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Author for correspondence:
Vikas S. Krishnamurthy
e-mail: vikas.krishnamurthy2@gmail.com

A transformation between stationary point vortex equilibria

Vikas S. Krishnamurthy[1], Miles H. Wheeler[2], Darren G. Crowdy[3] and Adrian Constantin[1]

[1]Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
[2]Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, UK
[3]Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK

A new transformation between stationary point vortex equilibria in the unbounded plane is presented. Given a point vortex equilibrium involving only vortices with negative circulation normalized to -1 and vortices with positive circulations that are either integers, or half-integers, the transformation produces a new equilibrium with a free complex parameter that appears as an integration constant. When iterated the transformation can produce infinite hierarchies of equilibria, or finite sequences that terminate after a finite number of iterations, each iteration generating equilibria with increasing numbers of point vortices and free parameters. In particular, starting from an isolated point vortex as a seed equilibrium, we recover two known infinite hierarchies of equilibria corresponding to the Adler–Moser polynomials and a class of polynomials found, using very different methods, by Loutsenko [J. Phys. A: Math. Gen. 37, (2004)]. For the latter polynomials the existence of such a transformation appears to be new. The new transformation therefore unifies a wide range of disparate results in the literature on point vortex equilibria.

1. Introduction

The laws of vorticity and vortex motion were formulated by Helmholtz [1] more than a century and a half ago. Point vortices are weak solutions of the two-dimensional Euler equation, which governs the unsteady flow of an

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incompressible and inviscid fluid [2,3]. They provide a rich class of exact solutions to the Euler equation and, although they were discovered during the age of "classical mathematics" [4], substantial research interest has been devoted to them over the past several decades [3,5]. A historical survey of point vortex dynamics with a derivation of the equations of motion of singularities is given in [6]. Each point vortex corresponds to a singular Dirac delta vorticity distribution; its circulation remains constant under evolution according to the Helmholtz laws of vortex motion or Kelvin's circulation theorem [2].

Relative equilibria are special configurations of point vortices in which the vortices are stationary relative to each other [7] (the term "relative equilibrium" is given slightly different meanings in the literature, some of which are different from ours). Relative equilibria may be classified into three basic types: (i) rotating equilibria, where the configuration of vortices is rigidly rotating, (ii) translating equilibria, where the vortex configuration is in steady translation without change of form, and (iii) stationary equilibria, where no vortices move. In this paper we will focus exclusively on stationary equilibria.

The study of point vortex equilibria has implications for a wide variety of experimental studies. The patterns formed by magnetic disks in an external rotating magnetic field [8,9], vortices in rotating superfluid helium [10], vortices in Bose-Einstein condensates [11] and magnetised electron columns in Malmberg-Penning traps [12,13] are some examples of such experiments. Geophysical applications have motivated the experimental study of the formation of few-vortex equilibrium systems in rotating fluids, including monopoles, dipoles, tripoles and dipole pairs; for a recent review and discussion see [14]. Vortex crystals have been observed to emerge from a two-dimensional turbulent flow in experiments on magnetised electron columns [15] and in numerical studies of forced turbulence [16].

Vortex statics is the study of point vortex equilibria – sometimes called 'vortex crystals' [4,7]. A recent review, with a focus on Aref's contributions to the subject, lists several open problems [17]. In this article we focus our attention on the connections between planar equilibrium configurations and certain areas of mathematical physics, specifically systems of polynomials whose roots display the same patterns as that of vortex equilibria. Such connections have been the subject of many studies [18,19]. In this paper, however, we work at the level of rational functions – the aforementioned polynomials arise as their numerators and denominators – and use local expansions of these rational functions to study equilibria.

While a single vortex of any circulation is in stationary equilibrium, no stationary equilibria exist for two vortices. They are either translating or rotating equilibria, depending on whether the vortex circulations sum to zero or not. For three vortices, all stationary equilibria are necessarily colinear, and a general formula exists for the vortex positions [5]. For a given set of vortex circulations, it is known that there are exactly two stationary equilibria of four vortices, and a general formula exists that gives the vortex positions as functions of these circulations [20]. Surprising connections exist to various known polynomial systems for $M > 4$ vortices (but with restrictions on the values of M) and we focus on some of these connections in this paper [18,19].

Burchnall and Chaundy studied the conditions under which, given $P(z)$ and $Q(z)$ which are two polynomials in a complex variable z, both the rational functions P^2/Q^2 and Q^2/P^2 can be integrated to give another rational function [21]. They showed that this is equivalent to seeking polynomial solutions of the bilinear differential equation[1] $P''Q - 2P'Q' + PQ'' = 0$, where primes denote derivatives. They also showed how to construct such polynomials and obtained a Wronskian representation [21] using commutative-operator theory; also see [22]. The procedure for constructing these polynomials can be iterated to produce an infinite sequence of polynomials. The same polynomials arose in a completely different context, the study of rational solutions of the KdV equation [23], where they were constructed by Adler and Moser [24] using iterated Darboux–Crum transformations of a Schrödinger operator [25]. The Adler–Moser polynomials have been generalised to the case of the rational antiderivative of $P^{2/m}/Q^2$ and

[1]Burchnall and Chaundy [21] considered a more general problem, that of integrating P^m/Q^2 and Q^m/P^2 where m is some positive integer, and found the differential equation $P''Q - mP'Q' + PQ'' = 0$. We restrict ourselves to the case $m = 2$.

Q^{2m}/P^2 by Loutsenko [26], who has shown that the bilinear differential equation stated above generalises in this case to $P''Q - 2mP'Q' + m^2PQ'' = 0$ for $m = 1/2$ and $m = 2$. However an analogous construction of the Loutsenko polynomials through the Darboux–Crum process, such as exists for the Adler–Moser polynomials, has so far not been found. The new transformation given here throws light on such matters.

A system of $M \geq 4$ point vortices is generally not integrable [27], but the *equilibrium* configurations of point vortices can be the same as configurations of other systems such as a two-dimensional Coulomb gas. Consider for integer n, a system of $M_+ = n(n+1)/2$ vortices with circulations $+1$ and $M_- = n(n-1)/2$ vortices with circulations -1. We define the polynomials P and Q with degrees M_+ and M_- through the vortex positions i.e. the vortices are located at the roots of P and Q. It can then be shown that P and Q satisfy the bilinear differential equation $P''Q - 2P'Q' + PQ'' = 0$, which is called *Tkachenko's equation* in the context of vortex dynamics [7, 28]. Bartman [29] made the connection that the Adler–Moser polynomials provide polynomial solutions to Tkachenko's equation, and hence vortex equilibrium solutions for vortices of the same circulation but mixed sign. Bartman [29] also briefly discussed the case of vortex circulations 1 and -2, essentially the same vortex circulations corresponding to Loutsenko's polynomials; although not many details are provided, differential equations for the polynomials are written down. Campbell & Kadtke [30] found a subset of the Adler–Moser polynomials by generalising Tkachenko's method, see also [31].

For a given number of point vortices and using ideas from algebraic geometry, O'Neil [20,32] has calculated the number of stationary and translating equilibria for generic vortex circulations. Applying some methods used in the Newtonian four-body problem, a count of the number of rotating four-vortex equilibria can also be made, although the count is incomplete [33]. An infinite number of vortex equilibria, depending continuously on some parameter, can only exist for special values of the vortex circulations. Examples of such equilibria with a small number of vortices are provided in [32]. The Adler–Moser polynomials and the polynomials found by Loutsenko [26] also fall into this category, since at every stage in the iteration there is an additional complex-valued parameter so that the n^{th} polynomial in these hierarchies depends on n distinct complex-valued parameters.

As mentioned above, one method of describing relative equilibria uses generating polynomials, which are defined so that the point vortices are at their roots [34]. In the case of equilibria with more than one species of vortex i.e. with multiple values of vortex circulations, multiple polynomials are defined [18,19,29,30]. By using the conditions required for a point vortex equilibrium, differential equations are derived for these polynomials which are then used to study the equilibria and establish connections to various polynomial systems. For an alternate approach to point vortex equilibria that is based on matrix methods see [35,36].

The subject of relative equilibria may be approached in two general ways [7]: (i) the vortex circulations are specified and we ask for the vortex positions such that they are in equilibrium, or (ii) the vortex positions are given and the corresponding circulations need to be found so that they are in equilibrium. Most of the literature surveyed above falls into category (i). An example of (ii) is that of three vortices situated at the vertices of an equilateral triangle, then it is known that they are always in rotating or translating equilibrium, regardless of the vortex circulations. From a physical point of view it is more natural to be given the vortex circulations with the vortex positions to be worked out; this is a harder problem from the mathematical point of view. In this paper we present a transformation that takes a given equilibrium with given positions and circulations into a new equilibrium with new positions and circulations, *both* of which are determined by the transformation.

The idea that a given point vortex equilibrium can be related to a different equilibrium with a different number of point vortices is not necessarily new. For example, it forms the basis for numerical methods that have been used in the past to obtain a rotating $M+1$ vortex equilibrium by "growing" a vortex at co-rotating points of an M vortex equilibrium [37]. Our approach here is different in spirit, however, and involves an explicit and direct transformation to a different equilibrium in contrast to this continuation process of gradually growing additional vortices in an existing equilibrium.

This paper is organised as follows. We give a mathematical introduction to point vortices and relative equilibria in §2, and introduce the new transformation between stationary equilibria in §3. We discuss some of its properties in §4 and obtain conditions for it to yield non-trivial equilibria. In §5 we discuss various examples and, in particular, show that the Adler–Moser and Loutsenko hierarchies can be obtained from the transformation with the same simple seed, a single stationary vortex with different circulations. We end with a discussion of future directions, including the relationship between the present work and other work by the authors where a class of hybrid equilibria of the two-dimensional incompressible Euler equation have been found comprising a combination of Stuart-type vorticity with superposed point vortices [38].

2. Mathematical formulation

Consider a two-dimensional, incompressible and inviscid homogeneous fluid. Let (x, y) denote the Cartesian coordinates of a planar cross-section of the flow, $\boldsymbol{V}(x, y)$ the velocity field with components $\boldsymbol{V} = (u, v)$, $p(x, y)$ the pressure and ρ_0 the constant density of the fluid. The motion of the fluid is governed by the Euler equation [2]

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\frac{\nabla p}{\rho_0}, \tag{2.1}$$

where here $\nabla = (\partial/\partial x, \partial/\partial y)$ is the two-dimensional gradient operator. Since the flow is incompressible with $\nabla \cdot \boldsymbol{V} = 0$, we can define a streamfunction $\psi(x, y)$ via the equations

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}, \tag{2.2}$$

and, since the flow is two-dimensional, the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{V}$ has a single non-zero component

$$\zeta(x, y) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{2.3}$$

which is related to the streamfunction through the Poisson equation

$$\nabla^2 \psi = -\zeta. \tag{2.4}$$

For the planar flows that we are considering, it is advantageous to work in a complex $z = x + \mathrm{i}y$ plane. Point vortices are defined as solutions of (2.4) corresponding to the Dirac-delta vorticity [5]

$$\zeta = \sum_{j=1}^{M} \Gamma_j \delta(z - z_j), \tag{2.5}$$

where here the complex numbers $z_j = x_j + \mathrm{i}y_j$ are the locations and the real numbers Γ_j are the circulations of the M point vortices. Since by (2.5) the flow is irrotational everywhere except at M points in the plane, away from these points there exists a (multivalued) velocity potential φ such that $\boldsymbol{V} = \nabla\varphi$. Using the incompressibility condition $\nabla \cdot \boldsymbol{V} = 0$, we see that φ solves the Laplace equation $\nabla^2\varphi = 0$. Together with (2.4) and (2.5) this implies the existence of the *complex potential*

$$f(z) = \varphi + \mathrm{i}\psi = \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^{M} \Gamma_j \log(z - z_j). \tag{2.6}$$

The fluid velocity field $\xi(z) = u - \mathrm{i}v$ is simply given by the derivative[2] of the complex potential, $\xi(z) = f'(z)$. The velocity field due to the point vortices is [5]

$$\xi(z) = \overline{\frac{\mathrm{d}z}{\mathrm{d}t}} = \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^{M} \frac{\Gamma_j}{z - z_j}, \tag{2.7}$$

[2]Throughout this paper, primes denote derivatives with respect to the complex variable z.

where the overbar denotes complex conjugation. The locations of the point vortices are in general time-dependent and they move about under the influence of each other according to [6]

$$\overline{\frac{\mathrm{d}z_k}{\mathrm{d}t}} = \frac{1}{2\pi \mathrm{i}} \sum_{\substack{j=1 \\ j \neq k}}^{M} \frac{\Gamma_j}{z_k - z_j} \quad \text{for } k = 1, 2, \ldots, M. \tag{2.8}$$

The expression (2.8) is the non-self-induced velocity field at the location of a point vortex i.e. the finite part of the fluid velocity induced by all the other point vortices at this point vortex location.

With x_k and $\Gamma_k y_k$ as canonical conjugate variables, the system (2.8) is the canonical Hamiltonian system associated with the Hamiltonian

$$\mathcal{H} = -\frac{1}{2\pi} \sum_{\substack{j,k=1 \\ j<k}}^{M} \Gamma_k \Gamma_j \log |z_k - z_j|. \tag{2.9}$$

This Hamiltonian system is integrable for $M \leq 3$, due to the existence of three further integrals of motion: the linear impulse $\mathcal{X} + \mathrm{i}\mathcal{Y} = \sum_j \Gamma_j z_j$ and the angular impulse $\mathcal{I} = \sum_j \Gamma_j |z_j|^2$. However, for $M \geq 4$ it is generally chaotic [3,27]. Relative equilibria can be obtained as extrema of $\mathcal{H}$ subject to the constraints that $\mathcal{X}, \mathcal{Y}, \mathcal{I}$ are constant [7].

In a relative equilibrium the inter-vortex distances remain constant, so that the shape and size of the configuration remains fixed. The velocity field in this case takes the form [7]

$$\frac{\mathrm{d}z_k}{\mathrm{d}t} = \mathrm{i}\Omega z_k + U, \tag{2.10}$$

where the angular velocity Ω is a real parameter and the linear velocity U is a complex parameter. This paper is focused on studying stationary configurations of point vortices for which $\Omega = U = 0$. In this case (2.8) reduces to the M conditions on the vortex positions,

$$\sum_{\substack{j=1 \\ j \neq k}}^{M} \frac{\Gamma_j}{z_k - z_j} = 0 \quad \text{for } k = 1, 2, \ldots, M. \tag{2.11}$$

The algebraic equations (2.11) can be viewed as M conditions on the M unknowns $z_1, \ldots, z_M$, for given values of the circulations $\Gamma_1, \ldots, \Gamma_M$. We note that under the operations of scaling all the vortex circulations and scaling plus shifting all the vortex positions, a stationary equilibrium remains stationary.

3. The transformation

In this section we suppose that we are given a point vortex equilibrium with locations $z_1, \ldots, z_M$ and circulations $\Gamma_1, \ldots, \Gamma_M$ which satisfy the constraint

$$\Gamma_k = -1, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \qquad k = 1, \ldots, M, \tag{3.1}$$

i.e., each Γ_k is either -1 or a positive integer or half-integer. We can then define a rational function $h'(z)$ via the complex potential $f(z)$ as

$$h'(z) = A[\exp(2\pi \mathrm{i}\, f(z))]^2 = A \prod_{j=1}^{M} (z - z_j)^{2\Gamma_j}, \tag{3.2}$$

where A is a nonzero constant. The velocity field $\xi(z)$ in (2.7) is given in terms of $h(z)$ by

$$\xi(z) = \frac{1}{4\pi \mathrm{i}} \left(\log h'(z)\right)' = \frac{1}{4\pi \mathrm{i}} \frac{h''(z)}{h'(z)}. \tag{3.3}$$

Changing the value of A is equivalent to adding a constant to the complex potential and so does not affect the velocity field. The algebraic conditions (2.11) are equivalent to $\xi(z)$ having a Laurent series

$$\xi(z) = \frac{1}{2\pi \mathrm{i}} \frac{\Gamma_k}{z - z_k} + \mathcal{O}(z - z_k), \tag{3.4}$$

with vanishing constant term near each of its singularities.

We now show that the rational function $\widehat{h}'(z)$ defined, in terms of a rational function $h'(z)$ associated with a given equilibrium within the class just described, by the transformation

$$h'(z) \mapsto \widehat{h}'(z) = \widehat{A}\left[\frac{h'(z)}{(h(z))^2}\right]^{\alpha}, \tag{3.5}$$

also corresponds to a stationary point vortex equilibrium. Here $\widehat{A}$ is a nonzero constant, α is a nonzero real number, and $h(z)$ is any primitive of $h'(z)$. Equivalently, if the velocity field (3.3) corresponds to some stationary point vortex configuration, we claim that the velocity field

$$\widehat{\xi}(z) = \frac{1}{4\pi \mathrm{i}} \left(\log \widehat{h}'(z)\right)' = \frac{\alpha}{4\pi \mathrm{i}} \left[\frac{h''(z)}{h'(z)} - \frac{2\, h'(z)}{h(z)}\right] \tag{3.6}$$

corresponds to another, distinct stationary point vortex configuration. Necessary and sufficient conditions for this to be case are (i) $h(z)$ is rational, and (ii) the only singularities of $\widehat{\xi}(z)$ are simple poles at which the constant term in the Laurent series vanishes.

Observe that, assuming (i) holds, the singularities of $\widehat{\xi}(z)$ are precisely the zeros and poles of $\widehat{h}'(z)$. Looking at (3.5) or (3.6), we see that the possible singularities of $\widehat{\xi}(z)$ are either (a) zeros of $h'(z)$, (b) poles of $h'(z)$, or (c) zeros of $h(z)$. The poles of $h(z)$ and $h''(z)$ coincide with the poles of $h'(z)$, and so do not need to be checked separately.

(a) Proof that $h(z)$ is rational

We begin by showing (i), which is equivalent to $h'(z)$ having zero residue at each of its poles. Clearly all poles of $h'(z)$ are at point vortex locations z_k with negative circulations $\Gamma_k = -1$, due to the restriction (3.1) on the allowable vortex circulations. Near such a z_k, we rewrite

$$h'(z) = A(z - z_k)^{2\Gamma_k} H_k(z), \tag{3.7}$$

where we have defined the functions

$$H_k(z) = \prod_{\substack{j=1 \\ j \neq k}}^{M} (z - z_j)^{2\Gamma_j} \quad \text{for } k = 1, 2, \ldots, M. \tag{3.8}$$

Since the vortex positions are non-overlapping, $H_k(z_k)$ is finite and non-zero. The series representation for $h'(z)$ near z_k is

$$h'(z) = A\left(H_k(z_k)(z - z_k)^{2\Gamma_k} + H_k'(z_k)(z - z_k)^{2\Gamma_k + 1} + \frac{H_k''(z_k)}{2}(z - z_k)^{2\Gamma_k + 2} + \cdots\right). \tag{3.9}$$

In particular, since $2\Gamma_k + 1 = -1$, $h'(z)$ will have zero residue at z_k if and only if the coefficient $H_k'(z_k)$ vanishes. Combining (3.8) and (2.11) yields

$$\frac{H_k'(z_k)}{H_k(z_k)} = (\log H_k(z))'\Big|_{z = z_k} = \sum_{\substack{j=1 \\ j \neq k}}^{M} \frac{\Gamma_j}{z_k - z_j} = 0, \tag{3.10}$$

and hence $H_k'(z_k) = 0$ as desired. Similar arguments show that allowing for $\Gamma_k = -1/2$ in (3.1) would *always* lead to non-rational $h(z)$. Allowing for larger negative circulations, say $\Gamma_k = -\frac{3}{2}$, would require the corresponding coefficient $H_k''(z_k)$ to vanish, which is not true in general. On the other hand it can happen in specific examples, for instance the trivial example of a single point vortex.

(b) The singularities of $\widehat{\xi}(z)$ are stationary point vortices

Now we show (ii), which requires us to analyse the poles of the transformed velocity field (3.6). We will take the cases (a), (b), (c) introduced above in turn.

First, let z_k be a zero of $h'(z)$ which is not also a zero of $h(z)$, so that the second term $h'(z)/h(z)$ in (3.6) vanishes at z_k. Since the first term in (3.6) is proportional to the original velocity field (3.3), we have

$$\widehat{\xi}(z) = \frac{\alpha}{4\pi \mathrm{i}} \frac{h''(z)}{h'(z)} + \mathcal{O}(z - z_k) = \alpha \xi(z) + \mathcal{O}(z - z_k).$$

Thus we have a stationary vortex of circulation $\alpha \Gamma_k$ at z_k. Next, let z_k be a pole of $h'(z)$. By our assumption (3.1) we must have $\Gamma_k = -1$, and therefore z_k is a second order pole of $h'(z)$ and a first order pole of $h(z)$. Looking at (3.5), we see that $\widehat{h}'(z)$ is therefore analytic at z_k, and that point is not a singularity of $\widehat{\xi}(z)$. Alternatively, this can be checked by expanding (3.6) directly.

Now suppose that $\widehat{z}_j$ is a zero of $h(z)$, and further assume that it is a simple zero. Then, for some constants $a_0, a_1, \ldots$ with $a_0 \neq 0$, we have

$$h(z) = a_0(z - \widehat{z}_j) + a_1(z - \widehat{z}_j)^2 + a_2(z - \widehat{z}_j)^3 + \cdots, \tag{3.11a}$$

$$h'(z) = a_0 + 2a_1(z - \widehat{z}_j) + 3a_2(z - \widehat{z}_j)^2 + \cdots, \tag{3.11b}$$

$$h''(z) = 2a_1 + 6a_2(z - \widehat{z}_j) + \cdots. \tag{3.11c}$$

Substituting (3.11) into (3.6) we find the transformed velocity field near $\widehat{z}_j$ to be

$$\widehat{\xi}(z) = -\frac{1}{2\pi \mathrm{i}} \frac{\alpha}{z - \widehat{z}_j} + \mathcal{O}(z - \widehat{z}_j), \tag{3.12}$$

which is of the desired form for a stationary vortex of circulation $-\alpha$ at $\widehat{z}_j$.

Finally, suppose that $\widehat{z}_j$ is a multiple root of $h(z)$. Then $\widehat{z}_j$ must also be a root of $h'(z)$, and so $\widehat{z}_j = z_k$ for some $k = 1, \ldots, M$. Moreover, since z_k is a root of $h'(z)$ with multiplicity $2\Gamma_k$ by construction, it must be a root of $h(z)$ with multiplicity $2\Gamma_k + 1$. Thus we can write

$$h(z) = (z - z_k)^{2\Gamma_k + 1} G_k(z) \tag{3.13a}$$

for some rational function $G_k(z)$ with $G_k(z_k) \neq 0$. Differentiating (3.13*a*) yields

$$h'(z) = (z - z_k)^{2\Gamma_k} \big((2\Gamma_k + 1) G_k(z) + (z - z_k) G_k'(z) \big), \tag{3.13b}$$

and hence that $G_k(z)$ is related to the function $H_k(z)$ defined in (3.8) via

$$(2\Gamma_k + 1) G_k(z) + (z - z_k) G_k'(z) = H_k(z). \tag{3.13c}$$

Differentiating (3.13*b*) once more to calculate $h''(z)$ and expanding near z_k, a calculation similar to the one in the previous paragraph shows that

$$\widehat{\xi}(z) = \frac{\alpha}{4\pi i} \left[-\frac{2(\Gamma_k + 1)}{z - z_k} - \frac{2\Gamma_k}{2\Gamma_k + 1} \frac{G_k'(z_k)}{G_k(z_k)} + \mathcal{O}(z - z_k) \right]. \tag{3.14}$$

Differentiating (3.13*c*) and substituting $z = z_k$, we find

$$(2\Gamma_k + 2) G_k'(z_k) = H_k'(z_k) = 0, \tag{3.15}$$

where the last equality follows from (2.11) exactly as in (3.10). In particular, since $\Gamma_k \neq -1$, we deduce that $G_k'(z_k) = 0$. Thus (3.14) is of the desired form for a stationary point vortex at z_k with circulation $-\alpha(\Gamma_k + 1)$.

(c) Collapse configurations

While in the above we have focused on a single fixed primitive $h(z)$ of $h'(z)$, it is instructive to consider the whole family of primitives $h(z) + C$ where C is a complex integration constant. For generic values of C, the rational function $h(z) + C$ has only simple roots, depending continuously

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on C and corresponding to stationary point vortices of circulation $-\alpha$ as shown by (3.12). Clearly, the only possible multiple roots are points z_k where

$$C = -h(z_k) \quad \text{and} \quad h'(z_k) = 0. \tag{3.16}$$

In this case, the arguments in §3(b) show that this root has multiplicity $2\Gamma_k + 1$ and corresponds to a point vortex with circulation $-\alpha(\Gamma_k + 1)$ as shown in (3.14). In the limit as $C \to -h(z_k)$, then, $2\Gamma_k + 1$ "movable" vortices of circulation $-\alpha$ and a single "fixed" vortex of circulation $\alpha\Gamma_k$ *collapse* to form a new vortex of circulation $-\alpha(2\Gamma_k + 1) + \alpha\Gamma_k = -\alpha(\Gamma_k + 1)$. See §5(a) and figures 1 and 2 for simple examples of these collapse scenarios. Note that the poles of $h(z) + C$ coincide with the poles of $h'(z)$ and are therefore independent of C.

Of course the values of the constant C where $h(z) + C$ has multiple roots can also be found by setting the discriminant of its numerator equal to zero, yielding a polynomial equation in C. By contrast, the roots z_k of $h'(z)$ are the known locations of the positive strength point vortices in the given equilibrium (3.2), and so calculating C from (3.16) is trivial. Moreover, it clarifies that there is exactly one collapse scenario for each of these positive vortices.

4. Iterated transformations

Under certain circumstances, starting from some seed equilibrium $h_0'(z)$, the transformation (3.5) can be repeated to produce a sequence of equilibria defined by

$$h_{n+1}'(z) = A_{n+1}\left[\frac{h_n'(z)}{(h_n(z))^2}\right]^{\alpha_n} \qquad n \geq 0, \tag{4.1}$$

where the α_n are real constants. As long as the vortex strengths in $h_n'(z)$ satisfy the constraints (3.1), $h_{n+1}'(z)$ represents a new point-vortex equilibrium.

Here, as in (3.5), $h_n(z)$ is any primitive of $h_n'(z)$, but we can consider the family of primitives $h_n(z) + C_n$ where the C_n $(n \geq 0)$ are complex integration constants. The comments in §3(c) about collapse configurations are still applicable and hold for each n. That is, the non-generic values of the constant C_n are given by $C_n = -h_n(z_k)$, where z_k is a root of $h_n'(z)$. Note that the roots of $h_n'(z)$ and hence the non-generic values of C_n depend on $C_0, \ldots, C_{n-1}$. Thus the set of collapse configurations becomes larger (and richer) as n increases. We do not explore this aspect of collapse configurations in detail; see figures 3–6 for a few select examples.

(a) Convention for the constants A_n and C_n

Suppose that $h_n'(z)$ has a rational primitive $N(z)/D(z)$ where $N(z)$ and $D(z)$ are polynomials. Polynomial long division gives

$$\frac{N(z)}{D(z)} = P(z) + \frac{R(z)}{D(z)},$$

where $P(z)$ and $R(z)$ are polynomials and the degree of $R(z)$ is strictly less than that of $D(z)$. We then define

$$h_n(z) = P(z) - P(0) + \frac{R(z)}{D(z)}. \tag{4.2}$$

This amounts to setting the constant term in this representation equal to zero. That constant term can then be added back in explicitly, and we will call it C_n. We can then rewrite (4.1) as

$$h_{n+1}'(z) = A_{n+1}\left[\frac{h_n'(z)}{(h_n(z) + C_n)^2}\right]^{\alpha_n} \qquad n \geq 0. \tag{4.3}$$

It only remains to fix A_{n+1} which, we recall from (3.3), has no physical meaning since it is simply an additive constant in the complex potential. We therefore choose A_{n+1} so that the numerator and denominator of the rational function $h_{n+1}'(z)$ are both monic polynomials in z. Moreover we always begin with a seed equilibrium $h_0'(z)$ which has monic numerator and denominator polynomials.

(b) The special case $\alpha_n = 1$

To study the role of the parameter α_n in the transformation (4.1), we now look at the general theory in the case when $\alpha_n = 1$ for $n \geq 0$. We show that while this choice of α_n produces a new equilibrium at the first stage ($n = 0$), no new equilibria are produced in subsequent stages ($n \geq 1$) of the transformation. Up to a reparametrisation of the integration constants, the family of equilibria produced for each $n \geq 1$ is exactly the same as the family at the previous stage. We call such a transformation *trivial*, if it produces an equilibrium with the same number and circulations of vortices as at the previous stage, but with reparametrised constants.

Let $h_0'(z)$ represent a point vortex equilibrium that is transformed by (4.3) and $\alpha_0 = 1$ into

$$h_1'(z) = A_1 \left(\frac{1}{h_0(z) + C_0} \right)' \quad \Longrightarrow \quad h_1(z) = \frac{A_1}{h_0(z) + C_0} + C_1, \tag{4.4}$$

where C_0 and A_1, C_1 are constants chosen according to the convention described in §4(a). We have used a slightly different form of the transformation (4.3) in (4.4), but it is easy to check that they are the same upto a multiplicative constant, which is absorbed into A_n. From §3 we know that if all the negative vortex circulations in $h_0'(z)$ are -1 (in other words $h_0'(z)$ has only second order poles), then $h_0(z)$ is a rational function with generically simple zeros, and hence $h_1'(z)$ is a new point vortex equilibrium. Further from (4.4) we see that these conditions on $h_0'(z)$ and $h_0(z)$ are sufficient to ensure that $h_1'(z)$ and $h_1(z)$ also satisfy the same conditions. Hence we can consider the sequence of n transformations from $h_0'(z)$ to $h_n'(z)$, choosing $\alpha_n = 1$ at each stage, and each stage being a point vortex equilibrium. Now we see from (4.4) that $h_1(z)$ is a Möbius transformation of $h_0(z)$, and since $\alpha_n = 1$ at each stage, $h_n(z)$ is a Möbius transformation of $h_{n-1}(z)$. We can express $h_n(z)$ in terms of $h_0(z)$ in the form of a finite continued fraction:

$$h_n(z) = C_n + \cfrac{A_n}{C_{n-1} + \cfrac{A_{n-1}}{\ddots + \cfrac{A_1}{C_0 + h_0(z)}}}, \tag{4.5}$$

where $A_0, A_1, \ldots, A_n$ and $C_0, C_1, \ldots, C_n$ are constants. The function $h_n(z)$ is a Möbius transformation of $h_0(z)$,

$$h_n(z) = \frac{E_n h_0(z) + F_n}{\hat{E}_n h_0(z) + \hat{F}_n}, \tag{4.6}$$

for some constants E_n, $\hat{E}_n$, F_n and $\hat{F}_n$ which can be expressed in terms of the constants A's and C's. Taking a derivative of (4.6), we find that the form of the equilibrium after n transformations is

$$h_n'(z) = \left(\frac{E_n \hat{F}_n - F_n \hat{E}_n}{\hat{E}_n^2} \right) \left(\frac{h_0'(z)}{(h_0(z) + \hat{F}_n / \hat{E}_n)^2} \right). \tag{4.7}$$

Thus $h_n'(z)$ is a trivial transformation of $h_1'(z)$ if $\alpha_n = 1$ for all n.

5. Classes of equilibria generated by the transformation

In this section, we look at examples of stationary equilibria produced via the transformation (3.5). First, in §5(a), we consider single-stage transformations in which a given point vortex equilibrium (the seed) is transformed into a new point vortex equilibrium by (3.5). For our examples we choose seed equilibria from among the O'Neil equilibria [32] discussed in the context of stationary equilibria of point vortices with non-identical circulations. We have seen in §4(b) that if we consider the iterated transformation (4.1) with $\alpha_n = 1$ for all n, then a new

	$\alpha_n=-1$	$\alpha_n=\begin{cases}-2 \text{ for } n \text{ even}\\ -1/2 \text{ for } n \text{ odd}\end{cases}$	$\alpha_n=\begin{cases}-1/2 \text{ for } n \text{ even}\\ -2 \text{ for } n \text{ odd}\end{cases}$
$\Gamma=1$	Adler–Moser	Logarithms	Logarithms
$\Gamma=1/2$	Terminating	Loutsenko ($i\leq 0$)	Not point vortices
$\Gamma=2$	Special case of Adler–Moser	Special case of Loutsenko ($i\leq 0$)	Loutsenko ($i\geq 0$)

Table 1. The behaviour of the transformation (4.3) with the seed equilibrium $h_0'(z)=z^{2\Gamma}$ for different choices of Γ and α_n. In different cases we obtain either an entire known hierarchy or else a special case of the hierarchy where some of the free constants have been fixed. The entries marked "terminating" are finite length iterated equilibria, which end when the transformation yields logarithms. In other cases, we obtain logarithms from $1/z$ terms in the seed, or rational functions in $\sqrt{z}$ which do not correspond to point vortices.

equilibrium is only produced in the first stage and the subsequent transformations are all trivial. By making various other choices for α_n, we show that non-trivial hierarchies of equilibria can be produced from the same simple seed equilibrium $h_0'(z)=z^{2\Gamma}$, for different values of the seed circulation Γ (see table 1). In this way, we can reproduce known hierarchies of stationary equilibria: the Adler–Moser polynomials [24] are discussed in §5(b) and the two heirarchies of Loutsenko polynomials [26] are discussed in §5(c). We can also produce hierarchies that terminate after a finite number of stages; see §5(d).

The constants A_n and C_n in all the cases below, including the single-stage and terminating cases, are set according to the conventions described in §4(a). The values of the constants C_n corresponding to the special 'collapse configurations' may be found directly using the method in §3(c). To illustrate this method we provide complete details of the collapse configurations for the single-stage transformations in §5(a). The corresponding constants for the examples in §§5(b)–(d) are obtained in a completely analogous manner and are recorded in table 3.

(a) Single-stage transformations

In our first examples, we look at three- and four-vortex equilibria that are transformed by (3.5) into equilibria with higher numbers of vortices.

(i) From three to eleven vortices

Consider then three vortices with circulations $\Gamma_1=3$, $\Gamma_2=3/2$, $\Gamma_3=-1$ located at $z_1=-2$, $z_2=1$, $z_3=0$ respectively [32]; as shown in figure 1(I). The function $h'(z)$ constructed from (3.2) is

$$h'(z)=\frac{(z+2)^6(z-1)^3}{z^2}, \tag{5.1}$$

where we have set $A=1$. Since the vortex circulations satisfy the constraint (3.1), $h(z)$ is also a rational function. Indeed a simple calculation shows

$$h(z)=\frac{z^8}{8}+\frac{9z^7}{7}+\frac{9z^6}{2}+3z^5-18z^4-36z^3+24z^2+144z+\frac{64}{z}. \tag{5.2}$$

The transformation (3.5) produces a new equilibrium as shown in figure 1(T), with

$$\widehat{h}'(z)=\frac{(z+2)^6(z-1)^3}{(z^9+\frac{72}{7}z^8+36z^7+24z^6-144z^5-288z^4+192z^3+1152z^2+8Cz+512)^2}, \tag{5.3}$$

where C is a constant and we have chosen $\alpha=1$, $\widehat{A}=-1/64$. Comparing (5.3) with (3.2) we see that, for generic choices of the integration constant C, (5.3) corresponds to an equilibrium

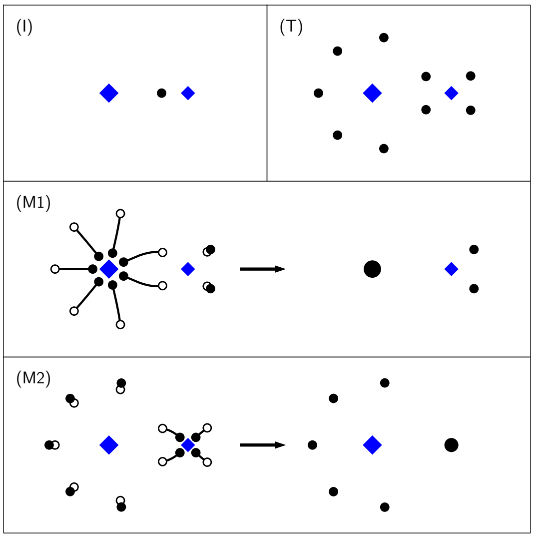

Figure 1. New stationary equilibria produced from known stationary equilibria via the transformation (3.5). Positive vortices are represented by blue diamonds (◆), negative vortices by black disks (●), and the size of the markers represents the vortex circulation. The three-vortex equilibrium (I) given by (5.1) is transformed into an eleven-vortex equilibrium (T) given by (5.3). In fact, there is a family of transformed equilibria parameterized by an integration constant C which is set equal to 0 in (T). For two special values $C_1^{\text{col}} = 1152/7$, $C_2^{\text{col}} = -10467/56$ of C, some of the negative point vortices collapse onto the positive point vortices. The left of (M1) shows the approach to a collapsed configuration as C is varied from 0 to C_1^{col}: open disks mark vortex locations at $C = 0$, filled disks mark vortex locations as we approach C_1^{col}, and every set of corresponding points on the solid curves marks an intermediate equilibrium configuration. The limiting four-vortex configuration is shown to the right; see (5.4) for the explicit vortex locations, which agree with the formulas in the literature [20]. Panel (M2) similarly shows the quite different collapse as C approaches C_2^{col}, this time leading to a seven-vortex equilibrium.

of eleven vortices: two vortices with circulations $\Gamma_1 = 3$ and $\Gamma_2 = 3/2$ located at $z_1 = -2$ and $z_2 = 1$ respectively, and nine vortices with circulations -1 each, located at the roots of the ninth degree polynomial in the denominator of (5.3). The locations of the negative vortices depend continuously on the complex parameter C, and several examples are shown in figure 1.

It is worth repeating here that the complex potential $\widehat{f}(z)$ and velocity field $\widehat{\xi}(z)$ are obtained from any of the $\widehat{h}'(z)$ by the simple formulas $4\pi\mathrm{i}\,\widehat{f}(z) = \log \widehat{h}'(z)$ and $4\pi\mathrm{i}\,\widehat{\xi}(z) = (\log \widehat{h}'(z))'$.

Collapse configurations. From the discussion in §3(c), we see that there are exactly two special values $C_1^{\text{col}}, C_2^{\text{col}}$ of C for which the number and circulations of the point vortices changes: one for each of the positive vortices at $z_1 = -2$, $z_2 = 1$. Indeed, since the zeros of $h'(z)$ are precisely z_1 and z_2, these are the only possible locations for the multiple roots of $h(z) + C$. The special values of C are therefore $C_1^{\text{col}} = -h(z_1)$ and $C_2^{\text{col}} = -h(z_2)$. While C_1^{col} and C_2^{col} can also be found by setting the discriminant of the denominator polynomial in (5.3) equal to zero, the method given above is clearly simpler to use.

First consider $C_1^{\text{col}} = -h(z_1) = 1152/7$. As $C \to C_1^{\text{col}}$, $2\Gamma_1 + 1 = 7$ of the vortices with circulation -1 collapse onto the vortex at z_1, creating a new vortex of circulation $-(\Gamma_1 + 1) = -4$ there when $C = C_1^{\text{col}}$. The corresponding rational function $\widehat{h}'(z)$ is

$$\widehat{h}'(z) = \frac{(z-1)^3}{(z+2)^8(z^2 - \frac{26}{7}z + 4)^2}, \tag{5.4}$$

with one vortex of circulation Γ_2 at z_2, one vortex of circulation -4 at z_1, and two vortices of circulation -1 each, located at $z = (13 \pm 3\sqrt{3}\mathrm{i})/7$; see figure 1(M1). After a simple shifting and

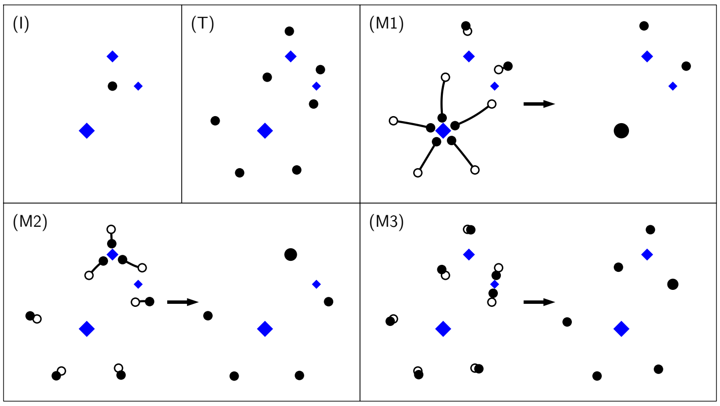

Figure 2. New stationary equilibria produced from known stationary equilibria [32] just as in figure 1 but starting from the four-vortex equilibrium (I). The four-vortex equilibrium (I) in (5.6) is transformed into a ten-vortex equilibrium (T) in (5.8). This is part of a family parametrized by the integration constant C, which is set equal to 0 in (T). As C approaches three particular values $C = C_1^{\rm col}, C_2^{\rm col}, C_3^{\rm col}$ given in (5.9), (T) collapses into the five-, seven- and eight-vortex equilibria (M1), (M2) and (M3) respectively.

scaling, these vortex locations agree with the general formula for four-vortex equilibria given in §8 of [20].

For the second collapse we get $C_2^{\rm col} = -h(z_2) = -10467/56$. As $C \to C_2^{\rm col}$, $2\Gamma_2 + 1 = 4$ of the vortices of circulation -1 each collapse onto the vortex at z_2, combining to form a new vortex of circulation $-(\Gamma_2 + 1) = -\frac{5}{2}$ when $C = C_2^{\rm col}$. The corresponding rational function $\widehat{h}'(z)$ is

$$\widehat{h}'(z) = \frac{(z+2)^6}{(z-1)^5(z^5 + \frac{100}{7}z^4 + \frac{610}{7}z^3 + \frac{2036}{7}z^2 + \frac{3869}{7}z + 512)^2}, \tag{5.5}$$

with one vortex of circulation Γ_1 at z_1, one vortex of circulation $-5/2$ at z_2, and five vortices of circulation -1 each located at the roots of the degree-five polynomial in (5.5); see figure 1(M2). Since all the coefficients in this polynomial are real, the five vortices are symmetrically located about the x-axis. Although it might appear at first sight from figure 1(M2) that the five vortices are arranged on a circle centred at z_2, an inspection of the roots reveals that this is not the case.

(ii) From four to ten vortices

Next consider four vortices with circulations $\Gamma_1 = 2$, $\Gamma_2 = 1$, $\Gamma_3 = 1/2$, $\Gamma_4 = -1$ located at $z_1 = -\sqrt{3} - 3\mathrm{i}$, $z_2 = 2\mathrm{i}$, $z_3 = \sqrt{3}$, $z_4 = 0$ respectively, as shown in figure 2(I). They form an equilibrium [32] with

$$h'(z) = \frac{(z + \sqrt{3} + 3\mathrm{i})^4(z-2\mathrm{i})^2(z-\sqrt{3})}{z^2}. \tag{5.6}$$

Since the vortex strengths satisfy the constraints (3.1), $h(z)$ is a rational function, given by

$$h(z) = \frac{z^6}{6} + \frac{1}{5}(3\sqrt{3} + 8\mathrm{i})z^5 - (1 - 3\sqrt{3}\mathrm{i})z^4 + 4(2\sqrt{3} + 3\mathrm{i})z^3 - 12(1 - 3\sqrt{3}\mathrm{i})z^2 + 24(\sqrt{3} - 9\mathrm{i})z + \frac{288(\sqrt{3} + 3\mathrm{i})}{z}. \tag{5.7}$$

The equilibrium (5.6) is transformed by (3.5), with $\alpha = 1$ and $\widehat{A} = -1/36$, into the ten-vortex equilibrium shown in figure 2(T), with

$$\widehat{h}'(z) = \frac{(z + \sqrt{3} + 3\mathrm{i})^4(z-2\mathrm{i})^2(z-\sqrt{3})}{(6z(h(z) + C))^2}, \tag{5.8}$$

where C is a constant and the denominator is a monic polynomial.

Collapse configurations. As in the previous example, we can completely characterise the collapse configurations using the method of §3(c). There are three special values

$$C_1^{\text{col}} = \frac{144}{5}(33 - \sqrt{3}\text{i}), \quad C_2^{\text{col}} = \frac{32}{15}(-436 + 111\sqrt{3}\text{i}), \quad C_3^{\text{col}} = \frac{9}{10}(-453 - 286\sqrt{3}\text{i}) \tag{5.9}$$

of the integration constant C, at which negative point vortices collapse onto each of the three positive point vortices at z_1, z_2, z_3 forming five-, seven- and eight-vortex equilibria, respectively. These collapsed equilibria are displayed in figure 2(M1) – (M3). The corresponding rational functions $\widehat{h}'(z)$ are respectively

$$\frac{(z-2\text{i})^2(z-\sqrt{3})}{(z+\sqrt{3}+3\text{i})^6(p_1(z))^2}, \quad \frac{(z+\sqrt{3}+3\text{i})^4(z-\sqrt{3})}{(z-2\text{i})^4(p_2(z))^2}, \quad \frac{(z+\sqrt{3}+3\text{i})^4(z-2\text{i})^2}{(z-\sqrt{3})^3(p_3(z))^2}, \tag{5.10a}$$

where the polynomials $p_1(z), p_2(z), p_3(z)$ are

$$p_1(z) = z^2 - \frac{7\sqrt{3}}{5}z - \frac{27\text{i}}{5}z - 6(1-\sqrt{3}\text{i}), \tag{5.10b}$$

$$p_2(z) = z^4 + \frac{6}{5}(3\sqrt{3}+13\text{i})z^3 + \frac{6}{5}(-73+33\sqrt{3}\text{i})z^2 - \frac{4}{5}(183\sqrt{3}+343\text{i})z + 216(3-\sqrt{3}\text{i}), \tag{5.10c}$$

$$p_3(z) = z^5 + \frac{4}{5}(7\sqrt{3}+12\text{i})z^4 + \frac{3}{5}(41+62\sqrt{3}\text{i})z^3 + \frac{6}{5}(67\sqrt{3}+222\text{i})z^2 + \frac{9}{5}(187+354\sqrt{3}\text{i})z + 576(\sqrt{3}+3\text{i}). \tag{5.10d}$$

(b) The Adler–Moser polynomials

Consider the sequence of transformations (4.3) with the seed equilibrium

$$h_0'(z) = z^2 \quad \text{and} \quad \alpha_n = -1 \text{ for } n \geq 0. \tag{5.11}$$

With the conventions in §4(a), the first few rational functions in this sequence are

$$h_1'(z) = \frac{\left(z^3+3C_0\right)^2}{z^2}, \tag{5.12a}$$

$$h_2'(z) = \frac{\left(z^6+15C_0z^3+5C_1z-45C_0^2\right)^2}{(z^3+3C_0)^2}, \tag{5.12b}$$

$$h_3'(z) = \frac{\left(z^{10}+45C_0z^7+35C_1z^5+7C_2z^3-525C_0C_1z^2+4725C_0^3z+21C_0C_2-\frac{175}{3}C_1^2\right)^2}{\left(z^6+15C_0z^3+5C_1z-45C_0^2\right)^2}. \tag{5.12c}$$

The polynomials in (5.12) are the Adler–Moser polynomials constructed by Adler and Moser in [24] utilising Darboux–Crum transformations. The two can be compared by identifying the constants in the Adler–Moser polynomials with the constants in (5.12) as $\tau_2 = 3C_0$, $\tau_3 = 5C_1$ and $\tau_4 = 7C_2$.

For generic values of the integration constants, the polynomials in (5.12) have only simple roots, and hence the rational functions correspond to equilibria of point vortices of the same circulation but opposite signs. The equilibria corresponding to $h_1'(z), \dots, h_4'(z)$ are shown in figure 3. The constants $C_0, \dots, C_3$ are chosen to be real in panels (S1) – (S4) so that the equilibria are symmetric with respect to the real axis. By choosing these constants to be complex, we can obtain asymmetric equilibria as shown in panels (A1) – (A4). As in the single-stage examples, there are special values of the constants for which some of the vortices collapse into a single vortex. However, since there are now multiple integration constants which can be varied simultaneously, the collapse scenarios are more complicated. We show some examples in (M1) – (M5); see table 3 for the exact values of the constants used in producing these panels. The locations of the point vortices in the simple collapse configuration (M1) is given in table 2.

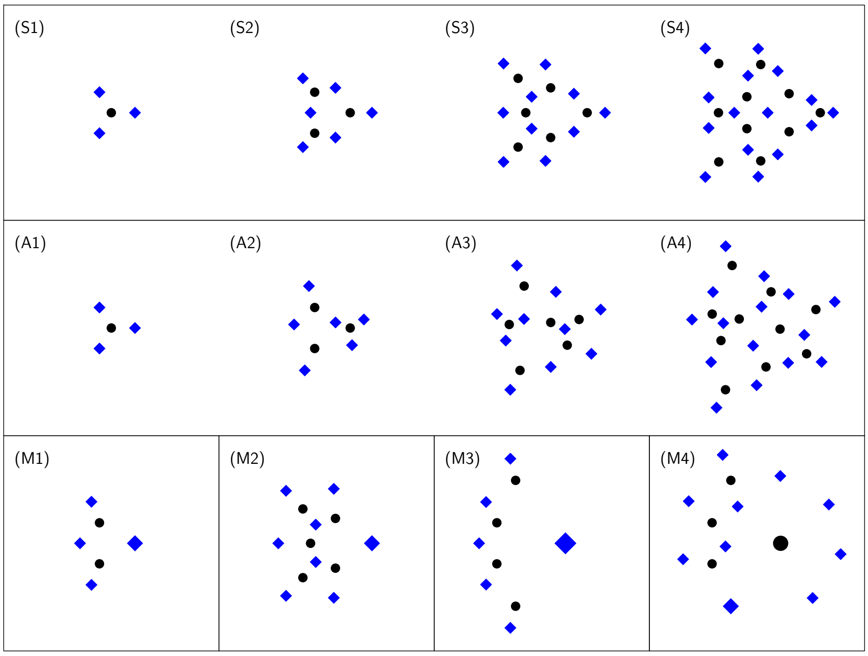

Figure 3. Point vortex equilibria at the roots of the Adler–Moser polynomials [24], obtained via the transformation (4.3) from the seed (5.11) and given by the rational functions (5.12). Panels (S1) – (S4) show symmetric equilibria, (A1) – (A4) show asymmetric equilibria, (M1) – (M5) show various collapsed equilibria. The values of all constants in the figures are given in table 3.

(c) The polynomials due to Loutsenko [26]

First hierarchy. Consider the sequence of transformations (4.3) for $n \geq 0$ with the seed equilibrium

$$h_0'(z) = z \quad \text{and} \quad \alpha_n = \begin{cases} -2 & \text{for } n \text{ even,} \\ -1/2 & \text{for } n \text{ odd.} \end{cases} \tag{5.13}$$

With the conventions in §4(a), the first few rational functions in this sequence are

$$h_1'(z) = \frac{\left(z^2 + 2C_0\right)^4}{z^2}, \tag{5.14a}$$

$$h_2'(z) = \frac{z^8 + \frac{56}{5}C_0 z^6 + 56C_0^2 z^4 + 224C_0^3 z^2 + 7C_1 z - 112C_0^4}{\left(z^2 + 2C_0\right)^2}, \tag{5.14b}$$

$$h_3'(z) = \frac{\left(z^7 + 14C_0 z^5 + 140C_0^2 z^3 + 5C_2 z^2 - 280C_0^3 z + 10C_0C_2 - \frac{35}{2}C_1\right)^4}{\left(z^8 + \frac{56}{5}C_0 z^6 + 56C_0^2 z^4 + 224C_0^3 z^2 + 7C_1 z - 112C_0^4\right)^2}. \tag{5.14c}$$

The polynomials in (5.14) are the polynomials studied by Loutsenko in [26], in particular the branch described in his notation by $i \leq 0$. Loutsenko's constants are labelled τ_i, t_i and they are identified with our constants as $\tau_{-1} = 2C_0, t_{-2} = 7C_1, \tau_{-2} = 5C_2$ and so on. We see from (5.14) that $h_1'(z)$ is an equilibrium of two vortices of circulations $+2$ each and one vortex of circulation -1; $h_2'(z)$ is an equilibrium of eight vortices of circulations $+1/2$ each and two vortices of circulations -1 each; and so on. The choice of α_n in (5.13) is made to ensure that the negative vortices at each step have circulations -1. The circulations of the positive vortices oscillate between $+2$ and $+1/2$, in contrast to the Adler–Moser polynomials where the positive vortices always have circulation

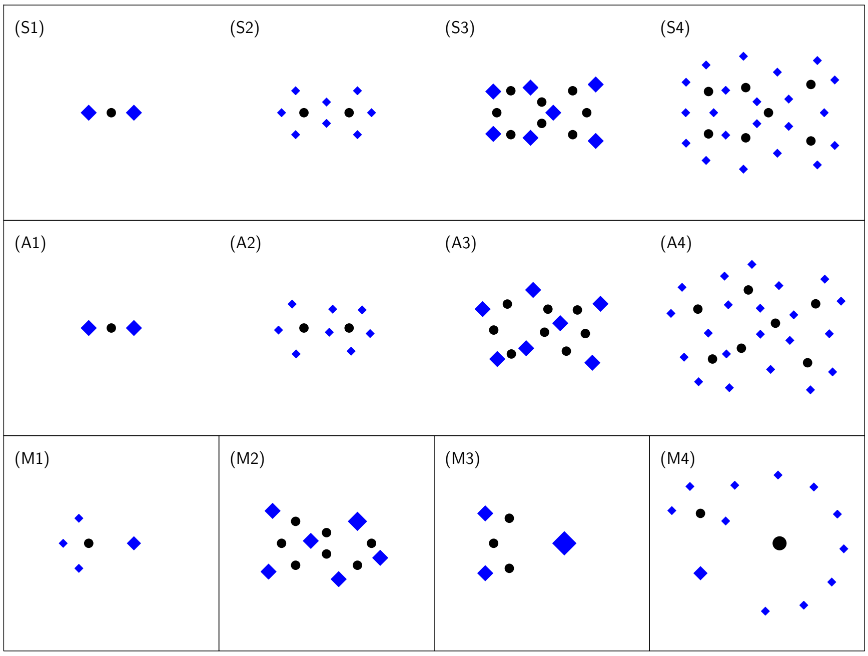

Figure 4. Point vortex equilibria at the roots of the Loutsenko ($i \leq 0$) polynomials, produced by the iterated transformations (4.3) with the seed equilibrium (5.13), and given by the rational functions (5.14). The panels are analogous to those in figure 3, and vortex locations and circulations for the collapsed configurations (M1) and (M3) are provided in table 2. See table 3 for the values of the integration constants.

+1. Then from §3 we have that the $h_n(z)$ are rational functions and $h_n'(z)$ are stationary equilibria for all n.

Examples of the equilibria $h_1'(z), \ldots, h_4'(z)$ are shown in figure 4. Point vortex locations and circulations for the—particularly simple—five-vortex equilibrium (M1) and six-vortex equilibrium (M3) are given in table 2. The constants C_n for all the equilibria are given in table 3. The equilibria in panels (S2) and (A2) can be recognised as figure 3 of [38], where these configurations are obtained as limits of hybrid smooth Stuart vortex and point vortex equilibria. The function $h_0'(z)$ can be identified with the function $h'(z)$ defined in (3.8) of [38] with the choice $C_0 = -1/2$.

Second hierarchy. Now consider the sequence of transformations (4.3) with the seed equilibrium

$$h_0'(z) = z^4 \quad \text{and} \quad \alpha_n = \begin{cases} -1/2 & \text{for } n \text{ even} \\ -2 & \text{for } n \text{ odd}. \end{cases} \tag{5.15}$$

With the conventions in §4(a), the first few rational functions in this sequence are

$$h_1'(z) = \frac{z^5 + 5C_0}{z^2}, \quad h_2'(z) = \frac{\left(z^5 + 4C_1 z - 20C_0\right)^4}{\left(z^5 + 5C_0\right)^2}, \quad h_3'(z) = \frac{p(z)}{\left(z^5 + 4C_1 z - 20C_0\right)^2}, \tag{5.16a}$$

where the numerator $p(z)$ is

$$p(z) = z^{16} + \frac{176}{7}C_1 z^{12} - 160C_0 z^{11} + 352C_1^2 z^8 - \frac{42240}{7}C_0 C_1 z^7 + 35200C_0^2 z^6 + 11C_2 z^5$$
$$- 2816C_1^3 z^4 + 28160C_0 C_1^2 z^3 - 140800C_0^2 C_1 z^2 + 352000C_0^3 z - \frac{2816}{5}C_1^4 + 55C_0 C_2. \tag{5.16b}$$

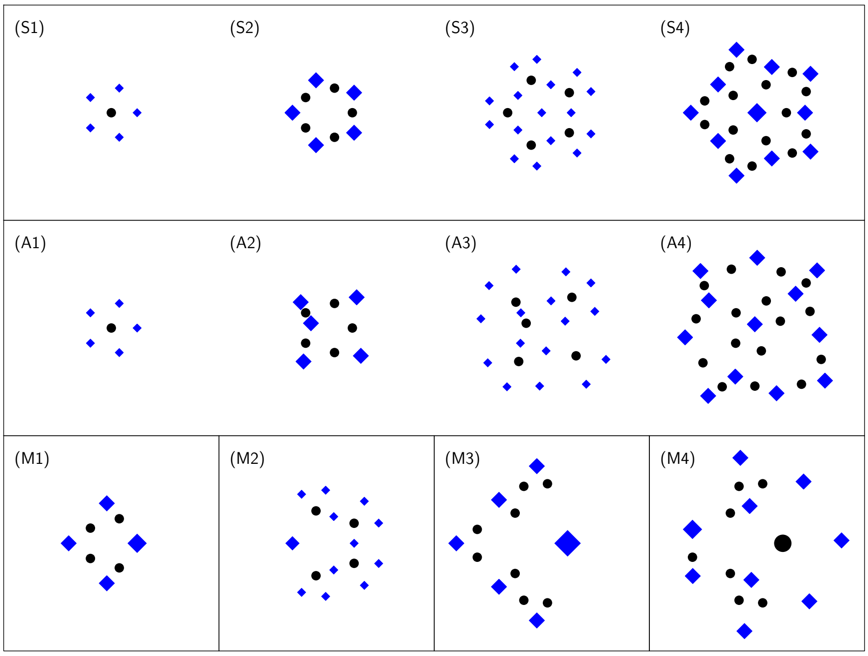

Figure 5. Point vortex equilibria at the roots of the Loutsenko ($i \geq 0$) polynomials, produced by the iterated transformations (4.3) with the seed equilibrium (5.15), and given by the rational functions (5.16). The vortex circulations and locations for the collapsed configuration (M1) are given in table 2. Table 3 gives the values of all the integration constants.

The polynomials in (5.16) are the second hierarchy found by Loutsenko [26] ($i \geq 0$ in his notation), the constants can be compared by setting his $t_1 = 5C_0$ and $\tau_2 = 4C_1$. We see from (5.16) that the negative vortices all have circulation -1 and the theory in §3 applies. The positive vortices oscillate between circulations $+1/2$ and $+2$ just as in the Loutsenko ($i \leq 0$) hierarchy, but in this case they begin in the hierarchy at $+1/2$ instead of $+2$. The choice of α_n in (5.15) is once again made so that the negative vortices always have circulations -1. Examples of the equilibria in the second Loutsenko hierarchy are shown in figure 5. Also see tables 2 and 3.

Figure 3(M1)	Circulations	2	-1	1
	Locations	1	$-1/2 \pm \sqrt{3}\mathrm{i}/2$	roots of $z^3 + 3z^2 + 6z + 5$
Figure 4(M1)	Circulations	3/2	-1	1/2
	Locations	1	-1	roots of $z^3 + 5z^2 + \frac{47}{5}z + 7$
Figure 4(M3)	Circulations	5	2	-2
	Locations	1	$-5/2 \pm \sqrt{7}\mathrm{i}/2$	roots of $z^3 + 5z^2 + \frac{47}{5}z + 7$
Figure 5(M1)	Circulations	3	-1	2
	Locations	1	roots $\neq 1$ of $z^5 - 1$	roots of $z^3 + 2z^2 + 3z + 4$

Table 2. Vortex circulations and locations for selected simple equilibria in figures 3–5; see §§5(b)–(c) for details.

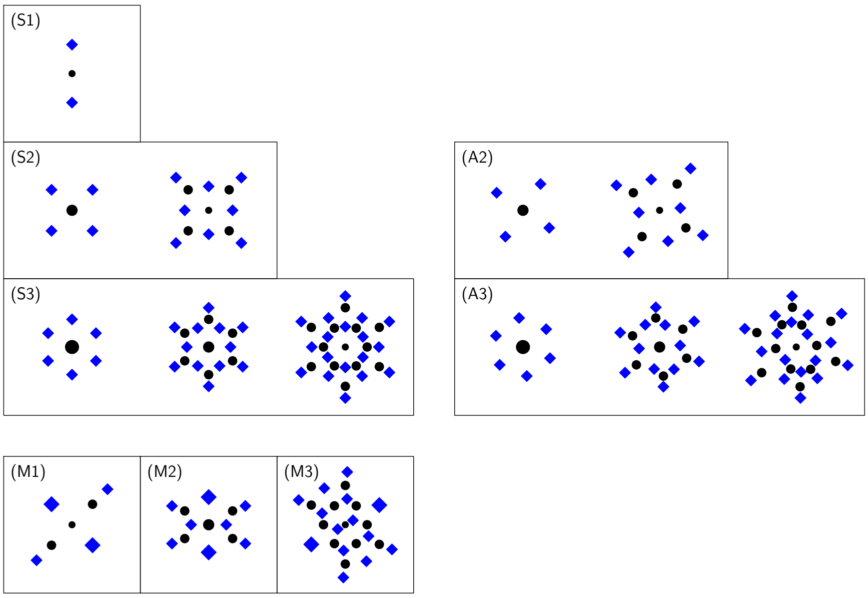

Figure 6. Terminating sequences of point vortex equilibria produced by the iterated transformations (4.3) with the seed (5.17), given by the rational functions (5.18). Symmetric equilibria for $\Gamma = 1/2, 3/2, 5/2$ are shown in (S1) – (S3), and asymmetric equilibria for $\Gamma = 3/2, 5/2$ in (A2), (A3). Panel (M1) is a collapsed version of the second column of (S2), while (M2) and (M3) are collapsed versions of the last two columns of (S3). See table 3 for the values of the C_n.

(d) Terminating sequences of stationary equilibria

Consider the seed equilibrium $h_0'(z) = z^{2\Gamma}$ for half-integer Γ,

$$h_0'(z) = z, z^3, z^5, \dots \quad \text{and} \quad \alpha_n = -1. \tag{5.17}$$

With this seed we find that the transformation (4.3) produces sequences of equilibria which terminate after a finite number of steps. If $\Gamma = 1/2$, then the equilibria terminate after one stage, after two stages if $\Gamma = 3/2$, after three stages if $\Gamma = 5/2$, and so on. The iteration terminates due to a simple pole term that appears in $h_n'(z)$, which leads to a logarithmic term in $h_n(z)$. At the first stage, the circulation of the point vortex at the origin switches sign to become $-\Gamma$. At each subsequent stage, it increases by 1 until it becomes $-1/2$ and the iteration terminates. Examples of terminating equilibria are shown in figure 6. The rational functions for $\Gamma = 1/2$ are

$$h_0'(z) = z, \quad h_1'(z) = \frac{\left(z^2 + 2C_0\right)^2}{z}. \tag{5.18a}$$

The rational functions for $\Gamma = 3/2$ are

$$h_0'(z) = z^3, \quad h_1'(z) = \frac{\left(z^4 + 4C_0\right)^2}{z^3}, \quad h_2'(z) = \frac{\left(z^8 + 24C_0 z^4 + 6C_1 z^2 - 48C_0^2\right)^2}{z\left(z^4 + 4C_0\right)^2}. \tag{5.18b}$$

The rational functions for $\Gamma = 5/2$ are

$$\begin{aligned} h_0'(z) &= z^5, & h_2'(z) &= \frac{\left(z^{12} + 48C_0 z^6 + 8C_1 z^4 - 72C_0^2\right)^2}{z^3\left(z^6 + 6C_0\right)^2}, \\ h_1'(z) &= \frac{\left(z^6 + 6C_0\right)^2}{z^5}, & h_3'(z) &= \frac{(p(z))^2}{z\left(z^{12} + 48C_0 z^6 + 8C_1 z^4 - 72C_0^2\right)^2}, \end{aligned} \tag{5.18c}$$

		C_0	C_1	C_2	C_3
Figure 3	S1–S4	$-1/3$	-1	20	80
	A1–A4	$-1/3$	$2-\mathrm{i}$	$8-8\mathrm{i}$	$40+120\mathrm{i}$
	M1	$-1/3$	$9/5$		
	M2	$-1/3$	-1	63.8065	
	M3	$-1/3$	$9/5$	$-225/7$	$9800/9$
	M4	$-1/3$	$9/5$	$-225/7$	$-574.64+6344.3\mathrm{i}$
Figure 4	S1–S4	$-1/2$	0	6	40
	A1–A4	$-1/2$	$3+3\mathrm{i}$	$6-12\mathrm{i}$	$20+20\mathrm{i}$
	M1	$-1/2$	$128/35$		
	M2	$-1/2$	0	$-11.350+6.3767\mathrm{i}$	
	M3	$-1/2$	$128/35$	$-56/5$	
	M4	$-1/2$	$128/35$	$-56/5$	$(-2.959+7.618\mathrm{i})\times 10^5$
Figure 5	S1–S4	$-1/5$	0	0	0
	A1–A4	$-1/5$	$(3+\mathrm{i})/2$	$1000-2000\mathrm{i}$	$100+100\mathrm{i}$
	M1	$-1/5$	$-5/4$		
	M2	$-1/5$	0	$(1600/11)\times 2^{2/5}$	
	M3	$-1/5$	$-5/4$	$-12800/77$	$440/7$
	M4	$-1/5$	$-5/4$	$-12800/77$	$-385.79-120.55\mathrm{i}$
Figure 6	S1–S3	1	0	0	
	A2–A3	$1+\mathrm{i}$	$10\mathrm{i}$	-8	
	M1	1	$32\mathrm{i}/3$		
	M2	1	$27\sqrt[3]{6}/4$		
	M3	1	0	$193.30-334.81\mathrm{i}$	

Table 3. Values of the constants $C_0, \ldots, C_3$ in figures 3–6. The constants are calculated according to the method in §3(c) and §4. Figure panels (S1)–(S4) and (A1)–(A4) which share the same values of the constants are grouped together. For example in the first row corresponding to figure 3, (S1) has $C_0 = -1/3$, (S2) has $C_0 = -1/3$, $C_1 = -1$ and so on. All decimal values given are numerical approximations.

where the numerator $p(z)$ of $h_3'(z)$ is

$$p(z) = z^{18} + 216C_0z^{12} + 80C_1z^{10} + 10C_2z^8 - 4320C_0^2z^6 - 960C_0C_1z^4 + z^2\left(60C_0C_2 - \frac{320}{3}C_1^2\right) - 4320C_0^3. \qquad (5.18d)$$

Here, as always, we observe the conventions in §4(a) for the constants A_n and C_n. We note that a similar, finite, sequence of polynomials are discussed in a different context in [39].

6. Summary and future directions

We have presented a general transformation linking two distinct stationary point vortex equilibria. It allows us to find a new equilibrium from any given equilibrium, as long as all the negative vortex circulations in the given equilibrium are −1 and the positive circulations are all integers or half-integers. If some of the negative vortex circulations are different, then the theory presented in §3 needs to be modified. We have shown that the transformation can be iterated to reproduce the Adler–Moser hierarchy and the hierarchies due to Loutsenko, along with finite length sequences of equilibria that appear to be new. All of these equilibria can be produced from a simple seed equilibrium by changing a couple of parameters; see table 1.

Our transformation (4.3) can be viewed as a generalisation of the Darboux–Crum transformation [25]. Given a seed $h_0'(z)$, if we pick $\alpha_n = -1$ for $n \geq 0$ and define functions $\phi_n(z)$ via $h_n'(z) = (\phi_n(z))^2$ for $n \geq 0$, then (4.3) reduces to the iterated Darboux-Crum transformation. A deeper investigation of this topic takes us into the theory of Schrödinger potentials; this is a separate topic that we intend to take up in another paper. For further discussion of this in the context of vortex dynamics, see [19] and the references therein. We also note that [19] lists finding polynomial solutions to several differential equations arising in the context of vortex equilibria as open problems. In particular, for $m = 2$ the equation $P''Q - 2mP'Q' + m^2PQ'' = 0$ leads to the polynomials found by Loutsenko, but it is not known whether it possesses polynomial solutions for $m > 2$.

As remarked in the introduction, it is known from numerical exploration that families of rotating equilibria exist which do not appear to have been captured analytically so far [37]. The type of analysis used in the present paper might be of some applicability here, particularly since it captures both symmetric and asymmetric configurations. Similar in spirit to growing new point vortex equilibria from existing equilibria [37], exact solutions have been constructed in [40] with two vortex patches grown at the co-rotating stagnation points of a rotating point vortex pair equilibrium. The latter solution of point vortex and vortex patch equilibria builds on the mathematical ideas in [41], in which a multipolar stationary equilibrium of point vortices and vortex patches is constructed.

Finally, the authors have recently constructed stationary equilibrium solutions of the steady Euler equation which they refer to as hybrid equilibria comprising a combination of Stuart-type vorticity with superposed point vortices [38]. There is a close relationship between generalizations of those solutions and the stationary point vortex equilibria presented here: hybrid solutions of this kind turn out to continuously interpolate and extrapolate between the various stationary point vortex equilibria exhibited in this paper. A detailed description of all these matters is in preparation and will be published elsewhere.

Data Accessibility. This article presents the theory of a mathematical transformation and contains no external data.

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References

1. Helmholtz H. 1858 Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. *J. Reine Angew. Math.* **55**, 25–55.
2. Saffman PG. 1992 *Vortex dynamics*. Cambridge University Press.
3. Marchioro C, Pulvirenti M. 1994 *Mathematical theory of incompressible nonviscous fluids* vol. 96*Applied Mathematical Sciences*. Springer-Verlag, New York.
4. Aref H. 2007 Point vortex dynamics: A classical mathematics playground. *J. Math. Phys.* **48**, 065401.
5. Newton PK. 2001 *The N-vortex problem: Analytical techniques* vol. 145 *Applied Mathematical Sciences*. Springer-Verlag, New York.
6. Llewellyn Smith SG. 2011 How do singularities move in potential flow?. *Phys. D* **240**, 1644–1651.
7. Aref H, Newton PK, Stremler MA, Tokieda T, Vainchtein DL. 2003 Vortex Crystals. *Adv. Appl. Mech.* **39**, 1–79.

8. Mayer AM. 1878 Floating magnets. *Nature* **18**, 258–260.
9. Grzybowski BA, Stone HA, Whitesides GM. 2000 Dynamic self-assembly of magnetized, millimetre-sized objects rotating at a liquid-air interface. *Lett. to Nat.* **405**, 1033–1036.
10. Yarmchuk EJ, Gordon MJV, Packard RE. 1979 Observation of stationary vortex arrays in rotating superfluid helium. *Phys. Rev. Lett.* **43**, 214–217.
11. Abo-Shaeer JR, Raman C, Vogels JM, Ketterle W. 2001 Observation of vortex lattices in Bose-Einstein condensates. *Science* **292**, 476–479.
12. Durkin D, Fajans J. 2000a Experimental dynamics of a vortex within a vortex. *Phys. Rev. Lett.* **85**, 4052–4055.
13. Durkin D, Fajans J. 2000b Experiments on two-dimensional vortex patterns. *Phys. Fluids* **12**, 289–293.
14. Carnevale GF, Kloosterziel RC, Orlandi P. 2016 Equilibration of centrifugally unstable vortices: A review. *Eur. J. Mech. B/Fluids* **55**, 246–258.
15. Fine KS, Cass AC, Flynn WG, Driscoll CF. 1995 Relaxation of 2D turbulence to vortex crystals. *Phys. Rev. Lett.* **75**, 3277–3280.
16. Jiménez J, Guegan A. 2007 Spontaneous generation of vortex crystals from forced two-dimensional homogeneous turbulence. *Phys. Fluids* **19**, 085103.
17. Newton PK. 2014 Point vortex dynamics in the post-Aref era. *Fluid Dyn. Res.* **46**, 031401.
18. Aref H. 2007 Vortices and polynomials. *Fluid Dyn. Res.* **39**, 5–23.
19. Clarkson PA. 2009 Vortices and polynomials. *Stud. Appl. Math.* **123**, 37–62.
20. O'Neil KA. 1987 Stationary configurations of point vortices. *Trans. Amer. Math. Soc.* **302**, 383–425.
21. Burchnall JL, Chaundy TW. 1930 A set of differential equations which can be solved by polynomials. *Proc. London Math. Soc. (2)* **30**, 401–414.
22. Burchnall JL, Chaundy TW. 1928 Commutative Ordinary Differential Operators. *Proc. R. Soc. A* **118**, 557–583.
23. Airault H, McKean HP, Moser J. 1977 Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem. *Comm. Pure Appl. Math.* **30**, 95–148.
24. Adler M, Moser J. 1978 On a class of polynomials connected with the Korteweg-deVries equation. *Commun. Math. Phys.* **30**, 1–30.
25. Crum MM. 1955 Associated Sturm-Liouville systems. *Quart. J. Math. Oxford Ser. (2)* **6**, 121–127.
26. Loutsenko I. 2004 Equilibrium of charges and differential equations solved by polynomials. *J. Phys. A: Math. Gen.* **37**, 1309–1321.
27. Aref H. 1983 Integrable, chaotic, and turbulent vortex motion in two-dimensional flows. *Annu. Rev. Fluid Mech.* **15**, 345–389.
28. Tkachenko VK. 1964 . (Unpublished thesis) Institute of Physical Problems, Moscow.
29. Bartman AB. 1984 A new interpretation of the Adler-Moser KdV polynomials: interaction of vortices. In *Nonlinear and turbulent processes in physics, Vol. 3 (Kiev, 1983)* pp. 1175–1181. Harwood Academic Publ., Chur.
30. Campbell LJ, Kadtke JB. 1987 Stationary configurations of point vortices and other logarithmic objects in two dimensions. *Phys. Rev. Lett.* **58**, 670–673.
31. Kadtke JB, Campbell LJ. 1987 Method for finding stationary states of point vortices. *Phys. Rev. A* **36**, 4360–4370.
32. O'Neil KA. 2006 Minimal polynomial systems for point vortex equilibria. *Phys. D* **219**, 69–79.
33. Hampton M, Moeckel R. 2009 Finiteness of stationary configurations of the four-vortex problem. *Trans. Amer. Math. Soc.* **361**, 1317–1332.
34. Aref H. 2011 Relative equilibria of point vortices and the fundamental theorem of algebra. *Proc. R. Soc. Lond. A* **467**, 2168–2184.
35. Barreiro A, Bronski J, Newton PK. 2010 Spectral gradient flow and equilibrium configurations of point vortices. *Proc. R. Soc. A* **466**, 1687–1702.
36. Newton PK, Ostrovskyi V. 2012 Stationary equilibrium singularity distributions in the plane. *Nonlinearity* **25**, 495–511.
37. Aref H, Vainchtein DL. 1998 Point vortices exhibit asymmetric equilibria. *Nature* **392**, 769–770.
38. Krishnamurthy VS, Wheeler MH, Crowdy DG, Constantin A. 2019 Steady point vortex pair in a field of Stuart-type vorticity. *J. Fluid Mech.* **874**, R1.
39. Duistermaat JJ, Grünbaum FA. 1986 Differential equations in the spectral parameter. *Comm. Math. Phys.* **103**, 177–240.
40. Crowdy DG, Marshall J. 2004 Growing vortex patches. *Phys. Fluids* **16**, 3122–3130.
41. Crowdy DG. 1999 A class of exact multipolar vortices. *Phys. Fluids* **11**, 2556–2564.