Linear types and approximation

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We study continuous lattices with maps that preserve all suprema rather than only directed
ones. We introduce the (full) subcategory of FS-lattices, which turns out to be ∗-autonomous,
and in fact maximal with this property. FS-lattices are studied in the presence of
distributivity and algebraicity. The theory is extremely rich with numerous connections to
classical Domain Theory, complete distributivity, Topology and models of Linear Logic.

1. Introduction

The work reported in this paper derives its motivation from at least three different direc-
tions. First, there is the theory of *autonomous* (or *symmetric monoidal closed*) categories as
described extensively in Eilenberg and Kelly (1966). These are abstractions of the frequent
phenomenon in algebra of the set of homomorphisms between two structures being again
a structure of the same kind but without the internal hom functor interacting with the
product in the usual way. The correspondence as it is expressed in Linear Algebra, then, is
between *bilinear* maps and *tensor products* rather than between linear maps and products.
In Barr (1979), the abstract theory of symmetric monoidal closed categories is extended
with a duality derived from a dualizing object ⊥. Again, algebra provides a number of
motivating examples. One of these is the category **SUP** of complete lattices and sup-
preserving functions[¶]. In the present paper we augment the objects of this category with
a notion of 'approximation' in the sense of Domain Theory (Abramsky and Jung 1994).
We show that the full subcategory **CL** of continuous lattices is not closed, and one of
our main results characterizes the largest closed full subcategory of **CL** (under one extra
condition). The result is reminiscent of similar theorems for cartesian closed categories
(Smyth 1983; Jung 1990); it would be very interesting to find a deeper reason for this
similarity.

[¶] In fact, Barr works with infima rather than suprema but this difference is immaterial.

From a different perspective, this paper introduces a new model for Classical Linear Logic (Girard 1987). On the surface, this construction seems fairly straightforward, given the general theory of ∗-autonomous categories as presented in Barr (1991). We choose the modality ! to be that of all Scott-closed subsets of the lattice with the goal in mind of getting Scott-continuous maps in the corresponding co-Kleisli category. Rather pleasingly, the dual modality ? has a meaningful interpretation in its own right, rather than just being the de Morgan dual of !; it yields precisely the so-called Smyth-powerdomain (Smyth 1978). One may see this as a vindication of the move to approximated lattices, as such a characterization is not available in the bigger category **SUP**. (Abramsky and Jung (1994) contains other instances of this phenomenon.)

Finally, one may see this paper as an attempt to achieve a linear decomposition of Scott-continuous functions along the lines of Girard's original construction of coherence spaces and stable maps. It is then interesting to see that certain concepts of Domain Theory still apply, certifying their robustness and generality.

The structure of the paper is as follows. We recall the algebraic tradition that led to the theory of ∗-autonomous categories in Section 2. In Section 3 we give some details of Barr's example **SUP** for a ∗-autonomous category consisting of complete lattices and suprema preserving functions. It is the ambient category for the remainder of the paper. Section 4 introduces the main objects of study, linear FS-lattices. They are defined in analogy to FS-domains (Jung 1990), and, as in the Scott-continuous setting, they provide a closed category of approximated objects. In fact, we are able to show that they are a maximal choice when a certain further condition (called 'leanness') is assumed. FS-lattices are subsequently augmented with two (independent) properties: distributivity (Section 5) and algebraicity (Section 7). In both cases, we obtain additional information: distributive FS-lattices turn out to be completely distributive and they form not only a ∗-autonomous but a compact closed category. Algebraic FS-lattices are shown to be exactly the bifinite ones (in the linear sense), and a fairly involved argument in Subsection 7.3 shows that algebraic FS-lattices are the maximal ∗-autonomous full subcategory of **SUP** whose objects are algebraic. A number of parallels between the Scott-continuous and the linear setting are pointed out in the remainder of Section 7.

Meanwhile, in Section 6, we show how to build a Benton-model of Linear Logic with the ingredients of Domain Theory. The development is extremely smooth and we would like to claim that the model is a natural yet non-trivial one. We were particularly pleased to find the connection between modalities and powerdomains mentioned before. Although Section 6 refers to distributivity at some points, it can be read directly after Section 4.

Section 8 indicates how the theory could be extended from lattices to Scott-domains. For the sake of brevity, we have refrained from a detailed exposition. Section 9 refers to further interesting discoveries about FS-lattices, which were made more recently.

We follow Abramsky and Jung (1994) in our notation for domain-theoretic concepts; relevant background information on continuous lattices can be found there as well as in Gierz *et al.* (1980).

2. Categorical preliminaries

If $\mathcal{K}$ is a class of algebraic structures and A, B, C are objects in $\mathcal{K}$, one calls a map $\phi: A \times B \to C$ a *bihomomorphism* if for every $a \in A$, $b \in B$ the functions

$$\phi(a, \_): y \mapsto \phi(a, y), \text{and}$$
$$\phi(\_, b): x \mapsto \phi(x, b)$$

are homomorphisms of $\mathcal{K}$. The prime example is vector spaces and bilinear maps.

A category is an abstract version of 'class of structures of the same kind and their homomorphisms'. However, the definition of a bihomomorphism seems to require an explicit reference to elements. Also, the map ϕ itself is certainly external to the category at hand.

A slight redefinition of bihomomorphism is more amenable to a categorical treatment. Instead of $\phi: A \times B \to C$, we consider $\phi': A \to (B \to C)$ given by $\phi'(x)(y) := \phi(x, y)$. If we assume that the set $(B \to C)$ of homomorphisms is itself a structure of the same kind as A, B and C through a pointwise definition of the operations, then bihomomorphisms $\phi: A \times B \to C$ and *homomorphism* $\phi': A \to (B \to C)$ are in one-to-one correspondence. These two conditions are indeed satisfied for vector spaces and, also, for the objects under consideration here, complete lattices with sup-preserving maps.

Categorically, one requires an object $\top$ and an *internal hom-functor* $(\_ \to \_)$, contravariant in the first and covariant in the second argument, to model the requirement that the set of homomorphisms qualifies as a structure. In order to recognize the object $(A \to B)$ as the set of homomorphisms from A to B, one requires certain natural transformations and equivalences, to wit

$$
\begin{aligned}
(\top \to A) &\cong A \\
\top &\overset{\cdot}{\longrightarrow} (A \to A) \\
(B \to C) &\overset{\cdot}{\longrightarrow} ((A \to B) \to (A \to C))
\end{aligned}
$$

subject to a number of axioms (Eilenberg and Kelly 1966). A category with these properties is called *closed*. In a closed category we may replace 'bihomomorphism' with 'morphism from A to $(B \to C)$'. See Banaschewski and Nelson (1976) for an in-depth discussion.

A closed category is called *symmetric closed* if $(A \to (B \to C))$ and $(B \to (A \to C))$ are naturally isomorphic.

From Linear Algebra we know that bilinear maps $A \times B \to C$ are in one-to-one correspondence with *linear* maps $A \otimes B \to C$, where $\_ \otimes \_$ denotes the *tensor product* of vector spaces. Abstractly, then, the presence of a 'tensor product' gives us an alternative way of coding bihomomorphisms. To make this precise, one stipulates that $\_ \otimes \_$ be a bifunctor for which $\_ \otimes B$ is left adjoint to $(B \to \_)$, or, equivalently, $(A \otimes B \to C)$ and $(A \to (B \to C))$ are naturally isomorphic. In addition to this, the abstract tensor product is required to be associative and to have a unit I subject to a number of coherence axioms (Eilenberg and Kelly 1966; Mac Lane 1971). With this additional data, we arrive at a *monoidal closed category*. In a monoidal closed category, which is also symmetric in the sense above, the tensor product is commutative, $A \otimes B \cong B \otimes A$. Together, one speaks of a *symmetric monoidal closed* or *autonomous* category.

We should make one last remark: not every algebraic theory allows us to internalize the hom-functor (non-Abelian groups are an example), and even if it does, a suitable tensor product may not exist. Beyond these two obstacles, a further one needs to be overcome for a category to be *cartesian* closed, namely, it must be the case that bihomomorphisms are already homomorphisms. The category **SET** qualifies for trivial reasons; in the case of **DCPO** (directed-complete partial orders and Scott-continuous functions) this is one of the fundamental lemmas of its theory (Abramsky and Jung 1994, Lemma 3.2.6).

In Barr (1979), Michael Barr studies the situation where an autonomous category is equipped with an internal duality, that is, where there exists an object $\perp$ such that A and $((A \to \perp) \to \perp)$ are naturally isomorphic for all objects A. Writing $A^\perp$ for $(A \to \perp)$, one gets the following equivalences:

$$(A \to B) \;\cong\; (B^\perp \to A^\perp) \tag{1}$$

$$A \otimes B \;\cong\; (A \to B^\perp)^\perp \tag{2}$$

without making any further assumptions. A category with these properties, dubbed *$*$-autonomous* by Barr, provides a model for the multiplicative part of Linear Logic (Girard 1987; Barr 1991).

3. SUP as a model of Linear Logic

The category **SUP** of complete lattices and suprema preserving maps was mentioned as an example of a $*$-autonomous category in Barr (1979). For our purposes below, it will be necessary to have some understanding of the concrete structure of the various connectives in **SUP**. We will also have to adjust the categorical notation to this particular setting.

Definition 3.1. Let A and B be complete lattices and f be a map from A to B. We call f *linear* if it preserves all suprema, $f(\bigvee X) = \bigvee f(X)$, $X \subseteq A$. We write $f : A \multimap B$ in this situation. The set of all linear maps between A and B, ordered pointwise, is denoted by $(A \multimap B)$.

Complete associativity of the supremum operation in lattices (Abramsky and Jung 1994, Proposition 2.1.4(3)) entails that the function space $(A \multimap B)$ is again a complete lattice.

Every linear map $f : A \multimap B$ has an upper adjoint $f^* : B \to A$ (Abramsky 1994, Section 3.1.3; Gierz *et al.* 1980, Chapter IV). It is given by

$$f^*(y) := \bigvee \{x \mid f(x) \leqslant y\} \,.$$

Alternatively, the correspondence between f and f^* may be encoded in the equivalence

$$f(x) \leqslant y \iff x \leqslant f^*(y) \,. \tag{3}$$

From this we glean that the assignment $f \mapsto f^*$ is order reversing[†]. Hence, if we view f^* as a map from B^{op} to A^{op}, we get a linear function and the correspondence $f \mapsto f^* : (A \multimap B) \to (B^{\mathrm{op}} \multimap A^{\mathrm{op}})$ is in fact an order isomorphism.

There is only one possibility for a dualizing object in **SUP**, and this is the two-element lattice **2**. For the dual $A^{\perp}$ of a complete lattice A with respect to $\perp = \mathbf{2}$, we have

$$A^{\perp} = (A \multimap \mathbf{2}) \cong (\mathbf{2}^{\mathrm{op}} \multimap A^{\mathrm{op}}) \cong (\mathbf{2} \multimap A^{\mathrm{op}}) \cong A^{\mathrm{op}} \,,$$

where the last isomorphism holds because the bottom element of **2** must be mapped onto the bottom element of A^{op} by any linear function and the top element can be mapped onto any element of A^{op} whatsoever.

From now on, we will write A^{op} instead of $A^{\perp}$ and **2** instead of $\perp$ to avoid confusion with the established notation for the least element of a domain. Also, we will use the symbols, $\leqslant$, $\vee$, *etc.*, as they apply to A even when we speak of A^{op}.

For the tensor product we take equivalence (2) as the (necessary) definition: $A \otimes B := (A \multimap B^{\mathrm{op}})^{\mathrm{op}}$. Concretely, a linear map r from A to B^{op} corresponds to an antitone map from A to B that translates suprema into infima. The upper adjoint $r^* : B^{\mathrm{op}} \to A$, if viewed as a function from B to A, has exactly the same property. Together, (r, r^*) form a *Galois-connection* between A and B. Any pair of maps between complete lattices satisfying

$$r(x) \geqslant y \iff x \leqslant s(y) \,, \qquad x \in A, y \in B \tag{4}$$

is of this kind.

The de Morgan dual of $\otimes$, denoted $\parr$ ('par'), is given by the set of linear functions from A^{op} to B. Maps $r : A^{\mathrm{op}} \multimap B$ together with their adjoints $r^* : B^{\mathrm{op}} \multimap A$ form pairs (r, r^*) which are completely characterized by the equivalence

$$r(x) \leqslant y \iff x \geqslant s(y) \,, \qquad x \in A, y \in B \,.$$

As noted in Barr (1979), $A \parr B$ can be different from $A \otimes B$, even for finite lattices A and B. In fact, it is distributivity, not finiteness, that renders $\parr$ and $\otimes$ equal, as we will see in Section 5.

It is quite enjoyable to explore what the abstract equivalences of a $*$-autonomous category amount to in the case of **SUP**. For example, the symmetry of the tensor product is effected by switching to the other half of a Galois-connection. The natural isomorphism between $(A \otimes B \multimap C)$ and $(A \multimap (B \multimap C))$ is encoded in the equation

$$\phi^*(c)(a) \;=\; \psi(a)^*(c)$$

$$\phi \in (A \otimes B \multimap C) \qquad \psi \in (A \multimap (B \multimap C))$$

in which one side completely determines the other.

Besides the multiplicatives of Linear Logic, which are all faithfully modelled because **SUP** is $*$-autonomous, we can also study the additives $\&$ and $\oplus$. In **SUP**, these are both modelled by cartesian product (which is also the coproduct because $(A \times B)^{\mathrm{op}} \cong A^{\mathrm{op}} \times B^{\mathrm{op}}$), with the one-element lattice representing the units.

[†] Assume $f \leqslant g$. From $g^*(y) \leqslant g^*(y)$ we get $g(g^*(y)) \leqslant y$ and hence $f(g^*(y)) \leqslant y$. Therefore $g^*(y) \leqslant f^*(y)$.
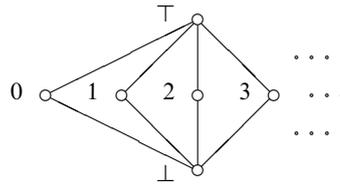
Fig. 1. The lattice M_∞.

Since the interpretations of $\&$ and $\oplus$ coincide, our model satisfies *all* distributivity laws of the form $A\,m(B\,a\,C)$, where $m \in \{\otimes, \otimes\}$, $a \in \{\&, \oplus\}$, that is, it is *fully distributive*. This property has already been noted in Huth (1995b) and has recently been studied from a proof-theoretic point of view in Leneutre (1998).

4. Adding approximation

We come to the main objective of this paper, which is to enrich the objects of Barr's category **SUP** with a domain-theoretic notion of approximation; that is, to consider *continuous lattices*. We are faced with an immediate difficulty, because the category **CL** of continuous lattices and linear maps is not closed.

Example 4.1. Let M_∞ be the lattice of the discretely ordered set of natural numbers extended with a least and a largest element (see Figure 1). In the linear function space $(M_\infty \multimap M_\infty)$ we look at the identity id. Because all maps of this space are sup-preserving, there is only one function below id, namely, the constant bottom function. If $(M_\infty \multimap M_\infty)$ were continuous, id would have to be a compact element. However, we have the following chain of maps whose supremum exceeds id without any of its elements being above id:

$$f_n \colon M_\infty \to M_\infty, n \in \mathbb{N}$$
$$f_n(\bot) = \bot, f_n(\top) = \top$$
$$f_n(m) = \begin{cases} \top, & \text{if } m \leqslant n; \\ m+1, & \text{otherwise.} \end{cases}$$

A similar problem arises in Domain Theory. There one has the cartesian closed category **DCPO** whose full subcategories of continuous, respectively, algebraic, domains are not closed. By restricting these categories further, one recovers closedness. Examples are Scott-domains, SFP-domains, *etc.*, see Abramsky and Jung (1994, Chapter 4) for more details. In the same vein, we will now exhibit a full subcategory of **CL** that is closed.

Definition 4.1 (Jung 1990). A function $f \colon A \to A$ on a partially ordered set A is said to be *finitely separated* from id_A if there exists a finite subset M of A such that for all $x \in A$ there exists $m \in M$ with $f(x) \leqslant m \leqslant x$.

For a complete lattice A to be an *FS-lattice* we require the existence of a directed family $\mathscr{D}$ of *linear* finitely separated functions on A whose supremum equals id_A. Let **FS** denote the full subcategory of **SUP** whose objects are FS-lattices.

This definition is formulated in close analogy to a similar one for domains (Jung 1990). Because the setting is now that of complete lattices we can immediately reformulate it in a number of ways.

Proposition 4.1. For a complete lattice A the following are equivalent:

(i) A is an FS-lattice.
(ii) There exists some family of linear finitely separated functions on A whose supremum equals id_A.
(iii) The supremum of all linear finitely separated functions below id_A equals id_A.

Proof. Observe that the pointwise supremum of a finite set of linear finitely separated functions is again linear and finitely separated from id_A.   □

Obviously, every finite lattice is in **FS** because we can choose $\mathscr{D} = \{\mathsf{id}\}$ in this case. As for infinite examples, we will see in Section 5 below that every completely distributive lattice is in **FS**. At this point, however, it is necessary to justify our definition by showing that FS-lattices are indeed continuous. We let $[A \to B]$ denote the complete lattice of all Scott-continuous functions $f : A \to B$ in the pointwise order. Note that $(A \multimap B)$ is a subset of $[A \to B]$ closed under all suprema.

Lemma 4.1. Let A be a complete lattice. If a Scott-continuous function $f \in [A \to A]$ is finitely separated from id_A, then $f(x) \ll x$ for all $x \in A$.

Proof. Let M be the finite subset of A that separates f from id_A. Given $x \in A$ and a directed set $D \subseteq A$ with $x \leqslant \bigvee^{\uparrow} D$, let $D_m := \{d \in D \mid f(d) \leqslant m \leqslant d\}$, $m \in M$. By assumption, we have $D = \bigcup_{m \in M} D_m$ and so at least one D_{m_0} must be cofinal in D. Hence we get $f(x) \leqslant f(\bigvee^{\uparrow} D) = f(\bigvee^{\uparrow} D_{m_0}) = \bigvee^{\uparrow} f(D_{m_o}) \leqslant m_0 \leqslant d$ for any $d \in D_{m_0}$.   □

Corollary 4.1. FS-lattices are continuous.

Let us now show that **FS** carries enough structure to model all of Linear Logic. As we know from Section 3, the whole structure of a $*$-autonomous category is derived from the function space. The following is therefore crucial.

Lemma 4.2. Let A and B be FS-lattices. Then $(A \multimap B)$ is also an FS-lattice.

Proof. Let $\mathscr{D} \subseteq (A \multimap A)$ and $\mathscr{E} \subseteq (B \multimap B)$ be directed sets with $\bigvee^{\uparrow} \mathscr{D} = \mathsf{id}_A$ and $\bigvee^{\uparrow} \mathscr{E} = \mathsf{id}_B$ such that all $f \in \mathscr{D}$ and $g \in \mathscr{E}$ are finitely separated from the respective identities. For $f \in \mathscr{D}$, $g \in \mathscr{E}$ and M_f, M_g the respective finite separating sets, we will show that $\phi_{f,g}^2$, where $\phi_{f,g}(h) = g \circ h \circ f$, is finitely separated from $\mathsf{id}_{(A \multimap B)}$. This suffices to prove the result because $\bigvee^{\uparrow} \phi_{f,g}^2$ is equal to $\mathsf{id}_{(A \multimap B)}$. So let $f \in \mathscr{D}$, $g \in \mathscr{E}$ be given. We define an equivalence relation $\sim$ on $(A \multimap B)$ by

$$h_1 \sim h_2 \; :\Leftrightarrow \; \forall m \in M_f. \, {\uparrow}g(h_1(m)) \cap M_g = {\uparrow}g(h_2(m)) \cap M_g.$$

As M_f and M_g are finite, there are only finitely many equivalence classes on $(A \multimap B)$. Let K be a set of representatives of these classes. We claim that the finite set $\phi_{f,g}(K)$ separates

$\phi_{f,g}^2$ from $\mathsf{id}_{(A \multimap B)}$. Given $h \in (A \multimap B)$, let k_h be the corresponding representative in K. For $a \in A$, we compute

$$
\begin{aligned}
h(a) &\geqslant h(m_f) && \text{for some } m_f \in M_f \text{ with } f(a) \leqslant m_f \leqslant a \\
&\geqslant m_g && \text{for some } m_g \in M_g \text{ with } g(h(m_f)) \leqslant m_g \leqslant h(m_f) \\
&\geqslant g(k_h(m_f)) && \text{as } g(h(m_f)) \leqslant m_g \text{ and } h \sim k_h \\
&\geqslant g(k_h(f(a))) && \text{as } f(a) \leqslant m_f.
\end{aligned}
$$

By symmetry, we obtain $k_h \geqslant \phi_{f,g}(h)$, so $h \geqslant \phi_{f,g}(k_h) \geqslant \phi_{f,g}^2(h)$. $\qquad\square$

A similar proof, for *FS-domains*, appeared first in Jung (1990).

Theorem 4.1. FS is a ∗-autonomous full subcategory of **SUP**. Furthermore, it is closed under cartesian products.

Remember that the order dual of a lattice, A^{op}, can be expressed as a linear function space: $A^{\mathrm{op}} \cong (A \multimap 2)$, so the preceding theorem says, in particular, that with A we automatically have that A^{op} is an FS-lattice again.

Let us now attempt to show that **FS** is indeed the largest full subcategory of continuous lattices of **SUP** that is closed. Finiteness, which is part of the definition of an FS-lattice, will have to come from a compactness argument. In other words, we will have to work with topological concepts as well as order theoretic ones. The topology that is appropriate for our purposes is the *patch-* or *Lawson-topology*, because it is compact Hausdorff on a continuous lattice (Gierz *et al.* 1980, Theorem III-1.10). It is a refinement of the Scott-topology and generated by Scott-open subsets and complements of Scott-compact upper subsets.

Now, for a complete lattice A it is easy to see that every Scott-compact upper set $C \subseteq A$ is closed with respect to the Scott-topology on A^{op} because a downward directed set $(x_i)_{i \in I}$ gives rise to a directed collection $(A \setminus \downarrow x_i)_{i \in I}$ of Scott-open sets, resulting in a compactness argument if the infimum of $(x_i)_{i \in I}$ is assumed not to be in C. The converse is not necessarily true: consider the lattice M_∞ from Example 4.1; every upper set in M_∞ is closed with respect to $\sigma_{M_\infty^{\mathrm{op}}}$ but only finite upper sets are compact with respect to σ_{M_∞}. Let us say that a complete lattice A is *lean* if every $\sigma_{A^{\mathrm{op}}}$-closed subset is σ_A-compact.

Somewhat surprisingly, leanness is a self-dual concept in our setting.

Lemma 4.3. Let A be a bicontinuous lattice. Then A is lean if and only if A^{op} is lean.

Proof. Let us denote the join of the two Scott-topologies by σ^2. It is a refinement of both Lawson-topologies λ_A and $\lambda_{A^{\mathrm{op}}}$. Under the assumption of continuity, the Lawson-topology is compact Hausdorff. In this setting, for A to be lean means nothing else but $\lambda_A = \sigma^2$. So assuming A to be lean renders σ^2 a compact Hausdorff refinement of the compact Hausdorff topology $\lambda_{A^{\mathrm{op}}}$. It is a standard topological result that the two topologies must coincide in this case. $\qquad\square$

Remark 4.1. The previous lemma holds already if A and A^{op} are assumed to be sober spaces in their Scott-topologies, because the so-called patch topologies are then compact Hausdorff. We will, however, not need this generality.

Lemma 4.4. FS-lattices are lean.

Proof. Let C be a $\sigma_{A^{\mathrm{op}}}$-closed subset of the FS-lattice A and let $(f_i)_{i\in I}$ be an approximating family of finitely separated linear maps. For each $i \in I$ let M_i be the finite separating set. We have that C is contained in $\uparrow N_i$ where $N_i = \{m \in M_i \mid \exists x \in C.f_i(x) \leqslant m \leqslant x\}$. Each $\uparrow N_i$ is σ_A-compact as it is generated by a finite set. The intersection C' of all $\uparrow N_i$, $i \in I$, contains C and is σ_A-compact again because A is a complete lattice (Abramsky and Jung 1994, Theorem 4.2.18). All we need to show is that $C' = C$.

To this end, let a be in the $\sigma_{A^{\mathrm{op}}}$-open set $A \setminus C$. Since the family of upper adjoints $(f_i^*)_{i\in I}$ is approximating from above, there exists $i_0 \in I$ such that $f_{i_0}^*(a) \in A \setminus C$. The corresponding f_{i_0} maps C into $A \setminus \downarrow a$ because $f_{i_0}(x) \leqslant a$ implies $x \leqslant f_{i_0}^*(a)$. It follows that $\uparrow N_{i_0}$ does not contain a. $\qquad\square$

After these preliminaries, let us now press on towards the promised maximality result.

Lemma 4.5. Let A be a complete lattice and $f \ll g$ in $(A \multimap A)$. Then $f(a) \ll g(a)$ for all $a \in A$.

Proof. Let $g(a) \leqslant \bigvee^{\uparrow}_{i\in I} x_i$ be given. Define

$$f_i(x) := \begin{cases} \bot_A, & x = \bot_A; \\ x_i, & x \leqslant a; \\ \top_A, & \text{otherwise.} \end{cases}$$

Then $(f_i)_{i\in I}$ is directed in $(A \multimap A)$ and $g \leqslant \bigvee^{\uparrow}_{i\in I} f_i$. Since $f \ll g$ in $(A \multimap A)$, we have $f \leqslant f_j$ for some $j \in I$ and $f(a) \leqslant f_j(a) = x_j$, as desired. $\qquad\square$

Corollary 4.2. Let A be a complete lattice such that $(A \multimap A)$ is continuous. Then both A and A^{op} are continuous.

Proof. For A this follows directly from the previous lemma. It is true for A^{op} as well because $(A \multimap A)$ and $(A^{\mathrm{op}} \multimap A^{\mathrm{op}})$ are isomorphic. $\qquad\square$

Lemma 4.6. Let A be a lean continuous lattice with continuous linear function space $(A \multimap A)$. If f is way-below id_A in $(A \multimap A)$, then f is finitely separated from id_A.

Proof. The continuity of $(A \multimap A)$ and the Scott-continuity of composition imply the existence of some $g \ll \mathrm{id}_A$ with $f \leqslant g \circ g$. As $h \mapsto h^* : (A \multimap A) \to (A^{\mathrm{op}} \multimap A^{\mathrm{op}})$ is an order isomorphism, we obtain $g^* \ll \mathrm{id}_{A^{\mathrm{op}}}$ in $(A^{\mathrm{op}} \multimap A^{\mathrm{op}})$. By the previous lemma, $g^*(a) \ll a$ in A^{op} for all $a \in A$. Thus, $O_a := \{b \in A^{\mathrm{op}} \mid g^*(a) \ll b \text{ in } A^{\mathrm{op}}\}$ contains a and is Scott-open in A^{op}. Since A is lean, this set is also λ_A-open. The continuity of A ensures that $U_a := \{e \in A \mid g(a) \ll e \text{ in } A\}$ is Scott-open in A; again, it contains a. Thus, $V_a := O_a \cap U_a$ is a λ_A-open set containing a.

The topology λ_A is compact, as A is continuous. Therefore, the open cover $\bigcup_{a\in A} V_a$ of A has a finite subcover $A = \bigcup_{m\in M} V_m$. For $a \in A$, we have $a \in V_m$ for some $m \in M$. In particular, this guarantees the inequalities $g(m) \leqslant a$ and $a \leqslant g^*(m)$. The latter is equivalent to $g(a) \leqslant m$, so $f(a) \leqslant g(g(a)) \leqslant g(m) \leqslant a$ shows that $g(M)$ is a finite set separating f from id_A. $\qquad\square$

As a direct consequence of this lemma, we get our first main result.

Theorem 4.2. FS is the largest (full) ∗-autonomous subcategory of **SUP** whose objects are lean and continuous.

It is slightly unsatisfactory that we need to refer to leanness in the statement of this theorem. Indeed, in Section 7.3 we dispense with this condition in the special case of algebraic lattices. The proof, as we will see, is rather technical and makes vital use of the abundance of compact elements. It would be desirable to have a more conceptual account of this result, which – one hopes – would then also apply to continuous lattices. We leave this as an open problem.

5. Distributivity

The aim of this section is to study the subcategory **CD** of **SUP** whose objects are completely distributive lattices. Before we do so, we need to record some fundamental properties of these lattices.

It was discovered very early in the history of continuous lattices that there is a strong connection between the notions of approximation and distributivity (Scott 1972; Gierz *et al.* 1980, Theorem I-2.3). In the case of completely distributive lattices this connection was noted even earlier in the work of G. N. Raney (Raney 1953). Let us review the main points.

Definition 5.1. Let x, y be elements of a complete lattice A. We say that a' is *completely below* a (and write $a' \lll a$) if for every subset X of A we have that $a \leqslant \bigvee X$ implies $a' \leqslant x$ for some $x \in X$.

This, of course, is the same as the definition of the way-below relation with arbitrary subsets replacing the directed ones. The elementary properties of $\lll$ are the same as for $\ll$ and their proofs are completely analogous (and simpler).

Proposition 5.1. For any complete lattice A and $a, a', b, b' \in A$ the following are true:

(i) $a' \lll a$ implies $a' \leqslant a$.
(ii) $a' \leqslant a \lll b \leqslant b'$ implies $a' \lll b'$.
(iii) $\bot \lll a$ if and only if $\bot \neq a$.

We can now define a complete lattice A to be *super-continuous* if every element of A is the supremum of elements completely below it. However, super-continuity is equivalent to complete distributivity.

Theorem 5.1 (Raney). A complete lattice A is completely distributive if and only if for all $a \in A$, $a = \bigvee \{a' \in A \mid a' \lll a\}$ holds.

Corollary 5.1.

(i) A complete lattice A is super-continuous if and only if A^{op} is super-continuous.
(ii) Completely distributive lattices are bicontinuous.

The corollary says that we get approximation from both sides automatically in super-continuous lattices. Observe, however, that the relations $\lll_A$ and $(\lll_{A^{\mathrm{op}}})^{-1}$ are different in general.

We will also make use of the following observation, which is a consequence of Raney's work on tight Galois connections (Raney 1960).

Theorem 5.2 (Raney). A complete lattice A is completely distributive if and only if for every $a \in A$ we have $a = \bigwedge_{a' \nleqslant a} \bigvee_{a'' \ngeqslant a'} a''$.

Proof. 'If': It is easy to see that for every $a' \nleqslant a$ the element $x := \bigvee_{a'' \ngeqslant a'} a''$ is completely above a. Hence A^{op} is super-continuous.

'Only if': Since a is always among the a'' of which we take the supremum in $\bigvee_{a'' \ngeqslant a'} a''$, we have $y := \bigwedge_{a' \nleqslant a} \bigvee_{a'' \ngeqslant a'} a'' \geqslant a$. Assume that y is strictly above a. Then, by super-continuity, we have an element y' completely below y but not below a. This y' is one of the a' in the formula, and it follows that $y' \lll y \leqslant \bigvee_{a'' \ngeqslant y'} a''$; hence there exists $a'' \ngeqslant y'$ which is above y', which is clearly absurd. $\square$

Approximation, rather than distributivity, is used to show the following lemma.

Lemma 5.1. Let A and B be complete lattices and $m : A \to B$ be monotone.

(i) If A is continuous, then the largest continuous function $\check{m}$ below m is given by $\check{m}(x) = \bigvee^{\uparrow} \{m(y) \mid y \ll x\}$. The assignment $m \mapsto \check{m}$ is continuous as a function from the monotone function space to the continuous function space.

(ii) If A is super-continuous, then the largest linear function $\mathring{m}$ below m is given by $\mathring{m}(x) = \bigvee \{m(y) \mid y \lll x\}$. The assignment $m \mapsto \mathring{m}$ is linear as a function from the monotone function space to the linear function space.

If m has finite image within B, then so do $\check{m}$ and $\mathring{m}$, respectively.

We need to refine this lemma somewhat for our purposes.

Lemma 5.2. Let A, B be continuous lattices and let $m : A \to B$ be a $\vee$-homomorphism that also maps $\perp_A$ to $\perp_B$. Then $\check{m} = \mathring{m}$.

Proof. Since any supremum can be written as a combination of directed supremum and finite suprema, $\bigvee X = \bigvee^{\uparrow}_{F \subseteq_{\mathrm{fin}} X} F$, it suffices to show that $\check{m}$ is still a $\vee$-homomorphism.

We always have $\check{m}(a \vee a') \geqslant \check{m}(a) \vee \check{m}(a')$ by monotonicity. For the converse, assume $b \ll \check{m}(a) \vee \check{m}(a')$. The set $\{y \vee y' \mid y \ll \check{m}(a), y' \ll \check{m}(a')\}$ is directed with supremum $\check{m}(a) \vee \check{m}(a')$, so for some $y \ll \check{m}(a)$ and $y' \ll \check{m}(a')$ we have $b \leqslant y \vee y'$. The definition of $\check{m}$ gives us $x \ll a$ and $x' \ll a'$ such that $y \leqslant m(x)$ and $y' \leqslant m(x')$. Now, $x \vee x' \ll a \vee a'$, and hence $\check{m}(a \vee a') \geqslant m(x \vee x') = m(x) \vee m(x') \geqslant y \vee y' \geqslant b$. Thus we have shown that every element way below $\check{m}(a) \vee \check{m}(a')$ is also below $\check{m}(a \vee a')$, and so $\check{m}(a \vee a') \leqslant \check{m}(a) \vee \check{m}(a')$ follows as B is continuous. $\square$

Besides approximation from below, continuous lattices also enjoy a representation from above: every element x is the infimum of $\wedge$-irreducible elements (Gierz *et al.* 1980, Theorem I-3.10). If the lattice is bicontinuous, this infimum may be taken over the subset of $\wedge$-irreducible elements that are way-below x in A^{op}. In a distributive lattice there is no

difference between $\wedge$-irreducible and $\wedge$-prime elements. Finally, an element y that is both $\vee$-prime and way-below x is actually completely below x. These observations prove the following theorem.

Theorem 5.3 (Gierz *et al.* 1980). A complete lattice is completely distributive if, and only if, it is bicontinuous and distributive. In that case, every element is the supremum of $\vee$-primes way-below it.

Let us now put these preliminaries to work in our setting.

Lemma 5.3. Every completely distributive lattice is an FS-lattice.

Proof. Let A be a completely distributive lattice; it is bicontinuous by Corollary 5.1 and so every element is the supremum of $\vee$-prime elements below it. For every finite subset F of $\vee$-primes, define $m_F : A \to A$, $m_F(x) := \bigvee \{a \in F \mid a \leqslant x\}$. Then m_F preserves finite suprema and the conditions of Lemma 5.2 are satisfied. Hence $\check{m}_F$ is linear.

Every $\check{m}_F$ has a finite image and so is finitely separated from id_A. The identity is equal to the directed supremum of all m_F and since it itself is continuous, it is also the directed supremum of the $\check{m}_F$ by Lemma 5.1(1). $\square$

Theorem 5.4. A complete lattice is completely distributive if and only if it is a distributive FS-lattice.

Proof. The proof follows from Lemma 5.3, Corollary 4.1, Theorems 4.1 and 5.3. $\square$

Lemma 5.4. The category **CD** of completely distributive lattices and linear maps is closed.

Proof. The lattices 2 and $\top$ are objects in **CD**. By the preceding theorem, we already know that the linear function space $(A \multimap B)$ of two completely distributive lattices is FS, and we only need to show distributivity. To this end, observe that the supremum of elements in $(A \multimap B)$ is calculated pointwise; even the finite pointwise infimum, however, is not sup-preserving in general. Hence the infimum is given by Lemma 5.1:

$$(f \wedge g)(a) = \bigvee \{f(a') \wedge g(a') \mid a' \lll a\} \, .$$

Now, given $f, g, h : A \multimap B$, we will always have $(f \wedge g) \vee (f \wedge h) \leqslant f \wedge (g \vee h)$. For the converse, fix $a \in A$ and assume $b \lll (f \wedge (g \vee h))(a)$. By what we just said about infima in $(A \multimap B)$, there must exist $a' \lll a$ such that $b \leqslant f(a') \wedge (g(a') \vee h(a'))$. Distributivity at the element level gives us $b \leqslant (f(a') \wedge g(a')) \vee (f(a') \wedge h(a'))$, and the latter is a term that occurs in the calculation of $((f \wedge g) \vee (f \wedge h))(a)$. $\square$

Theorem 5.5. **CD** is the largest closed full subcategory of **SUP** whose objects are distributive and continuous.

It follows that **CD** gives us another, smaller model of Linear Logic. Besides its objects being more regular than those of **FS**, we find that in **CD** the interpretation of tensor and its de Morgan dual, par, coincide.

Theorem 5.6. Let A and B be complete lattices and let one of them be completely distributive. Then $(A \multimap B^{\mathrm{op}}) \cong (A^{\mathrm{op}} \multimap B)^{\mathrm{op}}$, that is, $A \otimes B \cong A \,\invamp\, B$.

Proof. (Note that all operations and relation symbols in this proof refer to the original lattices, not their order duals.) Given complete lattices A and B, define

$$\Phi : (A \multimap B^{\mathrm{op}}) \quad \to \quad (A^{\mathrm{op}} \multimap B), \quad \Phi(r)(x) := \bigvee_{x' \not\leqslant x} r(x')$$

$$\Psi : (A^{\mathrm{op}} \multimap B) \quad \to \quad (A \multimap B^{\mathrm{op}}), \quad \Psi(s)(x) := \bigwedge_{x' \not\geqslant x} s(x') \, .$$

It is clear that Φ and Ψ are antitone. More important is well-definedness:

$$\begin{aligned} \Phi(r)(\textstyle\bigwedge X) \quad &= \quad \bigvee_{x' \not\leqslant \bigwedge X} r(x') \\ &= \quad \bigvee_{x \in X} \bigvee_{x' \not\leqslant x} r(x') \text{ by the definition of } \textstyle\bigwedge X, \end{aligned}$$

and dually for Ψ. The maps Φ and Ψ are mutual inverses of each other. Let $s : A^{\mathrm{op}} \multimap B$. Then

$$\Phi(\Psi(s))(x) = \bigvee_{x' \not\leqslant x} \Psi(s)(x') = \bigvee_{x' \not\leqslant x} \bigwedge_{x'' \not\geqslant x'} s(x'') =: t(x) \, .$$

It is clear that $t(x) \leqslant s(x)$ because x is always one of the x'' in the formula. For the converse we use complete distributivity of A, which entails $x = \bigwedge_{a \gg x} a$ and $x = \bigwedge_{x' \not\leqslant x} \bigvee_{x'' \not\geqslant x'} x''$ (Theorem 5.2). Now, for $a \gg x$ we get $\exists x' \not\leqslant x. \bigvee_{x'' \not\geqslant x'} x'' \leqslant a$, that is, $\exists x' \not\leqslant x \, \forall x'' \not\geqslant x'. x'' \leqslant a$. Since s is antitone, this translates as $\exists x' \not\leqslant x \, \forall x'' \not\geqslant x'. s(x'') \geqslant s(a)$, and hence $t(x) \geqslant s(a)$. Since s translates infima into suprema, we get $s(x) = s(\bigwedge_{a \gg x} a) = \bigvee_{a \gg x} s(a) \leqslant t(x)$.

Note that we have used complete distributivity of A alone. Complete distributivity of B would also suffice since we can always switch to the other half of a Galois-connection. $\square$

In Barr's terminology, what we have shown is the following corollary.

Corollary 5.2. The category **CD** is compact closed.

We conclude this section with an observation that is easy to justify at this point, though it will be used only in Section 7.3.

Lemma 5.5. Let A and B be bicontinuous lattices and let $F \subseteq (A \multimap B)$ be filtered. Then the infimum of F in $(A \multimap B)$ equals the infimum of F in $[A \to B]$.

Proof. Given a filtered family $F \subseteq (A \multimap B)$, we consider the pointwise infimum $m(x) := \bigwedge_{f \in F} f(x)$. It is not only monotone but also preserves the least element and binary suprema. This is because B^{op} is also continuous and on a continuous lattice the binary infimum is a continuous operation. Now we can apply Lemma 5.2 and we get that $\breve{m}$, which is the infimum of m in $[A \to B]$, is linear, and hence the infimum in $(A \multimap B)$. $\square$

6. The modalities

So far we have ignored the modalities of Linear Logic and it is high time to study how they can be added to our framework. Some general comments may be in place here. From the viewpoint of *-autonomous categories, modalities require a further piece of structure in the form of a comonad. First Seely (Seely 1989), and later Benton, Bierman, de Paiva and Hyland (Benton *et al.* 1993b; Benton *et al.* 1993a; Bierman 1995), worked out the

precise conditions that need to be imposed on the comonad in order to get the desired close correspondence between proof theory and categorical semantics.

More recently, Benton (Benton 1994) came up with a quite different notion of categorical model, where one has a cartesian closed category (the *intuitionistic category*) and a ∗-autonomous category (the *linear category*) linked by a monoidal adjunction. The attractions of Benton's approach are twofold. First, the set of axioms is small and uses well-established concepts only. Second, the free parameters in a Benton model of Linear Logic are clearly visible; neither does the linear category determine the intuitionistic one, nor the other way round; and once the two categories are fixed, there may still be some variability in terms of which adjunction to choose.

These general benefits are augmented with some specific advantages in our setting. Since we can choose the intuitionistic category independently from the linear category, we have the opportunity to bring classical categories of domains into the picture. In other words, we are not forced to work with complete lattices alone. This ought to facilitate the application of our results to Denotational Semantics.

Although the definition of a Benton model is very neat, the number of diagrams to check is still quite daunting. We are helped by the following general result from Kelly (1974) (which was also noted in Benton (1994)).

Theorem 6.1. Let $(\mathbf{C}; \otimes_C, I_C) \xrightarrow{G} (\mathbf{D}; \otimes_D, I_D) \xrightarrow{F} (\mathbf{C}; \otimes_C, I_C)$ be an adjunction between (symmetric) monoidal categories and let

$$\mathsf{n}: F(A) \otimes_C F(B) \xrightarrow{\;\cdot\;} F(A \otimes_D B) \qquad \mathsf{p}: I_C \to F(I_D)$$

be a natural transformation (respectively, a morphism) making the left adjoint F monoidal. Then the following are equivalent:

(i) The whole adjunction is monoidal.
(ii) All arrows $\mathsf{n}_{A,B}$ and p are isomorphisms.

In the spirit of Denotational Semantics and Domain Theory, the natural partner for Barr's linear category **SUP** is **DCPO**, the category of directed-complete partial orders and Scott-continuous functions. **DCPO** is cartesian closed and is the ambient category for many of the more refined concepts in Domain Theory. Our choice of adjunction is informed by our wish to decompose the maps of **DCPO**. Consider the definitions

$$\mathsf{H}D := \{X \subseteq D \mid X \text{ Scott-closed}\},$$

where D is a dcpo and the order on $\mathsf{H}D$ is subset inclusion, and

$$\mathsf{i}_D: D \to \mathsf{H}D, \ d \mapsto \downarrow d.$$

(We chose the notation H because HA is almost the Hoare-powerdomain of A, except that for the latter the empty set is usually excluded.) The functions i_D are Scott-continuous. Furthermore, we have the following lemma.

Lemma 6.1. Let D be a dcpo and B be a complete lattice. For every Scott-continuous function $f: D \to B$ there is a unique linear function $\hat{f}: \mathsf{H}D \multimap B$ such that $f = \hat{f} \circ \mathsf{i}_D$.

Proof. The equality $f = \hat{f} \circ i_D$ forces the following definition of $\hat{f}$:

$$\hat{f}(X) := \bigvee \{f(x) \mid \downarrow x \subseteq X\} .$$

For linearity, let $(X_i)_{i \in I}$ be a collection of Scott-closed subsets of D. Note that in $\mathsf{H}D$ the supremum is calculated as

$$\bigvee_{i \in I} X_i = \mathsf{cl}(\bigcup_{i \in I} X_i) ,$$

where $\mathsf{cl}(\cdot)$ denotes the closure of a subset in the Scott-topology. We need to show that $\hat{f}(\bigvee_{i \in I} X_i) \leqslant \bigvee_{i \in I} \hat{f}(X_i)$, the other inequality being satisfied trivially. Consider the Scott-closed subset $\downarrow \bigvee_{i \in I} \hat{f}(X_i)$ of B. Its pre-image under f is Scott-closed by the Scott-continuity of f and contains all X_i's, hence $\bigvee_{i \in I} X_i$ as well. So we get $f(\bigvee_{i \in I} X_i) \subseteq \downarrow \bigvee_{i \in I} \hat{f}(X_i)$, and consequently $\hat{f}(\bigvee_{i \in I} X_i) = \bigvee \{f(x) \mid x \in \bigvee_{i \in I} X_i\} \leqslant \bigvee_{i \in I} \hat{f}(X_i)$. $\square$

From the lemma above we obtain that **SUP** is a reflective subcategory of **DCPO**, the reflection being given by

$$
\begin{array}{rcl}
D & \mapsto & \mathsf{H}D \\
f : D \to E & \mapsto & \widehat{i_E \circ f} .
\end{array}
$$

In order to show that the adjunction is monoidal, we check the conditions of Theorem 6.1. First of all, $I_{SUP} = 2$ is clearly isomorphic to $\mathsf{H}I_{DCPO} = \mathsf{H}1$. We get the desired natural isomorphism between $\mathsf{H}A \otimes \mathsf{H}B$ and $\mathsf{H}(A \times B)$ from the following functional description of H[†]:

$$\mathsf{H}A \cong [A \to 2]^{\mathrm{op}}.$$

The calculation runs as follows

$$
\begin{array}{rcl}
\mathsf{H}A \otimes \mathsf{H}B & = & (\mathsf{H}A \multimap (\mathsf{H}B)^{\mathrm{op}})^{\mathrm{op}} \\
& \cong & [A \to (\mathsf{H}B)^{\mathrm{op}}]^{\mathrm{op}} \\
& \cong & [A \to [B \to 2]]^{\mathrm{op}} \\
& \cong & [A \times B \to 2]^{\mathrm{op}} \\
& \cong & \mathsf{H}(A \times B) .
\end{array}
$$

We also need to establish that these isomorphisms commute in a suitable way with the transformations that correspond to the associativity, symmetry and unit laws of the symmetric monoidal structure. For this we need a more explicit description of the above isomorphism.

For $a \in A, b \in B$ define a Galois-map $(a \nearrow b) : A \to B$ by

$$(a \nearrow b) := \bigwedge \{r \in A \otimes B \mid r(a) \geqslant b\}$$

or, explicitly,

$$
(a \nearrow b)(x) := \left\{
\begin{array}{ll}
\top_B, & \text{if } x = \bot_A; \\
b, & \text{if } x \in \downarrow a \setminus \{\bot_A\}; \\
\bot_B, & \text{if } x \notin \downarrow a.
\end{array}
\right.
$$

[†] As Paola Maneggia has pointed out to us, this representation of H is no coincidence: whenever H is a monoidal reflection from a Cartesian closed category to a $*$-autonomous subcategory with dualizing object $\bot$, one has $\mathsf{H}A \cong (\mathsf{H}A \multimap \bot) \multimap \bot \cong [A \to \bot] \multimap \bot$.

The other half of this Galois-map is just $(b \nearrow a)$, as one can see from the characterization in Formula (4). Furthermore, we have $r = \bigvee_{a \in A}(a \nearrow r(a))$ for all $r \in A \otimes B$, because r itself is an element of the set of which the infimum is taken in the definition of $(a \nearrow r(a))$. Also note that $(\bot_A \nearrow b)$ and $(a \nearrow \bot_B)$ equal $(\bot_A \nearrow \top_B)$, the smallest element in $A \otimes B$.

Using this information, we can describe the isomorphism between $\mathsf{H}A \otimes \mathsf{H}B$ and $\mathsf{H}(A \times B)$ explicitly by

$$(\downarrow a \nearrow \downarrow b) \leqslant r \iff (a, b) \in C$$

where $r \in \mathsf{H}A \otimes \mathsf{H}B$ and $C \in \mathsf{H}(A \times B)$. The diagrams for the monoidicity of $\mathsf{H} : \mathbf{DCPO} \to \mathbf{SUP}$ now become easy exercises. For example, commutativity of

$$
\begin{array}{ccc}
\mathsf{H}A \otimes \mathsf{H}B & \multimap & \mathsf{H}(A \times B) \\
\Big| & & \Big| \\
\mathsf{s}_{SUP} & & \mathsf{Hs}_{DCPO} \\
\circ & & \circ \\
\mathsf{H}B \otimes \mathsf{H}A & \multimap & \mathsf{H}(B \times A)
\end{array}
$$

is argued as follows. For $r \in \mathsf{H}A \otimes \mathsf{H}B$ we have $(\downarrow a \nearrow \downarrow b) \leqslant r \iff (a, b) \in C \iff (b, a) \in \mathsf{Hs}_{DCPO}(C) \iff (\downarrow b \nearrow \downarrow a) \leqslant \mathsf{s}_{SUP}(r)$. Leaving the remaining diagrams as exercises, we arrive at the following theorem.

Theorem 6.2. The categories **DCPO** and **SUP**, linked by the reflection $\mathsf{H} : \mathbf{DCPO} \to \mathbf{SUP}$, form a Benton model of Linear Logic.

The theorem implies that there is a natural transformation $A \times B \xrightarrow{\cdot} A \otimes B$. This, of course, is nothing other than the assignment $(a, b) \mapsto (a \nearrow b)$; it is linear in both variables separately.

The setup of Theorem 6.2 can be restricted on both sides to approximated objects. Since the Scott-topology of a continuous domain is a completely distributive lattice (Abramsky and Jung 1994, Theorem 7.2.28), we get a very small model by pairing Scott-domains on the intuitionistic side with completely distributive lattices on the linear side. At the other end, a maximal Benton model within approximated ordered structures is given by FS-domains paired with FS-lattices.

Now the desired decomposition of the Scott-continuous function space $[A \to B]$ into $(\mathsf{H}A \multimap B)$ was the motivation for our choice of the modality $!A$ as the lattice of all Scott-closed subsets of A, ordered by set inclusion. While $!A$ owes its definition to a topological notion, the nature of $?A$ is then completely determined by the structure of the ambient linear category **SUP**: $?A$ has to be naturally isomorphic to $(!A^{\mathrm{op}})^{\mathrm{op}}$. This, in turn, is naturally isomorphic to $\sigma_{A^{\mathrm{op}}}$, the Scott-topology on A^{op}. This already works on the level of **DCPO** and **SUP**. In the approximated case we can give a good deal more information about $?$. Recall that a subset of a topological space is called *saturated* if it equals the intersection of its neighbourhoods. The set of all compact saturated subsets of a space X, ordered by revered inclusion, is denoted by κ_X.

Proposition 6.1. If A is a lean complete lattice, then $?A$ and κ_A are isomorphic, where the isomorphism can be viewed as the identity at the level of sets.

Proof. We have remarked before that a compact upper set is necessarily closed with respect to $\sigma_{A^{\mathrm{op}}}$, that is, a member of $\mathsf{H}(A^{\mathrm{op}})$. The converse is exactly the definition of leanness. $\qquad\square$

The proposition above entails that $?A \cong \kappa_A$ holds for all FS-lattices A. Now, except for the empty set, κ_A is exactly the Smyth-powerdomain of A if A is continuous (Smyth 1978; Abramsky and Jung 1994). Hence in our domain-theoretic model of Linear Logic the two modalities are just the two fundamental powerdomains.

7. Algebraicity

The category **FS** has plenty of algebraic lattices as objects. Theorem 5.4 assures us that **FS** contains at least all completely distributive algebraic lattices; moreover, every finite lattice is certainly algebraic and FS. In this section we will explore the world of algebraic FS-lattices in more detail. As we will see, a lot of the theory is in close analogy to that of algebraic domains and Scott-continuous functions, but there are a few surprises. In the following, we will frequently refer to the classical theory of domains, so we should alert the reader to the fact that she will find FS-*domains* next to FS-*lattices* and *Scott-continuous* functions next to *linear* ones in our proofs. It will be crucial that every linear function is also Scott-continuous.

7.1. *Algebraic FS-lattices*

FS-lattices are defined with reference to finitely separated (linear) functions. There are two strengthenings of this concept that we will make use of here: a function below the identity is called a *deflation* if it has finite image. A deflation may or may not be idempotent. Scott-continuous deflations are familiar from the study of bifinite domains (Plotkin 1976; Abramsky and Jung 1994); here, of course, we require them to be linear.

Lemma 7.1. Let f be a finitely separated function on a complete lattice A. Then some finite iterate of f is an idempotent deflation.

Proof. The statement follows from the fact that in a sequence $x > f(x) > f^2(x) > \ldots$ a different separating element is needed at least every other step. Hence such a sequence can never be longer than $2l$ where l is the cardinality of the finite separating set. It follows that f^{2l} is idempotent. The iterated function has finite image because it remains finitely separated. $\qquad\square$

Proposition 7.1. A complete lattice A is an algebraic FS-lattice if and only if the identity id_A is the directed supremum of idempotent linear deflations.

Proof. 'If': The image of an idempotent deflation consists wholly of compact elements. So A must be algebraic if there exists a directed family of idempotent deflations approximating id_A. Since deflations are finitely separated (by their image), the lattice must also be FS.

'Only if': Given a compact element c of A there exists a finitely separated function f that fixes c. By the previous lemma, some iterate of f is an idempotent deflation. This

iterate still fixes c. This shows that the supremum of all idempotent deflations equals id_A. The supremum is directed because the pointwise supremum of idempotent deflations is another such function.   □

This characterization of algebraic FS-lattices allows us to prove easily that the linear function space of two algebraic FS-lattices is again of the same kind. This closure property is sufficient to give the following theorem.

Theorem 7.1. The category **aFS** of algebraic FS-lattices and linear maps is ∗-autonomous.

By analogy with the Scott-continuous case, one can define *linear bifinite lattices* as the bilimits of finite lattices with respect to *linear* embedding projection pairs. The following characterization is then proved exactly as for bifinite domains (Jung 1989, Theorem 1.26).

Proposition 7.2. A complete lattice A is linearly bifinite if and only if there exists a directed collection of idempotent deflations whose supremum equals id_A.

To summarize, we have the following theorem.

Theorem 7.2. For a complete lattice A the following are equivalent:

(i) A is an algebraic FS-lattice.
(ii) A is linearly bifinite.
(iii) A has a directed collection of idempotent linear deflations whose supremum equals id_A.
(iv) A has a collection of idempotent linear deflations whose supremum equals id_A.
(v)   The supremum of all idempotent linear deflations on A equals id_A.

7.2. *Retracts of bifinite lattices*

As we will see in the next subsection, it is often useful to be able to pass to retracts without leaving the ambient category. We therefore collect a few basic results about retracts of various kinds of FS-lattice.

Proposition 7.3. The category **FS** is closed under forming retracts.

Proof. For $A \in \mathbf{FS}$, $B \in \mathbf{SUP}$, let $r : A \multimap B$ and $e : B \multimap A$ be linear maps with $r \circ e = \mathrm{id}_B$. If f is finitely separated in $(A \multimap A)$ by a set M, then $r \circ f \circ e$ is easily seen to be finitely separated in $(B \multimap B)$ by the set $r(M)$. If the supremum of the set $\mathscr{D}$ of linear finitely separated functions on A equals id_A, then the supremum of the set of functions $r \circ f \circ e$, $f \in \mathscr{D}$, equals id_B, because r is linear and the supremum of linear functions is calculated pointwise.   □

Corollary 7.1. Retracts of linear bifinite lattices are FS-lattices.

As in the Scott-continuous case, retracts of linear bifinite lattices can be characterised functionally as follows.

Theorem 7.3. A complete lattice B is a linear retract of some linear bifinite lattice if, and only if, its identity is the directed supremum of deflations in $(B \multimap B)$.

The question arises whether every FS-lattice is the retract of an algebraic FS-lattice (= linear bifinite lattice). We do not know if this is the case. The situation is exactly as with bifinite domains and FS-domains (Abramsky and Jung 1994, Proposition 4.2.12), although we do not see any general reason for this analogy.

If we combine distributivity with algebraicity, then the problem does not arise.

Theorem 7.4. Every distributive FS-lattice is the linear retract of a distributive linear bifinite lattice.

Proof. A distributive FS-lattice A is automatically completely distributive by Theorem 5.3. Now, if A is in **CD**, let B be the lattice of lower sets of $\vee$-prime elements in A ordered by inclusion. Then B is completely distributive and algebraic. The maps $r: B \to A$, $L \mapsto \bigvee L$, and $e: A \to B$, $x \mapsto \{r \mid r \ll x, r \ \vee\text{-prime}\}$, are linear with $r \circ e = \mathsf{id}_A$ due to Theorem 5.3. $\qquad\square$

7.3. *Maximality of* **aFS**

In the case of continuous lattices, our proof techniques required lattices to be lean in order to realize **FS** as a maximal *-autonomous subcategory of continuous lattices in **SUP**, Lemma 4.6 and Theorem 4.2. This topological assumption can be eliminated in the algebraic setting (Huth 1995a) as follows.

Theorem 7.5. Let A be an algebraic lattice with continuous linear function space $(A \multimap A)$. Then A is an FS-lattice.

Corollary 7.2. **aFS** is the largest (full) *-autonomous subcategory of **SUP** such that every object is algebraic.

The proof of the theorem above is tailored for the structural properties of algebraic lattices; it remains unclear whether it has a suitable abstraction allowing one to prove its continuous version. We leave this as an open problem: if $(A \multimap A)$ is a continuous lattice, is A necessarily lean?

Since A is algebraic in the theorem above, we know that id_A is the directed supremum of idempotent, *Scott-continuous* deflations. Thus, it suffices to show that any such function d has a *linear* deflation p above it. We will reason the existence of such a p in a number of steps. In the discussion below, we fix an algebraic lattice A such that $(A \multimap A)$ is continuous and d is an arbitrary Scott-continuous idempotent deflation on A.

Step 1: *A is bicontinuous.* This follows directly from Corollary 4.2.

Step 2: *Obtaining a candidate linear deflation.* Any candidate linear deflation above d has to be in the set $\mathscr{U} = \{f \in (A \multimap A) \mid d \leqslant f \leqslant \mathsf{id}\}$. This set contains id and is closed under composition as composition is monotone and d and id are idempotent. The combination of these two facts establishes that $\mathscr{U}$ is a filtered subset of $(A \multimap A)$, and by Lemma 5.5 we may conclude that its filtered infimum p in $(A \multimap A)$ is actually the one in $[A \to A]$, using the bicontinuity of A secured in Step 1. Thus, p has to be above d. Since id is in $\mathscr{U}$, we get $p \leqslant \mathsf{id}$. From this, the minimality of p in $\mathscr{U}$, and the fact that $\mathscr{U}$ is closed under composition, we infer that p is idempotent. In summary, p is the minimal idempotent

linear function above d and below id. Since the order on such functions is given by the inclusion of their image, we conclude that there is a *linear* deflation above d if, and only if, the image of p is finite.

From now on we write B for the image of p, and $i: B \to A$, $q: A \to B$ for the decomposition of p into inclusion and projection part.

Step 3: $(B \multimap B)$ *is continuous.* The pair (q, i) realizes B as a linear retract of A. Using the internal hom ($\multimap$) on the pairs (q, i) and (i, q), we obtain $(B \multimap B)$ as a linear retract of $(A \multimap A)$. Since the Scott-continuous retract of a continuous lattice is continuous (Gierz *et al.* 1980; Abramsky and Jung 1994), we infer that $(B \multimap B)$ is continuous.

Step 4: *The identity is compact in* $(B \multimap B)$. The deflation d is in $\mathsf{K}[A \to A]$, so $\mathscr{W} = \{h \in (A \multimap A) \mid d \leqslant h\}$ is Scott-open in $(A \multimap A)$ as directed suprema are the same in $[A \to A]$ and $(A \multimap A)$. Thus, p is a minimal element of the Scott-open set $\mathscr{W}$ and the continuity of $(A \multimap A)$ makes p compact in $(A \multimap A)$. Using this compactness, one may now compute that $q \circ i$ is compact in $(B \multimap B)$, but $q \circ i$ is just id_B.

Step 5: *B satisfies the ascending (ACC) and descending chain condition (DCC).* We already know that the identity of B is compact in $(B \multimap B)$. By Lemma 4.5, we get that every $b \in B$ is compact. Since $(B \multimap B)$ is isomorphic to $(B^{\mathrm{op}} \multimap B^{\mathrm{op}})$, we also get $\mathrm{id} \in \mathsf{K}(B^{\mathrm{op}} \multimap B^{\mathrm{op}})$ and may use the same lemma to infer that every $b \in B$ is compact in B^{op}. These two properties ensure that B satisfies (ACC) and (DCC).

To summarize the discussion so far, we have arrived at a bicontinuous lattice B with continuous linear function space $(B \multimap B)$, where B satisfies (ACC) and (DCC). Let us say that any lattice C with these properties has *property F*. Our aim is to demonstrate that property F is nothing but that of being a finite lattice.

Step 6: *Property F is inherited by principal lower and upper sets.* Note that C has property F if C^{op} has property F and *vice versa*. This is due to the isomorphism $(C \multimap C) \cong (C^{\mathrm{op}} \multimap C^{\mathrm{op}})$. Thus, given C with property F, we only have to show such a closure for a principal lower set $\downarrow x$. The retraction $\mathrm{ret}_x: C \to C$ that leaves $\downarrow x$ fixed and maps all other elements to x realizes $\downarrow x$ as a linear retract of C. As before, we obtain $(\downarrow x \multimap \downarrow x)$ as a linear retract of $(C \multimap C)$. In particular, $(\downarrow x \multimap \downarrow x)$ is continuous. Since $\downarrow x$ evidently inherits (ACC) and (DCC) from C, we only need to establish that $\downarrow x$ is bicontinuous; but this follows from Corollary 4.2.

Because an interval $[\underline{x}, \overline{x}] = \{y \in P \mid \underline{x} \leqslant y \leqslant \overline{x}\}$ in a poset P can be realized as the principal lower set $\downarrow \overline{x}$ in a principal upper set $\uparrow \underline{x}$, property F is also inherited by all intervals in B.

Step 7: *B is finite.* We use proof by contradiction: let us assume that B, the image of p, is indeed infinite. Our goal is to argue that M_∞ (Example 4.1) is sitting inside B.

Step 7.1: *Finding infinite anti-chains.* Consider the poset P of all infinite subintervals of B, ordered by inclusion. It contains B by assumption. As a poset, P satisfies (DCC) because an infinite chain of smaller and smaller intervals would produce either an infinite ascending chain in B (considering the lower endpoints) or an infinite descending chain in B (upper endpoints), and we already know that B is free of both. We can conclude that B contains a minimal infinite subinterval. By Step 6 it will also have property F,

so we might as well assume that B equals that minimal infinite subinterval. Under this assumption, we have the following properties in addition to property F:

(i) $\downarrow x$ is finite for all $x < \top$ in B,

(ii) $\uparrow x$ is finite for all $\bot < x$ in B.

Since B satisfies (DCC), we get $B \setminus \{\bot\} = \uparrow T$, where T is the set of minimal elements in $B \setminus \{\bot\}$. Dually, the condition (ACC) guarantees that $B \setminus \{\top\} = \downarrow S$, with S being the set of maximal elements in $B \setminus \{\top\}$. Since B is infinite, item (i) implies that S is an infinite anti-chain. Dually, item (ii) implies that T is an infinite anti-chain as well.

Step 7.2: *Carving out M_∞.* We use items (i) and (ii) above together with the two infinite anti-chains S and T to construct M_∞ as a linear retract of B. We define inductively a family of elements $(x_i)_{i \in \mathbb{N}}$ in T and a family $(S_i)_{i \in \mathbb{N}}$ of subsets of S: pick any x_0 in T and define S_0 as $\uparrow x_0 \cap S$. By item (ii) above, we see that S_0 is finite. Thus, item (i) entails that $\downarrow S_0 \cap T$ is finite as well. Since T is infinite, we may pick some x_1 in $T \setminus \downarrow S_0$ and repeat this process by picking a new element x_{i+1} in the complement of $\bigcup_{1 \leq j \leq i} \downarrow S_j$ in T. Suppose that $x_i \vee x_{i+k} < \top$ for some $i < i+k$. Then $x_i \vee x_{i+k}$ has to be below some $s \in S$. Then $x_i \leq s$ means $s \in S_i$, and $x_{i+k} \leq s$ renders $x_{i+k} \in \downarrow S_i$, contradicting the choice of the element x_{i+k}. Thus, $x_i \vee x_j = \top$ for all $i \neq j$. This ensures that $\{x_i \mid i \geq 0\} \cup \{\bot, \top\}$ is closed under all suprema and infima in B and isomorphic to M_∞. Therefore, we have an injective map $e : M_\infty \to B$ preserving all infima and all suprema. Because of the former, e has a lower adjoint $l : B \to M_\infty$. The injectivity of e implies $l \circ e = \mathrm{id}_{M_\infty}$. Since lower adjoints preserve suprema, we have realized M_∞ as a linear retract of B. Again, this entails that $(M_\infty \multimap M_\infty)$ is a linear retract of $(B \multimap B)$, whence $(M_\infty \multimap M_\infty)$ has to be continuous, contradicting Example 4.1. Hence the assumption that B be infinite is false.

To summarize, we have shown that there is a linear idempotent deflation above every Scott-continuous idempotent deflation in A, and the proof that A is an FS-lattice is complete.

7.4. *Internal characterization*

We have seen in Section 7.1 that algebraic FS-lattices are in fact bifinite, and we have characterized them in terms of idempotent deflations. So far, this is very much parallel to the theory of domains and Scott-continuous functions; in fact, the proofs of these facts for the linear case are virtually the same as for the continuous case. We will now attempt to push the analogy further to the internal characterization of bifinite domains and lattices.

Recall that bifinite domains can be characterized by the structure of their subposet of compact elements (Plotkin 1981; Abramsky and Jung 1994). Essentially, this is achieved by a study of the fine structure of the images of idempotent deflations. One observes that such an image must consist of compact elements and that the image is closed under the formation of minimal upper bounds of finite subsets.

In the present setting we will try to proceed similarly. From the continuous case we inherit the information that the image of a *linear* idempotent deflation must consist of compact elements, and, consequently, the internal characterization will refer to compact

elements only. The study of minimal upper bounds, however, is trivial for complete lattices as every subset has a supremum, and closing a finite set of compact elements with all suprema will always yield a finite set of compact elements. Hence *continuous* idempotent deflations abound. Our problem is to ensure that there are enough *linear* ones.

We will not study the preservation of suprema directly, but instead generate a deflation *together with an upper adjoint*. Linearity will then be automatic. To start off in this direction, let us record a few observations about adjoints, which can all be proved from the characterizing equivalence 3 in Section 3.

Proposition 7.4. Let A be a complete lattice and $f : A \multimap A$ a linear function. The following relationships hold between f and its upper adjoint f^*:

(i) $f \leqslant \text{id}_A \iff f^* \geqslant \text{id}_A$;
(ii) $f \circ f = f \iff f^* \circ f^* = f^*$;
(iii) f has finite image $\iff f^*$ has finite image.

Corollary 7.3. If f is a linear projection (idempotent deflation) on the complete lattice A, then f^* is a linear projection (idempotent deflation) on A^{op}.

The following lemma will be the key to our characterization. It holds without assuming finite image.

Lemma 7.2. Let f be a linear projection on a complete lattice A, and let x be in $\text{im}(f)$, the image of f. Then x creates a partition of A with the classes $\mathsf{U}_x = {\uparrow}x$ and $\mathsf{L}_x = A \setminus {\uparrow}x$, which is respected by both f and f^*, that is,

$$f(\mathsf{U}_x) \subseteq \mathsf{U}_x , \quad f^*(\mathsf{U}_x) \subseteq \mathsf{U}_x ,$$
$$f(\mathsf{L}_x) \subseteq \mathsf{L}_x , \quad f^*(\mathsf{L}_x) \subseteq \mathsf{L}_x .$$

Furthermore, $\mathsf{L}_x = {\downarrow}f^*(\mathsf{L}_x)$.

Proof. Assume $y \geqslant x$. Then $f(y) \geqslant f(x) = x$ because f is idempotent, and hence f restricts to U_x. The upper adjoint trivially restricts to U_x because we have $f^* \geqslant \text{id}_A$ by Proposition 7.4(1) and U_x is an upper set. For the same reason, f restricts to the lower set L_x. Lastly, let $y \ngeqslant x$ and assume $f^*(y) \geqslant x$. Then $y \geqslant f(x)$ by adjointness. However, $f(x) = x$ as x belongs to the image of f and we get a contradiction.

The additional claim about L_x follows from what we just proved and the fact that $f^* \geqslant \text{id}_A$. $\qquad\qquad\square$

Proposition 7.5. Let f be a linear projection on a complete lattice A, and let X be a subset of $\text{im}(f)$. Then the maximal elements of $\mathsf{L}_X = A \setminus {\uparrow}X$ all belong to $\text{im}(f^*)$.

Proof. We have that f^* restricts to $\mathsf{L}_X = \bigcap_{x \in X} \mathsf{L}_x$ by the previous lemma, and that f^* is above id_A by Proposition 7.4(1). Hence a maximal element of L_X must remain fixed under f^*. $\qquad\qquad\square$

This last result allows us to characterize images of linear projections.

Theorem 7.6. The set of linear projections on a complete lattice A is in one-to-one correspondence with pairs of subsets (M, N) that have the following properties:

P1 $\forall X \subseteq M.\ \max(A \setminus {\uparrow}X) \subseteq N$;

P2 $\forall Y \subseteq N.\ \min(A \setminus {\downarrow}Y) \subseteq M$;

P3 $\forall X \subseteq M\ \forall a \in A \setminus {\uparrow}X\ \exists n \in N \setminus {\uparrow}X.\ a \leqslant n$;

P4 $\forall Y \subseteq N\ \forall a \in A \setminus {\downarrow}Y\ \exists m \in M \setminus {\downarrow}Y.\ b \geqslant m$.

The correspondence assigns to a linear projection f the pair $(\mathrm{im}(f), \mathrm{im}(f^*))$ and to a pair (M, N) the function $f : a \mapsto \bigvee({\downarrow}a \cap M)$.

Proof. Given a linear projection f, we have $(\mathrm{im}(f), \mathrm{im}(f^*))$ has the four properties listed because of Lemma 7.2 and Proposition 7.5. Conversely, given a pair of subsets with these properties, we let f be as stated and $g : a \mapsto \bigwedge({\uparrow}a \cap N)$. It is clear that f is idempotent and below id_A.

Before we can show that f is linear, we need to establish that M is indeed all of $\mathrm{im}(f)$. For this, let $x \in \mathrm{im}(f)$, that is $x = \bigvee({\downarrow}x \cap M)$. For every $a \not\geqslant x$ there must exist $m_a \in {\downarrow}x \cap M$ not below a. By Property **P3**, there is some $n \in N$ above a and not above m_a. Hence $A \setminus {\uparrow}x = {\downarrow}(N \setminus {\uparrow}x)$. Since x is maximal in $A \setminus {\downarrow}(N \setminus {\uparrow}x)$, it belongs to M by Property **P2**. Properties **P1** and **P4** are used to show that N is all of $\mathrm{im}(g)$.

We prove that f is linear by showing that f and g are adjoint. Assume $x \not\leqslant g(y)$. We have just shown that $g(y) \in N$, so by Property **4** there exists $m \in M$ with $m \leqslant x$ and $m \not\leqslant g(y)$. By the definition of f, this entails $f(x) \not\leqslant g(y)$. Since $y \leqslant g(y)$ we cannot have $f(x) \leqslant y$. So $f(x) \leqslant y$ implies $x \leqslant g(y)$. The other direction follows by duality.

We have had to show already that starting with a pair (M, N), constructing f from it and taking $(\mathrm{im}(f), \mathrm{im}(f^*))$ will give back (M, N). For the other identity, start with a projection f. If follows (even in the monotone case) that f is recovered from $\mathrm{im}(f)$ in the way stated. □

For projections with finite image the characterization is even simpler.

Theorem 7.7. Let A be a complete lattice. The set of linear idempotent deflations is in one-to-one correspondence with pairs of finite subsets (M, N) that have the properties **P1** and **P2** from the previous theorem plus

P3′ $M \subseteq \mathsf{K}(A)$;

P4′ $N \subseteq \mathsf{K}(A^{\mathrm{op}})$.

The correspondence is established as before.

Proof. We know from Corollary 7.3 that every linear idempotent deflation has an adjoint that is a linear idempotent deflation on A^{op}. We also know that the image of a linear idempotent deflation consists of compact elements only. For the converse we need that **P3′** and **P4′** (together with **P1** and **P2**) imply their counterparts in Theorem 7.6. This is very easy: for every $X \subseteq M$, the set $A \setminus {\uparrow}X$ is σ_A-closed by **P3**. Hence every element of this set is below a maximal element. The maximal elements of $A \setminus {\uparrow}X$, however, all belong to N by **P1**. □

We need to be able to extend every finite set M of compact elements to an image of a linear idempotent deflation if we want that a given algebraic lattice belongs to **FS**. By the previous theorem, the smallest extension (if it exists) is generated by turning conditions

(1) and (2) into mutually dependent closure operators:

$$
\begin{aligned}
M^0 &:= M \\
M^{k+1} &:= \bigcup_{Y \subseteq N^k} \min(A \setminus {\downarrow}Y) \\
N^{k+1} &:= \bigcup_{X \subseteq M^k} \max(A \setminus {\uparrow}X) \\
M^* &:= \bigcup_{k \in \mathbb{N}} M^k \\
N^* &:= \bigcup_{k \in \mathbb{N}} N^k.
\end{aligned}
$$

Theorem 7.8. An algebraic lattice is an FS-lattice if and only if for every finite subset M of compact elements the sets M^* and N^* are finite and consist of compact elements of A and A^{op}, respectively.

It is instructive to consider in what ways the generation process can fail to lead to a linear idempotent deflation. First, we observe that for a finite set X of compact elements, the set ${\uparrow}X$ is both open and compact. Because of the former, the complement $A \setminus {\uparrow}X$ has a maximal element above every member. The latter implies that $A \setminus {\uparrow}X$ is open in A^{op}. If we *assume* that A^{op} is algebraic as well, then each maximal element in $A \setminus {\uparrow}X$ is compact with respect to A^{op}. Hence, assuming that A is bialgebraic will guarantee that M^* and N^* consist of compact elements only.

Second, we need that the generation process does not lead to an infinite set. For this, we observe the following.

Proposition 7.6. Let A be bialgebraic. Then A is lean if and only if for every C compact open in A^{op}, the set $A \setminus C$ is compact open in A.

Proof. A set C that is compact saturated in A^{op} is closed in A. Hence its complement is open in A. As C is open in A^{op}, its complement is closed in A^{op}. The complement is then compact in A by the definition of leanness.

For the converse, let C be closed in A^{op}. For every $x \in A \setminus C$ there is an A^{op}-compact element above it. Given a finite set X of A^{op}-compact elements in $A \setminus C$, the set ${\downarrow}X$ is compact open in A^{op}. By assumption, its complement (which contains C) is compact open in A. It follows that C is the filtered intersection of compact open sets in A. Since algebraic lattices are sober (Abramsky and Jung 1994, Proposition 7.2.27), C is compact as well (Abramsky and Jung 1994, Corollary 7.2.11). $\qquad\square$

As an illustration, consider the non-lean bialgebraic lattice M_∞ from Example 4.1. Here the generation process, when started on any element different from $\top$ or $\bot$, leads immediately to infinite subsets.

Unfortunately, however, leanness is not sufficient for the generation process to succeed. Figure 2 shows a bialgebraic lean lattice that is not FS. As a third condition, in addition to bialgebraic and lean, we therefore need to stipulate that the generation process terminates after finitely many iterations. This is in surprising analogy to the classical theory of bifinite *domains*. There, too, 'two thirds' of being bifinite is captured topologically (compactness of the Lawson-topology), but the remaining third is formulated with reference to a generation process.
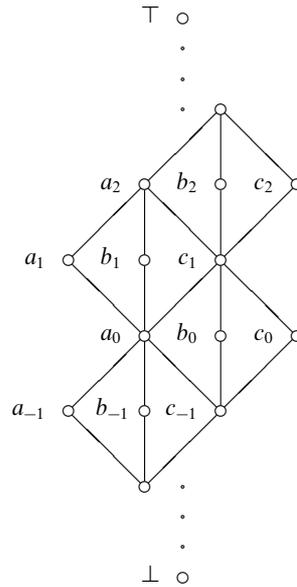
Fig. 2. A bialgebraic lean lattice which is not an FS-lattice.

8. Extensions to Scott-domains

If we drop the requirement that objects A be isomorphic to $(A \multimap 2) \multimap 2$, we may consider the category **BC** of bounded complete dcpos and maps $f : A \to B$ preserving all existing suprema: the existence of $\bigvee X$ for $X \subseteq A$ implies that $\bigvee f(X)$ exists in B and equals $f(\bigvee X)$. Since **SUP** is a full subcategory of **BC**, we have a concrete forgetful functor with a left adjoint given by $(\_ \multimap 2) \multimap 2$ (Huth 1995b). The tight connection between these categories is corroborated at the level of objects: A embeds into $(A \multimap 2) \multimap 2$ such that its image is a lower set closed under all suprema existing in A. So, while morphisms in **BC** do not have an upper adjoint in general, one could define the other linear types in **BC** using the connections above such that the forgetful functor becomes symmetric monoidal.

Instead of providing the details, we briefly discuss the aspect of approximation in **BC**. If we restrict attention to continuous (Scott)-domains, the resulting subcategory is not closed since **CL** is not. We may define approximative objects A such that their double dual is an FS-lattice, but one may equivalently define such objects directly as we did for FS-lattices. It is not hard to see that this leads to a full symmetric monoidal closed subcategory of continuous Scott-domains in **BC**. One may transfer our maximality results *à la* Theorems 4.2, 5.5, and 7.5; yet we can only define leanness indirectly by stipulating that a bounded complete continuous domain A be 'lean' if $(A \multimap 2) \multimap 2$ is lean in the sense we defined earlier. The Scott-domains obtained in this fashion were first introduced in Huth (1994). As for distributivity, the domains A for which $(A \multimap 2) \multimap 2$ is a completely distributive algebraic lattice are exactly Glynn Winskel's prime-algebraic domains (Winskel 1988; Huth 1995b).

9. Related and future work

In Huth and Mislove (1994) one finds another, rather astonishing, external characterization of FS-lattices. Since the inclusion of $(A \multimap B)$ into $[A \to B]$ is linear, it has an upper adjoint, which is just the restriction of $m \mapsto \mathring{m}$ to $[A \to B]$ as a domain of definition in Lemma 5.1. If A equals B and is continuous, then A is an FS-lattice (completely distributive) if, and only if, this upper adjoint is Scott-continuous (linear).

In Heckmann and Huth (1998a; 1998b) one finds a duality theory with which one can show that the more general continuous function space $[X \to B]$ for a sober space X is an FS-lattice (completely distributive) if, and only if, X is a continuous space – essentially a continuous domain – and B an FS-lattice (completely distributive) (Heckmann *et al.* 1999).

Elements in bicontinuous lattices are infima of $\wedge$-irreducible elements and suprema of $\vee$-irreducible elements. Since these elements determine the fine-structure of such lattices, it is desirable to know whether such elements have descriptions that reflect the type constructors, such as $[\_ \to \_]$ and $(\_ \multimap \_)$, in adequate ways for FS-lattices. While one can use the natural isomorphism $(\mathsf{H}A \multimap B) \cong [A \to B]$ to arrive at such notions for the space $[A \to B]$, no identifications of such elements in $(A \multimap B)$ have yet been made if neither A nor B are distributive. The difficulty in obtaining a characterization, say, of $\vee$-irreducible elements in $(A \multimap B)$ is linked to the open problems mentioned in this paper.

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