Approximate Load Models for Conic OPF Solvers

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***Abstract*—The global optimum of the optimal power flow (OPF) problem can be sought in various practical settings by adopting the conic relaxations, such as the second order cone programs (SOCPs) and semi-definite programs (SDPs). However, the ZIP (constant impedance, constant current, constant power) and exponential load models are not directly amenable with these conic solvers. Thus, these are mostly treated as constant power loads in the literature. In this letter, we propose two simple methods to approximate these static loads with good accuracy. The proposed methods perform much better than the traditional constant power approximation.**

***Index Terms*—ZIP load model, Exponential load model, Second order cone programming, Semi-definite programming, Optimal power flow.**

I. Introduction

THE load flow model is the main source of non-convexity for the traditional optimal power flow (OPF) problem, and thus making the problem NP-hard. Recently, various relaxations have been proposed to convexify the load flow model. The DistFlow model can be convexified by either adopting linear relaxations [1] or the SOCP relaxations [2], whereas the bus injection model can be convexified through the SDP relaxations [3]. However, these convexifications are not able to accurately capture the voltage dependence of the loads. This leads to the adoption of simple constant power load models in most of the cases. It is noteworthy that a load model having both the constant power and constant impedance terms, referred to as ZP, can be handled by the above relaxed solvers equally efficiently.

The voltage dependence of the loads plays an important part in the net nodal injections. The distribution automation project at the B.C. Hydro reported a response of 1.6% and 3.1% respectively in the active and reactive power demand per 1% voltage change in one of their spring day trials [4]. The accurate load models are more important for studies pertaining to voltage regulation, and particularly to conservation voltage reduction, where the voltages are held near the lower bounds to decrease the demand on the substation transformers.

In this letter, we propose two methods to approximate the static voltage dependent loads to high accuracy. Both the methods seek to find an equivalent ZP model for the original load. The ZP nature of the equivalent model makes it amenable with the convex relaxations of the OPF problem. We use the Binomial approximation and the linear regression analysis for deriving the equivalent models. Marti *et al* used an approach on the basis of curve fitting for deriving their equivalent models [5]. However, their focus was to solve the linear power flow problem and their final model consisted of constant power and constant impedance terms.

The first attempt to address this issue was reported in [6], where a rank relaxation was used to allow approximate representation of the ZIP loads with the SDP solver. However, this approach can not be extended to the SOCP and the linear DistFlow solvers. Whereas, our approach works equally well for all the mentioned solvers and is able to take into account both the ZIP and the exponential loads. It is also reported that the solver times increase by 20% to 30% in the method of [6]. Whereas, the numerical results show that the solver times do not increase in our method.

In this letter, we give the expressions considering the real power demand only. However, the proposed equivalent models are easily extendable to the reactive power loads by using the same procedure.

II. Binomial Approximation Method

The Binomial series, in terms of generalised Binomial coefficients, is explicitly written as:

$$(1+x)^{\delta} = \sum_{k=0}^{\infty} \binom{\delta}{k} x^k \tag{1}$$

The infinite series in (1) can be approximated to the first two terms, with high accuracy, provided $|x| << 1$ and $|\delta x| << 1$. The proposed Binomial approximation method (BAM) makes use of this assumption and the fact that under normal operating conditions the voltage magnitudes of all the buses in a network stay close to 1 in the per unit system. This allows us to write the monomial exponent of the node voltage magnitude as follows:

$$v^{\eta} = (1+\Delta v)^{\eta} \approx 1 + \eta \Delta v \tag{2}$$

Here, v denotes the node voltage magnitude and η is such that $|\eta \Delta v| << 1$. We also denote the square of the node voltage magnitude by the auxiliary variable u, and let $u = 1 + \Delta u$. Setting $\eta = 2$ in (2), we have the following key relation:

$$\Delta u \approx 2\Delta v \tag{3}$$

A. ZIP Loads

The ZIP static load model is defined by a second-order polynomial as given below:

$$p = \alpha^p + \alpha^z v^2 + \alpha^i v \tag{4}$$

α^p, α^z and α^i are scalar coefficients representing the parameters of this model, and p denotes the real power demand.

We propose to approximate the ZIP model in (4) with an equivalent ZP model, whose parameters can be easily obtained

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from the original ZIP model parameters. The proposed model is derived below, starting from (4) and using the relation (3).

$$p = \alpha^p + \alpha^z u + \alpha^i (1 + \Delta v) \tag{5}$$

$$p \approx \alpha^p + \alpha^z u + \alpha^i (1 + \frac{\Delta u}{2}) \tag{6}$$

$$p \approx \alpha^p + \alpha^z u + \alpha^i (1 + \frac{u-1}{2}) \tag{7}$$

$$p \approx \underbrace{(\alpha^p + \frac{\alpha^i}{2})}_{\alpha_1^p} + \underbrace{(\alpha^z + \frac{\alpha^i}{2})}_{\alpha_1^z} u \tag{8}$$

The relation (8) defines the proposed equivalent ZP model whose parameters are α_1^p and α_1^z. It is evident from this relation that the constant current parameter of the original ZIP model gets split equally between the constant power and constant impedance parameters of the equivalent model.

B. Exponential Loads

This static load model expresses the power demand as an exponential function of the voltage magnitude, as given below:

$$p = p_0 v^\beta \tag{9}$$

The equivalent ZP model, with the parameters as α_2^p and α_2^z, for (9) is derived by employing (2) and the procedure inline with the ZIP models as given in (5) - (8), and is written as:

$$p \approx \underbrace{p_0(1 - \frac{\beta}{2})}_{\alpha_2^p} + \underbrace{\frac{p_0 \beta}{2}}_{\alpha_2^z} u \tag{10}$$

III. Linear Regression Method

Linear regression method (LRM) estimates the parameters of the equivalent ZP model based on minimizing the summation of the squared residuals. Let us suppose that the parameters of the estimated model are $\widehat{\alpha^p}$ and $\widehat{\alpha^z}$. Also, we sample the voltage magnitude, in a specified range of interest, to collect n data points. Let $\mathbf{v}$ be the real valued vector which represents these voltage magnitude data points, and $\mathbf{v_m}$ be the vector obtained by $\mathbf{m}$ times multiplying the vector $\mathbf{v}$ by itself in an element-wise fashion. This implies that each element in $\mathbf{v_2}$ is the square of the corresponding element in $\mathbf{v}$. Then we can write the following:

$$\mathbf{V} \begin{pmatrix} \widehat{\alpha^p} \\ \widehat{\alpha^z} \end{pmatrix} = \mathbf{p} + \mathbf{r} \tag{11}$$

Here $\mathbf{V} \in \mathbb{R}^{n \times 2}$ is defined as the matrix $[\mathbf{1} \ \mathbf{v_2}]$, $\mathbf{p}$ is the vector of the real power values corresponding to $\mathbf{v}$, and $\mathbf{r}$ denotes the residual vector. Note that we employ the notation of a bold face lower case letter representing a vector and a bold face upper case letter representing a matrix. The LRM aims to minimize the sum of squares of the residual vector in order to have the best possible equivalent ZP model. The solution of the LRM estimator will, thus, be:

$$\begin{pmatrix} \widehat{\alpha^p} \\ \widehat{\alpha^z} \end{pmatrix} \approx (\mathbf{V}^T \mathbf{V})^{-1} \mathbf{V}^T \mathbf{p} \tag{12}$$

A. ZIP Loads

The ZIP load definition given by (4) results in the following transformation when used with the LRM estimator of (12).

$$\begin{pmatrix} \widehat{\alpha^p} \\ \widehat{\alpha^z} \end{pmatrix} \approx \underbrace{(\mathbf{V}^T \mathbf{V})^{-1} \mathbf{V}^T \mathbf{W}}_{\mathbf{C}} \begin{pmatrix} \alpha^p \\ \alpha^z \\ \alpha^i \end{pmatrix} \tag{13}$$

Here, $\mathbf{W} \in \mathbb{R}^{n \times 3}$ is defined as the matrix $[\mathbf{1} \ \mathbf{v_2} \ \mathbf{v}]$. The transformation matrix $\mathbf{C}$ relates the equivalent model space with the actual ZIP space, and it takes the following shape.

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & c^p(\mathbf{v}) \\ 0 & 1 & c^z(\mathbf{v}) \end{pmatrix} \tag{14}$$

Where $c^p(\mathbf{v}) = \frac{\overline{\mathbf{v}}\,\overline{\mathbf{v_4}} - \overline{\mathbf{v_2}}\,\overline{\mathbf{v_3}}}{\overline{\mathbf{v_4}} - (\overline{\mathbf{v_2}})^2}$ and $c^z(\mathbf{v}) = \frac{\overline{\mathbf{v_3}} - \overline{\mathbf{v}}\,\overline{\mathbf{v_2}}}{\overline{\mathbf{v_4}} - (\overline{\mathbf{v_2}})^2}$. The overline denotes mean value operator for a vector, thus $\overline{\mathbf{v}}$ is the mean value of the vector $\mathbf{v}$. It is evident that these coefficients, $c^p(\mathbf{v})$ and $c^z(\mathbf{v})$, are adjustable and their value depends on the vector of the collected voltage magnitude samples only. The detailed derivation of $\mathbf{C}$ is given in the appendix.

B. Exponential Loads

The non-linear nature of (9) does not allow a general expression of the equivalent ZP model for the exponential loads.Therefore, the following procedure should be followed to get the equivalent ZP parameters:

- Sample the voltage magnitude in the desired range and obtain the voltage vector $\mathbf{v}$. This vector is used to calculate the real valued matrix $\mathbf{V} = [\mathbf{1} \ \mathbf{v_2}]$.
- For each sampled value in $\mathbf{v}$, obtain a corresponding load value using (9). The resultant vector is designated as $\mathbf{p}$.
- Finally, leverage (12) to estimate the parameters of the exponential model.

IV. Results

The two proposed approaches for the modelling of static loads are tested for their effectiveness on both the nodal and the network levels. On the nodal level, the accuracy of the node injections for typical values of the load parameters is recorded. While on the network level, the effect of the proposed load models on the voltage profile of the network is observed.

A. Accuracy of the Proposed Load Models

In this set of tests, the real power demand at a node is studied for a typical voltage range considering the different load models. Fig. 1 and Fig. 2 show the comparison of these different approaches. The parameters of the ZIP model are taken from [5], where $\alpha^p = 0.466$, $\alpha^z = 0.025$, and $\alpha^i = 0.51$. While the parameters for the exponential model are taken from [7], in which the most prevalent value of β throughout the world is estimated to be 0.7. The value of p_0 is 1. For the LRM approach, the vector $\mathbf{v}$ is computed by taking linearly spaced samples with a step length of 0.01 in the range of 0.7 to 1.3. Thus, the values of $c^p(\mathbf{v})$ and $c^z(\mathbf{v})$ are respectively 0.4877 and 0.4969. It is obvious from the Fig. 1 and Fig. 2 that the traditional method of

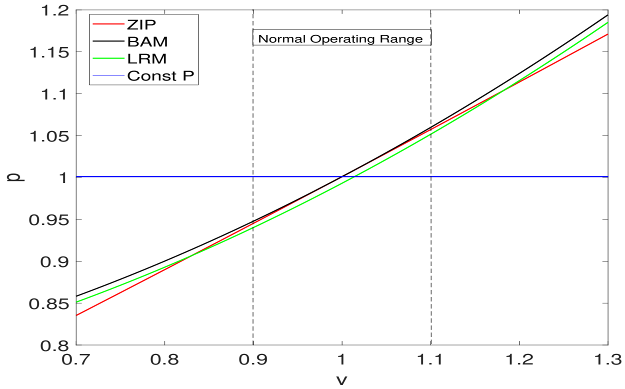

Fig. 1. Accuracy of Various Models for the ZIP Loads.

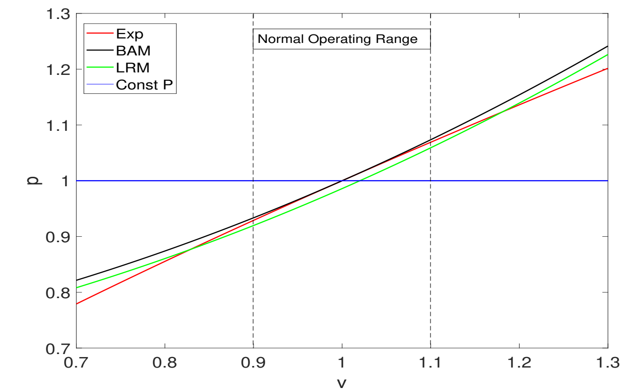

Fig. 2. Accuracy of Various Models for the Exponential Loads.

approximating the static loads with the constant power (Const-P) type results in appreciable discrepancy in the net injection, which is efficiently reduced by the proposed BAM and LRM approaches. However, the BAM approach works very well for the normal operating range, while as the LRM approach should be employed when the network is under voltage stress. This is because the LRM method is more accurate when the voltage is outside the normal operating range.

B. Impact on the Network Voltage Profile

The load flow study is carried on the UKGDS-95 bus system, whose data is taken from [8]. However, the active and reactive power loads are slightly modified such that they are split equally between the constant power, constant current and constant impedance parts at the unity voltage. The load flow studies are carried out through the current injection algorithm [9] by utilizing the full ZIP models. This study is carried in MATLAB and its solution acts as the benchmark. In order to demonstrate the effectiveness of the proposed load modelling approaches for the OPF based solvers, the load flow studies are also carried by setting up a second order conic optimization framework in-line with [2], which is solved through the CPLEX optimizer [10] interfaced with MATLAB. We separately test the Const-P, BAM and LRM concepts through this optimization based load flow study. Fig. 3 shows

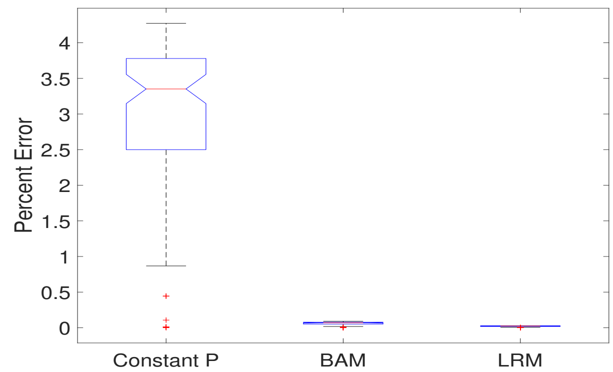

Fig. 3. Percentage Normalized Error in the Network Voltage Magnitudes with Different Load Modelling Approaches for the UKGDS-95 Bus System.

TABLE I
AVERAGE TIME TAKEN FOR THE CONST-P, BAM AND LRM APPROACHES TO CONVERGE

Approach	Const P	BAM	LRM
Average CPU Time (sec)	0.0878	0.0873	0.0881

the percentage normalized error of these load flow studies as against the benchmark. It is clear that the BAM and the LRM approaches can effectively reduce the voltage errors which are present in the traditional constant power modelling approach.

C. Computational Performance

The computational tests are performed on a 3.5 GHz Intel Xeon E5 processor with 64 GB of RAM, where the second order cone programming based load flow routine, along the lines of [2], is coded in MATLAB, and then subsequently solved using the CPLEX 12.7 optimization studio. These load flow studies are carried on the UKGDS-95 bus system, with the same data as in the previous subsection. Here again the Const-P, BAM and LRM approaches are tested separately. We run the load flow ten times for each approach, and then average the time taken in the ten runs, which is given in the Table I. It is evident from this table that the convergence time by the BAM and the LRM approaches is almost same as the Const-P approach. This means that the proposed approaches do not increase the solver times.

V. CONCLUSION

This letter presents two different methods for approximating the voltage dependence of the static loads for the conic OPF solvers. Under normal operating conditions BAM yields highly accurate results, however the LRM performs better when the network in under voltage stress.

APPENDIX A
DERIVATIONS PERTAINING TO THE LRM

Supposing that the voltage range is sampled to collect n samples in the vector $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & . & . & v_n \end{bmatrix}^T$. Thus, the

LRM estimator for the ZIP loads satisfies the following:

$$\underbrace{\begin{pmatrix}1 & v_1^2\\ 1 & v_2^2\\ . & .\\ . & .\\ 1 & v_n^2\end{pmatrix}}_{\mathbf{V}}\begin{pmatrix}\widehat{\alpha^p}\\ \widehat{\alpha^z}\end{pmatrix}\approx\underbrace{\begin{pmatrix}1 & v_1^2 & v_1\\ 1 & v_2^2 & v_2\\ . & . & .\\ . & . & .\\ 1 & v_n^2 & v_n^2\end{pmatrix}}_{\mathbf{W}}\begin{pmatrix}\alpha^p\\ \alpha^z\\ \alpha^i\end{pmatrix} \tag{15}$$

This implies:

$$\mathbf{V}^T\mathbf{V}=\begin{pmatrix}1 & 1 & . & . & 1\\ v_1^2 & v_2^2 & . & . & v_n^2\end{pmatrix}\begin{pmatrix}1 & v_1^2\\ 1 & v_2^2\\ . & .\\ . & .\\ 1 & v_n^2\end{pmatrix} \tag{16}$$

$$=\begin{pmatrix}n & \sum v^2\\ \sum v^2 & \sum v^4\end{pmatrix} \tag{17}$$

Also evaluate the following matrix product:

$$\mathbf{V}^T\mathbf{W}=\begin{pmatrix}1 & 1 & . & . & 1\\ v_1^2 & v_2^2 & . & . & v_n^2\end{pmatrix}\begin{pmatrix}1 & v_1^2 & v_1\\ 1 & v_2^2 & v_2\\ . & . & .\\ . & . & .\\ 1 & v_n^2 & v_n\end{pmatrix} \tag{18}$$

$$=\begin{pmatrix}n & \sum v^2 & \sum v\\ \sum v^2 & \sum v^4 & \sum v^3\end{pmatrix} \tag{19}$$

Thus, the transformation matrix is given as:

$$\mathbf{C}=(\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T\mathbf{W} \tag{20}$$

$$=\frac{1}{n\sum v^4-(\sum v^2)^2}\begin{pmatrix}\sum v^4 & -\sum v^2\\ -\sum v^2 & n\end{pmatrix} \tag{21}$$
$$\begin{pmatrix}n & \sum v^2 & \sum v\\ \sum v^2 & \sum v^4 & \sum v^3\end{pmatrix}$$

$$=\frac{1}{n\sum v^4-(\sum v^2)^2}\left(\begin{matrix}n\sum v^4-(\sum v^2)^2...\\ -n\sum v^2+n\sum v^2...\end{matrix}\right. \tag{22}$$
$$\left.\begin{matrix}\sum v^4\sum v^2-\sum v^2\sum v^4 & \sum v\sum v^4-\sum v^2\sum v^3\\ -(\sum v^2)^2+n\sum v^4 & -\sum v\sum v^2+n\sum v^3\end{matrix}\right)$$

$$=\begin{pmatrix}1 & 0 & \dfrac{\sum v\sum v^4-\sum v^2\sum v^3}{n\sum v^4-(\sum v^2)^2}\\ 0 & 1 & \dfrac{-\sum v\sum v^2+n\sum v^3}{n\sum v^4-(\sum v^2)^2}\end{pmatrix} \tag{23}$$

Divide the numerator and the denominator of (23) by n^2, we get:

$$\mathbf{C}=\begin{pmatrix}1 & 0 & \dfrac{\overline{\mathbf{v}}\,\overline{\mathbf{v}}_\mathbf{4}-\overline{\mathbf{v}}_\mathbf{2}\overline{\mathbf{v}}_\mathbf{3}}{\overline{\mathbf{v}}_\mathbf{4}-(\overline{\mathbf{v}}_\mathbf{2})^2}\\ 0 & 1 & \dfrac{\overline{\mathbf{v}}_\mathbf{3}-\overline{\mathbf{v}}\,\overline{\mathbf{v}}_\mathbf{2}}{\overline{\mathbf{v}}_\mathbf{4}-(\overline{\mathbf{v}}_\mathbf{2})^2}\end{pmatrix} \tag{24}$$

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