AN APPLICATION OF WALL-CROSSING TO NOETHER–LEFSCHETZ LOCI

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ABSTRACT. Consider a smooth projective 3-fold X satisfying the Bogomolov–Gieseker conjecture of Bayer-Macrì-Toda (such as $\mathbb{P}^3$, the quintic threefold or an abelian threefold).

Let L be a line bundle supported on a very positive surface in X. If $c_1(L)$ is a primitive cohomology class then we show it has very negative square.

1. INTRODUCTION

Let $(X, \mathcal{O}(1))$ be a smooth polarised complex threefold. For the strongest results we take $\mathcal{O}(1)$ to be primitive. Set $H := c_1(\mathcal{O}(1))$, though we do not require it to be effective.

Weak stability conditions on the derived category $\mathcal{D}(X)$ were introduced by Bayer-Macrì-Toda [BMT14]. Together with their Bogomolov-Gieseker Conjecture 3.1 below they constitute the main technique for producing Bridgeland stability conditions on threefolds.

We only need certain weakenings of the conjecture described in **(BG1)**, **(BG2)** below. They are known to hold for many threefolds [BMS16, Ko18a, Ko18b, Li19b, Li19a, MP16, Ma14, Sc14] such as $\mathbb{P}^3$ or the quintic 3-fold. We apply them to certain weak-semistable objects of $\mathcal{D}(X)$ as we move through the space of weak stability conditions. Combined with wall-crossing techniques this proves results about line bundles on surfaces in $|\mathcal{O}(n)|$.

Theorem 1.1. *Fix any irreducible divisor[1] $D \subset X$ in $|\mathcal{O}(n)|$ and any line bundle L on D with $c_1(L) \neq 0$ in $H^2(D, \mathbb{Q})$ and $c_1(L).H = 0$.*

(A) *If* **(BG1)** *holds on X and $n \geq 4$ then $L^2 \leq \dfrac{-2n}{3}$.*

(B) *If* **(BG2)** *holds on X and $n \geq 10$ then $L^2 \leq -2n + 4$.*

See below for consequences of **(B)** on $\mathbb{P}^3$, for the observation that it is sharp, and for stronger inequalities for line bundles $L = \mathcal{L}|_D$ which are restricted from X.

It is the classes on D which are *not* restricted from X that most interest us. One obvious source of such classes is the *vanishing cycles* of D — the (co)homology classes of the Lagrangian two-spheres in D that are contracted to nodes as we deform D inside $|\mathcal{O}(n)|$ to a nodal surface. These classes all have square $-2 > \frac{-2n}{3}$ so Theorem 1.1 tells us they can never be the class of a line bundle L on D.

Corollary 1.2. *The vanishing cycles of $D \in |\mathcal{O}(n)|$ have empty Noether-Lefschetz loci. In fact any sum of m disjoint vanishing cycles has empty Noether-Lefschetz locus when*

- *X satisfies* **(BG1)**, *$n \geq 4$ and $m \leq \lfloor \frac{n-1}{3} \rfloor$, or*
- *X satisfies* **(BG2)**, *$n \geq 10$ and $m \leq n - 3$.*

In other words, if we look for irreducible $D \in |\mathcal{O}(n)|$ where our vanishing class has Hodge type $(1, 1)$ we should find only singular D on which our cohomology class has ceased to exist (or, considered as a homology class, some part of it has vanished).

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[1]D may be singular. The results also apply to D reducible, so long as L is *slope semistable* on D.

So not all classes in $H^2(D, \mathbb{Z})$ become $(1,1)$ under some deformation inside $|\mathcal{O}(n)|$, even though those which do generate $H^2(D, \mathbb{Z})$ over $\mathbb{Z}$ by [Vo07, p19].

Method. To prove Theorem 1.1 we move in a space of weak stability conditions on $\mathcal{D}(X)$, and show that if $L^2 > -2n/3$ then the Bogomolov-Gieseker inequality **(BG1)** implies $\iota_* L$ is unstable in certain regions, where $\iota \colon D \hookrightarrow X$ is the inclusion. We find the wall on which it becomes unstable, where we show it is destabilised by a map from $\iota_* L$ to $T(-n)[1]$, for some line bundle T with torsion $c_1(T)$. Thus by relative Serre duality for the map ι,

$$(1) \qquad \mathrm{Hom}_X(\iota_* L, T(-n)[1]) \; = \; \mathrm{Hom}_D(L, T|_D) \; \neq \; 0,$$

which means $L^* \otimes T|_D$ is effective.[2] Since $L.H = 0$ this implies $L = T|_D$, so, in particular $c_1(L) = 0$ in $H^2(D, \mathbb{Q})$.

Projective space. There are two different ways to saturate the inequality **(B)** on $\mathbb{P}^3$ and hence deduce it is sharp.

Firstly, we can take D to contain disjoint lines $L_1, L_2 \subset \mathbb{P}^3$. Their normal bundles inside D are $\mathcal{O}_{\mathbb{P}^1}(-n+2)$, so $L := \mathcal{O}_D(L_1 - L_2)$ satisfies $L.H = 0$ and $L^2 = -2n + 4$.

Secondly, if an irreducible $D \in |\mathcal{O}_{\mathbb{P}^3}(n)|$, $n \geq 10$, contains disjoint degree $d \neq 1$ *plane curves* C_1, C_2, then **(B)** applied to $\mathcal{O}_D(C_1 - C_2)$ proves $n \geq d + 2$. Thus **(B)** is saturated if $n = d + 2$, and it is indeed easy to construct $D \supset C_1$, C_2 of any degree $n \geq d + 2$.

More generally if $D \in |\mathcal{O}_{\mathbb{P}^3}(n)|$ contains disjoint degree d curves C_1, C_2 of genus g_1, g_2 then **(B)** applied to $\mathcal{O}_D(C_1 - C_2)$ gives $g_1 + g_2 \leq (n-4)(d-1)$ for $n \geq 10$.

Line bundles restricted from X. When $L = \mathcal{L}|_D$ extends to a line bundle $\mathcal{L}$ on X with $\mathcal{L}.H^2 = 0$ then **(A)** is trivial on *any X*. In fact $L^2 = n\mathcal{L}^2.H$ is divisible by n and < 0 by the Hodge index theorem, so

$$(2) \qquad\qquad\qquad\qquad L^2 \; \leq \; -n.$$

But then if **(BG2)** holds, **(B)** gives $\mathcal{L}^2.nH \leq -2n + 4$, i.e. any line bundle $\mathcal{L}$ on X satisfies

$$\mathcal{L}.H^2 \; = \; 0 \quad \Longrightarrow \quad \mathcal{L}^2.H \; \leq \; -2.$$

This appears to be nontrivial, but not very (the Hodge index theorem already gives ≤ -1). Plugging it back into the argument that gave (2) strengthens it to

$$(3) \qquad\qquad\qquad\qquad L^2 \; \leq \; -2n.$$

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[2]This also shows that if $c_1(L)$ is torsion then it lifts to X. For D smooth this follows already from the Lefschetz hyperplane theorem: X is made from D by attaching $(n \geq 3)$-cells, so $H^3(X, D)$ is torsion free.

similar methods to prove the famous OSV conjecture from physics; see Section 9 for a brief discussion. The second author acknowledges support from EPSRC grant EP/R013349/1.

CONTENTS

2. WEAK STABILITY CONDITIONS

In this section, we review the notion of a weak stability condition on the derived category of coherent sheaves on a smooth threefold. The main references are [BMT14, BMS16].

Let $(X, \mathcal{O}(1))$ be a smooth polarised complex threefold, and $H = c_1(\mathcal{O}(1))$. Denote the bounded derived category of coherent sheaves on X by $\mathcal{D}(X)$ and its Grothendieck group by $K(X) := K(\mathcal{D}(X))$. We define the μ_H-slope of a coherent sheaf E on X to be

$$\mu_H(E) := \begin{cases} \frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3} & \text{if } \mathrm{ch}_0(E) \neq 0, \\ +\infty & \text{if } \mathrm{ch}_0(E) = 0. \end{cases}$$

Associated to this slope every sheaf E has a Harder-Narasimhan filtration. Its graded pieces have slopes whose maximum we denote by $\mu_H^+(E)$ and minimum by $\mu_H^-(E)$.

For any $b \in \mathbb{R}$, let $\mathcal{A}(b) \subset \mathcal{D}(X)$ denote the abelian category of complexes

(4) $$\mathcal{A}(b) = \left\{ E^{-1} \xrightarrow{d} E^0 : \mu_H^+(\ker d) \leq b, \, \mu_H^-(\mathrm{coker}\, d) > b \right\}.$$

Then $\mathcal{A}(b)$ is the heart of a t-structure on $\mathcal{D}(X)$ by [Br08, Lemma 6.1]. Let $w \in \mathbb{R} \setminus \{0\}$. On $\mathcal{A}(b)$ we have the slope function[3]

$$N_{b,w}(E) := \begin{cases} \frac{w\,\mathrm{ch}_2^{bH}(E).H - \frac{1}{6}w^3\,\mathrm{ch}_0(E)H^3}{w^2\,\mathrm{ch}_1^{bH}(E).H^2} & \text{if } \mathrm{ch}_1^{bH}(E).H^2 \neq 0, \\ +\infty & \text{if } \mathrm{ch}_1^{bH}(E).H^2 = 0, \end{cases}$$

where $\mathrm{ch}^{bH}(E) := \mathrm{ch}(E)e^{-bH}$. When $w > 0$ this defines a Harder-Narasimhan filtration on $\mathcal{A}(b)$ by [BMT14, Lemma 3.2.4]. It will be convenient to replace this with

(5) $$\nu_{b,w} := \sigma N_{b,\sigma} + b, \quad \text{where } \sigma := \sqrt{6(w - b^2/2)},$$

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[3]This is called $\nu_{b,w}$ in [BMT14, Equation 7], but we reserve $\nu_{b,w}$ for its rescaling (5).

for $w > b^2/2$. This is because

$$(6) \qquad \nu_{b,w}(E) \ = \ \begin{cases} \frac{\mathrm{ch}_2(E).H - w\,\mathrm{ch}_0(E)H^3}{\mathrm{ch}_1^{bH}(E).H^2} & \text{if } \mathrm{ch}_1^{bH}(E).H^2 \neq 0, \\ +\infty & \text{if } \mathrm{ch}_1^{bH}(E).H^2 = 0 \end{cases}$$

has a denominator that is linear in b and numerator linear in w, so the walls of $\nu_{b,w}$-instability will turn out to be *linear*; see Proposition 4.1. Note that if $\mathrm{ch}_i(E).H^{n-i} = 0$ for $i = 0, 1, 2$, the slope $\nu_{b,w}(E)$ is defined by (6) to be $+\infty$. Since (5) only rescales and adds a constant, it defines the same Harder-Narasimhan filtration as $N_{b,\sigma}$, so it too defines a weak stability condition on $\mathcal{A}(b)$.

Definition 2.1. Fix $w > \frac{b^2}{2}$. We say $E \in \mathcal{D}(X)$ is $\nu_{b,w}$-(semi)stable if and only if

- $E[k] \in \mathcal{A}(b)$ for some $k \in \mathbb{Z}$, and
- $\nu_{b,w}(E[k]) \ (\leq) \ \nu_{b,w}(F)$ for all non-trivial quotients $E[k] \twoheadrightarrow F$ in $\mathcal{A}(b)$.

Here $(\leq)$ denotes $<$ for stability and $\leq$ for semistability.

Remark 2.2. Given $(b, w) \in \mathbb{R}^2$ with $w > \frac{b^2}{2}$, the argument in [Br07, Propostion 5.3] describes $\mathcal{A}(b)$. It is generated by the $\nu_{b,w}$-stable two-term complexes $E = \{E^{-1} \to E^0\}$ in $\mathcal{D}(X)$ satisfying the following conditions on the denominator and numerator of $\nu_{b,w}$ (6):

- (a) $\mathrm{ch}_1^{bH}(E).H^2 \geq 0$, and
- (b) $\mathrm{ch}_2(E).H - w\,\mathrm{ch}_0(E)H^3 \geq 0$ if $\mathrm{ch}_1^{bH}(E).H^2 = 0$.

That is, $\mathcal{A}(b)$ is the extension-closure of the set of these complexes.

3. BOGOMOLOV-GIESEKER INEQUALITY

We recall the conjectural strong Bogomolov-Gieseker inequality of [BMT14, Conjecture 1.3.1], rephrased in terms of the rescaling (5).

Conjecture 3.1. *For $\nu_{b,w}$-semistable $E \in \mathcal{A}(b)$ with $\mathrm{ch}_2^{bH}(E).H = \left(w - \frac{b^2}{2}\right)\mathrm{ch}_0(E)H^3$,*

$$\mathrm{ch}_3^{bH}(E) \ \leq \ \left(\frac{w}{3} - \frac{b^2}{6}\right)\mathrm{ch}_1^{bH}(E).H^2.$$

Although this conjecture is known not to hold for all classes on all threefolds [Sc17], it is possible it always holds for objects of the classes $\mathrm{ch}(\iota_*L)$ that we consider. In Theorem 1.1 we only need the conjecture in special cases, namely

(BG1) Conjecture 3.1 holds for sheaves of class $\mathrm{ch}(\iota_*L)$ and stability parameters $\left(-\frac{n}{2}, w\right)$ for any $w > \frac{n^2}{4} - \frac{1}{H^3}$ for fixed $n \geq 4$.

(BG2) Conjecture 3.1 holds for both

- sheaves of class $\mathrm{ch}(\iota_*L)$ and stability parameters $\left(-\frac{n}{2}, w\right)$ for any $w > \frac{n^2}{4} - \frac{3}{H^3}$ and fixed $n \geq 10$, and
- torsion-free sheaves F with $\mathrm{ch}_0(F) = 1$, $\mathrm{ch}_1(F).H^2 = 0$, $\mathrm{ch}_2(F).H \in \{-1, -2\}$, and stability parameters (b^*, w^*) with $b^* = \mathrm{ch}_2(F).H - \frac{1}{2H^3}$, $w^* = (b^*)^2 + \frac{\mathrm{ch}_2(F).H}{H^3}$.

Conjecture 3.1 is a special case of [BMS16, Conjecture 4.1], which has now been proved for

- X is projective space $\mathbb{P}^3$ [Ma14], the quadric threefold [Sc14] or, more generally, any Fano threefold of Picard rank one [Li19a],
- X an abelian threefold [MP16], a Calabi-Yau threefold of abelian type [BMS16], a Kummer threefold [BMS16], or a product of an abelian variety and $\mathbb{P}^n$ [Ko18a],
- X with nef tangent bundle [Ko18b], and
- X is a quintic threefold and (b, w) are described below [Li19b].

Theorem 3.2. [Li19b, Theorem 2.8] *Let X be a smooth quintic threefold. Then Conjecture 3.1 is true for (b, w) satisfying*

$$(7) \qquad w \;>\; \frac{1}{2}b^2 + \frac{1}{2}\big(b - \lfloor b \rfloor\big)\big(\lfloor b \rfloor - b + 1\big).$$

In particular **(BG1)** *and* **(BG2)** *hold on X.*

Proof. Using the notation (α, β) for (w, b), [Li19b, Theorem 2.8] proves that (7) implies [BMS16, Conjecture 4.1]. This gives Conjecture 3.1, so we are left with checking that the parameters in **(BG1)**, **(BG2)** satisfy (7).

For **(BG1)** we take $n \geq 4$, $b = -\frac{n}{2}$ and $w > \frac{n^2}{4} - \frac{1}{H^3}$. Then certainly $n^2 > \frac{8}{H^3} + 1$, which can be rearranged to give

$$(8) \qquad \frac{n^2}{8} + \frac{1}{8} \;<\; \frac{n^2}{4} - \frac{1}{H^3} \;<\; w.$$

But since $b = -\frac{n}{2}$ we have

$$(9) \qquad \frac{1}{2}b^2 + \frac{1}{2}\big(b - \lfloor b \rfloor\big)\big(\lfloor b \rfloor - b + 1\big) \;\leq\; \frac{n^2}{8} + \frac{1}{8}$$

which by (8) gives (7).

For **(BG2)** we take $n \geq 10$, $b = -\frac{n}{2}$ and $w > \frac{n^2}{4} - \frac{3}{H^3}$. Then certainly $n^2 > \frac{24}{H^3} + 1$, which can be rearranged to give

$$\frac{n^2}{8} + \frac{1}{8} \;<\; \frac{n^2}{4} - \frac{3}{H^3} \;<\; w.$$

By (9) this gives (7).

For the second part of **(BG2)**, use the obvious inequality $(2\epsilon - x)(\epsilon - x) + (\epsilon - 1)x > 0$ for $\epsilon \in \{1, 2\}$ and $x \in (0, 1)$. By rearranging this is equivalent to

$$\frac{1}{2}\left(-\epsilon - \frac{x}{2}\right)^2 - \epsilon x \;>\; \frac{x}{4}\left(1 - \frac{x}{2}\right).$$

Substituting in $\epsilon = -\operatorname{ch}_2(F).H$, $x = \frac{1}{H^3}$ and $b^* = \operatorname{ch}_2(F).H - \frac{1}{2H^3}$ makes this

$$\frac{(b^*)^2}{2} + \frac{\operatorname{ch}_2(F).H}{H^3} \;>\; \frac{1}{4H^3}\left(1 - \frac{1}{2H^3}\right).$$

For $w^* = (b^*)^2 + \frac{\operatorname{ch}_2(F).H}{H^3}$ this is

$$w^* - \frac{(b^*)^2}{2} \;>\; \frac{1}{2}\left(1 - \frac{1}{2H^3}\right)\frac{1}{2H^3} \;=\; \frac{1}{2}\big(b^* - \lfloor b^* \rfloor\big)\big(\lfloor b^* \rfloor - b^* + 1\big),$$

i.e. the inequality (7) for (b^*, w^*) as required.   □

4. Wall and chamber structure

In Figure 1 we plot the (b, w)-plane simultaneously with the image of the projection map

$$\Pi\colon\ K(X) \setminus \big\{E\colon \mathrm{ch}_0(E) = 0\big\}\ \longrightarrow\ \mathbb{R}^2,$$
$$E\ \longmapsto\ \left(\frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3}\,,\ \frac{\mathrm{ch}_2(E).H}{\mathrm{ch}_0(E)H^3}\right).$$
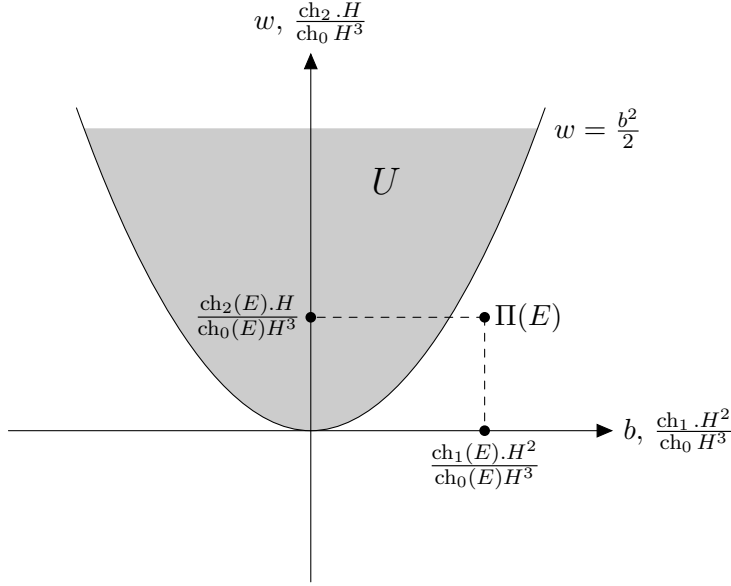

FIGURE 1. (b, w)-plane and the projection $\Pi(E)$

Note that for any weak stability condition $\nu_{b,w}$, the pair (b, w) is in the shaded open subset

$$(10) \qquad\qquad U\ :=\ \left\{(b, w) \in \mathbb{R}^2 \colon w > \frac{b^2}{2}\right\}.$$

Conversely, the image $\Pi(E)$ of $\nu_{b,w}$-semistable objects E with $\mathrm{ch}_0(E) \neq 0$ is *outside U*,

$$\left(\frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3}\right)^2 - 2\,\frac{\mathrm{ch}_2(E).H}{\mathrm{ch}_0(E)H^3}\ \geq\ 0,$$

by the classical Bogomolov-Gieseker-type inequality of [BMS16, Theorem 3.5],

$$(11) \qquad\quad \Delta_H(E)\ :=\ \big(\mathrm{ch}_1(E).H^2\big)^2 - 2(\mathrm{ch}_0(E)H^3)(\mathrm{ch}_2(E).H)\ \geq\ 0,$$

for the H-discriminant $\Delta_H(E)$ of a $\nu_{b,w}$-semistable object E.[4]

Proposition 4.1 (**Wall and chamber structure**). *Fix an object $E \in \mathcal{D}(X)$ such that the vector $\big(\mathrm{ch}_0(E), \mathrm{ch}_1(E).H^2, \mathrm{ch}_2(E).H\big) \neq 0$ is non-zero. There exists a locally finite collection of lines $\{\ell_i\}_{i \in I}$ in $\mathbb{R}^2$ (called "walls") which satisfies the following conditions:*

(a) *Any line ℓ_i passes through the point $\Pi(E)$ if $\mathrm{ch}_0(E) \neq 0$, or has fixed slope $\frac{\mathrm{ch}_2(E).H}{\mathrm{ch}_1(E).H^2}$ if $\mathrm{ch}_0(E) = 0$.*

(b) *The $\nu_{b,w}$-(semi)stability of E is unchanged as (b,w) varies within any connected component (called a "chamber") of $U \setminus \bigcup_{i \in I} \ell_i$.*

(c) *For any wall ℓ_i there exists $k_i \in \mathbb{Z}$ and a map $f \colon F \to E[k_i]$ in $\mathcal{D}(X)$ such that*
- *for any $(b,w) \in \ell_i \cap U$, the objects $E[k_i], F$ lies in the heart $\mathcal{A}(b)$,*
- *$E[k_i]$ is $\nu_{b,w}$-semistable with $\nu_{b,w}(E) = \nu_{b,w}(F) = \mathrm{slope}\,(\ell_i)$ constant on $\ell_i \cap U$, and*
- *f is an injection $F \subset E[k_i]$ in $\mathcal{A}(b)$ which strictly destabilises $E[k_i]$ for (b,w) in one of the two chambers adjacent to the wall ℓ_i.*
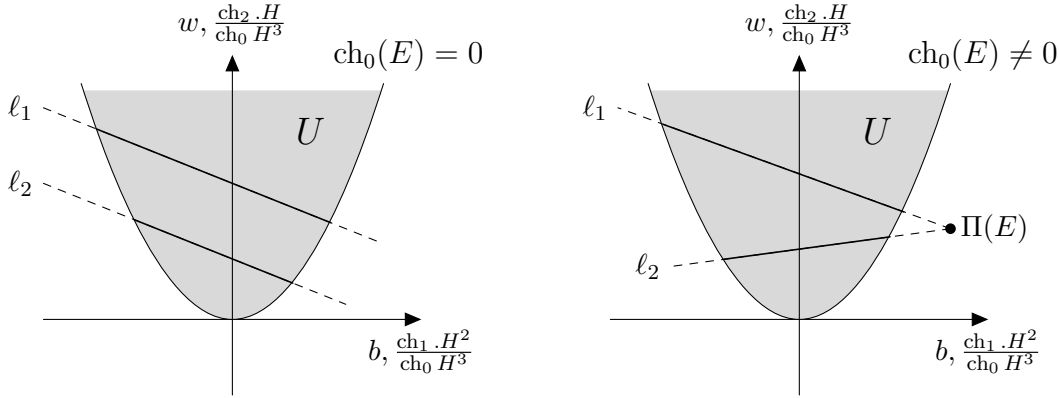

FIGURE 2. The line segments $\ell_i \cap U$ are walls for E.

Proof. For $E \in \mathcal{D}(X)$ the existence of a locally finite set of walls in the (b, w) plane follows from the arguments in [Br08, Proposition 9.3] or [BMS16, Proposition 12.5].

Suppose that E is $\nu_{b,w}$-strictly semistable. Then there is a $k \in \mathbb{Z}$ such that $E[k] \in \mathcal{A}(b)$ and a $\nu_{b,w}$-stable destabilising object $F \subset E[k]$ in $\mathcal{A}(b)$. The condition that $\nu_{b,w}(E[k]) = \nu_{b,w}(F)$ is

$$
(12) \qquad \frac{w - \frac{\mathrm{ch}_2(E[k]).H}{\mathrm{ch}_0(E[k])H^3}}{b - \frac{\mathrm{ch}_1(E[k]).H^2}{\mathrm{ch}_0(E[k])H^3}} = \frac{w - \frac{\mathrm{ch}_2(F).H}{\mathrm{ch}_0(F)H^3}}{b - \frac{\mathrm{ch}_1(F).H^2}{\mathrm{ch}_0(F)H^3}} \quad \text{if} \quad \mathrm{ch}_0(E[k]) \neq 0 \neq \mathrm{ch}_0(F),
$$

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[4][BMS16, Theorem 3.5] state (11) with ch replaced by ch^{bH}, but the result is still $\Delta_H(E)$. We use the stronger Bogomolov inequality $\mathrm{ch}_1(E)^2.H - 2\,\mathrm{ch}_0(E)(\mathrm{ch}_2(E).H) \geq 0$ for μ_H-semistable sheaves in (25).

or

$$(13) \qquad \frac{w - \frac{\mathrm{ch}_2(E[k]).H}{\mathrm{ch}_0(E[k])H^3}}{b - \frac{\mathrm{ch}_1(E[k]).H^2}{\mathrm{ch}_0(E[k])H^3}} \;=\; \frac{\mathrm{ch}_2(F).H}{\mathrm{ch}_1(F).H^2} \quad \text{if} \;\; \mathrm{ch}_0(E[k]) \neq 0 = \mathrm{ch}_0(F),$$

or

$$(14) \qquad \frac{\mathrm{ch}_2(E[k]).H}{\mathrm{ch}_1(E[k]).H^2} \;=\; \frac{w - \frac{\mathrm{ch}_2(F).H}{\mathrm{ch}_0(F)H^3}}{b - \frac{\mathrm{ch}_1(F).H^2}{\mathrm{ch}_0(F)H^3}} \quad \text{if} \;\; \mathrm{ch}_0(E[k]) = 0 \neq \mathrm{ch}_0(F).$$

As we move through the (b, w) plane, (12) is the equation of the straight line joining $\Pi(E)$ and $\Pi(F)$, (13) is the straight line though $\Pi(E)$ of slope $\frac{\mathrm{ch}_2(F).H}{\mathrm{ch}_1(F).H^2}$, and (14) is the line through $\Pi(F)$ of slope $\frac{\mathrm{ch}_2(E[k]).H}{\mathrm{ch}_1(E[k]).H^2}$. In each case the slopes of $E[k]$ and F are constant on the wall, and satisfy strict (and opposite) inequalities on the two sides of the wall. This explains the shape of the walls of instability.

If $\mathrm{ch}_0(E[k]) = 0 = \mathrm{ch}_0(F)$ we do not get a wall since both slopes remain constant as we move throughout the whole of U in the (b, w) plane.

Finally, if we move along a wall, the $\nu_{b,w}$-slopes of all the Jordan-Hölder factors of $E[k]$ coincide and remain constant. So long as they're finite, Remark 2.2 implies that the Jordan-Hölder factors remain in the heart $\mathcal{A}(b)$, and so $E[k]$ does too. If they're infinite the wall is vertical, and the category $\mathcal{A}(b)$ is constant, so the conclusion is the same. $\qquad \square$

5. Large volume limit

As usual we consider a line bundle L on $D \in |\mathcal{O}(n)|$ such that $L.H = 0$. The Chern character of its push-forward is

$$(15) \qquad \mathrm{ch}(\iota_* L) \;=\; \left(0, \, nH, \, \iota_*(c_1(L)) - \frac{n^2}{2}H^2, \, \frac{1}{2}L^2 + \frac{n^3}{6}H^3 \right).$$

To move through the space U (10) of weak stability conditions, we begin in the large volume region $w \gg 0$. We use the fact that L is slope stable on D since it has no proper saturated subsheaves when D is irreducible. (The results of this paper also hold for reducible D if we *assume* that $\iota_* L$ is slope semistable.)

Lemma 5.1. *The sheaf $\iota_* L$ is $\nu_{b,w}$-semistable for any $b \in \mathbb{R}$ and $w \gg 0$.*

Proof. We sketch the proof, which is very similar to [Br08, Proposition 14.2]. The key point is that a sheaf $\iota_* E$ pushed forward from D has rank 0 so its $\nu_{b,w}$-slope (6),

$$(16) \qquad \nu_{b,w}(\iota_* E) \;=\; \frac{\mathrm{ch}_2(\iota_* E).H}{\mathrm{ch}_1(\iota_* E).H^2} \;=\; \frac{\mathrm{ch}_1(E).H}{\mathrm{ch}_0(E)H^2} - \frac{n}{2} \;=\; \mu_H(E) - \frac{n}{2},$$

is independent of $(b, w) \in \mathbb{R}^2$ and essentially reduces to the ordinary slope of E on D. Here the intersections take place on X in the second term and on D in the third term. (On reducible D the denominator $\mathrm{ch}_0(E)H^2$ would be replaced by the leading coefficient of the Hilbert polynomial of E.)

Fix a real number $b \in \mathbb{R}$. The sheaf $\iota_* L$ is in the heart $\mathcal{A}(b)$. Fix a subobject E_1 of $\iota_* L$ in $\mathcal{A}(b)$ with quotient E_2. Then the ordinary cohomology sheaves $\mathcal{H}^i$ of these objects sit in a long exact sequence

$$0 \longrightarrow \mathcal{H}^{-1}(E_2) \longrightarrow \mathcal{H}^0(E_1) \longrightarrow \iota_* L \longrightarrow \mathcal{H}^0(E_2) \longrightarrow 0.$$

In particular E_1 is a sheaf. Suppose first that $\operatorname{rank}(E_1) \neq 0$. Since $E_1 \in \mathcal{A}(b)$ we know $\mu_H^-(E_1) > b \Rightarrow \mu_H(E_1) > b \Rightarrow \operatorname{ch}_1^{bH}(E_1).H^2 > 0$. By (6) therefore, $+\infty > \nu_{b,w}(E_1) \to -\infty$ as $w \to \infty$, so E_1 does not destabilise $\iota_* L$ for $w \gg 0$. As in [Br08, Proposition 14.2] one can in fact make the bound on w (so that E_1 does not destabilise) uniform in E_1.

If $\operatorname{rank}(E_1) = 0$ then $\mathcal{H}^{-1}(E_2) = 0$ because $E_2 \in \mathcal{A}(b)$ implies that $\mathcal{H}^{-1}(E_2)$ is a torsion-free sheaf. Therefore E_1 is a subsheaf of $\iota_* L$, which by (16) and the slope semistability of L cannot strictly $\nu_{b,w}$-destabilise $\iota_* L$. $\square$

6. THE FIRST WALL

From now on we work in one of the situations

(i) suppose **(BG1)** holds, $n \geq 4$ and $L^2 \geq \lfloor \frac{-2n}{3} \rfloor + 1$, or
(ii) suppose **(BG2)** holds, $n \geq 10$ and $L^2 \geq -2n + 5$.

Then moving in the space U of weak stability conditions we will try to show that $c_1(L)$ is a torsion class in $H^2(D, \mathbb{Z})$. This will prove Theorem 1.1.

By Proposition 4.1 the walls of instability for $\iota_* L$ are all lines of slope $-\frac{n}{2}$ in the (b, w) plane; see Figure 3. The lowest such line which intersects $\overline{U}$ is $w = -\frac{n}{2} b - \frac{n^2}{8}$, which is tangent to ∂U at $\left(-\frac{n}{2}, \frac{n^2}{8}\right)$. Therefore the vertical line

$$(17) \qquad b \equiv b_0 := -\frac{n}{2}$$

intersects all the possible walls of instability of $\iota_* L$. We will move down this vertical line from the large volume region $w \gg 0$.

By (15), $\operatorname{ch}_2^{b_0 H}(\iota_* L).H = 0 = \operatorname{ch}_0(\iota_* L)$ on the line $b = b_0$, so we can apply the Bogomolov-Gieseker Conjecture 3.1 for stability parameters $\left(-\frac{n}{2}, w\right)$. That is, if $\iota_* L$ is $\nu_{b_0, w}$-semistable then
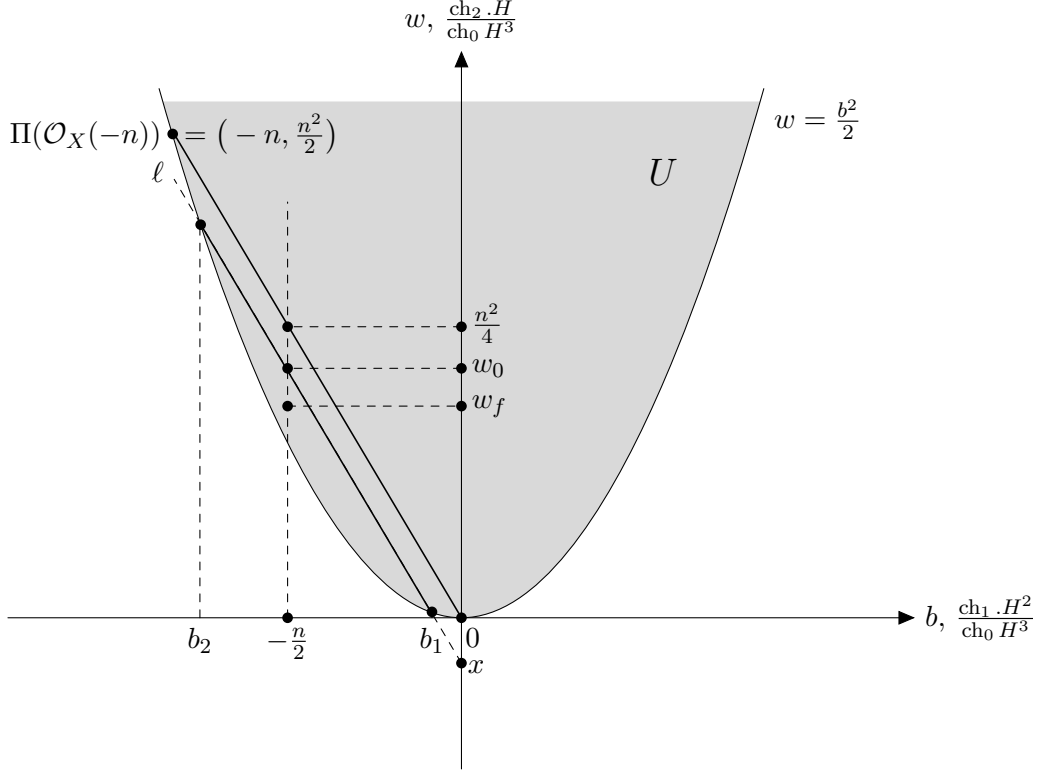
$$\operatorname{ch}_3^{b_0 H}(\iota_* L) \leq \left(\frac{w}{3} - \frac{b_0^2}{6}\right) \operatorname{ch}_1^{b_0 H}(\iota_* L).H^2.$$

Using (15) and rearranging gives

$$(18) \qquad w \geq w_f := \frac{n^2}{4} + \frac{3L^2}{2nH^3}.$$

Note that case (i) gives $w_f > \frac{n^2}{4} - \frac{1}{H^3}$, while case (ii) gives $w_f > \frac{n^2}{4} - \frac{3n-6}{nH^3} > \frac{n^2}{4} - \frac{3}{H^3}$. In both cases then, $w_f > \frac{b_0^2}{2} = \frac{n^2}{8}$, so (b_0, w_f) lies inside U.

Therefore, when we move down the line $b = -\frac{n}{2}$, we find there is a point $w_0 \geq w_f$ where $\iota_* L$ is first destabilised. We next show that in fact $w_0 \in \left[w_f, \frac{n^2}{4}\right]$.

FIGURE 3. Walls for $\iota_* L$

Proposition 6.1. *There is a wall of slope $-\frac{n}{2}$ for $\iota_* L$ that bounds the large volume chamber $w \gg 0$. It passes through a point (b_0, w_0), where $w_0 \in \left[w_f, \frac{n^2}{4} \right]$.*

In the destabilising sequence $F_1 \hookrightarrow \iota_* L \twoheadrightarrow F_2$ in $\mathcal{A}(b_0)$, we have $\dim \operatorname{supp} \mathcal{H}^0(F_2) \le 1$, the object F_1 is a rank one sheaf with $\operatorname{ch}_1(F_1).H^2 = 0$ and, in cases (i), (ii),

(i) $\operatorname{ch}_2(F_1).H = 0$,
(ii) $\operatorname{ch}_2(F_1).H \in \{0, -1, -2\}$.

Proof. By Proposition 4.1 and (18), $\iota_* L$ is ν_{b_0, w_0}-destabilised by a sequence $F_1 \hookrightarrow \iota_* L \twoheadrightarrow F_2$ in $\mathcal{A}(b_0)$ for $b_0 = -\frac{n}{2}$ and some $w_0 \ge w_f$. The corresponding wall is denoted by ℓ in Figure 3. It has equation $w = -\frac{n}{2}b + x$, where

$$ x \;=\; w_0 - \frac{n^2}{4} \;\ge\; w_f - \frac{n^2}{4} \;=\; \frac{3L^2}{2nH^3} $$

satisfies

(19)
$$ x \;>\; \begin{cases} -\frac{1}{H^3} & \text{in case (i),} \\ -\frac{3}{H^3} & \text{in case (ii).} \end{cases} $$

Let $b_2 < b_1$ be the values of b at the intersection points of ℓ and the boundary $w = \frac{b^2}{2}$ of U,

$$b_1 \;=\; \sqrt{\frac{n^2}{4} + 2x} - \frac{n}{2}, \qquad b_2 \;=\; -\sqrt{\frac{n^2}{4} + 2x} - \frac{n}{2}\,.$$

We claim that

(20) $$b_1 \; > \; -\frac{1}{2H^3} \quad \text{and} \quad b_2 + n \; < \; \frac{1}{2H^3}\,.$$

Both are equivalent to $\sqrt{\frac{n^2}{4} + 2x} > \frac{n}{2} - \frac{1}{2H^3}$, and therefore to $2x > \frac{1}{4(H^3)^2} - \frac{n}{2H^3}$. Since $x \geq \frac{3L^2}{2nH^3}$ it is sufficient to show

$$\frac{3L^2}{n} \;\geq\; \frac{1}{4H^3} - \frac{n}{2}\,.$$

For (i) this follows from $L^2 \geq \left\lfloor -\frac{2n}{3} \right\rfloor + 1 \geq -\frac{2n}{3} + \frac{1}{3}$ and the inequality $-2 + \frac{1}{n} > -\frac{n}{2} + \frac{1}{4H^3}$ that holds for $n \geq 4$. For (ii) it follows from $L^2 \geq -2n + 5$ and the inequality $\frac{n}{2} > 6 - \frac{15}{n} + \frac{1}{4H^3}$ that holds for all $n \geq 10$.

Taking cohomology from the destabilising sequence $F_1 \hookrightarrow \iota_* L \twoheadrightarrow F_2$ gives the long exact sequence of coherent sheaves

(21) $$0 \longrightarrow \mathcal{H}^{-1}(F_2) \longrightarrow \mathcal{H}^0(F_1) \longrightarrow \iota_* L \longrightarrow \mathcal{H}^0(F_2) \longrightarrow 0.$$

In particular, the destabilising subobject F_1 is a coherent sheaf. As we saw in the proof of Proposition 4.1, if it had rank 0 then its slope would be constant throughout U, like that of $\iota_* L$, so we would not have a wall. Thus $\mathrm{ch}_0(F_1) > 0$ so (21) gives

$$\mathrm{ch}_0(\mathcal{H}^{-1}(F_2)) \;=\; \mathrm{ch}_0(F_1) \; > \; 0.$$

As in Proposition 4.1, $\Pi(F_1)$ and $\Pi(F_2)$ lie on the line ℓ. All along $\ell \cap U$ (i.e. for $b \in (b_2, b_1)$) the objects F_1 and F_2 lie in the heart $\mathcal{A}(b)$ and (semi)destabilise $\iota_* L$. Therefore by the definition (4) of $\mathcal{A}(b)$ and the inequalities (20),

(22) $$\mu_H^+(\mathcal{H}^{-1}(F_2)) \;\leq\; b_2 \; < \; -n + \frac{1}{2H^3} \quad \text{and} \quad \mu_H^-(F_1) \;\geq\; b_1 \; > \; \frac{-1}{2H^3}\,.$$

Thus dividing $\left(\mathrm{ch}_1(\iota_* L) - \mathrm{ch}_1(\mathcal{H}^0(F_2))\right).H^2 = \left(\mathrm{ch}_1(F_1) - \mathrm{ch}_1(\mathcal{H}^{-1}(F_2))\right).H^2$ by $\mathrm{ch}_0(F_1)H^3$ gives

(23) $$\frac{n}{\mathrm{ch}_0(F_1)} - \frac{\mathrm{ch}_1(\mathcal{H}^0(F_2)).H^2}{\mathrm{ch}_0(F_1)H^3} \;=\; \mu_H(F_1) - \mu_H(\mathcal{H}^{-1}(F_2))$$

$$\geq\; \mu_H^-(F_1) - \mu_H^+(\mathcal{H}^{-1}(F_2)) \;\geq\; b_1 - b_2 \; > \; n - \frac{1}{H^3}\,.$$

Since $\mathcal{H}^0(F_2)$ has rank zero, $\mathrm{ch}_1(\mathcal{H}^0(F_2)).H^2 \geq 0$ so $\frac{n}{\mathrm{ch}_0(F_1)} \geq$ (23). Thus the inequalities imply $\mathrm{ch}_0(F_1) = 1$ and $\mathrm{ch}_1(\mathcal{H}^0(F_2)).H^2 = 0$. In particular, $\mathcal{H}^0(F_2)$ is supported in dimension ≤ 1.

Hence $\mu_H(F_1) = \frac{\mathrm{ch}_1(F_1).H^2}{H^3}$ is an integer multiple of $\frac{1}{H^3}$, so the inequality (22) implies that $\mu_H(F_1) \geq 0$. Similarly (22) gives $\mu_H(\mathcal{H}^{-1}(F_2)) \leq -n$ while (23) gives $\mu_H(\mathcal{H}^{-1}(F_2)) \geq -n$.

The upshot is that $\mu_H(F_1) = 0$ and $\mu_H(\mathcal{H}^{-1}(F_2)) = -n$. Hence $\mathrm{ch}_1(F_1).H^2 = 0$ and $\Pi(F_1)$ lies on the w-axis. But it also lies on the wall ℓ given by $w = \frac{-n}{2}b + x$, so

$$(24) \qquad x \;=\; \frac{\mathrm{ch}_2(F_1).H}{\mathrm{ch}_0(F_1)H^3} \;=\; \frac{\mathrm{ch}_2(F_1).H}{H^3} \,.$$

Since the sheaf F_1 is ν_{b_0,w_0}-semistable, $\Pi(F_1)$ lies outside U by (11). Thus $x \le 0$ which is $w_0 \le \frac{n^2}{4}$, as claimed. Combining this with (19) and (24) gives, finally,

$$0 \;\ge\; \mathrm{ch}_2(F_1).H \;>\; \begin{cases} -1 & \text{in case (i),} \\ -3 & \text{in case (ii).} \end{cases} \qquad\qquad \square$$

Proposition 6.2. *Under the assumptions of Proposition 6.1, the destabilising subobject F_1 of ι_*L satisfies $\mathrm{ch}_2(F_1).H = 0$. That is, $x = 0$, $w_0 = \frac{n^2}{4}$, and the wall bounding the large volume chamber is the line of slope $-\frac{n}{2}$ through the origin.*

Proposition 6.1 proves this in case (i). We will prove Proposition 6.2 in case (ii) in Section 8 by applying the Bogomolov-Gieseker conjecture 3.1 to F_1 and F_2. This gives upper bounds for $\mathrm{ch}_3(F_1)$ and $\mathrm{ch}_3(F_2)$ respectively. In turn the latter gives a lower bound for $\mathrm{ch}_3(F_1)$. If we work only at (b_0, w_0), as in [To12], the bounds are not optimal, but by working at more general points of the (b, w)-plane we get stronger bounds which together force $\mathrm{ch}_2(F_1).H = 0$.

Lemma 6.3. *Under the assumptions of Proposition 6.1, $\dim \operatorname{supp} \mathcal{H}^0(F_2) = 0$ and*

$$\mathrm{ch}_1(\mathcal{H}^{-1}(F_2)) \;=\; -nH \quad \text{in } H^2(X, \mathbb{Q}).$$

Proof. By Proposition 6.1, F_2 has rank 1 and lies in $\mathcal{A}(b_0)$ (4), so $\mathcal{H}^{-1}(F_2)$ is a torsion-free rank one sheaf. Therefore it is μ_H-semistable and the classical Bogomolov inequality says

$$(25) \qquad \mathrm{ch}_1(\mathcal{H}^{-1}(F_2))^2.H - 2\,\mathrm{ch}_2(\mathcal{H}^{-1}(F_2)).H \;\ge\; 0.$$

From the exact sequence (21) we calculate $\mathrm{ch}_i(\mathcal{H}^{-1}(F_2)) = \mathrm{ch}_i(F_1) - \mathrm{ch}_i(\iota_*L) + \mathrm{ch}_i(\mathcal{H}^0(F_2))$. Taking $i = 2$ and intersecting with H, Proposition 6.2 kills the first term while (15) and $L.H = 0$ calculate the second, yielding

$$(26) \qquad \mathrm{ch}_2(\mathcal{H}^{-1}(F_2)).H \;=\; \frac{n^2 H^3}{2} + \mathrm{ch}_2(\mathcal{H}^0(F_2)).H.$$

Taking $i = 1$ and intersecting with H^2, Proposition 6.1 kills the first and third terms, giving

$$\mathrm{ch}_1(\mathcal{H}^{-1}(F_2)).H^2 \;=\; -nH^3.$$

So by the Hodge index theorem

$$(27) \qquad n^2 H^3 \;=\; \frac{\left(\mathrm{ch}_1(\mathcal{H}^{-1}(F_2)).H^2\right)^2}{H^3} \;\ge\; \mathrm{ch}_1(\mathcal{H}^{-1}(F_2))^2.H,$$

with equality if and only if $\mathrm{ch}_1(\mathcal{H}^{-1}(F_2))$ is a multiple of H in $H^2(X, \mathbb{Q})$.

Combining (25), (26) and (27) gives

$$(28) \qquad\qquad\qquad -2\,\mathrm{ch}_2(\mathcal{H}^0(F_2)).H \;\ge\; 0.$$

But Proposition 6.1 also showed that $\mathcal{H}^0(F_2)$ is supported in dimension ≤ 1, so (28) shows it must have 0-dimensional support and (28, 27) are equalities. Thus $\mathrm{ch}_1(\mathcal{H}^{-1}(F_2))$ is a multiple of H in $H^2(X, \mathbb{Q})$.

To determine the multiple we calculate from the sequence (21) that $\mathrm{ch}_1(\mathcal{H}^{-1}(F_2)).H^2 = \mathrm{ch}_1(F_1).H^2 - \mathrm{ch}_1(\iota_*L).H^2$. The former is zero by Proposition 6.1 and the second is nH^3. $\square$

So $\mathcal{H}^0(F_2)$ is supported in dimension 0 and is a quotient of ι_*L by (21). Thus there is a 0-dimensional subscheme $Z \subset D$ with ideal sheaf I_Z on D such that (21) simplifies to

$$(29) \qquad 0 \longrightarrow \mathcal{H}^{-1}(F_2) \longrightarrow F_1 \longrightarrow \iota_*(L \otimes I_Z) \longrightarrow 0,$$

where $\mathcal{H}^{-1}(F_2)$ and F_1 are rank 1 torsion free sheaves. By Lemma 6.3 there is a dim ≤ 1 subscheme $C \subset X$ such that

$$(30) \qquad \mathcal{H}^{-1}(F_2) \;\cong\; T(-n) \otimes I_C$$

for some line bundle T with $c_1(T) = 0 \in H^2(X, \mathbb{Q})$. Rotating the exact triangle (29), we get a short exact sequence in $\mathcal{A}(b_0)$:

$$(31) \qquad 0 \longrightarrow F_1 \longrightarrow \iota_*(L \otimes I_Z) \longrightarrow T(-n) \otimes I_C[1] \longrightarrow 0.$$

In fact any rank zero sheaf such as $\iota_*(L \otimes I_Z)$ lies in the heart $\mathcal{A}(b_0)$. Since $T(-n)$ is a line bundle, it is a μ_H-semistable sheaf of the same slope as $\mathcal{H}^{-1}(F_2)$, and thus its shift by [1] lies in $\mathcal{A}(b_0)$ because F_2 does. By the same reasoning,

$$(32) \qquad 0 \longrightarrow T(-n) \otimes \mathcal{O}_C \to T(-n) \otimes I_C[1] \longrightarrow T(-n)[1] \longrightarrow 0$$

is also a short exact sequence in $\mathcal{A}(b_0)$.

7. Proof of main Theorem

We are now ready to prove Theorem 1.1. We compose the $\mathcal{A}(b_0)$-surjections (the third arrows) of (31) and (32) to give

$$\iota_*(L \otimes I_Z) \longrightarrow T(-n)[1].$$

Since this is a surjection in $\mathcal{A}(b_0)$, it is a nonzero element of

$$(33) \qquad \mathrm{Ext}^1(\iota_*(L \otimes I_Z), T(-n)) \;\cong\; \mathrm{Ext}^1(\iota_*L, T(-n)) \;\cong\; \mathrm{Hom}(L, T|_D).$$

(The first isomorphism follows from $\mathrm{Ext}^{<3}(\mathcal{O}_Z, T(-n)) = 0$, by dim $Z = 0$, and the second from relative Serre duality for ι.) Thus $L^* \otimes T|_D$ is effective. Since $L.H = 0$ this implies $L = T|_D$. In particular, $c_1(L) = 0$ in $H^2(D, \mathbb{Q})$. $\square$

Remark 7.1. In fact, calculating $\mathrm{ch}_2(F_1).H$ from (29) and (30) gives $-H.C$, which by Proposition 6.2 is zero. Therefore both C and Z are 0-dimensional and the $\nu_{b,w}$ slopes of $T(-n) \otimes I_C$ and $\iota_*(L \otimes I_Z)$ are the same as those of $T(-n)$ and ι_*L respectively. Thus the map $\iota_*L \to T(-n)[1]$ produced in (33) also destabilises in $\mathcal{A}(b)$ on the first wall. That is,

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

– tensored with T and rotated – gives the destabilising short exact sequence in $\mathcal{A}(b)$.

8. DESTABILISING OBJECTS IN CASE (ii)

What remains is to prove Proposition 6.2 in case (ii). So we assume **(BG2)** holds, $n \geq 10$ and $L^2 \geq -2n + 5$. By Proposition 6.1, in $\mathcal{A}(b)$ there is a destabilising sequence $F_1 \hookrightarrow \iota_* L \twoheadrightarrow F_2$ for $\iota_* L$ along the wall ℓ with equation

$$w \;=\; -\frac{n}{2}b + \frac{\mathrm{ch}_2(F_1).H}{H^3}\,.$$

Moreover rank $F_1 = 1 = -\operatorname{rank} F_2$, and, by Proposition 6.1,

$$\mathrm{ch}_1(F_1).H^2 = 0 \quad \text{and} \quad \mathrm{ch}_2(F_1).H \in \{0, -1, -2\}.$$

We will assume that $\mathrm{ch}_2(F_1).H \neq 0$ and apply the Bogomolov-Gieseker inequality to F_1 and F_2 to get a contradiction.

It will be convenient to work with $b = b_1 := -\frac{1}{H^3}$ because then, by (6),

$$\nu_{b_1,w}(E) \;=\; \frac{\mathrm{ch}_2(E).H - w\,\mathrm{ch}_0(E)H^3}{\mathrm{ch}_1(E).H^2 + \mathrm{ch}_0(E)}$$

has a denominator $D_1(E) := \mathrm{ch}_1(E).H^2 + \mathrm{ch}_0(E)$ which

- is integral and ≥ 0 for $E \in \mathcal{A}(b_1)$,
- is additive on K-theory classes: $D_1(E_1 + E_2) = D_1(E_1) + D_1(E_2)$, and
- takes the minimal nonzero value 1 on F_1.

This means that in $\mathcal{A}(b_1)$ the object F_1 can only be destabilised by objects with denominator $D_1 = 0$.[5] Such objects have $\nu = +\infty$ so, in particular, F_1 can never be semi-destabilised: it is either stable or strictly unstable, and has no walls of instability. Since it is semistable on ℓ, and this intersects $b = b_1$ at the point

$$w_1 \;=\; \frac{n}{2H^3} + \frac{\mathrm{ch}_2(F_1).H}{H^3}$$

which defines a weak stability condition in U by

$$w_1 - \frac{b_1^2}{2} \;=\; \frac{n}{2H^3} - \frac{1}{2(H^3)^2} + \frac{\mathrm{ch}_2(F_1).H}{H^3} \;\geq\; \frac{nH^3 - 1 - 4H^3}{2(H^3)^2} \;>\; 0,$$

we conclude the following.

Lemma 8.1. *The destabilising sheaf F_1 is $\nu_{b_1,w}$-stable for any $w > \frac{b_1^2}{2}$.* $\square$

Similarly if we work with $b = b_2 := -n + \frac{1}{H^3}$ then the denominator of $\nu_{b_2,w}$ is

$$D_2(E) \;:=\; \mathrm{ch}_1(E(n)).H^2 - \mathrm{ch}_0(E).$$

This has the same properties as $D_1(E)$, except the third is replaced now by $D_2(F_2) = 1$ being minimal. Again ℓ intersects $b = b_2$ in a point

$$w_2 \;=\; \frac{n^2}{2} - \frac{n}{2H^3} + \frac{\mathrm{ch}_2(F_1).H}{H^3}$$

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[5]This argument is familiar from the analogous fact that rank 1 sheaves can only be destabilised by rank 0 torsion sheaves when working with slope (for which the denominator is rank).

inside the space U of weak stability conditions, by

$$w_2 - \frac{b_2^2}{2} \;=\; \frac{n}{2H^3} - \frac{1}{2(H^3)^2} + \frac{\mathrm{ch}_2(F_1).H}{H^3} \;\geq\; \frac{nH^3 - 1 - 4H^3}{2(H^3)^2} \;>\; 0.$$

So the same argument as for Lemma 8.1 gives the following.

Lemma 8.2. *The destabilising quotient F_2 is $\nu_{b_2,w}$-stable for any $w > \frac{b_2^2}{2}$.* $\qquad\square$

Proposition 8.3. $\mathrm{ch}_3(F_1) \;\leq\; \dfrac{2}{3}\,\mathrm{ch}_2(F_1).H\left(\mathrm{ch}_2(F_1).H - \dfrac{1}{2H^3}\right).$

Proof. Recall the line $\{b = b_1\} \cap U$ used in Lemma 8.1. Its base on $w = \frac{b_2^2}{2}$ is the point $\left(-\frac{1}{H^3}, \frac{1}{2(H^3)^2}\right)$. Let ℓ_2 denote the line connecting this point to $\Pi(F_1) = \left(0, \frac{\mathrm{ch}_2(F_1).H}{H^3}\right)$,

$$(34) \qquad\qquad w \;=\; \left(\mathrm{ch}_2(F_1).H - \frac{1}{2H^3}\right)b + \frac{\mathrm{ch}_2(F_1).H}{H^3}\,.$$
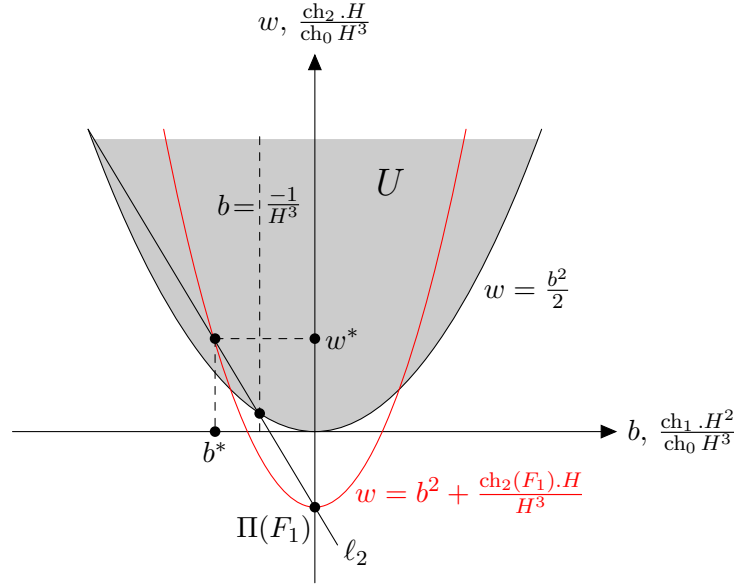

FIGURE 4. The first wall for the sheaf F_1

By the description of the walls of instability (Proposition 4.1), the $w \downarrow \frac{b_1^2}{2}$ limit of Lemma 8.1 therefore shows that F_1 is $\nu_{b,w}$-semistable for any $(b,w) \in \ell_2 \cap U$; see Figure 4.

To apply the Bogomolov-Gieseker Conjecture 3.1 to F_1 on ℓ_2 we need to find a point of $\ell_2 \cap U$ satisfying $\mathrm{ch}_2^{bH}(F_1).H = \left(w - \frac{b^2}{2}\right)\mathrm{ch}_0(F_1)H^3$, i.e.

$$\frac{\mathrm{ch}_2(F_1).H}{H^3} + \frac{b^2}{2} \;=\; w - \frac{b^2}{2}\,.$$

This intersects ℓ_2 (34) at the point (b^*, w^*), where

$$b^* \ = \ \mathrm{ch}_2(F_1).H - \frac{1}{2H^3} \quad \text{and} \quad w^* \ = \ \big(\mathrm{ch}_2(F_1).H\big)^2 + \frac{1}{4(H^3)^2}\,.$$

ν_{b^*,w^*} is a weak stability condition since $w^* - \frac{(b^*)^2}{2} = \frac{1}{2}\big(\mathrm{ch}_2(F_1).H + \frac{1}{2H^3}\big)^2 > 0$, so by **(BG2)** we may apply Conjecture 3.1 to give

$$\mathrm{ch}_3(F_1) - b^*\,\mathrm{ch}_2(F_1).H - \frac{(b^*)^3 H^3}{6} \ \leq \ \frac{1}{3}\left(w^* - \frac{(b^*)^2}{2}\right)(-b^* H^3)$$

$$= \ \frac{1}{3}\left(\frac{\mathrm{ch}_2(F_1).H}{H^3} + \frac{(b^*)^2}{2}\right)(-b^* H^3).$$

Simplifying gives

$$\mathrm{ch}_3(F_1) \ \leq \ \frac{2}{3}\,b^*\,\mathrm{ch}_2(F_1).H. \hspace{4cm} \square$$

Proposition 8.4. $\mathrm{ch}_3(F_2(n)) \ \leq \ \dfrac{2}{3}\,\mathrm{ch}_2(F_2(n)).H\left(\mathrm{ch}_2(F_2(n)).H + \dfrac{1}{2H^3}\right).$

Proof. By Lemma 8.2, $F_2 \in \mathcal{A}(b_2)$ is $\nu_{b_2,w}$-semistable for $w \gg 0$. Thus $F_2(n) \in \mathcal{A}(b_2+n) = \mathcal{A}(-b_1)$ is $\nu_{-b_1,w}$-semistable for $w \gg 0$. Therefore, by [BMT14, Lemma 5.1.3(b)] the shifted derived dual $F_2(n)^{\vee}[1]$ lies in an exact triangle

$$F \ \hookrightarrow \ F_2(n)^{\vee}[1] \ \twoheadrightarrow \ Q[-1],$$

with Q a zero-dimensional sheaf and F a $\nu_{b_1,w}$-semistable object of $\mathcal{A}(b_1)$ for $w \gg 0$. Since rank $F = 1$ it is a torsion-free sheaf by [BMS16, Lemma 2.7]. We also have $\mathrm{ch}_1(F).H^2 = \mathrm{ch}_1(F_2(n)).H^2 = 0$. Thus F has all the properties of F_1 used in Lemma 8.1 and Proposition 8.3, so the latter gives

$$\mathrm{ch}_3(F) \ \leq \ \frac{2}{3}\,\mathrm{ch}_2(F).H\left(\mathrm{ch}_2(F).H - \frac{1}{2H^3}\right).$$

Since $\mathrm{ch}_2(F_2(n)).H = -\,\mathrm{ch}_2(F).H$ and $\mathrm{ch}_3(F_2(n)) = \mathrm{ch}_3(F_2(n)^{\vee}[1]) = \mathrm{ch}_3(F) - \mathrm{ch}_3(Q) \leq \mathrm{ch}_3(F)$ the claim follows. $\square$

Proof of Proposition 6.2. Set $c := \mathrm{ch}_2(F_1).H \in \{0, -1, -2\}$, so by Proposition 8.3,

$$(35) \hspace{3cm} \mathrm{ch}_3(F_1) \ \leq \ \frac{2c}{3}\left(c - \frac{1}{2H^3}\right).$$

Using $\mathrm{ch}_0(F_1) = 1$, $\mathrm{ch}_1(F_1).H^2 = 0$ and the exact triangle $F_1 \to \iota_*L \to F_2$ we compute

$$\mathrm{ch}_1(F_2(n)).H^2 \ = \ 0, \quad \mathrm{ch}_2(F_2(n)).H \ = \ -c \quad \text{and} \quad \mathrm{ch}_3(F_2(n)) \ = \ -nc - \mathrm{ch}_3(F_1) + \frac{L^2}{2}\,.$$

The inequality of Proposition 8.4 therefore becomes

$$-nc - \mathrm{ch}_3(F_1) + \frac{L^2}{2} \ \leq \ -\frac{2c}{3}\left(-c + \frac{1}{2H^3}\right).$$

Combined with (35) and our assumption $L^2 > -2n + 4$ this gives

$$-n(c+1) + 2 \; < \; -nc + \frac{L^2}{2} \; \leq \; \mathrm{ch}_3(F_1) - \frac{2c}{3}\left(-c + \frac{1}{2H^3}\right) \; \leq \; \frac{4c}{3}\left(c - \frac{1}{2H^3}\right).$$

If $c = -1$ this gives the contradiction $2 < \frac{4}{3} + \frac{2}{3H^3}$. If $c = -2$ we get $n + 2 < \frac{16}{3} + \frac{4}{3H^3} < 7$ but $n \geq 10$. So $c = 0$. $\qquad\square$

9. Curve counting

The results of this paper are a special case of the results in [FT19], which in turn builds on [GST14]. Consider 2-dimensional torsion sheaves of the form $\iota_*(L \otimes I_C)$, where $D \in |\mathcal{O}(n)|$ and $I_C \subset \mathcal{O}_D$ is the ideal sheaf of a subscheme of dimension ≤ 1. We take $L.H = 0$ and n is sufficiently large as in this paper; the main difference in [FT19] is that we allow nonempty C.

We show the moduli space of slope semistable sheaves in the class of $\iota_*(L \otimes I_C)$ is isomorphic to the product of $\mathrm{Pic}_{\mathrm{tors}}(X)$[6] – the line bundles on X with torsion c_1 – and the moduli space of Joyce-Song pairs

(36) $$\mathcal{O}(-n) \xrightarrow{\ s\ } \mathcal{I}_C.$$

Here $\mathcal{I}_C \subset \mathcal{O}_X$ is an ideal sheaf on X and $s \in H^0(\mathcal{O}(n))$ is a nonzero section with zero divisor $D \supset C$. The correspondence takes the cokernel of (36) and tensors it with a line bundle L with torsion $c_1(L)$ to get a sheaf of the form $\iota_*(L \otimes I_C)$.

For $n \gg 0$ the moduli space of pairs (36) is a projective bundle over the moduli space of ideal sheaves $\mathcal{I}_C$. The fibre $\mathbb{P}\big(H^0(\mathcal{I}_C(n))\big)$ has Euler characteristic $\chi(\mathcal{I}_C(n))$. If X is a Calabi-Yau 3-fold with $H^1(\mathcal{O}_X) = 0$ this gives the relation

$$\#(\text{2-dimensional sheaves}) \; = \; \#H^2(X, \mathbb{Z})_{\mathrm{tors}} \cdot \chi(\mathcal{I}_C(n)) \cdot \#(\text{ideal sheaves}).$$

The first term is a DT invariant counting Gieseker stable sheaves[7] of the same topological type as $\iota_*(L \otimes I_C)$. The next two terms are topological constants. The final term is the DT invariant counting ideal sheaves of the topological type of $\mathcal{I}_C$.

The set of all of these DT invariants counting ideal sheaves is equivalent, by the MNOP conjecture [MNOP] (proved for most Calabi-Yau 3-folds in [PP17]), to the set of Gromov-Witten invariants of X. The upshot is that the Gromov-Witten invariants of X are governed by counts of 2-dimensional sheaves. In turn the generating series of the latter are conjectured by physicists to be mock modular forms due to *S-duality*.

Both this paper and [FT19] use very similar methods to those employed so impressively by Toda [To12] to prove the famous OSV conjecture on Calabi-Yau threefolds X with $\mathrm{Pic}\, X = \mathbb{Z}$ satisfying the Bogomolov-Gieseker conjecture. Toda also considers slope stable sheaves of dimension two and follows them down the wall $b = b_0$ (17), using the Bogomolov-Gieseker inequality to find the first wall of instability ℓ. One difference between the papers is that in Proposition 6.1 we analyse the destabilising objects F_1, F_2 along the wall ℓ, and

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[6]Note this is *not* the set of torsion line bundles, though it contains it of course.

[7]We show that for $n \gg 0$, slope semistability is equivalent to slope stability and to Gieseker stability.

use the fact that they lie in $\mathcal{A}(b)$ at its endpoints $\ell \cap \partial U$ to constrain $\mathrm{ch}(F_i)$. Toda works only on $b = b_0$ and uses different arguments to analyse $\mathrm{ch}(F_i)$. A similar comment applies to the work in Section 8 to prove Proposition 6.2, as described in the discussion below Proposition 6.2.

The main difference between our work and Toda's is that we consider subtly different Chern characters. In [FT19] we consider two dimensional sheaves with $\mathrm{ch}_1 = nH$ and

$$\mathrm{ch}_2 \;=\; -\beta - \frac{n^2}{2}H^2$$

for $n \gg 0$ and some curve class β (in this paper ultimately $\beta = 0 \in H^4(X, \mathbb{Q})$). Toda also considers $\mathrm{ch}_1 = nH$ but

$$\mathrm{ch}_2 \;=\; -\beta,$$

for fixed β and $n \gg 0$. To apply his methods to our class would require a bound like $\beta.H \geq \frac{1}{2}n^2H^3$, while his paper works in the opposite regime $\beta.H < \epsilon n^2$. As a result he manages to express counts of 2-dimensional sheaves in terms of both ideal sheaves and stable pairs, whereas for us the stable pairs are absent and the results rather different.

APPENDIX A. THE CASE OF $\mathbb{P}^3$
By Claire Voisin

When $X = \mathbb{P}^3$ we can prove a very similar result to (B) by more classical methods.

Theorem A.1. *Let D be a smooth surface of degree $n \geq 4$ in $\mathbb{P}^3$. Any nontrivial line bundle L on D with $c_1(L).H = 0$ satisfies $L^2 \leq -2n + 5$.*

Proof. The $K3$ case $n = 4$ is trivial: Riemann-Roch gives $h^0(L) + h^0(L^{-1}) = h^1(L) + 2 + \frac{L^2}{2}$ so if L is nontrivial with $L.H = 0$ this gives $0 = h^1(L) + 2 + \frac{L^2}{2}$ and so $L^2 \leq -4$.

So we can take $n \geq 5$. By Riemann-Roch,

$$(37) \qquad h^0(L) + h^0(K_D \otimes L^{-1}) \;\geq\; \chi(L) \;=\; \chi(\mathcal{O}_D) + \frac{1}{2}L^2 - \frac{1}{2}K_D.L.$$

We assume for a contradiction that L is nontrivial and $L^2 \geq -2n + 6$. Using $K_D = \mathcal{O}_D(n-4)$, $L.H = 0$ and $h^1(\mathcal{O}_D) = 0$, (37) gives

$$(38) \qquad h^0(L^{-1}(n-4)) \;\geq\; h^0(\mathcal{O}_D(n-4)) - (n-4).$$

Let $C := H \cap D$ be a smooth plane section and $L_C := L|_C$. Then the exact sequences $0 \to L^{-1}(i-1) \to L^{-1}(i) \to L_C^{-1}(i) \to 0$ give

$$(39) \qquad h^0\big(L^{-1}(i)\big) - h^0\big(L^{-1}(i-1)\big) \;\leq\; h^0\big(L_C^{-1}(i)\big).$$

Since $h^0(L^{-1}) = 0$, summing over $1 \leq i \leq n-4$ gives

$$(40) \qquad h^0\big(L^{-1}(n-4)\big) \;\leq\; \sum_{i=1}^{n-4} h^0\big(L_C^{-1}(i)\big).$$

Replacing L^{-1} by $\mathcal{O}_D$ gives equality in (39) for $1 \le i \le n-4$ by Kodaira vanishing, and so

$$(41) \qquad h^0\big(\mathcal{O}_D(n-4)\big) - 1 \; = \; \sum_{i=1}^{n-4} h^0\big(\mathcal{O}_C(i)\big).$$

Comparing (38), (40) and (41) shows that $h^0\big(L_C^{-1}(i)\big) \ge h^0\big(\mathcal{O}_C(i)\big)$ for some $1 \le i \le n-4$. Since $\deg L_C = 0$ this implies $L_C = \mathcal{O}_C$ by [Ha86, Theorem 2.1, 2(b)].

By standard methods, this now implies the contradiction $L = \mathcal{O}_D$. For instance, consider the blow up $\pi \colon \widehat{D} \to D$ of D in the baselocus of a pencil of Cs, giving a fibration $p \colon \widehat{D} \to \mathbb{P}^1$. Then π^*L is trivial on the fibres, so is the pullback from $\mathbb{P}^1$ of the line bundle $p_*(\pi^*L) \cong \mathcal{O}_{\mathbb{P}^1}(d)$. Restricting π^*L to (the proper transform of) another plane section (a multisection of p) and using $L.H = 0$ shows that $d = 0$. $\qquad\square$

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