associated valuation $v_x = -\log(|\cdot|_x)$ on $\mathcal{K}(\xi_x)^\times$. The completed residue field $\mathcal{H}(x)$ is the completion of $\mathcal{K}(\xi_x)$ with respect to $|\cdot|_x$.

As a topological space, the analytification X^{an} is equipped with the weakest topology such that the forgetful map

$$\ker\colon X^{\mathrm{an}} \to X, \quad x = (\xi_x, |\cdot|_x) \mapsto \ker(x) \coloneqq \xi_x,$$

is continuous, and, for any Zariski open $U \subseteq X$ and any regular function f on U, the map

$$\ker^{-1}(U) \ni x \mapsto |f(x)| \coloneqq |f(\xi_x)|_x$$

is continuous.

(2.1.1.6) We denote by $X^{\mathrm{bir}} \subseteq X^{\mathrm{an}}$ the subset of *birational points*, i.e. the ker-preimage of the generic point of X. Equivalently X^{bir} is the subspace of valuations on the function field of X extending the given valuation on $\mathcal{K}$.

Further, X^{an} may be equipped with a sheaf of analytic functions that makes $\ker\colon X^{\mathrm{an}} \to X$ into a morphism of locally $\mathcal{K}$-ringed spaces. Given a coherent sheaf $\mathcal{F}$ on X, the pullback $\mathcal{F}^{\mathrm{an}} := \ker^*(\mathcal{F})$ is a coherent sheaf on X^{an}, called the *analytification* of $\mathcal{F}$.

Definition 2.1.1.7. Suppose $\mathcal{K}$ is trivially-valued. If $X = \operatorname{Spec}(A)$ is affine, we define the $\beth$-*analytification* $X^\beth$ of X as the subspace of X^{an}

$$X^\beth = \{x \in X^{\mathrm{an}} \colon |f(x)| \leqslant 1 \text{ for all } f \in A\}.$$

For a valuation $x \in X^\beth$, the prime ideal corresponding to $\{f \in A \colon |f(x)| < 1\}$ defines the centre of x in X, which is denoted by $c_X(x)$.

For a $\mathcal{K}$-variety X, the $\beth$-*analytification* $X^\beth$ of X in the sense of [Thu07] is the compact analytic domain of X^{an} consisting of valuations that admit a centre on X (so $X^\beth = X^{\mathrm{an}}$ when X is proper).

We observe that there is a $\mathbb{R}_+$-action on $X^\beth$: for $a \in \mathbb{R}_+$ and $x \in X^\beth$, the point $a \cdot x \in X^\beth$ given by $|f(a \cdot x)| := |f(x)|^a$ for $f \in \mathcal{K}(x)$. In terms of valuations, the action is $v_{a \cdot x} = a \cdot v_x$. Moreover, if $a > 0$, then $c_X(a \cdot x) = c_X(x)$.

2.1.2. Models

Let $\mathcal{K} = K$ be discretely-valued.

Definition 2.1.2.1. Given a variety X over K, a *model* for X over R (or more classically, a *degeneration* of X) is a normal, flat, separated scheme $\mathscr{X}$ of finite type over R, endowed with an isomorphism of K-schemes $\mathscr{X}_K \simeq X$. The model is called *semistable* if the special fibre is reduced.

Further, $\mathscr{X}$ is an *snc model* of X if it is regular, and the special fibre $\mathscr{X}_0$ is an snc divisor on $\mathscr{X}$. We note that, when K is of residue characteristic zero, snc models always exist by Hironaka's theorem on resolution of singularities.

We say that $(\mathscr{X}, \mathscr{D})$ is a model for the pair (X, D) over R if $\mathscr{X}$ is a model for X over R, $\mathscr{D} = \overline{D} + \mathscr{X}_{0,\mathrm{red}}$, and $K_{\mathscr{X}} + \mathscr{D}_{\mathrm{red}}$ is $\mathbb{Q}$-Cartier when $K_X + D_{\mathrm{red}}$ is so.

Definition 2.1.2.2. Let X be a K-variety and $\mathscr{X}$ be a model of X over R. The (analytic) generic fibre $\widehat{\mathscr{X}_\eta}$ is a compact analytic domain of X^{an} constructed in [Ber96, §1]. It consists of the points $x \in X^{\mathrm{an}}$ such that the K-morphism $\operatorname{Spec}(\mathcal{H}(x)) \to X$ extends to a R-morphism $\operatorname{Spec}(\mathcal{H}(x)^\circ) \to \mathscr{X}$. If this occurs, the image of the closed point via $\operatorname{Spec}(\mathcal{H}(x)^\circ) \to \mathscr{X}$ is the *centre* (or *reduction*) of x on $\mathscr{X}$, and we say that x admits a centre on $\mathscr{X}$. The map

$$\mathrm{red}_{\mathscr{X}} : \widehat{\mathscr{X}_\eta} \to \mathscr{X}_0,$$

which sends $x \in \widehat{\mathscr{X}_\eta}$ to its centre on $\mathscr{X}$, is anti-continuous and it is called the *reduction map* with respect to $\mathscr{X}$. When $\mathscr{X} = \operatorname{Spec}(\mathcal{A})$ is affine, then

$$\widehat{\mathscr{X}_\eta} = \{x \in X^{\mathrm{an}} \colon |f(x)| \leqslant 1 \text{ for all } f \in \mathcal{A}\}.$$

2.1.3. Divisorial and monomial valuations

For $x \in X^{\mathrm{an}}$, set $s(x) \coloneqq \mathrm{tr.deg}(\widetilde{\mathcal{H}(x)}/\widetilde{\mathcal{K}})$ and $t(x) \coloneqq \dim_{\mathbb{Q}}(\sqrt{|\mathcal{H}(x)^{\times}|}/\sqrt{|\mathcal{K}^{\times}|})$. Assume X is normal.

Definition 2.1.3.1. A point $x \in X^{\mathrm{an}}$ is *divisorial* if the following condition holds.

- When $\mathcal{K}$ is trivially-valued, if $s(x) = \dim(X) - 1$ and $t(x) = 1$. If $x \in X^{\beth}$, then [ZS60, VI, §14, Theorem 31] shows that this is equivalent to the following geometric criterion: there exists a constant $c > 0$, a proper birational morphism $h\colon Y \to X$ from a normal $\mathcal{K}$-variety Y, and a prime divisor $E \subseteq Y$ such that

  $$|f(x)| = e^{-c\,\mathrm{ord}_E(h^* f)}$$

  for $f \in \mathcal{K}(X)$; in this case, we say that the triple $(c, Y \xrightarrow{h} X, E)$ determines x or is a divisorial representation of x.

- When $\mathcal{K}$ is discretely-valued, if $s(x) = \dim(X)$ and $t(x) = 0$. If $x \in X^{\mathrm{an}}$ admits a centre on some model of X, then there is a corresponding geometric criterion: there exists a model $\mathscr{X}$ of X and an irreducible component $E \subseteq \mathscr{X}_0$ such that

  $$|f(x)| = |\varpi|^{\mathrm{ord}_E(f)/\,\mathrm{ord}_E(\varpi)}$$

  for $f \in \mathcal{K}(X)$, where ϖ is a uniformizer of $\mathcal{K}$; in this case, we say that the pair $(\mathscr{X}, E)$ determines x or is a divisorial representation of x.

We write $X^{\mathrm{div}} \subseteq X^{\mathrm{an}}$ for the subset of divisorial points. If $\mathrm{char}(\widetilde{\mathcal{K}}) = 0$, then X^{div} is dense; see e.g. [JM12, Remark 4.11] in the trivially-valued setting and [MN15, Proposition 2.4.9] in the discretely-valued case. Divisorial points belong to a larger class of valuations in X^{an}, defined as follows.

Definition 2.1.3.2. A point $x \in X^{\mathrm{an}}$ is *(quasi-)monomial* if the following condition holds.

- When $\mathcal{K}$ is trivially-valued, a point $x \in X^{\beth}$ is *quasi-monomial* if there exists a proper birational morphism $h\colon Y \to X$ from a normal $\mathcal{K}$-variety Y, a regular system of parameters $(y_1, \ldots, y_r)$ at a regular point ξ of Y, and an r-tuple $(\alpha_1, \ldots, \alpha_r) \in \mathbb{R}_+^r$ such that x is the unique minimal real valuation with $v_x(y_i) = \alpha_i$; see [JM12, Proposition 3.1].

- When $\mathcal{K}$ is discretely-valued, a point $x \in X^{\mathrm{an}}$ is *monomial* if there exists an snc model $\mathscr{X}$ of X, an r-tuple $(E_1, \ldots, E_r)$ of distinct irreducible components of $\mathscr{X}_0$, local equations y_i for E_i at a generic point ξ of $E_1 \cap \ldots \cap E_r$ for each $i \in \{1, \ldots, r\}$, and an r-tuple $(\alpha_1, \ldots, \alpha_r) \in \mathbb{R}_+^r$ where $\sum_{i=1}^r \alpha_i \,\mathrm{ord}_{E_i}(\varpi) = 1$ and ϖ is a uniformizer of $\mathcal{K}$, such that x is the unique minimal real valuation with $v_x(y_i) = \alpha_i$; see [MN15, Proposition 2.4.4].

Such tuples $(Y \xrightarrow{h} X, (y_1, \dots, y_r), (\alpha_1, \dots, \alpha_r), \xi)$ and $(\mathscr{X}, (E_1, \dots, E_r), (\alpha_1, \dots, \alpha_r), \xi)$ determine the point x and are called monomial representations of x. We denote by X^{mon} the set of (quasi-)monomial points, and we note that $X^{\mathrm{div}} \subseteq X^{\mathrm{mon}} \subseteq X^{\mathrm{bir}} \subseteq X^{\mathrm{an}}$.

(2.1.3.3) Let X be an integral variety and let $\mathfrak{a}$ be a coherent sheaf of fractional ideals on X. If v is a valuation on a residue field of X that admits a centre ζ on X, in the trivially-valued case, or on a model $\mathscr{X}$, in the discretely-valued case, then we set

$$v(\mathfrak{a}) := \min_{f \in \mathfrak{a}_\zeta} v(f).$$

For a $\mathbb{Q}$-Cartier divisor D on X, set $v(D) := \frac{1}{m} v(\mathcal{O}_X(-mD))$, where $m \in \mathbb{Z}_{>0}$ is such that mD is Cartier.

2.2. Reminders on simplicial complexes

Definition 2.2.0.1. The *dual (intersection) complex* of a (pure-dimensional) simple normal crossing variety D is the cell complex $\mathcal{D}(D)$ whose vertices are in correspondence with the irreducible components D_i of D, and whose r-dimensional cells correspond to strata of codimension $r+1$. The attaching maps are prescribed by the inclusion relation. See [dFKX17, Definition 12] for further details.

Definition 2.2.0.2. Given a topological space A, the *cone* over A, denoted $\mathrm{Cone}(A)$, is the quotient of $A \times \mathbb{R}_+$ under the identification $(a_1, 0) \sim (a_2, 0)$ for any a_1, $a_2 \in A$. The vertex of $\mathrm{Cone}(A)$ is the image of $A \times \{0\}$ under the quotient map. The group $\mathbb{R}_+^*$ acts by rescaling on the second factor and descends to $\mathrm{Cone}(A)$. If C is a topological space homeomorphic to $\mathrm{Cone}(A)$, then A is homeomorphic to the quotient of the punctured cone $C^* := C \setminus \{\text{vertex}\}$ by the $\mathbb{R}_+^*$-action.

Definition 2.2.0.3. Given two topological spaces A and B, the *join* of A and B, denoted $A * B$, is the quotient of the space $A \times B \times I$, where $I = [0, 1]$, under the identifications

$$\begin{aligned} (a, b_1, 0) &\sim (a, b_2, 0) \quad \forall a \in A, b_1, b_2 \in B, \\ (a_1, b, 1) &\sim (a_2, b, 1) \quad \forall a_1, a_2 \in A, b \in B. \end{aligned}$$

In other words, the join is the space of all segments joining points in A and B, with two segments meeting only at common endpoints.

(2.2.0.4) The homeomorphism $A \times B \times I \times \mathbb{R}_+ \to A \times \mathbb{R}_+ \times B \times \mathbb{R}_+$ given by $(a, b, t, r) \mapsto (a, r(1-t), b, rt)$ descends to a $\mathbb{R}_+^*$-equivariant homeomorphism

$$\mathrm{Cone}(A * B) \simeq \mathrm{Cone}(A) \times \mathrm{Cone}(B), \tag{2.2.0.5}$$

where the cones are endowed with the $\mathbb{R}_+^*$-action defined in Definition 2.2.0.2, and the product has the diagonal action.

2.3. Reminders on birational geometry

When working in the birational equivalence class of a variety, the aim is to reduce the variety to some building blocks with simpler geometry: such a procedure is called a minimal model program (MMP). Performing some birational modifications, we contract divisors or subvarieties of codimension bigger than 1, and we possibly end up with a more singular variety than the original one. The singularities that can occur are classified and a class which is particularly relevant in our context is the class of divisorially log terminal (dlt) singularities. Indeed, these appear as the end product of those minimal model programs whose inputs are snc pairs [KM08, Corollary 3.44].

Definition 2.3.0.1. We say that a pair (X, Δ_X) is *log canonical (lc)* if, for every log resolution $f\colon Z \to X$, in the formula

$$K_Z + \Delta_Z = f^*(K_X + \Delta_X) + \sum a_D D,$$

where Δ_Z is the round up of the strict transform of Δ_X plus the reduced exceptional divisor, all the a_D are non-negative.

(2.3.0.2) In fact the quantity a_D, called the *log discrepancy of* D with respect to (X, Δ_X), depends only on the valuation corresponding to D, and this condition needs only be tested on a single log resolution. A closed subset $Y \subset X$ is called a *log canonical centre* if, for some (equivalently any) log resolution, Y is the image of a divisor D with $a_D = 0$.

Definition 2.3.0.3. We say that a pair (X, Δ_X) is *divisorially log terminal (dlt)* if it is lc and every log canonical centre Y has non-empty intersection with the snc locus X^{snc} of X, which is the largest open subset in X such that the pair (X, Δ_X) restricts to an snc pair.

Example 2.3.0.4. Let X be the cone over a rational normal curve of degree d. Consider the blow-up $\pi : Y \to X$ of the vertex of the cone. Then π is a log resolution and the exceptional divisor E is isomorphic to $\mathbb{P}^1$ with $E^2 = -d$. We write

$$K_Y + E = \pi^* K_X + a_E E,$$

and we compute that $-2 = \deg K_{\mathbb{P}^1} = \deg K_E = (K_Y + E) \cdot E = \pi^* K_X \cdot E + a_E E^2 = -a_E d$. Thus, $a_E = 2/d$ and we conclude that the pair $(X, \emptyset)$ is dlt; in fact it satisfies even a stronger condition, namely that all log discrepancies are strictly positive.

Example 2.3.0.5. Let X be the cone over a cubic curve C. Consider the blow-up $\pi : Y \to X$ of the vertex v of the cone. Then π is a log resolution and the exceptional divisor E is isomorphic to C. We write $K_Y + E = \pi^* K_X + a_E E$, and we compute that $0 = \deg K_C = \deg K_E = (K_Y + E) \cdot E = \pi^* K_X \cdot E + a_E E^2 = a_E E^2$. Thus, $a_E = 0$ and the image $\pi(E) = v$ is not contained in the snc locus of X. We conclude that the pair $(X, \emptyset)$ is lc but not dlt.

Definition 2.3.0.6. Let $(X, \Delta_X = \sum \alpha_i \Delta_i)$ be a dlt pair, and let $\Delta_X^{=1}$ be the set of irreducible components of Δ_X whose coefficient α_i is equal to one. The dual complex of (X, Δ_X) is the dual complex of $\Delta_X^{=1}$, i.e.

$$\mathcal{D}(X, \Delta_X) \coloneqq \mathcal{D}(\Delta_X^{=1}).$$

Remark 2.3.0.7. By [dFKX17, §2], the definition of the dual complex of a dlt pair is well-posed as the dual complex $\mathcal{D}(\Delta_X^{=1})$ coincides with the dual complex $\mathcal{D}(\Delta_X^{=1}|_{X^{\text{snc}}})$ of the restriction of $\Delta_X^{=1}$ to the snc locus X^{snc}. Indeed, the strata of $\Delta_X^{=1}$ are lc centres, hence they are contained in X^{snc} by definition of dlt.

2.4. Reminders on logarithmic geometry

Logarithmic geometry is a geometric theory which was founded by Fontaine and Illusie, then developed by Kato [Kat89]. It generalizes the theory of schemes by introducing the new notion of *log structure*. A log structure is an additional structure on a scheme that can encode geometric information of compactifications or degenerations of the scheme. In particular, it provides a suitable framework to incorporate the theories of toroidal embeddings and of differential forms with logarithmic poles [KKMSD73]. For a more extended dissertation of the topic we refer to [GR04; Kat89]. We will briefly review the basic definitions and focus on log-regular log schemes [Kat94].

2.4.1. Log structure

The algebra of monoids plays a central role in the theory of log schemes. We recall the main definitions.

Definition 2.4.1.1. A *monoid* is a commutative semi-group with a neutral element. A morphism of monoids is a map that respects internal laws and neutral elements. Given a monoid M, we denote by M^{gp} the groupification of M where

$$M^{\text{gp}} = M \times M/\sim, \quad (a,b) \sim (c,d) \Leftrightarrow \exists m \in M, m+a+d = m+b+c.$$

For example, the groupification of the monoid $\mathbb{N}$ is $\mathbb{Z}$.

Definition 2.4.1.2. A *submonoid* N of a monoid M is a subset of M such that the neutral element belongs to N, and N is closed under the monoid operation, that is for any $n_1, n_2 \in N$ one has $n_1 + n_2 \in N$.

For example, $\mathbb{N} \oplus \{0\} \subseteq \mathbb{N} \oplus \mathbb{N}$ and $\{0, 2, 3, 4, 5, \dots\} \subseteq \mathbb{N}$ are submonoids.

Definition 2.4.1.3. An *ideal* P of a monoid M is a subset of M such that for any $p \in P$ and $m \in M$ one has $p + m \in P$. A *prime ideal* P of M is an ideal such that the neutral element does not belong to P, and if $p_1 + p_2 \in P$ then $p_1 \in P$ or $p_2 \in P$.

For example, $P_n \coloneqq \{n' \in \mathbb{N} | n' \geqslant n\}$ is an ideal of $\mathbb{N}$, but is prime only for $n = 1$.

Definition 2.4.1.4. A monoid M is called *integral* if the morphism $M \to M^{\mathrm{gp}}$ is injective; it is called *fine* if M is integral and finitely generated; it is called *saturated* if it is integral and, whenever $m \in M^{\mathrm{gp}}$ and $nm \in M$ for some positive integer n, then $m \in M$. We use the abbreviation fs for fine and saturated monoids.

Definition 2.4.1.5. Given a scheme $\mathcal{X}$, a *log structure* on $\mathcal{X}$ consists of a sheaf of monoids $\mathcal{M}_{\mathcal{X}}$ with respect to the Zariski topology, together with a morphism of sheaves of monoids $\alpha : \mathcal{M}_{\mathcal{X}} \to (\mathcal{O}_{\mathcal{X}}, \times)$ such that $\alpha^{-1}(\mathcal{O}_{\mathcal{X}}^{\times}) \simeq \mathcal{O}_{\mathcal{X}}^{\times}$. We call and denote by

$\mathcal{X}^+ = (\mathcal{X}, \mathcal{M}_{\mathcal{X}})$ the logarithmic scheme defined by the pair $(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$,

$\mathcal{C}_{\mathcal{X},x} = \mathcal{M}_{\mathcal{X},x}/\mathcal{O}_{\mathcal{X},x}^{\times}$ the characteristic monoid of $\mathcal{X}^+$ at the point $x \in \mathcal{X}$,

$\mathcal{I}_{\mathcal{X},x}$ the ideal in $\mathcal{O}_{\mathcal{X},x}$ generated by $\mathcal{M}_{\mathcal{X},x} \setminus \mathcal{O}_{\mathcal{X},x}^{\times}$ at the point $x \in \mathcal{X}$.

Example 2.4.1.6. Let $\mathcal{X}$ be a scheme with an effective divisor D. Then the sheaf

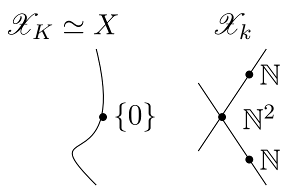
$$U \mapsto \mathcal{M}_D(U) = \{f \in \mathcal{O}_{\mathcal{X}}(U) \mid \quad f|_{\mathcal{X} \setminus D} \text{ is invertible}\},$$

together with the inclusion morphism in $\mathcal{O}_{\mathcal{X}}$, defines a log structure on $\mathcal{X}$. The log scheme consisting of $\mathcal{X}$ and $\mathcal{M}_D$ is denoted by $(\mathcal{X}, D)$, and is called the *divisorial log scheme* associated to the divisor D.

Example 2.4.1.7. Let $\mathscr{X}$ be an snc model over R of a smooth and proper variety X over K. We denote by $\mathscr{X}^+ = (\mathscr{X}, \mathscr{X}_k)$ the divisorial log scheme induced by the special fibre $\mathscr{X}_k$. Given a point $x \in \mathscr{X}$, we consider the irreducible components $E_1, \ldots, E_r$ of $\mathscr{X}_k$ passing through x, and a local equation z_i for each E_i. Then

$$\mathcal{M}_{\mathscr{X},x} = \left\{ u \cdot \prod_{i=1}^{r} z_i^{a_i} \mid u \in \mathcal{O}_{\mathscr{X},x}^{\times}, a_i \in \mathbb{N} \right\} \quad \text{and} \quad \mathcal{C}_{\mathscr{X},x} \simeq \mathbb{N}^r.$$

In the case of a curve over K, the characteristic sheaf is locally isomorphic to



2.4.2. Log-regularity

Definition 2.4.2.1. A log scheme is *log-regular* at a point x if the following two conditions are satisfied:

(i) $\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{\mathcal{X},x}$ is a regular local ring,

(ii) $\dim \mathcal{O}_{\mathcal{X},x} = \dim \mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{\mathcal{X},x} + \operatorname{rank} \mathcal{C}_{\mathcal{X},x}^{\mathrm{gp}}$.

Example 2.4.2.2. A toric variety with divisorial log structure induced by the toric boundary is log-regular. More generally, working over perfect fields, log-regular varieties correspond to toroidal embeddings (without self-intersections).

We recall two main properties of log-regular log scheme.

Proposition 2.4.2.3. *Let $\mathcal{X}^+$ be a log-regular log scheme.*

(i) [Kat94, Theorem 11.6] The locus where the log structure is non-trivial is a divisor, that we will denote by $D_{\mathcal{X}}$, and the log structure on $\mathcal{X}^+$ is the divisorial log structure induced by $D_{\mathcal{X}}$.

(ii) [GR04, Proposition 12.3.24] For log schemes over a perfect field, log-regularity is preserved under base change (in the category of fine and saturated log schemes) and composition.

2.5. Kato fans

2.5.1. Definition

Definition 2.5.1.1. [Kat94, Definition 9.1] A *monoidal space* $(T, \mathcal{M}_T)$ is a topological space T endowed with a sharp sheaf of monoids $\mathcal{M}_T$, where *sharp* means that $\mathcal{M}_{T,t}^{\times} = \{1\}$ for every $t \in T$. We often denote the monoidal space simply by T.

A morphism of monoidal spaces is a pair $(f, \varphi) : (T, \mathcal{M}_T) \to (T', \mathcal{M}_{T'})$ such that $f : T \to T'$ is a continuous function of topological spaces and $\varphi : f^{-1}(\mathcal{M}_T) \to \mathcal{M}_{T'}$ is a sheaf homomorphism such that $\varphi_t^{-1}(\{1\}) = \{1\}$ for every $t \in T$.

Example 2.5.1.2. If $\mathcal{X}^+$ is a log scheme then the Zariski topological space $\mathcal{X}$ is equipped with a sheaf of sharp monoids $\mathcal{C}_{\mathcal{X}}$, namely the characteristic sheaf of $\mathcal{X}^+$. Thus $(\mathcal{X}, \mathcal{C}_{\mathcal{X}})$ is a monoidal space. Moreover, morphisms of log schemes induce morphisms of characteristic sheaves, hence morphisms of monoidal spaces. We therefore obtain a functor from the category of log schemes to the category of monoidal spaces.

Example 2.5.1.3. Given a monoid M, we may associate to it a monoidal space called the *spectrum* of M. As a set, $\operatorname{Spec} M$ is the set of all prime ideals of M. The topology is characterized by the basis open sets $D(f) = \{P \in \operatorname{Spec} M | f \notin P\}$ for any $f \in M$. The monoidal sheaf is defined by

$$\mathcal{M}_{\operatorname{Spec} M}(D(f)) = S_f^{-1}M/(S_f^{-1}M)^{\times}$$

where $S_f = \{f^n | n \geqslant 0\}$.

For instance, when $M = \mathbb{N}$ the set of prime ideals of $\mathbb{N}$ is $\operatorname{Spec} \mathbb{N} = \{\emptyset, P_1\}$, with $P_1 = \{1, 2, 3, \ldots\}$. For $f = 0$, $D(0) = \operatorname{Spec} \mathbb{N}$ and $\mathcal{M}_{\operatorname{Spec} \mathbb{N}}(D(0)) = \mathbb{N}$. For any $f \neq 0$, $D(f) = \{\emptyset\}$ and $\mathcal{M}_{\operatorname{Spec} \mathbb{N}}(D(f)) = \{0\}$.

Definition 2.5.1.4. A monoidal space isomorphic to the monoidal space $\operatorname{Spec} P$ for some monoid P is called an affine Kato fan. A monoidal space is called a *Kato fan* if it has an open covering consisting of affine Kato fans. In particular, we call a Kato fan integral, saturated, of finite type or fs if it admits a cover by the spectra of monoids with the respective properties.

(2.5.1.5) A morphism of fs Kato fans $F' \to F$ is called a *subdivision* if it has finite fibres and the morphism

$$\operatorname{Hom}(\operatorname{Spec} \mathbb{N}, F') \to \operatorname{Hom}(\operatorname{Spec} \mathbb{N}, F)$$

is a bijection. By allowing subdivisions, a Kato fan might take the following shape.

Proposition 2.5.1.6. *[Kat94, Proposition 9.8] Let F be an fs Kato fan. Then there is a subdivision $F' \to F$ such that F' has an open cover $\{U_i'\}$ by affine Kato fans with $U_i' \simeq \operatorname{Spec} \mathbb{N}^{r_i}$.*

The strategy of the proof of Proposition 2.5.1.6 goes back to [KKMSD73] and relies on a sequence of particular subdivisions of the Kato fan, the so-called star and barycentric subdivisions [ACMUW16, Example 4.10].

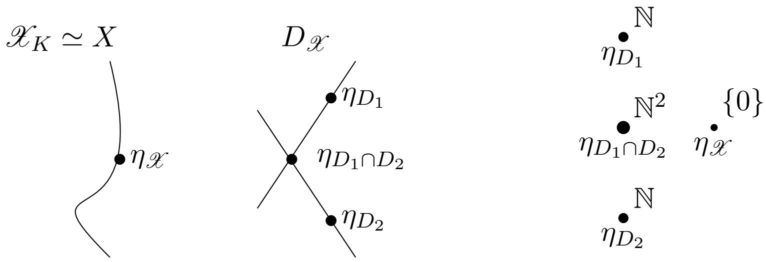
2.5.2. Kato fans associated to log-regular log schemes.

Theorem 2.5.2.1. *[Kat94, Proposition 10.2] Let $\mathcal{X}^+$ be a log-regular log scheme. Then there is an initial strict morphism $(\mathcal{X}, \mathcal{C}_{\mathcal{X}}) \to F$ to a Kato fan in the category of monoidal spaces. Explicitly, there exist a Kato fan F and a morphism $\varrho : (\mathcal{X}, \mathcal{C}_{\mathcal{X}}) \to F$ such that $\varrho^{-1}(\mathcal{M}_F) \simeq \mathcal{C}_{\mathcal{X}}$ and any other morphism from $(\mathcal{X}, \mathcal{C}_{\mathcal{X}})$ to a Kato fan factors through ϱ.*

The Kato fan F in Theorem 2.5.2.1 is called the Kato fan associated to $\mathcal{X}^+$. Concretely, it is the topological subspace of $\mathcal{X}$ consisting of the points x such that the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{\mathcal{X},x}$ is equal to $\mathcal{I}_{\mathcal{X},x}$, and $\mathcal{M}_F$ is the inverse image of $\mathcal{C}_{\mathcal{X}}$ on F; henceforth we write $\mathcal{C}_F$ for $\mathcal{M}_F$.

Example 2.5.2.2. Assume that $\mathscr{X}$ is regular, of finite type over S and $\mathscr{X}_k$ is a divisor with strict normal crossings. Then $\mathscr{X}^+$ is log-regular and F is the set of generic points of intersections of irreducible components of $\mathscr{X}_k$. For each point x of F, the stalk of $\mathcal{C}_F$ is isomorphic to $(\mathbb{N}^r, +)$, with r the number of irreducible components of $\mathscr{X}_k$ that pass through x.

For instance, consider the case of a curve X over K which admits a model $\mathscr{X}$ whose special fibre consists of two irreducible components meeting transversally. Let $\mathscr{X}^+$ be the log-regular log scheme induced by the special fibre. We denote by $\eta_.$ the generic point of an intersection of irreducible components of $D_{\mathscr{X}}$. We obtain the following Kato fan:



This example admits the following partial generalization.

Lemma 2.5.2.3. *Let $\mathcal{X}^+$ be a log-regular log scheme. Then the fan F consists of the generic points of intersections of irreducible components of $D_{\mathcal{X}}$.*

Proof. First, we show that every such generic point is a point of F. Let $E_1, \ldots, E_r$ be irreducible components of $D_{\mathcal{X}}$ and let x be a generic point of the intersection $E_1 \cap \ldots \cap E_r$. We set $d = \dim \mathcal{O}_{\mathcal{X},x}$. Since $\mathcal{X}^+$ is log-regular, we know that $\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{\mathcal{X},x}$ is regular and that

$$d = \dim \mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{\mathcal{X},x} + \operatorname{rank} \mathcal{C}^{\mathrm{gp}}_{\mathcal{X},x}. \tag{2.5.2.4}$$

We denote by $V(\mathcal{I}_{\mathcal{X},x})$ the vanishing locus of the ideal $\mathcal{I}_{\mathcal{X},x}$ in $\operatorname{Spec}(\mathcal{O}_{\mathcal{X},x})$. We want to prove that $\mathcal{I}_{\mathcal{X},x} = \mathfrak{m}_x$. We assume the contrary, hence that $\mathcal{I}_{\mathcal{X},x} \subsetneq \mathfrak{m}_x$. This assumption implies that there exists j such that $V(\mathcal{I}_{\mathcal{X},x}) \nsubseteq E_j$: indeed, if the vanishing locus is contained in each irreducible component E_i, that is,

$$V(\mathcal{I}_{\mathcal{X},x}) \subseteq E_1 \cap \ldots \cap E_r \subseteq \overline{\{x\}},$$

then $\mathcal{I}_{\mathcal{X},x} \supseteq \mathfrak{m}_x$. From the assumption of log-regularity it follows that the vanishing locus $V(\mathcal{I}_{\mathcal{X},x})$ is a regular subscheme, and, moreover, that $\mathcal{X}^+$ is Cohen–Macaulay by [Kat94], Theorem 4.1. Thus, there exists a regular sequence $(f_1, \ldots, f_l)$ in $\mathcal{I}_{\mathcal{X},x}$, where l is the codimension of $V(\mathcal{I}_{\mathcal{X},x})$, that is,

$$\dim \mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{\mathcal{X},x} = d - l.$$

Moreover by the equality (2.5.2.4), $\operatorname{rank} \mathcal{C}^{\mathrm{gp}}_{\mathcal{X},x} = l$.

We claim that the residue classes of these elements f_i in $\mathcal{C}^{\mathrm{gp}}_{\mathcal{X},x}$ are linearly independent. Assume the contrary. If $l = 1$ the proposition holds because f_1 is not a unit. Otherwise, $l \geqslant 2$ and, up to renumbering the f_i, there exist an integer e with $1 < e \leqslant l$, non-negative integers $a_1, \ldots, a_l$, not all zero, and a unit u in $\mathcal{O}_{\mathcal{X},x}$ such that

$$f_1^{a_1} \cdot \ldots \cdot f_{e-1}^{a_{e-1}} = u \cdot f_e^{a_e} \cdot \ldots \cdot f_l^{a_l}.$$

This contradicts the fact that $(f_1, \ldots, f_l)$ is a regular sequence in $\mathcal{I}_{\mathcal{X},x}$. Thus, the classes $\overline{f_1}, \ldots, \overline{f_l}$ are independent in $\mathcal{C}^{\mathrm{gp}}_{\mathcal{X},x}$. As we also have the equality $\operatorname{rank} \mathcal{C}^{\mathrm{gp}}_{\mathcal{X},x} = l$, it follows that these classes generate $\mathcal{C}^{\mathrm{gp}}_{\mathcal{X},x} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let g_j be a non-zero element of the ideal $\mathcal{I}_{\mathcal{X},x}$ that vanishes along E_j: it necessarily exists as otherwise E_j is not a component of the divisor $D_{\mathcal{X}}$. Then g_j satisfies

$$g_j^N = v \cdot f_1^{b_1} \cdot \ldots \cdot f_l^{b_l}$$

with $b_i \in \mathbb{Z}$, v is a unit in $\mathcal{O}_{\mathcal{X},x}$ and N is a positive integer. As g_j vanishes along the irreducible component E_j, at least one of the functions $f_1, \ldots, f_l$ has to vanish along E_j: assume it is f_1.

On the one hand, as f_1 is identically zero on E_j, the trace of E_j on $V(\mathcal{I}_{\mathcal{X},x})$ has codimension at most $l-1$ in E_j at the point x. On the other hand, we assumed that $V(\mathcal{I}_{\mathcal{X},x})$ is not contained in E_j and it has codimension l in $\mathcal{O}_{\mathcal{X},x}$. Then the trace of E_j on $V(\mathcal{I}_{\mathcal{X},x})$ has codimension l in E_j at x. This is a contradiction. We conclude that the ideal $\mathcal{I}_{\mathcal{X},x}$ is equal to the maximal ideal $\mathfrak{m}_x$, therefore x is a point of F.

It remains to prove the converse implication: every point x of the fan F is a generic point of an intersection of irreducible components of $D_{\mathcal{X}}$. Let x be a point of F: by construction of Kato fan F, the maximal ideal of $\mathcal{O}_{\mathcal{X},x}$ is equal to $\mathcal{I}_{\mathcal{X},x}$, thus it is generated by elements in $\mathcal{M}_{\mathcal{X},x}$. The zero locus of such an element is contained in $D_{\mathcal{X}}$ by definition of the logarithmic structure on $\mathcal{X}^+$. Therefore, the zero locus of a generator of $\mathfrak{m}_x$ in $\mathcal{M}_{\mathcal{X},x}$ is a union of irreducible components of the trace of $D_{\mathcal{X}}$ on $\operatorname{Spec} \mathcal{O}_{\mathcal{X},x}$ and x is a generic point of the intersection of all such irreducible components. □

Remark 2.5.2.5. By convention, the generic point of the empty intersection of irreducible components is the generic point of $\mathcal{X}$. By definition, this point is also included in the Kato fan F. Thus, for example, the Kato fan associated to S^+ consists of two points: the generic point of S that corresponds to the empty intersection, and the closed point s corresponding to the unique irreducible component of the logarithmic divisorial structure.

Moreover, Example 2.5.2.2 also leads to the following characterization.

Proposition 2.5.2.6. *[GR04, Corollary 12.5.35] Let $\mathscr{X}^+$ be a log-regular log scheme over S^+ and F its associated Kato fan. The following statements are equivalent:*

1. *for every $x \in F$, $\mathcal{C}_{F,x} \simeq \mathbb{N}^{r(x)}$,*
2. *the underlying scheme $\mathscr{X}$ is regular.*

If this is the case, then the special fibre $\mathscr{X}_k$ is an snc divisor.

(2.5.2.7) The construction of the Kato fan of a log scheme defines a functor from the category of log-regular log schemes to the category of Kato fans. Indeed, given a morphism of log schemes $\mathcal{X}^+ \to \mathcal{Y}^+$, we consider the embedding of the associated Kato fan $F_{\mathcal{X}}$ in $\mathcal{X}^+$ and the canonical morphism $\mathcal{Y}^+ \to F_{\mathcal{Y}}$: the composition

$$F_{\mathcal{X}} \hookrightarrow \mathcal{X}^+ \to \mathcal{Y}^+ \to F_{\mathcal{Y}}$$

functorially induces a map between associated Kato fans. Moreover, this association preserves strict morphisms [Uli17, Lemma 4.9].

2.5.3. Resolutions of log schemes via Kato fan subdivisions.

Proposition 2.5.3.1. *[Kat94, Proposition 9.9] Let $\mathcal{X}^+$ be a log-regular log scheme and let F be its associated Kato fan. Let $F' \to F$ be a subdivision of fans. Then there exist a log scheme $\mathcal{X}'^+$, a morphism of log schemes $\mathcal{X}'^+ \to \mathcal{X}^+$ and a commutative diagram*

$$\begin{array}{ccc} (\mathcal{X}', \mathcal{C}_{\mathcal{X}'}) & \xrightarrow{p} & F' \\ \downarrow & & \downarrow \\ (\mathcal{X}, \mathcal{C}_{\mathcal{X}}) & \xrightarrow{\pi_{\mathcal{X}}} & F \end{array}$$

such that $p^{-1}(\mathcal{M}_{F'}) \simeq \mathcal{C}_{\mathcal{X}'}$; they define a final object in the category of such diagrams and the refinement $F' \to F$ is functorially induced by the morphism of log-regular log schemes $\mathcal{X}'^+ \to \mathcal{X}^+$.

(2.5.3.2) It follows that given any subdivision $F' \to F$ of the Kato fan F associated with a log-regular log scheme $\mathcal{X}^+$, we can construct a log scheme over $\mathcal{X}^+$ with prescribed associated Kato fan F'. Combining this fact with Proposition 2.5.1.6 and Proposition 2.5.2.6 yields a construction of resolutions of log schemes in the following sense: for any log-regular log scheme $\mathscr{X}^+$ over S^+ we can find a birational modification by a regular log scheme $\mathscr{X}'^+$ with snc special fibre. Moreover, the morphism of log schemes $\mathscr{X}'^+ \to \mathscr{X}^+$ is obtained by a log blow-up [Niz06, Theorem 5.8].

2.5.4. Fibred products and associated Kato fans.

(2.5.4.1) Given morphisms of fs log schemes $f_1 : \mathcal{X}_1^+ \to \mathcal{Y}^+$ and $f_2 : \mathcal{X}_2^+ \to \mathcal{Y}^+$, their fibred product exists in the category of log schemes. It is obtained by endowing the usual fibred product of schemes

$$\begin{array}{ccc} \mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2 & \xrightarrow{p_1} & \mathcal{X}_1 \\ \downarrow{\scriptstyle p_2} & \searrow^{p_{\mathcal{Y}}} & \downarrow{\scriptstyle f_1} \\ \mathcal{X}_2 & \xrightarrow{f_2} & \mathcal{Y} \end{array} \tag{2.5.4.2}$$

with the log structure associated to $p_1^{-1}\mathcal{M}_{\mathcal{X}_1} \oplus_{p_{\mathcal{Y}}^{-1}\mathcal{M}_{\mathcal{Y}}} p_2^{-1}\mathcal{M}_{\mathcal{X}_2}$. If $u_1 : P \to Q_1$ and $u_2 : P \to Q_2$ are charts for the morphisms f_1 and f_2 respectively, then the induced morphism $\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2 \to \operatorname{Spec} \mathbb{Z}[Q_1 \oplus_P Q_2]$ is a chart for $\mathcal{X}_1^+ \times_{\mathcal{Y}^+} \mathcal{X}_2^+$.

(2.5.4.3) In general, the fibred product is not fs, but the category of fs log schemes also admits fibred products. Keeping the same notation, the following is a local description of the fibred product in the category of fine and saturated log schemes

$$\mathcal{X}_1^+ \times^{\mathrm{fs}}_{\mathcal{Y}^+} \mathcal{X}_2^+ = (\mathcal{X}_1^+ \times_{\mathcal{Y}^+} \mathcal{X}_2^+) \times_{\mathbb{Z}[Q_1 \oplus_P Q_2]} \mathbb{Z}[(Q_1 \oplus_P Q_2)^{\mathrm{sat}}]$$

in terms of the charts for f_1 and f_2 [Bul15, §3.6.16]. We remark that the two fibred products above may not only have different log structures, but also the underlying schemes may differ. Nevertheless, this obviously does not occur when the monoid $Q_1 \oplus_P Q_2$ is saturated.

(2.5.4.4) Log-smoothness is preserved under fs base change and composition [GR04, Proposition 12.3.24]. In particular, if $f_1 : \mathcal{X}_1^+ \to \mathcal{Y}^+$ is log-smooth and $\mathcal{X}_2^+$ is log-regular, then $\mathcal{X}_1^+ \times^{\mathrm{fs}}_{\mathcal{Y}^+} \mathcal{X}_2^+$ is log-regular, by [Kat94, Theorem 8.2].

Consider log-smooth morphisms of fs log schemes $\mathcal{X}_1^+ \to \mathcal{Y}^+$ and $\mathcal{X}_2^+ \to \mathcal{Y}^+$. The sheaves of logarithmic differentials are related by the isomorphism

$$p_1^*\Omega^{\log}_{\mathcal{X}_1^+/\mathcal{Y}^+} \oplus p_2^*\Omega^{\log}_{\mathcal{X}_2^+/\mathcal{Y}^+} \simeq \Omega^{\log}_{\mathcal{X}_1^+ \times^{\mathrm{fs}}_{\mathcal{Y}^+} \mathcal{X}_2^+/\mathcal{Y}^+} \tag{2.5.4.5}$$

by [GR04, Proposition 12.3.13]. Furthermore, by assumption of log-smoothness over S^+ the logarithmic differential sheaves are locally free of finite rank by [Kat94, Proposition 3.10], and we can consider their determinants; they are called log canonical bundles and denoted by $\omega^{\log}$. The following isomorphism is a direct consequence of (2.5.4.5)

$$p_1^*\omega^{\log}_{\mathcal{X}_1^+/\mathcal{Y}^+} \otimes p_2^*\omega^{\log}_{\mathcal{X}_2^+/\mathcal{Y}^+} \simeq \omega^{\log}_{\mathcal{X}_1^+ \times^{\mathrm{fs}}_{\mathcal{Y}^+} \mathcal{X}_2^+/\mathcal{Y}^+}. \tag{2.5.4.6}$$

(2.5.4.7) Similarly to the construction of fibred products of fs log schemes, the category of fs Kato fans admits fibred products: on affine Kato fans $F = \operatorname{Spec} P$ and $G = \operatorname{Spec} Q$ over $H = \operatorname{Spec} T$, $F \times_H G$ is the spectrum of the amalgamated sum $(P \oplus_T Q)^{\mathrm{sat}}$ in the category of fs monoids (see [Uli16, Proposition 2.4]), and on the underlying topological spaces this coincides with the usual fibred product.

We seek to compare the Kato fan associated to the fibred product of log-regular log schemes with the fibred product of associated Kato fans.

Proposition 2.5.4.8. *[Sai04, Lemma 2.8] Given $\mathcal{T}^+$ a log-regular log scheme, let $\mathcal{X}^+$ and $\mathcal{Y}^+$ be log-smooth log schemes over $\mathcal{T}^+$. We denote by $\mathcal{Z}^+$ the fs fibred product $\mathcal{X}^+ \times^{fs}_{\mathcal{T}^+} \mathcal{Y}^+$. Then the natural morphisms $F_{\mathcal{Z}} \to F_{\mathcal{X}}$ and $F_{\mathcal{Z}} \to F_{\mathcal{Y}}$ induce a morphism of Kato fans*

$$F_{\mathcal{Z}} \to F_{\mathcal{X}} \times_{F_{\mathcal{T}}} F_{\mathcal{Y}} \tag{2.5.4.9}$$

that is locally an isomorphism.

(2.5.4.10) For any pair of points (x, y) in $F_{\mathcal{X}} \times_{F_{\mathcal{T}}} F_{\mathcal{Y}}$, we denote by $n(x, y)$ the number of preimages of (x, y) in the Kato fan of $\mathcal{Z}^+$ under the local isomorphism (2.5.4.9).

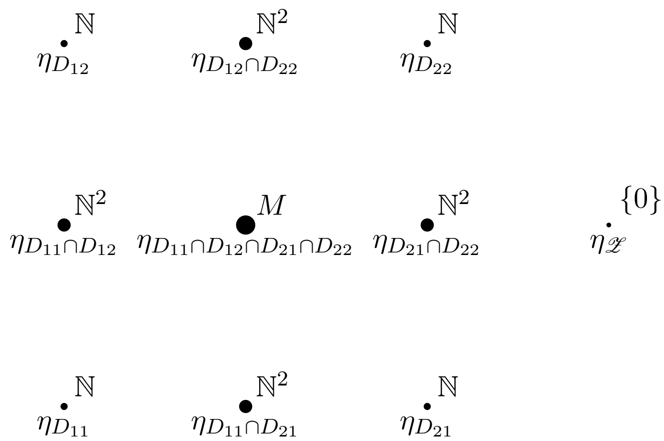
Lemma 2.5.4.11. *If x' is in the closure of x, and y' in the closure of y, then $n(x', y') \geqslant n(x, y)$.*

Proof. Let z' be a preimage of the pair (x', y'). By Proposition 2.5.4.8, there exists an open neighborhood $U_{z'}$ of z' such that the restriction of $F_{\mathcal{Z}} \to F_{\mathcal{X}} \times_{F_{\mathcal{T}}} F_{\mathcal{Y}}$ to $U_{z'}$ is an isomorphism onto its image. In particular, (x, y) lies in this image. Thus, there exists a unique preimage of (x, y) that is contained in $U_{z'}$. It follows that $n(x', y') \geqslant n(x, y)$. □

Example 2.5.4.12. Let $R = \mathbb{C}[[t]]$, $\mathscr{X}^+ = \left(\operatorname{Spec} \frac{R[x_1,x_2]}{(x_1x_2=t)}, \mathscr{X}_0 = \{x_1x_2 = 0\}\right)$ and $\mathscr{Y}^+ = \left(\operatorname{Spec} \frac{R[y_1,y_2]}{(y_1y_2=t)}, \mathscr{Y}_0 = \{y_1y_2 = 0\}\right)$. The fs fibred product is

$$\mathscr{Z}^+ = \left(\operatorname{Spec} \frac{R[x_1, x_2, y_1, y_2]}{(x_1x_2 = t = y_1y_2)}, \mathscr{Z}_0 = \sum D_{ij}\right),$$

where $D_{ij} = \{x_i = y_j = 0\}$ for $i, j \in \{1, 2\}$. We denote by $\eta_.$ the generic point of an intersection of irreducible components of $D_{\mathscr{Z}}$. We obtain the following Kato fan for $\mathscr{Z}^+$:



where M is the submonoid in $\mathbb{N}^3$ generated by $(1, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 0)$. Indeed, at the generic point $\eta_{D_{11}\cap D_{12}\cap D_{21}\cap D_{22}}$, the orders of vanishing a_{ij} of a regular function f along the divisors D_{ij} are such that $a_{11} + a_{22} = a_{12} + a_{21}$. Recalling Example 2.5.2.2, we observe that in this case the morphism $F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$ is an isomorphism; this is indeed an example of the result of Proposition 3.2.4.4.

3

Skeletons for a product of degenerations

3.1. Introduction

Let R be a discrete valuation ring with quotient field K and residue field k, and let X be a smooth proper variety over K. While there may be no way to extend X to a smooth proper variety over R, in $\mathrm{char}(k) = 0$ resolution of singularities guarantees that we can always produce an R-model $\mathscr{X}$ where the special fibre $\mathscr{X}_k$ is a strict normal crossings (snc) divisor. Given such a model, we associate the dual complex $\mathcal{D}(\mathscr{X}_k)$, which is the dual intersection complex of the components of the special fibre.

The dual complex of the special fibre of a such degeneration reflects the geometry of the generic fibre. If the generic fibre is rationally connected, then the dual complex of the special fibre is contractible [dFKX17]. For Calabi-Yau varieties, degenerations are classified by the action of monodromy on the cohomology. The principle is that the degenerations with maximally unipotent actions have the richest combinatorial structure in the dual complex. In the case of strict Calabi-Yau varieties, the dual complex is always a $\mathbb{Q}$-homology sphere, and Kollár and Xu [KX16] show that it is a sphere if $n \leqslant 3$ or $n \leqslant 4$ and the special fibre of a minimal divisorially log terminal (dlt) model is snc.

The goal of this chapter is to understand the dual complex of a model for the product of two smooth proper varieties over K. We consider this problem from two perspectives.

3.1.1. Skeletons of Berkovich spaces

The first is via the theory of Berkovich spaces. In this setting we assume that K is complete with respect to the valuation induced by R, which gives rise to a non-archimedean norm on K. In [Ber90], Berkovich develops a theory of analytic geometry over K. He associates a K-analytic space to X; each point corresponds to a real valuation on the residue field of a point of X, extending the discrete valuation on K. This space, denoted by X^{an}, is called the Berkovich space associated to X. See Definition 2.1.1.5 for further details.

From any snc model $\mathscr{X}$ of X one can construct a subspace of X^{an}, called the Berkovich skeleton of $\mathscr{X}$ and denoted by $\mathrm{Sk}(\mathscr{X})$: it is homeomorphic to the dual intersection complex of the divisor $\mathscr{X}_k$ [MN15]. The Berkovich skeletons turn out to be relevant in the study of the topology of X^{an}. They shape the Berkovich space, as X^{an} is homeomorphic to the inverse limit $\varprojlim \mathrm{Sk}(\mathscr{X})$ where $\mathscr{X}$ runs through all snc models of X. Also, the homotopy type of X^{an} is determined by any snc model $\mathscr{X}$: indeed, Berkovich and Thuillier prove that $\mathrm{Sk}(\mathscr{X})$ is a strong deformation retract of X^{an} [Ber90; Thu07].

3.1.2. The dual complex of a dlt model

The other approach to the study of the dual complexes comes from birational geometry. In this setting, we consider a pair $(\mathscr{X}, \Delta_{\mathscr{X}})$ over the germ of a curve. In the log general type case, running the minimal model program (MMP) distinguishes a canonical model for the degeneration $(\mathscr{X}, \Delta_{\mathscr{X}})$ [Ale96; HMX18; KSB88], at the cost of possibly introducing worse singularities. If we are willing to tolerate some ambiguity in our choice of model, we can choose instead to produce a minimal dlt model; see Definition 2.3.0.3. One advantage of dlt models is that they are expected to exist for all pairs admitting a log pluricanonical form. The singularities are mild enough that it is possible to define the dual complex as the dual intersection complex of divisors of coefficient 1, denoted $\mathcal{D}(\mathscr{X}, \Delta_{\mathscr{X}}^{=1})$. In [dFKX17], de Fernex, Kollár, and Xu investigate how the dual complex is affected by the operations of the minimal model program. They show, under mild hypotheses, that every step of the MMP induces a homotopy equivalence between dual complexes. Moreover, $\mathcal{D}(\mathscr{X}, \Delta_{\mathscr{X}}^{=1})$ is a piecewise linear (PL) invariant under log crepant birational maps.

3.1.3. The essential skeleton

Recently there has been much interest in a synthesis of the two approaches. Kontsevich and Soibelmann [KS06] define a version of the skeleton of a variety with trivial canonical bundle, which detects the locus of simple poles along the special fibre of the distinguished canonical form. Mustaţă and Nicaise [MN15] extend their definition to any variety with non-negative Kodaira dimension. The key technical tool is the definition, for a rational pluricanonical form, of a weight function on the Berkovich space. The essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X)$ is the union over all regular pluricanonical forms of the minimality locus of the associated weight functions.

Thus the essential skeleton has the advantage of being intrinsic to the variety X, with no dependence on a choice of model. As the weight function is closely related to the log discrepancy from birational geometry, it is natural to expect that the essential skeleton in some way encodes some of the minimal model theory of X. Nicaise and Xu [NX16] show, when X is a smooth projective variety with K_X semiample, and $\mathscr{X}$ is a good minimal dlt model, that the dual complex of $\mathscr{X}_k$ can be identified with the essential skeleton of X. While it is in general a difficult problem to produce good minimal dlt models, Kollár, Nicaise and Xu [KNX17] show that for any smooth projective X with K_X semiample, X extends to a good minimal dlt model over a finite extension of the valuation ring.

3.1.4. Skeletons for log-regular models

To produce nice models of the product, we work in the context of logarithmic geometry. To any log-regular scheme $\mathscr{X}^+$, in [Kat94] Kato attaches a combinatorial structure $F_{\mathscr{X}}$ called a fan: if we denote by $D_{\mathscr{X}}$ the locus where the log structure is non-trivial, then the fan $F_{\mathscr{X}}$ consists of the set of the generic points of intersections of irreducible components of $D_{\mathscr{X}}$, equipped with a sheaf of monoids. See Section 2.5 for the details on Kato fans.

In Section 3.2 we define a logarithmic version of the Berkovich skeleton for a log-regular model $\mathscr{X}^+$ of X over R: it gives rise to a polyhedral complex in X^{an} whose faces correspond to the points of $F_{\mathscr{X}}$.

Given two log-smooth log schemes $\mathscr{X}^+$ and $\mathscr{Y}^+$ over R, their product $\mathscr{Z}^+$ in the category of fine and saturated log schemes is naturally log-regular, hence $\mathscr{Z}^+$ has an associated skeleton, and it is a model of the product $\mathscr{X}_K \times_K \mathscr{Y}_K$ of the generic fibres. If one of the two underlying schemes $\mathscr{X}$ or $\mathscr{Y}$ is semistable, which means it has reduced special fibre, then we show that the skeleton of the product $\mathscr{Z}^+$ is the product of the skeletons, with the projection maps given by restricting the valuation to the corresponding function fields (Proposition 3.2.5.3).

3.1.5. Skeletons for pairs

Working in the logarithmic setting, we may also allow a non-trivial log structure over the generic fibre. Geometrically this corresponds to adding horizontal divisors to the special fibre and yields the addition of some unbounded faces to the skeleton. In [GRW16] Gubler, Rabinoff, and Werner construct a skeleton for strictly semistable snc models with suitable horizontal divisors. Both constructions recover the Berkovich skeleton when there is no horizontal component and the special fibre is snc.

Pairs arise frequently in the minimal model program. We say that a pair (X, Δ_X) is fractional snc if Δ_X is an effective $\mathbb{Q}$-divisor such that $\Delta_X = \sum a_i \Delta_{X,i}$ with $0 \leqslant a_i \leqslant 1$ and the round-up $(X, \lceil \Delta_X \rceil)$ is snc. Taking advantage of a construction that admits horizontal components, the definition of the essential skeleton extends to the case of a fractional snc pair (X, Δ_X) over K, and to pluricanonical forms of some positive index r with divisor of poles no worse than $r\Delta_X$. We extend to pairs the result of Mustaţă and Nicaise [MN15] on the birational invariance of the essential skeleton (Proposition 3.4.1.5), as well as Nicaise and Xu's result [NX16] that the essential skeleton is homeomorphic to the dual complex of a good minimal dlt model (Proposition 3.4.1.7). It follows from these results that we can define the notion of essential skeleton for a dlt pair.

3.1.6. Main Result

Our main result establishes the behavior of essential skeletons under products.

Theorem 3.1.6.1 (Theorem 3.4.2.2). *Assume that the residue field k is algebraically closed. Let (X, Δ_X) and (Y, Δ_Y) be fractional snc pairs. Suppose that both pairs have non-negative Kodaira–Iitaka dimension and admit semistable log-regular models $\mathscr{X}^+$ and $\mathscr{Y}^+$ over S^+. Then the PL homeomorphism of skeletons induces a PL homeomorphism of essential skeletons*

$$\mathrm{Sk}^{\mathrm{ess}}(Z, \Delta_Z) \xrightarrow{\sim} \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) \times \mathrm{Sk}^{\mathrm{ess}}(Y, \Delta_Y)$$

where Z and Δ_Z are the respective products.

Semistability is a key assumption; without it the projection map might fail to be injective; see Example 3.2.5.4. As expected, we get a corresponding result for dual complexes of semistable good minimal dlt models (Theorem 3.4.3.5). Unfortunately semistability is not well behaved under birational transformations so it seems possible that a degeneration admits a semistable good minimal dlt model but no semistable log-regular model.

3.1.7. Structure of the chapter

Section 3.2 contains the definition and properties of the skeleton of a log-regular scheme, and it requires no hypothesis on the characteristic of K. In Section 3.3 we introduce the weight function and essential skeleton from [MN15], extending their definition to the case of a pair. In Section 3.4 we give connections to birational geometry, along with proofs of the main theorems.

3.2. The skeleton of a log-regular log scheme

3.2.1. Construction

(3.2.1.1) Let $\mathscr{X}^+$ be a log-regular log scheme. Let x be a point of the associated Kato fan F. Denote by $F(x)$ the set of points y of F such that x lies in the closure of $\{y\}$, and by $\mathcal{C}_{F(x)}$ the restriction of $\mathcal{C}_F$ to $F(x)$. Denote by $\operatorname{Spec} \mathcal{C}_{\mathscr{X},x}$ the spectrum of the monoid $\mathcal{C}_{\mathscr{X},x} = \mathcal{C}_{F,x}$. Then there exists a canonical isomorphism of monoid spaces

$$(F(x), \mathcal{C}_{F(x)}) \to \operatorname{Spec} \mathcal{C}_{\mathscr{X},x} : y \mapsto \{s \in \mathcal{C}_{\mathscr{X},x} \mid s(y) = 0\}$$

where the expression $s(y) = 0$ means that $s'(y) = 0$ for any representative s' of s in $\mathcal{M}_{\mathscr{X},x}$. In particular, we obtain a bijective correspondence between the faces of the monoid $\mathcal{C}_{\mathscr{X},x}$ and the points of $F(x)$, and for every point y of $F(x)$, a surjective cospecialization morphism of monoids

$$\tau_{x,y} : \mathcal{C}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{X},y}$$

which induces an isomorphism of monoids

$$S_y^{-1}\mathcal{C}_{\mathscr{X},x}/(S_y^{-1}\mathcal{C}_{\mathscr{X},x})^{\times} \cong \mathcal{C}_{\mathscr{X},x}/S_y \xrightarrow{\sim} \mathcal{C}_{\mathscr{X},y}$$

where S_y denotes the monoid of elements s in $\mathcal{C}_{\mathscr{X},x}$ such that $s(y) \neq 0$.

(3.2.1.2) Assume that $\mathscr{X}^+$ is a log-regular log scheme over S^+. For each point x in F, we denote by σ_x the set of morphisms of monoids

$$\alpha : \mathcal{C}_{\mathscr{X},x} \to (\mathbb{R}_{\geqslant 0}, +)$$

such that $\alpha(\pi) = 1$ for every uniformizer π in R. We endow σ_x with the topology of pointwise convergence, where $\mathbb{R}_{\geqslant 0}$ carries the usual Euclidean topology. Note that σ_x is a polyhedron, possibly unbounded, in the real affine space

$$\{\alpha : \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} \to (\mathbb{R}, +) \,|\, \alpha(\pi) = 1 \text{ for every uniformizer } \pi \text{ in } R\}.$$

If y is a point of $F(x)$, then the surjective cospecialization morphism $\tau_{x,y}$ induces a topological embedding $\sigma_y \to \sigma_x$ that identifies σ_y with a face of σ_x.

(3.2.1.3) We denote by T the disjoint union of the topological spaces σ_x with x in F. On the topological space T, we consider the equivalence relation $\sim$ generated by couples of the form $(\alpha, \alpha \circ \tau_{x,y})$ where x and y are points in F such that x lies in the closure of $\{y\}$ and α is a point of σ_y.

The skeleton of $\mathscr{X}^+$ is defined as the quotient of the topological space T by the equivalence relation $\sim$. We denote this skeleton by $\mathrm{Sk}(\mathscr{X}^+)$. It is clear that $\mathrm{Sk}(\mathscr{X}^+)$ has the structure of a polyhedral complex with cells $\{\sigma_x,\, x \in F\}$, so it comes equipped with a PL structure, and that the faces of a cell σ_x are precisely the cells σ_y with y in $F(x)$.

(3.2.1.4) We note that σ_x is empty for any point x that does not lie in the special fibre $\mathscr{X}_k$: indeed, outside the special fibre any uniformizer is an invertible element, so it is trivial in $\mathcal{C}_{\mathscr{X},x}$ and is mapped to 0 by any morphism of monoids. Therefore, the construction of the skeleton associated to $\mathscr{X}^+$ only concerns the points in the Kato fan F that lie in the special fibre. Moreover, given a generic point $x \in \mathscr{X}_k$ of an intersection of components of $D_{\mathscr{X}}$, where at least one component is not in the special fibre, the corresponding face σ_x is unbounded.

In other words, the skeleton associated to a log-regular scheme $\mathscr{X}^+$, where $D_{\mathscr{X}}$ allows horizontal components, generalizes Berkovich's skeletons by admitting unbounded faces in the direction of the horizontal components as well as by allowing singularities in the special fibre. It also generalizes the construction performed by Gubler, Rabinoff and Werner in [GRW16] of a skeleton associated to a strictly semistable snc pair.

3.2.2. Embedding the skeleton in the non-archimedean generic fibre.

(3.2.2.1) Let $\mathscr{X}^+$ be a log-regular log scheme over S^+. Let x be a point of the associated Kato fan F. As the log structure on $\mathscr{X}^+$ is of finite type, the characteristic monoid $\mathcal{C}_{\mathscr{X},x}$ is of finite type too, and thus $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$ is a free abelian group of finite rank. Hence there exists a section

$$\zeta : \mathcal{M}^{\mathrm{gp}}_{\mathscr{X},x}/\mathcal{M}^{\times}_{\mathscr{X},x} \to \mathcal{M}^{\mathrm{gp}}_{\mathscr{X},x}.$$

The section ζ restricts to $\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x}$; indeed, if $x \in \mathcal{M}_{\mathscr{X},x}$ then $\zeta(\overline{x}) - x \in \mathcal{M}^{\times}_{\mathscr{X},x}$. Therefore we may choose a section

$$\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x} \tag{3.2.2.2}$$

of the projection homomorphism

$$\mathcal{M}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{X},x}$$

and use this section to view $\mathcal{C}_{\mathscr{X},x}$ as a submonoid of $\mathcal{M}_{\mathscr{X},x}$. Note that $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$ generates the ideal $\mathcal{I}_{\mathscr{X},x}$ of $\mathcal{O}_{\mathscr{X},x}$.

We propose a generalization of [MN15, §2.4.4].

Lemma 3.2.2.3. *Let A be a Noetherian ring, let I be an ideal of A and let $(y_1, \dots, y_m)$ be a system of generators for I. We denote by $\hat{A}$ the I-adic completion of A. Let B be a subring of A such that the elements $y_1, \dots, y_m$ belong to B and generate the ideal $B \cap I$ in B. Then, in the ring $\hat{A}$, every element f of B can be written as*

$$f = \sum_{\beta \in \mathbb{Z}^m_{\geqslant 0}} c_\beta y^\beta \tag{3.2.2.4}$$

where the coefficients c_β belong to $((A \setminus I) \cap B) \cup \{0\}$.

Proof. Let f be an element of B; we construct an expansion for f of the form (3.2.2.4) by induction. If f belongs to the complement of I, the conclusion trivially holds. Otherwise, f belongs to I and we can write f as a linear combination of the elements $y_1, \dots, y_m$ with coefficients in B:

$$f = \sum_{j=1}^{m} b_j y_j, \quad b_j \in B.$$

By the induction hypothesis, we suppose that i is a positive integer and that we can write every f in B as a sum of an element f_i of the form (3.2.2.4) and a linear combination of degree-i monomials in the elements $y_1, \dots, y_m$ with coefficients in B. We apply this assumption to the coefficients b_j, hence

$$b_j = b_{j,i} + \sum_{\substack{\beta \in \mathbb{Z}^m_{\geqslant 0} \\ |\beta| = i}} b_{j,\beta} y^\beta, \quad b_{j,\beta} \in B.$$

Then we can write f as a sum of an element f_{i+1} of the form (3.2.2.4) and a linear combination of degree-$(i+1)$ monomials in the elements $y_1, \ldots, y_m$ with coefficients in B

$$f = \underbrace{\sum_{j=1}^{m} b_{j,i} y_j}_{f_{i+1}} + \sum_{j=1}^{m} \Big(\sum_{\substack{\beta \in \mathbb{Z}^m_{\geqslant 0} \\ |\beta| = i}} b_{j,\beta} y^{\beta} \Big) y_j$$

such that f_i and f_{i+1} have the same coefficients in degree less than or equal to i. Iterating this construction, we finally find an expansion of f of the required form. □

(3.2.2.5) Let f be an element of $\mathcal{O}_{\mathscr{X},x}$. Considering $A = B = \mathcal{O}_{\mathscr{X},x}$, $I = \mathfrak{m}_x$, and a system of generators for $\mathfrak{m}_x$ in $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$, by Lemma 3.2.2.3 we can write f as a formal power series

$$f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma \tag{3.2.2.6}$$

in $\widehat{\mathcal{O}}_{\mathscr{X},x}$, where each coefficient c_γ is either zero or a unit in $\mathcal{O}_{\mathscr{X},x}$. We call this formal series an *admissible expansion* of f. We set

$$\Gamma = \{\gamma \in \mathcal{C}_{\mathscr{X},x} \,|\, c_\gamma \neq 0\} \tag{3.2.2.7}$$

and we denote by $\Gamma_x(f)$ the set of elements of Γ that lie on a compact face of the convex hull of $\Gamma + \mathcal{C}_{\mathscr{X},x}$ in $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} \otimes_{\mathbb{Z}} \mathbb{R}$. We call $\Gamma_x(f)$ the *initial support* of f at x, notation which is justified by the next proposition.

Proposition 3.2.2.8.

1. *The element*

$$f_x = \sum_{\gamma \in \Gamma_x(f)} c_\gamma(x) \gamma \quad \in k(x)[\mathcal{C}_{\mathscr{X},x}]$$

depends on the choice of the section (3.2.2.2)*, but not on the expansion* (3.2.2.6)*.*

2. *The subset $\Gamma_x(f)$ of $\mathcal{C}_{\mathscr{X},x}$ only depends on f and x, and not on the choice of the section* (3.2.2.2) *or the expansion* (3.2.2.6)*.*

Proof. If we denote by I the ideal of $k(x)[\mathcal{C}_{\mathscr{X},x}]$ generated by $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$, then it follows from [Kat94] that there exists an isomorphism of $k(x)$-algebras

$$\mathrm{gr}_I\, k(x)[\mathcal{C}_{\mathscr{X},x}] \to \mathrm{gr}_{\mathfrak{m}_x} \mathcal{O}_{\mathscr{X},x}. \tag{3.2.2.9}$$

Using this result and following the argument of [MN15, Proposition 2.4.4], we show now that f_x does not depend on the expansion of f. Let

$$f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c'_\gamma \gamma$$

be another admissible expansion of f with associated set $\Gamma_x(f)'$ and element f'_x. Then

$$0 = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} (c_\gamma - c'_\gamma)\gamma = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} d_\gamma \gamma$$

where the right-hand side is an admissible expansion obtained by choosing admissible expansions for the elements $c_\gamma - c'_\gamma$ that do not lie in $\mathcal{O}^\times_{\mathscr{X},x} \cup \{0\}$. In particular $d_\gamma(x) = c_\gamma(x) - c'_\gamma(x)$ for any γ in $\Gamma_x(f) \cup \Gamma_x(f)'$. The isomorphism of graded algebras in (3.2.2.9) implies that the elements d_γ must all vanish, hence $\Gamma_x(f) = \Gamma_x(f)'$ and $f_x = f'_x$.

Point (2) follows from the fact that the coefficients c_γ of f_x are independent of the chosen section up to multiplication by a unit in $\mathcal{O}_{\mathscr{X},x}$, so that the support $\Gamma_x(f)$ of f_x only depends on f and x. □

Proposition 3.2.2.10. *Let x be a point of F and let*

$$\alpha : \mathcal{C}_{\mathscr{X},x} \to (\mathbb{R}_{\geqslant 0}, +)$$

be an element of σ_x. Then there exists a unique minimal real valuation

$$v : \mathcal{O}_{\mathscr{X},x} \setminus \{0\} \to \mathbb{R}_{\geqslant 0}$$

such that $v(m) = \alpha(\overline{m})$ for each element m of $\mathcal{M}_{\mathscr{X},x}$.

Proof. We will prove that the map

$$v : \mathcal{O}_{\mathscr{X},x} \setminus \{0\} \to \mathbb{R} : f \mapsto \min\{\alpha(\gamma) \mid \gamma \in \Gamma_x(f)\} \tag{3.2.2.11}$$

satisfies the requirements in the statement. We fix a section

$$\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x}.$$

It is straightforward to check that $(f \cdot g)_x = f_x \cdot g_x$ for all f and g in $\mathcal{O}_{\mathscr{X},x}$. This implies that v is a valuation. It is obvious that $v(m) = \alpha(\overline{m})$ for all m in $\mathcal{M}_{\mathscr{X},x}$, since we can write m as the product of an element of $\mathcal{C}_{\mathscr{X},x}$ and a unit in $\mathcal{O}_{\mathscr{X},x}$.

Now we prove minimality. Consider any real valuation

$$w : \mathcal{O}_{\mathscr{X},x} \to \mathbb{R}$$

such that $w(m) = \alpha(\overline{m})$ for each element m of $\mathcal{M}_{\mathscr{X},x}$, and let f be an element of $\mathcal{O}_{\mathscr{X},x}$. We must show that $w(f) \geqslant v(f)$.

We set

$$C_\alpha = \mathcal{C}_{\mathscr{X},x} \setminus \alpha^{-1}(0).$$

We denote by I the ideal in $\mathcal{O}_{\mathscr{X},x}$ generated by C_α and by A the I-adic completion of

$\mathcal{O}_{\mathscr{X},x}$. By Lemma 3.2.2.3, we see that we can write f in A as

$$\sum_{\beta \in C_\alpha \cup \{1\}} d_\beta \beta \tag{3.2.2.12}$$

where d_β is either zero or contained in the complement of I in $\mathcal{O}_{\mathscr{X},x}$.

Since $\alpha(\beta) > 0$ for every $\beta \in C_\alpha$, we can find an integer $N > 0$ such that $w(g) > w(f)$ for every element g in I^N. So we have

$$w(f) \geqslant \min\{\alpha(\beta) \,|\, d_\beta \neq 0\}$$

recalling that $w(\beta) = \alpha(\beta)$ for all β in $\mathcal{C}_{\mathscr{X},x}$.

We consider the coefficients in the expansion (3.2.2.12) of f. Applying Lemma 3.2.2.3 as in paragraph (3.2.2.5), we can write admissible expansions of these coefficients in $\widehat{\mathcal{O}}_{\mathscr{X},x}$ as

$$d_\beta = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_{\gamma,\beta}\gamma, \quad c_{\gamma,\beta} \in \mathcal{O}^\times_{\mathscr{X},x} \cup \{0\},$$

with $\alpha(\gamma) = 0$ in the expansions of d_β that belong to $\mathfrak{m}_x \setminus I$.

Therefore we obtain an admissible expansion of f

$$f = \sum_{\substack{\beta \in C_\alpha \cup \{1\} \\ \gamma \in \mathcal{C}_{\mathscr{X},x}}} c_{\gamma,\beta}\, \gamma\beta$$

and we have $v(f) = \min\{\alpha(\gamma\beta) \,|\, c_{\gamma,\beta} \neq 0\} = \min\{\alpha(\beta) \,|\, d_\beta \neq 0\} \leqslant w(f)$. □

Remark 3.2.2.13. In the definition (3.2.2.11) of the valuation v, we compute the minimum over the terms in the initial support of f: these elements are a finite number and they only depend on x and f by Proposition 3.2.2.8. Therefore, this minimum provides a well defined function on $\mathcal{O}_{\mathscr{X},x} \setminus \{0\}$. Nevertheless, it is equivalent to consider the minimum over all the terms of an admissible expansion of f, that is, for any admissible expansion $f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma$

$$\min\{\alpha(\gamma) \,|\, \gamma \in \Gamma_x(f)\} = \min\{\alpha(\gamma) \,|\, \gamma \in \Gamma\},$$

where $\Gamma = \{\gamma \in \mathcal{C}_{\mathscr{X},x} \,|\, c_\gamma \neq 0\}$ as in (3.2.2.7). Indeed, any element that belongs to Γ can be written as a sum of an element of the initial support of f and an element of $\mathcal{C}_{\mathscr{X},x}$. Since the morphism α is additive and takes positive real values, the minimum is necessarily attained by the elements in the initial support.

(3.2.2.14) We will denote the valuation v from Proposition 3.2.2.10 by $v_{x,\alpha}$. Since $v_{x,\alpha}$ induces a real valuation on the function field of $\mathscr{X}_K$ that extends the discrete valuation v_K on K, it defines a point of the K-analytic space $\mathscr{X}_K^{\mathrm{an}}$, which we will denote by the same symbol $v_{x,\alpha}$. We now show that the characterization of $v_{x,\alpha}$ in Proposition 3.2.2.10 implies that

$$v_{y,\alpha'} = v_{x,\alpha' \circ \tau_{x,y}}$$

for every y in $F(x)$ and every α' in σ_y.

First we note that $\mathcal{O}_{\mathscr{X},y}$ is the localization of $\mathcal{O}_{\mathscr{X},x}$ with respect to the elements $m \in \mathcal{M}_{\mathscr{X},x}$ in the kernel of $\tau_{x,y}$. Indeed, by construction of $\tau_{x,y}$, the kernel is given by

$$\ker(\tau_{x,y}) = \{s \in \mathcal{C}_{\mathscr{X},x} | s(y) \neq 0\}.$$

To obtain $\mathcal{O}_{\mathscr{X},y}$ from $\mathcal{O}_{\mathscr{X},x}$, we localize by

$$S_y = \{a \in \mathcal{O}_{\mathscr{X},x} | a(y) \neq 0\}.$$

Therefore we can identify the set of elements of $\mathcal{M}_{\mathscr{X},x}$ whose reduction is in $\ker(\tau_{x,y})$ with the set S_y, recalling that, for points in the Kato fan, $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$ generates the maximal ideal of $\mathcal{O}_{\mathscr{X},x}$. Therefore we are dealing with these two morphisms:

$$\mathcal{O}_{\mathscr{X},x} \hookrightarrow S_y^{-1}\mathcal{O}_{\mathscr{X},x} = \mathcal{O}_{\mathscr{X},y},$$

$$\mathcal{C}_{\mathscr{X},x} \twoheadrightarrow \mathcal{C}_{\mathscr{X},x}/S_y = \mathcal{C}_{\mathscr{X},y}.$$

Let f be an element of $\mathcal{O}_{\mathscr{X},x}$. In the notation of Lemma 3.2.2.3, we apply the lemma to $A = \mathcal{O}_{\mathscr{X},y}$ and $B = \mathcal{O}_{\mathscr{X},x}$, choosing a system of generators of $\mathfrak{m}_y$ in $\mathcal{C}_{\mathscr{X},x}$: we can find an admissible expansion of f of the form

$$f = \sum_{\delta \in \mathcal{C}_{\mathscr{X},y}} d_\delta \delta \quad \text{with } d_\delta \in (\mathcal{O}_{\mathscr{X},x} \cap \mathcal{O}^{\times}_{\mathscr{X},y}) \cup \{0\}.$$

Admissible expansions of coefficients d_δ induce an admissible expansion for f by

$$f = \sum_{\delta \in \mathcal{C}_{\mathscr{X},y}} (\sum_{\gamma \in S} c_{\gamma\delta}\gamma)\delta \quad \text{with } c_{\gamma\delta} \in \mathcal{O}^{\times}_{\mathscr{X},x} \cup \{0\},$$

where γ runs through the set S_y since $d_\delta \in \mathcal{O}^{\times}_{\mathscr{X},y}$. Thus we have

$$\begin{aligned} v_{y,\alpha'}(f) &= \min\{\alpha'(\delta) \,|\, \delta \in \Gamma_y(f)\} \\ &= \min\{\alpha' \circ \tau_{x,y}(\gamma\delta) \,|\, \delta \in \Gamma_y(f), \gamma \in S\} \\ &= \min\{\alpha' \circ \tau_{x,y}(\gamma\delta) \,|\, \gamma\delta \in \Gamma_x(f)\} \\ &= v_{x,\alpha' \circ \tau_{x,y}}(f). \end{aligned}$$

Hence, we obtain a well defined map

$$\iota : \mathrm{Sk}(\mathscr{X}^+) \to \mathscr{X}_K^{\mathrm{an}}$$

by sending α to $v_{x,\alpha}$ for every point x of F and every $\alpha \in \sigma_x$.

Proposition 3.2.2.15. *The map*

$$\iota : \mathrm{Sk}(\mathscr{X}^+) \to \mathscr{X}_K^{\mathrm{an}}$$

is a topological embedding.

Proof. First, we show that ι is injective. Let x be a point of F and α an element of σ_x. Let y be the point of $F(x)$ corresponding to the face $\mathcal{C}_{\mathscr{X},x} \setminus \alpha^{-1}(0)$ of $\mathcal{C}_{\mathscr{X},x}$. Then α factors through an element

$$\alpha' : \mathcal{C}_{\mathscr{X},y} \to \mathbb{R}_{\geqslant 0}$$

of σ_y. Note that $\alpha = \alpha'$ in $\mathrm{Sk}(\mathscr{X}^+)$ because $\alpha = \alpha' \circ \tau_{x,y}$. Moreover, since $(\alpha')^{-1}(0) = \{1\}$, the centre of the valuation $v_{y,\alpha'}$ is the point y, so that $\mathrm{red}_{\mathscr{X}}(v_{y,\alpha'}) = y$. Thus we can recover y from $v_{y,\alpha'}$. Then we can also reconstruct α' by looking at the values of $v_{y,\alpha'}$ at the elements of $\mathcal{M}_{\mathscr{X},y}$. We conclude that ι is injective.

Now, we show that ι is a homeomorphism onto its image. For any valuation v in $\mathrm{Sk}(\mathscr{X}^+)$ and any small open neighborhood U of $\iota(v)$ in $\mathscr{X}_K^{\mathrm{an}}$, there exists a closed subset C in $\mathrm{Sk}(\mathscr{X}^+)$ such that $U \cap \iota(\mathrm{Sk}(\mathscr{X}^+)) \subseteq \iota(C)$ and, up to subdivisions, we can assume that the C is a closed cell of $\mathrm{Sk}(\mathscr{X}^+)$. Therefore, it suffices to prove that the restriction of ι to any closed cell σ_x of $\mathrm{Sk}(\mathscr{X}^+)$ is a homeomorphism. The restriction $\iota_{|\sigma_x}$ is an injective map from a compact set to the Hausdorff space $\mathscr{X}_K^{\mathrm{an}}$, so we reduce to showing that $\iota_{|\sigma_x}$ is continuous, to conclude that $\iota_{|\sigma_x}$ is a homeomorphism. By definition of the Berkovich topology, it is enough to prove that the map

$$\sigma_x \to \mathbb{R} : \alpha \mapsto v_{x,\alpha}(f)$$

is continuous for every f in $\mathcal{O}_{\mathscr{X},x}$. This is obvious from the formula (3.2.2.11). □

(3.2.2.16) From now on, we will view $\mathrm{Sk}(\mathscr{X}^+)$ as a topological subspace of $\mathscr{X}_K^{\mathrm{an}}$ by means of the embedding ι in Proposition 3.2.2.15. If $\mathscr{X}$ is regular over R and $\mathscr{X}_k$ is a divisor with strict normal crossings, the skeleton $\mathrm{Sk}(\mathscr{X}^+)$ was described in [MN15, §3.1].

3.2.3. Contracting the generic fibre to the skeleton.

(3.2.3.1) We denote by $D_{\mathscr{X},\mathrm{hor}}$ the component of $D_{\mathscr{X}}$ not contained in the special fibre $\mathscr{X}_k$. The inclusion $\iota : \mathrm{Sk}(\mathscr{X}^+) \to \mathscr{X}_K^{\mathrm{an}}$ is actually an inclusion in $(\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}}$ and it admits a continuous retraction

$$\rho_{\mathscr{X}} : (\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X}^+)$$

constructed as follows. Let x be a point of $(\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}}$ and consider the reduction map

$$\mathrm{red}_{\mathscr{X}} : (\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}} \to \mathscr{X}_k.$$

Let $E_1, \ldots, E_r$ be the irreducible components of $D_{\mathscr{X}}$ passing through the point $\mathrm{red}_{\mathscr{X}}(x)$. We denote by ξ the generic point of the connected component of $E_1 \cap \ldots \cap E_r$ that contains $\mathrm{red}_{\mathscr{X}}(x)$. By Lemma 2.5.2.3, ξ is a point in the associated Kato fan F. We set α to be the morphism of monoids

$$\alpha : \mathcal{C}_{\mathscr{X},\xi} \to \mathbb{R}_{\geqslant 0}$$

such that $\alpha(\overline{m}) = v_x(m)$ for any element m of $\mathcal{M}_{\mathscr{X},\xi}$. In particular $\alpha(\pi) = v_x(\pi) = 1$ as we assumed the normalization of all valuations in the Berkovich space. Then $\rho_{\mathscr{X}}(x)$ is the point of $\mathrm{Sk}(\mathscr{X}^+)$ corresponding to the couple (ξ, α). By construction $\rho_{\mathscr{X}}$ is continuous and right inverse to the inclusion ι.

(3.2.3.2) Given a dominant morphism $f : \mathscr{X}^+ \to \mathscr{Y}^+$ of integral flat separated log-regular log schemes over S, it induces a map between the set of birational points $\mathrm{Bir}(\mathscr{X}_K) \to \mathrm{Bir}(\mathscr{Y}_K)$. As $\mathrm{Bir}(\mathscr{X}_K) \subseteq (\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}}$, we can employ the retraction ρ to define a map of skeletons as follows

$$\begin{array}{ccc} (\mathscr{X}_K)^{\mathrm{bir}} & \xrightarrow{\widehat{f}} & (\mathscr{Y}_K)^{\mathrm{bir}} \\ {\scriptstyle \iota_{\mathscr{X}}} \uparrow \downarrow {\scriptstyle \rho_{\mathscr{X}}} & & \downarrow {\scriptstyle \rho_{\mathscr{Y}}} \\ \mathrm{Sk}(\mathscr{X}^+) & \dashrightarrow & \mathrm{Sk}(\mathscr{Y}^+). \end{array}$$

This association makes the skeleton construction $\mathrm{Sk}(\mathscr{X}^+)$ functorial in $\mathscr{X}^+$ with respect to dominant morphisms.

3.2.4. Semistability and Kato fans associated to the fibred products.

(3.2.4.1) We recall that a log-regular log scheme $\mathscr{X}^+$ is said to be semistable if the special fibre is reduced. We will see that semistability is a sufficient condition to establish injectivitiy of the local isomorphism (2.5.4.9).

(3.2.4.2) Given a log-regular log scheme $\mathscr{X}^+$ over S^+, the morphism $f : \mathscr{X}^+ \to S^+$ is called saturated if for any $x \in \mathscr{X}$ the morphism $\mathcal{C}_{S,f(x)} \to \mathcal{C}_{\mathscr{X},x}$ on the stalks of the characteristic sheaves is a saturated morphism of monoids; that is, if for any x and any morphism $u : \mathcal{C}_{S,f(x)} \to P$ of fs monoids, the amalgamated sum $\mathcal{C}_{\mathscr{X},x} \oplus_{\mathcal{C}_{S,f(x)}} P$ is still a saturated monoid.

Following the work by T. Tsuji in an unpublished 1997 preprint, Vidal in [Vid04] defines the notion of saturation index for morphisms of log schemes. In the case of a log-regular log scheme $\mathscr{X}^+$ over S^+, the saturation index at a point $x \in \mathscr{X}_k$ is the least common multiple of the multiplicities in $\mathscr{X}_k$ of the prime components of $\mathscr{X}_k$ passing through x. The following criterion holds.

Lemma 3.2.4.3. *[Vid04, §1.3] The morphism $\mathscr{X}^+ \to S^+$ is saturated if and only if the saturation index at any point of $\mathscr{X}_k$ is equal to 1.*

Proposition 3.2.4.4. *Assume that the residue field k is algebraically closed. Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log schemes over S^+. Let $\mathscr{Z}^+$ be their fs fibred product. If $\mathscr{X}^+$*

is semistable, then for any pair of points (x, y) in $F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$ whose closures intersect the special fibres $\mathscr{X}_k$ and $\mathscr{Y}_k$ respectively, the morphism $F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$, induced by the projections $\mathscr{Z}^+ \to \mathscr{X}^+$ and $\mathscr{Z}^+ \to \mathscr{Y}^+$, is a bijection above the pair (x, y), namely $n(x, y) = 1$.

Proof. Since $\mathscr{X}^+$ is a semistable log-regular log scheme over S^+, the saturation index of $\mathscr{X}^+ \to S^+$ at any point x in $\mathscr{X}_k$ is 1. Thus, by Lemma 3.2.4.3 the morphism $\mathscr{X}^+ \to S^+$ is saturated, and it follows that the fibred product in the category of log schemes coincides with the fibred product in the category of fs log schemes. This fact is crucial in the sequel of the proof to describe explicitly the divisor $D_{\mathscr{Z}}$, and to characterize the points of $\mathscr{Z}_k$.

We write $D_{\mathscr{X}} = \Delta_{\mathscr{X}} + \mathscr{X}_k$, where $\Delta_{\mathscr{X}} = \sum_i \Delta_{\mathscr{X},i}$ is the horizontal part and $\mathscr{X}_k = \sum_i E_{\mathscr{X},i}$; and $D_{\mathscr{Y}} = \Delta_{\mathscr{Y}} + \mathscr{Y}_{k,\mathrm{red}}$, with $\Delta_{\mathscr{Y}} = \sum_j \Delta_{\mathscr{Y},j}$ and $\mathscr{Y}_{k,\mathrm{red}} = \sum_j E_{\mathscr{Y},j}$. The divisor $D_{\mathscr{Z}}$, associated to the log structure of the fs fibred product, is given by $D_{\mathscr{Z}} = \Delta_{\mathscr{Z}} + \mathscr{Z}_{k,\mathrm{red}}$, where $\Delta_{\mathscr{Z}} = \Delta_{\mathscr{X}} \times \mathscr{Y} + \mathscr{X} \times \Delta_{\mathscr{Y}} = \sum_l \Delta_{\mathscr{Z},l}$ and $\mathscr{Z}_{k,\mathrm{red}} = \sum_h E_{\mathscr{Z},h}$. As k is algebraically closed, the irreducible components of $\mathscr{Z}_k = \mathscr{X}_k \times_k \mathscr{Y}_k$ are given by the products of irreducible components of $\mathscr{X}_k$ and $\mathscr{Y}_k$; that is, for any h we have $E_{\mathscr{Z},h} = E_{\mathscr{X},i_h} \times_k E_{\mathscr{Y},j_h}$.

The points in the special fibre $\mathscr{Z}_k$ are characterized as follows:

$$z = (x, y, s, \mathfrak{p}) \text{ and } \mathcal{O}_{\mathscr{Z},z} = (\mathcal{O}_{\mathscr{X},x} \otimes_R \mathcal{O}_{\mathscr{Y},y})_{\mathfrak{p}}$$

where x and y are points of $\mathscr{X}^+$ and $\mathscr{Y}^+$ both mapped to s, while $\mathfrak{p}$ is a prime ideal of the tensor product of residue fields $\kappa(x) \otimes_k \kappa(y)$. We assume now that z is a point in $F_{\mathscr{Z}} \cap \mathscr{Z}_k$, and see the restrictions on x, y and $\mathfrak{p}$ that follow from this assumption.

1. We claim that the projections x and y are points of the respective Kato fans. By Lemma 2.5.2.3, z is a generic point of an intersection of irreducible components of $D_{\mathscr{Z}}$.

   If every irreducible component of $D_{\mathscr{Z}}$ containing z is in the special fibre, then the intersection is given by

   $$\cap_{h=1}^{r} E_{\mathscr{Z},h} = \cap_{h=1}^{r} E_{\mathscr{X},i_h} \times E_{\mathscr{Y},j_h} = (\cap_{h=1}^{r} E_{\mathscr{X},i_h}) \times (\cap_{h=1}^{r} E_{\mathscr{Y},j_h}).$$

   The projections x and y of z are generic points of strata in $\mathscr{X}_k$ and $\mathscr{Y}_k$, hence lie in the respective Kato fans.

   Otherwise, let $\Delta_{\mathscr{Z},1}, \ldots, \Delta_{\mathscr{Z},r'}$ be the horizontal components of $D_{\mathscr{Z}}$ passing through z. We consider the subscheme $\mathscr{D} = \cap_{l=1}^{r'} \Delta_{\mathscr{Z},l}$ of $\mathscr{Z}$, endowed with the divisorial log structure $D_{\mathscr{D}} = \mathscr{D}_{k,\mathrm{red}}$. Then $\mathscr{D}^+ = (\mathscr{D}, D_{\mathscr{D}})$ is a log-regular log scheme and z is a point in the Kato fan of $\mathscr{D}^+$ that corresponds to the generic point of an intersection of irreducible components of $D_{\mathscr{D}}$, that are all contained in the special fibre. Since

   $$\mathscr{D} = \cap_{l=1}^{r'} \Delta_{\mathscr{Z},l} = (\cap_{l=1}^{r''} \Delta_{\mathscr{X},l} \times \mathscr{Y}) \cap (\cap_{l=r''+1}^{r'} \mathscr{X} \times \Delta_{\mathscr{Y},l}) = (\cap_{l=1}^{r'} \Delta_{\mathscr{X},l}) \times (\cap_{l=r''+1}^{r'} \Delta_{\mathscr{Y},l})$$

is itself a fibred product of log-smooth log schemes, we conclude by the previous case.

2. We claim that the ideal $\mathfrak{p}$ is minimal. Since z lies in $F_{\mathscr{Z}}$, we have that $\dim \mathcal{O}_{\mathscr{Z},z} = \mathrm{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z}$. At the level of characteristic sheaves

$$\mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z} = \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} + \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Y},y} - 1,$$

holds, since the stalk at z of the characteristic sheaf of $\mathscr{Z}^+$ is equal to $\mathcal{C}_{\mathscr{Z},z} = \mathcal{C}_{\mathscr{X},x} \oplus_{\mathbb{N}} \mathcal{C}_{\mathscr{Y},y}$ and the morphism $\mathbb{N} \to \mathcal{C}_{\mathscr{X},x}$ is saturated. As x and y are both points in the associated Kato fans, the equality between the dimension of the local ring and the rank of the groupification of the characteristic sheaves holds for x and y as well, and we obtain

$$\dim \mathcal{O}_{\mathscr{Z},z} = \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z} = \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} + \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Y},y} - 1 = \dim \mathcal{O}_{\mathscr{X},x} + \dim \mathcal{O}_{\mathscr{Y},y} - 1.$$

By log-regularity of $\mathscr{Z}^+$ we have the inequality $\dim \mathcal{O}_{\mathscr{Z},z'} \geqslant \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z'}$ at any point z', hence the equality that holds for z necessarily implies that $\mathfrak{p}$ is a minimal prime ideal of $\kappa(x) \otimes_k \kappa(y)$.

Let (x, y) be a pair of points of $F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$ that lie in the respective special fibres. In order to determine the number $n(x, y)$ of preimages of (x, y) in $F_{\mathscr{Z}}$, we need to study the number of minimal prime ideals of the tensor product $\kappa(x) \otimes_k \kappa(y)$. Since the residue field k is an algebraically closed field, the tensor product $\kappa(x) \otimes_k \kappa(y)$ is a domain; in particular, it has a unique minimal prime ideal, namely 0. We obtain that $n(x, y) = 1$.

Let (x, y) be a pair of points in $F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$ whose closures intersect the special fibres, namely there exist $x' \in F_{\mathscr{X}} \cap \mathscr{X}_k$ and $y' \in F_{\mathscr{Y}} \cap \mathscr{Y}_k$ such that x' is in the closure of x and y' in the closure of y. Then, by the previous part of the proof and by Lemma 2.5.4.11, we have $n(x, y) \leqslant n(x', y') = 1$. □

3.2.5. Skeleton of an fs fibred product.

(3.2.5.1) Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log schemes over S^+, and let $\mathscr{Z}^+$ be their fs fibred product. Let

$$\mathrm{Sk}(\mathscr{Z}^+) \to \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

be the continuous map of skeletons functorially associated to the projections $\mathrm{pr}_{\mathscr{X}} : \mathscr{Z}^+ \to \mathscr{X}^+$ and $\mathrm{pr}_{\mathscr{Y}} : \mathscr{Z}^+ \to \mathscr{Y}^+$. We denote this map by $(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})})$ and we recall that it is constructed considering the following diagram.

$$\begin{array}{ccc} (\mathscr{Z}_K)^{\mathrm{bir}} & \xrightarrow{(\widehat{\mathrm{pr}_{\mathscr{X}}}, \widehat{\mathrm{pr}_{\mathscr{Y}}})} & (\mathscr{X}_K)^{\mathrm{bir}} \times (\mathscr{Y}_K)^{\mathrm{bir}} \\ {\scriptstyle \iota_{\mathscr{Z}}} \uparrow \downarrow {\scriptstyle \rho_{\mathscr{Z}}} & & \downarrow {\scriptstyle (\rho_{\mathscr{X}}, \rho_{\mathscr{Y}})} \\ \mathrm{Sk}(\mathscr{Z}^+) & \xrightarrow{(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})})} & \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+). \end{array} \tag{3.2.5.2}$$

Proposition 3.2.5.3. *Assume that the residue field k is algebraically closed. If $\mathscr{X}^+$ is semistable, then the map $(pr_{\mathrm{Sk}(\mathscr{X})}, pr_{\mathrm{Sk}(\mathscr{Y})})$ is a PL homeomorphism.*

Proof. We first provide an explicit description of the map (respectively of $\mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}$) in diagram (5.3.6.4). Then we show that the map $(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})})$ is injective and surjective.

Let $v_{z,\varepsilon}$ be the valuation in $\mathrm{Sk}(\mathscr{Z}^+)$ corresponding to a couple (z, ε) with $z \in F_{\mathscr{Z}} \cap \mathscr{Z}_k$ and $\varepsilon \in \sigma_z$. We consider the morphism of associated Kato fans

$$F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$$

as established in Proposition 2.5.4.8. We denote respectively by $\mathrm{pr}_{F_{\mathscr{X}}}$ and $\mathrm{pr}_{F_{\mathscr{Y}}}$ the projection to the first and second factor. Then $\mathrm{pr}_{F_{\mathscr{X}}}(z)$ is a point in the associated Kato fan $F_{\mathscr{X}}$, which we denote by x. We consider the morphism of monoids

$$i_x : \mathcal{C}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{Z},z}$$

and the composition

$$\begin{array}{lccccc} \mathrm{pr}_{\mathscr{X}}(\varepsilon): & \mathcal{C}_{\mathscr{X},x} & \xrightarrow{i_x} & \mathcal{C}_{\mathscr{Z},z} = (\mathcal{C}_{\mathscr{X},x} \oplus_{\mathbb{N}} \mathcal{C}_{\mathscr{Y},y})^{\mathrm{sat}} & \xrightarrow{\varepsilon} & \mathbb{R}_{\geqslant 0} \\ & a & \longmapsto & [a,1] & \longmapsto & \varepsilon([a,1]). \end{array}$$

It trivially satisfies $\varepsilon \circ i_x(\pi) = 1$. In order to conclude that it correctly defines a point in the skeleton $\mathrm{Sk}(\mathscr{X}^+)$, we need to check the compatibility with respect to the equivalence relation $\sim$. Indeed, suppose that $\varepsilon = \varepsilon' \circ \tau_{z,z'}$ for some $z \in \overline{\{z'\}}$. We denote by x' the projection of z' under the local isomorphism of associated Kato fans. The diagram

$$\begin{array}{ccccc} \mathcal{C}_{\mathscr{X},x} & \xrightarrow{i_x} & \mathcal{C}_{\mathscr{Z},z} & \overset{\varepsilon}{\searrow} & \\ \downarrow{\scriptstyle \tau_{x,x'}} & & \downarrow{\scriptstyle \tau_{z,z'}} & & \mathbb{R}_{\geqslant 0} \\ \mathcal{C}_{\mathscr{X},x'} & \xrightarrow{i_{x'}} & \mathcal{C}_{\mathscr{Z},z'} & \underset{\varepsilon'}{\nearrow} & \end{array}$$

is commutative as it is made up of a commutative square and a commutative triangle of arrows. Therefore, by commutativity

$$\mathrm{pr}_{\mathscr{X}}(\varepsilon) = \mathrm{pr}_{\mathscr{X}}(\varepsilon') \circ \tau_{x,x'}$$

and this implies that $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$ defines a well defined point $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ of $\mathrm{Sk}(\mathscr{X}^+)$.

We claim that $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ is indeed the image of $v_{z,\varepsilon}$ under the map $\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}$. We recall that the projection $\widehat{\mathrm{pr}_{\mathscr{X}}}$, in diagram (5.3.6.4), is such that a valuation v on the function field $K(\mathscr{Z}_K)$ maps to the composition $v \circ i$ where $i : K(\mathscr{X}_K) \hookrightarrow K(\mathscr{Z}_K)$. Thus, we need to prove that the equality in the following inner diagram holds

$$\begin{array}{ccc}
(\mathscr{Z}_K)^{\mathrm{bir}} & \xrightarrow{\qquad\qquad\widehat{\mathrm{pr}_{\mathscr{X}}}\qquad\qquad} & (\mathscr{X}_K)^{\mathrm{bir}} \\
{\scriptstyle \iota_{\mathscr{Z}}}\uparrow\downarrow{\scriptstyle \rho_{\mathscr{Z}}} & \begin{array}{ccc} v_{z,\varepsilon} & \longmapsto & v_{z,\varepsilon}\circ i \\ \updownarrow & & \updownarrow \\ v_{z,\varepsilon} & \longmapsto & \rho_{\mathscr{X}}(v_{z,\varepsilon}\circ i) = v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)} \end{array} & \downarrow{\scriptstyle \rho_{\mathscr{X}}} \\
\mathrm{Sk}(\mathscr{Z}^+) & \xrightarrow{\qquad\qquad\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}\qquad\qquad} & \mathrm{Sk}(\mathscr{X}^+).
\end{array}$$

We denote $\rho_{\mathscr{X}}(v_{z,\varepsilon}\circ i)$ by (x,α) as a point of $\mathrm{Sk}(\mathscr{X}^+)$. By definition of the retraction $\rho_{\mathscr{X}}$, the morphism α is characterized by the fact that $\alpha(\overline{m}) = (v_{z,\varepsilon}\circ i)(m)$ for any m in $\mathcal{M}_{\mathscr{X},x}$ and then we have

$$\alpha(\overline{m}) = (v_{z,\varepsilon}\circ i)(m) = v_{z,\varepsilon}(m) = \varepsilon(\overline{m}).$$

On the other hand, for any m in $\mathcal{M}_{\mathscr{X},x}$

$$v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}(m) = \mathrm{pr}_{\mathscr{X}}(\varepsilon)(\overline{m}) = \varepsilon(\overline{m})$$

hence we obtain that α coincides with the morphism $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$. This means that their associated points $\rho_{\mathscr{X}}(v_{z,\varepsilon}\circ i)$ and $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ coincide in $\mathrm{Sk}(\mathscr{X}^+)$.

We now prove injectivity and surjectivity of the map $(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})})$. Given a pair of points in $\mathrm{Sk}(\mathscr{X}^+)\times\mathrm{Sk}(\mathscr{Y}^+)$, they are of the form $(v_{x,\alpha}, v_{y,\beta})$, with $x\in F_{\mathscr{X}}\cap\mathscr{X}_k$, $y\in F_{\mathscr{Y}}\cap\mathscr{Y}_k$, $\alpha\in\sigma_x$ and $\beta\in\sigma_y$. The assumptions of semistability of $\mathscr{X}^+$ and algebraic closedness of k guarantee that there is a unique z in $F_{\mathscr{Z}}$ in the fibre of x and y, by Proposition 3.2.4.4 and Remark 3.2.1.4. We claim that we can construct a unique $\varepsilon\in\sigma_z$ such that $(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})})(v_{z,\varepsilon}) = (v_{x,\alpha}, v_{y,\beta})$. We set

$$\begin{aligned}
\varepsilon:\quad \mathcal{C}_{\mathscr{Z},z} = (\mathcal{C}_{\mathscr{X},x}\oplus_{\mathbb{N}}\mathcal{C}_{\mathscr{Y},y})^{\mathrm{sat}} &\longrightarrow \mathbb{R}_{\geqslant 0}\\
[a,1] &\longrightarrow \alpha(a)\\
[1,b] &\longrightarrow \beta(b);
\end{aligned}$$

this is well defined as $\varepsilon([\pi,1]) = \alpha(\pi) = 1 = \beta(\pi) = \varepsilon([1,\pi])$. By construction, the image of $v_{z,\varepsilon}$ is $(v_{x,\alpha}, v_{y,\beta})$, so the map $(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})})$ is surjective. Moreover, we can uniquely reconstruct ε by looking at the values of $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ at the elements of $\mathcal{M}_{\mathscr{X},x}$ and of $v_{y,\mathrm{pr}_{\mathscr{Y}}(\varepsilon)}$ at the elements of $\mathcal{M}_{\mathscr{Y},y}$. Thus $(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})})$ is injective. □

The assumption of semistability is crucial in the result of Proposition 3.2.5.3. To see this, it is helpful to consider an example.

Example 3.2.5.4. Let q be the equation of a generic quartic curve in $\mathbb{P}^2_{\mathbb{C}((t))}$. Then $\mathscr{X}: tq + x^2y^2 = 0$ gives the equation of a family of genus 3 curves, degenerating to two double lines. The dual complex $\mathcal{D}(\mathscr{X}_{\mathbb{C}})$ of the special fibre $\mathscr{X}_{\mathbb{C}}$ is a line segment and $\mathscr{X}$ has four singularities of type A_1 in each component of the special fibre, corresponding to the base points of the family. In this case taking a semistable model of $\mathscr{X}_{\mathbb{C}((t))}$ requires an order two base change R' of $R = \mathbb{C}[[t]]$, which induces coverings branched at each of these singular points (see [HM98, p. 133] for details). Let $\mathscr{Y}$ be such a semistable reduction. Thus the special fibre of $\mathscr{Y}$ consists of two elliptic curves, call them E_1 and E_2, which intersect in two points, p_A and p_B, which are the preimages of the point $(0:0:1)$. The dual complex $\mathcal{D}(\mathscr{Y}_{\mathbb{C}})$ of the special fibre $\mathscr{Y}_{\mathbb{C}}$ is isomorphic to S^1.

We will compare the dual complex of $(\mathscr{X} \times_R \mathscr{X})_{\mathbb{C}}$ with that of $(\mathscr{Y} \times_{R'} \mathscr{Y})_{\mathbb{C}}$. The models $\mathscr{X}$ and $\mathscr{Y}$ are not log-regular at every point, but from our perspective it is enough that they are log-regular at the generic point of each stratum. For the product with a semistable model, the dual complex is the product of the dual complexes, and $\mathcal{D}((\mathscr{Y} \times_{R'} \mathscr{Y})_{\mathbb{C}})$ is therefore a real 2-torus $S^1 \times S^1$.

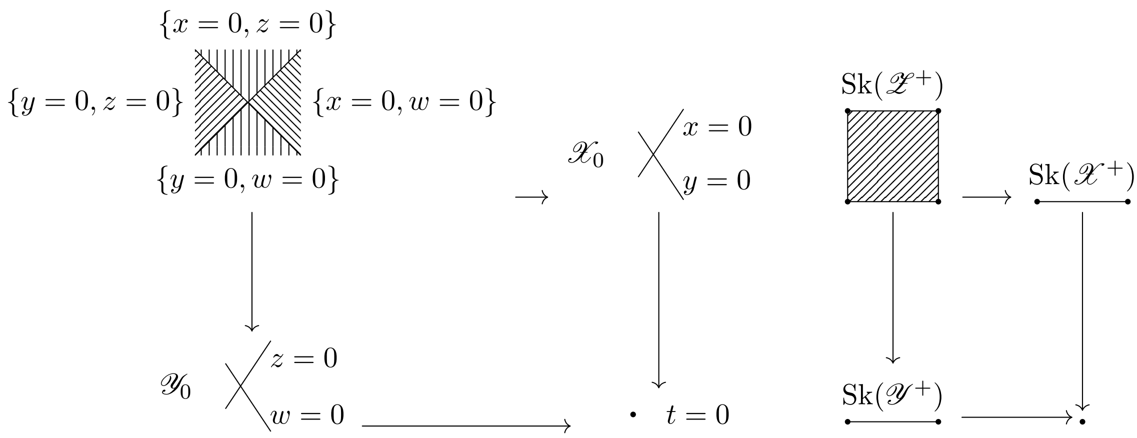
On the other hand, the dual complex of the product $(\mathscr{X} \times_R \mathscr{X})_{\mathbb{C}}$ is given by a quotient of $S^1 \times S^1$ by the action of $\mathbb{Z}/2\mathbb{Z}$. $\mathcal{D}((\mathscr{Y} \times_{R'} \mathscr{Y})_{\mathbb{C}})$ has the structure of a cell complex, whose cells correspond to ordered pairs of strata in $\mathcal{D}(\mathscr{Y}_{\mathbb{C}})$, so

zero-dimensional strata:	$(E_1, E_2), (E_1, E_1), (E_2, E_1), (E_2, E_2)$
one-dimensional strata:	$(E_1, p_A), (E_1, p_B), (E_2, p_A), (E_2, p_B)$
	$(p_A, E_1), (p_B, E_1), (p_A, E_2), (p_B, E_2)$
two-dimensional strata:	$(p_A, p_A), (p_A, p_B), (p_B, p_A), (p_B, p_B)$.

The action of $\mathbb{Z}/2\mathbb{Z}$ fixes E_1 and E_2, while switching p_A and p_B. Therefore it fixes exactly the zero-dimensional strata while acting freely on the other points. The quotient, the complex $\mathcal{D}((\mathscr{X} \times_R \mathscr{X})_{\mathbb{C}})$, is piecewise linearly homeomorphic to the sphere S^2. In particular, it is not isomorphic to the product of two line segments.

Moreover, we show through an example that the fibred product of snc models is not snc in general. This motivates the introduction of the language of log-regular models.

Example 3.2.5.5. Let $R = \mathbb{C}[[t]]$ and $\mathscr{X}^+ = \left(\operatorname{Spec} \frac{R[x,y]}{(xy=t)}, \mathscr{X}_0 = \{xy = 0\}\right)$ and $\mathscr{Y}^+ = \left(\operatorname{Spec} \frac{R[z,w]}{(zw=t)}, \mathscr{Y}_0 = \{zw = 0\}\right)$. Their special fibres are snc, have two irreducible components intersecting in a point, so the associated skeletons are both isomorphic to the closed unit interval. The special fibre of the product $\mathscr{Z}$ consists of four irreducible components and is not an snc divisor: indeed, the four components intersect in a codimension three stratum. Thus, the dual complex is not well defined.



3.3. The weight function

From now on we assume $\mathrm{char}(K) = \mathrm{char}(k) = 0$.

3.3.1. Weight function associated to a logarithmic pluricanonical form.

(3.3.1.1) Let X be a connected, smooth and proper K-variety of dimension n. We introduce the following notation: for any log-regular model $\mathscr{X}^+$ of X, for any point $x = (\xi_x, |\cdot|_x) \in X^{\mathrm{an}}$ and for any $\mathbb{Q}$-Cartier divisor D on $\mathscr{X}^+$ whose support does not contain ξ_x, we set

$$v_x(D) = -\ln |f(x)|^{\frac{1}{m}}$$

where f is any element of $K(X)^\times$ such that $mD = \mathrm{div}(f)$ locally at $\mathrm{red}_{\mathscr{X}}(x)$, and $m \in \mathbb{Z}_{>0}$ is such that mD is Cartier.

(3.3.1.2) Let (X, Δ_X) be an snc pair as in §2.1.1, so that Δ_X is an effective $\mathbb{Q}$-divisor such that $\Delta_X = \sum a_i \Delta_{X,i}$ has $0 \leqslant a_i \leqslant 1$, and the round-up $X^+ = (X, \lceil \Delta_X \rceil)$ is snc. Let ω be a regular m-pluricanonical form on X^+ with poles of order at most ma_i along $\Delta_{X,i}$, for some m such that $ma_i \in \mathbb{N}$ for any i. Thus, such a form is a section of $\mathcal{O}_X(m(K_X + \Delta_X))$, and we call it a Δ_X-logarithmic m-pluricanonical form.

Given a log-regular model $\mathscr{X}^+$ of X^+, where $D_{\mathscr{X}} = \overline{\lceil \Delta_X \rceil} + \mathscr{X}_{k,\mathrm{red}}$, we can view any Δ_X-logarithmic m-pluricanonical forms as rational sections of the logarithmic m-pluricanonical bundle $(\omega^{\log}_{\mathscr{X}^+/S^+})^{\otimes m}$, that is, of

$$(\omega^{\log}_{\mathscr{X}^+/S^+})^{\otimes m} = \omega^{\otimes m}_{\mathscr{X}/S} \otimes \mathcal{O}_{\mathscr{X}}(m(\overline{\lceil \Delta_X \rceil} + \mathscr{X}_{k,\mathrm{red}} - \mathrm{div}(\pi))).$$

The form ω, viewed as a rational section of $(\omega^{\log}_{\mathscr{X}^+/S^+})^{\otimes m}$, defines a divisor $\mathrm{div}_{\mathscr{X}^+}(\omega)$ on $\mathscr{X}^+$. Note that the multiplicity in $\mathrm{div}_{\mathscr{X}^+}(\omega)$ of the closure $\overline{\Delta_{X,i}}$ in $\mathscr{X}$ of $\Delta_{X,i}$ is at least $m(1 - a_i)$.

(3.3.1.3) Given a Δ_X-logarithmic m-pluricanonical form ω, we can consider it as a rational section of $\omega^{\otimes m}_{X/K}$. Hence, we can associate to ω the weight function wt_ω as in [MN15].

The following lemma gives an interpretation of the weight function associated to ω in terms of logarithmic differentials, which we will use in the sequel.

Lemma 3.3.1.4. *Let $\mathscr{X}^+$ be a log-regular model of X^+. Then for every point x of* $\mathrm{Sk}(\mathscr{X}^+)$

$$\mathrm{wt}_\omega(x) = v_x(div_{\mathscr{X}^+}(\omega)) + m.$$

Proof. It suffices to prove the equality for the divisorial points in $\mathrm{Sk}(\mathscr{X}^+)$, since they are dense in the skeleton, and both $\mathrm{wt}_\omega(\cdot)$ and $v.(\mathrm{div}_{\mathscr{X}^+}(\omega))$ are continuous functions on the skeleton $\mathrm{Sk}(\mathscr{X}^+)$.

Let x be a divisorial point in $\mathrm{Sk}(\mathscr{X}^+)$. If x corresponds to a component of the special fibre $\mathscr{X}_k$, then the centre $\mathrm{red}_{\mathscr{X}}(x)$ of x does not contain the closure of any components of Δ_X. Thus, locally around $\mathrm{red}_{\mathscr{X}}(x)$ the log schemes $\mathscr{X}^+$ and $(\mathscr{X}, \mathscr{X}_{k,\mathrm{red}})$ are isomorphic and, in particular, $\mathrm{div}_{\mathscr{X}^+}(\omega) = \mathrm{div}_{(\mathscr{X},\mathscr{X}_{k,\mathrm{red}})}(\omega)$. The computation in [NX16, §3.2.2] shows that $\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{(\mathscr{X},\mathscr{X}_{k,\mathrm{red}})}(\omega)) + m$, so we obtain the required equality.

Otherwise, we consider the blow-up $h : \mathscr{X}'^+ \to \mathscr{X}^+$ at the closure of the centre $\mathrm{red}_{\mathscr{X}}(x)$ of x in $\mathscr{X}^+$. By [KM08, Lemma 2.45][1] iterating this procedure a finite number of times, we obtain a log-regular model $\mathscr{Y}^+$ such that the x corresponds to a component E of the special fibre $\mathscr{Y}_k$. By reduction to the previous case, it is enough to check that the value $v_x(\mathrm{div}_{\mathscr{X}^+}(\omega))$ does not change under such a blow-up h.

The morphism h is the blow-up of a stratum of the log boundary $\overline{\lceil \Delta_X \rceil} + \mathscr{X}_{k,\mathrm{red}}$, hence h is a log étale morphism. By [Kat89, Proposition 3.12] we have $h^*(\omega^{\log}_{\mathscr{X}^+/S^+}) = \omega^{\log}_{\mathscr{X}'^+/S^+}$. It follows that $\mathrm{div}_{\mathscr{X}'^+}(\omega) = h^*(\mathrm{div}_{\mathscr{X}^+}(\omega))$ and this concludes the proof. □

(3.3.1.5) We recall from [MN15, §4.7] that the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X, \Delta_X, \omega)$ is the closure in $\mathrm{Bir}(X)$ of the set of divisorial points of X^{an} where the weight function wt_ω reaches its minimal weight, namely

$$\mathrm{wt}_\omega(X, \Delta_X) = \inf\{\mathrm{wt}_\omega(x) \,|\, x \in \mathrm{Div}(X)\} \in \mathbb{R} \cup \{-\infty\}.$$

A priori the weight function associated to a rational pluricanonical form may have minimal weight $-\infty$, hence the corresponding Kontsevich–Soibelman skeleton would be empty. We prove that this does not occur for Δ_X-logarithmic pluricanonical forms.

Proposition 3.3.1.6. *Given a Δ_X-logarithmic m-pluricanonical form ω, for any log-regular model $\mathscr{X}^+$ of X^+ the inclusion* $\mathrm{Sk}(X, \Delta_X, \omega) \subseteq \mathrm{Sk}(\mathscr{X}^+)$ *holds.*

Proof. Let $\mathscr{X}^+$ be a log-regular model of X^+ and let y be a divisorial point of X^{an}. It suffices to prove that

$$\mathrm{wt}_\omega(y) \geqslant \mathrm{wt}_\omega(\rho_{\mathscr{X}}(y))$$

[1] The proof in [KM08] considers the case of a variety over a field, but it generalizes to varieties of finite type over a discrete valuation ring.

and that the equality holds if and only if y is in $\mathrm{Sk}(\mathscr{X}^+)$. As in the proof of Lemma 3.3.1.4, we consider the blow-up of $\mathscr{X}^+$ at the closure of $\mathrm{red}_{\mathscr{X}}(y)$: iterating this procedure a finite number of times, we obtain a log-regular model $\mathscr{Y}^+$ such that $y \in \mathrm{Sk}(\mathscr{Y}^+)$.

Let $h : \mathscr{Z}^+ \to \mathscr{W}^+$ be a morphism of this sequence. If $\overline{\{\mathrm{red}_{\mathscr{W}}(y)\}}$ is a stratum of $D_{\mathscr{W}}$, then the morphism h induces a subdivision of the skeleton $\mathrm{Sk}(\mathscr{W}^+)$, so $\rho_{\mathscr{Z}}(y) = \rho_{\mathscr{W}}(y)$.

Otherwise, $\overline{\{\mathrm{red}_{\mathscr{W}}(y)\}}$ is strictly contained in a stratum V of $\mathcal{D}_{\mathscr{W}}$. Let E be the exceptional divisor of h, $r = r_h + r_v$ be the codimension of V in $\mathscr{W}$, where r_h and r_v are the number of irreducible components of $\overline{\lceil \Delta_X \rceil}$ and respectively of the special fibre $\mathscr{W}_k$, containing V. Let $r + j$ be the codimension of $\overline{\{\mathrm{red}_{\mathscr{W}}(y)\}}$, where $j \geqslant 1$. We denote the projections onto S^+ by $s_{\mathscr{W}} : \mathscr{W}^+ \to S^+$ and $s_{\mathscr{Z}} : \mathscr{Z}^+ \to S^+$ and by π a uniformizer in R. Then we have that

$$\begin{aligned} h^*(\omega^{\log}_{\mathscr{W}^+/S^+}) &= h^*(\omega_{\mathscr{W}/R} \otimes \mathcal{O}_{\mathscr{W}}(\overline{\lceil \Delta_X \rceil}_{,\mathrm{red}} + \mathscr{W}_{k,\mathrm{red}} - s^*_{\mathscr{W}} \operatorname{div}(\pi))) \\ &= \omega_{\mathscr{Z}/R} \otimes \mathcal{O}_{\mathscr{Z}}((1 - r - j)E + \overline{\lceil \Delta_X \rceil}_{,\mathrm{red}} + r_h E + \mathscr{Z}_{k,\mathrm{red}} + (r_v - 1)E - s^*_{\mathscr{Z}} \operatorname{div}(\pi)) \\ &= \omega^{\log}_{\mathscr{Z}^+/S^+} \otimes \mathcal{O}_{\mathscr{Z}}(-jE). \end{aligned}$$

It follows that $\operatorname{div}_{\mathscr{Z}^+}(\omega) = h^*(\operatorname{div}_{\mathscr{W}^+}(\omega)) + mjE$, so

$$\begin{aligned} v_{\rho_{\mathscr{Z}}(y)}(\operatorname{div}_{\mathscr{Z}^+}(\omega)) &= v_{\rho_{\mathscr{Z}}(y)}(h^*(\operatorname{div}_{\mathscr{W}^+}(\omega)) + mjE) \\ &\geqslant v_{\rho_{\mathscr{W}}(y)}(\operatorname{div}_{\mathscr{W}^+}(\omega)) + mj v_{\rho_{\mathscr{W}}(y)}(E) \\ &> v_{\rho_{\mathscr{W}}(y)}(\operatorname{div}_{\mathscr{W}^+}(\omega)) \end{aligned}$$

where for the first inequality we apply [MN15, Proposition 3.1.6] while the second strict inequality holds as $j > 0$ and $v_{\rho_{\mathscr{W}}(y)}(E) > 0$ since the centre of the valuation $\rho_{\mathscr{W}}(y)$ is contained in E. Therefore, for any such morphism h, the weight is strictly increasing, namely $\mathrm{wt}_\omega(\rho_{\mathscr{Z}}(y)) > \mathrm{wt}_\omega(\rho_{\mathscr{W}}(y))$. This concludes the proof. □

Only the components of Δ_X with coefficient $a_i = 1$ determine strata that are contained in the Kontsevich–Soibelman skeletons. The introduction of Δ_X-log pluricanonical forms allows us to construct non-empty Kontsevich–Soibelman skeletons even for varieties with Kodaira dimension $-\infty$, as in the following examples.

Example 3.3.1.7. Let X be the projective line $\mathbb{P}^1_K$ with affine coordinates x and y, and $\Delta_X = (0:1) + (1:0)$. Then $a_i = 1$ for any i and there exist Δ_X-logarithmic canonical forms. For example, we consider

$$\omega = \frac{dx}{x} = -\frac{dy}{y}.$$

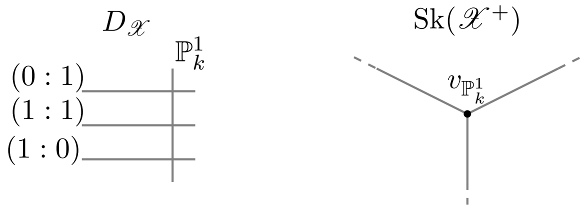
Let $\mathscr{X} = \mathbb{P}^1_R$ and $D_{\mathscr{X}} = (0:1) + (1:0) + \mathbb{P}^1_k$. The log scheme $\mathscr{X}^+ = (\mathscr{X}, D_{\mathscr{X}})$ is a log-regular model of $X^+ = (X, \lceil \Delta_X \rceil)$ and the associated skeleton $\mathrm{Sk}(\mathscr{X}^+)$ looks like this:

Since $\mathrm{div}_{\mathscr{X}^+}(\omega) = 0$, the weight associated to ω is minimal at any point of the skeleton $\mathrm{Sk}(\mathscr{X}^+)$. Thus $\mathrm{Sk}(X, \Delta_X, \omega) = \mathrm{Sk}(\mathscr{X}^+) \simeq \mathbb{R}$.

Example 3.3.1.8. Let $X = \mathbb{P}^1_K$ and $\Delta_X = \frac{2}{3}(0:1) + \frac{2}{3}(1:0) + \frac{2}{3}(1:1)$. So $a_i = \frac{2}{3}$ for any i and there exist Δ_X-logarithmic 3-pluricanonical forms. We set

$$\omega = \frac{1}{(x-1)^2} \cdot \frac{1}{x^2}(dx)^3 = -\frac{1}{(1-y)^2} \cdot \frac{1}{y^2}(dy)^3.$$

We consider $\mathscr{X} = \mathbb{P}^1_R$ and $D_{\mathscr{X}} = (0:1) + (1:0) + (1:1) + \mathbb{P}^1_k$; then $\mathscr{X}^+ = (\mathscr{X}, D_{\mathscr{X}})$ is a log-regular model of $X^+ = (X, \lceil \Delta_X \rceil)$ and $\mathrm{Sk}(\mathscr{X}^+)$ is



Since $\mathrm{div}_{\mathscr{X}^+}(\omega) = (0:1) + (1:0) + (1:1)$, the weight associated to ω is minimal at the divisorial point $v_{\mathbb{P}^1_k}$ corresponding to $\mathbb{P}^1_k$ and is strictly increasing with slope 1 along the unbounded edges, when we move away from the point $v_{\mathbb{P}^1_k}$. Therefore, $\mathrm{Sk}(X, \Delta_X, \omega) = \{v_{\mathbb{P}^1_k}\}$.

3.3.2. Weight function and Kontsevich–Soibelman skeleton for products.

(3.3.2.1) Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-regular models over S^+ of $X^+ = (X, \lceil \Delta_X \rceil)$ and $Y^+ = (Y, \lceil \Delta_Y \rceil)$ respectively. Then the fs fibred product $\mathscr{Z}^+ = \mathscr{X}^+ \times^{\mathrm{fs}}_{S^+} \mathscr{Y}^+$ is a log-regular model of $Z^+ := X^+ \times^{\mathrm{fs}}_K Y^+$. Therefore, given ω_{X^+} and ω_{Y^+} Δ_X-logarithmic and Δ_Y-logarithmic m-pluricanonical forms on (X, Δ_X) and (Y, Δ_Y) respectively, the form

$$\varpi = \mathrm{pr}^*_{X^+} \omega_{X^+} \otimes \mathrm{pr}^*_{Y^+} \omega_{Y^+}$$

is a Δ_Z-logarithmic m-pluricanonical form on (Z, Δ_Z), where $\Delta_Z = X \times_K \Delta_Y + \Delta_X \times_K Y$. Viewing these forms as rational sections of logarithmic m-pluricanonical bundles, we see that $\mathrm{div}_{\mathscr{Z}^+}(\varpi) = \mathrm{pr}^*_{\mathscr{X}^+}(\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})) + \mathrm{pr}^*_{\mathscr{Y}^+}(\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+}))$ according to (2.5.4.6).

(3.3.2.2) Let z be a point of $F_{\mathscr{Z}} \cap \mathscr{Z}_k$; as before, we denote by x and y the images of z under the local isomorphism $F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$. Any morphism $\varepsilon \in \sigma_z$ defines a point $v_{z,\varepsilon}$ in $\mathrm{Sk}(\mathscr{Z}^+)$. For the sake of convenience, we simply denote the valuations by the corresponding morphism and we denote $\alpha = \mathrm{pr}_{\mathscr{X}}(\varepsilon)$ and $\beta = \mathrm{pr}_{\mathscr{Y}}(\varepsilon)$. We aim to relate the valuation $v_\varepsilon(\mathrm{div}_{\mathscr{Z}^+}(\varpi))$ to the values

$$v_\alpha(\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+}))\ ,\ v_\beta(\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})).$$

(3.3.2.3) Let $f_x \in \mathcal{O}_{\mathscr{X},x}$ be a local equation of $\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})$ around x. In order to evaluate $v_{x,\alpha}$ on f_x, we consider an admissible expansion of f_x as in (3.2.2.6)

$$f_x = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma.$$

Furthermore, this expansion also induces an expansion of $\mathrm{pr}^*_{\mathscr{X}}(f_x)$ by

$$\mathrm{pr}^*_{\mathscr{X}}(f_x) = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} \mathrm{pr}^*_{\mathscr{X}}(c_\gamma)\gamma$$

as formal power series in $\widehat{\mathcal{O}}_{\mathscr{Z},z}$, since the morphism of characteristic sheaves $\mathcal{C}_{\mathscr{X},x} \hookrightarrow \mathcal{C}_{\mathscr{Z},z}$ is injective. Following the same procedure for a local equation $f_y \in \mathcal{O}_{\mathscr{Y},y}$ of $\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})$ around y, we get an expansion of f_y that extends to $\mathrm{pr}^*_{\mathscr{Y}}(f_y)$:

$$f_y = \sum_{\delta \in \mathcal{C}_{\mathscr{Y},y}} d_\delta \delta.$$

(3.3.2.4) A local equation of ϖ around z is determined by $\mathrm{pr}^*_{\mathscr{X}}(f_x)\ \mathrm{pr}^*_{\mathscr{Y}}(f_y)$. Thus

$$v_\varepsilon(\mathrm{div}_{\mathscr{Z}^+}(\varpi)) = v_\varepsilon(\mathrm{pr}^*_{\mathscr{X}}(f_x)\ \mathrm{pr}^*_{\mathscr{Y}}(f_y))$$

and by multiplicativity of the valuation v_ε

$$v_\varepsilon(\mathrm{pr}^*_{\mathscr{X}}(f_x)\ \mathrm{pr}^*_{\mathscr{Y}}(f_y)) = v_\varepsilon(\mathrm{pr}^*_{\mathscr{X}}(f_x)) + v_\varepsilon(\mathrm{pr}^*_{\mathscr{Y}}(f_y)).$$

Recalling Remark 3.2.2.13, the valuation can be computed as follows

$$v_\varepsilon(\mathrm{pr}^*_{\mathscr{X}}(f_x)) = \min\{\varepsilon(\gamma) \,|\, c_\gamma \neq 0\};$$

as the elements γ belong to $\mathcal{C}_{\mathscr{X},x}$ and α is defined to be $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$, we have

$$\min\{\varepsilon(\gamma) \,|\, c_\gamma \neq 0\} = \min\{\alpha(\gamma) \,|\, c_\gamma \neq 0\} = v_{x,\alpha}(f_x).$$

Hence, we conclude that

$$\begin{aligned} v_\varepsilon(\mathrm{div}_{\mathscr{Z}^+}(\varpi)) &= v_\varepsilon(\mathrm{pr}^*_{\mathscr{X}}(f_x)) + v_\varepsilon(\mathrm{pr}^*_{\mathscr{Y}}(f_y)) \\ &= v_\alpha(f_x) + v_\beta(f_y) \\ &= v_\alpha(\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})) + v_\beta(\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})). \end{aligned} \tag{3.3.2.5}$$

(3.3.2.6) This result turns out to be advantageous to compute the weight function wt_ϖ on divisorial points of $\mathrm{Sk}(\mathscr{Z}^+)$:

$$\begin{aligned} \mathrm{wt}_\varpi(\varepsilon) &= v_\varepsilon(\mathrm{div}_{\mathscr{Z}^+}(\varpi)) + m \\ &= v_\alpha(\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})) + v_\beta(\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})) + m \\ &= \mathrm{wt}_{\omega_{X^+}}(\alpha) + \mathrm{wt}_{\omega_{Y^+}}(\beta) - m. \end{aligned} \tag{3.3.2.7}$$

(3.3.2.8) In the notation of the previous paragraphs, our computations lead to the following result.

Theorem 3.3.2.9. *Suppose that the residue field k is algebraically closed and that $\mathscr{X}^+$ is semistable. Then the PL homeomorphism of skeletons*

$$\mathrm{Sk}(\mathscr{Z}^+) \xrightarrow{\sim} \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

given in Proposition 3.2.5.3 restricts to a PL homeomorphism of Kontsevich–Soibelman skeletons

$$\mathrm{Sk}(Z, \Delta_Z, \varpi) \xrightarrow{\sim} \mathrm{Sk}(X, \Delta_X, \omega_{X^+}) \times \mathrm{Sk}(Y, \Delta_Y, \omega_{Y^+}).$$

Proof. This follows immediately from equality (3.3.2.7), which shows that a point in $\mathrm{Sk}(\mathscr{Z}^+)$ has minimal value $\mathrm{wt}_\varpi(Z, \Delta_Z)$ if and only if its projections have minimal value $\mathrm{wt}_{\omega_{X^+}}(X, \Delta_X)$ and $\mathrm{wt}_{\omega_{Y^+}}(Y, \Delta_Y)$. □

3.4. The essential skeleton of a product

For this section, we assume the residue field k is algebraically closed. We recall that π denotes a uniformiser in R, and $S = \mathrm{Spec}(R)$.

3.4.1. Essential skeleton of a pair.

(3.4.1.1) Let X be a proper variety over K and (X, Δ_X) an snc pair as in §2.1.1. Let $\mathscr{X}^+$ be a log-regular model of $(X, \lceil \Delta_X \rceil)$ over S^+. Let ω be a non-zero regular Δ_X-logarithmic m-pluricanonical form on (X, Δ_X). There exist minimal positive integers d and n such that the divisor $\mathrm{div}_{\mathscr{X}^+}(\omega^{\otimes d}\pi^{-n})$ is effective and the multiplicity of some component of the special fibre is zero: we denote this divisor by $D_{\min}(\mathscr{X}, \omega)$. It follows from the properties of the weight function (see [MN15, Proposition 4.5.5]) that for any $x \in \mathrm{Sk}(\mathscr{X}^+)$

$$v_x(D_{\min}(\mathscr{X}, \omega)) + dm = \mathrm{wt}_{\omega^{\otimes d}\pi^{-n}}(x) = d \cdot \mathrm{wt}_\omega(x) + v_x(\pi^{-n}) = d \cdot \mathrm{wt}_\omega(x) - n.$$

Lemma 3.4.1.2. *Let $v_{x,\alpha}$ be a divisorial point in $\mathrm{Sk}(\mathscr{X}^+)$. Then $v_{x,\alpha} \in \mathrm{Sk}(X, \Delta_X, \omega)$ if and only if $D_{\min}(\mathscr{X}, \omega)$ does not contain x.*

Proof. We denote $v_{x,\alpha}$ simply by α. By the above series of equalities, the weight function wt_ω reaches its minimum at α if and only if $v_\alpha(D_{\min}(\mathscr{X}, \omega))$ is minimal, hence, in particular, equal to zero.

Let $h: \mathscr{Y}^+ \to \mathscr{X}^+$ be a sequence of blow-up morphisms of strata of $D_{\mathscr{X}}$ such that α corresponds to an irreducible component E of $D_{\mathscr{Y}}$. As in the proof of Proposition 3.3.1.6

$$h^*(D_{\min}(\mathscr{X}, \omega)) = h^*(\mathrm{div}_{\mathscr{X}^+}(\omega^{\otimes d}\pi^{-n})) = \mathrm{div}_{\mathscr{Y}^+}(\omega^{\otimes d}\pi^{-n}).$$

Therefore we have that $v_\alpha(D_{\min}(\mathscr{X}, \omega)) > 0$ if and only if $E \subseteq \mathrm{div}_{\mathscr{Y}^+}(\omega^{\otimes d}\pi^{-n})$, and this holds if and only if $x \in D_{\min}(\mathscr{X}, \omega)$. □

(3.4.1.3) We define the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$ of an snc pair (X, Δ_X) as the union of all Kontsevich–Soibelman skeletons $\mathrm{Sk}(X, \Delta_X, \omega)$, where ω ranges over all regular Δ_X-logarithmic pluricanonical forms. In the case of an empty boundary, this recovers the notions introduced in [MN15].

The further reason to define the essential skeleton this way is that it behaves nicely under birational morphisms. Let $f\colon X' \to X$ be a log resolution. Then there is a $\mathbb{Q}$-divisor Γ' with snc support, and no coefficient exceeding 1, such that $K_{X'} + \Gamma' = f^*(K_X + \Delta_X)$. Take $\Delta_{X'}$ to be the positive part of Γ' and write

$$K_{X'} + \Delta_{X'} = f^*(K_X + \Delta_X) + N.$$

For any m, pullback along with multiplication by the divisor of discrepancies N induces an isomorphism of vector spaces

$$H^0(X, mK_X + m\Delta_X) \cong H^0(X', mK_{X'} + m\Delta_{X'}). \tag{3.4.1.4}$$

Let ω and ω' be corresponding forms via this isomorphism.

Proposition 3.4.1.5. *Under the identification of the birational points of X with those of X', $\mathrm{Sk}(X, \Delta_X, \omega)$ is identified with $\mathrm{Sk}(X', \Delta_{X'}, \omega')$.*

Proof. The Kontsevich–Soibelman skeleton $\mathrm{Sk}(X, \Delta_X, \omega)$ is contained in the skeleton associated to any log-regular model by Proposition 3.3.1.6. Thus, we choose log-regular models $\mathscr{X}^+$ and $\mathscr{X}'^+$ so that f extends to a log resolution $f_R\colon \mathscr{X}' \to \mathscr{X}$ and the pair $(\mathscr{X}, \Delta_{\mathscr{X}})$ is snc, where $\Delta_{\mathscr{X}} = \overline{\Delta_X} + \mathscr{X}_{k,\mathrm{red}}$. Likewise we denote $\Delta_{\mathscr{X}'} = \overline{\Delta_{X'}} + \mathscr{X}'_{k,\mathrm{red}}$, and $(\mathscr{X}', \Delta_{\mathscr{X}'})$ is snc as f_R is a log resolution. It suffices to check the proposition for divisorial valuations. By Lemma 3.4.1.2 a divisorial valuation v of $\mathrm{Sk}(\mathscr{X}^+)$ is in $\mathrm{Sk}(X, \Delta_X, \omega)$ if and only if it is a log canonical centre of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and the divisor $D_{\min}(\mathscr{X}, \omega)$ does not contain the centre of v.

Suppose v is a divisorial point in $\mathrm{Sk}(X, \Delta_X, \omega)$. Without loss of generality, we can assume that the divisor $\mathrm{div}_{\mathscr{X}^+}(\omega)$ in $\mathscr{X}^+$ does not contain the centre of v. As $(\mathscr{X}, \Delta_{\mathscr{X}})$ is an snc pair, it is dlt (2.3.0.3) and we have that

$$K_{\mathscr{X}'} + \Delta_{\mathscr{X}'} = f_R^*(K_{\mathscr{X}} + \Delta_{\mathscr{X}}) + M$$

where M is effective, thus $\mathrm{div}_{\mathscr{X}'^+}(\omega') = f_R^*(\mathrm{div}_{\mathscr{X}^+}(\omega)) + mM$. As v is a log canonical centre of $(\mathscr{X}, \Delta_{\mathscr{X}})$, M does not vanish along v, so neither does $\operatorname{div}_{\mathscr{X}'^{+}}(\omega')$. Likewise v is a log canonical centre of $(\mathscr{X}', \Delta_{\mathscr{X}'})$. It follows that $v \in \mathrm{Sk}(X', \Delta_{X'}, \omega')$.

Conversely, if v is a divisorial point in $\mathrm{Sk}(X', \Delta_{X'}, \omega')$, it is a log canonical centre of $(\mathscr{X}', \Delta_{\mathscr{X}'})$ and the divisor $D_{\min}(\mathscr{X}', \omega')$ does not contain the centre of v. As a result, v is also a log canonical centre of $(\mathscr{X}, \Delta_{\mathscr{X}})$, and the divisor $D_{\min}(\mathscr{X}, \omega)$ does not contain the centre of v since its pullback does not. □

We define the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X, \Delta_X, \omega)$ of a dlt pair (X, Δ_X) as the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X', \Delta_{X'}, \omega')$ where $(X', \Delta_{X'})$ is any log resolution of (X, Δ_X), and ω' is the form corresponding to ω under the isomorphism (3.4.1.4): Proposition 3.4.1.5 guarantees that this is well defined. It follows that we can define the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$ of a dlt pair (X, Δ_X) as the essential skeleton of any log resolution of (X, Δ_X).

Moreover, notice that our construction works more generally for log canonical pairs, hence the notions of Kontsevich–Soibelman skeleton and essential skeleton generalize to such pairs.

(3.4.1.6) Suppose that (X, Δ_X) is a proper dlt pair over K, such that $K_X + \Delta_X$ is semiample. Suppose also that $\mathscr{X}$ is a good minimal dlt model of (X, Δ_X) over R and let $\Delta_{\mathscr{X}} = \overline{\Delta_X} + \mathscr{X}_{k,\mathrm{red}}$. We denote by $\mathcal{D}_0(\mathscr{X}, \Delta_{\mathscr{X}}^{=1})$ the dual complex of the strata of the coefficient-1 part of $\Delta_{\mathscr{X}}$ that lie in the special fibre.

We consider a log resolution $f : X' \to X$ that extends to a log resolution of $(\mathscr{X}, \Delta_{\mathscr{X}})$. We write $K_{X'} + \Gamma' = f^*(K_X + \Delta_X)$. Let $\Delta_{X'}$ be the positive part of Γ'. We may embed the open dual complex $\mathcal{D}_0(\mathscr{X}, \Delta_{\mathscr{X}}^{=1})$ into the birational points of X.

Proposition 3.4.1.7. *This embedding identifies* $\mathcal{D}_0(\mathscr{X}, \Delta_{\mathscr{X}}^{=1})$ *with* $\mathrm{Sk}^{\mathrm{ess}}(X', \Delta_{X'})$.

Proof. Choose a regular R-model $\mathscr{X}'$ for X' which is a log resolution of $(\mathscr{X}, \Delta_{\mathscr{X}})$ extending f and let $\Delta_{\mathscr{X}'} = \overline{\Delta_{X'}} + \mathscr{X}'_{k,\mathrm{red}}$, namely we have the log resolutions

$$\begin{aligned} &(X', \Delta_{X'}) \to (X, \Delta_X) && \text{where } (X, \Delta_X) \text{ dlt and } \Delta_{X'} \text{ is the positive part of } \Gamma' \\ &(\mathscr{X}', \Delta_{\mathscr{X}'}) \to (\mathscr{X}, \Delta_{\mathscr{X}}) && \text{where } \Delta_{\mathscr{X}} = \overline{\Delta_X} + \mathscr{X}_{k,\mathrm{red}} \text{ and } \Delta_{\mathscr{X}'} = \overline{\Delta_{X'}} + \mathscr{X}'_{k,\mathrm{red}}. \end{aligned}$$

As in the previous proof, it suffices to check the proposition for divisorial valuations. Let v be a divisorial valuation, and suppose $v \in \mathcal{D}_0(\mathscr{X}, \Delta_{\mathscr{X}}^{=1})$. Then v is a log canonical centre for $(\mathscr{X}, \Delta_{\mathscr{X}})$, so v is also a log canonical centre of $(\mathscr{X}', \Delta_{\mathscr{X}'})$. For a sufficiently divisible index, we may find a Δ_X-logarithmic pluricanonical form on (X, Δ_X) whose associated divisor in $\mathscr{X}$ has vanishing locus C such that C is a divisor not containing the centre of v. After pullback, we get a $\Delta_{X'}$-logarithmic pluricanonical form ω' whose associated divisor in $\mathscr{X}'$ is supported on the strict transform of C and the exceptional divisors of positive log discrepancy. But none of these contain v. Thus, $v \in \mathrm{Sk}(X', \Delta_{X'}, \omega')$.

Conversely, if v is a divisorial point in $\mathrm{Sk}^{\mathrm{ess}}(X', \Delta_{X'})$, then v is a log canonical centre of $(\mathscr{X}', \Delta_{\mathscr{X}'})$, so v is a log canonical centre of $(\mathscr{X}, \Delta_{\mathscr{X}})$, hence an element of the open dual complex $\mathcal{D}_0(\mathscr{X}, \Delta_{\mathscr{X}}^{=1})$. □

Remark 3.4.1.8. Proposition 3.4.1.7 compares the essential skeleton of (X, Δ_X) with the skeleton of a good minimal dlt model of (X, Δ_X). Thus, the result can be restated as follows: if (X, Δ_X) is a dlt pair with $K_X + \Delta_X$ semiample and $(\mathscr{X}, \Delta_{\mathscr{X}})$ is a good minimal dlt model of (X, Δ_X) over R, then $\mathcal{D}_0(\mathscr{X}, \Delta_{\mathscr{X}}^{=1}) = \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$. This generalizes [NX16, Theorem 3.3.3] to dlt pairs.

3.4.2. Essential skeletons and products of log-regular models.

(3.4.2.1) We say that a proper dlt pair (X, Δ_X) has non-negative Kodaira–Iitaka dimension if some multiple of the line bundle $K_X + \Delta_X$ has a regular section. We also define products for pairs: if (X, Δ_X) and (Y, Δ_Y) are pairs over K, then we define their product to be (Z, Δ_Z), where $Z = X \times_K Y$ and $\Delta_Z = \Delta_X \times_K Y + X \times_K \Delta_Y$. If (X, Δ_X) and (Y, Δ_Y) have semistable models $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ over R, then their product is $(\mathscr{Z}, \Delta_{\mathscr{Z}})$, where $\mathscr{Z} = \mathscr{X} \times_R \mathscr{Y}$ and $\Delta_{\mathscr{Z}}$ is the sum of the strict transform of Δ_Z with the special fibre $\mathscr{Z}_k$, which is reduced.

Theorem 3.4.2.2. *Let (X, Δ_X) and (Y, Δ_Y) be proper dlt pairs such that $X^+ = (X, \lceil \Delta_X \rceil)$ and $Y^+ = (Y, \lceil \Delta_Y \rceil)$ are log-regular log schemes over K. Suppose that both pairs have non-negative Kodaira–Iitaka dimension and both admit semistable log-regular models $\mathscr{X}^+$ and $\mathscr{Y}^+$ over S^+. Then the PL homeomorphism of skeletons*

$$\mathrm{Sk}(\mathscr{Z}^+) \xrightarrow{\sim} \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

of Proposition 3.2.5.3 induces a PL homeomorphism of essential skeletons

$$\mathrm{Sk}^{\mathrm{ess}}(Z, \Delta_Z) \xrightarrow{\sim} \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) \times \mathrm{Sk}^{\mathrm{ess}}(Y, \Delta_Y)$$

where $\mathscr{Z}^+$, Z and Δ_Z are the respective products.

Proof. It follows immediately from Theorem 3.3.2.9 that the inclusion $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) \times \mathrm{Sk}^{\mathrm{ess}}(Y, \Delta_Y) \subseteq \mathrm{Sk}^{\mathrm{ess}}(Z, \Delta_Z)$ holds. Thus, we reduce to proving the following statement. Let $v_{z,\varepsilon}$ be a divisorial point in $\mathrm{Sk}(\mathscr{Z}^+)$ and $(v_{x,\alpha}, v_{y,\beta})$ be the corresponding pair in $\mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$ under the isomorphism of Proposition 3.2.5.3; if $v_{z,\varepsilon}$ lies in the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(Z, \Delta_Z)$, then $v_{x,\alpha}$ lies in $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$.

Assume that $v_{z,\varepsilon}$ lies in the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(Z, \Delta_Z)$. Then there exists a non-zero regular Δ_Z-logarithmic m-pluricanonical form ω on Z^+, such that $v_{z,\varepsilon} \in \mathrm{Sk}(Z, \Delta_Z, \omega)$. By Lemma 3.4.1.2, $D_{\min}(\mathscr{Z}, \omega)$ does not contain z.

Let E be an irreducible component of $\mathscr{Y}_k$ containing y and denote by ξ_E the generic point of E. Then the point in the Kato fan of $\mathscr{Z}^+$ corresponding to (x, ξ_E) is not contained in $D_{\min}(\mathscr{Z}, \omega)$, as otherwise z would be contained in it.

As k is algebraically closed, we can choose a k-rational point p in E such that p is contained in no other components of $D_{\mathscr{Y}}$ and $D_{\min}(\mathscr{Z}, \omega)$ does not contain the locus $\overline{\{x\}} \times_R \{p\}$. By Hensel's lemma and the assumption of semistability, p can be lifted to an

R-rational point of $\mathscr{Y}$. The pullback of $\mathscr{Z}^+$ along this R-rational point is an embedding $i : \mathscr{X}^+ \to \mathscr{Z}^+$, so we have the following diagram.

$$\begin{array}{ccccc} \mathscr{X}^+ & \xrightarrow{i} & \mathscr{Z}^+ & \xrightarrow{\mathrm{pr}_{\mathscr{X}}} & \mathscr{X}^+ \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{\mathscr{Y}}} & & \downarrow \\ S & \longrightarrow & \mathscr{Y}^+ & \longrightarrow & S. \end{array}$$

Since S has trivial normal bundle in $\mathscr{Y}$, we have that

$$\omega^{\log}_{X^+/K} = i^*(\omega^{\log}_{Z^+/K}),$$

so $i^*(\omega)$ is a non-zero pluricanonical form on X and, in particular, is a regular Δ_X-logarithmic m-pluricanonical form. Moreover, $D_{\min}(\mathscr{X}, i^*(\omega)) = i^*(D_{\min}(\mathscr{Z}, \omega))$. Finally, x is not contained in $D_{\min}(\mathscr{X}, i^*(\omega))$, as otherwise $i(x) = \{x\} \times_R \{p\}$ would be contained in $D_{\min}(\mathscr{Z}, \omega)$. By Lemma 3.4.1.2, x is a point of $\mathrm{Sk}(X, \Delta_X, i^*(\omega))$ and this concludes the proof. □

Remark 3.4.2.3. Consider the case where the line bundles $K_X + \Delta_X$ and $K_Y + \Delta_Y$ are semiample, that is some multiple of them is base point free. It follows from the arguments of [NX16, Theorem 3.3.3] that the essential skeleton of (Z, Δ_Z) is a finite union of Kontsevich–Soibelman skeletons where the union runs through a generating set of global sections of a sufficiently large multiple of $K_Z + \Delta_Z$. We can construct such a set from generating sets of global sections of multiples of $K_X + \Delta_X$ and $K_Y + \Delta_Y$ respectively, via tensor product. Then, in this case, the result of Theorem 3.4.2.2 follows directly from Theorem 3.3.2.9.

3.4.3. Essential skeletons and products of dlt models.

(3.4.3.1) We need a combinatorial lemma to understand the formally local behavior of products of semistable dlt models.

Lemma 3.4.3.2. *Let M be the monoid generated by $r_1 \dots r_{n_1}$, $s_1 \dots s_{n_2}$ with the single relation $\sum_{i=1}^{n_1} r_i = \sum_{j=1}^{n_2} s_j$. Then any small $\mathbb{Q}$-factorialization of the affine toric variety $W = \mathrm{Spec}(k[M])$ associated to M is a log resolution.*

Proof. We calculate the fan of W. Let N be the dual lattice of M. The fan associated to W is the cone of elements of $N \otimes \mathbb{R}$ which are non-negative on M. We consider these as linear functions l on the vector space spanned by the r_i and s_i, subject to the restriction that $l(\sum_{i=1}^{n_1} r_i) = l(\sum_{j=1}^{n_2} s_j)$. Let x_{ij} be the function which is 1 on r_i and s_j and 0 on all others. Then the fan of W is given by the single cone C_W spanned by the x_{ij}.

Any $\mathbb{Q}$-factorialization $\widetilde{W}$ corresponds to a simplicial subdivision of the cone C_W (see [Ful93, p. 65]). We now check that every choice of $\widetilde{W}$ is non-singular.

A maximal cone of $\widetilde{W}$ is spanned by $n = n_1 + n_2 - 1$ independent rays of C_W. Each ray of C_W corresponds to a choice of x_{ij}, and we can index these by edges of the complete bipartite graph B on the r_i and s_j. These x_{ij} are independent if and only if the corresponding edges form a spanning tree. Let $w_1 \dots w_n$ span a maximal cone of $\widetilde{W}$. On this affine chart, $\widetilde{W}$ is smooth if and only if the w_i generate N as a lattice. We have shown already that the x_{ij} generate N. But every x_{ij} is either one of the w_i, or it completes a cycle in B, so that it is a $\mathbb{Z}$-linear combination of the w_i. □

Proposition 3.4.3.3. *Let $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ be semistable projective good minimal dlt pairs over the germ of a pointed curve $\mathscr{C}$, such that both $\Delta_{\mathscr{X}}$ and $\Delta_{\mathscr{Y}}$ contain their respective special fibres. The product $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is a log canonical pair, $K_{\mathscr{Z}} + \Delta_{\mathscr{Z}}$ is semiample, and the log canonical centres of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ are strata of the coefficient-1 part of $\Delta_{\mathscr{Z}}$.*

Proof. We first show the product $\mathscr{Z}$ is normal. As $\mathscr{X}$ and $\mathscr{Y}$ are semistable, the special fibres $\mathscr{X}_k$ and $\mathscr{Y}_k$ are reduced. The dlt condition guarantees that both $\mathscr{X}$ and $\mathscr{Y}$ are Cohen–Macaulay, hence the Cartier divisors $\mathscr{X}_k$ and $\mathscr{Y}_k$ are Cohen–Macaulay. The special fibre $\mathscr{Z}_k$ is the product $\mathscr{X}_k \times_k \mathscr{Y}_k$. Since k is algebraically closed, $\mathscr{Z}_k$ is reduced, so $\mathscr{Z}$ is regular in codimension 1. The special fibre $\mathscr{Z}_k$ is the product of Cohen–Macaulay varieties over a field, so it is Cohen–Macaulay by [BK02]. As $\mathscr{Z}_k$ is a Cartier divisor in $\mathscr{Z}$, $\mathscr{Z}$ must be Cohen–Macaulay and, in particular, it is S_2.

The divisor $K_{\mathscr{Z}} + \Delta_{\mathscr{Z}}$ is semiample by pullback of semiample divisors. Let $(\widetilde{\mathscr{X}}, \Delta_{\widetilde{\mathscr{X}}})$ and $(\widetilde{\mathscr{Y}}, \Delta_{\widetilde{\mathscr{Y}}})$ be log resolutions of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta\mathscr{Y})$, respectively. Then we have

$$K_{\widetilde{\mathscr{X}}} + \Delta_{\widetilde{\mathscr{X}}} = f_{\mathscr{X}}^*(K_{\mathscr{X}} + \Delta_{\mathscr{X}}) + \sum a_i E_{\mathscr{X},i}$$
$$K_{\widetilde{\mathscr{Y}}} + \Delta_{\widetilde{\mathscr{Y}}} = f_{\mathscr{Y}}^*(K_{\mathscr{Y}} + \Delta_{\mathscr{Y}}) + \sum b_j E_{\mathscr{Y},j}$$

where the coefficients a_i and b_j are non-negative. Let $\widetilde{\mathscr{Z}}$ be a toroidal log resolution of the fs product $\widetilde{\mathscr{X}} \times_{\mathscr{C}}^{\mathrm{fs}} \widetilde{\mathscr{Y}}$. In particular, $\widetilde{\mathscr{Z}}$ is a log resolution of $\mathscr{Z}$ and we can write

$$K_{\widetilde{\mathscr{Z}}} + \Delta_{\widetilde{\mathscr{Z}}} = f_{\mathscr{Z}}^*(K_{\mathscr{Z}} + \Delta_{\mathscr{Z}}) + \sum c_h E_{\mathscr{Z},h}$$

where $\Delta_{\widetilde{\mathscr{Z}}}$ is effective. Over the generic fibre, $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is dlt, so we need only compute discrepancies over the special fibre, namely study the positivity of the coefficients c_h.

Let Γ be a divisor of $\widetilde{\mathscr{Z}}$ over the special fibre, denote by v_Γ the corresponding divisorial valuation in $\mathrm{Sk}(\widetilde{\mathscr{Z}}^+)$, and by $\Gamma_{\mathscr{X}}$, $\Gamma_{\mathscr{Y}}$ and $\Gamma_{\mathscr{Z}}$ its images in $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$. The projections of v_Γ in $\mathrm{Sk}(\widetilde{\mathscr{X}}^+)$ and $\mathrm{Sk}(\widetilde{\mathscr{Y}}^+)$ are divisorial valuations. Up to subdivisions of the skeletons, we can assume without loss of generality that the projections correspond to divisors $\Gamma_{\widetilde{\mathscr{X}}}$ and $\Gamma_{\widetilde{\mathscr{Y}}}$.

Choose $\Delta_{\mathscr{X}}$-logarithmic and $\Delta_{\mathscr{Y}}$-logarithmic pluricanonical forms $\omega_{\mathscr{X}}$ on $\mathscr{X}$ and $\omega_{\mathscr{Y}}$ on $\mathscr{Y}$ respectively, such that the divisors $\mathrm{div}_{\mathscr{X}^+}(\omega_{\mathscr{X}})$ and $\mathrm{div}_{\mathscr{Y}^+}(\omega_{\mathscr{Y}})$ do not contain $\Gamma_{\mathscr{X}}$ and $\Gamma_{\mathscr{Y}}$ respectively, where $\mathscr{X}^+ = (\mathscr{X}, \lceil \Delta_{\mathscr{X}} \rceil)$ and $\mathscr{Y}^+ = (\mathscr{Y}, \lceil \Delta_{\mathscr{Y}} \rceil)$. Then the divisor $\mathrm{div}_{\mathscr{Z}^+}(\omega_{\mathscr{Z}})$, associated to the wedge product $\omega_{\mathscr{Z}}$ of the pullbacks $\omega_{\mathscr{X}}$ and $\omega_{\mathscr{Y}}$ to $\mathscr{Z}$, does not contain $\Gamma_{\mathscr{Z}}$. Denote by $\omega_{\widetilde{\mathscr{X}}}$, $\omega_{\widetilde{\mathscr{Y}}}$ and $\omega_{\widetilde{\mathscr{Z}}}$ the pullback of the respective forms to $\widetilde{\mathscr{X}}$,

$\widetilde{\mathscr{Y}}$ and $\widetilde{\mathscr{Z}}$. Then we have

$$\begin{aligned}\operatorname{div}_{\widetilde{\mathscr{X}}+}(\omega_{\widetilde{\mathscr{X}}}) &= f^*_{\mathscr{X}}(\operatorname{div}_{\mathscr{X}+}(\omega_{\mathscr{X}})) + \sum a_i E_{\mathscr{X},i}\\ \operatorname{div}_{\widetilde{\mathscr{Y}}+}(\omega_{\widetilde{\mathscr{Y}}}) &= f^*_{\mathscr{Y}}(\operatorname{div}_{\mathscr{Y}+}(\omega_{\mathscr{Y}})) + \sum b_j E_{\mathscr{Y},j}\\ \operatorname{div}_{\widetilde{\mathscr{Z}}+}(\omega_{\widetilde{\mathscr{Z}}}) &= f^*_{\mathscr{Z}}(\operatorname{div}_{\mathscr{Z}+}(\omega_{\mathscr{Z}})) + \sum c_h E_{\mathscr{Z},h}.\end{aligned}$$

As $\operatorname{div}_{\mathscr{X}+}(\omega_{\mathscr{X}})$, $\operatorname{div}_{\mathscr{Y}+}(\omega_{\mathscr{Y}})$ and $\operatorname{div}_{\mathscr{Z}+}(\omega_{\mathscr{Z}})$ do not contain $\Gamma_{\mathscr{X}}$, $\Gamma_{\mathscr{Y}}$ and $\Gamma_{\mathscr{Z}}$ respectively, we have

$$\begin{aligned}v_{\Gamma_{\widetilde{\mathscr{X}}}}(\operatorname{div}_{\widetilde{\mathscr{X}}+}(\omega_{\widetilde{\mathscr{X}}})) &= \sum a_i v_{\Gamma_{\widetilde{\mathscr{X}}}}(E_{\mathscr{X},i}) \geqslant 0\\ v_{\Gamma_{\widetilde{\mathscr{Y}}}}(\operatorname{div}_{\widetilde{\mathscr{Y}}+}(\omega_{\widetilde{\mathscr{Y}}})) &= \sum b_j v_{\Gamma_{\widetilde{\mathscr{Y}}}}(E_{\mathscr{Y},j}) \geqslant 0\\ v_{\Gamma}(\operatorname{div}_{\widetilde{\mathscr{Z}}+}(\omega_{\widetilde{\mathscr{Z}}})) &= \sum c_h v_{\Gamma}(E_{\mathscr{Z},h}).\end{aligned}$$

From formula 3.3.2.5, $v_{\Gamma}(\operatorname{div}_{\widetilde{\mathscr{Z}}+}(\omega_{\widetilde{\mathscr{Z}}})) = v_{\Gamma_{\widetilde{\mathscr{Y}}}}(\operatorname{div}_{\widetilde{\mathscr{Y}}+}(\omega_{\widetilde{\mathscr{Y}}})) + v_{\Gamma_{\widetilde{\mathscr{X}}}}(\operatorname{div}_{\widetilde{\mathscr{X}}+}(\omega_{\widetilde{\mathscr{X}}}))$, hence we obtain that the log discrepancy of Γ with respect to the pair $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is non-negative. Moreover, it is zero if and only if the log discrepancies of $\Gamma_{\widetilde{\mathscr{X}}}$ and $\Gamma_{\widetilde{\mathscr{Y}}}$ are both zero, namely if and only if $\Gamma_{\mathscr{X}}$ and $\Gamma_{\mathscr{Y}}$ are log canonical centres of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ respectively. Since for dlt pairs the log canonical centres are the strata of the coefficient-1 part of the boundary, it follows that any log canonical centre of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is a product of such strata, hence a stratum of the coefficient-1 part of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$. □

(3.4.3.4) Let (X, Δ_X) and (Y, Δ_Y) be dlt pairs over the germ of a punctured curve C, and (Z, Δ_Z) their product, where $\Delta_Z = X \times_C \Delta_Y + \Delta_X \times_C Y$. Let $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ be semistable projective good minimal dlt models over the pointed curve.

Theorem 3.4.3.5. *The product (Z, Δ_Z) has a semistable projective good minimal dlt model $(\mathscr{Z}', \Delta_{\mathscr{Z}'})$ and $\mathcal{D}_0(\Delta^{=1}_{\mathscr{Z}'}) \simeq \mathcal{D}_0(\Delta^{=1}_{\mathscr{X}}) \times \mathcal{D}_0(\Delta^{=1}_{\mathscr{Y}})$.*

Proof. Let $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ be the product of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ as in Proposition 3.4.3.3. Thus $K_{\mathscr{Z}} + \Delta_{\mathscr{Z}}$ is semiample and log canonical, so any minimal dlt model over $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ will also have semiample log canonical divisor.

Let $\vartheta : \mathscr{W} \to \mathscr{Z}$ be a log resolution given by iterated blow-ups at centres of codimension at least 2. Then there exists an effective divisor D supported on all of the exceptional divisors, such that $-D$ is ϑ-ample.

Claim 3.4.3.6. *There exist $B_{\mathscr{X}}$ and $B_{\mathscr{Y}}$ effective divisors on $\mathscr{X}$ and $\mathscr{Y}$ whose respective supports contain no log canonical centres of $\Delta_{\mathscr{X}}$ and $\Delta_{\mathscr{Y}}$, and such that $B_{\mathscr{X}} - \epsilon\Delta^{=1}_{\mathscr{X}}$ and $B_{\mathscr{Y}} - \epsilon\Delta^{=1}_{\mathscr{Y}}$ are ample, for ϵ small and rational.*

Choose ϵ small and rational. Then $\Gamma_{\mathscr{X}} = \Delta_{\mathscr{X}} + B_{\mathscr{X}} - \epsilon\Delta^{=1}_{\mathscr{X}}$ and $\Gamma_{\mathscr{Y}} = \Delta_{\mathscr{Y}} + B_{\mathscr{Y}} - \epsilon\Delta^{=1}_{\mathscr{Y}}$ are effective, and $(\mathscr{X}, \Gamma_{\mathscr{X}})$ and $(\mathscr{Y}, \Gamma_{\mathscr{Y}})$ are klt. Let $\Gamma'_{\mathscr{W}}$ be the log pullback to $\mathscr{W}$ of the product $\Gamma_{\mathscr{Z}}$ of $\Gamma_{\mathscr{X}}$ and $\Gamma_{\mathscr{Y}}$, namely $K_{\mathscr{W}} + \Gamma'_{\mathscr{W}} = \vartheta^*(K_{\mathscr{Z}} + \Gamma_{\mathscr{Z}})$; we denote by $\Gamma_{\mathscr{W}}$ the positive part of $\Gamma'_{\mathscr{W}}$. Since $(\mathscr{Z}, \Gamma_{\mathscr{Z}})$ was klt, so is $(\mathscr{W}, \Gamma_{\mathscr{W}})$. For sufficiently small δ, $(\mathscr{W}, \Gamma_{\mathscr{W}} + \delta D)$ is still klt.

Let α_i be arbitrary small rational coefficients, one for each divisor $\Delta^{=1}_{\mathscr{Z},i}$ of $\Delta_{\mathscr{Z}}$ with coefficient 1 and in the special fibre. We will recover the requested dlt model by running an MMP with scaling on the pair $(\mathscr{W}, \Gamma_{\mathscr{W}} + \delta D - \sum \alpha_i \Delta^{=1}_{\mathscr{Z},i})$, scaling with respect to an ample divisor A equivalent to $-D$. By [BCHM10] this MMP terminates in a log terminal model $\phi\colon \mathscr{W} \dashrightarrow \mathscr{Z}'$. Moreover, as long as the α_i are small relative to δ, when the MMP terminates it must be the case that every exceptional divisor is contracted. As running MMP induces birational contractions, the morphism $\psi\colon \mathscr{Z}' \to \mathscr{Z}$ is small. Hence $(\mathscr{Z}', \psi^*\Delta_{\mathscr{Z}})$ is log canonical, and every log canonical centre dominates a stratum of the coefficient-1 part of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$.

By construction of $\mathscr{Z}'$, the divisor $-\sum \alpha_i \Delta^{=1}_{\mathscr{Z},i} + \mu\phi_* A$ is ψ-ample, where μ is arbitrarily small. But for μ small enough, we can absorb $\mu\phi_* A$ into the term δD. Thus in fact $-\sum \alpha_i \Delta^{=1}_{\mathscr{Z},i}$ is ψ-ample. As a result we can represent

$$\mathscr{Z}' = \operatorname{Proj}_{\mathscr{Z}} \bigoplus_{m \geqslant 0} \mathcal{O}(-m(\textstyle\sum \alpha_i \Delta^{=1}_{\mathscr{Z},i})).$$

At this point we may take an arbitrarily large Veronese subring and assume the α_i are all integers.

Now we show that $(\mathscr{Z}', \psi^*\Delta_{\mathscr{Z}})$ is a dlt pair by looking at a formal toric model. Indeed, by Proposition 3.4.3.3 $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is formally locally toric at the log canonical centres, and the condition of being dlt is a formally local property. Moreover, we reduce to checking this property at the log canonical centres of $(\mathscr{Z}', \psi^*\Delta_{\mathscr{Z}})$ that lie in the special fibre, as the generic fibres of $\mathscr{Z}'$ and $\mathscr{Z}$ are isomorphic and the latter is dlt.

Let z be the generic point of the image in $\mathscr{Z}$ of a log canonical centre of $(\mathscr{Z}', \psi^*\Delta_{\mathscr{Z}})$, hence z is the generic point of a log canonical centre of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ and by Proposition 3.4.3.3 it is a stratum of the coefficient-1 part of $\Delta_{\mathscr{Z}}$. Let x and y be the generic points of the corresponding strata of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$. Let E_x and E_y be the monoids of effective Cartier divisors supported on the strata near x and y respectively. Then the corresponding monoid for z is $\langle E_x \oplus E_y \rangle / (t_x = t_y)$, where t_x and t_y are the respective sums of local equations of strata in the special fibres. This monoid has the form $M \oplus \mathbb{N}^l$, where l is the number of horizontal divisors containing z, and M is a monoid of the type considered in Lemma 3.4.3.2. The toric variety $T_{M,k} = \operatorname{Spec} k[M \oplus \mathbb{N}^l]$ is a formal local model for $\mathscr{Z}$ near z, so it suffices to consider

$$T' = \operatorname{Proj}_{T_{M,k}} \bigoplus_{m \geqslant 0} \mathcal{O}(-m(\textstyle\sum \alpha_i \Delta_{T,i})),$$

where the divisors $\Delta_{T,i}$ range over the torus invariant Weil divisors corresponding to the $\Delta^{=1}_{\mathscr{Z},i}$.

The divisors corresponding to the l generators of $\mathbb{N}^l$ are Cartier, so their contribution to T is trivial and we can reduce to the case $l = 0$. For sufficiently general choices for the

α_i, the toric variety T' is simplicial, its fan being induced by the simplicial subdivision of the fan of $T_{M,k}$ such that the α_i induce a strictly convex PL function. Thus, T' is a small $\mathbb{Q}$-factorialization of $T_{M,k}$. By Lemma 3.4.3.2, for any such model, T' is a smooth toric variety, hence its invariant divisors are snc. Thus $(\mathscr{Z}', \psi^*\Delta_{\mathscr{Z}})$ is dlt.

Finally, we compute the dual complex of the coefficient-1 part of $(\mathscr{Z}', \psi^*\Delta_{\mathscr{Z}})$ by looking at formal toric models again. Locally at the generic points of the log canonical centres, $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ are snc, hence log-regular, and are also semistable. By Proposition 3.2.4.4 there is a bijective correspondence between pairs of points in the Kato fans of $\mathscr{X}$ and $\mathscr{Y}$ locally around the log canonical centres and the points in the Kato fan of their product. This induces, by Proposition 3.2.5.3, a PL homeomorphism between the product of skeletons around the log canonical centres and the skeleton of their product, namely a bijective correspondence between pairs of points in the dual complexes $\Delta_{\mathscr{X}}^{=1}$ and $\Delta_{\mathscr{Y}}^{=1}$, and points in the skeleton of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ whose projections map to $\Delta_{\mathscr{X}}^{=1}$ and $\Delta_{\mathscr{Y}}^{=1}$. A face σ_z of the skeleton of $\mathscr{Z}$, where z is a log canonical centre corresponding to (x, y), corresponds to a prime ideal of the monoid $E_z = \langle E_x \oplus E_y \rangle / (t_x = t_y)$. Thus, we may reduce to considering such faces in the formal toric model $T_{M,k}$, and to studying the dual complex of the coefficient-1 part of $(\mathscr{Z}', \psi^*\Delta_{\mathscr{Z}})$ in the formal model T'. But T' is obtained by a simplicial subdivision of T. Thus, we conclude that the dual complex $\mathcal{D}_0((\psi^*\Delta_{\mathscr{Z}})^{=1})$ is identified with the product of the dual complexes of the coefficient-1 part of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$. □

Proof of Claim. Let $(\mathscr{X}, \Delta_{\mathscr{X}})$ be a good projective minimal dlt model of (X, Δ_X) and let $A_{\mathscr{X}}$ be an ample divisor on $\mathscr{X}$. Let $\mathcal{J}$ be the ideal sheaf of $\Delta_{\mathscr{X}}^{=1}$ and let

$$\chi : \mathscr{X}' = \mathrm{Bl}_{\mathcal{J}} \mathscr{X} \to \mathscr{X}$$

be the blow-up of $\mathscr{X}$ with respect to $\mathcal{J}$. Then the transform $-((\Delta_{\mathscr{X}}^{=1})' + E)$ is a χ-ample Cartier divisor, where $(\Delta_{\mathscr{X}}^{=1})'$ denotes the strict transform of $\Delta_{\mathscr{X}}^{=1}$ and E an effective divisor supported on the exceptional divisor of χ. For ε small positive rational $\chi^*(A_{\mathscr{X}}) - \varepsilon((\Delta_{\mathscr{X}}^{=1})' + E)$ is ample. We choose such an ε. For a sufficiently large integer n, we can find

$$G_{\mathscr{X}'} \sim n\chi^*(A_{\mathscr{X}}) - n\varepsilon((\Delta_{\mathscr{X}}^{=1})' + E)$$

such that $G_{\mathscr{X}'}$ is effective and contains no log canonical centres. Then the pushforward $G_{\mathscr{X}} \sim nA_{\mathscr{X}} - n\varepsilon\Delta_{\mathscr{X}}^{=1}$ of $G_{\mathscr{X}'}$ is effective and contains no log canonical centres of $(\mathscr{X}, \Delta_{\mathscr{X}})$. In particular $G_{\mathscr{X}}$ is Cartier at the log canonical centres of $(\mathscr{X}, \Delta_{\mathscr{X}})$.

We can run the same construction for the ideal sheaf of the divisor $G_{\mathscr{X}}$. Notice that the blow-up with respect to $G_{\mathscr{X}}$ is an isomorphism at the log canonical centres of $(\mathscr{X}, \Delta_{\mathscr{X}})$ as there $G_{\mathscr{X}}$ is Cartier. We obtain that, for sufficiently small positive rationals δ and then sufficiently large integers m, we can find $H_{\mathscr{X}} \sim mA_{\mathscr{X}} - m\delta G_{\mathscr{X}}$ such that it is effective and does not contain any log canonical centre of $(\mathscr{X}, \Delta_{\mathscr{X}})$. Then

$$\frac{1}{\delta}A_{\mathscr{X}} - \frac{1}{m\delta}H_{\mathscr{X}} \sim G_{\mathscr{X}} \sim nA_{\mathscr{X}} - n\varepsilon\Delta_{\mathscr{X}}^{=1}$$

implies that

$$\Big(\frac{1}{\delta} - n\Big)A_{\mathscr{X}} \sim \frac{1}{m\delta}H_{\mathscr{X}} - n\varepsilon\Delta_{\mathscr{X}}^{=1}.$$

For δ sufficiently small the term $(\frac{1}{\delta} - n)$ is positive, and then for m sufficiently large $B_{\mathscr{X}} = \frac{1}{m\delta}H_{\mathscr{X}}$, which concludes the proof. □

4

Application to hyperkähler varieties

4.1. Introduction

As an application of the main results in Chapter 3, we study certain degenerations of hyperkähler varieties. One way to produce hyperkähler varieties is by taking the Hilbert scheme of points on a K3 surface. Another is to extend the Kummer construction to higher-dimensional abelian varieties. Aside from two other examples found by O'Grady in dimensions 6 [O'G03] and 10 [O'G99] there are no other known examples, up to deformation equivalence.

4.1.1. Kulikov degenerations

Degenerations of 2-dimensional hyperkähler varieties, namely $K3$ surfaces, are treated in the work of Kulikov, Persson and Pinkham. In [Kul77] and [PP81], they consider more generally surfaces such that some power of the canonical bundle is trivial. They prove that, after base change and birational transformations, any degeneration can be arranged to be semistable with trivial canonical bundle, namely a *Kulikov degeneration*. Then they classify the possible special fibres of Kulikov degenerations according to the type of degeneration.

We recall that the monodromy operator T on $\mathrm{H}^2(X_t, \mathbb{Q})$ of the fibres X_t of a Kulikov degeneration is unipotent, so we denote by ν the nilpotency index of $\log(T)$, namely the positive integer such that $\log(T)^{\nu} = 0$ and $\log(T)^{(\nu-1)} \neq 0$. The type of the Kulikov degeneration is defined as the nilpotency index ν and called type I, II or III accordingly.

It follows from [Kul77, Theorem II] that the dual complex of the special fibre of a

Kulikov degeneration of a K3 surface is a point, a closed interval or the sphere S^2 according to the respective type. For a degeneration of abelian surfaces, the dual complex of the special fibre is homeomorphic to a point, the circle S^1 or the torus $S^1 \times S^1$ according to the three types (see an overview of these results in [FM83]). In all cases, the dimension of the dual complex is equal $\nu - 1$, hence determined by the type.

4.1.2. Remarks on degenerations of hyperkähler varieties

Just as for surfaces with trivial canonical bundle, also for a semistable degeneration of hyperkähler manifolds it is possible to define the type as the nilpotency index of the monodromy operator on the second cohomology group. This yields a classification of degenerations of hyperkähler varieties in three types (I, II, III).

In [KLSV17], Kollár, Laza, Saccà and Voisin study the essential skeleton of a degeneration of hyperkähler manifolds in terms of the type. More precisely, in their Theorem 0.10, given a minimal dlt degeneration of $2n$-dimensional hyperkähler manifolds, firstly they prove that the dual complex of the special fibre has dimension $(\nu - 1)n$, where ν denotes the type of the degeneration. Secondly, they prove that, in the type II case the dual complex has the rational homology type of a point, and in the type III case of a complex projective space.

Gulbrandsen, Halle, Hulek and Zhang [GHH15; GHH16; GHHZ18] consider degenerations of nth order Hilbert schemes arising from some type II degenerations of $K3$ surfaces, and show that the dual complex is an n-simplex. Their approach is based on the method of *expanded degenerations*, which first appeared in [Li01], and on the construction of suitable GIT quotients, in order to obtain an explicit minimal dlt degeneration for the associated family of Hilbert schemes.

4.1.3. Main results

The following theorems confirm and strengthen the results mentioned in Section 4.1.2, for the specific cases of Hilbert schemes and generalised Kummer varieties. In particular, they turn the rational cohomological description of the essential skeleton [KLSV17, Theorem 0.10(ii)] into a topological characterization.

Theorem 4.1.3.1 (§ 4.3). *Assume that the residue field k is algebraically closed. Let S be a $K3$ surface over K. If S admits a semistable log-regular model or a semistable good minimal dlt model, then the essential skeleton of the Hilbert scheme of n points on S is isomorphic to the nth symmetric product of the essential skeleton of S*

$$\mathrm{Sk}^{\mathrm{ess}}(\mathrm{Hilb}^n(S)) \xrightarrow{\sim} \mathit{Sym}^n(\mathrm{Sk}^{\mathrm{ess}}(S)).$$

Computing these complexes gives a single point in the type I case, an n-simplex in the type II case, and $\mathbb{CP}^n$ in the type III case. The same types arise in the Kummer case.

Theorem 4.1.3.2 (§ 4.4). *Assume the residue field k is algebraically closed. Let A be an abelian surface over K. Suppose that A admits a semistable log-regular model or a semistable good minimal dlt model. If the essential skeleton of A is homeomorphic to a point, the circle S^1 or the torus $S^1 \times S^1$, then the essential skeleton of the nth generalized Kummer variety $K_n(A)$ is isomorphic to a point, the standard n-simplex or $\mathbb{CP}^n$ respectively.*

Analysis of the weight function gives a powerful yet accessible approach to controlling the skeletons of these varieties. In both cases we use Theorem 3.4.2.2 to establish that the skeleton of the hyperkähler variety is a finite quotient of the n-fold product of the skeleton of the original surface under the action of a symmetric group. In the case of Hilbert schemes we can get a complete description of the action using functoriality of the projection maps, but in the Kummer case we additionally need to understand the restriction of the multiplication map to the essential skeleton of an abelian surface ([Ber90; HN17; Tem16]).

To our knowledge these are the first examples of type III degenerations of hyperkähler varieties where the PL homeomorphism type of the dual complex is known.

4.1.4. Mirror symmetry viewpoint

The structure of the essential skeleton of a degeneration of hyperkähler manifolds is relevant in the context of mirror symmetry and in view of the work of Kontsevich and Soibelman [KS01; KS06]. The Strominger–Yau–Zaslow (SYZ) fibration [SYZ96] is a conjectural geometric explanation for the phenomenon of mirror symmetry and, roughly speaking, asserts the existence of a special Lagrangian fibration, such that mirror pairs of manifolds with trivial canonical bundle should admit fiberwise dual special Lagrangian fibrations. Moreover, the expectation is that, for type III degenerations of $2n$-dimensional hyperkähler manifolds, the base of the SYZ fibration is $\mathbb{CP}^n$. See Sections 1.1 and 1.3 for an extended introduction to these ideas and existing outcomes.

The most relevant fact from our perspective is that Kontsevich and Soibelman predict that the base of the Lagrangian fibration of a type III degeneration is homeomorphic to the essential skeleton. So, the results on the topology of the essential skeleton we obtain in Theorems 4.1.3.1 and 4.1.3.2 match the predictions of mirror symmetry about the occurrence of $\mathbb{CP}^n$ in the type III case.

4.2. Weight functions and skeletons for finite quotients

(4.2.0.1) Let X be a connected, smooth and proper K-variety and let G be a finite group acting on X. Let X^{an} be the analytification of X. We recall that any point of X^{an} is a pair $x = (\xi_x, |\cdot|_x)$ with $\xi_x \in X$ and $|\cdot|_x$ an absolute value on the residue field $\kappa(\xi_x)$ that extends the absolute value on K. For any point ξ_x of X, an element g of the group G induces an isomorphism between the residue fields $\kappa(\xi_x)$ and $\kappa(g.\xi_x)$, which we still denote by g. Then the action of G extends to X^{an} in the following way

$$g.\,(\xi_x, |\cdot|_x) = (g.\,\xi_x, |\cdot|_x \circ g^{-1}).$$

In particular the action preserves the sets of divisorial and birational points of X.

Let $f : X \to Y = X/G$ be the quotient map of K-schemes, let $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the map of Berkovich spaces induced by functoriality and let $\tilde{f} : X^{\mathrm{an}} \to X^{\mathrm{an}}/G$ be the quotient map of topological spaces.

Proposition 4.2.0.2. *[Ber95, Corollary 5] In the above notation, there is a canonical homeomorphism between X^{an}/G and Y^{an} such that $\tilde{f}$ and f^{an} are identified.*

Lemma 4.2.0.3. *Let ω be an m-pluricanonical rational form on X. If ω is G-invariant, then the weight function associated to ω on the set of birational points is stable under the action of G.*

Proof. Let x be a birational point of X and g an element of G. There exist snc models $\mathscr{X}$ and $\mathscr{X}'$ over R such that $x \in \mathrm{Sk}(\mathscr{X})$ and $g.\,x \in \mathrm{Sk}(\mathscr{X}')$. By replacing them by an snc model $\mathscr{Y}$ that dominates both $\mathscr{X}$ and $\mathscr{X}'$, we can assume that both points lies in $\mathrm{Sk}(\mathscr{Y})$. The weights of ω at x and $g.\,x$ are such that

$$\begin{aligned} \mathrm{wt}_\omega(g.\,x) &= v_{g.x}(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m = v_x((g^{-1})^*\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m \\ &= v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m = \mathrm{wt}_\omega(x) \end{aligned}$$

as ω is a G-invariant form. Thus we see that birational points in the same G-orbit have the same weight with respect to ω. □

Corollary 4.2.0.4. *Let ω be a G-invariant pluricanonical rational form on X. Then the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X, \omega)$ is stable under the action of G.*

Proof. This follows immediately from Lemma 4.2.0.3. □

(4.2.0.5) Let y be a divisorial point of Y^{an} and consider a normal R-model $\mathscr{Y}$ of Y adapted to y; this means that y is the divisorial valuation associated to $(\mathscr{Y}, E)$ for some irreducible component E of $\mathscr{Y}_k$, and the model $\mathscr{Y}$ is regular in a neighborhood of the generic point of E. We denote by $\mathscr{X}$ the normalization of $\mathscr{Y}$ inside $K(X)$, where $K(\mathscr{Y}) = K(Y) = K(X)^G \hookrightarrow K(X)$. As X is normal and the quotient map $f : X \to Y$ is finite, we obtain that $\mathscr{X}$ is an R-model of X. Moreover, by normality $\mathscr{X}$ is regular at the generic points of the special fibre $\mathscr{X}_k$.

(4.2.0.6) We assume $\mathrm{char}(k) = 0$. We denote respectively by R and by B the codimension-1 components of the ramification locus and of the branch locus of $f : X \to Y$. We set $X^+ = (X, R_{\mathrm{red}})$ and $Y^+ = (Y, B_{\mathrm{red}})$. Then f extends to a morphism of log schemes $f^+ : X^+ \to Y^+$.

Lemma 4.2.0.7. *The reflexive sheaves $\omega^{\log}_{X^+/K}$ and $\omega^{\log}_{Y^+/K}$ are identified via the pullback $(f^+)^*$ along the smooth locus of Y.*

Proof. From a generalization of [Har77, Proposition IV.2.3] to higher dimension, we have that $\omega_{X/K} = f^*(\omega_{Y/K}) \otimes \mathcal{O}_X(R)$ along the smooth locus of Y. It follows that

$$\omega^{\log}_{X^+/K} = \omega_{X/K} \otimes \mathcal{O}_X(R_{\text{red}}) = f^*(\omega_{Y/K}) \otimes \mathcal{O}_X(R + R_{\text{red}}).$$

In order to study the divisor $R + R_{\text{red}}$, we consider one irreducible component of B. Let D_B be an irreducible component of B, denote by e the ramification index of f at D_B, and by D_R the support of the preimage of D_B. As $\text{mult}_{D_R}(R) = e - 1$, $\text{mult}_{D_R}(f^*(D_B)) = e$ and $\text{mult}_{D_R}(R_{\text{red}}) = 1$, we conclude that $R + R_{\text{red}} = f^*(B)$ and finally $\omega^{\log}_{X^+/K} = (f^+)^*(\omega^{\log}_{Y^+/K})$. □

In particular, Lemma 4.2.0.7 implies that G-equivariant logarithmic pluricanonical forms on X correspond to logarithmic pluricanonical forms on Y via pullback.

Proposition 4.2.0.8. *Let ω be a G-invariant R_{red}-logarithmic m-pluricanonical form on X^+ and let ϖ be the corresponding B_{red}-logarithmic form on Y^+ via pullback. Let y be a divisorial point of Y^{an}. Then, for any divisorial point $x \in (f^{\text{an}})^{-1}(y)$, the weights of ω at x and of ϖ at y coincide.*

Proof. Let $\mathscr{Y}$ be a model of Y over R such that y has divisorial representation $(\mathscr{Y}, E)$ and the model is regular at the generic point of E. Let $\mathscr{X}$ be the normalization of $\mathscr{Y}$ in $K(X)$: as we observed in paragraph 4.2.0.5, it is a model of X, regular at generic points of the special fibre $\mathscr{X}_k$. We denote as well by $f : \mathscr{X} \to \mathscr{Y}$ the extension of f. The preimage of E coincides with the pullback of the Cartier divisor E on $\mathscr{X}$, hence $f^{-1}(E)$ still defines a codimension-1 subset on $\mathscr{X}$. We denote by F_i the irreducible components of $f^{-1}(E)$ and we associate to the F_i their corresponding divisorial valuations $x_i = (\mathscr{X}, F_i)$. By Lemma 4.2.0.3, it is enough to prove the result for one of the x_i. We denote it by $x = (\mathscr{X}, F)$ and we compare the weights at y and x.

We recall that for log-étale morphisms the sheaves of logarithmic differentials are stable under pullback ([Kat89, Proposition 3.12]). Furthermore, it suffices to check that, locally around the generic point of F, the morphism $\mathscr{X}^+ \to \mathscr{Y}^+$ is a log-étale morphism of divisorial log structures, to conclude that the weights coincide. For this purpose, we will apply Kato's criterion for log-étaleness ([Kat89, Theorem 3.5]) to log schemes with respect to the étale topology.

We denote by ξ_F the generic point of F and by ξ_E the generic point of E. The divisorial log structures on $\mathscr{X}^+$ and $\mathscr{Y}^+$ have charts $\mathbb{N}$ at ξ_F and ξ_E. In the étale topology, the normalization morphism $\mathscr{X}^+ \to \mathscr{Y}^+$ admits a chart induced by $u : \mathbb{N} \to \mathbb{N}$ where $1 \mapsto m$ for some positive integer m as follows.

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O}_{\mathscr{X},\xi_F} & \longrightarrow & \operatorname{Spec} \mathbb{Z}[\mathbb{N}] \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O}_{\mathscr{Y},\xi_E} & \longrightarrow & \operatorname{Spec} \mathbb{Z}[\mathbb{N}] \end{array}$$

Firstly, by the universal property of the fibre product, we have a morphism

$$\operatorname{Spec} \mathcal{O}_{\mathscr{X},\xi_F} \to \operatorname{Spec} \mathcal{O}_{\mathscr{Y},\xi_E} \times_{\operatorname{Spec} \mathbb{Z}[\mathbb{N}]} \operatorname{Spec} \mathbb{Z}[\mathbb{N}]$$

and it corresponds to

$$\mathcal{O}_{\mathscr{Y},\xi_E} \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[\mathbb{N}] \to \mathcal{O}_{\mathscr{X},\xi_F}.$$

This is a morphism of finite type with finite fibres between regular rings and by [Liu02, Lemma 4.3.20] and [Now97] it is flat and unramified, hence étale. One of the two conditions in Kato's criterion for log-étaleness is then fulfilled. Secondly, the chart $u : \mathbb{N} \mapsto \mathbb{N}$ induces a group homomorphism $u^{\mathrm{gp}} : \mathbb{Z} \mapsto \mathbb{Z}$; in particular, it is injective and it has finite cokernel. Then u satisfies the second condition of Kato's criterion for log-étaleness. Therefore we conclude that $\mathrm{wt}_{\overline{\omega}}(y) = \mathrm{wt}_{\omega}(x)$. □

Proposition 4.2.0.9. *Let ω be a G-invariant R_{red}-logarithmic pluricanonical form on X and let $\overline{\omega}$ be the corresponding B_{red}-logarithmic form. Then the canonical homeomorphism between X^{an}/G and Y^{an} of Proposition 4.2.0.2 induces the homeomorphism*

$$\mathrm{Sk}(X,\omega)/G \simeq \mathrm{Sk}(X/G,\overline{\omega}).$$

Proof. This follows immediately from Corollary 4.2.0.4 and Proposition 4.2.0.8. □

(4.2.0.10) Let X be a smooth K-variety and let ω_X be a pluricanonical form on X. Let $\mathrm{pr}_j : X^n \to X$ be the j-th canonical projection. Then

$$\omega = \bigwedge_{1 \leqslant j \leqslant n} \mathrm{pr}_j^* \omega_X$$

is a pluricanonical form on X^n and, moreover, a tensor product $\omega^{\otimes d}$ of ω is invariant under the action of $\mathfrak{S}_n$. We denote by $\overline{\omega}$ the form on the quotient $X^n/\mathfrak{S}_n$ induced by $\omega^{\otimes d}$ as in Lemma 4.2.0.7.

Proposition 4.2.0.11. *Assume that the residue field k is algebraically closed. If X admits a semistable log-regular model or a semistable good minimal dlt model, then the Kontsevich–Soibelman skeleton of the n-th symmetric product of X associated to $\overline{\omega}$ is PL homeomorphic to the n-th symmetric product of the Kontsevich–Soibelman skeleton of X associated to ω_X.*

Proof. Iterating the result of Theorems 3.4.2.2 and 3.4.3.5, we have that the projection map defines a PL homeomorphism of Kontsevich–Soibelman skeletons

$$\mathrm{Sk}(X^n,\omega) \xrightarrow{\sim} \mathrm{Sk}(X,\omega_X) \times \ldots \times \mathrm{Sk}(X,\omega_X).$$

Thus, applying Proposition 4.2.0.9 with the group $\mathfrak{S}_n$ acting on the product X^n, we obtain that

$$\mathrm{Sk}(X^n/\mathfrak{S}_n, \overline{\omega}) \simeq \mathrm{Sk}(X^n, \omega^{\otimes d})/\mathfrak{S}_n \simeq \mathrm{Sk}(X^n, \omega)/\mathfrak{S}_n \simeq \mathrm{Sk}(X, \omega_X)^n/\mathfrak{S}_n,$$

where the middle homeomorphism follows from the equality $\mathrm{wt}_{\omega^{\otimes d}} = d \cdot \mathrm{wt}_\omega$ in [MN15, Proposition 4.5.5(v)]. Since the action on the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X^n, \omega)$ is induced from the symmetric action on X^n, and the projections $\mathrm{pr}_j : X^n \to X$ functorially induce the projections $\overline{\mathrm{pr}_j} : \mathrm{Sk}(X, \omega_X)^n \to \mathrm{Sk}(X, \omega)$, the action of $\mathfrak{S}_n$ on $\mathrm{Sk}(X, \omega)^n$ is exactly by permutations of the components. Thus,

$$\mathrm{Sk}(X, \omega_X)^n/\mathfrak{S}_n \simeq \mathrm{Sym}^n(\mathrm{Sk}(X, \omega_X))$$

and we conclude. □

4.3. The essential skeleton of Hilbert schemes of a K3 surface

(4.3.0.1) Let S be an irreducible regular surface. We consider $\mathrm{Hilb}^n(S)$ the Hilbert scheme of n points on S: by [Fog68] it is an irreducible regular variety of dimension $2n$. Moreover, the morphism

$$\rho_{HC} : \mathrm{Hilb}^n(S) \to S^n/\mathfrak{S}_n$$

that sends a zero-dimensional scheme $Z \subseteq S$ to its associated zero-cycle $[Z]$ is a birational morphism, called the Hilbert-Chow morphism.

(4.3.0.2) Let S be a K3 surface over K, namely S is a complete non-singular variety of dimension 2 such that $\Omega^2_{S/K} \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$. In particular S is a variety with trivial canonical line bundle.

Corollary 4.3.0.3. *Assume that the residue field k is algebraically closed. Suppose that S admits a semistable log-regular model or a semistable good minimal dlt model. Then the essential skeleton of the Hilbert scheme of n points on S is PL homeomorphic to the n-th symmetric product of the essential skeleton of S*

$$\mathrm{Sk}^{\mathrm{ess}}(\mathrm{Hilb}^n(S)) \xrightarrow{\sim} \mathit{Sym}^n(\mathrm{Sk}(S)).$$

Proof. This follows immediately from Corollary 4.2.0.11 and the birational invariance of the essential skeleton [MN15, Proposition 4.10.1]. □

Proposition 4.3.0.4. *If the essential skeleton of S is PL homeomorphic to a point, a closed interval or the two-dimensional sphere, then the essential skeleton of* $\mathrm{Hilb}^n(S)$ *is PL homeomorphic to a point, the standard n-simplex or $\mathbb{CP}^n$ respectively.*

Proof. Applying Corollary 4.3.0.3, we reduce to the computation of the symmetric product of a point, a closed interval or the sphere S^2. Then the result is trivially true in the first two cases, and follows from [Hat02], Section 4K in the third case. □

4.4. The essential skeleton of generalised Kummer varieties

(4.4.0.1) Let A be an abelian surface over K, namely a complete non-singular, connected group variety of dimension 2. Since A is a group variety, the canonical line bundle is trivial and the group structure provides a multiplication morphism $m_{n+1} : A \times A \times \ldots \times A \to A$ that is invariant under the permutation action of $\mathfrak{S}_{n+1}$, hence it induces a morphism

$$\Sigma_{n+1} : \mathrm{Hilb}^{n+1}(A) \xrightarrow{\rho_{HC}} \mathrm{Sym}^{n+1}(A) \to A$$

by composition with the Hilbert-Chow morphism. Then $\mathrm{K}_n(A) = \Sigma_{n+1}^{-1}(1)$ is called the nth generalised Kummer variety and is a hyper-Kähler manifold of dimension $2n$ ([Bea83]).

(4.4.0.2) In [HN17, Proposition 4.3.2] Halle and Nicaise, using Temkin's generalization of the weight function ([Tem16]), prove that the essential skeleton of an abelian variety A over K coincides with the construction of a skeleton of A done by Berkovich in [Ber90, §6.5]. It follows from this identification and [Ber90, Theorem 6.5.1] that the essential skeleton of A has a group structure, compatible with the group structure on A^{an} under the retraction ρ_A of A^{an} onto the essential skeleton, so the diagram

$$\begin{array}{ccc} (A^{\mathrm{an}})^{n+1} & \xrightarrow{m_{n+1}^{\mathrm{an}}} & A^{\mathrm{an}} \\ \Big\downarrow (\rho_A)^{n+1} & & \Big\downarrow \rho_A \\ \mathrm{Sk}(A)^{n+1} & \xrightarrow{\mu_{n+1}} & \mathrm{Sk}(A) \end{array}$$

commutes, where μ denotes the multiplication of $\mathrm{Sk}(A)$.

Proposition 4.4.0.3. *Assume that the residue field k is algebraically closed. Suppose that A admits a semistable log-regular model or a semistable good minimal dlt model. Then the essential skeleton of the nth generalised Kummer variety is PL homeomorphic to the symmetric quotient of the kernel of the morphism μ, namely*

$$\mathrm{Sk}^{\mathrm{ess}}(K_n(A)) \simeq \mathrm{Sk}\,(m_{n+1}^{-1}(1)/\mathfrak{S}_{n+1}) \simeq \mu_{n+1}^{-1}(1)/\mathfrak{S}_{n+1}.$$

Proof. The first homeomorphism follows from the birational invariance of the essential skeleton ([MN15, Proposition 4.10.1]). We write

$$L = m_{n+1}^{-1}(1) \quad \text{and} \quad \Lambda = \mu_{n+1}^{-1}(1).$$

For any choice of an $\mathfrak{S}_{n+1}$-invariant generating canonical form on L, it follows from Proposition 4.2.0.9 that $\mathrm{Sk}(L/\mathfrak{S}_{n+1}) \simeq \mathrm{Sk}(L)/\mathfrak{S}_{n+1}$. We reduce to studying the quotients $\mathrm{Sk}(L)/\mathfrak{S}_{n+1}$ and $\Lambda/\mathfrak{S}_{n+1}$.

Let $\mathfrak{S}'_n$ and $\mathfrak{S}''_n$ be the subgroups of $\mathfrak{S}_{n+1}$ of the permutations that fix n and $n+1$ respectively. Then $\mathfrak{S}_{n+1}$ is generated by the two subgroups, so its action on $\mathrm{Sk}(L)$ and Λ

is completely determined by the actions of these subgroups. We consider the isomorphisms

$$\begin{aligned} f_n : & L \xrightarrow{\sim} A^n \quad (z_1, \ldots, z_{n+1}) \mapsto (z_1, \ldots, z_{n-1}, z_{n+1}) \\ f_{n+1} : & L \xrightarrow{\sim} A^n \quad (z_1, \ldots, z_{n+1}) \mapsto (z_1, \ldots, z_{n-1}, z_n). \end{aligned}$$

Then f_n is $\mathfrak{S}'_n$-equivariant, f_{n+1} is $\mathfrak{S}''_n$-equivariant and the morphism ψ

$$\begin{array}{ccccc} & & L & & \\ & \swarrow^{f_{n+1}} & & \searrow^{f_n} & \\ A^n & & \xrightarrow{\quad\psi\quad} & & A^n \\ (z_1, \ldots, z_{n-1}, z_n) & & \longmapsto & & (z_1, \ldots, \prod_{i=1}^n z_i^{-1}). \end{array}$$

is equivariant with respect to the action of $\mathfrak{S}''_n$ on the source and of $\mathfrak{S}'_n$ on the target. Hence, we obtain a commutative diagram of equivariant isomorphisms. We denote by $\overline{f}_n$, $\overline{f}_{n+1}$ and $\overline{\psi}$ the isomorphisms induced on the essential skeletons. By Theorem 3.4.2.2 and Theorem 3.4.3.5 we can identify $\mathrm{Sk}(A^n)$ with $\mathrm{Sk}(A)^n$. Thus, we have the commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Sk}(L) & & \\ & \swarrow^{\overline{f}_{n+1}} & & \searrow^{\overline{f}_n} & \\ \mathrm{Sk}(A)^n & & \xrightarrow{\quad\overline{\psi}\quad} & & \mathrm{Sk}(A)^n \\ (v_1, \ldots, v_{n-1}, v_n) & & \longmapsto & & (v_1, \ldots, \prod_{i=1}^n v_i^{-1}). \end{array}$$

Then the action of $\mathfrak{S}_{n+1}$ on $\mathrm{Sk}(L)$ is induced by the isomorphisms $\overline{f}_n$ and $\overline{f}_{n+1}$ from the actions of $\mathfrak{S}''_n$ and $\mathfrak{S}'_n$ on $\mathrm{Sk}(A)^n$ and these actions are compatible as $\overline{\psi}$ is equivariant.

In a similar way, Λ is isomorphic to n copies of $\mathrm{Sk}(A)$ and comes equipped with an action of $\mathfrak{S}_{n+1}$. So we have equivariant projections g_n and g_{n+1} with respect to $\mathfrak{S}'_n$ and $\mathfrak{S}''_n$. The equivariant morphism that completes and makes the diagram commutative is $\overline{\psi}$. Finally, we have the equivariant commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Sk}(L) & & \\ & \swarrow^{\overline{f}_{n+1}} & & \searrow^{\overline{f}_n} & \\ \mathfrak{S}''_n \circlearrowleft \mathrm{Sk}(A)^n & & \xrightarrow{\quad\overline{\psi}\quad} & & \mathrm{Sk}(A)^n \circlearrowright \mathfrak{S}'_n \\ & \nwarrow_{g_{n+1}} & & \nearrow_{g_n} & \\ & & \Lambda & & \end{array}$$

and we conclude that the quotients $\mathrm{Sk}(L)/\mathfrak{S}_{n+1}$ and $\Lambda/\mathfrak{S}_{n+1}$ are homeomorphic. □

Proposition 4.4.0.4. *If the essential skeleton of A is PL homeomorphic to a point, the circle S^1 or the torus $S^1 \times S^1$, then the essential skeleton of $K_n(A)$ is PL homeomorphic to a point, the standard n-simplex or $\mathbb{CP}^n$ respectively.*

Proof. The case of the point is trivial. For the circle S^1, it follows directly from [Mor67, Theorem]. To prove the result for the torus $S^1 \times S^1$, we apply [Loo76, Theorem 3.4]: the action of the symmetric group corresponds to the root system of A_n, the highest root is the sum of the simple roots, each with coefficient 1, and so the quotient is the complex projective space of dimension n. □

5

Skeletons for pairs

5.1. Introduction

Degeneration and compactification of algebraic varieties are powerful tools in algebraic geometry: they recast the study of non-proper varieties into that of proper varieties and their invariants. General theorems on resolutions of singularities ensure the existence of simple normal crossing (snc) degenerations and compactifications. To such a divisor D on a variety X, we associate a regular Δ-complex $\mathcal{D}(D)$, namely the dual intersection complex of D (see Definition 2.2.0.1), which captures aspects of the geometry of $X \setminus D$. For instance, it follows from Deligne [Del71] that there exists a correspondence between the reduced rational homology of $\mathcal{D}(D)$ and the top dimensional pieces of the weight filtration on the cohomology of $X \setminus D$. See also [Ber00, Theorem 1.1.(c)] and [Pay13, Theorem 4.4].

Many conjectures in the theory of singularities, tropical geometry, mirror symmetry, or even non-abelian Hodge theory involve understanding the homotopy or homeomorphism type of particular dual complexes. The goal of Chapters 5 to 7 is to tackle certain of these problems by reframing the study of dual complexes in terms of non-archimedean geometry: in Chapter 5 we develop the necessary techniques we will employ in Chapters 6 and 7.

5.1.1. Skeletons over a trivially-valued field

We consider an snc pair (X, D) (more generally, a log-regular pair) over a trivially-valued field of characteristic zero, e.g. the complex numbers $\mathbb{C}$ equipped with the trivial norm. Following [Uli17, §6], [Thu07, §3] and Chapter 3, one constructs a skeleton $\mathrm{Sk}(X, D) \subseteq X^{\beth}$ which has the structure of a cone complex, with the vertex corresponding to the trivial valuation, which coincides with the quasi-monomial valuations in D of [JM12], and is homeomorphic to the cone over $\mathcal{D}(D)$; see Section 5.3. In particular, this construction behaves well under the operation of taking product of pairs; we refer to Section 5.3.6 for more details.

5.1.2. Weight functions and essential skeleton

Along the lines of [MN15] and Chapter 3, we construct a weight function $\mathrm{wt}_\eta\colon X^\beth \to \mathbb{R}\cup\{\pm\infty\}$ for each pluricanonical form η with poles along D. This is built in terms of certain log discrepancy functions and using the language of metrics on Berkovich spaces and analytification of line bundle; see Section 5.2.1 for a brief introduction to this subject.

The minimality locus $\mathrm{Sk}(X, D, \eta)$ of wt_η is called the Kontsevich–Soibelman skeleton of η, and the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, D)$ is the union of the Kontsevich–Soibelman skeletons for all $\eta \in H^0(X, \mathcal{O}_X(m(K_X+D)))$, the so-called regular D-logarithmic pluricanonical forms. As in Proposition 3.3.1.6, we prove (Proposition 5.3.7.4) that these Kontsevich–Soibelman skeletons are contained in $\mathrm{Sk}(X, D)$, and so $\mathrm{Sk}^{\mathrm{ess}}(X, D)$ is as well. In other words, these weight functions of (X, D) cut out certain essential faces from $\mathrm{Sk}(X, D)$, whose union defines the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, D)$.

Moreover, we establish an additional compatibility result (Proposition 5.3.8.5) between the weight functions in the trivially-valued and discretely-valued settings. This implies that the essential skeleton in the former setting is a cone over the essential skeleton in the latter, as stated in our first main result below; see also Figure 5.3.8.1.

Theorem 5.1.2.1 (Proposition 5.3.8.9). *Let $\mathcal{K}$ be a trivially-valued field of characteristic zero. Let $\mathscr{X}$ be a degeneration over $\mathcal{K}[[\varpi]]$ that arises as the base change of $X \to C$ along $\mathrm{Spec}(\widehat{\mathcal{O}}_{C,0}) \to C$, where C is the germ of a smooth $\mathcal{K}$-curve, $0 \in C(\mathcal{K})$, and $\widehat{\mathcal{O}}_{C,0} \simeq \mathcal{K}[[\varpi]]$. Suppose that the generic fibre $\mathscr{X}_{\mathcal{K}((\varpi))}$ of $\mathscr{X}$ is smooth, and X is a normal, flat, projective C-scheme such that the special fibre X_0 is reduced. If (X, X_0) is log canonical and $K_X + X_0$ is semiample, then*

$$\mathrm{Sk}^{\mathrm{ess}}(\mathscr{X}_{\mathcal{K}((\varpi))}) = \mathrm{Sk}^{\mathrm{ess}}(X, X_0) \cap \mathscr{X}^{\mathrm{disc}},$$

where $\mathscr{X}^{\mathrm{disc}} \subseteq X^\beth$ is the $\mathcal{K}((\varpi))$-analytic generic fibre of $\mathscr{X}$.

The end result is a collection of tools (namely the weight functions, skeletons of log-regular pairs, Kontsevich–Soibelman skeletons, and the essential skeleton) which extend and unify different techniques used to study interesting subspaces of the Berkovich analytification, both in the trivially-valued and discretely-valued settings.

5.1.3. Closure of skeletons

Let (X, D) be an snc pair over a non-archimedean field $\mathcal{K}$ that is either trivially or discretely-valued. For any regular D-logarithmic pluricanonical form η on X, and any stratum W of D, we write $\mathrm{Res}_W(\eta)$ for the residue form of η along W, and $(W, \sum_{j\colon W\not\subseteq D_j} D_j|_W)$ for the induced log-regular structure on W (see Proposition 5.4.1.2 for a precise definition).

We describe the closure of $\mathrm{Sk}(X, D, \eta)$ in terms of the Kontsevich–Soibelman skeletons of the residue forms $\mathrm{Res}_W(\eta)$ of η along the various strata W of D.

Theorem 5.1.3.1 (Proposition 5.4.3.4). *If D is an snc divisor on X and η is a non-zero regular D-logarithmic pluricanonical form on X, then the closure of the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X, D, \eta)$ in X^{an} lies in the disjoint union of the Kontsevich–Soibelman skeletons*

$$\bigsqcup_W \mathrm{Sk}\Big(W, \sum_{j\colon W \not\subseteq D_j} D_j|_W, \mathrm{Res}_W(\eta)\Big),$$

where the index runs over all strata W of D.

In addition, we show that the inclusion in Theorem 5.1.3.1 is an equality when $\mathcal{K}$ is trivially-valued (see Proposition 5.4.3.8), while it is false in the discretely-valued setting (see Example 5.4.3.10).

There are instances of similar decompositions that occur in the literature. For example, if X is the toric variety over $\mathcal{K}$ associated to a rational polyhedral fan Σ, then Σ admits a natural compactification, which is endowed with a decomposition indexed by the strata of the toric boundary divisor; see [Pay09, §3] and [Rab12, Proposition 3.4]. In [Thu07, §2], this compactification of the support of Σ is embedded into X^{an}, and the image is called the toric skeleton of X. In Corollary 5.4.2.7 we show that the toric skeleton can be realized as an essential skeleton.

5.1.4. Structure of the chapter

In Section 5.2, we recall the formalism of metrics on analytifications of line bundles, and we construct the weight metrics on the logarithmic pluricanonical bundles over trivially or discretely-valued fields of residue characteristic zero. In Section 5.3, we construct the skeleton of a log-regular pair over an arbitrary trivially-valued field and discuss its basic properties, mirroring the discretely-valued presentation of Chapter 3. In addition, we define the Kontsevich–Soibelman skeletons and the essential skeleton of a pair in characteristic zero; this section culminates in the proof of Theorem 5.1.2.1. The closure of the Kontsevich–Soibelman skeletons and of the essential skeleton is described in Section 5.4.

5.2. Weight metrics

(5.2.0.1) In this section, we introduce the notion of a weight function associated to a rational pluricanonical form on a variety defined over a trivially-valued field of characteristic zero. The weight functions are crucial to define and compute the essential skeleton of a pair in the trivially-valued setting.

To this end, we briefly recall the formalism of metrics on the analytification of a line bundle over an arbitrary non-archimedean field $\mathcal{K}$. We introduce the weight metric on the analytification of the pluricanonical bundle; in the discretely-valued case, the weight metric originates in [MN15] and it is studied further in [BM19; NX16; Tem16]. To do so, we assume that $\mathcal{K}$ has residue characteristic zero: this guarantees the divisorial points are dense in the Berkovich analytification (see Definition 2.1.3.1), a property that we employ in the construction of weight functions and metrics.

(5.2.0.2) Throughout the section, let X be a normal variety, and let D be a Weil $\mathbb{Q}$-divisor on X such that $K_X + D_{\mathrm{red}}$ is $\mathbb{Q}$-Cartier. For $m \in \mathbb{Z}_{>0}$ sufficiently divisible, the sections of the line bundle

$$\omega_{(X,D_{\mathrm{red}})}^{\otimes m} := \omega_X^{[m]}(mD_{\mathrm{red}}) = \mathcal{O}_X(m(K_X + D_{\mathrm{red}}))$$

are called *logarithmic m-pluricanonical forms of* (X, D), while the sections of the rank-1 reflexive sheaf

$$\omega_{(X,D)}^{\otimes m} := \omega_X^{[m]}(mD) = \mathcal{O}_X(m(K_X + D))$$

are called *D-logarithmic m-pluricanonical forms on X*.

5.2.1. Metrics on non-archimedean line bundles

Assume that X is defined over a non-Archimedean field $\mathcal{K}$.

Definition 5.2.1.1. Given a line bundle L on X, a *metric* ϕ on its analytification L^{an} is the data of a function $\phi(\cdot, x)\colon L_x^{\mathrm{an}} \to \overline{\mathbb{R}}$ for each $x \in X^{\mathrm{an}}$ such that for any $s \in L_x^{\mathrm{an}}$ and $f \in \mathcal{O}_{X^{\mathrm{an}},x}$, we have

$$\phi(fs, x) = v_x(f) + \phi(s, x). \tag{5.2.1.2}$$

A metric ϕ is *continuous* if for any open subset $U \subseteq X^{\mathrm{an}}$ and any section $s \in \Gamma(U, L^{\mathrm{an}})$, the function

$$U \ni x \mapsto \phi(s, x) \in \overline{\mathbb{R}}$$

is continuous. This definition coincides with the continuous metrics of [CL11], provided that $\phi(\cdot, x) \not\equiv +\infty$ for all $x \in X^{\mathrm{an}}$. Similarly, one defines *upper-semicontinuous* (usc) and *lower-semicontinuous* (lsc) metrics.

It is easy to check that the (semi)continuity of a metric on L^{an} can be verified on algebraic sections of L, as opposed to on all analytic sections of L^{an} as in Definition 5.2.1.1. This is often a more convenient condition to check for metrics that are defined in terms of algebro-geometric data.

Remark 5.2.1.3. Let $x \in X^{\mathrm{an}}$. From the isomorphism $L_x^{\mathrm{an}} \simeq L_{\ker(x)} \otimes_{\mathcal{O}_{X,\ker(x)}} \mathcal{O}_{X^{\mathrm{an}},x}$ it follows that the function $\phi(\cdot, x)\colon L_x^{\mathrm{an}} \to \overline{\mathbb{R}}$ is determined by the restriction $\phi(\cdot, x)\colon L_{\ker(x)} \subseteq L_x^{\mathrm{an}} \to \overline{\mathbb{R}}$. Indeed, if $s \in L_{\ker(x)}$ is an $\mathcal{O}_{X,\ker(x)}$-module generator, then $s \otimes 1 \in L_x^{\mathrm{an}}$ is an $\mathcal{O}_{X^{\mathrm{an}},x}$-module generator, so the values of $\phi(\cdot, x)$ on L_x^{an} are completely determined by the value $\phi(s \otimes 1, x)$ and the formula Eq. (5.2.1.2).

Furthermore, if $\mathcal{K}$ is trivially-valued, then we may consider metrics on $L^{\beth} := L^{\mathrm{an}}|_{X^{\beth}}$. In this setting, the function $\phi(\cdot, x)$ at a point $x \in X^{\beth}$ is determined by its restriction to

$L_{c_X(x)}$; indeed, the localization map $\mathcal{O}_{X,c_X(x)} \hookrightarrow \mathcal{O}_{X,\ker(x)}$ gives rise to an isomorphism $L_{\ker(x)} \simeq L_{c_X(x)} \otimes_{\mathcal{O}_{X,c_X(x)}} \mathcal{O}_{X,\ker(x)}$ and we argue as before.

(5.2.1.4) In the literature, it is common to write a metric ϕ on line bundles in the 'multiplicative' notation $\|\cdot\| = r^{\phi}$ for some $r \in (0,1)$, as opposed to the 'additive' notation introduced in Definition 5.2.1.1. That is, a metric on L^{an} can also be defined as a collection of functions $\|\cdot\|_x \colon L_x^{\mathrm{an}} \to \mathbb{R}_+$ such that $\|f \cdot s\|_x = |f(x)| \cdot \|s\|_x$ for $s \in L_x^{\mathrm{an}}$ and $f \in \mathcal{O}_{X^{\mathrm{an}},x}$. See [CL11] for further details. The multiplicative notation is adopted especially in Chapter 6, whereas the additive notation is more convenient elsewhere.

5.2.2. The trivial metric

Definition 5.2.2.1. Assume $\mathcal{K}$ is trivially-valued. Given a line bundle L on X, the *trivial metric* $\phi_{\mathrm{triv},L}$ on $L^{\beth}$ assigns to a point $x \in X^{\beth}$ and a local section $s \in L_{c_X(x)}$ the number

$$\phi_{\mathrm{triv},L}(s,x) = v_x(f), \tag{5.2.2.2}$$

where s is given by the function $f \in \mathcal{O}_{X,c_X(x)}$ locally at $c_X(x)$. Said differently, pick any $\mathcal{O}_{X,c_X(x)}$-module generator $\delta \in L_{c_X(x)}$, and write $s = f\delta$ in $L_{c_X(x)}$. The expression Eq. (5.2.2.2) is independent of the choice of generator δ, since any two generators differ by a unit $u \in \mathcal{O}^{\times}_{X,c_X(x)}$, and $v_x(u) = 0$.

(5.2.2.3) The trivial metric $\phi_{\mathrm{triv},L}$ allows us to identify a function $\varphi\colon X^{\beth} \to \overline{\mathbb{R}}$ with a metric $\varphi + \phi_{\mathrm{triv},L}$ on $L^{\beth}$; that is, to a point $x \in X^{\beth}$ and a local section $s \in L_{c_X(x)}$, the metric $\varphi + \phi_{\mathrm{triv},L}$ assigns the number

$$(\varphi + \phi_{\mathrm{triv},L})(s,x) \coloneqq \varphi(x) + v_x(f),$$

where, locally at $c_X(x)$, s is given by the function $f \in \mathcal{O}_{X,c_X(x)}$. In fact, every metric on $L^{\beth}$ arises in this manner. See [BJ18b, §2.8] for further details.

Remark 5.2.2.4. If X is proper over $\mathcal{K}$, the trivial metric $\phi_{\mathrm{triv},L}$ is the non-Archimedean metric on $L^{\beth}$ associated to the trivial test configuration of (X,L), in the sense of [BHJ17, Remark 3.3]. The relationship between test configurations and non-Archimedean metrics yields new insights in the study of K-stability; see [BJ18a] for an overview.

5.2.3. The weight metric over a discretely-valued field

(5.2.3.1) Suppose that $\mathcal{K}$ is a discretely-valued field with residue characteristic zero. Generalizing an idea of Kontsevich and Soibelman, to any rational pluricanonical forms η of X, one can construct a function on the analytification X^{an}, called the *weight function associated to* η and denoted wt_{η}. We briefly recall the definition (see [BM19; MN15; NX16] for further details), and prove a maximality property for the weight function.

(5.2.3.2) Let η be a rational section of $\omega_{(X,D)}^{\otimes m}$. The definition of the weight function associated to η on divisorial points is as follows. If $x \in X^{\mathrm{div}}$ has a divisorial representation on a model $\mathscr{X}$ of X, then we may assume that $(\mathscr{X}, D_{\mathscr{X}})$ is a log-regular model of (X, D_{red}), where $D_{\mathscr{X}} = \overline{D}_{\mathrm{red}} + (\mathscr{X}_0)_{\mathrm{red}}$. Then, we set

$$\mathrm{wt}_\eta(x) := v_x(\mathrm{div}_{(\mathscr{X}, D_{\mathscr{X}} - \mathrm{div}(\pi))}(\eta)) + m, \tag{5.2.3.3}$$

where $\mathrm{div}_{(\mathscr{X}, D_{\mathscr{X}} - \mathrm{div}(\pi))}(\eta)$ denotes the divisor on $\mathscr{X}$ determined by η, thought of as a rational section of the line bundle $\mathcal{O}_{\mathscr{X}}(m(K_{\mathscr{X}/K^\circ} + D_{\mathscr{X}} - \mathrm{div}(\pi)))$. By [BM19, Lemma 4.1.4], Eq. (5.2.3.3) is equivalent to the original definition of [MN15, §4.3].

Theorem 5.2.3.4. *Suppose X is smooth. For any rational section η of $\omega_{(X,D)}^{\otimes m}$, there is a unique maximal lower-semicontinuous extension $\mathrm{wt}_\eta \colon X^{\mathrm{an}} \to \overline{\mathbb{R}}$ of the weight function $\mathrm{wt}_\eta \colon X^{\mathrm{div}} \to \mathbb{R}$.*

The extension was produced by Mustață and Nicaise in [MN15, §4.4], and the maximality property is demonstrated below. This property is presumably well-known to experts, but the author is not aware of a proof appearing in the literature.

Proof. Pick a smooth compactification $X \subseteq \overline{X}$ of X, so $X^{\mathrm{bir}} = \overline{X}^{\mathrm{bir}}$. The construction of a lsc extension $\mathrm{wt}_{X,\eta} \colon X^{\mathrm{an}} \to \overline{\mathbb{R}}$ is made in [MN15, §4.4], and similarly we have an extension $\mathrm{wt}_{\overline{X},\eta} \colon \overline{X}^{\mathrm{an}} \to \overline{\mathbb{R}}$. By [MN15, Proposition 4.5.5], $\mathrm{wt}_{X,\eta} = \mathrm{wt}_{\overline{X},\eta}$ on X^{an}. We now prove that $\mathrm{wt}_{X,\eta}$ is maximal: given another lsc extension $W \colon X^{\mathrm{an}} \to \overline{\mathbb{R}}$ of wt_η on X^{div}, we must show the inequality

$$W(x) \leqslant \mathrm{wt}_{X,\eta}(x) \tag{5.2.3.5}$$

for all $x \in X^{\mathrm{an}}$. To this end, we first prove Eq. (5.2.3.5) for $x \in X^{\mathrm{mon}}$, and then for any $x \in X^{\mathrm{an}}$ by approximating x by monomial points.

Step 1. If $x \in X^{\mathrm{mon}}$, pick a sequence (x_j) of divisorial points that converges to x, all of whom lie in the skeleton of a fixed snc model of X. By the lower-semicontinuity of W, we have

$$W(x) \leqslant \liminf_j W(x_j) = \liminf_j \mathrm{wt}_{X,\eta}(x_j) = \mathrm{wt}_{X,\eta}(x),$$

where the final equality $\liminf_j \mathrm{wt}_{X,\eta}(x_j) = \mathrm{wt}_{X,\eta}(x)$ follows from the continuity of the weight function on a fixed skeleton, as in [MN15, Proposition 4.4.3].

Step 2. If $x \in X^{\mathrm{an}}$, then [BFJ16, Corollary 3.2] implies that $x = \lim_{\overline{\mathscr{X}}} \rho_{\overline{\mathscr{X}}}(x)$, where the limit runs over all snc models $\overline{\mathscr{X}}$ of $\overline{X}$ and $\rho_{\overline{\mathscr{X}}} \colon \overline{X}^{\mathrm{an}} \to \mathrm{Sk}(\overline{\mathscr{X}})$ denotes the retraction onto the skeleton from [MN15, §3.1]. As $\rho_{\overline{\mathscr{X}}}(x) \in \overline{X}^{\mathrm{mon}} = X^{\mathrm{mon}}$ for all snc models

$\overline{\mathscr{X}}$, the lower-semicontinuity of W shows that

$$\begin{aligned} W(x) \leqslant \liminf_{\mathscr{X}} W(\rho_{\overline{\mathscr{X}}}(x)) &\leqslant \liminf_{\mathscr{X}} \mathrm{wt}_{X,\eta}(\rho_{\overline{\mathscr{X}}}(x)) \\ &= \liminf_{\mathscr{X}} \mathrm{wt}_{\overline{X},\eta}(\rho_{\overline{\mathscr{X}}}(x)) \\ &\leqslant \sup_{\overline{\mathscr{X}}} \mathrm{wt}_{\overline{X},\eta}(\rho_{\overline{\mathscr{X}}}(x)) = \mathrm{wt}_{\overline{X},\eta}(x) = \mathrm{wt}_{X,\eta}(x). \end{aligned}$$

The uniqueness of the extension follows from the maximality, and we write it simply as $\mathrm{wt}_\eta = \mathrm{wt}_{X,\eta}$. □

Definition 5.2.3.6. The *weight metric* $\mathrm{wt}_{\mathrm{disc}}$ is the metric on $(\omega^{\otimes m}_{(X,D_{\mathrm{red}})})^{\mathrm{an}}$ satisfying

$$\mathrm{wt}_{\mathrm{disc}}(\eta, x) = \mathrm{wt}_\eta(x) \tag{5.2.3.7}$$

for any $x \in X^{\mathrm{an}}$ and rational section η of $\omega^{\otimes m}_{(X,D_{\mathrm{red}})}$ that is regular at $\ker(x)$. By Theorem 5.2.3.4, $\mathrm{wt}_{\mathrm{disc}}$ is the maximal lower-semicontinuous metric on $(\omega^{\otimes m}_{(X,D_{\mathrm{red}})})^{\mathrm{an}}$ such that Eq. (5.2.3.7) holds on X^{div}. Write $\|\cdot\|_{\mathrm{wt}_{\mathrm{disc}}}$ for the weight metric in multiplicative notation, as in 5.2.1.4.

5.2.4. The weight metric over a trivially-valued field

(5.2.4.1) Suppose that $\mathcal{K}$ is a trivially-valued field of characteristic zero, and assume that $K_X + D$ is $\mathbb{Q}$-Cartier. The following definition is a standard numerical invariant of a divisorial valuation that arises in birational geometry; see e.g. [Kol13, Definition 2.4].

Definition 5.2.4.2. Let $x \in X^{\mathrm{div}} \cap X^{\beth}$ be the divisorial point determined by the triple $(c, Y \xrightarrow{h} X, E)$. Pick canonical divisors K_Y on Y and K_X on X such that $h_*(K_Y) = K_X$. The *log discrepancy* $A_{(X,D)}(x)$ of x is the value

$$A_{(X,D)}(x) \coloneqq c\left(1 + \mathrm{ord}_E\left(K_Y - \frac{1}{m}h^*(m(K_X + D))\right)\right) \tag{5.2.4.3}$$

for $m \in \mathbb{Z}_{>0}$ sufficiently divisible. The pair (X, D) is *log canonical* if $A_{(X,D)}(x) \geqslant 0$ for all $x \in X^{\mathrm{div}} \cap X^{\beth}$.

It is easy to verify that the log discrepancy $A_{(X,D)}(x)$ depends only on x, and not on the choice of m or of the birational model Y of X where the centre of x is a divisor.

(5.2.4.4) There is a maximal lower-semicontinuous extension $A_{(X,D)} \colon X^{\beth} \to \overline{\mathbb{R}}$ of the log discrepancy on the divisorial points $X^{\mathrm{div}} \cap X^{\beth}$; it is given by

$$A_{(X,D)}(x) = \sup_{U \ni x} \inf_{y \in U \cap X^{\mathrm{div}}} A_{(X,D)}(y), \tag{5.2.4.5}$$

where the supremum runs over all open neighborhoods U of x in $X^{\beth}$. The extension $A_{(X,D)}$, which we also refer to as the log discrepancy function is $\mathbb{R}_+$-homogeneous, and it is non-negative when (X, D) is log canonical. The restriction to $X^{\mathrm{bir}} \cap X^{\beth}$ admits an alternative characterization; see [Blu18, §3.2].

The log discrepancy function is well studied in the literature: when X is smooth and $D = \emptyset$, it is introduced in [JM12, §5] as a function $A_X \colon X^{\mathrm{bir}} \cap X^{\beth} \to \overline{\mathbb{R}}_+$. The same holds for normal varieties by [BdFFU15]. The function A_X is extended to all of $X^{\beth}$ when X is smooth in [BJ18a, Appendix A], and it is constructed in positive characteristic in [Can17, §3].

Definition 5.2.4.6. For a rational section η of $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$ that is regular on the Zariski open $U \subseteq X$, the *weight function* $\mathrm{wt}_\eta \colon U^{\beth} \to \overline{\mathbb{R}}$ of η is given by

$$\mathrm{wt}_\eta(x) = mA_{(X,D_{\mathrm{red}})}(x) + \phi_{\mathrm{triv},\omega_{(X,D_{\mathrm{red}})}^{\otimes m}}(\eta, x).$$

The *weight metric* $\mathrm{wt}_{\mathrm{triv}}$ is the metric on $(\omega_{(X,D_{\mathrm{red}})}^{\otimes m})^{\beth}$ satisfying

$$\mathrm{wt}_{\mathrm{triv}}(\eta, x) = \mathrm{wt}_\eta(x) \tag{5.2.4.7}$$

for any $x \in X^{\beth}$ and rational section η of $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$ that is regular at $\ker(x)$. By 5.2.4.4, $\mathrm{wt}_{\mathrm{triv}}$ is the the maximal lower-semicontinuous metric on $(\omega_{(X,D_{\mathrm{red}})}^{\otimes m})^{\beth}$ such that Eq. (5.2.4.7) holds on $X^{\mathrm{div}} \cap X^{\beth}$. Write $\| \cdot \|_{\mathrm{wt}_{\mathrm{triv}}}$ for the weight metric in multiplicative notation.

Remark 5.2.4.8. There is another construction in [MN15, §6.1] of a weight function in the trivially-valued setting, which is distinct from the weight function of Definition 5.2.4.6 (indeed, it does not take a pluricanonical section as an argument).

5.2.5. Alternative expressions for the weight function over a trivially-valued field.

(5.2.5.1) Assume that $\mathcal{K}$ is a trivially-valued field of characteristic zero. For a rational section η of $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$, set

$$D_\eta \coloneqq D_{\mathrm{red}} - \mathrm{div}_{(X,D_{\mathrm{red}})}(\eta),$$

where $\mathrm{div}_{(X,D_{\mathrm{red}})}(\eta)$ denotes the divisor of η, thought of as a rational section of the line bundle $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$. In the following proposition, we provide an alternative expression for the weight function associated to η, which is purely in terms of a log discrepancy function.

Proposition 5.2.5.2. *For any $x \in X^{\beth}$, we have $\mathrm{wt}_\eta(x) = mA_{(X,D_\eta)}(x)$.*

Proof. By 5.2.4.4, it suffices to check the equality on divisorial points. If $x \in X^{\mathrm{div}} \cap X^{\beth}$

is determined by the triple $(c, Y \xrightarrow{h} X, E)$, then we have that

$$\begin{aligned}
\mathrm{wt}_\eta(x) &= mA_{(X,D_{\mathrm{red}})}(x) + \phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta, x) \\
&= mc\left(1 + \mathrm{ord}_E\left(K_Y - \frac{1}{m}h^*(m(K_X + D_{\mathrm{red}}))\right)\right) + c\,\mathrm{ord}_E(h^*\mathrm{div}_{(X,D_{\mathrm{red}})}(\eta)) \\
&= mA_{(X,D_{\mathrm{red}}-\mathrm{div}_{(X,D_{\mathrm{red}})}(\eta))}(x) \\
&= mA_{(X,D_\eta)}(x),
\end{aligned}$$

as required. □

Corollary 5.2.5.3. *If $x \in X^{\mathrm{div}} \cap X^{\beth}$ is the divisorial point determined by the triple $(c, Y \xrightarrow{h} X, E)$, then*

$$\mathrm{wt}_\eta(x) = v_x(\mathrm{div}_{(Y,D_Y)}(h^*\eta)),$$

where $D_Y = \widetilde{D}_{\mathrm{red}} + \sum_i E_i$, $\widetilde{D}_{\mathrm{red}}$ denotes the strict transform of D_{red} via h, and the E_i's are the irreducible h-exceptional divisors on Y.

Corollary 5.2.5.3 shows that the weight function wt_η on $X^{\mathrm{div}} \cap X^{\beth}$ can be computed much as in the discretely-valued setting; indeed, this result is the analogue of [BM19, Lemma 4.1.4]. Moreover, Corollary 5.2.5.3 can be deduced from Proposition 5.2.5.2, but we find enlightening to provide a different proof of the statement using a local calculation.

Proof. By definition of weight function and of the log discrepancy function, we have that

$$\begin{aligned}
\mathrm{wt}_\eta(x) &= mA_{(X,D_{\mathrm{red}})}(x) + \phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta, x) \\
&= mc\left(1 + \mathrm{ord}_E\left(K_Y - \frac{1}{m}h^*(m(K_X + D_{\mathrm{red}}))\right)\right) + \phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta, x) \\
&= c\,\mathrm{ord}_E\left((\omega^{\otimes m}_{(Y,D_Y)})^{-1} \otimes h^*\omega^{\otimes m}_{(X,D_{\mathrm{red}})}\right) + \phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta, x),
\end{aligned}$$

where the last equality uses the convention in 2.1.3.3. Let $\xi = c_X(x)$ be the centre of x on X, and let ξ' be the generic point of E in Y. Consider a $\mathcal{O}_{X,\xi}$-module generator δ of $\omega^{\otimes m}_{(X,D_{\mathrm{red}}),\xi}$. Then, locally at ξ, we write the section η as $\eta = f\delta$ for some $f \in \mathrm{Frac}(\mathcal{O}_{X,\xi})$, so that

$$\phi_{\mathrm{triv},\omega^{\otimes m}_{(X,D_{\mathrm{red}})}}(\eta, x) = v_x(f) = c\,\mathrm{ord}_E(h^*f).$$

Consider now a $\mathcal{O}_{Y,\xi'}$-module generator α of $\left(\omega^{\otimes m}_{(Y,D_Y)} \otimes (h^*\omega^{\otimes m}_{(X,D_{\mathrm{red}})})^{-1}\right)_{\xi'}$. It follows that $\alpha \otimes h^*\delta$ is a $\mathcal{O}_{Y,\xi'}$-module generator of $(\omega^{\otimes m}_{(Y,D_Y)})_{\xi'}$, and we write $h^*\eta = (\alpha^{-1}h^*f) \cdot \alpha \otimes h^*\delta$ locally at ξ'. It follows that

$$\begin{aligned}
v_x(\mathrm{div}_{(Y,D_Y)}(h^*\eta)) = v_x(\alpha^{-1}h^*f) &= v_x(\alpha^{-1}) + v_x(h^*f) \\
&= c\,\mathrm{ord}_E\left((\omega^{\otimes m}_{(Y,D_Y)})^{-1} \otimes h^*\omega^{\otimes m}_{(X,D_{\mathrm{red}})}\right) + c\,\mathrm{ord}_E(h^*f) \\
&= \mathrm{wt}_\eta(x),
\end{aligned}$$

which concludes the proof. □

5.3. Skeletons over a trivially-valued field

(5.3.0.1) In this section, we construct a skeleton associated to a log-regular log scheme over a trivially valued field k. This generalizes the construction of the simplicial cones of quasi-monomial valuations in [JM12, §3], and it is a trivially-valued analogue of the skeletons of [BM19, §3]. The outcome coincides with the skeleton of [Uli17, §6], but the descriptions of the points in the two constructions are slightly different: our realization of the skeleton, inspired by [MN15], enables us to describe the minimality loci of the weight functions of Section 5.2.4, and ultimately to define the essential skeleton of a pair over k, when k has characteristic zero.

(5.3.0.2) Throughout the section, let k be a trivially-valued field, X a normal variety over k, and D an effective $\mathbb{Q}$-divisor on X such that $K_X + D_{\mathrm{red}}$ is $\mathbb{Q}$-Cartier, and assume that the log scheme $X^+ = (X, D_{\mathrm{red}})$ is log-regular, in particular $D_{X^+} = D_{\mathrm{red}}$. Note that, under these assumptions, the pair (X, D_{red}) is log canonical.

5.3.1. The faces of the skeleton of a log-regular scheme.

(5.3.1.1) In the following proposition, we construct the valuations that will form the skeleton of X^+. Over a perfect field, the log scheme X^+ has toroidal singularities, and the valuations of its skeleton are the toric or monomial valuations of the local toric model, parametrized by the realification of the cocharacter lattice ([Thu07]). For an arbitrary log-regular pair X^+ the valuations are expressed in terms of the log-geometric data.

Proposition 5.3.1.2. *For any $x \in F_{X^+}$ and $\alpha \in \mathrm{Hom}(\mathcal{C}_{X^+,x}, \overline{\mathbb{R}}_+)$, there exists a unique minimal semivaluation*

$$v_\alpha \colon \mathcal{O}_{X,x} \backslash \{0\} \to \overline{\mathbb{R}}_+$$

such that

1. *v_α extends the trivial valuation v_0 on $k \hookrightarrow \mathcal{O}_{X,x}$;*
2. *for any $f \in \mathcal{M}_{X^+,x}$, we have $v_\alpha(f) = \alpha(\overline{f})$.*

Moreover, v_α is a valuation if and only if $\alpha \in \mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbb{R}_+)$.

Proof. The proof is identical to Proposition 3.2.2.10, and we recall the construction here. Pick a multiplicative section $\sigma \colon \mathcal{C}_{X^+,x} \to \mathcal{M}_{X^+,x}$ of the quotient map $\mathcal{M}_{X^+,x} \to \mathcal{C}_{X^+,x}$. By [BM19, Lemma 3.2.3], any $f \in \mathcal{O}_{X,x}$ can be expressed as

$$f = \sum_{\gamma \in \mathcal{C}_{X^+,x}} a_\gamma \cdot \sigma(\gamma)$$

as an element of the $\mathfrak{m}_x$-adic completion $\widehat{\mathcal{O}}_{X,x}$, where $a_\gamma \in \mathcal{O}_{X,x}^\times \cup \{0\}$. Such an expression will be referred to as an *admissible expansion* of f. Now, set

$$v_\alpha(f) := \inf_{\gamma \in \mathcal{C}_{X,x}} v_0(a_\gamma) + \alpha(\gamma). \tag{5.3.1.3}$$

Following [BM19, Proposition 3.2.10], one can show that $v_\alpha(f)$ is independent of the choice of admissible expansion of f or of the choice of section σ, the infimum is in fact a minimum, and v_α defines a semivaluation on $\mathcal{O}_{X,x}$ that satisfies the desired properties. □

(5.3.1.4) For any $x \in F_{X^+}$, consider the subset

$$\mathrm{Sk}_x(X^+) := \{v_\alpha \colon \alpha \in \mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbb{R}_+)\}$$

of $X^\beth$, equipped with the subspace topology inherited from $X^\beth$. Alternatively, $\mathrm{Sk}_x(X^+)$ can be equipped with the topology of pointwise convergence inherited from the identification with the space $\mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbb{R}_+)$; that is, for $(\alpha_n)_{n=1}^\infty$ and α in $\mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbb{R}_+)$, we have $v_{\alpha_n} \to v_\alpha$ in $\mathrm{Sk}_x(X^+)$ if and only if $\alpha_n(\gamma) \to \alpha(\gamma)$ for all $\gamma \in \mathcal{C}_{X^+,x}$. These two topologies are compared below.

Lemma 5.3.1.5. *The topology of pointwise convergence on $\mathrm{Sk}_x(X^+)$ coincides with the subspace topology inherited from $X^\beth$.*

Proof. Given $(\alpha_n)_{n=1}^\infty$ and α in $\mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbb{R}_+)$, it suffices to show that $v_{\alpha_n}(f) \to v_\alpha(f)$ for all $f \in \mathcal{O}_{X,x}$ if and only if $\alpha_n(\gamma) \to \alpha(\gamma)$ for all $\gamma \in \mathcal{C}_{X^+,x}$. Granted the latter assumption, the convergence $v_{\alpha_n}(f) \to v_\alpha(f)$ follows by (5.3.1.3). Conversely, for any lift $\widetilde{\gamma} \in \mathcal{M}_{X^+,x}$ of γ, we have that $\alpha_n(\gamma) = v_{\alpha_n}(\widetilde{\gamma}) \to v_\alpha(\widetilde{\gamma}) = \alpha(\gamma)$. □

Lemma 5.3.1.6. *For any $x \in F_{X^+}$, the closure $\overline{\mathrm{Sk}}_x(X^+)$ of $\mathrm{Sk}_x(X^+)$ in $X^\beth$ coincides with the subset*

$$\{v_\alpha \colon \alpha \in \mathrm{Hom}(\mathcal{C}_{X^+,x}, \overline{\mathbb{R}}_+)\}. \tag{5.3.1.7}$$

In addition, $\overline{\mathrm{Sk}}_x(X^+) \cap X^{\mathrm{bir}} = \mathrm{Sk}_x(X^+)$

Proof. Denote by Z_x the subset of $X^\beth$ defined in Eq. (5.3.1.7). It is clear that $\mathrm{Sk}_x(X^+) \subseteq Z_x$; thus, we need to show that Z_x is contained in $\overline{\mathrm{Sk}}_x(X^+)$ and Z_x is closed in $X^\beth$.

Consider a net $(v_{\alpha_\epsilon})_\epsilon$ in Z_x such that $v_{\alpha_\epsilon} \to v$ for some $v \in X^\beth$. For any $f \in \mathcal{O}_{X,x}^\times$, $v(f) = \lim_\epsilon v_{\alpha_\epsilon}(f) = 0$, so the restriction of v to $\mathcal{M}_{X^+,x}$ descends to a monoid morphism $\alpha \colon \mathcal{C}_{X^+,x} \to \overline{\mathbb{R}}_+$. Arguing as in Lemma 5.3.1.5, one sees that $\alpha_\epsilon \to \alpha$ and hence $v_{\alpha_\epsilon} \to v_\alpha$ in $X^\beth$; thus, $v = v_\alpha$ lies in Z_x, so Z_x is contained in $\overline{\mathrm{Sk}}_x(X^+)$.

Now, following the proof of Lemma 5.3.1.5, we observe that the map

$$\begin{aligned} \mathrm{Hom}(\mathcal{C}_{X^+,x}, \overline{\mathbb{R}}_+) &\twoheadrightarrow Z_x \subseteq X^\beth \\ \alpha &\mapsto v_\alpha \end{aligned}$$

is continuous, so Z_x is the image of a compact space into a Hausdorff space via a continuous map, and hence Z_x is closed. It follows that $Z_x = \overline{\mathrm{Sk}}_x(X^+)$. Finally, $Z_x \cap X^{\mathrm{bir}} = \mathrm{Sk}_x(X^+)$ by Proposition 5.3.1.2. □

5.3.2. The skeleton of a log-regular scheme.

(5.3.2.1) The subsets $\mathrm{Sk}_x(X^+)$ of $X^{\beth}$, for $x \in F_{X^+}$, can be glued together compatibly with the relation of specialization in the Kato fan F_{X^+}; see 3.2.1.1 for the definition of this relation. Indeed, consider $x, y \in F_{X^+}$ where x is a specialization of y, i.e. $x \in \overline{\{y\}}$. The localization map $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_{X,y}$ descends to a surjective monoid morphism $\tau_{x,y} \colon \mathcal{C}_{X^+,x} \twoheadrightarrow \mathcal{C}_{X^+,y}$. In this case, the two subsets of $\mathrm{Sk}_x(X^+)$ and $\mathrm{Sk}_y(X^+)$ of $X^{\beth}$ are related as follows:

Lemma 5.3.2.2. *The map $\mathrm{Sk}_y(X^+) \to \mathrm{Sk}_x(X^+)$, given by $v_\alpha \mapsto v_{\alpha\circ\tau_{x,y}}$, is continuous and injective. Furthermore, this map identifies $\mathrm{Sk}_y(X^+)$ as a subspace of $\mathrm{Sk}_x(X^+)$ in $X^{\beth}$.*

Proof. The continuity is immediate from Lemma 5.3.1.5, and the injectivity follows from the surjectivity of $\tau_{x,y}$. Finally, note that v_α and $v_{\alpha\circ\tau_{x,y}}$ coincide as points of $X^{\beth}$ by the uniqueness in Proposition 5.3.1.2. □

Definition 5.3.2.3. The *skeleton* of X^+ is the subspace

$$\mathrm{Sk}(X^+) := \bigcup_{x\in F_{X^+}} \mathrm{Sk}_x(X^+) \subseteq X^{\beth},$$

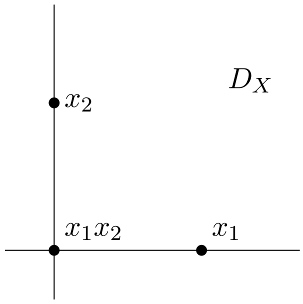
where $\mathrm{Sk}_y(X^+)$ is identified as a subset of $\mathrm{Sk}_x(X^+)$ whenever $x \in \overline{\{y\}}$ via Lemma 5.3.2.2.

By construction, $\mathrm{Sk}(X^+)$ has the structure of a polyhedral cone complex with vertex v_0 where

$$\{v_0\} = \mathrm{Hom}(\{0\}, \mathbb{R}_+) = \mathrm{Sk}_{\eta_X}(X^+)$$

and $\eta_X \in F_{X^+}$ is the generic point of X. The faces of $\mathrm{Sk}(X^+)$ are precisely the subsets $\mathrm{Sk}_x(X^+)$ for $x \in F_{X^+}$. Write $\overline{\mathrm{Sk}}(X^+)$ for the closure of $\mathrm{Sk}(X^+)$ in $X^{\beth}$. Lemma 5.3.1.6 shows that $\overline{\mathrm{Sk}}(X^+)$ is the union of the subsets $\overline{\mathrm{Sk}}_x(X^+)$ for $x \in F_{X^+}$ with the suitable identifications as in Lemma 5.3.2.2.

Example 5.3.2.4. Consider $X = \mathbb{A}^2_{\mathbb{C}}$ with coordinates x_1, x_2, and endow X with the divisorial log structure induced by $\{x_1x_2 = 0\}$. We denote by D_1, D_2 and D_{12} the non-empty intersections of irreducible components of D_X, and by $\eta_\cdot$ the corresponding generic points. At each point x in F_{X^+}, we determine the morphisms $\alpha : \mathcal{C}_{X^+,x} \to (\mathbb{R}_{\geqslant 0}, +)$ to define the point in the skeleton:



$\alpha : \mathcal{C}_{X^+,\eta_{D_i}} \simeq \mathbb{N} \to (\mathbb{R}_{\geqslant 0}, +)$

$\alpha : \mathcal{C}_{X^+,\eta_{D_{12}}} \simeq \mathbb{N}^2 \to (\mathbb{R}_{\geqslant 0}, +)$

$\alpha : \mathcal{C}_{X^+,\eta_X} \simeq \{0\} \to (\mathbb{R}_{\geqslant 0}, +)$

We obtain that the faces of the skeleton $\mathrm{Sk}(X^+)$ are $\mathrm{Sk}_{\eta_{D_i}}(X^+) \simeq \mathbb{R}_{\geqslant 0}$, $\mathrm{Sk}_{\eta_{D_{12}}}(X^+) \simeq (\mathbb{R}_{\geqslant 0})^2$, and $\mathrm{Sk}_{\eta_X}(X^+) \simeq \{pt\}$. Thus, the skeleton is isomorphic to $(\mathbb{R}_{\geqslant 0})^2$. Further explicit examples of skeletons are considered in Section 5.4.

(5.3.2.5) For any log-regular log scheme X^+ over k, [Kat94, Proposition 9.8] shows that there is a regular k-scheme X', a reduced snc divisor D' on X', and a morphism $X'^+ = (X', D') \to X^+$ of log schemes such that $F_{X'^+}$ is obtained from F_{X^+} via subdivisions. As subdivisions of the Kato fan do not change the associated skeleton, it follows that $X'^\beth \to X^\beth$ restricts to a homeomorphism $\mathrm{Sk}(X'^+) \simeq \mathrm{Sk}(X^+)$. Two consequences of this fact are detailed below:

- The skeleton $\mathrm{Sk}(X^+)$ coincides with the subspace $\mathrm{QM}(X', D_{X'}) \subseteq X^\beth$ of quasi-monomials valuations in $(X', D_{X'})$ constructed in [JM12, §3]. It follows that $\mathrm{Sk}(X^+)$ lies in the locus of quasi-monomial points of $X^\beth$.

- Under the identification $\mathrm{Sk}(X'^+) \simeq \mathrm{Sk}(X^+)$, the skeleton $\mathrm{Sk}(X^+)$ is endowed with the structure of a simplicial cone complex, and moreover with an integral piecewise affine structure (analogous to [JM12, §4.2]).

5.3.3. The retraction to the skeleton.

Proposition 5.3.3.1. *There is a continuous retraction map* $\rho_{X^+} \colon X^\beth \to \overline{\mathrm{Sk}}(X^+)$ *such that* $c_X(v) \in \overline{\{c_X(\rho_{X^+}(v))\}}$ *for all* $v \in X^\beth$. *Moreover,* ρ_{X^+} *restricts to a continuous retraction map* $X^{\mathrm{bir}} \cap X^\beth \to \mathrm{Sk}(X^+)$.

Proof. Given $v \in X^\beth$, the construction of the retraction $\rho_{X^+}(v)$ is described below. Write $x = c_X(v)$, and let y be a Kato point in F_{X^+} to which x specializes; that is, y is the generic point of a stratum of D_{X^+} to which x specializes. If $\mathcal{I}_{X^+,x}$ denotes the ideal of $\mathcal{O}_{X,x}$ generated by $\mathcal{M}_{X^+,x} \backslash \mathcal{O}^\times_{X^+,x}$, then the natural map $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_{X,y}$ is the localization at $\mathcal{I}_{X^+,x}$, and hence $\mathfrak{m}_y = \mathcal{I}_{X^+,x}\mathcal{O}_{X,y}$. For any multiplicative section $\sigma \colon \mathcal{C}_{X^+,x} \to \mathcal{M}_{X^+,x}$, [BM19, Lemma 3.2.3] shows that any $f \in \mathcal{O}_{X,x}$ can be expressed as

$$f = \sum_{\gamma \in \mathcal{C}_{X^+,x}} a_\gamma \cdot \sigma(\gamma) \tag{5.3.3.2}$$

as an element of $\widehat{\mathcal{O}}_{X,y}$, where $a_\gamma \in (\mathcal{O}_{X,x} \backslash \mathcal{I}_{X^+,x}) \cup \{0\}$. For any expansion of f as in Eq. (5.3.3.2), set

$$\widetilde{v}(f) \coloneqq \min_{\gamma \in \mathcal{C}_{X^+,x}} v_0(a_\gamma) + v(\sigma(\gamma)). \tag{5.3.3.3}$$

Following the proof of [BM19, Proposition 3.2.10], one can show that $\widetilde{v}$ is well defined and is a semivaluation $\widetilde{v}\colon \mathcal{O}_{X,x} \to \overline{\mathbb{R}}_+$. Further, it is clear that x specializes to $c_X(\widetilde{v})$ since $\widetilde{v}(f) \geqslant 0$ for all $f \in \mathcal{O}_{X,x}$.

We claim that $\widetilde{v} \in \overline{\mathrm{Sk}}_y(X^+)$. To see this, we construct a monoid morphism $\widetilde{\alpha} \in \mathrm{Hom}(\mathcal{C}_{X^+,y}, \overline{\mathbb{R}}_+)$ such that $\widetilde{v} = v_{\widetilde{\alpha}}$ as semivaluations, where $v_{\widetilde{\alpha}}$ is the semivaluation constructed in Proposition 5.3.1.2. Observe that any $f \in \mathcal{O}_{X,y}$ can be written as $f = g/h$ with $g \in \mathcal{O}_{X,x}$ and $h \in \mathcal{O}_{X,x} \backslash \mathcal{I}_{X^+,x}$, so $\widetilde{v}(h) = 0$ and hence $\widetilde{v}(f) = \widetilde{v}(g) \geqslant 0$. In addition, f is invertible in $\mathcal{O}_{X,y}$ if and only if g is, which is equivalent to $g \in \mathcal{O}_{X,x} \backslash \mathcal{I}_{X^+,x}$; in this case, $\widetilde{v}(f) = \widetilde{v}(g) = 0$ by construction. Thus, the restriction of $\widetilde{v}$ to $\mathcal{M}_{X^+,y}$ descends to a monoid morphism $\widetilde{\alpha}\colon \mathcal{C}_{X^+,y} \to \overline{\mathbb{R}}_+$. The uniqueness in Proposition 5.3.1.2 guarantees that $\widetilde{v} = v_{\widetilde{\alpha}}$; thus, set $\rho_{X^+}(v) \coloneqq \widetilde{v} \in \overline{\mathrm{Sk}}_y(X^+)$.

Note that if $v \in \overline{\mathrm{Sk}}(X^+)$, then we have $\widetilde{v} = v$. Indeed, if $x = c_X(v) \in F_{X^+}$, then the formula Eq. (5.3.3.3) defining $\widetilde{v}$ on elements of $\mathcal{O}_{X,x}$ coincides with Eq. (5.3.1.3). That is, ρ_{X^+} is a retraction of $X^\beth$ onto $\overline{\mathrm{Sk}}(X^+)$ for the inclusion $\overline{\mathrm{Sk}}(X^+) \to X^\beth$.

It remains to show that ρ_{X^+} is continuous. For each $w \in X^\beth$, consider the subset $U_w = c_X^{-1}(\overline{\{c_X(\rho_{X^+}(w))\}})$ of $X^\beth$, which is an open neighborhood of w since the centre map is anticontinuous. As $\{U_w\}_{w \in X^\beth}$ is an open cover of $X^\beth$, it suffices to show that the restriction $\rho_{X^+}|_{U_w}$ is continuous for each $w \in X^\beth$. Note that the image of $\rho_{X^+}|_{U_w}$ lies in $\overline{\mathrm{Sk}}_{c_X(\rho_{X^+}(w))}(X^+)$ because $c_X(\rho_{X^+}(w))$ is a Kato point to which $c_X(w')$ specializes for all $w' \in U_w$. The continuity of $\rho_{X^+}|_{U_w}$ is then a consequence of the following: for any $f \in \mathcal{O}_{X,c_X(\rho_{X^+}(w))}$, the map

$$\begin{aligned} U_w &\to \overline{\mathbb{R}}_+ \\ w' &\mapsto v_{\rho_{X^+}(w')}(f) \end{aligned}$$

is continuous. Indeed, if $f = \sum_\gamma a_\gamma \cdot \sigma(\gamma)$ is an admissible expansion in $\widehat{\mathcal{O}}_{X,c_X(\rho_{X^+}(w))}$, then

$$v_{\rho_{X^+}(w')}(f) = \min_\gamma v_0(a_\gamma) + w'(\sigma(\gamma))$$

is continuous in w'. Hence, $\rho_{X^+}|_{U_w}$ is continuous, which concludes the proof. □

(5.3.3.4) The retraction of Proposition 5.3.3.1 is related to other constructions in the literature.

- If $X^+ = (X, D_{X^+})$ is an snc pair, the retraction ρ_{X^+} of Proposition 5.3.3.1 restricts to the retraction $X^{\mathrm{bir}} \cap X^\beth \to \mathrm{Sk}(X^+)$ of [JM12, §4.3]. Note that [JM12] denotes the space $X^{\mathrm{bir}} \cap X^\beth$ by Val_X.

- After identifying $\overline{\mathrm{Sk}}(X^+)$ with the extended cone complex $\overline{\Sigma}_{X^+}$ following [Uli17, §6.1], ρ_{X^+} coincides with the tropicalization map $X^\beth \to \overline{\Sigma}_{X^+}$. In particular, [Uli17, Theorem 1.2] implies that ρ_{X^+} recovers Thuillier's (strong deformation) retraction map from [Thu07, §3.2].

5.3.4. Functoriality of the skeleton.

(5.3.4.1) Given log-regular log schemes X^+ and Y^+ over k and a morphism $\varphi\colon X \to Y$ of k-schemes, write $\varphi^{\beth}\colon X^{\beth} \to Y^{\beth}$ for the $\beth$-analytification. The retraction map of Proposition 5.3.3.1 shows that $\varphi^{\beth}$ restricts to a continuous map

$$\overline{\mathrm{Sk}}(X^+) \hookrightarrow X^{\beth} \xrightarrow{\varphi^{\beth}} Y^{\beth} \xrightarrow{\rho_{Y^+}} \overline{\mathrm{Sk}}(Y^+) \tag{5.3.4.2}$$

between the closures of the skeletons. If φ is a dominant map, then Eq. (5.3.4.2) restricts to a continuous map

$$\mathrm{Sk}(X^+) \hookrightarrow X^{\mathrm{bir}} \cap X^{\beth} \xrightarrow{\varphi^{\beth}} Y^{\mathrm{bir}} \cap Y^{\beth} \xrightarrow{\rho_{Y^+}} \mathrm{Sk}(Y^+). \tag{5.3.4.3}$$

That is, the formation of the skeleton is functorial with respect to dominant morphisms.

5.3.5. Comparison with the dual complex in the snc case

(5.3.5.1) In [MN15, Proposition 3.1.4], Mustaţă and Nicaise remark that, given a variety X over a discretely valued field, the skeleton associated to an snc model $\mathscr{X}$ of X over the valuation ring is homeomorphic to the dual intersection complex of the special fibre $\mathscr{X}_0$.

We treat now the trivially-valued field case: consider a log-regular pair $X^+ = (X, D_{X^+})$, where D_{X^+} is an snc divisor. Let $\mathcal{D}(D_{X^+})$ denote the dual intersection complex of D_{X^+} as in Definition 2.2.0.1. In the following proposition, we compare it with the skeleton $\mathrm{Sk}(X^+)$; this result is well-known to experts, but we include a proof for the sake of completeness.

Proposition 5.3.5.2. *There is a homeomorphism between* $\mathrm{Sk}(X^+)$ *and the cone over* $\mathcal{D}(D_{X^+})$.

In Section 7.2, we extend Proposition 5.3.5.2 to more singular pairs; see for instance Lemma 7.2.4.6.

Proof. A point $x \in F_{X^+}$ is the generic point of a stratum of D_{X^+} of codimension r, for some r; since D_{X^+} is snc, a choice of local equations for D_{X^+} at x yields an isomorphism $\mathcal{C}_{X^+,x} \simeq \mathbf{N}^r$. This induces an isomorphism $\mathrm{Sk}_x(X^+) \simeq \mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbb{R}_+) \simeq (\mathbb{R}_+)^r$ of topological monoids.

A face of $\mathcal{D}(D_{X^+})$ correspond to a stratum Z of D_{X^+} of codimension r for some r, and is isomorphic to the standard simplex Δ^{r-1}. Thus, the cone over this face is homeomorphic to $(\mathbb{R}_+)^r$, i.e. to $\mathrm{Sk}_x(X^+)$ where x is the generic point of Z.

As the gluing maps on the dual complex are compatible with the identifications on $\mathrm{Sk}(X^+)$, we conclude that the cone over the dual complex is homeomorphic to the skeleton of X^+. □

Definition 5.3.5.3. The *link* of the skeleton $\mathrm{Sk}(X^+)$ is the (topological) quotient $\mathrm{Sk}(X^+)^*/\mathbb{R}_+^*$ by the rescaling action of Definition 2.1.1.7.

Proposition 5.3.5.4. [Thu07, Proposition 4.7] *The spaces* $\mathrm{Sk}(X^+)^*/\mathbb{R}_+^*$ *and* $\mathcal{D}(D_{X^+})$ *are homeomorphic.*

Proof. The rescaling action on the punctured cone over $\mathcal{D}(D_{X^+})$ (as in Definition 2.2.0.2) makes the homeomorphism of Proposition 5.3.5.2 into an $\mathbb{R}_+^*$-equivariant one. The assertion follows by taking quotients by the $\mathbb{R}_+^*$-actions. □

It follows from Proposition 5.3.5.4 that $\mathrm{Sk}(X^+)^*/\mathbb{R}_+^*$ has the structure of a (compact) cell complex induced by the homeomorphism with $\mathcal{D}(D_{X^+})$.

5.3.6. The skeleton of a product

(5.3.6.1) Let k be a trivially-valued and algebraically closed field. Let $X^+ = (X, D_{X^+})$ and $Y^+ = (Y, D_{Y^+})$ be log-regular pairs over k. We denote by $Z^+ = (Z, D_{Z^+})$ the product in the category of fine and saturated log schemes. In particular, Z^+ is log-regular and $D_{Z^+} = D_{X^+} \times Y + X \times D_{Y^+}$. The goal of this section is to compare the skeleton associated to Z^+ with the product of skeletons of X^+ and Y^+ in the category of topological spaces.

Lemma 5.3.6.2. *The projection maps* $(\mathrm{pr}_X, \mathrm{pr}_Y)\colon Z \to X \times_k Y$ *induces an isomorphism* $F_{Z^+} \xrightarrow{\simeq} F_{X^+} \times F_{Y^+}$.

Proof. As any stratum of D_{Z^+} is of the form $D_x \times D_y$ for some $x \in F_{X^+}$ and $y \in F_{Y^+}$, we have a bijective correspondence between F_{Z^+} and $F_{X^+} \times F_{Y^+}$ that is compatible with the projections to the factors. Moreover, this bijection is actually an isomorphism of Kato fans, observing that

$$\mathcal{C}_{Z^+,z} \simeq \mathcal{C}_{X^+,x} \oplus \mathcal{C}_{Y^+,y}$$

when the Kato point $z \in F_{Z^+}$ maps to $(x, y) \in F_{X^+} \times F_{Y^+}$. □

(5.3.6.3) The projections $\mathrm{pr}_X : Z^+ \to X^+$ and $\mathrm{pr}_Y : Z^+ \to Y^+$ are dominant morphisms of log-regular log schemes, hence they induce a continuous map of skeletons

$$(\mathrm{pr}_{\mathrm{Sk}(X^+)}, \mathrm{pr}_{\mathrm{Sk}(Y^+)}) : \mathrm{Sk}(Z^+) \to \mathrm{Sk}(X^+) \times \mathrm{Sk}(Y^+)$$

that is constructed as in 5.3.4; that is, $(\mathrm{pr}_{\mathrm{Sk}(X^+)}, \mathrm{pr}_{\mathrm{Sk}(Y^+)})$ is the composition

$$\mathrm{Sk}(Z^+) \hookrightarrow Z^{\beth} \cap Z^{\mathrm{bir}} \xrightarrow{(\mathrm{pr}^{\beth}_{X^+}, \mathrm{pr}^{\beth}_{Y^+})} (X^{\beth} \cap X^{\mathrm{bir}}) \times (Y^{\beth} \cap Y^{\mathrm{bir}}) \xrightarrow{(\rho_{X^+}, \rho_{Y^+})} \mathrm{Sk}(X^+) \times \mathrm{Sk}(Y^+).$$

It follows that there is a commutative diagram

$$\begin{array}{ccc} Z^{\mathrm{bir}} \cap Z^{\beth} & \xrightarrow{(\mathrm{pr}^{\beth}_{X^+}, \mathrm{pr}^{\beth}_{Y^+})} & (X^{\mathrm{bir}} \cap X^{\beth}) \times (Y^{\mathrm{bir}} \cap Y^{\beth}) \\ \downarrow{\scriptstyle \rho_{Z^+}} & & \downarrow{\scriptstyle (\rho_{X^+}, \rho_{Y^+})} \\ \mathrm{Sk}(Z^+) & \xrightarrow{(\mathrm{pr}_{\mathrm{Sk}(X^+)}, \mathrm{pr}_{\mathrm{Sk}(Y^+)})} & \mathrm{Sk}(X^+) \times \mathrm{Sk}(Y^+). \end{array} \qquad (5.3.6.4)$$

In the following lemma, we show that the map $\mathrm{pr}_{\mathrm{Sk}(X^+)}\colon \mathrm{Sk}(Z^+) \to \mathrm{Sk}(X^+)$ is in fact induced by the restriction of morphisms of monoids.

Lemma 5.3.6.5. *Let $z = (x, y) \in F_Z^+$ be a Kato point and $\varepsilon \in \mathrm{Hom}(\mathcal{C}_{Z^+,z}, \mathbb{R}_+)$. If $i_{x,z}\colon \mathcal{C}_{X^+,x} \hookrightarrow \mathcal{C}_{Z^+,z}$ and $i_{y,z}\colon \mathcal{C}_{Y^+,y} \hookrightarrow \mathcal{C}_{Z^+,z}$ denote the inclusions of characteristic sheaves, then*

$$\mathrm{pr}_{\mathrm{Sk}(X^+)}(v_\varepsilon) = v_{\varepsilon\circ i_{x,z}} \textit{ and } \mathrm{pr}_{\mathrm{Sk}(Y^+)}(v_\varepsilon) = v_{\varepsilon\circ i_{y,z}}$$

Proof. It suffices to show the first equality. By definition (cf. 5.3.6.3), we have that

$$\mathrm{pr}_{\mathrm{Sk}(X^+)}(v_\varepsilon) = \rho_{X^+}(\mathrm{pr}^{\beth}_{X^+}(v_\varepsilon)).$$

Since $\mathrm{pr}_{\mathrm{Sk}(X^+)}(v_\varepsilon)$ is a point of $\mathrm{Sk}_x(X^+)$, there exists $\alpha \in \mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbb{R}_+)$ such that

$$\rho_{X^+}(\mathrm{pr}^{\beth}_{X^+}(v_\varepsilon)) = v_\alpha.$$

Hence, it suffices to show $\alpha = \varepsilon \circ i_{x,z}$. By Proposition 5.3.1.2, for any $m \in \mathcal{M}_{X,x}$ we have

$$\alpha(\overline{m}) = \mathrm{pr}^{\beth}_{X^+}(v_\varepsilon)(m)$$

and, since pr_X induces the inclusion of fraction fields $i : k(X) \hookrightarrow k(Z)$, we obtain that

$$\mathrm{pr}^{\beth}_{X^+}(v_\varepsilon)(m) = (v_\varepsilon \circ i)(m) = v_\varepsilon(m) = \varepsilon(\overline{m}).$$

On the other hand, for any $m \in \mathcal{M}_{X,x}$ we also have

$$v_{\varepsilon\circ i_{x,z}}(m) = (\varepsilon \circ i_{x,z})(\overline{m}) = \varepsilon(\overline{m}),$$

which concludes the proof. □

Similar to [BM19, Proposition 3.4.3], we prove that log-regular skeletons are well-behaved under products.

Proposition 5.3.6.6. *The skeleton $\mathrm{Sk}(Z^+)$ is homeomorphic to the product $\mathrm{Sk}(X^+) \times \mathrm{Sk}(Y^+)$ of the skeletons via the map $(\mathrm{pr}_{\mathrm{Sk}(X^+)}, \mathrm{pr}_{\mathrm{Sk}(Y^+)})$.*

Proof. It suffices to show that $(\mathrm{pr}_{\mathrm{Sk}(X^+)}, \mathrm{pr}_{\mathrm{Sk}(Y^+)})$ restricts to a homeomorphism $\mathrm{Sk}_z(Z^+) \simeq \mathrm{Sk}_x(X^+) \times \mathrm{Sk}_y(Y^+)$ for each $z = (x, y) \in F_{Z^+}$. By Lemma 5.3.6.5, this is equivalent to showing that the map

$$\mathrm{Hom}(\mathcal{C}_{Z^+,z}, \mathbb{R}_+) \to \mathrm{Hom}(\mathcal{C}_{X^+,x}, \mathbb{R}_+) \times \mathrm{Hom}(\mathcal{C}_{Y^+,y}, \mathbb{R}_+),$$

given by $\varepsilon \mapsto (\varepsilon \circ i_{x,z}, \varepsilon \circ i_{y,z})$, is a homeomorphism. It is clearly continuous and, if $q_{z,x}\colon \mathcal{C}_{Z^+,z} \to \mathcal{C}_{X^+,x}$ and $q_{z,y}\colon \mathcal{C}_{Z^+,z} \to \mathcal{C}_{Y^+,y}$ denote the projections, then $(\varepsilon_1, \varepsilon_2) \mapsto \varepsilon_1 \circ q_{z,x} + \varepsilon_2 \circ q_{z,y}$ is a continuous inverse. □

Proposition 5.3.6.7. *The link* $\mathrm{Sk}(Z^+)^*/\mathbb{R}_+^*$ *is homeomorphic to the join* $(\mathrm{Sk}(X^+)^*/\mathbb{R}_+^*)*(\mathrm{Sk}(Y^+)^*/\mathbb{R}_+^*)$.

Proof. Observe that the proof of Proposition 5.3.6.6 yields a $\mathbb{R}_+^*$-equivariant homeomorphism

$$\mathrm{Sk}(Z^+) \simeq \mathrm{Sk}(X^+) * \mathrm{Sk}(Y^+),$$

where the product is endowed with the diagonal action. By 2.2.0.4, there exists a $\mathbb{R}_+^*$-equivariant homeomorphism

$$\mathrm{Sk}(Z^+) \simeq \mathrm{Cone}((\mathrm{Sk}(X^+)^*/\mathbb{R}_+^*) * (\mathrm{Sk}(Y^+)^*/\mathbb{R}_+^*)).$$

The statement now follows from 2.2.0.2. □

5.3.7. The Kontsevich–Soibelman skeleton and the essential skeleton

(5.3.7.1) Assume now that k has characteristic zero and D is a Weil $\mathbf{Q}$-divisor on X such that $K_X + D_{\mathrm{red}}$ is $\mathbb{Q}$-Cartier. Following the approach of Kontsevich and Soibelman, for any rational D-logarithmic pluricanonical form η on X, we can construct a subset $\mathrm{Sk}(X, D, \eta)$ of $X^{\beth}$ as the set of birational points satisfying a minimality condition with respect to η. More precisely, we define

$$\mathrm{wt}_\eta(X, D) \coloneqq \inf\{\mathrm{wt}_\eta(x) \colon x \in X^{\beth}\} \in \overline{\mathbb{R}}.$$

Definition 5.3.7.2. The *Kontsevich–Soibelman skeleton* of the triple (X, D, η) is

$$\mathrm{Sk}(X, D, \eta) = \{x \in X^{\mathrm{bir}} \cap X^{\beth} \colon \mathrm{wt}_\eta(x) = \mathrm{wt}_\eta(X, D)\}.$$

In fact, as in [MN15, Theorem 4.7.5], $\mathrm{Sk}(X, D, \eta)$ is the closure in $X^{\mathrm{bir}} \cap X^{\beth}$ of the points $x \in X^{\mathrm{div}} \cup \{v_0\}$ such that $\mathrm{wt}_\eta(x) = \mathrm{wt}_\eta(X, D)$.

(5.3.7.3) Assume in addition that $X^+ = (X, D_{\mathrm{red}})$ is log-regular, hence log canonical. In this case, the function $A_{(X,D_{\mathrm{red}})}$ is non-negative on $X^{\beth}$, and it has value exactly 0 at any divisorial point in $\mathrm{Sk}(X^+)$, thus on $\mathrm{Sk}(X^+)$. In fact, the only $x \in X^{\mathrm{bir}} \cap X^{\beth}$ with $A_{(X,D_{\mathrm{red}})}(x) = 0$ are those in the skeleton by [Blu18, Proposition 3.2.5].

Proposition 5.3.7.4. *Suppose* X^+ *is as in 5.3.7.3. If* η *is a non-zero regular* D*-logarithmic pluricanonical form on* X *and* $x \in X^{\beth}$*, then*

$$\mathrm{wt}_\eta(x) \geqslant \mathrm{wt}_\eta(\rho_{X^+}(x)),$$

and if $x \in X^{\mathrm{bir}} \cap X^{\beth}$*, then equality holds if and only if* $x \in \mathrm{Sk}(X^+)$.

Proof. By maximal lower-semicontinuity of the weight function, it suffices to show the inequality on $X^{\mathrm{bir}} \cap X^{\beth}$ (or even on $X^{\mathrm{div}} \cap X^{\beth}$). Let $x \in X^{\mathrm{bir}} \cap X^{\beth}$. Denote by ξ and ξ' the centres of x and $\rho_{X^+}(x)$, respectively. By construction of the retraction ρ_{X^+}, we have that $\xi \in \overline{\{\xi'\}}$, and hence there exists a trivializing open $U \subseteq X$ for the logarithmic pluricanonical bundle $\omega_{(X,D_{\mathrm{red}})}^{\otimes m}$ that contains both ξ and ξ'. On such an open set U, the form $\eta|_U$ corresponds to a regular function f on U, and the weight functions can be computed as

$$\mathrm{wt}_\eta(x) = A_{(X,D_{\mathrm{red}})}(x) + v_x(f) \quad \text{and} \quad \mathrm{wt}_\eta(\rho_{X^+}(x)) = A_{(X,D_{\mathrm{red}})}(\rho_{X^+}(x)) + v_{\rho_{X^+}(x)}(f).$$

Locally at ξ', f has an admissible expansion of the form $f = \sum_{\gamma \in \mathcal{C}_{X^+,\xi'}} c_\gamma \gamma$. The ultrametric inequality gives

$$v_x(f) \geqslant \min_\gamma \{v_0(c_\gamma) + v_x(\gamma)\} = v_{\rho_{X^+}(x)}(f), \tag{5.3.7.5}$$

and $A_{(X,D_{\mathrm{red}})}(x) \geqslant 0 = A_{(X,D_{\mathrm{red}})}(\rho_{X^+}(x))$ by 5.3.7.3; adding this to Eq. (5.3.7.5), we get that $\mathrm{wt}_\eta(x) \geqslant \mathrm{wt}_\eta(\rho_{X^+}(x))$.

Assume, in addition, that the equality $A_{(X,D_{\mathrm{red}})}(x) + v_x(f) = v_{\rho_{X^+}(x)}(f)$ holds. As $v_x(f) \geqslant v_{\rho_{X^+}(x)}(f)$ and $A_{(X,D_{\mathrm{red}})}(v_x) \geqslant 0$, the assumption implies that $A_{(X,D_{\mathrm{red}})}(x) = 0$. Now, it follows from 5.3.7.3 that x lies in the skeleton $\mathrm{Sk}(X^+)$. □

Definition 5.3.7.6. The *essential skeleton* $\mathrm{Sk}^{\mathrm{ess}}(X,D)$ of (X,D) is the union of all Kontsevich–Soibelman skeletons $\mathrm{Sk}(X,D,\eta)$, where η runs over all non-zero regular D-logarithmic pluricanonical forms on X. In symbols,

$$\mathrm{Sk}^{\mathrm{ess}}(X,D) \coloneqq \bigcup_\eta \mathrm{Sk}(X,D,\eta).$$

(5.3.7.7) For any regular D-logarithmic pluricanonical form η, the function $\phi_{\mathrm{triv},\omega_{(X,D_{\mathrm{red}})}^{\otimes m}}(\eta,\cdot)$ is non-negative, and hence wt_η is as well. Further, if η is non-zero, $\mathrm{wt}_\eta(v_0) = 0$, where v_0 is the trivial valuation. It follows that $\mathrm{wt}_\eta(X,D) = 0$ and $v_0 \in \mathrm{Sk}(X,D,\eta)$ for every such form η. In particular, the essential skeleton of (X,D) is nonempty whenever there exists a non-zero regular D-logarithmic form on X.

(5.3.7.8) By arguing as in [MN15, Proposition 4.5.5(v)], one can show that $\mathrm{Sk}(X,D,\eta^{\otimes m}) = \mathrm{Sk}(X,D,\eta)$ for any $m \in \mathbb{Z}_{>0}$. In particular, $\mathrm{Sk}^{\mathrm{ess}}(X,D)$ can be computed as the union of Kontsevich–Soibelman skeletons of sections of $m(K_X + D)$ with $m \in \mathbb{Z}_{>0}$ sufficiently divisible.

Remark 5.3.7.9. There are two fundamental reasons why the essential skeleton is defined in terms of non-zero regular D-logarithmic pluricanonical forms. They are the following:

- If $\xi \in X$ and δ is a generating section of $\omega_{(X,D_{\mathrm{red}}),\xi}^{\otimes m}$, then any regular section η of $\omega_{(X,D)}^{\otimes m}$ can be written, locally at ξ, as $\eta = f\delta$ for some $f \in \mathcal{O}_{X,\xi}$. For any $x \in X^{\beth}$ such that f is regular at $c_X(x)$ and $c_X(\rho_{X^+}(x))$, we have $v_x(f) \geqslant v_{\rho_{X^+}(x)}(f)$, as in Proposition 5.3.7.4. In particular, the minimality locus of wt_η on $X^{\mathrm{bir}} \cap X^{\beth}$ (and hence the essential skeleton) lies in the log-regular skeleton $\mathrm{Sk}(X^+)$.

- The definition of the essential skeleton is in terms of D-logarithmic pluricanonical forms, as opposed to logarithmic pluricanonical forms. This is done so that the faces of $\mathrm{Sk}(X^+)$ corresponding to components of D with coefficient strictly less than 1 do not lie in the essential skeleton. This choice is compatible with the correspondence between the dual complex of a dlt boundary divisor and the essential skeleton in the discretely-valued setting, as explored in [NX16, Theorem 3.3.3] and [BM19, Proposition 5.1.7].

Furthermore, when (X, D) is a logCY pair, we will show in Proposition 7.2.4.2 that the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, D)$ in fact coincides with the skeleton $\mathrm{Sk}(X, D^{=1})$. This plays a crucial role in the proof of Theorem 1.4.3.1.

5.3.8. Compatibility between trivially-valued and discretely-valued setting

(5.3.8.1) This section explores a relationship between the weight functions in the trivially-valued and in the discretely-valued cases. To this end, we work in a setting where both the weight functions are defined and interact, namely on the total space of a degeneration. Proposition 5.3.8.9 shows that we can regard an essential skeleton, defined in the trivially-valued setting, as a cone over the essential skeleton in the discretely-valued setting.

(5.3.8.2) Let k be a trivially-valued field of characteristic zero. Let $\mathscr{X}$ be a degeneration over $k[[\varpi]]$, i.e. a normal, flat, separated scheme of finite type over $k[[\varpi]]$. The formal completion $\widehat{\mathscr{X}}$ of $\mathscr{X}$ along the special fibre $\mathscr{X}_0$ is a formal $k[[\varpi]]$-scheme of finite type, and the structure morphism $\widehat{\mathscr{X}} \to \mathrm{Spf}(k[[\varpi]])$ is a morphism of special formal k-schemes in the sense of [Ber96, §1]. This induces a morphism $\mathscr{X}^{\mathrm{triv}} \to D^1_k(0,1)$ on the analytic generic fibres, where $D^1_k(0,1)$ denotes the open unit disc over k. We can identify $D^1_k(0,1)$ with the interval $[0,1)$ by sending $r \in [0,1)$ to the ϖ-adic seminorm $|\cdot|_r$ on $k[[\varpi]]$ normalized so that $|\varpi|_r = r$. Under this identification, the fibre of $\mathscr{X}^{\mathrm{triv}} \to D^1_k(0,1)$ above $1/e$ is the generic fibre of $\mathscr{X}$, denoted $\mathscr{X}^{\mathrm{disc}}$, as an analytic space over the field $K := (k((\varpi)), |\cdot|_{1/e})$ (in the sense of Definition 2.1.2.2); see [Nic11, Lemma 4.2] for details.

Definition 5.3.8.3. We say that $\mathscr{X}$ is *defined over a curve* if there exists a germ of a smooth curve C over k, a closed point $0 \in C(k)$, an isomorphism $\widehat{\mathcal{O}}_{C,0} \simeq k[[\varpi]]$ (which we write as an equality from now on), and a normal, flat, separated scheme X over C such that

$$\mathscr{X} = X \times_C \mathrm{Spec}(\widehat{\mathcal{O}}_{C,0}).$$

For the rest of the section, fix a morphism $X \to C$ and $0 \in C(k)$ as in Definition 5.3.8.3. There is a cartesian square of analytic spaces over k given by

$$\begin{array}{ccccc} \mathscr{X}^{\mathrm{disc}} & \hookrightarrow & \mathscr{X}^{\mathrm{triv}} & \hookrightarrow & X^{\beth} \\ \downarrow & & \downarrow & & \downarrow \\ \{1/e\} \simeq \mathcal{M}(K) & \hookrightarrow & [0,1) \simeq D^1_k(0,1) & \hookrightarrow & C^{\beth}. \end{array}$$

(5.3.8.4) Let $X_0 \subseteq X$ denote the fibre above 0. Suppose that X_0 is reduced, $\mathscr{X}_{k((\varpi))}$ is smooth, and $K_X + X_0$ is $\mathbb{Q}$-Cartier. For any regular section η of $\omega_{(X,X_0)}^{\otimes m}$, write η_K for the *Gelfand–Leray form* associated to η: this is the regular section of $\omega_{\mathscr{X}_K}^{\otimes m}$ characterized by the property that $\eta_K \wedge d\varpi$ coincides with the pullback of η along $\mathscr{X} \to X$, or equivalently it is the contraction of η with the vector field $\partial/\partial\varpi$. See [NS07, Definition 9.5] for more details. We can define weight functions on $\mathscr{X}^{\mathrm{triv}}$ and $\mathscr{X}^{\mathrm{disc}}$ as follows:

- the weight function $\mathrm{wt}_{\eta_K}^{\mathrm{disc}} \colon \mathscr{X}^{\mathrm{disc}} \to \overline{\mathbb{R}}$ is defined as in Theorem 5.2.3.4, where we consider η_K as a regular section of $\omega_{(\mathscr{X},\mathscr{X}_0)}^{\otimes m}$;
- the weight function $\mathrm{wt}_{\eta}^{\mathrm{triv}} \colon \mathscr{X}^{\mathrm{triv}} \to \overline{\mathbb{R}}$ is the restriction of the weight function $\mathrm{wt}_\eta \colon X^{\beth} \to \overline{\mathbb{R}}$ defined as in Definition 5.2.4.6.

Note that the reason we assume that $\mathscr{X}$ is defined over a curve is that our definition of $\mathrm{wt}_\eta^{\mathrm{triv}}$ only holds on the $\beth$-analytification of a k-variety, but not on a general k-analytic space.

Proposition 5.3.8.5. *Let $m \in \mathbb{Z}_{>0}$ be such that $m(K_X + X_0)$ is Cartier. For $\eta \in H^0(X, m(K_X + X_0))$ and $x \in \mathscr{X}^{\mathrm{disc}}$, we have*

$$\mathrm{wt}_{\eta_K}^{\mathrm{disc}}(x) = \mathrm{wt}_{\eta}^{\mathrm{triv}}(x). \tag{5.3.8.6}$$

If in addition $X \to C$ is proper (and hence $\mathscr{X}^{\mathrm{disc}} = \mathscr{X}_K^{\mathrm{an}}$), then there is an inclusion of Kontsevich–Soibelman skeletons

$$\mathrm{Sk}(\mathscr{X}_K, \eta_K) \supseteq \mathrm{Sk}(X, X_0, \eta) \cap \mathscr{X}^{\mathrm{disc}}, \tag{5.3.8.7}$$

which is an equality provided that (X, X_0) is log canonical and that there is a component of X_0 along which η does not vanish identically.

Proof. We prove Eq. (5.3.8.6) in two steps.

Step 1. Assume that $x \in \mathscr{X}^{\mathrm{disc}} \cap X^{\mathrm{div}}$ and is determined by a prime divisor on a proper birational model $h\colon Y \to X$ of X, where h is an isomorphism away from X_0. Let $\mathscr{Y} := Y \times_X \mathscr{X}$; it is equipped with a proper birational morphism $\mathscr{Y} \to \mathscr{X}$, also denoted by h, that is an isomorphism outside of $\mathscr{X}_0$. In particular, $\mathscr{Y}$ is a model of $\mathscr{X}_K$.

Set $\xi = \mathrm{red}_{\mathscr{X}}(x)$ and take a $\mathcal{O}_{\mathscr{X},\xi}$-module generator δ of $\omega_{(\mathscr{X},\mathscr{X}_0),\xi}^{\otimes m}$. Locally at ξ, write the section η_K as $\eta_K = f\delta$ for some $f \in \mathcal{O}_{\mathscr{X},\xi}$. Consider the identity

$$\begin{aligned} m(K_{\mathscr{Y}/k[[\varpi]]} + \mathscr{Y}_{0,\mathrm{red}} - \mathrm{div}_{\mathscr{Y}}(\varpi)) - \Big(\sum_i ma(E_i)E_i + \mathrm{div}_{\mathscr{Y}}(h^*f) - m\mathrm{div}_{\mathscr{Y}}(\varpi)\Big) \\ = h^*(m(K_{\mathscr{X}/k[[\varpi]]} + \mathscr{X}_0) - \mathrm{div}_{\mathscr{X}}(f)), \end{aligned} \tag{5.3.8.8}$$

where E_i are the exceptional prime divisors of h, and $a(\cdot)$ is the log discrepancy function with respect to $(\mathscr{X}, \mathscr{X}_0)$. Note that $a(E_i) = A_{(X,X_0)}(\mathrm{ord}_{E_i})$.

We have the following series of equalities

$$\begin{aligned}
\mathrm{wt}^{\mathrm{disc}}_{\eta_K}(x) &= v_x(\mathrm{div}_{(\mathscr{Y},\mathscr{Y}_{0,\mathrm{red}}-\mathrm{div}_{\mathscr{Y}}(\varpi))}(\eta_K)) + m && \text{cf. Eq. (5.2.3.3)}\\
&= m\, a(x) + v_x(h^*f) - m v_x(\varpi) + m && \text{cf. Eq. (5.3.8.8)}\\
&= m A_{(X,X_0)}(x) + v_x(f) && \text{as } v_x(\varpi)=1\\
&= m A_{(X,X_0-\mathrm{div}_{(X,X_0)}(\eta))}(x) && \text{cf. [Kol13, Lemma 2.5]}\\
&= \mathrm{wt}^{\mathrm{triv}}_{\eta}(x) && \text{cf. Proposition 5.2.5.2.}
\end{aligned}$$

Thus, Eq. (5.3.8.6) holds for any $x \in \mathscr{X}^{\mathrm{disc}} \cap X^{\mathrm{div}}$. Note that $(\mathscr{X}_K)^{\mathrm{div}} = \mathscr{X}^{\mathrm{disc}} \cap X^{\mathrm{div}}$, since the blow-up of a formal ideal on $\mathscr{X}$ that is cosupported on $\mathscr{X}_0$ can be realized as the completion of an algebraic blow-up of X.

Step 2. To prove the equality Eq. (5.3.8.6) on all of $\mathscr{X}^{\mathrm{disc}}$, it suffices to check that both $\mathrm{wt}^{\mathrm{disc}}_{\eta_K}$ and $\mathrm{wt}^{\mathrm{triv}}_{\eta}$ are maximal lower-semicontinuous extensions on $\mathscr{X}^{\mathrm{disc}}$ of

$$\mathrm{wt}^{\mathrm{disc}}_{\eta_K}|_{(\mathscr{X}_K)^{\mathrm{div}}} = \mathrm{wt}^{\mathrm{triv}}_{\eta}|_{\mathscr{X}^{\mathrm{disc}} \cap X^{\mathrm{div}}}.$$

This follows immediately for $\mathrm{wt}^{\mathrm{disc}}_{\eta_K}$ from Theorem 5.2.3.4.

By Definition 5.2.4.6 and since the inclusion $\mathscr{X}^{\mathrm{triv}} \hookrightarrow X^{\beth}$ is an open immersion, the weight function $\mathrm{wt}^{\mathrm{triv}}_{\eta}$ is the maximal lower-semicontinuous extension of $\mathrm{wt}^{\mathrm{triv}}_{\eta}|_{\mathscr{X}^{\mathrm{triv}} \cap X^{\mathrm{div}}}$. By construction, $\mathrm{wt}^{\mathrm{triv}}_{\eta}$ is $\mathbb{R}^*_+$-homogeneous, i.e. $\mathrm{wt}^{\mathrm{triv}}_{\eta}(a \cdot x) = a \cdot \mathrm{wt}^{\mathrm{triv}}_{\eta}(x)$ for $a \in \mathbb{R}^*_+$. By homogeneity, the restriction of $\mathrm{wt}^{\mathrm{triv}}_{\eta}$ to $\mathscr{X}^{\mathrm{disc}}$ is the maximal lower-semicontinuous extension of $\mathrm{wt}^{\mathrm{triv}}_{\eta}|_{\mathscr{X}^{\mathrm{disc}} \cap X^{\mathrm{div}}}$.

This completes the proof of Eq. (5.3.8.6).

The inclusion Eq. (5.3.8.7) can be deduced from Eq. (5.3.8.6) as follows: it implies that $\mathrm{wt}^{\mathrm{triv}}_{\eta}(X, X_0) \leqslant \mathrm{wt}^{\mathrm{disc}}_{\eta_K}(\mathscr{X}_K)$. By Definition 5.3.7.2, $\mathrm{Sk}(X, X_0, \eta) \cap \mathscr{X}^{\mathrm{disc}}$ consists of those $x \in \mathscr{X}^{\mathrm{bir}}_K$ such that $\mathrm{wt}^{\mathrm{triv}}_{\eta}(x) = \mathrm{wt}^{\mathrm{triv}}_{\eta}(X, X_0)$. Thus, for such an x, we have

$$\mathrm{wt}^{\mathrm{disc}}_{\eta_K}(x) = \mathrm{wt}^{\mathrm{triv}}_{\eta}(x) = \mathrm{wt}^{\mathrm{triv}}_{\eta}(X, X_0) \leqslant \mathrm{wt}^{\mathrm{disc}}_{\eta_K}(\mathscr{X}_K) \leqslant \mathrm{wt}^{\mathrm{disc}}_{\eta_K}(x),$$

and hence these are equalities. It follows that $x \in \mathrm{Sk}(\mathscr{X}_K, \eta_K)$ by [MN15, Theorem 4.7.5].

We show equality in Eq. (5.3.8.7) under the additional hypotheses that (X, X_0) is log canonical and there is a component $E \subsetneq X_0$ such that $\mathrm{ord}_E(\mathrm{div}_{(X,X_0)}(\eta))$ is zero. The former assumption guarantees that $\mathrm{wt}^{\mathrm{triv}}_{\eta}(X, X_0) = 0$ by 5.3.7.7, and the latter implies that $\mathrm{wt}^{\mathrm{triv}}_{\eta}(\mathrm{ord}_E) = mA_{(X,X_0)}(\mathrm{ord}_E) = 0$. After rescaling ord_E, we find that there is a point $x \in \mathscr{X}^{\mathrm{disc}}$ such that $\mathrm{wt}^{\mathrm{disc}}_{\eta_K}(x) = 0$; in particular,

$$0 = \mathrm{wt}^{\mathrm{triv}}_{\eta}(X, X_0) \leqslant \mathrm{wt}^{\mathrm{disc}}_{\eta_K}(\mathscr{X}_K) \leqslant \mathrm{wt}^{\mathrm{disc}}_{\eta_K}(x) = 0.$$

Thus, both sides of the inclusion Eq. (5.3.8.7) consist of those $x \in (\mathscr{X}^{\mathrm{disc}})^{\mathrm{bir}}$ such that $\mathrm{wt}^{\mathrm{disc}}_{\eta_K}(x) = 0$, hence they coincide. □

Proposition 5.3.8.9. *If $X \to C$ is projective, then there is an inclusion of essential skeletons*

$$\mathrm{Sk}^{\mathrm{ess}}(\mathscr{X}_K) \supseteq \mathrm{Sk}^{\mathrm{ess}}(X, X_0) \cap \mathscr{X}^{\mathrm{disc}}, \tag{5.3.8.10}$$

which is an equality when (X, X_0) is log canonical and $K_X + X_0$ is semiample.

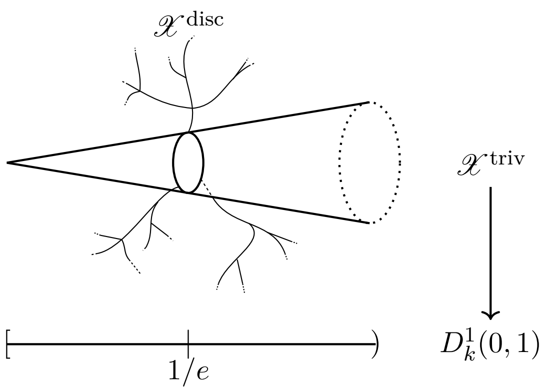


Figure 5.3.8.1: Consider the degeneration $\mathscr{X} := \{xyz + \varpi(x^3 + y^3 + z^3) = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}[[\varpi]]}$, where $\mathbb{P}^2_{\mathbb{C}[[\varpi]]}$ has homogeneous coordinates $[x\colon y\colon z]$. The equality of Eq. (5.3.8.10) can be illustrated for $\mathscr{X}$ as above: the cone is $\mathrm{Sk}^{\mathrm{ess}}(X, X_0)$, and its intersection with the fibre $\mathscr{X}^{\mathrm{disc}}$ is a circle (that is, the essential skeleton of the Tate elliptic curve $\mathscr{X}^{\mathrm{an}}_K$).

Proof. The inclusion Eq. (5.3.8.10) is immediate from Eq. (5.3.8.7) and 5.3.7.8. For the equality, assume now that (X, X_0) is log canonical and $K_X + X_0$ is semiample. Pick $m \in \mathbb{Z}_{>0}$ such that $m(K_X + X_0)$ is Cartier and globally generated, and pick global generators $\eta_1, \dots, \eta_N \in H^0(X, m(K_X + X_0))$ that do not vanish along all of X_0. As (X, X_0) is log canonical, [Kol13, Corollary 1.36] shows that there is a dlt pair $(X^{\mathrm{dlt}}, X_0^{\mathrm{dlt}})$, equipped with a crepant birational morphism $(X^{\mathrm{dlt}}, X_0^{\mathrm{dlt}}) \to (X, X_0)$ that is an isomorphism on the snc-locus of (X, X_0). In particular,

$$\mathscr{X}^{\mathrm{dlt}} := X^{\mathrm{dlt}} \times_C \mathrm{Spec}(\widehat{\mathcal{O}}_{C,0})$$

is a good minimal dlt model of $\mathscr{X}_K$ that dominates the model $\mathscr{X}$; this is a technical condition needed to apply the results of [NX16], and it is discussed further in Definition 7.2.6.6.

Write δ_i for the pullback of η_i to X^{dlt}, and $\delta_{i,K}$ for the restriction to $\mathscr{X}^{\mathrm{dlt}}_K = \mathscr{X}_K$. As $(X^{\mathrm{dlt}}, X_0^{\mathrm{dlt}}) \to (X, X_0)$ is crepant, the sections $\delta_1, \dots, \delta_N$ of $H^0(X^{\mathrm{dlt}}, m(K_{X^{\mathrm{dlt}}} + X_0^{\mathrm{dlt}}))$ are global generators of $m(K_{X^{\mathrm{dlt}}} + X_0^{\mathrm{dlt}})$. Then

$$\mathrm{Sk}^{\mathrm{ess}}(\mathscr{X}_K) = \bigcup_{i=1}^{N} \mathrm{Sk}(\mathscr{X}^{\mathrm{dlt}}_K, \delta_{i,K}) = \bigcup_{i=1}^{N} \mathrm{Sk}(\mathscr{X}_K, \eta_{i,K}), \tag{5.3.8.11}$$

where the first equality follows from [NX16, Theorem 3.3.3], and the second equality follows from [MN15, Proposition 4.7.2]. Observe that

$$\begin{aligned}
\mathrm{Sk}^{\mathrm{ess}}(X, X_0) \cap \mathscr{X}^{\mathrm{disc}} &\subseteq \mathrm{Sk}^{\mathrm{ess}}(\mathscr{X}_K) && \text{cf. Eq. (5.3.8.10)}\\
&= \textstyle\bigcup_{i=1}^{N} \mathrm{Sk}(\mathscr{X}_K, \eta_{i,K}) && \text{cf. Eq. (5.3.8.11)}\\
&= \textstyle\bigcup_{i=1}^{N} \mathrm{Sk}(X, X_0, \eta_i) \cap \mathscr{X}^{\mathrm{disc}} \\
&\subseteq \mathrm{Sk}^{\mathrm{ess}}(X, X_0) \cap \mathscr{X}^{\mathrm{disc}},
\end{aligned}$$

where the final equality follows from the case of equality in Eq. (5.3.8.7). This completes the proof. □

Remark 5.3.8.12. If (X, X_0) is not log canonical, then the equality in Eq. (5.3.8.10) does not necessarily hold. For example, take a semistable degeneration $X \to C$ of an elliptic curve to a cusp with $K_X + X_0$ trivial, such as $X = \{zy^2 = x^3 + \varpi z^3\}$ in $\mathbb{P}^2_k \times \mathrm{Spec}\,(k[\varpi])$. The pair (X, X_0) is not log canonical e.g. by [Kol13, Theorem 2.31], and hence $\mathrm{Sk}^{\mathrm{ess}}(X, X_0)$ is empty. However, $\mathrm{Sk}^{\mathrm{ess}}(\mathscr{X}_K)$ is the skeleton of the minimal regular model of $\mathscr{X}_K$, which is non-empty.

5.4. Closure of the skeleton of a log-regular pair

(5.4.0.1) The skeleton of a log-regular model of X, introduced in Section 5.3 and in [BM19], is a polyhedral complex in X^{bir} with (possibly) unbounded faces. The closure of the skeleton in the Berkovich analytification X^{an} has itself a decomposition into skeletons associated to the strata of the log-regular structure of X. This decomposition is treated in detail in [Thu07, Proposition 3.17] in the trivially-valued setting, and the case of a toroidal embedding is mentioned in [ACP15, Example 2.4.2 and Proposition 2.6.2]. In this section, we review and extend their description for a log-regular log scheme, in order to prove analogous results for the closure of the Kontsevich–Soibelman skeletons when the residue characteristic is zero.

(5.4.0.2) Let $\mathcal{X}^+$ be a log-regular log scheme over a trivially-valued field k, or a log-regular log scheme over a discretely valuation ring R. In order to treat the two cases simultaneously, we adopt the following table of notations:

$\mathcal{K}$	trivially-valued field k	discretely-valued field K
$\mathcal{K}^\circ$	valuation ring k of v_k	valuation ring R of v_K with uniformizer π
$\mathcal{S}^+ = (\mathcal{S}, s)$	trivial log scheme $\mathrm{Spec}(k)$	divisorial log scheme $S^+ = (S = \mathrm{Spec}(R), (\pi))$
$X^+ = (X, D_{X^+})$	log-regular log scheme over k	log-regular log scheme over K
$\mathcal{X}^+ = (\mathcal{X}, D_{\mathcal{X}^+})$	X^+	log-regular model $\mathscr{X}^+ = (\mathscr{X}, D_{\mathscr{X}^+})$ of X^+ over S^+
$\mathcal{X}^{\mathrm{an}}$	$X^{\beth}$	$\widehat{\mathscr{X}_\eta}$.

Moreover, we denote by

$D_{\mathcal{X}^+} = \sum_{i=1}^{N} D_{\mathcal{X}^+,i}$	the sum of the irreducible components of $D_{\mathcal{X}^+}$
$I_x \subseteq \{1, \dots, N\}$	the (possibly empty) index set such that $\overline{\{x\}}$ is the irreducible component of $\cap_{i \in I_x} D_{\mathcal{X}^+,i}$ with generic point $x \in F_{\mathcal{X}^+}$,
$\mathcal{D}_x = \overline{\{x\}}$	for each $x \in F_{\mathcal{X}^+}$,
$D_x = (\mathcal{D}_x)_{\mathcal{K}}$	the generic fibre of $\mathcal{D}_x$ for each $x \in F_{X^+}$.

Furthermore, for $x \in F_{X^+}$, let $\mathcal{D}_x^{\mathrm{an}}$ denote $D_x^{\beth}$ if $\mathcal{K}$ is trivially-valued, or the analytic generic fibre of $\mathcal{D}_x$ if $\mathcal{K}$ is discretely-valued. In both cases, the closed immersion $\mathcal{D}_x \hookrightarrow \mathcal{X}$ induces a closed immersion $\mathcal{D}_x^{\mathrm{an}} \hookrightarrow \mathcal{X}^{\mathrm{an}}$ of analytifications.

5.4.1. The decomposition of the closure of the skeleton

(5.4.1.1) In order to decompose the closure $\overline{\mathrm{Sk}}(\mathcal{X}^+)$ into a disjoint union of skeletons associated to the strata $\mathcal{D}_y^+$ for any $y \in F_{\mathcal{X}^+}$, we endow the subscheme $\mathcal{D}_y^+$ with the log-regular structure prescribed by the following proposition.

Proposition 5.4.1.2. *Let $x, y \in F_{\mathcal{X}^+}$ be such that $x \in \overline{\{y\}}$ and consider the submonoid*

$$\mathcal{I}_y = \{f \in \mathcal{M}_{\mathcal{X}^+,x} \colon f(y) = 0\}$$

of $\mathcal{M}_{\mathcal{X}^+,x}$. Then the log structure associated to $\mathcal{M}_{\mathcal{X}^+,x} \setminus \mathcal{I}_y \to \mathcal{O}_{\mathcal{X},x}/\mathcal{I}_y\mathcal{O}_{\mathcal{X},x}$ on the scheme $\mathrm{Spec}(\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_y\mathcal{O}_{\mathcal{X},x})$ *is log-regular.*

Proof. This follows immediately from [Kat94, Proposition 7.2]. □

(5.4.1.3) For each $y \in F_{\mathcal{X}^+}$, the log-regular structure obtained in Proposition 5.4.1.2 is called the *trace* of $\mathcal{X}^+$ on $\mathcal{D}_y$. More geometrically, the trace log structure on $\mathcal{D}_y$ is $(\mathcal{D}_y, \sum_{j \notin I_y} D_{\mathcal{X}^+,j}|_{\mathcal{D}_y})$. In particular, the Kato fan of $\mathcal{D}_y^+$ is given by

$$F_{\mathcal{D}_y^+} = F_{\mathcal{X}^+} \cap \mathcal{D}_y = \{x \in F_{\mathcal{X}^+} \colon x \in \overline{\{y\}}\},$$

and the characteristic sheaf of $\mathcal{D}_y^+$ at $x \in F_{\mathcal{D}_y^+}$ is $\mathcal{C}_{\mathcal{D}_y^+,x} = \{\gamma \in \mathcal{C}_{\mathcal{X}^+,x} \colon \gamma(y) \neq 0\}$. Thus, for any $x \in F_{\mathcal{D}_y^+}$, there is an injective monoid morphism

$$\mathcal{C}_{\mathcal{D}_y^+,x} \hookrightarrow \mathcal{C}_{\mathcal{X}^+,x}.$$

For any $y \in F_{\mathcal{X}^+}$ such that $\mathcal{D}_y^+$ dominates the base log scheme $\mathcal{S}^+$, we can construct the skeleton $\mathrm{Sk}(\mathcal{D}_y^+)$. In the case where $\mathcal{K} = K$ discretely-valued field and the scheme $\mathcal{D}_y^+$ is vertical (i.e. $\mathcal{D}_y^+$ is supported on the closed fibre of $\mathcal{X}^+$), we set $\mathrm{Sk}(\mathcal{D}_y^+) = \emptyset$.

Lemma 5.4.1.4. *For any $x \in F_{\mathcal{X}^+}$, the closure $\overline{\mathrm{Sk}}_x(\mathcal{X}^+)$ of $\mathrm{Sk}_x(\mathcal{X}^+)$ in $\mathcal{X}^{\mathrm{an}}$ coincides with the subset*

$$\{v_\alpha \colon \alpha \in \mathrm{Hom}(\mathcal{C}_{\mathcal{X}^+,x}, \overline{\mathbb{R}}_+) \text{ and if } \mathcal{K} = K,\ \alpha(\pi) = 1\} \tag{5.4.1.5}$$

of $\mathcal{X}^{\mathrm{an}}$. In particular, $\overline{\mathrm{Sk}}(\mathcal{X}^+) = \bigcup_{x \in F_{\mathcal{X}^+}} \{v_\alpha \colon \alpha \in \mathrm{Hom}(\mathcal{C}_{\mathcal{X}^+,x}, \overline{\mathbb{R}}_+)$ and if $\mathcal{K} = K,\ \alpha(\pi) = 1\}$.

Proof. In the trivially-valued case, the claim coincides with Lemma 5.3.1.6. Similarly, in the discretely-valued case, denote by Z_x the subset of $\mathcal{X}^{\mathrm{an}}$ as in Eq. (5.4.1.5). It is clear that $\mathrm{Sk}_x(\mathcal{X}^+) \subseteq Z_x$ and that Z_x is closed. Due to [BM19, Proposition 3.2.15], it follows that $Z_x \subseteq \overline{\mathrm{Sk}}_x(\mathcal{X}^+)$, hence we conclude that Z_x is the closure of $\mathrm{Sk}_x(\mathcal{X}^+)$. □

Proposition 5.4.1.6. *For any $x \in F_{\mathcal{X}^+}$, the closure $\overline{\mathrm{Sk}}_x(\mathcal{X}^+)$ of $\mathrm{Sk}_x(\mathcal{X}^+)$ in $\mathcal{X}^{\mathrm{an}}$ coincides with the disjoint union*

$$\overline{\mathrm{Sk}}_x(\mathcal{X}^+) = \coprod_{y \in F_{\mathcal{X}^+} \colon x \in \overline{\{y\}}} \mathrm{Sk}_x(\mathcal{D}_y^+).$$

Proof. A valuation in the closure of $\mathrm{Sk}_x(\mathcal{X}^+)$ is of the form v_α for a morphism $\alpha \in \mathrm{Hom}(\mathcal{C}_{\mathcal{X}^+,x}, \overline{\mathbb{R}}_+)$ by Lemma 5.4.1.4. If $\mathrm{Im}(\alpha) \subseteq \mathbb{R}_+$, then $v_\alpha \in \mathrm{Sk}_x(\mathcal{X}^+)$; otherwise, the subset

$$\mathcal{I}_\alpha = \{f \in \mathcal{O}_{\mathcal{X},x} \colon v_\alpha(f) = +\infty\}.$$

is non-empty, and it forms an ideal of $\mathcal{O}_{\mathcal{X},x}$.

Claim 5.4.1.7. *There exists $y \in F_{\mathcal{X}^+}$ such that $x \in \overline{\{y\}}$ and the vanishing locus $V(\mathcal{I}_\alpha) \subseteq \mathrm{Spec}(\mathcal{O}_{\mathcal{X},x})$ is $\overline{\{y\}}$.*

Proof of Claim 5.4.1.7. First, observe that $V(\mathcal{I}_\alpha) = \cap_{f \in \mathcal{I}_\alpha} V(f) = \cap_{f \in \mathcal{I}_\alpha \cap \mathcal{M}_{\mathcal{X}^+,x}} V(f)$. We just need to prove that $\cap_{f \in \mathcal{I}_\alpha} V(f) \supseteq \cap_{f \in \mathcal{I}_\alpha \cap \mathcal{M}_{\mathcal{X}^+,x}} V(f)$. Let $f \in \mathcal{I}_\alpha$, then any admissible expansion $\sum_{\gamma \in \mathcal{C}_{\mathcal{X}^+,x}} c_\gamma \gamma$ is such that, if $c_\gamma \neq 0$, then $\gamma \in \mathcal{I}_\alpha$. Therefore, for any $f \in \mathcal{I}_\alpha$

$$\bigcap_{\gamma \in \mathcal{I}_\alpha \cap \mathcal{M}_{\mathcal{X}^+,x}} V(\gamma) \subseteq \bigcap_{\gamma \colon c_\gamma \neq 0 \text{ in } f = \sum c_\gamma \gamma} V(\gamma) \subseteq V(f)$$

and we obtain the required inclusion. Therefore, $V(\mathcal{I}_\alpha)$ is a stratum of $D_{\mathcal{X}^+}$, since $\mathcal{I}_\alpha$ is a prime ideal and $V(f)$ is the union of irreducible components of $D_{\mathcal{X}^+}$ for any $f \in \mathcal{I}_\alpha \cap \mathcal{M}_{\mathcal{X}^+,x}$. By definition of a Kato point, we have that

$$\overline{\{x\}} = \bigcap_{f \in \mathcal{M}_{\mathcal{X}^+,x} \setminus \mathcal{O}_{\mathcal{X},x}^\times} V(f) \subseteq \bigcap_{\gamma \in \mathcal{I}_\alpha \cap \mathcal{M}_{\mathcal{X}^+,x}} V(\gamma) = \overline{\{y\}},$$

hence we conclude that $x \in \overline{\{y\}}$. This completes the proof of the claim. □

Now, let $y \in F_{\mathcal{X}^+}$ be such that $x \in \overline{\{y\}}$ and $V(\mathcal{I}_\alpha) = \overline{\{y\}}$. Any element γ of $\mathcal{C}_{\mathcal{X}^+,x}$ satisfies $\gamma \in \mathcal{C}_{\mathcal{D}_y^+,x}$ if and only if $\gamma(y) \neq 0$, or equivalently $\gamma \notin \mathcal{I}_\alpha$. Thus, the restriction

α_y of the morphism α to $\mathcal{C}_{\mathcal{D}_y^+,x}$ does not attain the value $+\infty$. To such a morphism we associate a valuation v_{α_y} that, by construction, lies in the skeleton $\mathrm{Sk}_x(\mathcal{D}_y^+)$. Therefore, by restriction of morphisms, we obtain an injective map

$$\overline{\mathrm{Sk}}_x(\mathcal{X}^+) \hookrightarrow \coprod_{y\in F_{\mathcal{X}^+}\,:\, x\in\overline{\{y\}}} \mathrm{Sk}_x(\mathcal{D}_y^+).$$

It remains to show the surjectivity of this map. Given a valuation $v_\beta \in \mathrm{Sk}_x(\mathcal{D}_y^+)$ for some $y \in F_{\mathcal{X}^+}$ with $x \in \overline{\{y\}}$ and $\beta \in \mathrm{Hom}(\mathcal{C}_{\mathcal{D}_y^+,x}, \mathbb{R}_+)$, we can extend β to a morphism $\widetilde{\beta}$ on $\mathcal{C}_{\mathcal{X}^+,x}$ by

$$\widetilde{\beta}(\gamma) := \begin{cases} \beta(\gamma), & \gamma \in \mathcal{C}_{\mathcal{D}_y^+,x}, \\ +\infty, & \text{otherwise}. \end{cases}$$

The associated valuation $v_{\widetilde{\beta}}$ lies in the closure $\overline{\mathrm{Sk}}_x(\mathcal{X}^+)$ in $\mathcal{X}^{\mathrm{an}}$, therefore we get

$$\coprod_{y\in F_{\mathcal{X}^+}\,:\, x\in\overline{\{y\}}} \mathrm{Sk}_x(\mathcal{D}_y^+) \hookrightarrow \overline{\mathrm{Sk}}_x(\mathcal{X}^+).$$

The two maps are inverse of each other by construction, which completes the proof. □

Proposition 5.4.1.8. *The closure of the skeleton* $\mathrm{Sk}(\mathcal{X}^+)$ *in* $\mathcal{X}^{\mathrm{an}}$ *admits the decomposition*

$$\overline{\mathrm{Sk}}(\mathcal{X}^+) = \coprod_{y\in F_{\mathcal{X}^+}} \mathrm{Sk}(\mathcal{D}_y^+),$$

where $\mathrm{Sk}(\mathcal{D}_y^+)$ *is viewed as a subset of* $\mathcal{X}^{\mathrm{an}}$ *by the inclusion* $\mathcal{D}_y^{\mathrm{an}} \hookrightarrow \mathcal{X}^{\mathrm{an}}$. *Further,* $\mathrm{Sk}(\mathcal{D}_y^+) = \overline{\mathrm{Sk}}(\mathcal{X}^+) \cap \ker^{-1}(y)$.

Proof. From Proposition 5.4.1.6, it follows that

$$\begin{aligned}\overline{\mathrm{Sk}}(\mathcal{X}^+) = \bigcup_{x\in F_{\mathcal{X}^+}} \overline{\mathrm{Sk}}_x(\mathcal{X}^+) &= \bigcup_{x\in F_{\mathcal{X}^+}} \coprod_{y\in F_{\mathcal{X}^+}\,:\, x\in\overline{\{y\}}} \mathrm{Sk}_x(\mathcal{D}_y^+) \\ &= \coprod_{y\in F_{\mathcal{X}^+}} \bigcup_{x\in F_{\mathcal{D}_y^+}} \mathrm{Sk}_x(\mathcal{D}_y^+) = \coprod_{y\in F_{\mathcal{X}^+}} \mathrm{Sk}(\mathcal{D}_y^+).\end{aligned}$$

For any $y \in F_{\mathcal{X}^+}$, if the skeleton $\mathrm{Sk}(\mathcal{D}_y^+)$ is non-empty, then it consists of birational points of $\mathcal{D}_y^{\mathrm{an}}$, hence of valuations whose image via the kernel map is the generic point of $\mathcal{D}_y$, thus y. Therefore, the kernel map distinguishes the different part of the disjoint union in $\overline{\mathrm{Sk}}(\mathcal{X}^+)$. □

5.4.2. The case of the toric varieties

(5.4.2.1) Let M be a finitely-generated free abelian group, let $N = \mathrm{Hom}(M, \mathbf{Z})$ be the cocharacter lattice, and let $\langle\cdot,\cdot\rangle\colon M\times N \to \mathbf{Z}$ be the evaluation pairing. Set $M_{\mathbb{R}} := M\otimes_{\mathbf{Z}}\mathbb{R}$ and $N_{\mathbb{R}} := N\otimes_{\mathbf{Z}}\mathbb{R}$. Let Σ be a rational polyhedral fan in $N_{\mathbb{R}}$. Given a cone $\sigma \in \Sigma$, consider the monoid $S_\sigma \coloneqq \sigma^\vee \cap M$.

Let X_Σ (resp. $\mathcal{X}_\Sigma$) be the normal toric variety over $\mathcal{K}$ (resp. model over S) associated to the fan Σ. Let D_Σ denote the (reduced) toric boundary divisor of X_Σ, and write $D_{\mathcal{X}^+}$ for D_Σ when $\mathcal{K}$ is trivially-valued, or for $\overline{D_\Sigma} + (\mathcal{X}_\Sigma)_{0,\mathrm{red}}$ when $\mathcal{K}$ is discretely-valued. The log scheme $\mathcal{X}^+ = (\mathcal{X}_\Sigma, D_{\mathcal{X}^+})$ is log-regular, and the goal of this section is to describe the closure in $\mathcal{X}_\Sigma^{\mathrm{an}}$ of the essential skeleton of (X_Σ, D_Σ).

(5.4.2.2) The support of the fan Σ admits a natural compactification $\overline{\Sigma}$, as in [Pay09, §3] and [Rab12, Proposition 3.4]; see also [ACMUW16, §7.2]. The construction is reviewed below. Given a cone $\sigma \in \Sigma$, we denote by $\overline{\sigma} \coloneqq \mathrm{Hom}(S_\sigma, \overline{\mathbb{R}}_+)$, equipped with the topology of pointwise convergence. The space $\overline{\sigma}$ admits a locally closed stratification by the quotient monoids σ/σ', for each face $\sigma' \preccurlyeq \sigma$, where the embedding $\sigma/\sigma' \hookrightarrow \overline{\sigma}$ is given by

$$u + \sigma' \mapsto \left[m \mapsto \begin{cases} \langle m, u \rangle, & m \in \sigma'^{\perp}, \\ +\infty, & \text{otherwise,} \end{cases} \right] \tag{5.4.2.3}$$

for $u \in \sigma$. For example, the natural inclusion $\sigma \hookrightarrow \overline{\sigma}$ coincides with the embedding $\sigma/\sigma' \hookrightarrow \overline{\sigma}$ associated to the face $\sigma' = 0$. If $\tau \preccurlyeq \sigma$, then the natural map $S_\sigma \to S_\tau$ induces an open embedding $\overline{\tau} \hookrightarrow \overline{\sigma}$; moreover, if $\sigma' \preccurlyeq \tau \preccurlyeq \sigma$, then the embedding $\overline{\tau} \hookrightarrow \overline{\sigma}$ restricts to the natural inclusion $\tau/\sigma' \hookrightarrow \sigma/\sigma'$. Consequently, the cones $\{\overline{\sigma} \colon \sigma \in \Sigma\}$ glue to give an extended cone

$$\overline{\Sigma} \coloneqq \bigcup_{\sigma \in \Sigma} \overline{\sigma} = \bigsqcup_{\sigma \in \Sigma} \bigcup_{\sigma \preccurlyeq \tau} \tau/\sigma, \tag{5.4.2.4}$$

which is a compactification of the support of Σ in $N \otimes_{\mathbb{Z}} \overline{\mathbb{R}}$. In [Thu07, §2], Thuillier constructs an embedding J_Σ of $\overline{\Sigma}$ into $\mathcal{X}_\Sigma^{\mathrm{an}}$, as well as a strong deformation retraction of $\mathcal{X}_\Sigma^{\mathrm{an}}$ onto the image of the embedding. The work of [Thu07] is over a trivially-valued field, but these constructions in fact hold over any field, as pointed out in [ACMUW16, Proposition 7.6]. The image of J_Σ in $\mathcal{X}_\Sigma^{\mathrm{an}}$ is called the *toric skeleton* of $\mathcal{X}_\Sigma$. Note that both the embedding J_Σ and the toric boundary D_Σ are completely determined by the choice of dense torus in X_Σ.

(5.4.2.5) For any cone $\sigma \in \Sigma$, the cones τ that contain σ as a face form a fan in $N/\langle\sigma\rangle \otimes_{\mathbb{Z}} \mathbb{R}$, whose associated toric S-scheme is the orbit closure $\mathcal{V}(\sigma)$ corresponding to the cone σ. Further, the subscheme $\mathcal{V}(\sigma)$ is a stratum of $D_{\mathcal{X}^+}$, so it can be endowed with the trace log structure $\mathcal{V}(\sigma)^+ = (\mathcal{V}(\sigma), D_{\mathcal{V}(\sigma)^+})$ as in 5.4.1.3. The stratification of the skeleton of $\mathcal{V}(\sigma)^+$ is related to the decomposition Eq. (5.4.2.4) by the embedding J_Σ, as demonstrated below.

Proposition 5.4.2.6. *For any cone σ of Σ, J_Σ restricts to a homeomorphism of $\bigcup_{\sigma \preccurlyeq \tau} \tau/\sigma$ onto $\mathrm{Sk}(\mathcal{V}(\sigma)^+)$.*

Proof. This follows from [Thu07, Proposition 2.13], [Uli17, Theorem 1.2 and §3.4], and Proposition 5.3.1.2. ☐

Corollary 5.4.2.7. *Assume $\mathcal{K}$ has residue characteristic zero. The closure of the essential skeleton of (X_Σ, D_Σ) in $\mathcal{X}_\Sigma^{\mathrm{an}}$ coincides with the toric skeleton; that is,*

$$J_\Sigma \colon \overline{\Sigma} = \bigsqcup_{\sigma \in \Sigma} \bigcup_{\sigma \preccurlyeq \tau} \tau/\sigma \xrightarrow{\simeq} \bigsqcup_{\sigma \in \Sigma} \mathrm{Sk}(\mathcal{V}(\sigma)^+) = \overline{\mathrm{Sk}}(\mathcal{X}^+) = \overline{\mathrm{Sk}}^{\mathrm{ess}}(X_\Sigma, D_\Sigma).$$

Proof. The homeomorphism between $\overline{\Sigma}$ and $\overline{\mathrm{Sk}}(\mathcal{X}^+)$ follows immediately from Corollary 5.4.1.8 and Corollary 5.4.2.6. As a toric variety with its toric boundary defines a logCY pair, the last homeomorphism will follow from Proposition 7.2.4.2 in the trivially-valued field case, and applying [BM19, Lemma 5.1.2] in the discretely-valued field case. □

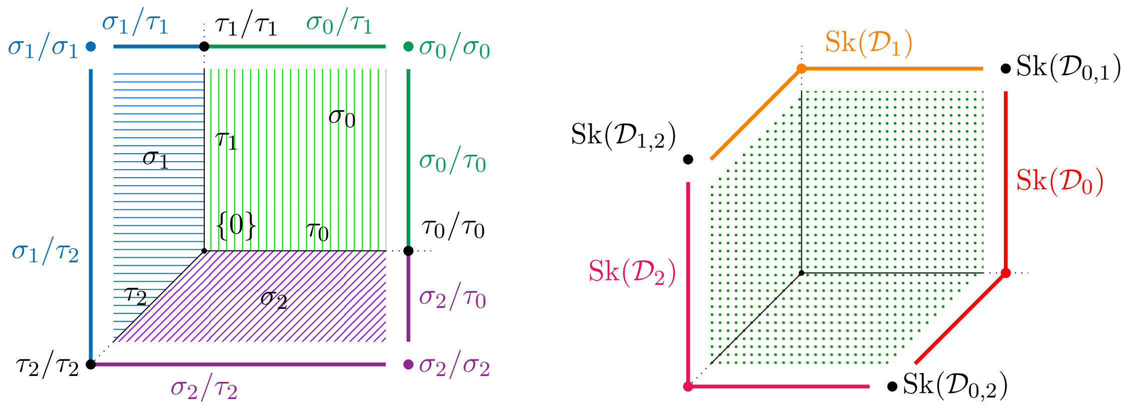


Figure 5.4.2.1: Let Σ be the usual fan in $\mathbb{R}^2$ associated to the $\mathcal{K}^\circ$-toric variety $\mathcal{X}_\Sigma = \mathbb{P}^2_{\mathcal{K}^\circ}$. In the picture on the left, we have the compactification $\overline{\Sigma}$ and its decomposition as in Eq. (5.4.2.4). In the picture on the right, there is the stratification of $\overline{\mathrm{Sk}}(\mathcal{X}^+)$ from Proposition 5.4.1.8.

5.4.3. The closure of a Kontsevich–Soibelman skeleton

(5.4.3.1) Assume that $\mathcal{K}$ has residue characteristic zero. Let (X, D) be a pair over $\mathcal{K}$ such that $D = \sum_i a_i D_i$ is a $\mathbb{Q}$-boundary divisor with snc support, the log scheme $X^+ = (X, D_{\mathrm{red}})$ is then log-regular, and $K_X + D_{\mathrm{red}}$ is $\mathbb{Q}$-Cartier. Let $\mathcal{X}$ be an snc model of X over $\mathcal{S}$. We set $D_{\mathcal{X}^+} = D_{\mathrm{red}}$ if $\mathcal{K}$ is trivially-valued, and $D_{\mathcal{X}^+} = \overline{D}_{\mathrm{red}} + \mathcal{X}_{0,\mathrm{red}}$ if it is discretely-valued. Then $\mathcal{X}^+ = (\mathcal{X}, D_{\mathcal{X}^+} = \sum_i D_{\mathcal{X}^+,i})$ is a log-regular model of (X, D_{red}) over $\mathcal{S}^+$.

Taking advantage of the decomposition of the closure of the skeleton $\mathrm{Sk}(\mathcal{X}^+)$ described in Proposition 5.4.1.8, we study the closure of Kontsevich–Soibelman skeletons. More precisely, in Proposition 5.4.3.4, we show that for any non-zero D-logarithmic pluricanonical form η, the valuations in the complement $\overline{\mathrm{Sk}}(X, D, \eta) \setminus \mathrm{Sk}(X, D, \eta)$ are minima of weight functions associated to suitable forms on the strata of $D_{X^+}(= D_{\mathrm{red}} = \sum_i D_i)$. In the trivially-valued setting, this characterization can be made even more precise (Proposition 5.4.3.8).

(5.4.3.2) As we assume that the divisor $D_{\mathcal{X}^+}$ is snc, the characteristic monoid $\mathcal{C}_{\mathcal{X}^+,x}$ at any Kato point x of $\mathcal{X}^+$ is a free monoid isomorphic to $\mathbb{N}^{|I_x|}$, where the isomorphism is determined by choosing local equations z_i of the components $D_{\mathcal{X}^+,i}$ at x. In this case, any $f \in \mathcal{O}_{\mathcal{X},x}$ at x has an admissible expansion of the form

$$f = \sum_{\gamma \in \mathbb{Z}_{\geqslant 0}^{|I_x|}} c_\gamma z^\gamma,$$

in the completion $\widehat{\mathcal{O}}_{\mathcal{X},x}$, where $c_\gamma \in \{0\} \cup \mathcal{O}_{\mathcal{X},x}^\times$.

(5.4.3.3) Let η be a non-zero regular D-logarithmic m-pluricanonical form on X, and let x be the generic point of an irreducible component D_i of D. The *residue* $\mathrm{Res}_{D_i}(\eta)$ of the form η along D_i is a (possibly zero) regular $\left(\sum_{j \neq i} a_j D_j\right)|_{D_i}$-logarithmic m-pluricanonical form on D_i, whose local description we review below. If the divisor D_i is locally defined at x by the equation $z_i = 0$, then η can locally be written at x as

$$\eta = \left(\frac{dz_i}{z_i}\right)^{\otimes m} \wedge \mu$$

for some local section μ of $\bigwedge^{n-1}(\Omega^1_{X/\mathcal{K}}(\log D))^{\otimes m}$, where n is the relative dimension of X over $\mathcal{K}$. The form $\mathrm{Res}_{D_i}(\eta)$ is a global section in $H^0\Big(D_i, \omega^{\otimes m}_{(D_i, \sum_{j\neq i} a_j D_j|_{D_i})}\Big)$ that is locally given by the restriction $\mu|_{D_i}$.

For a general Kato point $x \in F_{X^+}$, D_x is a stratum of D, i.e. a component of an intersection of $\{D_i \colon i \in I_x\}$, so we can iterate the above construction; that is, if z_i is a local equation of the component D_i at x for each $i \in I_x$, then write

$$\eta = \bigwedge_{i \in I_x} \left(\frac{dz_i}{z_i}\right)^{\otimes m} \wedge \mu$$

for some local section μ of $\bigwedge^{n-|I_x|}(\Omega^1_{X/\mathcal{K}}(\log D))^{\otimes m}$. The form $\mathrm{Res}_{D_x}(\eta)$ on D_x is locally given by $\eta|_{D_x}$. See [EV92, §2] for further details.

Proposition 5.4.3.4. *Under the same assumptions on (X, D), if η is a non-zero regular D-logarithmic pluricanonical form on X and $x \in F_{X^+}$, then there is an inclusion*

$$\overline{\mathrm{Sk}}(X, D, \eta) \cap \ker^{-1}(x) \subseteq \mathrm{Sk}(D_x, \textstyle\sum_{j \notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta)).$$

Proof. By Proposition 5.3.7.4 and Proposition 5.4.1.8,

$$\overline{\mathrm{Sk}}(X, D, \eta) \cap \ker^{-1}(x) \subseteq \overline{\mathrm{Sk}}(\mathcal{X}^+) \cap \ker^{-1}(x) = \mathrm{Sk}(\mathcal{D}_x^+) = \bigcup_{y \in F_{\mathcal{D}_x^+}} \mathrm{Sk}_y(\mathcal{D}_x^+),$$

$$\mathrm{Sk}(D_x, \textstyle\sum_{j \notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta)) \subseteq \mathrm{Sk}(\mathcal{D}_x^+) = \bigcup_{y \in F_{\mathcal{D}_x^+}} \mathrm{Sk}_y(\mathcal{D}_x^+).$$

Therefore, we may prove the desired inclusion for a valuation that lies in $\mathrm{Sk}_y(\mathcal{D}_x^+)$, for some $y \in F_{\mathcal{D}_x^+}$. In order to relate the closure of the Kontsevich–Soibelman skeleton $\mathrm{Sk}(X, D, \eta)$

to the Kontsevich–Soibelman skeleton $\mathrm{Sk}(D_x, \sum_{j\notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta))$, we will compute explicitely the associated weight functions on the faces $\mathrm{Sk}_y(\mathcal{X}^+)$ and $\mathrm{Sk}_y(\mathcal{D}_x^+)$. To that end, recall that the forms η and $\mathrm{Res}_{D_x}(\eta)$ respectively induce divisors $\mathrm{div}_{(\mathcal{X},D_{\mathcal{X}^+})}(\eta)$ and $\mathrm{div}_{(\mathcal{D}_x,D_{\mathcal{D}_x^+})}(\mathrm{Res}_{D_x}(\eta))$ on $\mathcal{X}$ and $\mathcal{D}_x$ when $\mathcal{K}$ is trivially-valued, and $\mathrm{div}_{(\mathcal{X},D_{\mathcal{X}^+}-\mathrm{div}(\pi))}(\eta)$ and $\mathrm{div}_{(\mathcal{D}_x,D_{\mathcal{D}_x^+}-\mathrm{div}(\pi))}(\mathrm{Res}_{D_x}(\eta))$ in the discretely-valued case. To compute the weight functions wt_η and $\mathrm{wt}_{\mathrm{Res}_{D_x}(\eta)}$ on $\mathrm{Sk}_y(\mathcal{X}^+)$ and $\mathrm{Sk}_y(\mathcal{D}_x^+)$, we first determine local equations for these divisors at y.

The Kato point $y \in F_{\mathcal{D}_x^+}$ satisfies $y \in \overline{\{x\}}$ and $I_x \subseteq I_y$. Let $z_1, \dots, z_n$ be local parameters at y such that z_i is a local equation of $D_{\mathcal{X}^+,i}$ for each $i \in I_y$. Then the forms η and $\mathrm{Res}_{D_x}(\eta)$ can be written locally at y as

$$\begin{aligned}
\eta &= fg^{-1} \cdot g \bigwedge_{i\in I_x} \left(\frac{dz_i}{z_i}\right)^{\otimes m} \bigwedge_{j\in I_y\setminus I_x} \left(\frac{dz_j}{z_j}\right)^{\otimes m} \bigwedge_{h\notin I_y} dz_h^{\otimes m},\\
\mathrm{Res}_{D_x}(\eta) &= \left(fg^{-1} \cdot g \bigwedge_{j\in I_y\setminus I_x} \left(\frac{dz_j}{z_j}\right)^{\otimes m} \bigwedge_{h\notin I_y} dz_h^{\otimes m}\right)\Big|_{D_x} =\\
&= fg^{-1}|_{D_x} \cdot g \bigwedge_{j\in I_y\setminus I_x} \left(\frac{dz_j}{z_j}\right)^{\otimes m} \bigwedge_{h\notin I_y} dz_h^{\otimes m}
\end{aligned}$$

for some $f \in \mathcal{O}_{\mathcal{X},y}$, with $g = \pi^m$ if $\mathcal{K}$ is discretely-valued and $g \in \mathcal{O}_{\mathcal{X},y}^\times$ if it is trivially-valued. Thus, fg^{-1} and $fg^{-1}|_{D_x}$ are the required local equations at y. An admissible expansion of f in $\widehat{\mathcal{O}}_{\mathcal{X},y}$ can be decomposed as

$$f = \sum_{\gamma\in\mathbb{Z}_{\geqslant 0}^{|I_y|}} c_\gamma z^\gamma = \sum_{\gamma\colon \exists i\in I_x, \gamma_i\neq 0} c_\gamma z^\gamma + \sum_{\gamma\colon \forall i\in I_x, \gamma_i=0} c_\gamma z^\gamma, \tag{5.4.3.5}$$

with $c_\gamma \in \{0\} \cup \mathcal{O}_{\mathcal{X},y}^\times$. It follows that

$$f|_{D_x} = \sum_{\gamma\colon \forall i\in I_x, \gamma_i=0} c_\gamma|_{D_x} z^\gamma \tag{5.4.3.6}$$

in $\widehat{\mathcal{O}}_{D_x,y}$ and $c_\gamma|_{D_x} \in \{0\} \cup \mathcal{O}_{D_x,y}^\times$, so Eq. (5.4.3.6) is an admissible expansion of $f|_{D_x}$ at y. Thus, for valuations $v_\alpha \in \mathrm{Sk}_y(\mathcal{X}^+)$ and $v_\beta \in \mathrm{Sk}_y(\mathcal{D}_x^+)$, we have

$$\begin{aligned}
\mathrm{wt}_\eta(v_\alpha) &= v_\alpha(f) = \min\left\{\min_{\gamma\colon \exists i\in I_x, \gamma_i\neq 0}\{v_\alpha(c_\gamma) + \alpha(z^\gamma)\}, \min_{\gamma\colon \forall i\in I_x, \gamma_i=0}\{v_\alpha(c_\gamma) + \alpha(z^\gamma)\}\right\},\\
\mathrm{wt}_{\mathrm{Res}_{D_x}(\eta)}(v_\beta) &= v_\beta(f|_{D_x}) = \min_{\gamma\colon \forall i\in I_x, \gamma_i=0}\{v_\beta(c_\gamma) + \beta(z^\gamma)\}.
\end{aligned} \tag{5.4.3.7}$$

Due to 5.4.3.7 the weight function wt_η extends to a continuous function on $\overline{\mathrm{Sk}}_y(\mathcal{X}^+)$ and restricts to $\mathrm{wt}_{\mathrm{Res}_{D_x}(\eta)}$ on $\mathrm{Sk}_y(\mathcal{D}_x^+)$. Indeed, if $v_\beta \in \mathrm{Sk}_y(\mathcal{D}_x^+)$, then $\beta(z^\gamma) = +\infty$ for any γ

such that $\gamma_i \neq 0$ for some $i \in I_x$. As a result we have that

$$\begin{aligned}
\mathrm{wt}_\eta(v_\beta) &= v_\beta(f) \\
&= \min\left\{\min_{\gamma\colon \exists i\in I_x, \gamma_i\neq 0}\{v_\beta(c_\gamma)+\beta(z^\gamma)\}, \min_{\gamma\colon \forall i\in I_x, \gamma_i=0}\{v_\beta(c_\gamma)+\beta(z^\gamma)\}\right\} \\
&= \min_{\gamma\colon \forall i\in I_x, \gamma_i=0}\{v_\beta(c_\gamma)+\beta(z^\gamma)\} \\
&= v_\beta(f|_{D_x}) = \mathrm{wt}_{\mathrm{Res}_{D_x}(\eta)}(v_\beta).
\end{aligned}$$

The minimality locus of wt_η along $\mathrm{Sk}_y(\mathcal{D}_x^+)$ are contained in the minimality locus of $\mathrm{wt}_{\mathrm{Res}_{D_x}(\eta)}$. Hence, we conclude that

$$\overline{\mathrm{Sk}}(X, D, \eta) \cap \ker^{-1}(x) \subseteq \mathrm{Sk}(D_x, \textstyle\sum_{j\notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta)).$$

□

When $\mathcal{K} = k$ is a trivially-valued field, the inclusion of Proposition 5.4.3.4 is in fact an equality.

Proposition 5.4.3.8. *Under the same assumptions on (X, D), suppose that $\mathcal{K} = k$ is trivially-valued. If η is a non-zero regular D-logarithmic pluricanonical form on X and $x \in F_{X^+}$, then*

$$\overline{\mathrm{Sk}}(X, D, \eta) \cap \ker^{-1}(x) = \mathrm{Sk}(D_x, \textstyle\sum_{j\notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta)).$$

Proof. Under the same assumption and notation of the proof of Proposition 5.4.3.4, consider $v_\beta \in \mathrm{Sk}_y(\mathcal{D}_x^+)$, where β is a morphism on $\mathcal{C}_{\mathcal{D}_x^+, y} \simeq \mathbb{N}^{|I_y \setminus I_x|}$. We construct a sequence of valuations $(v_{\alpha_n})_{n=1}^\infty$ in $\mathrm{Sk}_y(\mathcal{X}^+)$ that converge to v_β as follows:

$$\begin{cases} \alpha_n(z_i) = \beta(z_i), & i \in I_y \setminus I_x, \\ \alpha_n(z_i) = n, & i \in I_x. \end{cases}$$

We have $\alpha_n(z_i) \to +\infty$ as $n \to +\infty$ for any $i \in I_x$; moreover, for sufficiently large n, $v_{\alpha_n}(f)$ can be written as

$$\begin{aligned}
v_{\alpha_n}(f) = \min_{\gamma\colon \forall i\in I_x, \gamma_i=0}\{v_{\alpha_n}(c_\gamma)+\alpha_n(z^\gamma)\} &= \min_\gamma \alpha_n(z^\gamma) \\
&= \min_\gamma \beta(z^\gamma) = v_\beta(f|_{D_x}),
\end{aligned} \tag{5.4.3.9}$$

where the two right-hand minima range over all $\gamma \in \mathbb{Z}_{\geqslant 0}^{|I_y|}$ such that $c_\gamma \neq 0$ and such that for all $i \in I_x$, $\gamma_i = 0$. Thus, given any valuation v_β in $\mathrm{Sk}_y(\mathcal{D}_x^+)$, we can construct a sequence of valuations in $\mathrm{Sk}_y(\mathcal{X}^+)$ that converge to v_β, and moreover by (5.4.3.7) attaining the same weight with respect to $\mathrm{Res}_{D_x}(\eta)$ and η.

Assume now that $v_\beta \in \mathrm{Sk}(D_x, \sum_{j\notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta)) \cap \mathrm{Sk}_y(\mathcal{D}_x^+)$ and consider a sequence $(v_{\alpha_n})_{n=1}^{\infty}$ in $\mathrm{Sk}_y(\mathcal{X}^+)$ converging to v_β, as above. By 5.3.7.7, the minimal weight with respect to either form is zero and we know that

$$\mathrm{wt}_\eta(v_{\alpha_n}) = v_{\alpha_n}(f) = v_\beta(f|_{D_x}) = \mathrm{wt}_{\mathrm{Res}_{D_x}(\eta)}(v_\beta) = 0$$

by (5.4.3.7) and (5.4.3.9). It follows that $v_{\alpha_n} \in \mathrm{Sk}(X, D, \eta)$ for all n sufficiently large. In other words, any valuation in the Kontsevich–Soibelman skeleton $\mathrm{Sk}(D_x, \sum_{j \notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta))$ is the accumulation point of a sequence of valuations in $\mathrm{Sk}(X, D, \eta)$, hence we have the inclusion of $\mathrm{Sk}(D_x, \sum_{j \notin I_x} a_j D_j|_{D_x}, \mathrm{Res}_{D_x}(\eta))$ in $\overline{\mathrm{Sk}}(X, D, \eta) \cap \ker^{-1}(x)$, as required. □

Example 5.4.3.10. Over a discretely-valued field $\mathcal{K} = K$, the inclusion of Proposition 5.4.3.4 may be strict. Indeed, let

$$\mathscr{X} = \mathrm{Spec}\left(\frac{R[T_1, T_2, T_3]}{(\pi - T_1^2 T_2 T_3)}\right),$$

and D_i be the reduced vertical divisor on $\mathscr{X}$ given by the equation $T_i = 0$, for $i = 1, 2, 3$. Let D_4 be the horizontal divisor on $\mathscr{X}$ given by the equation $T_1 - a$, for some $a \in R \backslash \{0\}$. Consider the log scheme $\mathscr{X}^+ = (\mathscr{X}, D_{\mathscr{X}^+})$ with the divisorial log structure given by $D_{\mathscr{X}^+} = \sum_{i=1}^4 D_i$. In Figure 5.4.3.1 below, we picture the closure of $\mathrm{Sk}(\mathscr{X}^+)$, as well as the decomposition $\overline{\mathrm{Sk}}(\mathscr{X}^+) = \mathrm{Sk}(\mathscr{X}^+) \coprod \mathrm{Sk}(D_4^+)$ as in Proposition 5.4.1.8; there, the face of $\overline{\mathrm{Sk}}(\mathscr{X}^+)$ corresponding to the generic point of the intersection $\cap_{i \in J} D_i$, for $J \subseteq \{1, \ldots, 4\}$, is denoted by $x_{\cap_{i \in J} D_i}$.

Consider the logarithmic canonical forms given by

$$\eta = \frac{T_1^2 T_2^2 T_3^2}{T_1 - a} \frac{dT_2}{T_2} \wedge \frac{dT_3}{T_3} = 2\frac{T_1^2 T_2^2 T_3^2}{T_1 - a} \frac{dT_1}{T_1} \wedge \frac{dT_3}{T_3} = -2\frac{T_1^2 T_2^2 T_3^2}{T_1 - a} \frac{dT_1}{T_1} \wedge \frac{dT_2}{T_2},$$

$$\mathrm{Res}_{D_4}(\eta) = 2a T_2^2 T_3^2 \frac{dT_3}{T_3} = -2a T_2^2 T_3^2 \frac{dT_2}{T_2}.$$

The Kontsevich–Soibelman skeleton of η is $\mathrm{Sk}(\mathscr{X}_K, (D_4)_K, \eta) = \{v_{D_1}\}$, as

$$\mathrm{wt}_\eta(v_{D_i}) = v_{D_i}\left(\frac{T_1^2 T_2^2 T_3^2}{T_1 - a}\right) + 1 = \begin{cases} 2, & \text{for } i = 1, \text{ since the multiplicity of } D_1 \text{ is } 2, \\ 3, & \text{for } i = 2, 3. \end{cases}$$

However, the Kontsevich–Soibelman skeleton of $\mathrm{Res}_{D_4}(\eta)$ is the whole skeleton $\mathrm{Sk}(D_4^+)$. It follows that

$$\overline{\mathrm{Sk}}(\mathscr{X}_K, (D_4)_K, \eta) \cap \ker^{-1}(x_{D_4}) = \emptyset \quad \subsetneq \quad \mathrm{Sk}(D_4^+) = \mathrm{Sk}((D_4)_K, \emptyset, \mathrm{Res}_{D_4}(\eta)),$$

so the inclusion of Proposition 5.4.3.4 may be strict.

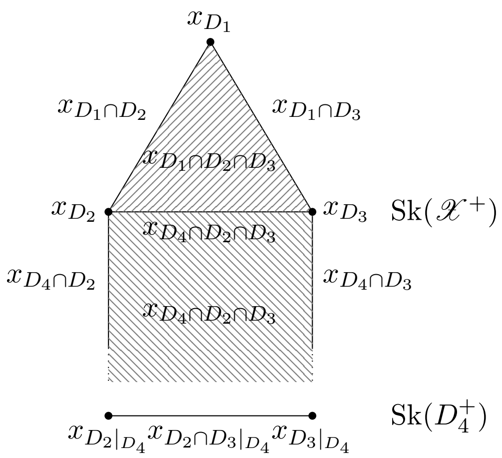


Figure 5.4.3.1: The closure $\overline{\mathrm{Sk}}(\mathscr{X}^+)$ in $\widehat{\mathscr{X}_\eta}$ of the skeleton of $\mathscr{X}^+$.

6

Comparison with Temkin's metric

6.1. Introduction

In this chapter the notion of weight function is studied in comparison with an intrinsic metric in non-archimedean analytic geometry, called Temkin's metric.

6.1.1. The weight metric

We recall that the weight functions constructed in Section 5.2 are built in terms of log discrepancy functions. The log discrepancy of a divisor is a ubiquitous notion in birational geometry, and it can be used to define a log discrepancy function on the space of divisorial valuations on a variety. Further, it extends to a lower-semicontinuous function on spaces of (semi)valuations by [BdFFU15; BJ18a; JM12].

We adopt the formalism of metrics on line bundles on $X^{\beth}$ in order to give a complementary perspective on the weight functions. If X is normal and $K_X + D$ is $\mathbb{Q}$-Gorenstein, then the collection of weight functions on $X^{\beth}$ determine weight metrics on the logarithmic pluricanonical bundles. Then we compare the weight metric (when X is smooth and D is empty) with an intrinsic metric on the pluricanonical bundles of $X^{\beth}$, constructed by Temkin in [Tem16].

Theorem 6.1.1.1 (§ 6.5). *If X is a smooth variety over a trivially-valued field $\mathcal{K}$ of characteristic zero, then Temkin's metric on $(\omega_{X/\mathcal{K}}^{\otimes m})^{\beth}$ coincides with the weight metric.*

Theorem 6.1.1.1 is the trivially-valued analogue of [Tem16, Theorem 8.3.3], which relates Temkin's metric and the weight metric over a discretely-valued field of residue characteristic zero. The parallel between Theorem 6.1.1.1 and [Tem16, Theorem 8.3.3] confirms that, in the case of empty boundary, our definition of weight function in the trivially-valued setting is the correct analogue of the weight function in the discretely-valued case. Note that [Tem16] does not treat the case of non-empty boundary.

In fact, Temkin's metric can be used to define the essential skeleton of a quasi-smooth analytic space over any field; this approach is adopted in [HN17, Proposition 4.3.2] and [KY18]. Theorem 6.1.1.1 shows that the definition of the essential skeleton of Chapter 5 coincides with this one, when both are defined. Moreover, it provides a concrete and computable description of both Temkin's metric and the essential skeleton.

6.1.2. Structure of the chapter

We first review Temkin's construction from [Tem16] of an intrinsic metric on the sheaves of differentials of an analytic space, as well as Temkin's comparison theorem [Tem16, Theorem 8.3.3] with the weight metric in the discretely-valued setting. Subsequently, we prove Theorem 6.1.1.1 by passing to a discretely-valued extension and applying Temkin's comparison result.

In contrast to the rest of the paper, the metrics in this section are written multiplicatively as in 5.2.1.4, following the conventions of [Tem16]. By doing so, one avoids changing the base of logarithms when passing between the trivially-valued and discretely-valued settings.

6.2. Temkin's metrization of the pluricanonical bundles

6.2.1. Seminorms on modules of Kähler differentials

(6.2.1.1) Let $(\mathcal{K}, |\cdot|_{\mathcal{K}})$ denote a non-Archimedean field. Let $(A, |\cdot|_A)$ be a seminormed $\mathcal{K}$-algebra, and let $\widehat{A}$ denote the separated completion of $(A, |\cdot|_A)$. Let $\Omega^1_{A/\mathcal{K}}$ be the (algebraic) module of Kähler differentials, which we equip with the seminorm

$$\|\eta\|_{A/\mathcal{K}} = \inf \max_i |a_i|_A \cdot |b_i|_A, \qquad \text{for } \eta \in \Omega^1_{A/\mathcal{K}},$$

where the infimum ranges over all finite expressions of the form $\eta = \sum_i a_i db_i$ with $a_i, b_i \in A$. By [Tem16, Lemma 4.1.3], $\|\cdot\|_{A/\mathcal{K}}$ is the maximal A-module seminorm such that the differential $d\colon A \to \Omega^1_{A/\mathcal{K}}$ is a non-expansive $\mathcal{K}$-module morphism.

The *completed module of Kähler differentials* $\widehat{\Omega}^1_{A/\mathcal{K}}$ of A is the separated completion of $(\Omega^1_{A/\mathcal{K}}, \|\cdot\|_{A/\mathcal{K}})$, and we write the resulting norm on $\widehat{\Omega}^1_{A/\mathcal{K}}$ also as $\|\cdot\|_{A/\mathcal{K}}$. In [Tem16], the seminorm $\|\cdot\|_{A/\mathcal{K}}$ on $\Omega^1_{A/\mathcal{K}}$ is referred to as the *Kähler seminorm*, and the norm $\|\cdot\|_{A/\mathcal{K}}$ on $\widehat{\Omega}^1_{A/\mathcal{K}}$ is known as the *Kähler norm*.

(6.2.1.2) There is an alternative, intrinsic description of the completed module of Kähler differentials. By [Tem16, Lemma 4.3.3], the composition $\widehat{d}\colon A \xrightarrow{d} \Omega^1_{A/\mathcal{K}} \to \widehat{\Omega}^1_{A/\mathcal{K}}$ is the universal non-expansive $\mathcal{K}$-derivation with values in a Banach $\widehat{A}$-module.

Furthermore, if A is a $\mathcal{K}$-affinoid algebra, then there is a natural isomorphism $\widehat{\Omega}^1_{A/\mathcal{K}} \simeq \mathcal{J}/\mathcal{J}^2$ of Banach A-modules, where $\mathcal{J}$ denotes the Banach A-module that arises as the kernel of the multiplication map $A\widehat{\otimes}_{\mathcal{K}}A \to A$.

Under this isomorphism, the derivation $\widehat{d}$ is induced by the non-expansive map $A \to \mathcal{J}$ of finite Banach A-modules given by $a \mapsto 1\widehat{\otimes}a - a\widehat{\otimes}1$. This approach is exposited in [Ber93, §3.3], and the equivalence of the definitions is discussed in [Tem16, Remark 4.3.4].

(6.2.1.3) For a good $\mathcal{K}$-analytic space Z, one can construct a coherent sheaf $\Omega^1_{Z/\mathcal{K}}$ of Kähler differentials on Z such that for any affinoid domain $V = \mathcal{M}(A)$ in Z, we have $\Gamma(V, \Omega^1_{Z/\mathcal{K}}) = \widehat{\Omega}^1_{A/\mathcal{K}}$. Strictly speaking, $\Omega^1_{Z/\mathcal{K}}$ is defined as a sheaf in the G-topology on Z, but there is no distinction by [Ber93, Proposition 1.3.4]. In addition, if X is a finite type $\mathcal{K}$-scheme, then there is a natural isomorphism $\Omega^1_{X^{\mathrm{an}}/\mathcal{K}} \simeq (\Omega^1_{X/\mathcal{K}})^{\mathrm{an}}$ (and similarly for the $\beth$-analytification when $\mathcal{K}$ is trivially-valued). For more details on the construction of the sheaf of differentials, see [Ber93, §3.3] and [Duc11, §5.1].

6.2.2. Temkin's metric

(6.2.2.1) Let Z be a good $\mathcal{K}$-analytic space. For each $z \in Z$, the stalk $(\Omega^1_{Z/\mathcal{K}})_z$ is the filtered colimit of $\widehat{\Omega}^1_{A/\mathcal{K}}$ over the affinoid neighborhoods $\mathcal{M}(A)$ of z. In particular, the stalk can be equipped with the colimit seminorm, which we denote by $\|\cdot\|_z$. The pair

$$((\Omega^1_{Z/\mathcal{K}})_z, \|\cdot\|_z)$$

is a seminormed $\mathcal{O}_{Z,z}$-module, and it is not complete in general. The data $\{\|\cdot\|_z\}_{z\in Z}$ of this seminorm on each stalk is known as *Temkin's metric on the sheaf* $\Omega^1_{Z/\mathcal{K}}$, and it gives $\Omega^1_{Z/\mathcal{K}}$ the structure of a seminormed sheaf of $\mathcal{O}_Z$-modules in the sense of [Tem16, §3.1].

(6.2.2.2) The stalks $(\Omega^1_{Z/\mathcal{K}})_z$ can be difficult to describe; for example, $(\Omega^1_{Z/\mathcal{K}})_z$ is not isomorphic to $\Omega^1_{\mathcal{O}_{Z,z}/\mathcal{K}}$ as normed $\mathcal{O}_{Z,z}$-algebras. Nevertheless, the completed fibres admit a much nicer description: for any affinoid neighborhood $\mathcal{M}(A)$ of z, the universal property of $\widehat{\Omega}^1_{Z/\mathcal{K}}$ yields a non-expansive map $\widehat{\Omega}^1_{A/\mathcal{K}} \to \widehat{\Omega}^1_{\mathcal{H}(z)/\mathcal{K}}$ of A-modules, and the universal property of the colimit gives rise to a commutative diagram

$$\begin{array}{ccc} \widehat{\Omega}^1_{A/\mathcal{K}} & & \\ \downarrow & \searrow & \\ (\Omega^1_{Z/\mathcal{K}})_z & \overset{\psi_z}{\dashrightarrow} & \widehat{\Omega}^1_{\mathcal{H}(z)/\mathcal{K}}. \end{array}$$

Now, [Tem16, Theorem 6.1.8] asserts that ψ_z identifies $\widehat{\Omega}^1_{\mathcal{H}(z)/\mathcal{K}}$ with the separated completion of the module $((\Omega^1_{Z/\mathcal{K}})_z, \|\cdot\|_z)$. In fact, ψ_z factors through the fibre $\Omega^1_{Z/\mathcal{K}}(z) \coloneqq (\Omega^1_{Z/\mathcal{K}})_z \otimes_{\mathcal{O}_{Z,z}} \mathcal{H}(z)$ (equipped with the tensor product seminorm), and this factorization identifies $\widehat{\Omega}^1_{\mathcal{H}(z)/\mathcal{K}}$ with the separated completion of $\Omega^1_{Z/\mathcal{K}}(z)$; see [Tem16, Corollary 6.1.9].

(6.2.2.3) Let Z be a quasi-smooth $\mathcal{K}$-analytic space (in the sense of [Duc11, Definition 5.2.4]) and let $\ell, m \in \mathbb{Z}_{>0}$. The exterior power $\Omega^\ell_{Z/\mathcal{K}} \coloneqq \bigwedge_{i=1}^{\ell} \Omega^1_{Z/\mathcal{K}}$ and the tensor power $(\Omega^\ell_{Z/\mathcal{K}})^{\otimes m}$ acquire metrics in the usual manner; see e.g. [Tem16, §3.2]. In particular, if

Z is of pure dimension n, then the *m-pluricanonical sheaf* $\omega_{Z/\mathcal{K}}^{\otimes m} := (\Omega_{Z/\mathcal{K}}^{n})^{\otimes m}$ is a line bundle on Z, and it carries a metric

$$\|\cdot\|_{\mathrm{Tem}} = \{\|\cdot\|_{\mathrm{Tem},z}\}_{z\in Z},$$

which we will also refer to as *Temkin's metric*. Moreover, for a fixed local section s of $\omega_{Z/\mathcal{K}}^{\otimes m}$, the function $\|s\|_{\mathrm{Tem}}$ is upper-semicontinuous on the locus where s is defined. Thus, in the notation of Section 5.2.1, Temkin's metric is lower-semicontinuous.

(6.2.2.4) When $\mathcal{K}$ is a nontrivially-valued field of residue characteristic zero, Temkin's metric $\|\cdot\|_{\mathrm{Tem}}$ on $\omega_{Z/\mathcal{K}}^{\otimes m}$ is the maximal lower-semicontinuous extension of its values on the divisorial points $Z^{\mathrm{div}} \subseteq Z$ (in the sense of [Tem16, §3.2.7]). This is shown in [Tem16, Corollary 8.2.10]. When $\mathcal{K}$ is trivially-valued of characteristic zero, one can show that Temkin's metric is determined by the set of divisorial points and by the trivial norm; this is done by reducing to the nontrivially-valued setting by means of the Gauss extensions (as in §6.3).

6.2.3. Temkin's metric and weight metric

(6.2.3.1) One of the main results of [Tem16] is a comparison theorem between Temkin's metric and the weight metric over a discretely-valued field of residue characteristic zero. Let $\mathcal{K} = K$ be such a field, and let π_K be an uniformizer of K.

To state Temkin's comparison theorem, we write the weight metric multiplicatively as in 5.2.1.4. For a normal K-variety X such that $\omega_{X/K}^{\otimes m}$ is invertible for $m \in \mathbb{Z}_{>0}$, recall that the weight metric $\|\cdot\|_{\mathrm{wt}_{\mathrm{disc}}}$ on the canonical bundle $(\omega_{X/K}^{\otimes m})^{\mathrm{an}} \simeq \omega_{X^{\mathrm{an}}/K}^{\otimes m}$ is defined as follows: for any $x \in X^{\mathrm{an}}$ and local section $s \in (\omega_{X/K}^{\otimes m})_{\ker(x)}$, set

$$\|s\|_{\mathrm{wt}_{\mathrm{disc},x}} := |\pi_K|^{\mathrm{wt}_s(x)}.$$

This formula determines the seminorm $\|\cdot\|_{\mathrm{wt}_{\mathrm{disc},x}}$ on all of the stalks $(\omega_{X^{\mathrm{an}}/K}^{\otimes m})_x$ as in §5.2.1. For a divisorial point $x \in X^{\mathrm{div}}$ corresponding to a K°-model $\mathscr{X}$ of X and an irreducible component $E \subseteq \mathscr{X}_0$, the weight metric admits a simple description: pick a $\mathcal{O}_{\mathscr{X},E}$-module generator δ of the stalk $(\omega_{\mathscr{X}/K^{\circ}}^{\otimes m})_E$ and write $s = f\delta$ for some $f \in K(X)$, then

$$\|s\|_{\mathrm{wt}_{\mathrm{disc},x}} = |f(x)| \cdot |g_E(x)|^m,$$

where g_E is a local equation of E at its generic point on $\mathscr{X}$. This expression is independent of the choice of δ since any two generators differ by a multiplicative factor $u \in \mathcal{O}_{\mathscr{X},E}^{\times}$ and $|u(x)| = 1$. Moreover, the construction of the weight metric can be made without reference to $(\mathscr{X}, E)$ as in [Tem16, §8.3.1].

Theorem 6.2.3.2. [Tem16, Theorem 8.3.3] *Let K be a discretely-valued field of residue characteristic zero with uniformizer π_K. For a smooth K-variety X and $m \in \mathbb{Z}_{>0}$, we* *have*

$$\|\cdot\|_{\mathrm{wt_{disc}}} = |\pi_K|^m \|\cdot\|_{\mathrm{Tem}}$$

as metrics on $(\omega_{X/K}^{\otimes m})^{\mathrm{an}} \simeq \omega_{X^{\mathrm{an}}/K}^{\otimes m}$.

The proof of Theorem 6.2.3.2, as outlined in [Tem16, Remark 8.3.4(i)], very much requires the description of the weight function as the maximal lower-semicontinuous extension of its values on divisorial points as in Theorem 5.2.3.4; combining this with 6.2.2.4, it suffices to check equality on divisorial points. Further, the proof of Theorem 6.2.3.2 uses that X is smooth in order to reduce to the case $m = 1$. It is not clear whether the assumptions can be weakened to assume only that X is $\mathbf{Q}$-Gorenstein.

6.3. Gauss extensions

(6.3.0.1) Let $\mathcal{K}$ be a non-Archimedean field and pick $r \in (0,1)\setminus\sqrt{|\mathcal{K}^*|}$. Consider the field extension $\mathcal{K}_r$ of $\mathcal{K}$ that consists of those bi-infinite series $\sum_{j\in\mathbb{Z}} a_j \varpi^j$ with $a_j \in \mathcal{K}$ such that $\lim_{|j|\to+\infty} |a_j| r^j = 0$. The field $\mathcal{K}_r$ is complete with respect to the norm

$$|\sum_{j\in\mathbb{Z}} a_j \varpi^j|_r := \max_{j\in\mathbb{Z}} |a_j|_{\mathcal{K}} r^j.$$

Introduced in [Ber90, §2.1], the extension $\mathcal{K}_r/\mathcal{K}$ of non-Archimedean fields is often referred to as a *Gauss extension* in the literature. If $\mathcal{K}$ is trivially-valued, then $\mathcal{K}_r$ is a Laurent series field $\mathcal{K}((\varpi))$ over $\mathcal{K}$ with the ϖ-adic norm satisfying $|\varpi|_r = r$.

(6.3.0.2) Let Z be a $\mathcal{K}$-analytic space, and write $p_r\colon Z_r := Z \times_{\mathcal{K}} \mathcal{K}_r \to Z$ for the ground field extension. For any $z \in Z$, the fibre $p_r^{-1}(z) \subseteq Z_r$ is naturally identified with $\mathcal{M}(\mathcal{H}(z)\hat{\otimes}_{\mathcal{K}}\mathcal{K}_r)$. If the tensor product norm on $\mathcal{H}(z)\widehat{\otimes}_{\mathcal{K}}\mathcal{K}_r$ is multiplicative, then it defines the unique Shilov point $\sigma_r(z)$ of $\mathcal{M}(\mathcal{H}(z)\hat{\otimes}_{\mathcal{K}}\mathcal{K}_r)$. In fact, [Ber93, Lemma 2.2.5] shows that $\sigma_r(z)$ is well defined for any $z \in Z$, and it gives a continuous section $\sigma_r\colon Z \to Z_r$ of p_r. Further, if $z \in Z$, then the natural map $\mathcal{H}(z)\widehat{\otimes}_{\mathcal{K}}\mathcal{K}_r \to \mathcal{H}(\sigma_r(z))$ is an isometric isomorphism. See [Poi13, §3] or [BJ18b, §1.6] for additional discussion.

The proof of the trivially-valued analogue of Theorem 6.2.3.2 uses a Gauss extension to reduce to the discretely-valued setting. For this reason, we describe below the behaviour of Temkin's metric under Gauss extensions.

Lemma 6.3.0.3. *Let Z be a good $\mathcal{K}$-analytic space, $r \in (0,1)\setminus\sqrt{|\mathcal{K}^*|}$, and $\ell, m \in \mathbb{Z}_{>0}$. Then, for any $z \in Z$,*

$$\|\cdot\|_{\mathrm{Tem},z} = \|(p_r)_z^*(\cdot)\|_{\mathrm{Tem},\sigma_r(z)}$$

as seminorms on $(\Omega^\ell_{Z/\mathcal{K}})_z^{\otimes m}$, where $(p_r)_z^*\colon (\Omega^\ell_{Z/\mathcal{K}})_z^{\otimes m} \to (\Omega^\ell_{Z_r/\mathcal{K}_r})_{\sigma_r(z)}^{\otimes m}$ denotes the pullback map at z.

Proof. We may assume that $m = \ell = 1$. Consider the commutative diagram

$$\begin{array}{ccc}\widehat{\Omega}^1_{\mathcal{H}(z)/\mathcal{K}}\widehat{\otimes}_{\mathcal{H}(z)}(\mathcal{H}(z)\widehat{\otimes}_{\mathcal{K}}\mathcal{K}_r) & \longrightarrow & \widehat{\Omega}^1_{(\mathcal{H}(z)\widehat{\otimes}_{\mathcal{K}}\mathcal{K}_r)/\mathcal{K}_r} \\ \downarrow & & \downarrow \\ \widehat{\Omega}^1_{\mathcal{H}(z)/\mathcal{K}}\widehat{\otimes}_{\mathcal{H}(z)}\mathcal{H}(\sigma_r(z)) & \longrightarrow & \widehat{\Omega}^1_{\mathcal{H}(\sigma_r(z))/\mathcal{K}_r}\end{array}$$

Arguing as in [Tem16, Theorem 6.3.11], it suffices to show that the bottom horizontal map is an isometry. Indeed, the vertical maps are isometric isomorphisms because the natural map $\mathcal{H}(z)\widehat{\otimes}_{\mathcal{K}}\mathcal{K}_r \to \mathcal{H}(\sigma_r(z))$ is so, and the top horizontal map is an isometric isomorphism due to [Tem16, Lemma 4.2.6]. □

6.4. Divisorial points under Gauss extensions

Assume now that $\mathcal{K}$ is a trivially-valued field of characteristic zero, which we will denote by k for convenience. Let X be a normal k-variety, and let $x \in X^{\beth}$ be the divisorial point determined by the triple $(c, Y \xrightarrow{h} X, E)$. Without loss of generality, we may assume that X is quasi-projective, and Y and E are smooth. Following the notation of §6.3, for any $r \in (0,1)$, write $k_r = k((\varpi))$ for the Gauss extension of k with $|\varpi|_r = r$. For any such r, the point $\sigma_r(x) \in X^{\mathrm{an}}_{k_r}$ is divisorial by [Poi13, Lemme 4.6].

The goal of this section is to pick an $r \in (0,1)$ such that we can construct an explicit divisorial representation of $\sigma_r(x) \in X^{\mathrm{div}}_{k_r}$. This is done in three steps.

1. Consider the Laurent series extension $k((\varpi))$ of k and construct an explicit $k[[\varpi]]$-model of $X_{k((\varpi))}$, together with an irreducible component F of its special fibre, with the property that for any element $a \in k(X)$, the order of vanishing $\mathrm{ord}_E(a)$ of a along E coincides with the order of vanishing $\mathrm{ord}_F(a)$ of a along F.

2. Endow $k((\varpi))$ with the ϖ-adic norm $|\varpi|_r = r$ for a suitable choice of $r \in (0,1)$ so that the divisorial valuation $y_F \in X^{\mathrm{an}}_{k_r}$ determined by F satisfies $p_r(y_F) = x$.

3. Show that $\sigma_r(x) = y_F$.

The construction we will present is inspired by a similar phenomenon involving test configurations, as in [BHJ17; BJ18b]; this relationship is described further in Remark 6.4.3.2.

6.4.1. Step 1

(6.4.1.1) We may assume that h is projective by [KM08, Lemma 2.45], and so [Har77, II, Theorem 7.17] implies that there exists a coherent ideal $I \subseteq \mathcal{O}_X$ such that $Y = \mathrm{Bl}_I X$ and h is identified with the blow-up morphism. Let $\alpha \in \mathbb{Z}_{\geqslant 0}$ be the multiplicity of E in the exceptional locus of h.

Consider the fibre product $\mathcal{X} := X \times_k k[[\varpi]]$: this is the trivial $k[[\varpi]]$-model of $X_{k((\varpi))}$, and its special fibre $\mathcal{X}_0$ is naturally identified with X. We set $\mathcal{I} := (I, \varpi) \subseteq \mathcal{O}_{\mathcal{X}}$, which is a coherent ideal sheaf on $\mathcal{X}$ that is cosupported on $\mathcal{X}_0$. Let $\nu \colon \mathcal{Y} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $\mathcal{I}$. As the vanishing locus of $\mathcal{I}$ lies in $\mathcal{X}_0$, it follows that $\mathcal{Y}$ is again a model of

$X_{k((\varpi))}$. The strict transform of $\mathcal{X}_0$ via ν can be identified with Y by [Har77, II, Corollary 7.15]. Under this identification, $\mathcal{Y}_0$ contains a copy of the divisor E, which we write as $\widetilde{E}$. Further, let $\rho\colon \mathcal{Y} \dashrightarrow Y \times_k k[[\varpi]]$ be the birational map given by the composition of $\nu\colon \mathcal{Y} \to \mathcal{X}$ with the inverse of $Y \times_k k[[\varpi]] \to \mathcal{X}$. These objects are collected in the diagram below.

$$\begin{array}{ccccccccc}
Y \times_k k[[\varpi]] & \overset{\rho}{\dashleftarrow} & & & \mathcal{Y} = \mathrm{Bl}_{\mathcal{I}}\mathcal{X} & & \\
Y \times_k k[[\varpi]] \to Y = \mathrm{Bl}_I X \supseteq E & \to & \widetilde{E} \subseteq \mathcal{Y}_0 & \hookrightarrow & \mathcal{Y} = \mathrm{Bl}_{\mathcal{I}}\mathcal{X} & \hookleftarrow & X_{k((\varpi))} \\
\downarrow h & & \downarrow & & \downarrow \nu & & \downarrow \\
X & = & \mathcal{X}_0 & \hookrightarrow & \mathcal{X} = X \times_k k[[\varpi]] & \hookleftarrow & X_{k((\varpi))} = X \times_k k((\varpi)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(k) & = & \mathrm{Spec}(k) & \hookrightarrow & \mathrm{Spec}(k[[\varpi]]) & \hookleftarrow & \mathrm{Spec}(k((\varpi)))
\end{array}$$

(6.4.1.2) Write η (resp. $\widetilde{\eta}$) for the generic point of E (resp. $\widetilde{E}$) in Y (resp. $\mathcal{Y}$). We claim that the composition of ρ with the projection $Y \times_k k[[\varpi]] \to Y$ onto the special fibre sends $\widetilde{\eta}$ to η. Indeed, observe that the diagram

$$\begin{array}{ccc}
Y \times_k k[[\varpi]] & \overset{\rho}{\dashleftarrow} & \mathcal{Y} \\
\downarrow & & \uparrow \\
Y \supset E & \longrightarrow & \widetilde{E} \subset \mathcal{Y}_0
\end{array}$$

is commutative, and that the bottom arrow restricts to an isomorphism from $\widetilde{E}$ to E. Hence, it suffices to show that $\widetilde{E}$ is not contained in the indeterminacy locus of ρ, and we show this with the following local computation. Suppose $X = \mathrm{Spec}(A)$ and $I = (f_1, \dots, f_\ell)$, in which case an affine chart of Y is given by $U = \mathrm{Spec}(B)$, where

$$B = \frac{A[S_2, \dots, S_\ell]}{(f_1 S_i - f_i \colon i = 2, \dots, \ell)}.$$

There is a corresponding affine chart of $\mathcal{Y}$ given by $\mathcal{U} = \mathrm{Spec}(\mathcal{B})$, where

$$\mathcal{B} = \frac{\mathcal{A}[S_2, \dots, S_\ell, \widetilde{S}]}{(f_1\widetilde{S} - \varpi, f_1 S_i - f_i \colon i = 2, \dots, \ell)}$$

and $\mathcal{A} = A \otimes_k k[[\varpi]]$. The birational map $\rho : \mathcal{Y} \dashrightarrow Y \times_k k[[\varpi]]$ is given on these charts by the composition of the two top arrows in the diagram below:

$$\begin{array}{ccccc}
B \otimes_k k[[\varpi]] & \longrightarrow & (B \otimes_k k[[\varpi]])[\widetilde{S}] & \longrightarrow & \mathcal{B} = \frac{(B \otimes_k k[[\varpi]])[\widetilde{S}]}{(f_1\widetilde{S} - \varpi)} \\
\uparrow & & & & \downarrow \\
B & \twoheadleftarrow & & & \mathcal{B}/(\varpi)
\end{array}$$

Thus, the construction of the map $\mathcal{Y} \dashrightarrow Y \times_k k[[\varpi]] \to Y$ yields a ring morphism $\mathcal{O}_{Y,\eta} \to \mathcal{O}_{\mathcal{Y},\widetilde{\eta}}$, which sends $\widetilde{\eta}$ to η, as required.

(6.4.1.3) The irreducible subscheme $\widetilde{E}$ of $\mathcal{Y}$ is not a divisor (indeed, it has codimension 2 in $\mathcal{Y}$), so consider the blow-up $\mu\colon \mathcal{Z} \to \mathcal{Y}$ of $\widetilde{E}$. Note that $\mathcal{Z}$ is again a model of $X_{k((\varpi))}$. Write $F \subseteq \mathcal{Z}_0$ for the exceptional divisor of μ, which is irreducible since $\widetilde{E}$ is so.

We claim that $\operatorname{ord}_F(a) = \operatorname{ord}_E(a)$ for all $a \in k(X)$. It suffices to show the equality for $a \in \mathcal{O}_{Y,\eta}$. In the notation of 6.4.1.2, the exceptional divisors of h in the affine chart $U = \operatorname{Spec}(B)$ of Y is defined by f_1. Let g be a local equation of E at η. In the model $\mathcal{Y}$, $\widetilde{E}$ is locally cut out by g and the equations defining the strict transform of $\mathcal{X}_0$. Therefore, in $\mathcal{Z}$, g is a local equation of F at its generic point.

Write $a = ug^\lambda$ for $u \in \mathcal{O}_{Y,\eta}^\times$ and $\lambda \in \mathbb{Z}_{\geqslant 0}$ (so that $\operatorname{ord}_E(a) = \lambda$). The image of this expression for a via the map

$$\mathcal{O}_{Y,\eta} \to \mathcal{O}_{\mathcal{Y},\widetilde{\eta}} \to \mathcal{O}_{\mathcal{Z},F}$$

gives an expression for a in $\mathcal{O}_{\mathcal{Z},F}$. As u remains a unit in $\mathcal{O}_{\mathcal{Z},F}$, we deduce that

$$\operatorname{ord}_F(a) = \lambda \operatorname{ord}_F(g) = \lambda = \operatorname{ord}_E(a),$$

as required.

6.4.2. Step 2

We will find $r \in (0,1)$ such that the divisorial valuation $y_F \in X_{k_r}^{\mathrm{an}}$ determined by $(\mathcal{Z}, F)$ satisfies $p_r(y_F) = x$. To that end, we first compute the multiplicity of F in the special fibre of $\mathcal{Z}$. Working in an affine chart of the blowup as in 6.4.1.2, the composition $\mathcal{Z} \to \mathcal{Y} \to \mathcal{X} \to \operatorname{Spec}(k[[\varpi]])$ can locally be written as

$$k[[\varpi]] \to \mathcal{A} = A \otimes_k k[[\varpi]] \to \mathcal{B} = \frac{\mathcal{A}[S_2,\ldots,S_\ell,\widetilde{S}]}{(f_1\widetilde{S}-\varpi, f_1S_i-f_i\colon i=2,\ldots,\ell)} \to \frac{\mathcal{A}[S_2,\ldots,S_\ell,\widetilde{S},Q]}{(\widetilde{S}-gQ, f_1\widetilde{S}-\varpi, f_1S_i-f_i\colon i=2,\ldots,\ell)}.$$

In particular, we can write $\varpi = f_1 g Q$ at the generic point of F. As Q is a unit in $\mathcal{O}_{\mathcal{Z},F}$, we conclude that

$$\operatorname{ord}_F(\varpi) = \operatorname{ord}_F(f_1) + \operatorname{ord}_F(g) = \operatorname{ord}_E(f_1) + \operatorname{ord}_F(g) = \alpha + 1.$$

Set $r = e^{-c(\alpha+1)}$, where recall that x is determined by the triple $(c, Y \to X, E)$. For any $a \in k(X)$, we have that

$$|a(x)| = e^{-c\operatorname{ord}_E(a)} = r^{\frac{\operatorname{ord}_E(a)}{\alpha+1}} = r^{\frac{\operatorname{ord}_F(a)}{\operatorname{ord}_F(\varpi)}} = r^{v_{y_F}(a)} = |a(y_F)|.$$

That is, $p_r(y_F) = x$.

6.4.3. Step 3

It remains to show that $y_F = \sigma_r(x)$. This is done by appealing the following lemma.

Lemma 6.4.3.1. *Let $\mathcal{X}^{(\ell)} \to \mathcal{X}^{(\ell-1)} \to \ldots \to \mathcal{X}^{(1)} \to \mathcal{X}$ be a sequence of models of $X_{k((\varpi))}$, where each morphism $\mathcal{X}^{(i+1)} \to \mathcal{X}^{(i)}$ is the blow-up of a $k^\times$-invariant ideal (in the sense of [BJ18b, §1.6]), and $\mathcal{X}$ is the trivial model of $X_{k((\varpi))}$. If k is an infinite field and $r \in (0,1)$, then any divisorial point of $X_{k_r}^{\mathrm{an}}$ determined by an irreducible component of $\mathcal{X}_0^{(\ell)}$ lies in the image of $\sigma_r \colon X^{\mathrm{an}} \to X_{k_r}^{\mathrm{an}}$.*

Proof. This follows from [BJ18b, Proposition 1.6]; see also [BHJ17, Lemma 4.5]. □

The point $y_F \in X_{k_r}^{\mathrm{an}}$ satisfies the hypotheses of Lemma 6.4.3.1 by construction, so it lies in the image of σ_r. As σ_r is a section of the projection p_r and $p_r(y_F) = x$ by Step 2, we conclude that $\sigma_r(x) = y_F$.

Remark 6.4.3.2. The construction of the point y_F is inspired by one in the proof of [BHJ17, Proposition 4.11]. There, for any $r \in (0,1)$, the authors view the Gauss extension as a continuous map $\sigma_r \colon X^{\beth} \to (X \times_k \mathbf{A}_k^1)^{\beth}$, and one can show the following: if $x \in X^{\beth}$ is the divisorial point given by the triple $(-\log(r), Y \to X, E)$, then $\sigma_r(x)$ is a monomial valuation on the birational model $Y \times_k \mathbf{A}_k^1 \to X \times_k \mathbf{A}_k^1$ in the snc divisor $E \times_k \mathbf{A}_k^1 + Y \times_k \{0\}$.

The construction of $y_F = \sigma_r(x)$ can be rephrased in the above language. We first consider the blow-up ν of $X \times_k \mathbf{A}_k^1$ at $\overline{\{c_X(x)\}} \times \{0\}$, and then the blow-up μ of the intersection of $\mathrm{Exc}(\nu)$ and the strict transform of $X \times_k \{0\}$ via ν. The valuation $\sigma_r(x)$ is realized as an order of vanishing along $\mathrm{Exc}(\mu)$. The advantage of realizing $\sigma_r(x)$ in this manner is that the blow-ups occur only above the origin of $\mathbf{A}_k^1$.

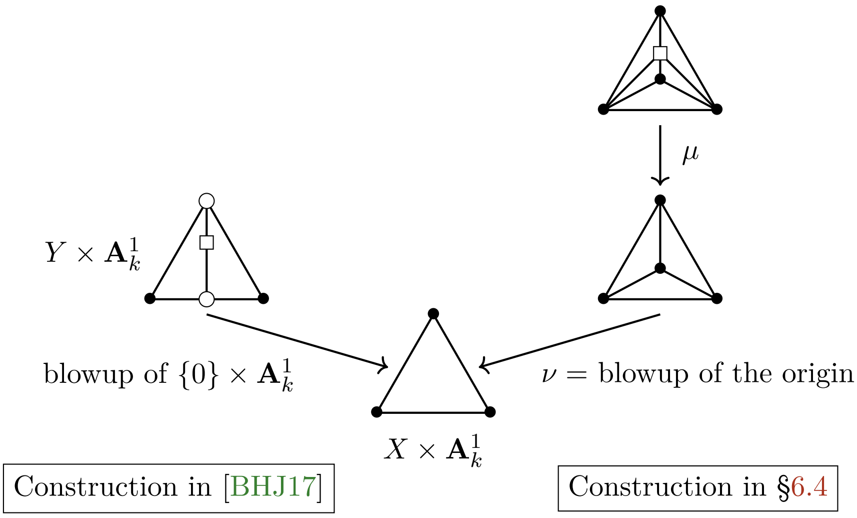


Figure 6.4.3.1: We illustrate the two approaches to the construction of $\sigma_r(x)$ in Remark 6.4.3.2 with a toric example. Consider $X = \mathbf{A}_k^2$ and the blow-up $Y \to X$ at the origin with exceptional divisor $E \subseteq Y$. Let $x \in X^{\beth}$ be the divisorial point determined by the triple $(-\log(r), Y \to X, E)$. In the above figure, the triangles represent a slice of the fans of the various toric blow-ups that occur in the two constructions. Following [BHJ17], $\sigma_r(x)$ is a monomial valuation in the divisors corresponding to the white nodes, which we picture as a square on the segment joining them. On the other side, according to §6.4, we extract a divisor corresponding to $\sigma_r(x)$ with a sequence of two blow-ups, and we mark this divisor with a square.

6.5. Proof of Theorem 6.1.1.1

The goal of this section is to prove Theorem 6.1.1.1, which is the trivially-valued analogue of Theorem 6.2.3.2. This justifies the definition of the weight metric in the trivially-setting from (5.2.4). The proof of Theorem 6.1.1.1 proceeds by reduction to Theorem 6.2.3.2.

Throughout this section, let k be a trivially-valued field of characteristic zero, and let X be a normal, $\mathbb{Q}$-Gorenstein k-variety. Fix $m \in \mathbb{Z}_{>0}$ such that $\omega_{X/k}^{\otimes m}$ is a line bundle. For $x \in X^{\beth}$, recall that we write

$$\|\cdot\|_{\mathrm{wt}_{\mathrm{triv}},x} = e^{-\mathrm{wt}_{\mathrm{triv}}(\cdot,x)}$$

for the multiplicative form of the weight metric on the stalk $(\omega_{X/k}^{\otimes m})^{\beth}_x$.

Proposition 6.5.0.1. *Let $x \in X^{\beth}$ be the divisorial point determined by the triple $(c, Y \xrightarrow{h} X, E)$. With notation as in § 6.4, for any rational section s of $\omega_{X/k}^{\otimes m}$, we have*

$$\|s\|_{\mathrm{wt}_{\mathrm{triv}},x} = r^{-m}\|q_r^* s\|_{\mathrm{wt}_{\mathrm{disc}},\sigma_r(x)}, \tag{6.5.0.2}$$

where $q_r \colon X_{k_r} \to X$ denotes the (algebraic) ground field extension (i.e. $p_r = q_r^{\mathrm{an}}$).

Proof. Set $\xi = c_X(x)$. Let s be a $\mathcal{O}_{X,\xi}$-module generator of the stalk $\omega_{X/k,\xi}^{\otimes m}$. It suffices to show Eq. (6.5.0.2) for s; indeed, any local section at ξ can be written as fs for some $f \in \mathcal{O}_{X,\xi}$, in which case both sides of Eq. (6.5.0.2) are multiplied by $|f(x)|$. By working locally around ξ, we may assume that $X = \mathrm{Spec}(A)$ is affine and s is globally-defined. In the notation as in Section 6.4.2, we have

$$\|s\|_{\mathrm{wt}_{\mathrm{triv}},x} = e^{-cm(1+(\ell-1)\alpha)}, \tag{6.5.0.3}$$

since $\mathrm{ord}_E(K_{Y/X}) = \mathrm{ord}_E(f_1^{\ell-1}) = (\ell-1)\alpha$. By [Liu02, Corollary 6.4.14], the stalk of the relative canonical sheaf $\omega_{\mathcal{Z}/\mathcal{X}}$ at the generic point of F can be viewed as the $\mathcal{O}_{\mathcal{Z},F}$-submodule of the function field of $X_{k((\varpi))}$; further, it is generated by $(gf_1^{\ell})^{-1}$. The m-th power of these generators multiplied by $q_r^* s$ thus gives a $\mathcal{O}_{\mathcal{Z},F}$-module generator of the stalk $(\omega_{\mathcal{Z}/k[[\varpi]]}^{\otimes m})_F$. It follows that

$$\|q_r^* s\|_{\mathrm{wt}_{\mathrm{disc}},\sigma_r(x)} = \|q_r^* s\|_{\mathrm{wt}_{\mathrm{disc}},y_F} = |(gf_1^{\ell})^m(y_F)| \cdot |g(y_F)|^m = |g(y_F)|^{m(2+l\alpha)} = e^{-cm(2+\ell\alpha)}. \tag{6.5.0.4}$$

Thus, combining Eq. (6.5.0.3) and Eq. (6.5.0.4), it follows that

$$r^{-m}\|q_r^* s\|_{\mathrm{wt}_{\mathrm{disc}},\sigma_r(x)} = e^{cm(\alpha+1)}e^{-cm(2+\ell\alpha)} = e^{-cm(1+(\ell-1)\alpha)} = \|s\|_{\mathrm{wt}_{\mathrm{triv}},x},$$

as required. □

Now, Proposition 6.5.0.1 is the key tool to prove the trivially-valued analogue of Theorem 6.2.3.2, which is stated as Theorem 6.1.1.1 in the introduction.

Theorem 6.5.0.5. *If X is a smooth k-variety, then $\|\cdot\|_{\mathrm{wt}_{\mathrm{triv}}} = \|\cdot\|_{\mathrm{Tem}}$ as metrics on $(\omega_{X/k}^{\otimes m})^{\beth} \simeq \omega_{X^{\beth}/k}^{\otimes m}$.*

Proof. By the maximality property of $A_{(X,\emptyset)}$ as in 5.2.4.4 and 6.2.2.4, it suffices to show the equality on the divisorial points of $X^{\beth}$. Fix $x \in X^{\mathrm{div}} \cap X^{\beth}$ and let $r' \in (0,1)$ be as in Section 6.4.2. It suffices to check equality on elements of the stalk $(\omega_{X/k}^{\otimes m})_{\ker(x)}$, i.e. on a rational section s of $\omega_{X/k}^{\otimes m}$. Now, applying Proposition 6.5.0.1, Theorem 6.2.3.2, and Lemma 6.3.0.3 we find that

$$\|s\|_{\mathrm{wt}_{\mathrm{triv}},x} = (r')^{-m}\|q_{r'}^{*}s\|_{\mathrm{wt}_{\mathrm{disc}},\sigma_{r'}(x)} = (r')^{-m}\left((r')^{m}\|q_{r'}^{*}s\|_{\mathrm{Tem},\sigma_{r'}(x)}\right) = \|s\|_{\mathrm{Tem},x},$$

which completes the proof. □

Remark 6.5.0.6. Let k, X, m be as above. To any Cartier divisor D on X, we can associate a canonical *singular metric* $\|\cdot\|_D$ on the line bundle $\mathcal{O}_X(D)^{\mathrm{an}}$ in the following manner: the divisor D induces an embedding ι_D of $\mathcal{O}_X(D)$ into the constant sheaf $k(X)$, and for any $x \in X^{\mathrm{an}}$ and $f \in \mathcal{O}_X(D)_{\ker(x)}$, set

$$\|f\|_{D,x} \coloneqq |\iota_D(f)(x)|.$$

Now, Temkin's metric $\|\cdot\|_{\mathrm{Tem}}$ and $\|\cdot\|_{D_{\mathrm{red}}}$ induce a tensor product metric on $(\omega_{(X,D_{\mathrm{red}})}^{\otimes m})^{\beth}$. By Theorem 6.5.0.5, this tensor product metric coincides with the weight metric $\mathrm{wt}_{\mathrm{triv}}$. It would be interesting if this metric $(\omega_{(X,D_{\mathrm{red}})}^{\otimes m})^{\beth}$ arises in a similar fashion as in [Tem16], or more generally in the singular case.

7

Evidence for the geometric P=W conjecture

7.1. Introduction

In this final chapter the construction of the essential skeleton of Chapters 3 and 5 is applied to compute the dual boundary complexes of certain character varieties.

Let G be either GL_n or SL_n. We consider the G-character variety M_G associated to a Riemann surface C of genus one, that is the GIT quotient of the fundamental group of C

$$\{(A, B) \in G \times G \colon AB = BA\} /\!\!/ G,$$

where G acts by conjugation on each factor. For example, when $G = \mathrm{GL}_n$, M_{GL_n} is isomorphic to the n-fold symmetric product $(\mathbb{C}^* \times \mathbb{C}^*)^{(n)}$ of the torus $\mathbb{C}^* \times \mathbb{C}^*$.

It is a fundamental result in non-abelian Hodge theory that M_G is isomorphic (as real analytic space) to a moduli space of Higgs bundles, called the Dolbeault moduli space M_{Dol}. A distinctive feature of the space M_{Dol} is that it comes equipped with a filtration, called Hitchin filtration, to an affine space. The geometric P=W conjecture relates the properties of a compactification of M_G with a geometric interpretation of the Hitchin fibration, when seen on the side of the character variety M_G. See Section 1.4 for a brief overview of this topic.

The geometric P=W conjecture predicts the existence of a nice compactification of M_G such that the dual boundary complex is well defined and, in particular, is homotopy equivalent to a sphere. Since M_G is a singular affine variety, it does not admit an snc compactification. Our solution is to consider a logCY dlt compactification, as one can associate a dual boundary complex $\mathcal{D}(\partial M_G)$ to it, whose homeomorphism type is an invariant of M_G.

We reinterpret $\mathcal{D}(\partial M_G)$ as the level set of a suitable weight function inside a Berkovich space. By way of this non-archimedean description, we provide the first non-trivial evidence for the geometric $P = W$ conjecture in the compact case.

Theorem 7.1.0.1 (§ 7.2-7.3). *The dual boundary complex $\mathcal{D}(\partial M_G)$ of a dlt logCY compactification of M_G has the homeomorphism type of $\mathbb{S}^{2n-1}$ if $G = \mathrm{GL}_n$, and of $\mathbb{S}^{2n-3}$ if $G = \mathrm{SL}_n$.*

7.2. Dual boundary complex of GL_n-character varieties of a genus one surface

The goal of this section is the study of the dual boundary complex $\mathcal{D}(\partial M_{\mathrm{GL}_n})$ of the GL_n-character variety M_{GL_n} associated to a Riemann surface of genus one. In particular, we prove of the following result.

Theorem 7.2.0.1. *The dual boundary complex $\mathcal{D}(\partial M_{\mathrm{GL}_n})$ of a dlt log Calabi–Yau compactification of M_{GL_n} has the homeomorphism type of $\mathbb{S}^{2n-1}$.*

We recall that M_{GL_n} is the n-fold symmetric product of the two-dimensional algebraic torus $\mathbb{C}^* \times \mathbb{C}^*$; see e.g. [FT16, Corollary 5.6]. Symmetric products of toric surfaces are natural candidates for compactifications of M_{GL_n}. However, these compactifications are not dlt, although log canonical and log Calabi–Yau (see Section 7.2.2). In Section 7.2.3 we adapt the strategy of [KX16] to prove Theorem 7.2.0.1 for $n = 2$. For higher n, this approach is not sufficient and we instead employ techniques from Berkovich geometry (Sections 7.2.4 and 7.2.6).

It is worth remarking that a related conjecture, known as cohomological $P = W$ conjecture, holds for a crepant resolution of M_{GL_n} thanks to [dCHM13]. For more details about this cohomological version, we refer the interested reader to [dCHM12] and to the excellent survey [Mig17].

Throughout this section, all varieties are defined over $\mathbb{C}$, which is thought of as a non-Archimedean field equipped with the trivial norm.

7.2.1. Dlt modifications and dual complexes

(7.2.1.1) Given a log canonical (lc) pair (X, Δ) (see Definition 5.2.4.2), a *lc centre* of the pair is the centre of a divisorial valuation $x \in X^{\beth}$ with $A_{(X,\Delta)}(x) = 0$. The *snc locus* X^{snc} is the largest open subset in X such that the pair (X, Δ) restricts to an snc pair. The pair (X, Δ) is said to be *divisorial log terminal* (dlt) if none of its lc centres are contained in $X \setminus X^{\mathrm{snc}}$; see [KM08, Definition 2.37] for more details.

(7.2.1.2) There are several advantages to working with dlt pairs over snc pairs. Most notably, we use the fact that any lc pair (X, Δ) is crepant birational to a (non-unique) dlt pair $(X^{\mathrm{dlt}}, \Delta^{\mathrm{dlt}})$, while the corresponding statement fails in general for snc pairs. Recall that two pairs (X, Δ_X) and (Y, Δ_Y) are *crepant birational* if X and Y are birational and $A_{(X,\Delta_X)} = A_{(Y,\Delta_Y)}$ as functions on $X^{\mathrm{bir}} = Y^{\mathrm{bir}}$. This fact is a consequence of the existence of *dlt modifications*, as in [Kol13, Corollary 1.36], which asserts that there exists a proper birational morphism $g : X^{\mathrm{dlt}} \to X$ with exceptional divisors $\{E_i\}_{i\in I}$ such that

1. (dlt) the pair $(X^{\mathrm{dlt}}, \Delta^{\mathrm{dlt}} := g_*^{-1}\Delta + \sum_{i\in I} E_i)$ is dlt, where $g_*^{-1}\Delta$ is the strict transform of Δ via g;

2. (crepant) $K_{X^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}} \sim_{\mathbb{Q}} g^*(K_X + \Delta)$.

(7.2.1.3) It is always possible to construct a dual intersection complex for a dlt pair (X, Δ) by following the same prescriptions as for snc pairs (while this is not in general possible for lc pairs). In fact, this coincides with the dual complex of the snc pair $(X^{\mathrm{snc}}, \Delta^{=1}|_{X^{\mathrm{snc}}})$ by [dFKX17, §2]. The dual complex of a lc pair (X, Δ) can be defined as the homeomorphism class of the dual complex of any dlt modification of (X, Δ), and it is denoted by $\mathcal{DMR}(X, \Delta)$; the notation is an abbreviation for Dual complex of a Minimal dlt partial Resolution. The homeomorphism class $\mathcal{DMR}(X, \Delta)$ is well defined, as it is independent of the choice of a dlt modification by [dFKX17, Definition 15].

7.2.2. Hilbert scheme of n points of a toric surface

(7.2.2.1) Let Z be a smooth, projective toric surface, and let Δ be its toric boundary. Let Σ_Z be a toric fan for Z, write $|\Sigma_Z|$ for its support, and $|\Sigma_Z|^* := |\Sigma_Z| \backslash \{0\}$. Note that:

1. $Z^+ := (Z, \Delta)$ is an snc logCY pair;

2. $Z \setminus \Delta \simeq \mathbb{C}^* \times \mathbb{C}^*$;

3. $\mathcal{D}(Z^+) \simeq \mathrm{Sk}^{\mathrm{ess}}(Z, \Delta)^*/\mathbb{R}_+^* = \mathrm{Sk}(Z^+)^*/\mathbb{R}_+^* = |\Sigma_Z|^*/\mathbb{R}_+^* \simeq (\mathbb{R}^2 \backslash \{0\})/\mathbb{R}_+^* \simeq \mathbb{S}^1$.

Denote by $Z^{[n]}$ the Hilbert scheme of n points of Z; see [Bea83, §6] for an overview of the construction. Recall that the Hilbert scheme of n points of a projective surface is a crepant resolution of its n-fold symmetric product. In a diagram, we have

$$\begin{array}{ccc} & & Z^n := \underbrace{Z \times \ldots \times Z}_{n-\text{times}} \\ & & \Big\downarrow q \\ Z^{[n]} & \xrightarrow{\rho_{\mathrm{HC}}} & Z^{(n)} := Z^n/\mathfrak{S}_n, \end{array}$$

where the crepant birational map ρ_{HC} is the Hilbert–Chow morphism, and q is the quotient of the product Z^n by the action of the symmetric group $\mathfrak{S}_n$ of degree n, which acts by permuting the factors of Z^n. This gives rise to the following diagram of lc logCY pairs:

$$\begin{array}{ccc} & & (Z^n, \Delta^n := \mathrm{pr}_1^*\Delta + \ldots + \mathrm{pr}_n^*\Delta) . \\ & & \big\downarrow q \\ (Z^{[n]}, \Delta^{[n]} := {\rho_{\mathrm{HC}}}^*\Delta^{(n)}) & \xrightarrow{\rho_{\mathrm{HC}}} & (Z^{(n)}, \Delta^{(n)} := q_*\Delta^n) \end{array}$$

The variety $Z^{(n)}$ is a compactification of $M_{\mathrm{GL}_n} \simeq (\mathbb{C}^* \times \mathbb{C}^*)^{(n)}$, as $\mathbb{C}^* \times \mathbb{C}^* \simeq Z \setminus \Delta \subseteq Z$. Further, since the lc pairs $(Z^{[n]}, \Delta^{[n]})$ and $(Z^{(n)}, \Delta^{(n)})$ are crepant birational, it follows from [dFKX17, Proposition 11] that

$$\mathcal{D}(\partial M_{\mathrm{GL}_n}) \simeq \mathcal{DMR}(Z^{(n)}, \Delta^{(n)}) \simeq \mathcal{DMR}(Z^{[n]}, \Delta^{[n]}) \simeq \mathcal{D}(\partial(\mathbb{C}^* \times \mathbb{C}^*)^{[n]}). \qquad (7.2.2.2)$$

Remark 7.2.2.3. Unfortunately, the pair $(Z^{(n)}, \Delta^{(n)} = Z^{(n)} \setminus (\mathbb{C}^* \times \mathbb{C}^*)^{(n)})$ fails to be dlt (or qdlt). In light of (7.2.2.2), one could eventually consider the Hilbert scheme $Z^{[n]}$, but even in that case the compactification is not dlt, as we show in the following. For simplicity, in this section we will focus our attention on the case $n = 2$.

Let $(\mathbb{C}^2_{x_1,x_2}, (x_1x_2 = 0))$ be a local toric chart for (Z, Δ). As above, consider the product pair

$$(\mathbb{C}^2_{x_1,x_2} \times \mathbb{C}^2_{y_1,y_2}, (x_1x_2y_1y_2 = 0)).$$

There is an involution which swaps x_1 and x_2 with y_1 and y_2 respectively. Via the change of coordinates $(u, v, r, s) = (x_1+y_1, x_2+y_2, x_1-y_1, x_2-y_2)$, the involution sends (u, v, r, s) to $(u, v, -r, -s)$. Hence, the previous diagram has the following form:

$$\begin{array}{ccc} & & \mathbb{C}^2_{u,v} \times \mathbb{C}^2_{r,s} \\ & & \big\downarrow q \\ \mathbb{C}^2_{u,v} \times \mathrm{Bl}_0(\mathbb{C}^2_{r,s}/(\mathbb{Z}/2\mathbb{Z})) & \xrightarrow{\rho_{HC}} & \mathbb{C}^2_{u,v} \times \mathbb{C}^2_{r,s}/(\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{C}^2_{u,v} \times \mathrm{Spec}\left(\frac{\mathbb{C}[x,y,z]}{(xz-y^2)}\right), \end{array}$$

where the maps q is given in coordinates by

$$q : (u, v, r, s) \mapsto (u, v, r^2, rs, s^2).$$

Consider the chart of the blowup $\mathrm{Bl}_0(\mathbb{C}^2_{r,s}/(\mathbb{Z}/2\mathbb{Z})) \subseteq \mathbb{C}^3_{x,y,z} \times \mathbb{P}^2_{[X:Y:Z]}$ given by

$$\begin{aligned} \mathbb{C}^2_{x',y'} &\hookrightarrow \mathrm{Bl}_0(\mathbb{C}^2_{r,s}/(\mathbb{Z}/2\mathbb{Z})) \subseteq \mathbb{C}^3_{x,y,z} \times \mathbb{P}^2_{[X:Y:Z]} \\ (x', y') &\mapsto ((x', x'y', x'y'^2), [1 : y' : y'^2]). \end{aligned}$$

In these local coordinates, the boundaries are given by the following equations:

1. $\Delta^2 = (x_1x_2y_1y_2 = 0) = ((u^2 - r^2)(v^2 - s^2) = 0)$;

2. $\Delta^{(2)} = ((u^2 - x)(v^2 - z) = 0)$;

3. $\Delta^{[2]} \overset{\text{loc}}{=} ((u^2 - x')(v^2 - x'y'^2) = 0)$.

One of the components of $\Delta^{(2)}$ and $\Delta^{[2]}$ is non-normal, and so none of the pairs $(X^{(2)}, \Delta^{(2)})$ and $(X^{[2]}, \Delta^{[2]})$ can be dlt (or qdlt) by [KM08, Corollary 5.52].

7.2.3. A proof of Theorem 7.2.0.1 for $n = 2$.

(7.2.3.1) The $n = 2$ case of Theorem 7.2.0.1 can be deduced from results in [KX16] and the Poincaré conjecture, as it is explained below. In the following lemma, the fundamental group of a variety refers to the topological fundamental group of the associated complex-analytic variety.

Lemma 7.2.3.2. *For $n \geq 2$, the dual boundary complex $\mathcal{D}(\partial(\mathbb{C}^* \times \mathbb{C}^*)^{[n]})$ is simply connected, i.e.*

$$\pi_1(\mathcal{D}(\partial(\mathbb{C}^* \times \mathbb{C}^*)^{[n]})) = 1.$$

Proof. Consider a dlt modification $h : (Z^{[n],\mathrm{dlt}}, \Delta^{[n],\mathrm{dlt}}) \to (Z^{[n]}, \Delta^{[n]})$. By [KX16, Theorem 36], there is a surjection of fundamental groups

$$\pi_1((Z^{[n],\mathrm{dlt}})^{\mathrm{sm}}) \twoheadrightarrow \pi_1(\mathcal{DMR}(Z^{[n]}, \Delta^{[n]})),$$

where the superscript 'sm' denotes the restriction to the smooth locus. Since h is a birational contraction (that is, the exceptional locus of the inverse map h^{-1} has complex codimension $\geqslant 2$), there exists a surjection

$$\pi_1((Z^{[n]})^{\mathrm{sm}}) \twoheadrightarrow \pi_1((Z^{[n],\mathrm{dlt}})^{\mathrm{sm}})$$

by [KX16, Lemma 41]. However, $Z^{[n]}$ is smooth and rationally connected, and hence $\pi_1(Z^{[n]}) = 1$; see e.g. [Deb01, Corollary 4.18.(c)]. It follows that $\mathcal{DMR}(Z^{[n]}, \Delta^{[n]}) \simeq \mathcal{D}(\partial(\mathbb{C}^* \times \mathbb{C}^*)^{[n]})$ is simply connected. □

Proof of Theorem 7.2.0.1 for $n = 2$. By [KX16], $\mathcal{DMR}(Z^{[2]}, \Delta^{[2]})$ is a real 3-manifold with the rational homology of the 3-sphere $\mathbb{S}^3$. By Lemma 7.2.3.2, it is also simply connected, and hence the Poincaré conjecture implies that it is homeomorphic to the 3-sphere $\mathbb{S}^3$. By (7.2.2.2), the same holds for $\mathcal{D}(\partial M_{\mathrm{GL}_2})$. □

(7.2.3.3) The methods of the proof of Theorem 7.2.0.1 for the $n = 2$ case are not sufficient to prove the theorem in the general case. The problem is that they do not provide a control on the torsion of $H_i(\mathcal{D}(\partial(\mathbb{C}^* \times \mathbb{C}^*)^{(n)}), \mathbb{Z})$. In the sequel, we avoid this issue by constructing an explicit homeomorphism between $\mathcal{DMR}(Z^{(n)}, \Delta^{(n)})$ and the sphere $\mathbb{S}^{2n-1}$, which is a non-Archimedean avatar of the geometric construction of the Hilbert scheme via products and finite quotients.

7.2.4. The essential skeleton of a logCY pair

(7.2.4.1) The construction of the dual complex of a lc pair relies on the intersection poset of the strata of a dlt modification. It is convenient to think of these strata as the associated monomial valuations, suitably normalized, as defined in Proposition 5.3.1.2. The advantage of this viewpoint is that these valuations are independent of the choice of dlt modification, and they embed in a common ambient space, namely the $\beth$-analytification, with image equal to the essential skeleton of the pair (see Lemma 7.2.4.6).

As defined in Section 5.3.7, the essential skeleton of a pair (X, Δ_X) is given by the union of the minimality loci of a collection of weight functions in the $\beth$-analytification. In the proper logCY case, the weight functions associated to regular Δ_X-pluricanonical forms coincide with the log discrepancy $A_{(X,\Delta_X)}$, as we show in Proposition 7.2.4.2.

Proposition 7.2.4.2. *Let (X, Δ_X) be a proper log-regular logCY pair. If η is a regular section of $\omega^{\otimes m}_{(X,\Delta_X)}$, then $\mathrm{wt}_\eta = mA_{(X,\Delta_X)}$ as functions on $X^\beth$. Moreover, if $(X, \Delta_X^{=1})$ is log-regular, then*

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) = \mathrm{Sk}(X, \Delta_X^{=1}). \tag{7.2.4.3}$$

Proof. By properness, there exists a unique regular section η of $\omega^{\otimes m}_{(X,\Delta_X)}$ up to scaling, for $m \in \mathbb{Z}_{>0}$ sufficiently divisible. As a result, the weight functions are independent of the choice of a Δ_X-logarithmic m-pluricanonical section, and so $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) = \mathrm{Sk}(X, \Delta_X, \eta)$.

We now proceed as in the proof of Proposition 5.2.5.2 and Corollary 5.2.5.3. Let δ be a local generator of $\omega^{\otimes m}_{(X,\Delta_{X,\mathrm{red}})}$ and f be a local regular function such that $\eta = f\delta$. As η is a global generator of $\omega^{\otimes m}_{(X,\Delta_X)}$, then f provides a local equation for $m(\Delta_{X,\mathrm{red}} - \Delta_X)$. Hence, from Proposition 5.2.5.2, we get that

$$\mathrm{wt}_\eta(x) = A_{(X,\Delta_{X,\mathrm{red}}-(\Delta_{X,\mathrm{red}}-\Delta_X))} = A_{(X,\Delta_X)}(x).$$

We conclude that

$$\begin{aligned}\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) = \mathrm{Sk}(X, \Delta_X, \eta) &= \{x \in X^{\mathrm{bir}} : A_{(X,\Delta_X)}(x) = 0\} \\ &= \{x \in X^{\mathrm{bir}} : A_{(X,\Delta_X^{=1})}(x) = 0\} = \mathrm{Sk}(X, \Delta_X^{=1}),\end{aligned}$$

where the intermediate equality follows from the fact that the log centres of the pairs $(X, \Delta_X^{=1})$ and (X, Δ_X) coincide. Indeed, one can first assume that (X, Δ_X) is an snc pair by passing to a log resolution that, locally at the generic point of the strata of $\Delta_X^{=1}$, is given by a sequence of blow-ups induced by subdivisions of the corresponding Kato fan. One then applies [Kol13, Proposition 2.7]. □

In fact, if there exists a boundary $\Delta \leqslant \Delta_X$ such that (X, Δ_X) is a log-regular pair, then one can define a skeleton of (X, Δ) as in Section 5.3 by throwing away suitable faces of $\mathrm{Sk}(X, \Delta_X)$. With this definition, the equality Eq. (7.2.4.3) holds without the additional hypothesis that $(X, \Delta_X^{=1})$ is log-regular. Nonetheless, Proposition 7.2.4.2 suggests the following generalization of the definition of the essential skeleton to a proper lc logCY sub-pair, which agrees with the skeleton of [BJ17, Proposition 5.6] in the dlt case.

Definition 7.2.4.4. Let (X, Δ_X) be a proper lc logCY sub-pair. The *essential skeleton* of (X, Δ_X) is

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) := \{x \in X^{\mathrm{bir}} \cap X^{\beth} : A_{(X,\Delta_X)}(x) = 0\}.$$

Definition 7.2.4.5. As in Definition 5.3.5.3, the *link* of the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$ is the quotient of the punctured skeleton $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^* := \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) \setminus \{v_0\}$ via rescaling, denoted by

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^*/\mathbb{R}^*_+.$$

Lemma 7.2.4.6. *If the proper lc logCY sub-pairs (X, Δ_X) and (Y, Δ_Y) are crepant birational, then*

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) = \mathrm{Sk}^{\mathrm{ess}}(Y, \Delta_Y), \tag{7.2.4.7}$$

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^*/\mathbb{R}^*_+ = \mathrm{Sk}^{\mathrm{ess}}(Y, \Delta_Y)^*/\mathbb{R}^*_+, \tag{7.2.4.8}$$

$$\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^*/\mathbb{R}^*_+ \simeq \mathcal{DMR}(X, \Delta_X). \tag{7.2.4.9}$$

Proof. The equalities Eq. (7.2.4.7) and Eq. (7.2.4.8) follow from the fact that $A_{(X,\Delta_X)} = A_{(Y,\Delta_Y)}$ on $X^{\mathrm{bir}} = Y^{\mathrm{bir}}$. The equality Eq. (7.2.4.9) is a consequence of the existence of a (crepant) dlt modification (as in 7.2.1.1), and Proposition 5.3.5.4, once we restrict to the snc locus of the dlt modification. □

Remark 7.2.4.10. Given a lc logCY pair (X, Δ_X), $X^{\beth}$ admits a strong deformation retraction onto the closure of $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$. Indeed, $X^{\beth}$ retracts onto the closure of the skeleton of a pair $(Y, f^{-1}_*(\Delta_X) + \sum_i E_i)$ by [Thu07, Theorem 3.26], where $f : Y \to X$ is an snc modification of (X, Δ_X) and E_i are the exceptional divisors of f. Then $\mathrm{Sk}(Y, f^{-1}_*(\Delta_X) + \sum_i E_i)$ retracts onto $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$ by [dFKX17, Theorem 28.(2)]. While this is not needed in the sequel, the existence of this retraction affirms the use of the terminology 'skeleton' used in Definition 7.2.4.4.

Lemma 7.2.4.11. *Let (X, Δ_X) be a proper lc logCY pair. Let G be a finite group acting on X so that the quotient map $q : X \to X/G$ is quasi-étale, i.e. étale away from a subscheme of codimension $\geqslant 2$. Then*

$$\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G} := q_*\Delta_X) = q^{\beth}(\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)) \simeq \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)/G. \tag{7.2.4.12}$$

In particular,

$$\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})^*/\mathbb{R}^*_+ \simeq \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^*/(\mathbb{R}^*_+ \times G). \tag{7.2.4.13}$$

Proof. Observe that the skeleton $\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})$ is well defined since the pair $(X/G, \Delta_{X/G})$ is lc logCY. Indeed, $q^*(K_{X/G} + \Delta_{X/G}) = K_X + \Delta \sim_{\mathbb{Q}} 0$, because q is quasi-étale. In particular, [KM08, Proposition 5.20] implies that the pair $(X/G, \Delta_{X/G})$ is lc as (X, Δ_X) is so.

In order to show the first equality of Eq. (7.2.4.12), it is enough to show that the surjective map $q^\beth : X^\beth \to (X/G)^\beth$ restricts to a surjective map $q^\beth|_{\mathrm{Sk}^{\mathrm{ess}}} : \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X) \to \mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})$ on essential skeletons. To this end, we first prove that the image of $\mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$ via $q^\beth$ lies in $\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})$ and that $q^\beth|_{\mathrm{Sk}^{\mathrm{ess}}}$ is surjective on divisorial points.

Let $x \in X^{\mathrm{div}} \cap X^\beth$ be the divisorial point determined by the triple $(c, Y \xrightarrow{h} X, E)$. By [Kol13, Lemma 2.22], there exists a commutative diagram

$$\begin{array}{ccc} E \subset Y & \overset{q'}{\dashrightarrow} & F \subset Y' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} \\ X & \xrightarrow{\quad q \quad} & X/G \end{array}$$

where Y' is a normal variety and F is a divisor on Y' satisfying

- the morphism h and h' are birational;

- the map q' is rational and dominant;

- the image of the divisor E via q' is the divisor F.

Note that the image $q^\beth(x)$ is determined by the triple $(c \cdot r(E), Y' \xrightarrow{h'} X/G, F)$, where $r(E)$ is the ramification index of q' along E. Indeed, we have that

$$c \cdot \mathrm{ord}_E(f \circ q \circ h) = c \cdot \mathrm{ord}_E(f \circ h' \circ q') = c \cdot r(E)\, \mathrm{ord}_F(f \circ h')$$

for any rational function $f \in K(X/G)$. By [KM08, Proposition 5.20], $A_{(X/G, \Delta_{X/G})}(q^\beth(x))$ is zero if $A_{(X, \Delta_X)}(x)$ is zero, hence $q^\beth(x) \in \mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})$ for any divisorial point $x \in \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)$. Similarly, the proof of [KM08, Proposition 5.20] shows that $q^\beth|_{\mathrm{Sk}^{\mathrm{ess}}}$ is surjective on divisorial points.

In fact, $q^\beth|_{\mathrm{Sk}^{\mathrm{ess}}}$ is surjective onto the whole skeleton $\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})$. Indeed, since $q^\beth$ is equivariant with respect to the $\mathbb{R}_+^*$-action, it is enough to check that the induced map

$$q^\beth|_{\mathrm{Sk}^{\mathrm{ess}*}/\mathbb{R}_+^*} : \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)^*/\mathbb{R}_+^* \to \mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G})^*/\mathbb{R}_+^*$$

is surjective. If $q^\beth|_{\mathrm{Sk}^{\mathrm{ess}}}$ is surjective on divisorial points, then $q^\beth|_{\mathrm{Sk}^{\mathrm{ess}*}/\mathbb{R}_+^*}$ is a continuous map from a compact topological space to a Hausdorff space with dense image. Hence, $q^\beth|_{\mathrm{Sk}^{\mathrm{ess}}}$ is surjective.

Finally, the second equality of Eq. (7.2.4.12) follows from [Ber95, Corollary 5]. Since the actions of G and $\mathbb{R}_+^*$ commute and the homeomorphism $\mathrm{Sk}^{\mathrm{ess}}(X/G, \Delta_{X/G}) \simeq \mathrm{Sk}^{\mathrm{ess}}(X, \Delta_X)/G$ is $\mathbb{R}_+^*$-equivariant, we conclude that also Eq. (7.2.4.13) holds. $\square$

7.2.5. Proof of Theorem 7.2.0.1

Proof of Theorem 7.2.0.1. The desired homeomorphism is obtained by applying the preceding sequence of lemmas as follows:

$$\begin{aligned}
\mathcal{D}(\partial M_{\mathrm{GL}_n}) &\simeq \mathcal{DMR}(Z^{(n)}, \Delta^{(n)}) \\
&\simeq \mathrm{Sk}^{\mathrm{ess}}(Z^{(n)}, \Delta^{(n)})^*/\mathbb{R}_+^* && \text{cf. Lemma 7.2.4.6} \\
&\simeq \mathrm{Sk}^{\mathrm{ess}}(Z^n, \Delta^n)^*/(\mathbb{R}_+^* \times \mathfrak{S}_n) && \text{cf. Lemma 7.2.4.11} \\
&\simeq \Big((\mathrm{Sk}^{\mathrm{ess}}(Z, \Delta)^*/\mathbb{R}_+^*) * \ldots * (\mathrm{Sk}^{\mathrm{ess}}(Z, \Delta)^*/\mathbb{R}_+^*)\Big)/\mathfrak{S}_n && \text{cf. Lemma 5.3.6.7} \\
&\simeq (\mathbb{S}^1 * \ldots * \mathbb{S}^1)/\mathfrak{S}_n \\
&\simeq \mathbb{S}^{2n-1}/\mathfrak{S}_n \\
&\simeq \mathbb{S}^{2n-1} && \text{cf. Lemma 7.2.5.1}
\end{aligned}$$

□

We conclude the section by proving the topological lemma mentioned at the end of the proof of Theorem 7.2.0.1. It is presumably well-known, but the author is not aware of a reference.

Lemma 7.2.5.1. *Consider the linear action of the symmetric group $\mathfrak{S}_n$ that permutes the standard coordinates of $\mathbb{C}^n$. The quotient of the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ by this action is homeomorphic to the sphere $\mathbb{S}^{2n-1}$.*

Proof. Consider the finite morphism $q\colon \mathbb{C} \times \ldots \times \mathbb{C} \to \mathbb{C}^{(n)} \simeq \mathbb{C}[z]_{n,1}$ given by

$$(z_1, \ldots, z_n) \mapsto \prod_{i=1}^{n} (z - z_i),$$

where we identify the symmetric product $\mathbb{C}^{(n)}$ with the space $\mathbb{C}[z]_{n,1}$ of monic polynomials of degree n in one variable with complex coefficients. The restriction of q to the boundary of the closed unit polydisc $\mathbb{D}^{2n}$

$$q\colon \mathbb{S}^{2n-1} \simeq \partial\mathbb{D}^{2n} = \partial\mathbb{D}^1 * \ldots * \partial\mathbb{D}^1 \to q(\mathbb{S}^{2n-1}) \simeq \mathbb{S}^{2n-1}/\mathfrak{S}_n$$

is the given quotient map. The space $\mathbb{C}[z]_{n,1}$ is isomorphic to $\mathbb{C}^n$ through the identification $\psi : \mathbb{C}[z]_{n,1} \to \mathbb{C}^n$ of a monic polynomial with the n-uples of its coefficients; more explicitly, ψ is by

$$\psi\left(\prod_{i=1}^{n} (z - z_i)\right) = \psi(z^n + r_1 e^{i\theta_1} z^{n-1} + \ldots + r_n e^{i\theta_n}) = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}),$$

where $(r_j, \theta_j)_{1 \leqslant j \leqslant n}$ are polar coordinates on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Further, let $\varphi\colon \mathbb{C}^n \to \mathbb{C}^n$ be the homeomorphism given by

$$\varphi(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n}) = (r_1 e^{i\theta_1}, \sqrt[2]{r_2} e^{i\theta_1}, \dots, \sqrt[n]{r_n} e^{i\theta_n}).$$

We can restrict the composition $\frac{\varphi}{|\varphi|} \circ \psi \circ q : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\}$ to a morphism of spheres which factors through the symmetric quotient by construction, as shown in the following diagram

$$\begin{array}{ccc} \mathbb{S}^{2n-1} & & \\ \downarrow{\scriptstyle q} & \searrow^{\frac{\varphi}{|\varphi|}\circ\psi\circ q} & \\ \mathbb{S}^{2n-1}/\mathfrak{S}_n & \xrightarrow[\frac{\varphi}{|\varphi|}\circ\psi]{} & \mathbb{S}^{2n-1}. \end{array}$$

We claim that the map

$$\frac{\varphi}{|\varphi|} \circ \psi : \mathbb{S}^{2n-1}/\mathfrak{S}_n \to \mathbb{S}^{2n-1}$$

is a homeomorphism. Indeed, since it is a continuous map from a compact topological space to a Hausdorff space, it is enough to check that it is bijective. This is equivalent to show that the preimage of any point in $\mathbb{S}^{2n-1}$ via the map $\frac{\varphi}{|\varphi|} \circ \psi \circ q$ is a $\mathfrak{S}_n$-orbit. Alternatively, we need to prove that the preimage of any real half-line $\{(re^{i\theta_1}, \dots, re^{i\theta_n}) : r \in \mathbb{R}_+\} \subseteq \mathbb{C}^n$ via the map $\varphi \circ \psi \circ q$ is the orbit of a half-line $\{(rz_1, \dots, rz_n) : r \in \mathbb{R}_+\} \subseteq \mathbb{C}^n$. This follows from the fact that

$$\begin{aligned} (\varphi \circ \psi \circ q)^{-1}(re^{i\theta_1}, \dots, re^{i\theta_n}) &= (\psi \circ q)^{-1}(re^{i\theta_1}, r^2 e^{i\theta_2}, \dots, r^n e^{i\theta_n}) \\ &= \bigcup_{\sigma \in \mathfrak{S}_n} (rz_{\sigma(1)}, \dots, rz_{\sigma(n)}) \end{aligned}$$

for any $r \in \mathbb{R}_+$, where the values z_j are chosen in such a way that $q(z_1, \dots, z_n) = (e^{i\theta_1}, \dots, e^{i\theta_n})$. □

7.2.6. An alternative proof of Theorem 7.2.0.1.

(7.2.6.1) The proof of Theorem 7.2.0.1 is inspired by the results in Section 4.3. There, the dual complex of a degeneration of the Hilbert scheme of n points of K3 surfaces induced by a maximal unipotent semistable degeneration of K3 surfaces is homeomorphic to the complex projective space $\mathbb{CP}^n$. Both proofs crucially rely on the compatibility of the construction of the essential skeleton with products and finite quotients.

In this section, we exhibit a direct connection between the two results: we show how Theorem 7.2.0.1 can be deduced from [BM19, Proposition 6.2.4]. This alternate proof relies on the construction of an explicit degeneration of Calabi–Yau varieties (see Proposition 7.2.6.7), and a global-to-local argument (Lemma 7.2.6.4) that relates the dual complex of the degeneration to that a logCY pair. While the proof of Theorem 7.2.0.1 presented in Section 7.2.5 is technically more elementary, we expect both strategies to prove useful for future calculations of dual complexes. Furthermore, the existence of a degeneration as in Proposition 7.2.6.7 is of independent interest: loosely speaking, it realizes a character variety as a "limit" of compact hyper-Kähler manifolds.

(7.2.6.2) Let (X, Δ_X) be a dlt pair with $\Delta_X^{=1} := \sum_{i=1}^m \Delta_i$. For every stratum W of (X, Δ_X), there exists a $\mathbb{Q}$-divisor $\mathrm{Diff}_W^*(\Delta_X)$ such that $(K_X + \Delta_X)|_W \sim_{\mathbb{Q}} K_W + \mathrm{Diff}_W^*(\Delta_X)$; see [Kol13, §4.18]. By adjunction, we have that $\mathrm{Diff}_W^*(\Delta_X)^{=1}$ coincides with the trace of Δ_X on W (as defined in 5.4.1.3), i.e.

$$\mathrm{Diff}_W^*(\Delta_X)^{=1} = \sum_{W \nsubseteq \Delta_i} \Delta_i|_W.$$

In particular, any stratum W of a dlt (logCY) pair has an induced structure of (logCY) pair $(W, \mathrm{Diff}_W^*(\Delta_X))$ such that

$$\mathcal{D}(\mathrm{Diff}_W^*(\Delta_X)^{=1}) \simeq \mathcal{D}\Big(\sum_{W \nsubseteq \Delta_i} \Delta_i|_W\Big). \tag{7.2.6.3}$$

Lemma 7.2.6.4 (Global-to-local argument). *Let (X, Δ_X) be a dlt pair such that the dual complex of $\mathcal{D}(\Delta_X)$ is a topological manifold. Then $\mathcal{D}(\mathrm{Diff}_W^*(\Delta_X))$ is homeomorphic to a sphere for any stratum W of $\Delta_X^{=1}$.*

Proof. Up to baricentrical subdivisions, the link of a neighborhood of a cell associated to W in $\mathcal{D}(\Delta_X)$ is isomorphic to $\mathcal{D}(\mathrm{Diff}_W^*(\Delta_X))$. Since $\mathcal{D}(\Delta_X)$ is a topological manifold, this link is homeomorphic to a sphere. □

(7.2.6.5) We will construct a degeneration of Hilbert schemes of a K3 surface with a component of the special fibre that, paired with the different of the special fibre, is crepant birational to a dlt compactification of M_{GL_n}. This is then combined with the global-to-local argument to compute the dual complex in Theorem 7.2.0.1. The properties of the required degeneration are collected below.

Definition 7.2.6.6. A model $\mathscr{X}$ over $\mathbb{C}[[t]]$ is *good minimal dlt* if $\mathscr{X}$ is $\mathbb{Q}$-factorial, the pair $(\mathscr{X}, \mathscr{X}_{0,\mathrm{red}})$ is dlt, and $K_{\mathscr{X}} + \mathscr{X}_{0,\mathrm{red}}$ is semiample.

Proposition 7.2.6.7. *Let (X, Δ_X) be a lc logCY pair. Assume there exist*

(a) a maximal unipotent semistable good minimal dlt model $\mathscr{S}$ of a K3 surface S over $\mathbb{C}((t))$,

(b) a good minimal dlt model $\mathscr{S}^{[n],\mathrm{dlt}}$ of the Hilbert scheme of n points of S,

such that (X, Δ_X) is crepant birational to $(D, \mathrm{Diff}_D^*(\mathscr{S}^{[n],\mathrm{dlt}}))$ for some irreducible component D of the special fibre $\mathscr{S}_0^{[n],\mathrm{dlt}}$. Then $\mathcal{D}(\Delta_X)$ is homeomorphic to a sphere.

Proof. It follows from the combination of Lemma 7.2.6.4, [dFKX17, Proposition 11], [BM19, Proposition 6.2.4], and [NX16, Propositon 3.3.3]. Note that we only use [BM19, Proposition 6.2.4] to grant that the dual complex of the degeneration is a manifold, and not the fact that it is actually homeomorphic to a complex projective space. □

Proof of Theorem 7.2.0.1. Let $\mathscr{S}$ be a semistable good minimal (although not $\mathbb{Q}$-factorial) model over $\mathbb{C}[[t]]$ of a quartic surface S in $\mathbb{P}^3_{\mathbb{C}((t))}$, degenerating to the union of four hyperplanes $\mathscr{S}_0 = \sum_{i=0}^3 D_i$. For example, take the Dwork pencil

$$\mathscr{S} := \left\{ x_0x_1x_2x_3 + t\sum_{i=0}^{3} x_i^4 = 0 \right\} \subseteq \mathbb{P}^3_{[x_0:x_1:x_2:x_3]} \times \operatorname{Spec}(\mathbb{C}[[t]]).$$

The degeneration $\mathscr{S}$ is a model of the K3 surface S as in Proposition 7.2.6.7(a), and the proof proceeds in two steps: we construct a model $\mathscr{S}^{[n],\mathrm{dlt}}$ of $S^{[n]}$ as in Proposition 7.2.6.7(b), and then we identify a component of the special fibre $\mathscr{S}_0^{[n],\mathrm{dlt}}$ which, paired with the different of $\mathscr{S}_0^{[n],\mathrm{dlt}}$, is crepant birational to a dlt compactification of M_{GL_n}.

For the first step, let $(\mathscr{S}^{(n)}, \mathscr{S}_0^{(n)})$ and $(\mathscr{S}^{[n]}, \mathscr{S}_0^{[n]})$ be the pairs given by the relative n-fold symmetric product and the relative Hilbert scheme of n points on $\mathscr{S}$ respectively, together with their special fibres. Consider a log resolution of $(\mathscr{S}^{[n]}, \mathscr{S}_0^{[n]})$, written

$$g\colon (\mathscr{Y}, \Delta_{\mathscr{Y}} := g_*^{-1}\mathscr{S}_0^{[n]} + E) \to (\mathscr{S}^{[n]}, \mathscr{S}_0^{[n]}),$$

which is an isomorphism on the snc locus of $(\mathscr{S}^{[n]}, \mathscr{S}_0^{[n]})$, where E is the sum of the g-exceptional divisors. Note that the composition $g \circ \rho_{\mathrm{HC}}$ of g with the Hilbert–Chow morphism ρ_{HC} gives a log resolution of the pair $(\mathscr{S}^{(n)}, \mathscr{S}_0^{(n)})$ as well. The $(K_{\mathscr{Y}/\mathbb{C}[[t]]} + \Delta_{\mathscr{Y}})$-MMP with scaling terminates with a $\mathbb{Q}$-factorial, dlt, minimal model of $S^{[n]}$

$$h : (\mathscr{S}^{[n],\mathrm{dlt}}, \mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}} = h_*^{-1}\mathscr{S}_0^{[n]} + E') \to \mathscr{S}^{[n]},$$

where E' is the sum of the $(g \circ \rho_{\mathrm{HC}})$-exceptional divisors that lie in the special fibre, and $\mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}}$ is the reduced special fibre of $\mathscr{S}^{[n],\mathrm{dlt}}$. The existence of such a h follows from [Kol13, Corollary 1.36]; note that the degeneration $\mathscr{S}$ is defined over a curve (see Definition 5.3.8.3), so we can run a relative MMP as usual. Note also that the pair $(\mathscr{S}^{(n)}, \mathscr{S}_0^{(n)})$ is (reduced) lc logCY, since $(\mathscr{S}, \mathscr{S}_0)$ is so. The pair $(\mathscr{S}^{[n],\mathrm{dlt}}, \mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}})$ is logCY as well, as h is a crepant morphism of pairs (c.f. [Kol13, §1.35]). Hence, $\mathscr{S}^{[n],\mathrm{dlt}}$ is a good minimal dlt model of $S^{[n]}$, as required in order to apply Proposition 7.2.6.7.

Now, we show that there exist irreducible components Δ_i^{dlt} of $\mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}}$ such that the pairs

$$(\Delta_i^{\mathrm{dlt}}, \operatorname{Diff}^*_{\Delta_i^{\mathrm{dlt}}}(\mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}}))$$

are crepant birational to a dlt compactification of $M_{\mathrm{GL}_n} \simeq (\mathbb{C}^* \times \mathbb{C}^*)^{(n)}$, equivalently of $(\mathbb{C}^* \times \mathbb{C}^*)^{[n]}$. To this end, note that the special fibre $\mathscr{S}_0^{(n)}$ contains irreducible components $\Delta_i \simeq (\mathbb{P}^2)^{(n)}$, $i \in I \simeq \{0, \dots, 3\}$, which are the n-fold symmetric products of the hyperplanes D_i. Denote by Δ_i' and Δ_i^{dlt} the strict transform of Δ_i under ρ_{HC} and h, respectively. By [Kol13, Proposition 4.6], the following pairs are crepant birational:

$$(\Delta_i', \mathrm{Diff}_{\Delta_i'}(\mathscr{S}_{0,\mathrm{red}}^{[n]})) \sim (\Delta_i, \mathrm{Diff}_{\Delta_i}(\mathscr{S}_0^{(n)})) \sim (\Delta_i^{\mathrm{dlt}}, \mathrm{Diff}_{\Delta_i^{\mathrm{dlt}}}(\mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}})).$$

Further, the inclusion of $D_i \setminus \cup_{j\neq i} D_j \simeq \mathbb{C}^* \times \mathbb{C}^*$ into D_i induces the embedding of $\Delta_i^\circ := (D_i \setminus \cup_{j\neq i} D_j)^{[n]}$ into Δ_i', which is isomorphic to $(\mathbb{C}^* \times \mathbb{C}^*)^{[n]}$. We need the following technical lemma.

Lemma 7.2.6.8. $\mathrm{Diff}_{\Delta_i'}(\mathscr{S}_{0,\mathrm{red}}^{[n]}) = \Delta_i' \setminus \Delta_i^\circ$.

Proof. It is clear that $\mathrm{Diff}_{\Delta_i'}(\mathscr{S}_{0,\mathrm{red}}^{[n]}) \geqslant (\mathscr{S}_0^{[n]} \setminus \Delta_i')|_{\Delta_i'}$. For the equality, it is enough to prove that no divisor whose generic point is contained in Δ_i° belongs to the support of $\mathrm{Diff}_{\Delta_i'}(\mathscr{S}_{0,\mathrm{red}}^{[n]})$. By [Kol13, Proposition 4.5 (1)], it is sufficient to prove that $\mathscr{S}^{[n]}$ is regular along Δ_i°. To this aim, let $\xi \in \Delta_i^\circ$ be a scheme of length n in $\mathscr{S}_0$. Since the immersion of a (formal) neighborhood of ξ in $\mathscr{S}_0^{[n]}$ factors through Δ_i, the subscheme ξ is unobstructed by [Fog68, Theorem 2.4]; it follows from [Kol96, Theorem 2.10] that $\mathscr{S}^{[n]}$ is regular at ξ. □

Finally, we conclude that

$$\begin{aligned}
\mathcal{D}(\partial(M_{\mathrm{GL}_n}) &\simeq \mathcal{DMR}(\Delta_i', \mathrm{Diff}_{\Delta_i'}(\mathscr{S}_{0,\mathrm{red}}^{[n]})) && \text{cf. Lemma 7.2.6.8 and (7.2.2.2)}\\
&\simeq \mathcal{DMR}(\Delta_i^{\mathrm{dlt}}, \mathrm{Diff}_{\Delta_i^{\mathrm{dlt}}}(\mathscr{S}_{0,\mathrm{red}}^{[n],\mathrm{dlt}})) && \text{cf. [dFKX17, Proposition 11]}\\
&\simeq \mathbb{S}^{2n-1} && \text{cf. Proposition 7.2.6.7.}
\end{aligned}$$

□

7.3. Dual boundary complex of SL_n-character varieties of a genus one surface

In this section, we determine the homeomorphism class of the dual boundary complex of the SL_n-character variety M_{SL_n} associated to a Riemann surface of genus one.

Theorem 7.3.0.1. *The dual boundary complex $\mathcal{D}(\partial M_{\mathrm{SL}_n})$ of a dlt log Calabi–Yau compactification of M_{SL_n} has the homeomorphism type of $\mathbb{S}^{2n-3}$.*

Proof. Observe that M_{SL_n} is the fibre of the determinant morphism

$$(\mathbb{C}^* \times \mathbb{C}^*)^{(n)} \simeq M_{\mathrm{GL}_n} \to \mathbb{C}^* \times \mathbb{C}^* \tag{7.3.0.2}$$

$$((a_i, b_i))_{i=1}^n \simeq [(A, B)] \mapsto (\det A, \det B) = \left(\textstyle\prod_{i=1}^n a_i, \prod_{i=1}^n b_i\right), \tag{7.3.0.3}$$

where the pair (A, B) of matrices in GL_n represents a point in M_{GL_n}, and $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ are their eigenvalues; see for instance [BS19, Lemma 8.17]. We proceed in several steps.

Step 1. The character variety M_{SL_n} admits a lc logCY compactification $\overline{M}_{\mathrm{SL}_n}$. Indeed, consider the diagram

$$\begin{array}{ccccc} L := m_r^{-1}(\mathbf{1}) \simeq (\mathbb{C}^* \times \mathbb{C}^*)^{n-1} & \longrightarrow & (\mathbb{C}^* \times \mathbb{C}^*)^n & \xrightarrow{m_n} & \mathbb{C}^* \times \mathbb{C}^* \ni \mathbf{1} = (1,1) \\ \big\downarrow & & \big\downarrow{\scriptstyle q} & \nearrow & \\ M_{\mathrm{SL}_n} & \longrightarrow & (\mathbb{C}^* \times \mathbb{C}^*)^{(n)}, & & \end{array}$$

where m_n is the multiplication map, and q the quotient by the action of the symmetric group $\mathfrak{S}_n$, permuting the factors. The projective closure $\overline{L}$ of L in $(\mathbb{P}^2)^n$ is invariant with respect to the action

$$(\mathbb{C}^* \times \mathbb{C}^*)^{n-1} \times (\mathbb{P}^2)^n \to (\mathbb{P}^2)^n$$

given by

$$\begin{aligned} ((a_i, b_i))_{i=1}^{n-1} \cdot ([x_j : y_j : z_j])_{j=1}^n = \Big(&[a_1 x_1 : b_1 y_1 : z_1], \dots, [a_{n-1} x_{n-1} : b_{n-1} y_{n-1} : z_{n-1}], \\ &[\textstyle\prod_{i=1}^{n-1} a_i^{-1} \cdot x_n : \prod_{i=1}^{n-1} b_i^{-1} \cdot y_n : z_n] \Big). \end{aligned}$$

As $L \simeq (\mathbb{C}^* \times \mathbb{C}^*)^{n-1}$ is a dense orbit of this algebraic action, it follows that $\overline{L}$ is a toric compactification of L. In particular, the pair $(\overline{L}, \partial L := \overline{L} \setminus L)$ is a (normal) lc logCY pair. Since $\overline{L}$ is $\mathfrak{S}_n$-invariant and the restriction of the quotient map $q : (\mathbb{P}^2)^n \to (\mathbb{P}^2)^{(n)}$ to $\overline{L}$ is quasi-étale, then the projective closure $\overline{M}_{\mathrm{SL}_n} := \overline{L}/\mathfrak{S}_n$ of M_{SL_n} in $(\mathbb{P}^2)^{(n)}$ is a lc logCY compactification of M_{SL_n}. Thus, we can construct the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(\overline{M}_{\mathrm{SL}_n}, \partial \overline{M}_{\mathrm{SL}_n})$ as in Definition 7.2.4.4. Although $\overline{L}$ is a toric variety, it is worth pointing out that the embedding $\overline{L} \hookrightarrow (\mathbb{P}^2)^n$ is not toric.

Step 2. Let Δ be the toric boundary of $\mathbb{P}^2$ and N be the cocharacter lattice of the torus $\mathbb{C}^* \times \mathbb{C}^* \subseteq \mathbb{P}^2$. By Proposition 7.2.4.2, $\mathrm{Sk}^{\mathrm{ess}}(\mathbb{P}^2, \Delta)$ is the skeleton of the log-regular pair $(\mathbb{P}^2, \Delta)$, and hence the multiplication m_n induces a map $\alpha_n \colon \mathrm{Sk}^{\mathrm{ess}}(\mathbb{P}^2, \Delta)^n \to \mathrm{Sk}^{\mathrm{ess}}(\mathbb{P}^2, \Delta)$ by functoriality, as in 5.3.4.3. In particular, in this toric case, the essential skeleton $\mathrm{Sk}^{\mathrm{ess}}(\mathbb{P}^2, \Delta)$ can be identified with $N_{\mathbb{R}} \simeq \mathbb{R}^2$ (see Section 5.4.2), and α_n is given by the linear map

$$\begin{aligned} (N_{\mathbb{R}})^n \simeq \mathbb{R}^{2n} &\to N_{\mathbb{R}} \simeq \mathbb{R}^2, \\ (x_i, y_i)_{i=1}^n &\mapsto \Big(\textstyle\sum_{i=1}^n x_i, \sum_{i=1}^n y_i \Big). \end{aligned}$$

Finally, observe that the symmetric quotient of the kernel of α_n is isomorphic to the additive group $\mathbb{C}^{n-1}$, i.e.

$$\alpha_n^{-1}(\mathbf{0})/\mathfrak{S}_n \simeq \mathbb{C}^{n-1}.$$

This follows from the diagram below:

$$\begin{array}{ccccc}
\alpha_n^{-1}(\mathbf{0}) & \longrightarrow & \mathrm{Sk}^{\mathrm{ess}}((\mathbb{P}^2)^n, \Delta^n) \simeq \mathbb{C}^n \simeq \mathbb{R}^{2n} & \xrightarrow{\alpha_n} & \mathrm{Sk}^{\mathrm{ess}}(\mathbb{P}^2, \Delta) \simeq \mathbb{C} \simeq \mathbb{R}^2 \\
\downarrow & & \downarrow q & \nearrow_{\mathrm{pr}} & \\
\alpha_n^{-1}(\mathbf{0})/\mathfrak{S}_n & \longrightarrow & \mathrm{Sk}^{\mathrm{ess}}((\mathbb{P}^2)^{(n)}, \Delta^{(n)}) \simeq \mathbb{C}^n, & &
\end{array}$$

where the map pr is the linear projection to the $\mathfrak{S}_n$-invariant coordinate α_n.

Step 3. The essential skeleton of the pair $(\overline{M}_{\mathrm{SL}_n}, \partial\overline{M}_{\mathrm{SL}_n})$ is homeomorphic to the symmetric quotient of the fibre of α_n, namely

$$\mathrm{Sk}^{\mathrm{ess}}(\overline{M}_{\mathrm{SL}_n}, \partial\overline{M}_{\mathrm{SL}_n}) \simeq \alpha_n^{-1}(\mathbf{0})/\mathfrak{S}_n.$$

This statement can be shown following the same strategy of [BM19, Proposition 6.3.3]. Indeed, the latter relies on the functoriality of skeletons via finite quotients and products, which we have reproved in the trivially-valued setting in Proposition 5.3.6.6 and Lemma 7.2.4.11.

Step 4. In the same fashion as in §7.2.5, we conclude that

$$\mathcal{D}(\partial M_{\mathrm{SL}_n}) \simeq \mathcal{DMR}(\overline{M}_{\mathrm{SL}_n}, \partial\overline{M}_{\mathrm{SL}_n}) \simeq \mathbb{S}^{2n-3}.$$

□

7.3.1. An alternative proof of Theorem 7.3.0.1

(7.3.1.1) Following the same strategy as in Section 7.2.6, one can invoke a global-to-local argument to reduce the proof of Theorem 7.3.0.1 to the construction of a degeneration as in the following proposition. Observe that the role of the Hilbert scheme of a K3 surface in Proposition 7.2.6.7 is replaced by the generalised Kummer variety of an abelian surface.

Proposition 7.3.1.2. *There exists a good minimal dlt model $\mathscr{K}^{\mathrm{dlt}}_{n-1}$ of a generalised Kummer variety and an irreducible component Δ^{dlt} of the special fibre $\mathscr{K}^{\mathrm{dlt}}_{n-1,0}$ such that the pair $(\Delta^{\mathrm{dlt}}, \mathrm{Diff}_{\Delta^{\mathrm{dlt}}}(\mathscr{K}^{\mathrm{dlt}}_{n-1,0,\mathrm{red}}))$ is crepant birational to a lc logCY compactification of M_{SL_n}.*

(7.3.1.3) The proof of Proposition 7.3.1.2 relies on some local computations on the *Tate curve*. Following [DR73, VII], we recall that the Tate curve $\overline{\mathscr{G}}_m$ is a model over $\mathbb{C}[[t]]$ of the multiplicative group $\mathbb{G}_m$ with special fibre given by an infinite chain of $\mathbb{P}^1$'s; see Section 7.3.2 for the construction. In fact, $\overline{\mathscr{G}}_m$ is the universal cover of the minimal model of a Tate elliptic over $\mathbb{C}((t))$, as in [Tat95] (see also [Sil09, C §14]). Mind that $\overline{\mathscr{G}}_m$ is the completion of a $\mathbb{C}$-scheme that is only *locally* of finite type over $\mathbb{C}$.

The model $\mathscr{G}_m$, obtained from $\overline{\mathscr{G}}_m$ by removing the nodes of the special fibre, is the Néron model of $\mathbb{G}_m$ (cf. [DR73, VII, Example 1.2.c)]). In particular, $\mathscr{G}_m$ is endowed with a multiplication morphism

$$\mathscr{G}_m^n := \mathscr{G}_m \times \ldots \times \mathscr{G}_m \to \mathscr{G}_m,$$

which extends the multiplication $\mathbb{G}_m^n \to \mathbb{G}_m$ on the generic fibre. Let $\mathscr{V}_{n-1}$ denote the fibre of the identity section via the multiplication map $\mathscr{G}_m^n \to \mathscr{G}_m$. By a local computation in the coordinates of [DR73, VII], one can show that the pair $(\overline{\mathscr{V}}_{n-1}, \overline{\mathscr{V}}_{n-1,0})$, given by the closure of $\mathscr{V}_{n-1}$ in the fibre product $\overline{\mathscr{G}}_m^n$ together with its special fibre, is normal, reduced, and toric. Further, the intersection $\overline{\mathscr{V}}_{n-1} \cap (\overline{\mathscr{G}}_m^n \setminus \mathscr{G}_m^n)$ has codimension two in $\overline{\mathscr{V}}_{n-1}$. The proof of these facts appear in Section 7.3.2.

Proof of Proposition 7.3.1.2. Let E be an elliptic curve over $\mathbb{C}((t))$ with multiplicative reduction (c.f. [Liu02, Definition 10.2.2]), and $\mathscr{E}$ be a semistable good minimal snc model of E over $\mathbb{C}[[t]]$. In order to later run a MMP, assume further that $\mathscr{E}$ is defined over a curve in the sense of Definition 5.3.8.3. For example, take $\mathscr{E}$ to be the Dwork pencil of cubic curves that appears in Fig. 5.3.8.1. The Néron model $\mathscr{N}$ of $\mathscr{E}$ is the group scheme obtained from $\mathscr{E}$ by removing the nodes of the special fibre; see [Liu02, Theorem 10.2.14].

We first perform the classical construction of a singular generalised Kummer variety, as in [Bea83, §7], but in the relative setting. Let $\mathscr{X}_{n-1}$ be the fibre of the identity section of the multiplication morphism

$$m_n : (\mathscr{N} \times \mathscr{N})^n := (\mathscr{N} \times_{\mathbb{C}[[t]]} \mathscr{N}) \times_{\mathbb{C}[[t]]} \ldots \times_{\mathbb{C}[[t]]} (\mathscr{N} \times_{\mathbb{C}[[t]]} \mathscr{N}) \to (\mathscr{N} \times_{\mathbb{C}[[t]]} \mathscr{N}).$$

The closure $\overline{\mathscr{X}}_{n-1}$ of $\mathscr{X}_{n-1}$ in $(\mathscr{E} \times \mathscr{E})^n$ is invariant under the action of the symmetric group $\mathfrak{S}_n$, which acts by permuting the factors of $(\mathscr{E} \times \mathscr{E})^n$. As a result, the quotient

$$\mathscr{K}_{n-1}^{\mathrm{sing}} := \overline{\mathscr{X}}_{n-1}/\mathfrak{S}_n$$

is a model of the singular generalised Kummer variety K_{n-1}^{sing} associated to the abelian surface $E \times E$. Let $\mathscr{K}_{n-1,0}^{\mathrm{sing}}$ be the special fibre of $\mathscr{K}_{n-1}^{\mathrm{sing}}$.

Lemma 7.3.1.4. *The pair $(\mathscr{K}_{n-1}^{\mathrm{sing}}, \mathscr{K}_{n-1,0}^{\mathrm{sing}})$ is reduced lc logCY.*

Proof. We omit the subscript $n-1$ for brevity. Since the quotient map $\overline{\mathscr{X}} \to \mathscr{K}^{\mathrm{sing}}$ is quasi-étale, it is equivalent to check that the pair $(\overline{\mathscr{X}}, \overline{\mathscr{X}}_0)$ is reduced lc logCY. To this end, observe that the universal cover of $(\mathscr{E} \times \mathscr{E})^n$ is the fibre product $(\overline{\mathscr{G}}_m \times \overline{\mathscr{G}}_m)^n$ of Tate curves. Therefore, the pair $(\overline{\mathscr{X}}, \overline{\mathscr{X}}_0)$ is reduced lc, since it is étale-locally isomorphic to the pair $(\overline{\mathscr{V}} \times \overline{\mathscr{V}}, (\overline{\mathscr{V}} \times \overline{\mathscr{V}})_0)$, which is the fibre product of the reduced toric pair $(\overline{\mathscr{V}}, \overline{\mathscr{V}}_0)$ by Proposition 7.3.2.3.

In order to verify that $K_{\overline{\mathscr{X}}/\mathbb{C}[[t]]} + \overline{\mathscr{X}}_0$ is trivial, it suffices to check that its restriction

$$\left(K_{\overline{\mathscr{X}}/\mathbb{C}[[t]]} + \overline{\mathscr{X}}_0\right)|_{\mathscr{X}} = K_{\mathscr{X}/\mathbb{C}[[t]]} + \mathscr{X}_0 \sim K_{\mathscr{X}/\mathbb{C}[[t]]}$$

to $\mathscr{X}$ is trivial; indeed, $\overline{\mathscr{X}} \setminus \mathscr{X}$ has codimension two in $\overline{\mathscr{X}}$ by Proposition 7.3.2.3. Let

$N_{\mathscr{X}}((\mathscr{N} \times \mathscr{N})^n)$ denote the normal bundle of $\mathscr{X}$ in $(\mathscr{N} \times \mathscr{N})^n$. As $\mathscr{X}$ is a fibre of the locally trivial fibration m_n, it follows that $\det(N_{\mathscr{X}}((\mathscr{N} \times \mathscr{N})^n))$ is trivial; in particular, we have

$$K_{\mathscr{X}/\mathbb{C}[[t]]} \sim K_{(\mathscr{N}\times\mathscr{N})^n/\mathbb{C}[[t]]}|_{\mathscr{X}} \otimes \det(N_{\mathscr{X}}((\mathscr{N} \times \mathscr{N})^n)) \sim 0,$$

since $(\mathscr{N} \times \mathscr{N})^n$ is Calabi–Yau. Thus, the pair $(\overline{\mathscr{X}}, \overline{\mathscr{X}}_0)$ is logCY, as required. □

Restrict now the construction of the relative generalised Kummer variety to the identity component of the special fibre of $\mathscr{N}$, which is isomorphic to $\mathbb{C}^*$: this gives the construction of M_{SL_n} in 7.3.0.2. As a consequence, an irreducible component of the special fibre $\mathscr{K}^{\mathrm{sing}}_{n-1,0}$ is a lc logCY compatification of M_{SL_n} in $(\mathbb{P}^1 \times \mathbb{P}^1)^{(n)}$ (c.f. Step 1 in the proof of Theorem 7.3.0.1).

Finally, the good minimal dlt model $\mathscr{K}^{\mathrm{dlt}}_{n-1}$ of the generalised Kummer variety K_{n-1} associated to $E \times E$ can be obtained, following [Kol13, Corollary 1.38], by extracting the exceptional divisors of the Hilbert–Chow morphism $\rho_{\mathrm{HC}} \colon K_{n-1} \to K^{\mathrm{sing}}_{n-1}$. □

Proof of Theorem 7.3.0.1. It follows from Lemma 7.2.6.4, Proposition 7.3.1.2, and [BM19, Proposition 6.3.4]. □

7.3.2. Local computations on the Tate curve

The goal of this last section is to prove Proposition 7.3.2.3, which is a technical ingredient needed in the proof of Proposition 7.3.1.2. The former result involves the Tate curve of [DR73, VII], whose existence and basic properties were discussed in 7.3.1.3. We begin by recalling its construction.

(7.3.2.1) Let $(x_i)_{i\in\mathbb{Z}}$ be a collection of indeterminates. The Tate curve $\overline{\mathscr{G}}_m$ over the base $R := \mathbb{C}[[t]]$ is the union of the affine charts $(U_{i+1/2})_{i\in\mathbb{Z}}$ given by

$$U_{i+1/2} := \operatorname{Spec}\left(\frac{R[x_i, y_{i+1}]}{(x_i y_{i+1} - t)}\right).$$

For each $i \in \mathbb{Z}$, the charts $U_{i-1/2}$ and $U_{i+1/2}$ are glued along the open subscheme

$$\begin{aligned} T_i := U_{i-1/2} \cap U_{i+1/2} &= \operatorname{Spec}\left(\mathcal{O}(U_{i+1/2})[x_i^{-1}]\right) = \operatorname{Spec}\left(R[x_i, x_i^{-1}]\right) && (y_{i+1} = t/x_i) \\ &= \operatorname{Spec}\left(\mathcal{O}(U_{i-1/2})[y_i^{-1}]\right) = \operatorname{Spec}\left(R[y_i, y_i^{-1}]\right) && (x_{i-1} = t/y_i) \end{aligned}$$

via the identification $x_i y_i = 1$.

(7.3.2.2) The R-group scheme $\mathscr{G}_m := \bigcup_{i\in\mathbb{Z}} T_i$, obtained from $\overline{\mathscr{G}}_m$ by removing the nodes in the special fibre, is the Néron model of the multiplicative group $\mathbb{G}_m$, as explained in [DR73, Example 1.2.c]. In particular, the n-th multiplication map $\mathbb{G}_m \times \ldots \times \mathbb{G}_m \to \mathbb{G}_m$ extends to a homomorphism

$$\mu_n \colon \mathscr{G}_m^n := \mathscr{G}_m \times_R \ldots \times_R \mathscr{G}_m \to \mathscr{G}_m$$

of R-group schemes, which (when $n=2$) is given in local charts by

$$\begin{aligned} T_i \times_R T_j &\longrightarrow T_{i+j} \\ R[x_i, x_i^{-1}] \otimes_R R[x_j, x_j^{-1}] &\longleftarrow R[x_{i+j}, x_{i+j}^{-1}] \\ x_i \otimes x_j &\leftarrow\!\shortmid x_{i+j}. \end{aligned}$$

As $x_{i-1}y_i = t$ and $x_i y_i = 1$, it follows that $x_i = t^{-i}x_0$. In particular, the identity section $\mathcal{I}d$ of $\mathscr{G}_m$ is cut out in the chart T_i by the equation $x_i = t^{-i}$.

Let $\mathscr{V}_{n-1} := \mu_n^{-1}(\mathcal{I}d)$ be the fibre of the identity section $\mathcal{I}d$ via the n-th multiplication map μ_n, and let $\overline{\mathscr{V}}_{n-1}$ denote the closure of $\mathscr{V}_{n-1}$ in $\overline{\mathscr{G}}_m$. The proposition below describes the singularities of the pair $(\overline{\mathscr{V}}_{n-1}, \overline{\mathscr{V}}_{n-1,0})$, where $\overline{\mathscr{V}}_{n-1,0}$ is the special fibre of $\overline{\mathscr{V}}_{n-1}$.

Proposition 7.3.2.3. *The pair $(\overline{\mathscr{V}}_{n-1}, \overline{\mathscr{V}}_{n-1,0})$ is normal, reduced, and toric (i.e. it is Zariski-locally isomorphic to a normal toric scheme with its reduced toric boundary). Furthermore, the intersection $\overline{\mathscr{V}}_{n-1} \cap (\overline{\mathscr{G}}^n_m \setminus \mathscr{G}^n_m)$ has codimension two in $\overline{\mathscr{V}}_{n-1}$.*

The Proposition 7.3.2.3 is an immediate corollary of Lemma 7.3.2.4 below. Indeed, the assertions in Proposition 7.3.2.3 are local: we may work on the the open subsets

$$U_{\alpha+1/2} := U_{\alpha_1+1/2} \times_R \ldots \times_R U_{\alpha_n+1/2},$$

for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, since the $U_{\alpha+1/2}$'s cover $\overline{\mathscr{G}}^n_m$.

For brevity, we omit the subscript $n-1$ from now on; let $\mathscr{V}_\alpha$ be the restriction of $\mathscr{V}$ to $T_\alpha := T_{\alpha_1} \times_R \ldots \times_R T_{\alpha_n}$, $\overline{\mathscr{V}}_\alpha$ be its closure in $U_{\alpha+1/2}$, and $\overline{\mathscr{V}}_{\alpha,0}$ be the special fibre of $\overline{\mathscr{V}}_\alpha$. In local coordinates, we have

$$\begin{aligned} U_{\alpha+1/2} &= \operatorname{Spec}\left(\frac{R[x_{\alpha_1}, y_{\alpha_1+1}, \ldots, x_{\alpha_n}, y_{\alpha_n+1}]}{(x_{\alpha_i}y_{\alpha_i+1} - t)}\right), \\ T_\alpha &= \operatorname{Spec}\left(R[x^{\pm 1}_{\alpha_1}, \ldots, x^{\pm 1}_{\alpha_n}]\right) \subset U_{\alpha+1/2}, \\ \mathscr{V}_\alpha &= \left\{ \prod_{i=1}^n x_{\alpha_i} = t^{-\sum \alpha_i} \right\} \subset T_\alpha. \end{aligned}$$

Lemma 7.3.2.4. *For any $\alpha \in \mathbb{Z}^n$, the pair $(\overline{\mathscr{V}}_\alpha, \overline{\mathscr{V}}_{\alpha,0})$ is normal, reduced, and toric. Furthermore, the intersection $\overline{\mathscr{V}}_\alpha \cap (U_{\alpha+1/2} \setminus \mathscr{G}^n_m)$ has codimension two in $\overline{\mathscr{V}}_\alpha$.*

Proof. The proof is divided into cases depending on the sign of $|\alpha| := \sum_{i=1}^n \alpha_i$.

Case 1. Assume $|\alpha| > 0$. In this case, $\overline{\mathscr{V}}_\alpha$ is cut out of $U_{\alpha+1/2}$ by the equation $t^{|\alpha|}\prod_{i=1}^n x_{\alpha_i} = 1$, so t is invertible on $\overline{\mathscr{V}}_\alpha$. In particular, $\overline{\mathscr{V}}_\alpha$ and $\mathscr{V}_\alpha$ both coincide with the generic fibre of $U_{\alpha+1/2}$, which is isomorphic to the $\operatorname{Frac}(R)$-scheme $\mathbb{G}^n_m$. Thus, there is nothing to prove.

Case 2. Assume $|\alpha| = 0$. As $\prod_{i=1}^n x_{\alpha_i} = 1$ on $\overline{\mathscr{V}}_\alpha$, it follows that the x_{α_i}'s are invertible there, and hence the variables $y_{\alpha_i+1} = x^{-1}_{\alpha_i} x_{\alpha_1} y_{\alpha_1+1}$ can be eliminated. Thus, we have

$$\begin{aligned}
\overline{\mathscr{V}}_\alpha &= \operatorname{Spec}\left(R[x_{\alpha_1}^{\pm 1}, \ldots, x_{\alpha_{n-1}}^{\pm 1}, y_{\alpha_1+1}]\right) \simeq \mathbb{G}_{m,R}^{n-1} \times_R \mathbb{A}_R^1, \\
\overline{\mathscr{V}}_{\alpha,0} &= \{x_{\alpha_1} y_{\alpha_1+1} = 0\} = \{y_{\alpha_1+1} = 0\}, \\
\overline{\mathscr{V}}_\alpha \cap (U_{\alpha+1/2} \setminus \mathscr{G}_m^n) &\subseteq \overline{\mathscr{V}}_\alpha \cap \left\{\prod_{i=1}^n x_{\alpha_i} = 0\right\} = \emptyset.
\end{aligned}$$

It is clear from the above equations that $(\overline{\mathscr{V}}_\alpha, \overline{\mathscr{V}}_{\alpha,0})$ satisfies the required properties.

Case 3. Assume $|\alpha| < 0$. We will show that $\overline{\mathscr{V}}_\alpha$ is normal by showing the conditions S_2 and R_1, and in the process we deduce that $(\overline{\mathscr{V}}_\alpha, \overline{\mathscr{V}}_{\alpha,0})$ is toric and $\overline{\mathscr{V}}_{\alpha,0}$ is reduced.

Step 1. Observe that $\overline{\mathscr{V}}_\alpha$ is contained in the closed, toric subscheme $\mathscr{Z}_\alpha$ of $U_{\alpha+1/2}$ given by the equations

$$\begin{cases} t \cdot \prod_{i=1}^n x_{\alpha_i} = t^{-|\alpha|+1}, \\ x_{\alpha_1} y_{\alpha_1+1} = \ldots = x_{\alpha_n} y_{\alpha_n+1} = t \end{cases}$$

in $\operatorname{Spec}\left(R[x_{\alpha_1}, y_{\alpha_1+1}, \ldots, x_{\alpha_n}, y_{\alpha_n+1}]\right)$. The fibres of $\mathscr{Z}_\alpha$ over R are easily described: over the generic fibre, $\mathscr{Z}_\alpha$ coincides with $\overline{\mathscr{V}}_\alpha$; over the special fibre, it is $(U_{\alpha+1/2})_0$, hence given by the equations

$$x_{\alpha_1} y_{\alpha_1+1} = \ldots = x_{\alpha_n} y_{\alpha_n+1} = t = 0.$$

Recall that if a Gorenstein scheme of pure dimension d is a union of two closed subschemes of pure dimension n and one is Cohen-Macaulay, then the other is Cohen-Macaulay; see [Kol11, Lemma 7]. Thus, since $\mathscr{Z}_\alpha = \overline{\mathscr{V}}_\alpha \cup (U_{\alpha+1/2})_0$ and both $\mathscr{Z}_\alpha$ and $(U_{\alpha+1/2})_0$ are complete intersections, it follows that $\overline{\mathscr{V}}_\alpha$ is Cohen-Macaulay, hence S_2. In particular, the pair $(\overline{\mathscr{V}}_\alpha, \overline{\mathscr{V}}_{\alpha,0})$ is toric, as both $\overline{\mathscr{V}}_\alpha$ and $\overline{\mathscr{V}}_{\alpha,0}$ are torus-invariant subschemes of $\mathscr{Z}_\alpha$.

Step 2. It is enough to check the condition R_1 at the generic point of each irreducible component of $\overline{\mathscr{V}}_{\alpha,0}$. As $\overline{\mathscr{V}}_\alpha$ is a toric R-scheme, such components are toric strata of $(U_{\alpha+1/2})_0$ of dimension $n-1$. Let (J, j) be the datum of a non-empty subset $J \subseteq I := \{1, \ldots, n\}$, along with a distinguished element $j \in J$. Consider the $(n-1)$-dimensional stratum $Z_{(J,j)}$ of $\overline{\mathscr{V}}_{\alpha,0}$ given by the equations

$$Z_{(J,j)} := \begin{cases} x_{\alpha_i} = 0 & i \in J, \\ y_{\alpha_i+1} = 0 & i \in (I \setminus J) \cup \{j\}, \\ t = 0. \end{cases}$$

Up to relabeling of the indices, we can assume that $1 \in J$ and $j = 1$, in which case we write $Z_{(J,j)}$ simply as Z_J. After localizing at the generic point of Z_J, the functions $\{x_{\alpha_i} \colon i \in I \setminus J\}$, and $\{y_{\alpha_i+1} \colon i \in J \setminus \{1\}\}$ become invertible, and hence the variables

$$\begin{aligned} y_{\alpha_i+1} &= x_{\alpha_i}^{-1} x_{\alpha_1} y_{\alpha_1+1} && i \in I \setminus J, \\ x_{\alpha_i} &= y_{\alpha_i+1}^{-1} x_{\alpha_1} y_{\alpha_1+1} && i \in J \setminus \{1\}, \end{aligned}$$

can be eliminated. Thus, locally at the generic point of Z_J, we have

$$\begin{aligned} \overline{\mathscr{V}}_\alpha &\overset{\text{loc}}{=} \left\{ x_{\alpha_1} \cdot \left(\prod_{i \in J \setminus \{1\}} y_{\alpha_i+1}^{-1} \right) \cdot (x_{\alpha_1} y_{\alpha_1+1})^{|J|-1} \cdot \left(\prod_{i \in I \setminus J} x_{\alpha_i} \right) = (x_{\alpha_1} y_{\alpha_1+1})^{-|\alpha|} \right\} \\ &= \left\{ (\text{invertible}) \cdot (x_{\alpha_1})^{|J|+|\alpha|} (y_{\alpha_1+1})^{|J|+|\alpha|-1} = 1 \right\} \end{aligned}$$

in $\operatorname{Spec}\left(R[x_{\alpha_i}^{\pm 1}, y_{\alpha_l+1}^{\pm 1} : i \in I \setminus J, l \in J \setminus \{1\}][x_{\alpha_1}, y_{\alpha_1+1}]\right)$.

If $|J| + |\alpha| > 1$ or $|J| + |\alpha| < 0$, then $\overline{\mathscr{V}}_{\alpha,0}$ does not contain Z_J, and there is nothing to prove.

If $|J| + |\alpha| = 0$, then y_{α_1+1} is invertible and it is a function of $x_{\alpha_i}^{\pm 1}$ and $y_{\alpha_l+1}^{\pm 1}$ with $i \in I \setminus J$ and $l \in J \setminus \{1\}$, so that

$$\overline{\mathscr{V}}_\alpha \overset{\text{loc}}{=} \operatorname{Spec}\left(R[x_{\alpha_i}^{\pm 1}, y_{\alpha_l+1}^{\pm 1} : i \in I \setminus J, l \in J \setminus \{1\}][x_{\alpha_1}]\right) \simeq \mathbb{G}_{m,R}^{n-1} \times_R \mathbb{A}_R^1.$$

In particular, $\overline{\mathscr{V}}_{\alpha,0} = \{x_{\alpha_1} = 0\}$.

Finally, if $|J| + |\alpha| = 1$, then x_{α_1} is invertible and it is a function of $x_{\alpha_i}^{\pm 1}$ and $y_{\alpha_l+1}^{\pm 1}$ with $i \in I \setminus J$ and $l \in J \setminus \{1\}$, so that

$$\overline{\mathscr{V}}_\alpha \overset{\text{loc}}{=} \operatorname{Spec}\left(R[x_{\alpha_i}^{\pm 1}, y_{\alpha_l+1}^{\pm 1} : i \in I \setminus J, l \in J \setminus \{1\}][y_{\alpha_1+1}]\right) \simeq \mathbb{G}_{m,R}^{n-1} \times_R \mathbb{A}_R^1.$$

In particular, $\overline{\mathscr{V}}_{\alpha,0} = \{y_{\alpha_1+1} = 0\}$.

We conclude that $\overline{\mathscr{V}}_\alpha$ is a normal toric irreducible scheme locally of finite type. The local computation above shows also that the divisor $\overline{\mathscr{V}}_{\alpha,0}$ is reduced. Further, in order to prove that $\overline{\mathscr{V}}_\alpha \cap (U_{\alpha+1/2} \setminus \mathscr{G}_m^n)$ has codimension two in $\overline{\mathscr{V}}_{n-1}$, it is enough to check that this intersection does not contain the generic point of any stratum Z_J. A point in $(U_{\alpha+1/2} \setminus \mathscr{G}_m^n)$ is characterized by the property that a pair of coordinates $(x_{\alpha_i}, y_{\alpha_i+1})$ for $i \in I$ vanishes simultaneously. However, this cannot happen at the generic point of Z_J, as the local equations above show. This concludes the proof of Lemma 7.3.2.4.

□

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