Adaptive Output Regulation via Nonlinear Luenberger Observer-based Internal Models and Continuous-Time Identifiers

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Abstract

In Marconi et al. 2007, the theory of nonlinear Luenberger observers was exploited to prove that a solution to the asymptotic output regulation problem for minimum-phase normal forms always exists. The paper provided an existence result and a very general regulator structure, although unfortunately, no constructive method was given to design all the degrees of freedom of the regulator. In this paper, we complete this design by introducing an adaptive unit tuning the regulator online by employing system identification algorithms selecting the "best" parameters according to a certain optimization policy. Instead of focusing on a single identification scheme, we give general conditions under which an algorithm may be used in the framework, and we develop a particular least-squares identifier satisfying these requirements. Closed-loop stability results are given, and it is shown that the asymptotic regulation error is related to the prediction capabilities of the identifier evaluated along the ideal error-zeroing steady-state trajectories.

Keywords: Output Regulation; Adaptive Output Regulation; Identification for Control

1. Introduction

The goal of output regulation is to design a control law able to steer to zero a particular output of a plant, referred to as regulation error, despite the presence of exogenous disturbances, modeled as trajectories of an autonomous system (typically referred to as "exosystem"). In the context of linear systems, this problem was completely solved in [15, 16, 13] where the authors showed that any *robust* regulator necessarily embeds an *internal model* of the exogeneous signals able to generate the ideal error-zeroing control law in steady state. The design is then completed by adding a *stabilizer* which stabilizes the system around this steady-state.

The problem of output regulation for nonlinear systems is more challenging and still open. "Local designs" first appeared in [21, 19, 17, 6], strongly inspired by the linear setting. A fully nonlinear theory of output regulation emerged in [7, 9, 8] based on the theory of invariant manifolds. Regulators were proposed for single-input-single-output (SISO) normal forms, under an *immersion* assumption on the ideal steady-state control in [8, 18, 11] which was then relaxed in [25]. More precisely, in [25], it was shown that for the class of minimum-phase SISO normal forms a solution of the output regulation problem always exists. This result, however, is not *constructive*, in the sense that, although the existence of a regulator is guaranteed and the general structure fixed, no general procedure is given to choose all its degrees of freedom. By leveraging this existence result, in [24], some methods have been proposed to construct "approximate" regulators. Nevertheless, the construction of the regulator remains mostly impractical.

In this paper, we develop further the approach of [25] by endowing the regulator with an adaptive unit tuning automatically the regulator's degrees of freedom, whose correct value is unknown to the designer. Adaptation is cast as a user-defined *system identification* problem, formally expressed as an optimization task defined on the closed-loop available signals. Sufficient conditions are given under which an identification scheme can be embedded in the adaptive unit, and the main result relates the prediction capabilities of the identifier, evaluated along the ideal error-zeroing trajectories, to the asymptotic regulation performance.

Among the different adaptive designs available for linear and nonlinear systems, the proposed approach shares some similarities with [14, 5], as the requirements asked here to the identifiers refer to the same stability properties. However, unlike [5], our design is purely nonlinear, and unlike [14], we do not rely on a high-gain internal model and an actual regression is guaranteed to exist in steady-state between the two input signals of the identifier without any assumption on the steady-state error-zeroing input. Other adaptive designs for nonlinear systems can be found, for instance, in [30], under a linear immersion assumption, in [28] with high-gain internal models and under an immersion assumption into a linearly parametrized exosystem,

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and in [29], where the same regulator as in this paper is made adaptive by using a canonical Lyapunov-based adaptive control strategy. Compared to these works, here we propose a regulator in a more general setting, without immersion assumptions, and in which adaptation is approached in a broader sense as a system identification problem, without assuming that a correct model exists in the identifier's model set. An approximate regulation result is given, with the asymptotic bound of the regulation error that is directly related to the performances of the identifier.

The paper is organized as follows. In Section 2 we describe our nonlinear setting, we recall the main result of [25], and we highlight the contribution of the paper. The adaptive regulator is constructed in Section 3, and the main result of the paper is given in Section 4. Finally, in Section 5, we provide an example of least-square based identifier satisfying the requirements.

Notation: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ the set of naturals and $\mathbb{R}_+ := [0, \infty)$. $\mathbb{S}_{n_\theta}$ denotes the set of positive semi-definite symmetric matrices of dimension n_θ. If $S \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a closed set, $|x|_S := \inf_{s\in S} |x - s|$ denotes the distance of $x \in \mathbb{R}^n$ to S. $\mathcal{C}^1$ denotes the set of continuously differentiable functions. A continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of class-K ($f \in \mathcal{K}$) if it is strictly increasing and $f(0) = 0$. It is of class-K_∞ ($f \in \mathcal{K}_\infty$) if $f \in \mathcal{K}$ and $f(s) \to \infty$ as $s \to \infty$. A continuous function $\beta : \mathbb{R}^2_+ \to \mathbb{R}_+$ is of class-KL ($\beta \in \mathcal{KL}$) if $\beta(\cdot, t) \in \mathcal{K}$ for each $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is strictly decreasing to zero for each $s \in \mathbb{R}_+$. $\mathrm{Id} \in \mathcal{K}$ denotes the map $\mathrm{Id}(s) = s$. With $h : \mathbb{R}^n \to \mathbb{R}$ a $\mathcal{C}^1$ function in the arguments $x_1, \dots, x_n$, and $f : \mathbb{R}^n \to \mathbb{R}$, for each $i \in \{1, \dots, n\}$ we denote by $L_f^{(x_i)} h$ the map $x \mapsto L_{f(x)}^{(x_i)} h(x) := \partial h/\partial x_i(x) f(x)$. In the text "ISS" stands for Input-to-State Stability [31]. Finally, for a matrix M, $M^\dagger$ denotes its Moore-Penrose pseudo-inverse and $\mathrm{msv}(M)$ its minimal nonzero singular value, namely the smallest nonzero eigenvalue of $M^\top M$.

2. The Framework

2.1. Problem Statement

We consider systems of the form

$$\begin{array}{rcl} \dot z &=& f(w, z, y) \\ \dot y &=& q(w, z, y) + b(w, z, y)u, \end{array} \tag{1}$$

with state (z, y) taking values in $\mathbb{R}^{n_z} \times \mathbb{R}$, control input $u \in \mathbb{R}$, measured output $y \in \mathbb{R}$, and with $w \in \mathbb{R}^{n_w}$ an exogenous input that we suppose to belong to the set of solutions of an *exosystem* of the form

$$\dot w = s(w), \tag{2}$$

originating in a compact invariant subset W of $\mathbb{R}^{n_w}$. We suppose that $f : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \times \mathbb{R} \to \mathbb{R}^{n_z}$ is locally Lipschitz, $q : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \times \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \times \mathbb{R} \to \mathbb{R}$ are $\mathcal{C}^1$, and for any compact set $\mathrm{Z} \subset \mathbb{R}^{n_z}$ and $\mathrm{Y} \subset \mathbb{R}$, there exists $\underline{b} > 0$ such that

$$b(w, z, y) \geq \underline{b} \qquad \forall (w, z, y) \in \mathrm{W} \times \mathrm{Z} \times \mathrm{Y}\ . \tag{3}$$

Given a compact set $\mathrm{Z}_0 \times \mathrm{Y}_0 \subset \mathbb{R}^{n_z} \times \mathbb{R}$, the goal of this paper is to design a regulator

$$\left\{ \begin{array}{rcl} \dot x_c &=& f_c(x_c, y) \\ u &=& h_c(x_c, y), \end{array} \right. \tag{4}$$

with state $x_c \in \mathbb{R}^{n_c}$, and a set $\mathrm{X}_c \subset \mathbb{R}^{n_c}$, such that the solutions $\mathbf{x} := (w, z, y, x_c)$ of the closed-loop system (1),(2),(4) originating in $\mathrm{W} \times \mathrm{Z}_0 \times \mathrm{Y}_0 \times \mathrm{X}_c$ are defined on $[0, +\infty)$ and uniformly eventually equibounded[1]and satisfy $\limsup_{t\to\infty} |y(t)| \leq \mu_w^\star$, with $\mu_w^\star$ as small as possible according to an optimality index to be specified. In other words, for a particular solution w of (2), the positive scalar $\mu_w^\star$ represents the desired/tolerated asymptotic bound on the output y, thus allowing for regulation objectives milder than the usual *asymptotic output regulation* where $\mu_w^\star = 0$.

We consider the problem at hand under the following minimum-phase assumption.

A1) *There exists a $\mathcal{C}^1$ map π defined on an open neighborhood of W and with values in $\mathbb{R}^{n_z}$, satisfying*

$$L_{s(w)}^{(w)} \pi(w) = f(w, \pi(w), 0)$$

on its domain of definition, and such that the system

$$\begin{array}{rcl} \dot w &=& s(w) \\ \dot z &=& f(w, z, y) \end{array} \tag{5}$$

is ISS relative to the compact set

$$\mathcal{A} = \{(w, z) \in \mathrm{W} \times \mathbb{R}^{n_z}\ :\ z = \pi(w)\}$$

and with respect to the input y.

We observe that, by using the same arguments of [25], A1 could be weakened to a local asymptotic stability requirement of the set $\mathcal{A}$ for the *zero dynamics*

$$\dot w = s(w), \qquad \dot z = f(w, z, 0), \tag{6}$$

as long as the domain of attraction includes $\mathrm{W} \times \mathrm{Z}_0 \times \mathrm{Y}_0$. This, however, comes at the price of a more involved technical treatise without substantial conceptual added value. Assumption A1 is customary in the literature of output regulation (see e.g. [20, 27]). Necessary and sufficient conditions for the existence of a single-valued steady-state map π can be found in [27]. By definition, continuity of π holds whenever $\mathcal{A}$ is closed, while its differentiability has to be assumed.

To conclude, we observe that, although we have chosen to restrict our attention to systems with unitary relative degree, extension to higher relative degree is straightforward along the lines of [22, 25].

[1] That is, there exists a compact set $\mathrm{K} \subset \mathbb{R}^{n_z} \times \mathbb{R} \times \mathbb{R}^{n_c}$ and a $\tau \geq 0$ such that every solution $\mathbf{x} := (z, y, x_c)$ of (1)-(4) originating in $\mathrm{Z}_0 \times \mathrm{Y}_0 \times \mathrm{X}_c$ satisfies $\mathbf{x}(t) \in \mathrm{K}$ for all $t \geq \tau$.

2.2. The Marconi-Praly-Isidori Regulator

Let us define

$$u^\star(w,z,y) = -\frac{q(w,z,y)}{b(w,z,y)} \ . \tag{7}$$

From (1) and A1, we see that the input u should ideally take the value $u^\star(w,\pi(w),0)$ in steady state. The ability of the regulator to generate such an input is typically referred to as the "internal model property". Under A1, it is proved in [25] that this can always be done by means of a controller of the form

$$\begin{array}{rcl} \dot\eta &=& F\eta + Gu \\ u &=& \gamma(\eta) + K(y) \end{array} \tag{8}$$

with state η taking values in $\mathbb{R}^{n_\eta}$, with $n_\eta = 2(n_w+n_z+1)$, (F,G) a controllable pair with F a Hurwitz matrix, and with $\gamma : \mathbb{R}^{n_\eta} \to \mathbb{R}$ and $K : \mathbb{R} \to \mathbb{R}$ suitably defined continuous functions. More details about this design will be given in Lemma 1 below. Intuitively, the function γ is tuned to reproduce $u^\star(w,\pi(w),0)$ in steady state, while K plays the role of stabilizer. The latter, in [25], is selected as a high-gain feedback in order to steer y to small values regardless of the value of γ. Then, once y is close to zero, the term $\gamma(\eta)$ steers y to zero by approaching $u^\star(w,\pi(w),0)$, thus ensuring asymptotic regulation.

Although constructive insight is given in [25] for the design of F, G, and K, the map γ is only guaranteed to exist, without any analytical procedure to design it outside the class of linear systems. As a consequence, the construction of the regulator (8) is hardly applicable in practice and, even if some numerical methods to approximate γ have been proposed (see [24]), their implementation remains difficult.

2.3. Contribution of the Paper

Towards a constructive design solution which is effectively implementable, in this paper we propose a regulator that employs online adaptation to approximate the map γ at run time. In particular, we augment the regulator of [25] with an adaptive unit, called the *identifier*, whose aim it to produce and update an estimate $\hat\gamma$ of the function γ to employ. Specifically, the identifier is a system that solves a user-defined *system identification* problem [23] cast on the closed-loop signals and adapts its guess of γ on the basis of an optimization problem. Instead of proposing a particular design of the identifier, we give sufficient stability conditions characterizing a class of algorithms that can be used, thus leaving to the designer a further degree of freedom in the choice of the actual identification strategy.

The main result of the paper, in turn, is to relate the asymptotic bound $\mu_w^\star$ on the regulated variable y to the prediction performance of the chosen identifier, in particular leading to *asymptotic regulation* (i.e., $\mu_w^\star = 0$) whenever the correct map γ is in the *model set* of the employed identification algorithm. Finally, we show how least-square identifiers fit into the framework, thus providing a fully constructive design.

3. The Regulator Structure

We consider a controller of the form

$$\begin{array}{rcl} \dot\eta &=& F\eta + Gu \\ \dot\xi &=& \varphi(\xi,\eta,u) \\ \dot\zeta &=& \ell(\zeta,y) \\ \dot u &=& \kappa(y,\zeta) + \psi(\xi,\eta,u) \end{array} \tag{9}$$

with state $(\eta,\xi,\zeta,u) \in \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\zeta} \times \mathbb{R}$, input y, and output u. We refer to the subsystems η, ξ, ζ and u respectively as the *internal model unit*, the *identifier*, the *derivative observer* and the *stabilizer*. These subsystems are constructed in the next subsections.

3.1. The Internal Model Unit

In this section, we detail the choice of the pair (F,G) following the guidelines of [25]. We start by reporting the following result, formalizing the fact that the subsystem η provided by an appropriate output map γ has the internal model property.

Lemma 1 ([25]). *Suppose that A1 holds and pick $n_\eta = 2(n_w+n_z+1)$. Then there exist a Hurwitz matrix $F \in \mathbb{R}^{n_\eta\times n_\eta}$, a matrix $G \in \mathbb{R}^{n_\eta\times 1}$, and continuous maps $\tau : \mathbb{R}^{n_w} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_\eta}$ and $\gamma : \mathbb{R}^{n_\eta} \to \mathbb{R}$ such that*

$$\gamma \circ \tau(w,z) = u^\star(w,z,0) \qquad \forall (w,z) \in \mathcal{A} \tag{10}$$

with $u^\star$ defined in (7), and the system

$$\begin{array}{rcl} \dot w &=& s(w) \\ \dot z &=& f(w,z,y) \\ \dot\eta &=& F\,\eta + G(u^\star(w,z,y) + \delta_0). \end{array}$$

is ISS relative to the set

$$\mathcal{G} := \Big\{ (w,z,\eta) \in \mathcal{A} \times \mathbb{R}^{n_\eta} \ : \ \eta = \tau(w,\pi(w)) \Big\},$$

and with respect to the input (y,δ_0).

More precisely, according to [25], the pair (F,G) can be chosen as a real realization of any complex pair (F_c,G_c) of dimension n_w+n_z+1, with G_c a vector with non zero entries, and F_c such that its eigenvalues λ_i have sufficiently negative real part and $(\lambda_1,\dots,\lambda_{n_\eta/2})$ is outside a set of zero-Lebesgue measure in $\mathbb{R}^{n_\eta/2}$. As for the map K, it must be chosen such that $K(0) = 0$, $K(y)y < 0$ for all non zero $y \in \mathbb{R}$ and an appropriate small gain condition is verified. If $\mathcal{A}$ is also locally exponentially stable for (6), then K can be taken linear, with sufficiently large gain.

Intuitively, Lemma 1 says that $\gamma(\eta)$ gives a proxy for $u^\star(w,\pi(w),0)$ in steady state when (y,δ_0) is small. This is why it is used in (8) as key element of the internal model unit. However, since this map γ is not available, we are going to look for another proxy of $u^\star(w,\pi(w),0)$ and identify γ online.

3.2. The Identifier

The goal of the identifier is to estimate online the unknown map γ which, according to (10), relates the steady-state signals

$$\alpha^\star_{\text{in}} := \tau(w, \pi(w)), \quad \alpha^\star_{\text{out}} := u^\star(w, \pi(w), 0), \tag{11}$$

as $\alpha^\star_{\text{out}} = \gamma(\alpha^\star_{\text{in}})$. This latter relation is interpreted here as a regression model between the signals $\alpha^\star_{\text{in}}$ and $\alpha^\star_{\text{out}}$, and the estimation of γ cast as an identification problem. Unfortunately, the signals in question are unknown, and the identifier must be fed by some known "proxies" $(\alpha_{\text{in}}, \alpha_{\text{out}})$ instead. To that end, we observe that η is a good proxy for the quantity $\alpha^\star_{\text{in}}$ close to the attractor $\mathcal{G}$ introduced in Lemma 1, while u will be shown to provide a proxy for $\alpha^\star_{\text{out}}$ when $(y, \dot{y})$ is small. Therefore, we choose

$$\alpha_{\text{in}} := \eta\ , \qquad \alpha_{\text{out}} := u. \tag{12}$$

The design of the identifier then relies on the following steps.

Let $A_{\text{in}} \subset \mathbb{R}^{n_\eta}$ and $A_{\text{out}} \subset \mathbb{R}$ be compact sets verifying

$$\tau(\mathrm{W}, \pi(\mathrm{W})) \subseteq A_{\text{in}} \quad , \quad u^\star(\mathrm{W}, \pi(\mathrm{W}), 0) \subseteq A_{\text{out}}\ .$$

We start by considering a class $\mathcal{I}$ of functions with value in $A_{\text{in}} \times A_{\text{out}}$ containing the ideal regression signals defined in (11), namely satisfying

$$\begin{aligned}\mathcal{I} \supseteq \big\{ &(\alpha^\star_{\text{in}}, \alpha^\star_{\text{out}}) : \mathbb{R}_+ \to A_{\text{in}} \times A_{\text{out}} \\ &\alpha^\star_{\text{in}} = \tau(w, \pi(w)),\ \alpha^\star_{\text{out}} = u^\star(w, \pi(w), 0), \\ &\qquad w \textit{ solves } (2) \textit{ with } w(0) \in \mathrm{W} \big\}\ ,\end{aligned} \tag{13}$$

and a *model set*, namely a C^1 map $\hat{\gamma} : \mathbb{R}^{n_\theta} \times \mathbb{R}^{n_\eta} \to \mathbb{R}$ such that a parametrized function of the form $\hat{\gamma}(\theta, \cdot)$ is a good model for the ideal map $\gamma(\cdot)$ in some sense to be defined. More precisely, the ultimate goal is to identify an "optimal" θ (hereafter indicated with $\theta^\star$) minimizing the mismatch between $\hat{\gamma}(\theta^\star, \alpha^\star_{\text{in}})$ and $\alpha^\star_{\text{out}} = \gamma(\alpha^\star_{\text{in}})$. This minimization is made formal through the definition of a cost functional associated to the elements in $\mathcal{I}$, assigning to each value of θ and each $t \in \mathbb{R}_+$ a value to be minimized. More precisely, to each $\alpha^\star := (\alpha^\star_{\text{in}}, \alpha^\star_{\text{out}}) \in \mathcal{I}$, we associate a function $\mathcal{J}_{\alpha^\star} : \mathbb{R}^{n_\theta} \times \mathbb{R}_+ \to \mathbb{R}_+$ of the form

$$\mathcal{J}_{\alpha^\star}(\theta, t) := \int_0^t \mathrm{c}\big(\varepsilon(\alpha^\star(s), \theta),\, t,\, s\big) ds + \varrho(\theta) \tag{14}$$

where

$$\varepsilon(\alpha(s), \theta) := \alpha_{\text{out}}(s) - \hat{\gamma}(\theta, \alpha_{\text{in}}(s))$$

denotes the *prediction error* at time s of the model $\hat{\gamma}(\theta, \cdot)$ corresponding to a given choice of $\theta \in \mathbb{R}^{n_\theta}$, $\mathrm{c} : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ assigns to each error and each time instant s a "cost", and where $\varrho : \mathbb{R}^{n_\theta} \to \mathbb{R}_+$ is a *regularization* term.

The next step consists in choosing $n_\xi, n_\theta \in \mathbb{N}$, a continuous map $\varphi : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta} \times \mathbb{R}$, and a bounded function $h : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\theta}$, such that the following *identifier*

$$\begin{aligned}\dot{\xi} &= \varphi(\xi, \alpha_{\text{in}}, \alpha_{\text{out}}) \\ \theta &= h(\xi)\end{aligned} \tag{15}$$

asymptotically produces the optimal model parameter $\theta^\star$, when fed with ideal inputs $\alpha^\star := (\alpha^\star_{\text{in}}, \alpha^\star_{\text{out}}) \in \mathcal{I}$. In other words, system (15) then asymptotically converges to a trajectory $\xi^\star : \mathbb{R}_+ \to \mathbb{R}^{n_\xi}$ whose output $\theta^\star := h(\xi^\star)$ is such that the inferred *prediction model*

$$\alpha^\star_{\text{out}} \approx \hat{\gamma}(\theta^\star, \alpha^\star_{\text{in}}) \tag{16}$$

fits "at best" in the sense of the cost function $\mathcal{J}_{\alpha^\star}$.

As explained above, because the ideal input $(\alpha^\star_{\text{in}}, \alpha^\star_{\text{out}})$ defined in (11) is unknown, we will need to feed the identifier (15) with some proxy signals $(\alpha_{\text{in}}, \alpha_{\text{out}})$ defined in (12). Therefore, we must also ask that, whenever α becomes close to $\alpha^\star$, the identifier solutions ξ become close to the ideal one $\xi^\star$, whose output $\theta^\star := h(\xi^\star)$ is a pointwise solution to the minimization problem associated with (14). This is formalized via an ISS-like property in the following requirement.

Requirement 1. *The pair (φ, h) with h bounded is said to fulfill the* identifier requirement *relative to $\mathcal{I}$ if there exist $\beta_\xi \in \mathcal{KL}$, $\rho_\xi \in \mathcal{K}$, a scalar $\bar{t}$, and a compact set $\Xi \subset \mathbb{R}^{n_\xi}$, such that for each $\alpha^\star = (\alpha^\star_{\text{in}}, \alpha^\star_{\text{out}}) \in \mathcal{I}$, there exists a function $\xi^\star : \mathbb{R}_+ \to \mathbb{R}^{n_\xi}$ verifying $\xi^\star(t) \in \Xi$ for all $t \geq \bar{t}$ and such that the following properties hold:*

1. ***Optimality:*** *for each $t \geq \bar{t}$, the output $\theta^\star := h(\xi^\star)$ satisfies*
$$\theta^\star(t) \in \arg\min_{\theta \in \mathbb{R}^{n_\theta}} \mathcal{J}_{\alpha^\star}(\theta, t).$$
2. ***Stability:*** *for each $d = (d_{\text{in}}, d_{\text{out}}) : \mathbb{R}_+ \to \mathbb{R}^{n_\eta} \times \mathbb{R}$, the system*
$$\dot{\xi} = \varphi(\xi, \alpha^\star_{\text{in}} + d_{\text{in}}, \alpha^\star_{\text{out}} + d_{\text{out}})$$
satisfies
$$|\xi(t) - \xi^\star(t)| \leq \max\big\{\beta_\xi(|\xi(0) - \xi^\star(0)|, t),\ \rho_\xi(|d|_{[0,t)})\big\}$$
for all $t \geq 0$.
3. ***Regularity:*** *the map*
$$\lambda(\xi, \alpha_{\text{in}}, \alpha_{\text{out}}) := \lim_{\epsilon \to 0} \frac{h(\xi + \epsilon\varphi(\xi, \alpha_{\text{in}}, \alpha_{\text{out}})) - h(\xi)}{\epsilon}$$
is well-defined and continuous on $\Xi \times A_{\text{in}} \times A_{\text{out}}$.

An example of a least-square identifier that fulfills such conditions is shown in Section 5. In view of this discussion, we make the following assumption.

A2) *The pair (φ, h) fulfills the identifier requirement relative to $\mathcal{I}$.*

To make the dependency on w explicit, we associate with each solution of (2) a signal $\xi^\star_w$ defined as the optimal steady state of ξ introduced in the identifier requirement corresponding to the ideal inputs (11) and we let $\theta^\star_w(t) := h(\xi^\star_w(t))$ be the corresponding optimal parameter. With those definitions and according to (10), we obtain $\alpha^\star_{\text{out}} =$

$\gamma(\alpha^\star_{\rm in})$, so that we associate with each solution w of (2) the signal

$$\varepsilon^\star_w := \gamma\big(\tau(w,\pi(w))\big) - \hat\gamma\big(\theta^\star_w, \tau(w,\pi(w))\big), \qquad (17)$$

which represents the *optimal prediction error* (i.e. the prediction error attained by the optimal model) along the solution w.

We remark that, while knowledge of the maps (11) is not assumed in this paper, the choice of the identifier, and in particular of the structure and parametrization of the map $\hat\gamma$, can be guided by a priori qualitative and quantitative information that the designer may have on the ideal steady-state signals $(\alpha^\star_{\rm in}(t), \alpha^\star_{\rm out}(t))$. We stress, however, that such information is only needed for the purpose of setting up the identification problem. In turn, the choice of the structure of $\hat\gamma$, and thus of the corresponding identification algorithm, depends on the amount and quality of the available information on (11), and it may range from a very specific set of functions, such as linear regressions, to *universal approximators*, such as wavelet bases or neural networks. We also stress that the inferred parametrization of $\hat\gamma$ does not require any assumption on the structure of the map γ. Indeed, the fact that there may not exist any $\theta^\star \in \mathbb{R}^{n_\theta}$ such that $\gamma(\cdot) = \hat\gamma(\theta^\star,\cdot)$ just implies that the optimal prediction error (17) may not be zero at the steady state. In turn, the main result of the paper relates the asymptotic bound on the regulated variable y to the optimal prediction error (17) (i.e., to the best prediction performance of the identifier), thus resulting in an approximate regulation property, which can be strengthened to asymptotic regulation only if $\gamma(\cdot) = \hat\gamma(\theta^\star,\cdot)$.

Finally, we stress that the regression map γ between the input and output of the identifier is always guaranteed to exist in steady state according to Lemma 1. This differs from [14] where a regression assumption is made on the steady state input $u^\star$.

3.3. The Stabilizer

The role of the stabilizer is mainly to bring y close to zero (implying that η is a good proxy for $\alpha^\star_{\rm in}$ according to Lemma 1) regardless of the internal model. But it should also ensure that u behaves as $\alpha^\star_{\rm out}$ after this transient, in order for the identifier to work. For this, since q is $\mathcal{C}^1$, we immerse (1) into the system

$$\begin{array}{rcl} \dot z &=& f(w,z,\chi_1)\\ \dot\chi_1 &=& \chi_2\\ \dot\chi_2 &=& \Phi(w,z,\chi) + b(w,z,\chi_1)\dot u \end{array} \qquad (18)$$

with

$$\chi_1 = y$$
$$\chi_2 = q(w,z,y) + b(w,z,y)u,$$

new input $\dot u$, and

$$\Phi(w,z,\chi) = q'(w,z,\chi) + \frac{b'(w,z,\chi)}{b(w,z,\chi_1)}(\chi_2 - q(w,z,\chi_1)) \qquad (19)$$

where

$$q'(w,z,\chi) := \Big(L^{(w)}_{s(w)} + L^{(z)}_{f(w,z,\chi_1)} + L^{(\chi_1)}_{\chi_2}\Big)\, q(w,z,\chi_1),$$

and

$$b'(w,z,\chi) := \Big(L^{(w)}_{s(w)} + L^{(z)}_{f(w,z,\chi_1)} + L^{(\chi_1)}_{\chi_2}\Big)\, b(w,z,\chi_1).$$

From (7) and the definition of the new state variables χ, we have

$$u^\star(w,z,\chi_1) = u - \frac{\chi_2}{b(w,z,\chi_1)}. \qquad (20)$$

According to the minimum-phase assumption A1, z is close to $\pi(w)$ when χ_1 is small. We thus deduce from (20) that $\alpha_{\rm out} = u$ is indeed a good proxy for $\alpha^\star_{\rm in} = u^\star(w,\pi(w),0)$ when χ is small. The idea is therefore to design $\dot u$ as a stabilizer for both χ_1 and χ_2 around 0. This is done in the following, first assuming χ_2 is available for feedback.

As a first step, with $\hat\gamma$ the prediction model of the identifier given in (16), and with λ the map introduced in the identifier requirement, we define the continuous map $\hat\gamma' : \Xi \times A_{\rm in} \times A_{\rm out} \to \mathbb{R}$ as

$$\hat\gamma'(\xi,\alpha_{\rm in},\alpha_{\rm out}) := \Big(L^{(h(\xi))}_{\lambda(\xi,\alpha_{\rm in},\alpha_{\rm out})} + L^{(\alpha_{\rm in})}_{F\alpha_{\rm in}+G\alpha_{\rm out}}\Big)\,\hat\gamma(h(\xi),\alpha_{\rm in}). \qquad (21)$$

We then let $\psi : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta} \times \mathbb{R} \to \mathbb{R}$ be any bounded function that agrees with $\hat\gamma'$ on the compact set $\Xi \times A_{\rm in} \times A_{\rm out}$ and for which[2] there exists $\rho_\psi \in \mathcal{K}$ such that for all $(\xi,\alpha_{\rm in},\alpha_{\rm out}) \in \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta} \times \mathbb{R}$ and all $(\xi^\star,\alpha^\star_{\rm in},\alpha^\star_{\rm out}) \in \Xi \times A_{\rm in} \times A_{\rm out}$,

$$\begin{aligned} &|\psi(\xi,\alpha_{\rm in},\alpha_{\rm out}) - \hat\gamma'(\xi^\star,\alpha^\star_{\rm in},\alpha^\star_{\rm out})|\\ &\quad \le \rho_\psi\left(|\xi-\xi^\star| + |\alpha_{\rm in}-\alpha^\star_{\rm in}| + |\alpha_{\rm out}-\alpha^\star_{\rm out}|\right). \end{aligned} \qquad (22)$$

We then consider the following virtual system

$$\begin{array}{rcl} \dot w &=& s(w), \qquad \dot z = f(w,z,\chi_1)\\ \dot\eta &=& F\eta + G\Big(u^\star(w,z,\chi_1) + \Delta_1(w,z,\chi_1,\chi_2+\delta_1)\Big)\\ \dot\xi &=& \varphi\Big(\xi,\eta,u^\star(w,z,\chi_1) + \Delta_1(w,z,\chi_1,\chi_2+\delta_1)\Big)\\ \dot\chi_1 &=& \chi_2 + \delta_1\\ \dot\chi_2 &=& \Delta_2(w,z,\eta,\xi,\chi_1,\chi_2+\delta_1) + b(w,z,\chi_1)\kappa_0(\chi) + \delta_2. \end{array} \qquad (23)$$

with

$$\begin{aligned} \Delta_1(w,z,\chi) &= \frac{\chi_2}{b(w,z,\chi_1)}\\ \Delta_2(w,z,\eta,\xi,\chi) &= \Phi(w,z,\chi) - \Phi(w,\pi(w),0)\\ &\quad + b(w,z,\chi_1)\psi\Big(\xi,\eta,u^\star(w,z,\chi_1) + \Delta_1(w,z,\chi)\Big)\\ &\quad - b(w,\pi(w),0)\hat\gamma'\Big(\xi^\star_w, \tau(w,\pi(w)), u^\star(w,\pi(w),0)\Big), \end{aligned} \qquad (24)$$

[2] ψ can be chosen as any bounded uniformly continuous extension of $\hat\gamma'$ that always exists according to [26].

with input $\delta = (\delta_1, \delta_2)$, and with κ_0 a "nominal" stabilizing action to be fixed. For the sake of conciseness, we define

$$\widetilde{\mathcal{G}} := \left\{(w, z, \eta, \xi) \in \mathcal{G} \times \mathbb{R}^{n_\xi} \ : \ \xi = \xi^\star_w\right\}$$
$$\mathcal{B} := \left\{(w, z, \eta, \xi, \chi) \in \widetilde{\mathcal{G}} \times \mathbb{R}^2 \ : \ \chi = 0\right\}.$$

Then, the following result holds.

Lemma 2. *Suppose that A1 and A2 hold, then for each pair of compact sets* $\mathrm{Z}_0 \subset \mathbb{R}^{n_z}$ *and* $\mathrm{Y}_0 \subset \mathbb{R}$ *and for each positive scalar* $\bar{\delta}$*, there exist a* $\mathcal{C}^1$ *function* κ_0*, maps* $\beta_0 \in \mathcal{KL}$ *and* $\rho_0 \in \mathcal{K}$*, such that for every input* $\delta : \mathbb{R} \to \mathbb{R}^2$ *verifying* $|\delta| \leq \bar{\delta}$*, any solution of* (23) *originating in* $\mathrm{W} \times \mathrm{Z}_0 \times \mathbb{R}^{n_\eta} \times \mathrm{Y}_0 \times \mathbb{R}$ *is defined on* $[0, +\infty)$ *and verifies for all* $t \geq \bar{t}$*,*

$$|(w(t), z(t), \eta(t), \xi(t), \chi(t))|_{\mathcal{B}} \leq \max\left\{\rho_0(|\delta|_{[\bar{t},t)}),\ \beta_0\big(|(w(\bar{t}), z(\bar{t}), \eta(\bar{t}), \xi(\bar{t}), \chi(\bar{t}))|_{\mathcal{B}}, t - \bar{t}\big)\right\}.$$

Proof. First, A1 and Lemma 1 show that the subsystem (w, z, η) of (23) is ISS relative to the set $\mathcal{G}$ and with respect to the input (χ_1, Δ_1). Then, under A2, Item (2) of Requirement 1 applied with

$$d_{\text{in}} = \eta - \tau(w, \pi(w))$$
$$d_{\text{out}} = u^\star(w, z, \chi_1) - u^\star(w, \pi(w), 0) + \Delta_1(w, z, \chi)$$

shows that the subsystem ξ of (23) is ISS relative to $\xi^\star_w$ and with respect to the input $(\eta - \tau(w, \pi(w)), \chi_1, \Delta_1)$. It follows that the cascade (w, z, η, ξ) is ISS relative to $\widetilde{\mathcal{G}}$ with respect to the input (χ_1, Δ_1). Therefore, using standard high gain arguments ([25]), by boundedness of ψ, continuity of Φ and b, and lower-boundedness of b on compact sets according to (3), for any $\bar{\delta}$, there exist compact sets $\mathrm{Z} \subset \mathbb{R}^{n_z}$ and $\mathrm{Y} \subset \mathbb{R}$, and a map $\underline{\kappa_0}$ such that for all κ_0 verifying $|\kappa_0| \geq |\underline{\kappa_0}|$ and $\kappa_0(e)e < 0$ for all e, every solution of (23) initialized in $\mathrm{W} \times \mathrm{Z}_0 \times \mathbb{R}^{n_\eta} \times \mathrm{Y}_0 \times \mathbb{R}$ with input $|\delta| \leq \bar{\delta}$ is bounded, defined on $[0, +\infty)$ and such that $(z, y) \in \mathrm{Z} \times \mathrm{Y}$ at all times. Besides, from the identifier requirement, for all $t \geq \bar{t}$, $\xi^\star_w \in \Xi$ so that (22) holds along trajectories. Since $(z, y) \in \mathrm{Z} \times \mathrm{Y}$, b is lower-bounded with (3) and since Δ is linear in χ_2, by continuity of Φ and thanks to (22), there exists $\varpi \in \mathcal{K}$ such that along any of those trajectories, $|\Delta(w, z, \eta, \xi, \chi_1, \chi_2 + \delta_1)| \leq \varpi(|(w, z, \eta, \xi, \chi)|_{\mathcal{B}} + |\delta_1|)$ for all $t \geq \bar{t}$. Then, invoking arguments from [25], κ_0 can be chosen so that the subsystem χ is ISS with respect to the input $(|(w, z, \eta, \xi)|_{\widetilde{\mathcal{G}}}, \delta)$ after $\bar{t}$ and a small-gain condition holds in the interconnection with the subsystem (w, z, η, ξ), thus ensuring the result. □

Following the guidelines given in [25], κ_0 can for instance be taken linear of the form

$$\kappa_0(\chi) = -k(\chi_2 + a\chi_1) \tag{25}$$

with $a > 0$ and with $k > 0$ sufficiently large, if Δ is locally Lipschitz and if $\mathcal{A}$ defined in A1 is locally exponentially stable for the zero-dynamics (6). The design of the stabilizer is then concluded in the following section by choosing κ in (9) on the basis of the functions κ_0 defined here and by using the state estimate of χ_2 provided by the derivative observer in place of χ_2 itself.

3.4. The Derivative Observer

The goal of this section is to design the degrees of freedom (n_ζ, ℓ, ν), such that the unimplementable stabilizing control law claimed by Lemma 2 can be substituted by a control action which employs the quantity $\nu(\zeta)$ in place of the unmeasured derivative χ_2 of the output $y = \chi_1$. In other words, we use a separation principle for the stabilization problem (23) and we require that the new control law makes the interconnection $(w, z, \eta, \chi, \zeta)$ ISS relative to the set

$$\mathcal{D} := \left\{(w, z, \eta, \xi, \chi, \zeta) \in \mathcal{B} \times \mathbb{R}^{n_\zeta} \ : \ \nu(\zeta) = 0\right\}$$

and with respect to the input δ, thus complementing the result of Lemma 2.

A3) *For each pair of compact subsets* $\mathrm{Z}_0 \subset \mathbb{R}^{n_z}$ *and* $\mathrm{Y}_0 \subset \mathbb{R}$*, and for any positive scalar* $\bar{\delta}$*, there exist functions* $\kappa : \mathbb{R}^2 \to \mathbb{R}$*,* $\beta \in \mathcal{KL}$*,* $\rho \in \mathcal{K}$ *and a non-empty subset* $\mathrm{T}_0 \subset \mathbb{R}^{n_\zeta}$ *such that, for every input* $\delta = (\delta_1, \delta_2) : \mathbb{R} \to \mathbb{R}^2$ *verifying* $|\delta| \leq \bar{\delta}$*, every solution of the system*

$$\begin{aligned}
\dot{w} &= s(w), \qquad \dot{z} = f(w, z, \chi_1)\\
\dot{\eta} &= F\eta + G\Big(u^\star(w, z, \chi_1) + \Delta_1(w, z, \chi_1, \chi_2 + \delta_1)\Big)\\
\dot{\xi} &= \varphi\Big(\xi, \eta, u^\star(w, z, \chi_1) + \Delta_1(w, z, \chi_1, \chi_2 + \delta_1)\Big)\\
\dot{\chi}_1 &= \chi_2 + \delta_2\\
\dot{\chi}_2 &= \Delta_2(w, z, \eta, \xi, \chi_1, \chi_2 + \delta_1)\\
&\qquad + b(w, z, \chi_1)\kappa(\chi_1, \nu(\zeta)) + \delta_2\\
\dot{\zeta} &= \ell(\zeta, \chi_1)
\end{aligned} \tag{26}$$

originating in $\mathrm{W} \times \mathrm{Z}_0 \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\xi} \times \mathrm{Y}_0 \times \mathbb{R} \times \mathrm{T}_0$ *is defined on* $[0, +\infty)$ *and verifies for all* $t \geq \bar{t}$*,*

$$|(w(t), z(t), \eta(t), \xi(t), \chi(t), \zeta(t))|_{\mathcal{D}} \leq \max\Big\{\rho(|\delta|_{[\bar{t},t)}),\ \beta\big(|(w(\bar{t}), z(\bar{t}), \eta(\bar{t}), \xi(\bar{t}), \chi(\bar{t}), \zeta(\bar{t}))|_{\mathcal{D}}, t - \bar{t}\big)\Big\}.$$

To satisfy A3 from the result of Lemma 2, one can use separation principles appearing for instance in [32, 2, 1]). Because $\Delta = 0$ on $\mathcal{D}$, following for instance the separation principle exposed in [2], if Δ is locally Lipschitz, then ℓ can be chosen as a dirty high-gain observer of dimension 2, i.e.

$$\dot{\zeta}_1 = \zeta_2 + L(\chi_1 - \zeta_1)\ , \qquad \dot{\zeta}_2 = L^2(\chi_1 - \zeta_1)\ , \tag{27}$$

with L sufficiently large, $\mathrm{T}_0 = \mathbb{R}^2$, $\nu(\zeta) = \zeta_2$, and κ a saturated version of κ_0.

4. Main Result

We now consider the interconnection of the system (1)-(2) with the controller (9). For the sake of readability we define the set

$$\mathcal{M} := \big\{(w,z,y,\eta,\xi,\zeta,u) \ : \\ (w,z,\eta,\xi,\zeta) \in \mathcal{D}\,,\ y=0\,,\ \nu(\zeta)=0\,,\ u\in\mathbb{R}\big\}\ .$$

The main result reads as follows.

Proposition 1. *Suppose that A1, A2 and A3 hold. Then for each pair of compact subsets* $\mathrm{Z}_0 \subset \mathbb{R}^{n_z}$ *and* $\mathrm{Y}_0 \subset \mathbb{R}$ *there exist* $\kappa : \mathbb{R}^2 \to \mathbb{R}$, $\beta_{\mathbf{x}} \in \mathcal{KL}$, $\rho_{\mathbf{x}} \in \mathcal{K}$, *and a non-empty subset* $\mathrm{T}_0 \subset \mathbb{R}^{n_\zeta}$ *such that each solution* $\mathbf{x} := (w,z,y,\eta,\xi,\zeta,u)$ *of the closed-loop system (1)-(2)-(9) originating in* $\mathrm{W}\times \mathrm{Z}_0 \times \mathrm{Y}_0 \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\xi} \times \mathrm{T}_0 \times \mathbb{R}$ *is bounded, defined on* $[0,+\infty)$, *and satisfies*

$$|\mathbf{x}(t)|_{\mathcal{M}} \le \max\big\{\beta_{\mathbf{x}}(|\mathbf{x}(\bar t)|_{\mathcal{M}}, t-\bar t),\ \rho_{\mathbf{x}}(|\varepsilon_w^\star|_{[\bar t,t)})\big\},$$

for all $t \ge \bar t$. *Thus, each of those solutions satisfies*

$$\limsup_{t\to\infty} |y(t)| \le \rho_{\mathbf{x}}\Big(\limsup_{t\to\infty} |\varepsilon_w^\star(t)|\Big).$$

Proof. Consider a solution $\mathbf{x}$ of (1)-(2)-(9) initialized in $\mathrm{W}\times \mathrm{Z}_0 \times \mathrm{Y}_0 \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\xi} \times \mathrm{T}_0 \times \mathbb{R}$. Define $\chi := (y,\dot y)$. Then, $(w,z,\eta,\xi,\chi,\zeta,u)$ is solution to

$$\begin{array}{rcl} \dot w &=& s(w)\\ \dot z &=& f(w,z,\chi_1)\\ \dot\eta &=& F\eta + Gu\\ \dot\xi &=& \varphi(\xi,\eta,u)\\ \dot\chi_1 &=& \chi_2\\ \dot\chi_2 &=& \Phi(w,z,\chi) + b(w,z,\chi_1)(\kappa(\chi_1,\nu(\zeta)) + \psi(\xi,\eta,u))\\ \dot\zeta &=& \ell(\zeta,\chi_1)\\ \dot u &=& \kappa(\chi_1,\nu(\zeta)) + \psi(\xi,\eta,u) \end{array} \tag{28}$$

and verifies (20) at all times. Consider the change of coordinates $\chi \mapsto \tilde\chi$, where

$$\begin{aligned}\tilde\chi_1 &:= \chi_1\\ \tilde\chi_2 &:= \chi_2 + b(w,\pi(w),0)(u^\star(w,\pi(w),0) - \hat\gamma(\theta_w^\star,\tau(w,\pi(w))))).\end{aligned}$$

Noting that $u^\star(w,\pi(w),0) = \gamma(\tau(w,\pi(w)))$, (17) yields

$$\chi = \tilde\chi - Eb(w,\pi(w),0)\varepsilon_w^\star, \tag{29}$$

with $E := \mathrm{col}(0,1)$. Moreover, from (7),

$$\tilde\chi_2 = \chi_2 - q(w,\pi(w),0) - b(w,\pi(w),0)\hat\gamma(\theta_w^\star,\tau(w,\pi(w)))).$$

so that

$$\begin{array}{rcl} \dot{\tilde\chi}_1 &=& \tilde\chi_2 - b(w,\pi(w),0)\varepsilon_w^\star\\ \dot{\tilde\chi}_2 &=& \Lambda(w,z,\eta,\xi,\tilde\chi,u) + b(w,z,\chi_1)\kappa(\tilde\chi_1,\nu(\zeta)) \end{array}$$

where

$$\begin{aligned}\Lambda(\cdot) :=\ & \Phi(w,z,\tilde\chi - Eb(w,\pi(w),0)\varepsilon_w^\star) + b(w,z,\chi_1)\psi(\xi,\eta,u)\\ &- q'(w,\pi(w),0) - b'(w,\pi(w),0)(u^\star(w,\pi(w),0) - \varepsilon_w^\star)\\ &- b(w,\pi(w),0)\hat\gamma'\big(\xi_w^\star,\tau(w,\pi(w)),u^\star(w,\pi(w),0)\big).\end{aligned}$$

With (19), we thus get

$$\begin{aligned}\Lambda(\cdot) =\ & \Phi(w,z,\tilde\chi - Eb(w,\pi(w),0)\varepsilon_w^\star) + b(w,z,\chi_1)\psi(\xi,\eta,u)\\ &- \Phi(w,\pi(w),0) + b'(w,\pi(w),0)\varepsilon_w^\star\\ &- b(w,\pi(w),0)\hat\gamma'\big(\xi_w^\star,\tau(w,\pi(w)),u^\star(w,\pi(w),0)\big).\end{aligned}$$

We thus notice that, in view of (20) and (24), Λ can be written as $\Lambda(w,z,\eta,\xi,\tilde\chi,u) = \Delta_2(w,z,\eta,\xi,\tilde\chi_1,\tilde\chi_2+\delta_1) + \delta_2$, where

$$\delta_1 = -b(w,\pi(w),0)\varepsilon_w^\star \quad , \quad \delta_2 = b'(w,\pi(w),0)\varepsilon_w^\star\ .$$

Now by boundedness of h and continuity of γ, τ, π and $\hat\gamma$, there exists a nonnegative scalar $\overline{\varepsilon_w^\star}$ such that $|\varepsilon_w^\star(t)| \le \overline{\varepsilon_w^\star}$ for all t and for any solution w of (2) in W. The result then follows directly from A3 with

$$\bar\delta = \max_{w\in\mathrm{W}}\{|b(w,\pi(w),0)|\overline{\varepsilon_w^\star},\ |b'(w,\pi(w),0)|\}.$$

□

The regulation performance thus depends on the optimal prediction error $\varepsilon_w^\star$ defined in (17), namely on how well $\hat\gamma(\theta_w^\star,\cdot)$ approximates the map γ given by Lemma 1. The choice of $\hat\gamma$ is therefore crucial, and if there exists θ such that $\gamma = \hat\gamma(\theta,\cdot)$ then asymptotic regulation is achieved.

5. Least-square identifier

In this section, we present an example of identifier that fulfills the Requirement 1. Consider the set $\mathcal{I}$ of maps $\alpha^\star : \mathbb{R}_+ \to \mathbb{R}^{n_\eta}\times\mathbb{R}$ such that there exist compact sets $\mathrm{A_{in}} \subset \mathbb{R}^{n_\eta}$, $\mathrm{A_{out}} \subset \mathbb{R}$ such that $\alpha^\star(\mathbb{R}_+) \subseteq \mathrm{A_{in}}\times\mathrm{A_{out}}$ for all $\alpha^\star \in \mathcal{I}$. Consider a model set made of functions $\hat\gamma(\theta,\cdot)$ that are linearly parametrized in θ, namely such that there exists a C^1 *regression vector* $\sigma : \mathbb{R}^{n_\eta} \to \mathbb{R}^{n_\theta}$ such that

$$\hat\gamma(\theta,\cdot) = \sum_{i=0}^{n_\theta} \theta_i\sigma_i(\cdot) = \theta^\top\sigma(\cdot)\ . \tag{30}$$

In other words, we hope that along the ideal signals $(\alpha_{\mathrm{in}}^\star,\alpha_{\mathrm{out}}^\star)$, there exists $\theta^\star \in \mathbb{R}^{n_\theta}$ such that $\alpha_{\mathrm{out}}^\star \approx {\theta^\star}^\top\sigma(\alpha_{\mathrm{in}}^\star)$. Since we know that $\alpha_{\mathrm{out}}^\star = \gamma(\alpha_{\mathrm{in}}^\star)$, the family of functions $\{\sigma_i\}$ can typically be chosen as elements of a basis for a decomposition of functions, such as a wavelet expansion for instance [12]. Then, the dimension n_θ characterizes the refinement of the approximation of γ.

We consider as cost function (14) a weighted least-square norm of the past prediction errors of the form

$$\mathcal{J}_\alpha(\theta,t) := \mathrm{p}\int_0^t e^{-\mathrm{p}(t-s)}|\alpha_{\mathrm{out}}(s) - \theta^\top\sigma(\alpha_{\mathrm{in}}(s))|^2 ds + \theta^\top\Omega\theta\ , \tag{31}$$

with p a positive *forgetting factor* and Ω a positive semi-definite *regularization matrix* $\Omega \in \mathbb{S}_{n_\theta}$. We then build an identifier with state $\xi = (\xi_1, \xi_2) \in \mathbb{S}_{n_\theta} \times \mathbb{R}^{n_\theta}$ and dynamics $\varphi = (\varphi_1, \varphi_2)$ defined by

$$\left\{ \begin{array}{rcl} \dot{\xi}_1 &=& -\mathrm{p}\,\xi_1 + \mathrm{p}\ \mathrm{sat}_1\left(\sigma(\alpha_{\mathrm{in}})\sigma(\alpha_{\mathrm{in}})^\top\right) \\ \dot{\xi}_2 &=& -\mathrm{p}\,\xi_2 + \mathrm{p}\ \mathrm{sat}_2(\sigma(\alpha_{\mathrm{in}})\alpha_{\mathrm{out}}) \end{array} \right. \tag{32}$$

with input $(\alpha_{\mathrm{in}}, \alpha_{\mathrm{out}})$ and output

$$h(\xi) = \mathrm{sat}_3((\xi_1 + \Omega)^\dagger \xi_2) \tag{33}$$

and with sat_i bounded continuous maps to be defined.

For that, consider a positive scalar ϵ. Let $\overline{\xi}_1 > 0$ and $\overline{\xi}_2 > 0$ such that for all $(\alpha^\star_{\mathrm{in}}, \alpha^\star_{\mathrm{out}}) \in \mathrm{A}_{\mathrm{in}} \times \mathrm{A}_{\mathrm{out}}$,

$$|\sigma(\alpha^\star_{\mathrm{in}})\sigma(\alpha^\star_{\mathrm{in}})^\top| \leq \overline{\xi}_1 \quad , \quad |\sigma(\alpha^\star_{\mathrm{in}})\alpha^\star_{\mathrm{out}}| \leq \overline{\xi}_2$$

and define the compact set

$$\Xi = \{(\xi_1, \xi_2) \in \mathbb{S}_{n_\theta} \times \mathbb{R}^{n_\theta} \ : \ \mathrm{msv}(\xi_1 + \Omega) \geq \epsilon \\ |\xi^\star_1| \leq \overline{\xi}_1\,, \ |\xi^\star_2| \leq \overline{\xi}_2\} \ . \tag{34}$$

We have the following result.

Lemma 3. *Assume the maps* sat_1, sat_2 *and* sat_3 *are bounded, continuous and such that for all* $(\alpha^\star_{\mathrm{in}}, \alpha^\star_{\mathrm{out}}) \in \mathrm{A}_{\mathrm{in}} \times \mathrm{A}_{\mathrm{out}}$*, and for all* $\xi^\star \in \Xi$*,*

$$\mathrm{sat}_1\left(\sigma(\alpha^\star_{\mathrm{in}})\sigma(\alpha^\star_{\mathrm{in}})^\top\right) = \sigma(\alpha^\star_{\mathrm{in}})\sigma(\alpha^\star_{\mathrm{in}})^\top \tag{35a}$$

$$\mathrm{sat}_2(\sigma(\alpha^\star_{\mathrm{in}})\alpha^\star_{\mathrm{out}}) = \sigma(\alpha^\star_{\mathrm{in}})\alpha^\star_{\mathrm{out}} \tag{35b}$$

$$\mathrm{sat}_3((\xi^\star_1 + \Omega)^\dagger \xi^\star_2) = (\xi^\star_1 + \Omega)^\dagger \xi^\star_2 \ . \tag{35c}$$

If there exists a positive scalars $\bar{t}$ *such that for all* $\alpha^\star = (\alpha^\star_{\mathrm{in}}, \alpha^\star_{\mathrm{out}}) \in \mathcal{I}$ *and for all* $t \geq \bar{t}$*,*

$$\mathrm{msv}\left(\Omega + \int_0^t e^{-\mathrm{p}(t-s)}\sigma(\alpha^\star_{\mathrm{in}}(s))\sigma(\alpha^\star_{\mathrm{in}}(s))^\top ds\right) \geq \epsilon \ , \tag{36}$$

then the pair (φ, h) *fulfills the* identifier requirement *relative to* $\mathcal{I}$*.*

Proof. First, h is bounded. Then, take $\alpha^\star = (\alpha^\star_{\mathrm{in}}, \alpha^\star_{\mathrm{out}}) \in \mathcal{I}$ and define $\xi^\star = (\xi^\star_1, \xi^\star_2)$ by

$$\xi^\star_1(t) = \mathrm{p}\int_0^t e^{-\mathrm{p}(t-s)}\sigma(\alpha^\star_{\mathrm{in}}(s))\sigma(\alpha^\star_{\mathrm{in}}(s))^\top ds$$

$$\xi^\star_2(t) = \mathrm{p}\int_0^t e^{-\mathrm{p}(t-s)}\sigma(\alpha^\star_{\mathrm{in}}(s))\alpha^\star_{\mathrm{out}}(s) ds$$

which is solution to (32) with input $\alpha^\star$ according to (35a)-(35b). By (36) and definition of (32), $\xi^\star(t) \in \Xi$ for all $t \geq \bar{t}$. It is easy to see that at each time t,

$$\frac{\partial \mathcal{J}_{\alpha^\star}}{\partial \theta}(\theta, t) = 0 \quad \Longleftrightarrow \quad (\xi^\star_1(t) + \Omega)\theta = \xi^\star_2(t) \ ,$$

so that by (35c), $\theta^\star(t) := h(\xi^\star(t))$ minimizes $\mathcal{J}_{\alpha^\star}(\theta, t)$ for all $t \geq \bar{t}$. Besides, it is clear to see that the ISS property of the stability requirement holds thanks to the continuity of σ and the boundedness of the saturations.

Now, according to [10, Theorem 10.5.3], defining the map $\varphi_\dagger : \mathbb{R}^{n_\theta \times n_\theta} \times \mathbb{R}^{n_\theta \times n_\theta} \to \mathbb{R}^{n_\theta \times n_\theta}$ by

$$\varphi_\dagger(M, \dot{M}) := -M^\dagger \dot{M} M^\dagger + (I - M^\dagger M)\dot{M}^\top (M^\dagger)^\top M^\dagger \\ + M^\dagger (M^\dagger)^\top \dot{M}^\top (I - MM^\dagger) \ ,$$

the map λ defined in the regularity requirement writes as

$$\lambda(\xi, \alpha_{\mathrm{in}}, \alpha_{\mathrm{out}}) = (\xi_1 + \Omega)^\dagger \varphi_2(\xi, \alpha_{\mathrm{in}}, \alpha_{\mathrm{out}}) \\ + \varphi_\dagger(\xi_1 + \Omega, \varphi_1(\xi, \alpha_{\mathrm{in}}, \alpha_{\mathrm{out}}))\xi_2 \tag{37}$$

which is well-defined everywhere and continuous on $\Xi \times \mathrm{A}_{\mathrm{in}} \times \mathrm{A}_{\mathrm{out}}$. Therefore Requirement 1 is satisfied. □

The requirement (36) is a persistence of excitation condition on the ideal inputs $\alpha^\star_{\mathrm{in}}$, which requires that $\Omega + \xi^\star_1$ remains (uniformly) of constant rank after a certain time (and not necessarily invertible). It is important to note that this assumption is always satisfied if the regularization matrix Ω is chosen positive definite. However, in that case, the minimizer $\theta^\star$ of the cost (31) no longer minimizes the past prediction errors due to the regularization term $\theta^\top \Omega \theta$. In particular, if, by chance, γ is in the model set, i.e. there exists θ_0 such that $\gamma = \hat{\gamma}(\theta_0, \cdot)$, the algorithm will yield $\theta^\star \neq \theta_0$ thus resulting in a non-zero prediction error $\varepsilon^\star$ and approximate regulation. This mismatch can nevertheless be made small by choosing Ω small and has the advantage of numerically robustifying the algorithm. On the other hand, if asymptotic regulation is desired, Ω must be chosen equal to zero and the regularity of the identifier exclusively relies on the excitation power of the ideal input $\alpha^\star_{\mathrm{in}}$.

In our context, according to A2, $\alpha^\star_{\mathrm{out}}$ represents the ideal input $u^\star(w, \pi(w), 0)$ to apply in steady state, and $\alpha^\star_{\mathrm{in}}$ is the steady state of the filter

$$\dot{\eta} = F\eta + Gu^\star(w, \pi(w), 0) \ .$$

In other words, $\alpha^\star_{\mathrm{out}}$ is determined by the exosystem and the plant, while $\alpha^\star_{\mathrm{in}}$ also depends on F and G. Usually, the knowledge of the set W and of the plant, gives a bound on the ideal input $u^\star(w, \pi(w), 0)$ to apply in steady state, and thus enables to define $\mathrm{A}_{\mathrm{out}}$. Then, a bound for $\alpha^\star_{\mathrm{in}}$ can be deduced depending on the choice of F and G, thus leading to A_{in}. From there, the saturation maps sat_1 and sat_2 can be chosen as prescribed by Lemma 3. As for sat_3, it can either be chosen based on the compact set Ξ if the value of ϵ is known or based on a priori bounds on the parameter $\theta^\star$ needed to model γ. Then, along (21), we define

$$\hat{\gamma}'(\xi, \alpha_{\mathrm{in}}, \alpha_{\mathrm{out}}) = \lambda(\xi, \alpha_{\mathrm{in}}, \alpha_{\mathrm{out}})^\top \sigma(\alpha_{\mathrm{in}}) \\ + h(\xi)^\top \sigma'(\alpha_{\mathrm{in}})(F\alpha_{\mathrm{in}} + G\alpha_{\mathrm{out}})$$

which is continuous on $\Xi \times \mathrm{A}_{\mathrm{in}} \times \mathrm{A}_{\mathrm{out}}$. Finally, we can simply take[3] ψ of the form

$$\psi(\xi, \alpha_{\text{in}}, \alpha_{\text{out}}) = \text{sat}(\hat{\gamma}'(\xi, \alpha_{\text{in}}, \alpha_{\text{out}})) \tag{38}$$

with sat a bounded map such that $\psi = \hat{\gamma}'$ on $\Xi \times \mathrm{A}_{\text{in}} \times \mathrm{A}_{\text{out}}$.

All in all, a possible regulator is made of (9) with F, G chosen along the lines of Section 3.1, κ a saturated version of κ_0 defined in (25) with k sufficiently large to ensure stability in Lemma 2, ψ defined in (38), ℓ as a high-gain observer (27) with L sufficiently large to verify A3, and φ defined by the least-square dynamics (32). An example is provided in the following section.

6. Example

Consider the system

$$\begin{aligned} \dot{z} &= -2z + y + 2w_1 \\ \dot{y} &= w_2^2 + zy + u \end{aligned}$$

with y to be regulated towards zero, and a disturbance w generated by an exosystem of the form (2). Although the knowledge of the exosystem could (to some extent) guide the choice of the model set as explained in Section 3.2, we want to highlight in this example that the explicit expression of the map s is not used for the internal model design. Indeed, the adaptation provided by the identifier enables to *learn* online the steady-state input $u^\star$, as precisely as possible, depending on the prediction capabilities of the model set. The only thing we need to know is some bounds of the disturbance and its derivative, namely that $|w(t)| \leq \mathrm{w}_{\max}$ and $|\dot{w}(t)| \leq \mathrm{dw}_{\max}$ for all t.

From these bounds, we deduce that in steady state, $|u^\star(t)| \leq \mathrm{u}^\star_{\max} := {\mathrm{w}_{\max}}^2$ and $|\dot{u}^\star(t)| \leq \psi_{\max} := 2\mathrm{w}_{\max}\mathrm{dw}_{\max}$, which gives us the bound for ψ, since ψ equals $\dot{u}^\star$ in steady state. Then, we choose $n_\eta = 2(n_w + 1) = 6$ and for instance,

$$F := \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad G := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

It follows that in steady state, $|\eta(t)| \leq \eta_{\max} := \|F^{-1}\|\|G\|\mathrm{u}^\star_{\max}$. The identifier is then chosen as a least-square identifier of the kind presented in Section 5 with a simple linear regressor vector, i.e. $\sigma(\eta) = \eta$. Therefore, in (32), sat_1 can be chosen as a saturation by $\overline{\xi}_1 := n_\theta {\eta_{\max}}^2$ and sat_2 by $\overline{\xi}_2 := n_\theta \eta_{\max}\mathrm{u}^\star_{\max}$, with maybe an additional security margin. The regularization matrix is chosen as $\Omega = 10^{-6}I$ and the forgetting factor as $\mathrm{p} = 0.05$. As for θ, it can be saturated to $\pm\|\Omega^{-1}\|\overline{\xi}_2$. Finally, it remains to choose the parameters of the observer and the stabilizer: we empirically took $k = 5$ in (25), $L = 15$ in (27), and saturated the stabilizer κ at 250.

[3] (22) can be shown observing that Ξ has non-empty interior, and $\hat{\gamma}'$ is continuous on a compact inflation C of $\Xi \times \mathrm{A}_{\text{in}} \times \mathrm{A}_{\text{out}}$ with non-empty interior. Then, ρ_ψ exists on C and (22) holds outside of C by lower-bounding the distance to points in $\Xi \times \mathrm{A}_{\text{in}} \times \mathrm{A}_{\text{out}}$ and using the boundedness of ψ. See [3, Section A.2].

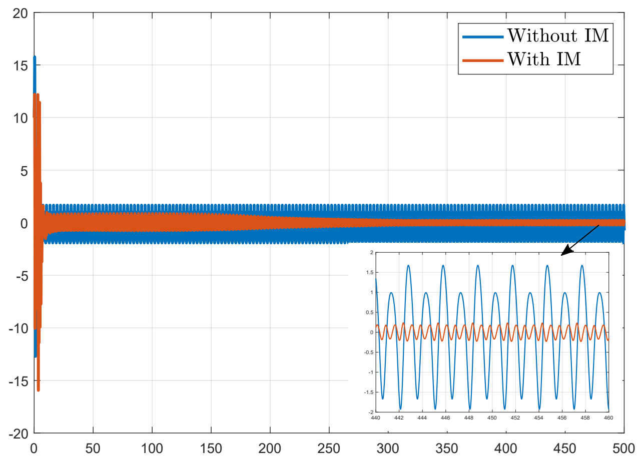

Figure 1: Trajectory $t \mapsto y(t)$ with exosystem (39) with and without internal model (IM), namely with and without ψ.

Once these parameters have been fixed, we present the results obtained with initial condition $(z, y)(0) = (1, 10)$ and two different exosystems defined by

$$\begin{array}{rcl} \dot{w}_1 &=& w_2 \\ \dot{w}_2 &=& -w_1 - w_1^3 \end{array} \tag{39}$$

and

$$\begin{array}{rcl} \dot{w}_1 &=& w_2 \\ \dot{w}_2 &=& (1 - w_1^2)w_2 - w_1 \end{array} \tag{40}$$

with initial condition $(0, 4)$. In each case, we compare the results with and without internal model, namely with and without the identified consistency term ψ. The trajectories of y are plotted in Figures 1 and 3: the adaptation provided by the identifier enables to reduce the asymptotic static error. Note that, when removing the internal model, the saturation of the stabilizer had to be doubled to preserve stability. This is why the peaking during the transient may seem smaller with the internal model in those particular simulations. The performance of the identifier are illustrated on Figures 2 and 4. The slow convergence is due to the low forgetting factor p, which allows the least square algorithm to keep a sufficient data history. Of course, choosing a larger model set or more complex (maybe discrete) identifiers could allow a better fitting (polynomial decompositions, wavelets, neural networks etc.) However, the numerical cost of running the identifier in continuous-time limits its dimension. Using discrete-time identifiers as in [4] would significantly reduce this load. Finally, the impact of the choice of structure and eigenvalues of F and G is not well understood at this point.

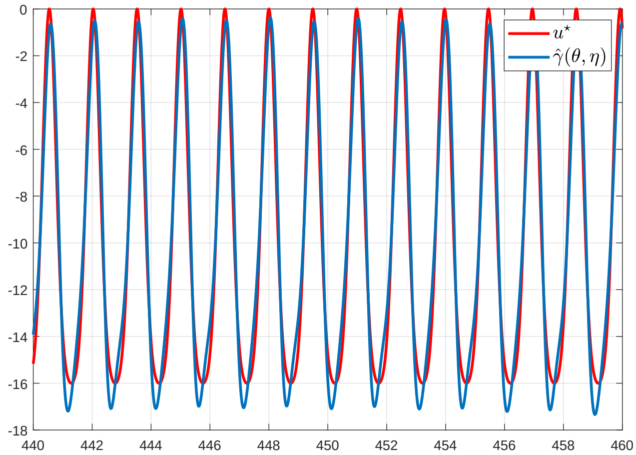

Figure 2: Comparison of the ideal steady state input $u^\star = -w_2^2$ and the one estimated by the identifier, namely $\hat{\gamma}(\theta, \eta) = \theta^\top \eta$ for the exosystem (39).

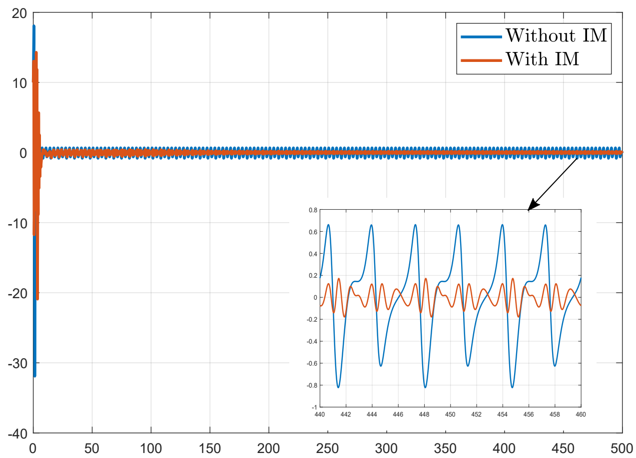

Figure 3: Trajectory $t \mapsto y(t)$ with exosystem (40) with and without internal model (IM), namely with and without ψ.

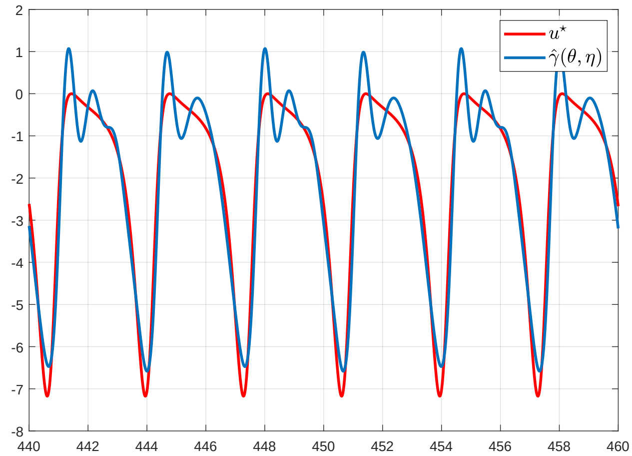

Figure 4: Comparison of the ideal steady state input $u^\star = -w_2^2$ and the one estimated by the identifier, namely $\hat{\gamma}(\theta, \eta) = \theta^\top \eta$ for the exosystem (40).

7. Conclusion

We have proposed a constructive design of a regulator solving the (approximate) output regulation problem for general single-input-single-output normal forms under a minimum-phase assumption. It extends the existence result of [25] by adding adaptation to tune online the unknown quantities characterizing the internal model unit. Adaptation is cast as a system identification problem, and the main result relates the asymptotic regulation performances to the prediction capabilities of the chosen identification algorithm. The proposed approach thus provides a constructive and systematic adaptive regulator design that yields an "optimal", and possibly asymptotic, regulation result. However, the controller still heavily relies on high gain to force the system to steady state, and on a saturation that may not be straightforward to choose. Parallel work presented in [4] develops an alternative "low-gain" strategy where the stabilizer's gain and saturation level are fixed beforehand and where handier discrete-time identifiers are used.

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