High-energy theory for close Randall Sundrum branes

Claudia de Rham* and Samuel Webster†

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, England

We obtain an effective theory for the radion dynamics of the two-brane Randall Sundrum model, correct to all orders in brane velocity in the limit of close separation, which is of interest for studying brane collisions and early Universe cosmology. Obtained via a recursive solution of the Bulk equation of motions, the resulting theory represents a simple extension of the corresponding low-energy effective theory to the high energy regime. The four-dimensional low-energy theory is indeed not valid when corrections at second order in velocity are considered. This extension has the remarkable property of including only second derivatives and powers of first order derivatives. This important feature makes the theory particularly easy to solve. We then extend the theory by introducing a potential and detuning the branes.

I. INTRODUCTION

Motivated by advances in string theory (in particular heterotic M-theory), there has recently been considerable interest in models where spacetime is effectively five-dimensional. In these models, matter fields are confined to three-branes, membrane-like surfaces embedded in the higher dimensional space, while gravity (and other bulk fields) can propagate in the whole of spacetime \(^1\). The cosmological consequences of these scenarios have been widely studied (for recent reviews see \(^\text{2,3,4}\)). The simplest such model is that of Randall and Sundrum \(^5,6\) where the bulk is assumed to be empty apart from a cosmological constant. This is a toy model through which braneworld ideas can be tested and, despite its simplicity, is rich in new physics.

In the low-energy limit, an effective four-dimensional theory can be derived on the branes \(^7,8,9,10\). This theory predicts that near the collision, the Hubble constant on each brane is related to the proper contraction or expansion velocity of the fifth dimension \(d\) by \(H_7^2 = \frac{d^2}{L^2}\), where \(L\) is the Anti-de Sitter (AdS) radius related to the bulk cosmological constant. However, an exact calculation gives the result \(H_7^2 = \frac{1}{L} \tanh^2 \left(\frac{d}{L}\right)\). As expected, this agrees with the four-dimensional effective theory only at low velocities. To lowest order in velocity, the low-energy limit gives an accurate result for the brane geometries. The aim of this work is to go beyond the low-energy limit and to develop a covariant formalism which describes exactly these velocity corrections in the small distance limit \(d \ll L\).

Braneworld cosmology offer a entirely new set of possibilities for the production of the large scale structure and in particular allows scenarios with high energies or for which the branes are moving apart at velocities which could be large \(^11,12,13,14,15,16,17,18,19\). We may therefore address the issue of the importance of the high-order terms in these regimes. In scenarios such as steep inflation \(^22,23\), the energy scales are important and facilitate the end of inflation. In the Cyclic Universe \(^24,25\) as another example, it is suggested that terms of fourth order in velocities should be considered in order to obtain a scale invariant spectrum after the bounce \(^26\). To work with such models, it seems important to understand the behaviour of these high-energy corrections and in particular to understand their consequences for the production of the large scale structure, as much in the context of an expanding as in a contracting Universe.

However, in order find a solvable theory when the low-energy constraint is relaxed, we need to work in another limit. The special limit in which we choose to work instead is the close brane limit where the distance \(d\) between the branes is small compared with the AdS radius \(L\): \(d \ll L\). This limit is interesting in cosmology for two main reasons. The first one is that there has been considerable interest in the interpretation of the Big Bang as a brane collision \(^24,25,26,27,28,29,30\). Therefore, at the beginning of our universe, there will be a regime where the distance between the branes is very small. Such a limit seems therefore to be relevant to study the production of the large scale structure.

The second reason is that it is in the limit where the distance between the branes is small that we expect the geometry on the brane to be well-described by a four-dimensional theory. Indeed, when the branes are far apart, the bulk degrees of freedom (leading to the Kaluza-Klein corrections on the branes) are more eas-

\*e-mail address: C.deRham@damtp.cam.ac.uk
†e-mail address: S.L.Webster@damtp.cam.ac.uk
ilily excited, making the theory on the branes non-local \[16, 31, 32, 33\].

Following these arguments, in the small distance limit, we therefore expect the branes to be well described by a four-dimensional theory which will be valid for any energy scale and will include higher-order derivatives. To work in this regime, we follow part of the idea of Shiromizu \textit{et al.} in \[34\], although our final result will be different. The main idea of this paper is to express the extrinsic curvature on the negative-tension brane as a Taylor expansion in terms of the extrinsic curvature on the positive-tension brane and its derivatives along the normal direction. We can then use the five-dimensional Einstein equations to express any second (or higher) order normal derivatives of the extrinsic curvature in terms of first normal derivatives and derivatives along the four transverse directions. Since the normal derivative of the extrinsic curvature on the positive-tension brane is known up to the induced Weyl tensor on that brane, this gives a formal equation for the induced Weyl tensor on the positive-tension brane, which is the only unknown information on the brane.

Although formally correct, this process can be complicated, in particular if matter is introduced on the branes. In order to understand this expansion we will therefore focus on the case of empty branes for which the extrinsic curvature on the branes is proportional to the metric. As it will appear later, the formalism becomes very simple in that context. As already mentioned, the second condition we impose is the close brane regime. More concretely, by imposing this condition we will keep only the terms at leading orders in the distance \(d\) between the branes at each order in the Taylor expansion. To do this, we will work in a recursive way. This will enable us to work out the exact form for the Weyl tensor on the positive-tension brane to leading order in \(d\).

As we might expect the exact expression for the Weyl tensor in the close brane limit has some higher derivative correction terms which are not present in the low-energy four-dimensional effective theory. However we may check that, at low velocities, this expression for the Weyl tensor is consistent with the one derived from the low-energy effective theory in the small \(d\) limit. Furthermore, as a non trivial check we may verify that our result gives precisely the right result to leading order in \(d\) for the background solution where it is possible to solve the five-dimensional geometry exactly. However the prescription is completely covariant and will allow us to study perturbations without needing to solve the full five-dimensional equations.

Having an exact expression for the induced Weyl tensor in the close-brane limit provides us with a unique modified Einstein equation on the brane describing the coupling of a scalar field (the radion) to the brane geometry. The equation of motion of the scalar field is then given by the requirement that the Weyl tensor be traceless. One of the most promising routes is the possibility of interpreting this scalar field as playing the role of the inflaton scalar field on the positive-tension brane \[35, 36, 37, 38, 39, 40\]. During inflation both branes could be moving apart. When the negative-tension brane moves towards infinity, its effect on the positive-tension brane becomes negligible which means that the scalar field decouples, giving an explanation of why such a field is not seen on the brane at the present time. In order to interpret the radion as a candidate for the inflaton a potential for the radion has to be introduced. So far the exact origin of that potential is not fully understood, but for the purpose of this study we shall introduce a potential by hand assuming some five-dimensional effects might explain its presence. We will therefore study how the theory we derived on the brane can be modified in order to include such a potential and how the equation of motion for the scalar field has to be modified in a consistent way in order to respect the conservation of energy and to be consistent with the low-energy theory. In order to do so we will review how the introduction of the potential is usually performed in the low-energy limit and then proceed in a similar way for the close-brane theory.

The paper is organised as follows. In Section II we review the Randall Sundrum model and derive the exact Friedmann equations on the branes for the background. We then give an overview of the low-energy four-dimensional effective theory and compare its predictions in the background with the exact solution. Since they disagree beyond the leading term in velocity, in Section III we derive a covariant expression for the Weyl tensor on each brane in the close brane limit. First we start with a toy model in order to understand the procedure. Then we present the five-dimensional model and derive the Weyl tensor on each brane using the small \(d\) approximation \(d \ll L\). We use this result in Section IV in order to find the exact theory on the branes, valid in the close brane limit. We check that this theory is consistent with the conservation of energy. Finally in Section V we consider extension of this theory, first by introducing a potential and then by detuning the brane tensions. We will then discuss the implications of our results in Section VI.

In Appendix A we present the technical details that allows us to derive the normal derivative of the extrinsic curvature in order to obtain the Weyl tensor on each brane. First we proceed in a special gauge where we suppose that the \(g_{yy}\) component of the metric can be taken to be independent of the fifth dimension. Then, we show that the same procedure is valid when this condition is relaxed and we obtain the same result.

In Appendix B we present the details of the derivation of the divergence of the stress-energy tensor and show how the equation of motion for the scalar field has to be modified when a potential is introduced.

Finally, for completeness, we derive the evolution equation for the Weyl tensor in Appendix C.
II. BACKGROUND BEHAVIOUR

In what follows, we shall be interested in the two brane Randall Sundrum model as a specific simple example of braneworld cosmologies. In that model, the spacetime is five dimensional, with a compact extra dimension having the topology of an $S_1/Z_2$ orbifold. The stress energy of the bulk is assumed to be from a pure negative cosmological constant $\Lambda = \kappa^2 M_n^2 L_n^2$, $\kappa = \pm 1$, $M_n$ the $n$-dimensional Planck mass. For simplicity, we set $\kappa = 1$ for the rest of this work. There are two boundary branes located at the fixed points of the $Z_2$ symmetry on which gauge and matter fields are confined. Apart from where stated otherwise, we assume, for simplicity, that the tension on each brane are fine-tuned to their canonical value $\lambda_{\pm} = \pm \frac{6}{T}$ and we don’t include any other kind of matter on the branes.

In this paper we use the index conventions that Greek indices are four dimensional, labeling the transverse $x^\mu$ directions, while Roman indices are fully five dimensional. We will denote by a “dot” any derivative with respect of the proper time unless specified otherwise.

A. Five-dimensional Theory

1. Frame where the bulk is static

In the fine-tuned case with cosmological symmetry (i.e. the three spatial directions are assumed to be homogeneous and isotropic) one can make considerable exact analytical progress. We work in the frame where the bulk is static, which exists as a consequence of Birkhoff’s theorem. In that frame, the most general geometry is Schwarzschild-Anti-de-Sitter (SAdS) with the parameter $C$ associated with the Black Hole mass [11]:

$$ds^2 = dY^2 - n^2(Y)dt^2 + a^2(Y)dx^2$$

with

$$a^2(Y) = e^{-2Y/L} + \frac{C}{4} e^{2Y/L}$$

$$n^2(Y) = L^2 a'(Y)^2 = a^2 - \frac{C}{a^2},$$

where we are assuming flat spatial sections for simplicity. In this frame, the branes are not static but have loci $Y = Y_{\pm}(T)$. The Israel matching conditions [12], associated with the $Z_2$-symmetry, impose the condition on the extrinsic curvature:

$$K_{\mu\nu}(Y = Y_{\pm}) = -\frac{1}{T} g_{\mu\nu}(Y = Y_{\pm}).$$

On the brane, the spatial components of the extrinsic curvature are:

$$K_{ij}(Y_{\pm}) = \left(1 - \frac{\dot{Y}_{\pm}^2}{n_{\pm}^2}\right)^{-1/2} \frac{a'(Y)}{a(Y)} g_{ij} \bigg|_{Y = Y_{\pm}}.$$

so the brane velocities must satisfy:

$$\dot{Y}_{\pm}^2 = \left(\frac{dY_{\pm}}{dT}\right)^2 = n_{\pm}^2 \left(1 - \frac{n_{\pm}^2}{a_{\pm}^2}\right),$$

where $a_{\pm}(T) \equiv a(Y = Y_{\pm}(T))$ is the value of the scale factor on each brane, and similarly for $n_{\pm}(T)$. The induced line element on the branes can be read off as

$$ds_{\pm}^2 = -(n_{\pm}^2 - \dot{Y}_{\pm}^2)dt^2 + a_{\pm}^2 dx^2$$

$$= a_{\pm}^2 (-d\sigma^2 + dx^2),$$

defining the conformal time on each brane:

$$d\tau_{\pm} \equiv \sqrt{n_{\pm}^2 - \dot{Y}_{\pm}^2} dT = \frac{n_{\pm}^2}{a_{\pm}^2} dT.$$

In terms of the conformal time, the scale factor on each brane evolves with constant velocity:

$$\left(\frac{da_{\pm}}{d\tau_{\pm}}\right)^2 = \dot{Y}_{\pm}^2 \left(\frac{dT}{d\tau_{\pm}}\right)^2 \frac{da(Y)}{dY} \bigg|_{Y = Y_{\pm}} = \frac{C}{L^2},$$

leading to the Hubble parameter on each brane:

$$H_{\pm}^2 = \frac{1}{a_{\pm}^2} \left(\frac{da_{\pm}}{d\tau_{\pm}}\right)^2$$

$$= \frac{1}{L^2} \left(1 - \frac{n_{\pm}^2}{a_{\pm}^2}\right) = \frac{C}{L^2 a_{\pm}^2}. $$

This result is a direct consequence of the projected Einstein equations on the brane (see Section III.C), which, for empty Friedmann Robertson Walkers (FRW) branes, gives the induced Ricci scalars as

$$R(\pm) = \frac{6}{a_{\pm}^2} \frac{d^2 a_{\pm}}{d\tau_{\pm}^2} = 0.$$

We may point out that in [S], the Hubble parameter on each brane is bounded by $L^2 H_{\pm}^2 \leq 1$, with equality for the negative-tension brane when this one touches the five-dimensional Black Hole horizon. This study is beyond the purpose of this paper and we consider both branes to be far away from the five dimensional singularity.

We now consider the specific case where the branes are moving apart after a collision at $\tau = 0$ and we denote by $a_0$ the value of the scale factor when the branes coincide (the situation where the branes are moving towards each other before the collision is completely analogous).

In contrast to the exact treatment above we now only consider the system in the limit of small brane separation, i.e. $a_+ \approx a_- \approx a_0$, $n_+ \approx n_- \approx n_0$ and $Y_+ \approx -Y_-$. Using (1), to linear order we have:

$$Y_+(T) = Y_0 - v(T - T_0) \left(1 + \mathcal{O}\left(\frac{T - T_0}{L}\right)\right)$$

$$Y_-(T) = Y_0 + v(T - T_0) \left(1 + \mathcal{O}\left(\frac{T - T_0}{L}\right)\right)$$

where

$$v = n_0 \sqrt{1 - \frac{n_0^2}{a_0^2},}$$

$$a_0 \equiv \frac{a(Y = Y_0)}{Y_0}.$$
where the collision happens at \( T = T_0 \) and \( Y = Y_0 \). When the branes are close, the proper distance between the branes goes as \( d \sim (T - T_0) \), so the corrections are of order \( O \left( \frac{(T - T_0)^2}{L^2} \right) = O \left( \frac{d^2}{L^2} \right) \). This is the limit in which we will work all through this paper.

2. Frame where the branes are quasi-static

So far we have worked in the frame where the bulk was static. However if we are interested in the proper distance between the branes, it is more intuitive to derive it from the frame where the branes are static. Such a frame is in general complicated, however, for the purpose of this study, it is enough to consider the frame in which the branes are “quasi-static”, or static to leading order in \( d/L \) or in \( (T - T_0)/L \). In order to work in such a frame, we may perform the gauge transformation \((Y, T) \rightarrow (y, t)\):

\[
T = T_0 + \frac{t}{n_0} \cosh \left( (2y - 1) \tanh^{-1} \left( \frac{v}{n_0} \right) \right) \tag{13}
\]

\[
Y = Y_0 + t \sinh \left( (2y - 1) \tanh^{-1} \left( \frac{v}{n_0} \right) \right). \tag{14}
\]

We can indeed check that at \( y = 0 \), \( Y = Y_0 - v(T - T_0) \) and at \( y = 1 \), \( Y = Y_0 + v(T - T_0) \). In this new frame, the branes are static to leading order in \( d \) and located at \( y = 0 \) and \( y = 1 \). In terms of the new coordinates \( y \) and \( t \) the bulk geometry in the limit of close separation is:

\[
ds^2 = A(t)^2 \, dy^2 - dt^2 + a^2(dx)^2, \tag{15}
\]

with \( A(t) = 2 t \tanh^{-1} \left( \frac{v}{n_0} \right) \).

In this limit, the induced metric on the branes is then

\[
ds_\pm^2 = -dt^2 + a(t)^2 dx^2, \tag{16}
\]

so that, to leading order, the expansion of the fifth dimension with respect to the proper time \( t \) is

\[
d = 2 \tanh^{-1} \left( \frac{v}{n_0} \right). \tag{17}
\]

This result is derived directly from the full five-dimensional equations subject only to the assumption that the branes are close. In \( \S \), we have already pointed out that the Hubble parameter was bounded \( L^2 H^2 < 1 \) which is consistent with \( \Box \).

We can compare this result with the analogous relation derived from the standard low-energy effective theory which we shall briefly review in the next Section. In Section \( \Box \) we will then compare this result with the one obtained from the small-\( d \) theory which we derived in this paper and we shall see that they agree perfectly.

B. Four-dimensional effective theory at low energy

1. Low-energy effective theory

At low energies the system is well-described by a four-dimensional effective theory \( \Box \). In this limit, braneworlds behave like conventional scalar-tensor theory of gravity where the bulk effects are represented by a single scalar field. In the low-energy approximation the induced metrics on each brane are conformally related:

\[
g_{\mu\nu}^{(-)} = \Psi^2 g_{\mu\nu}^{(+)}, \tag{21}
\]

and the field \( \Psi \) is related to the proper distance \( d \) between the branes by \( \Psi = e^{-d/L} \). The system is closed and described by the modified Einstein equation:

\[
E_{\mu\nu}^{(+)} = -G_{\mu\nu}^{(+)} \tag{22}
\]

\[
= \Psi^2 G_{\mu\nu}^{(+)} + 2 \partial_\mu \Psi \partial_\nu \Psi - 2 \Psi D_\mu D_\nu \Psi \tag{23}
\]

\[
+ \left( 2 \Psi \Box \Psi - (\partial \Psi)^2 \right) g_{\mu\nu}^{(+)},
\]

where all covariant derivatives are taken with respect to \( g_{\mu\nu}^{(+)} \). This is in complete agreement with the Gauss-Codacci equations in the low-energy limit. The equation of motion for the scalar field is given by the requirement that \( E_{\mu\nu}^{(+)} \) is traceless:

\[
\Box \Psi = 0. \tag{24}
\]

In terms of the proper distance between the branes, the low-energy effective theory becomes:

\[
G_{\mu\nu}^{(+)} = \frac{2 \Psi^2}{L^2} \left[ D_\mu D_\nu d \right. \frac{1}{L^2} \left( \partial_\nu d \partial_\mu d - \frac{1}{2} (\partial d)^2 g_{\mu\nu}^{(+)} \right) \tag{25}
\]

\[
\Box d = \left( \frac{\partial d}{d} \right)^2 \frac{1}{L}. \tag{26}
\]

2. Background behaviour of the effective theory

We may now examine the behaviour of this effective theory in the special case of a flat FRW background in
order to compare the results with those of Section 11 A
We consider the special case where the positive-tension brane expands. Since the Ricci scalar vanishes, the scale factor still satisfies:
\[
\begin{align*}
d^2a_+ &= 0, \\
a_+ &= v_+ \tau + a_0,
\end{align*}
\]
where \(a_0\) is the value of the scale factor at the collision and \(v_+\) is an arbitrary constant. Comparison with the five-dimensional result \((14)\) would identify \(v_+\) as \(\sqrt{C}/L\) but, considering only the four-dimensional effective theory, its value is not determined. The Friedmann equation obtained from \((29)\) is:
\[
(1 - \Psi^2) H^2_+ - 2\dot{\Psi} \ddot{\Psi} H_+ - \dddot{\Psi} = 0. \tag{29}
\]
This is a quadratic equation with the two solutions:
\[
H_+ = \frac{\dot{d}}{L(1 \pm \Psi^{-1})}. \tag{30}
\]
The two possible signs correspond to the branes moving either in the same \((-\)\) or opposite \((+\)\) directions, which can be seen as follows. From the conformal relation \((21)\) (i.e. \(a_- = \Psi a_+\)), the Hubble parameter on the negative-tension brane can be written in terms of \(H_+\) as:
\[
H_- = \frac{\dot{a}_-}{a_-} = -\frac{\dot{d}}{L} + H_+ = \mp e^{d/\hbar} H_+ = \mp H_+ (1 + \mathcal{O}(d/L)) . \tag{31}
\]
So a \(-\) sign in \((30)\) corresponds to the situation where \(H_- \sim H_+\) for which the branes are moving in the same direction. In that case, near the collision,
\[
\dot{d} \approx -H_+ d \approx -\frac{v_+}{a_0} d, \tag{32}
\]
so the branes take an infinite time to collide. Instead, we are interested in the situation where the collision happens at finite time and the branes are moving in opposite direction \(H_- \sim -H_+\), corresponding to a \(+\) sign in \((30)\).
In this case,
\[
H_\pm = \pm \frac{\dot{d}}{2L}, \quad \text{for } d \ll L, \tag{33}
\]
which implies
\[
\dot{d} \approx \frac{2Lv_+}{a_0^2}. \tag{34}
\]
In that case we notice that \(\dot{d}\) is of order \((d/L)^0\). We may compare this result with \((20)\), which is correct to all orders in \(\dot{d}\) for small \(d/L\). As expected, the low-energy theory reproduces only the leading term. The effective theory is therefore only valid to leading order in velocities (even for the background), but for any value of \(d\).

We may point out that in this result both \(LH_+\) and \(\dot{d}\) appear to be unbounded. This is only true in this low-energy regime for which the restrictions are very strong \(LH_+ \sim \dot{d} \ll 1\). As seen in \((20)\) when \(LH_+ \sim 1\) some new restrictions have to be imposed.

Since the low-energy effective theory only predicts the leading order of velocity, it might be possible to go beyond the low energy restriction and find a theory which would be valid to all order in velocities. In order to derive such a theory, we will work in a regime where the branes are close to each other. In the rest of this paper we show the existence and consistency of such a theory which successfully reproduces \((20)\) and agrees with the low-energy theory in the regime of small separation and velocity in which they are both valid.

### III. COVARIANT APPROACH IN SMALL \(d\) LIMIT

#### A. Toy model

In order to understand the procedure we will follow to work out the theory on the brane in the close brane limit, we first examine the following one-dimensional example. We consider a second order differential equation:
\[
f''(y) = U(y)f'(y) + V(y)f(y) + W(y), \tag{35}
\]
where \(U, V\) and \(W\) are some known function of \(y\).
We assume that the function \(f\) is known at \(y = 0\) and at \(y = 1\) and we wish to find the value \(f'(0)\) of its derivative at \(y = 0\). One way to do it would be to solve the differential equation exactly with the two boundary conditions for \(f\). Once \(f(y)\) is known for any \(y\), we can infer \(f'(0)\). But by doing so we extracted more information that we actually wanted; this method would be equivalent to solving the five-dimensional Einstein equations exactly in order to obtain the induced geometry on the brane. Although this is in theory possible it would be very hard to do. Instead we will summarise in this example the method used by \((34)\). The idea is not to solve the differential equation exactly but to differentiate it in order to use it in the Taylor expansion:
\[
f(y) = 1 = \sum_{n \geq 0} \frac{1}{n!} f^{(n)}(0). \tag{36}
\]
By differentiating equation \((35)\), we can find an expression for \(f^{(n)}(y)\):
\[
f^{(n)}(y) = U_{n-1}(y)f'(y) + V_{n-1}(y)f(y) + W_{n-1}(y). \tag{37}
\]
where \( U_n, V_n \) and \( W_n \) may be found in a recursive way:

\[
\begin{align*}
U_{n+1}(y) &= U'_n(y) + V_n(y) + U(y) U_n(y), \\
U_1 &= U(y), U_0 = 1, U_{-1} = 0
\end{align*}
\]

\[
\begin{align*}
V_{n+1}(y) &= V'_n(y) + V(y) U_n(y), \\
V_1 &= V(y), V_0 = 0, V_{-1} = 1
\end{align*}
\]

\[
\begin{align*}
W_{n+1}(y) &= W'_n(y) + W(y) U_n(y), \\
W_1 &= W(y), W_0 = 0, W_{-1} = 0.
\end{align*}
\]

Using these expressions, we may write \( f'(0) \) as:

\[
f'(0) = \frac{1}{\sum \frac{1}{n!} U_{n-1}(0)} \left( f(1) - \sum \frac{1}{n!} (V_{n-1} f - W_{n-1}) |_{y=0} \right). \tag{38}
\]

Knowing \( U_n, V_n \) and \( W_n \) in a recursive way, we can perform the sums and find an exact expression for \( f'(0) \). This will be very similar to the method we will use to find the induced Weyl tensor on the brane. Although the extrinsic curvature is known on the branes, its normal derivative (which involves the Weyl tensor) is not. We can derive, however, a second order differential equation for the extrinsic curvature which allows us to calculate this derivative in the same way as \ref{38}.

However, already in this linear one dimensional problem, the recursive relations are non-trivial. The five-dimensional problem is even harder since, unsurprisingly, the equations are non linear and formidably complicated. However, if we keep only the leading terms in \( d/L \), we find that the second order differential equation for \( K_{\mu\nu} \) is linear and, remarkably, the Taylor series corresponding to \ref{38} becomes tractable. We therefore end up with an expression, correct to leading order in \( d/L \), for the normal derivative of the extrinsic curvature on the positive-tension brane which enables us to write down the Einstein equations and obtain the close brane limit of the exact theory on the brane.

\[ \text{B. Regime of validity} \]

From now on we will work in a regime where the branes are very close \( d \ll L \). As already mentioned, there are two type of solutions for the background, depending whether the branes are moving in the same or in opposite directions. If the branes are moving in the same direction, we have seen in \ref{38} that, for the background, \( d \sim d/L \). It is subject to this assumption that \ref{38} was derived. In this regime, we may check that to leading order in \( d \), they recover the low-energy effective theory. Therefore, for this regime, the low-energy effective theory is valid to all order in velocities (at small \( d \)). However from \ref{38} we see that this is not valid when the branes are moving in opposite directions. In that case, \( d \sim (\frac{d}{L})^0 \sim 1 \). This will be the solution we will be interested in for the rest of this work and we will assume the relation \( \partial_\mu d \sim (\frac{d}{L})^0 \). Although this is strictly true only for the background, if we work with perturbations in a comoving gauge, this relation will still hold. In that gauge, \( \partial_\mu d \sim (\frac{d}{L})^0 \) will still be true covariantly.

Furthermore, for adiabatic perturbations, the perturbations behave the same way as the most general background solution. For the background there are two kinds of solutions, one where the branes move in the same direction for which \( \partial_\mu d \sim \frac{d}{L} \) and one where the collision happens at finite time, for which \( \partial_\mu d \sim (\frac{d}{L})^0 \). This is true for adiabatic perturbations as well. The adiabatic perturbations will follow a similar evolution to one of the two background solutions or a superposition of them.

Since the low-energy effective theory reproduces correctly one type of solution we may focus on perturbations that follow the other kind of behaviour for which \( \partial_\mu d \sim (\frac{d}{L})^0 \).

At the level of perturbations, one might think that this procedure might break down since the perturbations diverge. But this divergence is actually logarithmic in \( d \) and is therefore negligible compared to the terms in \( 1/d \) that we will find in the theory \ref{38}. Compared to \( L/d \), \( \log(d/L) \sim (d/L)^0 \). In \ref{38}, it is actually shown that in the right gauge, the perturbations remain “small” going towards the bounce.

\[ \text{C. Gauss Codacci formalism in the frame where the branes are static} \]

\[ \text{1. Formalism} \]

In this subsection we will follow the formalism of \ref{34}. We choose coordinates where the metric is of the form

\[
d s^2 = g_{ab} dx^a dx^b = A^2(y, x) dy^2 + q_{\mu\nu}(y, x) dx^\mu dx^\nu, \tag{39}
\]

with \( x^\mu \) the transverse coordinates and \( y \) parameterising the extra dimension. In this frame, the branes located at the fixed positions \( y = 0 \) and \( y = 1 \). \( q_{\mu\nu}(y, x) \) is the induced metric of a \( y = \bar{y} = \text{const} \) hypersurface. In particular, \( g_{\mu\nu}^{(+)}(x) = q_{\mu\nu}(y = 0, x) \) and \( g_{\mu\nu}^{(-)}(x) = q_{\mu\nu}(y = 1, x) \) are the induced metrics on both branes.

The Einstein tensor on a \( y = \text{const} \) hypersurface will be written simply as \( G_{\mu\nu} \), whereas the fully five-dimensional tensor will be denoted by \( (5)G_{\mu\nu} \). Four-dimensional quantities may be expressed in terms of five-dimensional quantities by means of the Gauss-Codacci formalism \ref{34}. \ref{35}. \ref{36}. Using the bulk Einstein equations

\[
(5)G_{ab} = -\frac{6}{L^2} g_{ab}, \tag{40}
\]

the modified four-dimensional Einstein equation is:

\[
G_{\mu\nu} = \frac{3}{L^2} q_{\mu\nu} + KK_{\mu\nu} - K_{\mu\alpha} K_{\nu}^\alpha - \frac{1}{2} (K^2 - K_{\beta} K_{\alpha}^\beta) q_{\mu\nu} - E_{\mu\nu}, \tag{41}
\]
where $E_{\mu\nu}$ is the Electric part of the five-dimensional Weyl tensor (see Appendix A). All indices are raised and lowered with respect to the four-dimensional metric $g_{\mu\nu}$. For a $\mathbb{Z}_2$-symmetric brane, the extrinsic curvature $K_{\mu\nu}$ can be uniquely determined on the brane using the Israël Matching conditions, which reduce to:

$$K_{\mu\nu}(y = 0) = K_{\mu\nu}(y = 1) = -\frac{1}{L} \delta_{\mu\nu}. \quad (42)$$

Substituting this into (41) gives the projected Einstein equation on the branes:

$$G_{\mu\nu}^{(\pm)} = -E_{\mu\nu}^{(\pm)}, \quad (43)$$

where $G_{\mu\nu}^{(+)} = G_{\mu\nu}(y = 0)$ and $G_{\mu\nu}^{(-)} = G_{\mu\nu}(y = 1)$ are the induced Einstein tensors on the two branes. The aim of this work is to find $E_{\mu\nu}^{(\pm)}$ exactly in the close brane approximation. As shown in [10, 45] the Weyl tensor of this work is to find the induced Einstein tensors on the two branes. The aim where $d$ distance between the branes is defined as

$$\tilde{q}_{\mu\nu} = \text{const}$$

by $\tilde{q}_{\mu\nu}(y, x) = \int_0^y A(y', x)dy'$. The proper distance between the branes is defined as $d(x) = \tilde{d}(y = 1, x)$.

The Lie derivative $L_\alpha \equiv \partial_\alpha \equiv \frac{1}{A} \partial_\nu$ is the derivative along the normal vector of any $y = \text{const}$ hypersurface. In particular the extrinsic curvature is the derivative along the normal vector of the induced metric on such a hypersurface: $K_{\mu\nu}(y, x) = \frac{1}{A} \partial_\nu \tilde{q}_{\mu\nu}(y, x)$.

In what follows we will use a similar procedure to the toy model. Knowing the extrinsic curvature on both branes, we write an expression for its derivative using the Taylor expansion:

$$K_{\mu\nu}(y = 1) = \sum_{n \geq 0} \frac{1}{n!} \partial_\nu^n K_{\mu\nu} \bigg|_{y=0}. \quad (45)$$

In what follows, we will use the notation: $Q' \equiv \partial_\nu Q$ and $Q'^{(n)} \equiv \partial_\nu^n Q$ for any quantity $Q(y, x)$ carrying indices only along the directions of the $y$ = const hypersurface.

To start with we will consider $A$ to be independent of $y$. This is indeed the case for the background geometry of the close-brane limit. We will see in Appendix A that this assumption does not affect the final answer. In that case $d(y, x) = yA(x)$, and in particular the proper distance between the branes is: $d(x) = A(x)$. We can now rewrite the expression (41) in the form:

$$K_{\mu\nu}' = -dE_{\mu\nu}' - D_{\mu}D_{\nu}d - dK_{\mu\nu}' + \frac{d}{L^2} \delta_{\mu\nu}. \quad (46)$$

In order to find the expression for $K_{\mu\nu}^{(n)}$ in (45) we need the derivative of the Weyl tensor and of the Christoffel symbol. The last one is given by:

$$\Gamma_{\mu\nu}' = D_{\mu}(d K_{\nu}^\alpha) + D_{\nu}(d K_{\mu}^\alpha) - D^\alpha(d K_{\mu\nu}). \quad (47)$$

For the Weyl tensor derivative, we may use the result derived in Appendix A:

$$E_{\mu\nu}' = d\left(2K_{\alpha}^\alpha E_{\mu}^\alpha - \frac{3}{2} K E_{\mu}^\nu \right) - \frac{1}{2} K_{\beta}^\beta E_{\nu}^\beta \delta_{\mu\nu} + C_{\alpha\nu\beta} K^\alpha \beta + (K^3)_{\nu}^\mu \frac{1}{2d} D^\alpha \left[d^2 D^\mu K_{\alpha\nu} + d^2 D_{\nu} K_{\mu}^\alpha - 2d^2 D_{\alpha} K_{\nu}^\mu \right].$$

where $(K^3)_{\nu}^\mu$ are some cubic terms in the traceless part of the extrinsic curvature which exact form will not be relevant for the purpose of this study (they vanish at $y = 0$ and $y = 1$). Both these relations (47) and (48) are valid for any $y$. On the brane they simplify considerably:

$$\Gamma_{\mu\nu}'(0) = -\frac{1}{L} \left(d_{\mu} \delta_{\nu}^\alpha + d_{\nu} \delta_{\mu}^\alpha - d^\alpha g_{\mu\nu}^{(+)} \right), \quad (49)$$

$$E_{\mu\nu}'(0) = \frac{4d}{L} E_{\nu}^\mu(0). \quad (50)$$

The important point is that the system is now closed. Writing $E_{\mu\nu}$ back in terms of the extrinsic curvature and its derivative, we obtain a second order differential equation for the extrinsic curvature which is non-linear but which involves four-dimensional quantities only. We may therefore apply the procedure of Section II A. In order to do so we will assume the branes to be close and keep only the leading order in $d$ in the expansion.

2. Small $d$ approximation

In what follow we denote by $0K_{\mu\nu}^{(n)}(x)$ the leading order in $d/L$ of $K_{\mu\nu}^{(n)}(y = 0, x)$, symbolically,

$$K^{(n)} = 0K^{(n)}(1 + O(d/L)). \quad (51)$$

Using the small distance approximation, we rewrite the expansion (45) as:

$$\sum_{n \geq 1} \frac{1}{n!} 0K_{\mu\nu}^{(n)} = 0, \quad (52)$$

where the $n = 0$ term cancel since the extrinsic curvature is the same on both branes. All the terms will be kept in the sum only if they are all of same order. We shall see in what follows that this is indeed the case. In order to do so, we will work in a recursive way.

3. Example for the $n=1$ and $n=2$ case

We first concentrate on the $n = 1$ and $n = 2$ cases in detail in order to gain insight; the technicalities of the general $n$ case are left for Appendix A.

For $n = 1$,

$$K_{\mu\nu}'(y = 0) = -dE_{\nu}^\mu - D_{\mu}D_{\nu}d |_{y=0}. \quad (53)$$
Since $\partial_\alpha \partial_\beta d \sim \partial_\alpha d \sim d^0$ the second term goes as $d^0$. In the effective theory, on the brane, $E_\mu^\nu \sim d^{-1}$ (using (23)). Although we have argued that at high energy the effective theory does not give the exact expression for the Weyl tensor, we have seen that (at least for the background) the behaviour is the same, differing only in corrections at higher order in the velocity. In particular $E_\mu^\nu$ should go as $d^{-1}$ at high energies as well (we will check later that this is indeed the case). We therefore have $K_\nu^\mu(y = 0) \sim d^0$.

For the second derivative we have:

$$K''_\nu^\mu(y) = -dE_\nu^\mu - q^{\mu\beta} D_\beta D_\nu d + q^{\mu\beta} \Gamma^\alpha_{\beta\nu} \partial_\alpha d - d \partial_\nu (K''_\nu^\mu).$$  

(54)

On the positive-tension brane we may compare these terms with the ones from $K''_\nu^\mu(y = 0)$:

- terms from $K''$:
  - $dE_\nu^\mu = \frac{d}{dL} E_\nu^\mu \ll dE_\nu^\mu$,

- $q^{\mu\beta} D_\beta D_\nu d = \frac{d}{dL} D_\nu D_\nu d \ll D_\nu D_\nu d$,

- $d \partial_\nu (K''_\nu^\mu) \ll K''_\nu^\mu$,

where we used $q^{\mu\beta} = d \partial_\nu q^{\mu\beta} = -2dK_\mu^{\beta\nu}$. When $d \ll L$, the only term which is not negligible in comparison to the terms present in $K''_\nu^\mu(y = 0)$ is the one coming from the derivative of the Christoffel symbol:

$$q^{\mu\beta} \Gamma^\alpha_{\beta\nu}(0) \partial_\alpha d = -\frac{1}{L} \left(2q^{\mu\beta} \partial_\alpha d - (\partial d)^2 \delta_\nu^\mu\right) (56) \sim K''_\nu^\mu(0).$$

This last term will give a contribution of the same order as $K''_\nu^\mu(y = 0)$.

4. General $n$ case

In the previous specific case, we had $0\ K_{(1)} \sim d^0$ and $0\ K_{(2)} \sim d^0$. In Appendix A we will show that the same is true for any $n$. In particular we will show that the leading contribution in $K_\nu^\mu(n)(y = 0)$ comes from the $(n - 1)^{th}$ derivative of this Christoffel symbol:

$$0\ K_\nu^\mu(n) = q^{\mu\beta} \Gamma^\alpha_{\beta\nu}(n-1) \partial_\alpha d,$$

(57)

and the leading contribution from the $(n - 1)^{th}$ derivative of the Christoffel symbol comes from the term:

$$\Gamma^\alpha_{\beta\nu}(n-1)(0) = d_\beta \ 0\ K_\nu^\alpha(n-2) + d_\nu \ 0\ K_\beta^\alpha(n-2) - d^{\alpha} \ 0\ K_{(n-2)}^\nu,$$

(58)

Using these two results, we therefore have:

$$0\ K_\nu^\mu(n) = d^{\mu} \ 0\ K_\nu^\alpha(n-2) + d_\nu \ 0\ K_\beta^\alpha(n-2) - d^{\alpha} \ 0\ K_{(n-2)}^\nu.$$

(59)

In the Taylor expansion, the term $0\ K_\nu^\mu(n)$ therefore comes in with the same contribution as the term $0\ K_\nu^\mu(n-2)$, so all terms have to be considered.

5. Expression for the Weyl tensor

We now use these results in the Taylor expansion. First, we may define the operator $\hat{O}$ such that:

$$\hat{O}Z_\nu^\mu = [d^{\mu} Z_\alpha^\nu + d_\nu Z^{\mu\alpha} - d^{\alpha} Z^{\nu\mu}] d_\alpha,$$

(60)

for any four-dimensional tensor $Z_\nu^\mu$. Using this notation, the Taylor expansion simplifies to:

$$\sum_{n \geq 0} \frac{1}{(2n + 1)!} \hat{O}^{(n)} 0K''_\nu^\mu = \sum_{n \geq 1} \frac{1}{(2n)!} \hat{O}^{(n)} 0K''_\nu^\mu$$

(61)

where may be symbolically written as:

$$\frac{1}{\sqrt{O}} \sinh \sqrt{O} \ 0K''_\nu^\mu = \frac{1}{L} \left( \cosh \sqrt{O} - 1 \right) \delta_\nu^\mu. $$

(62)

To leading order in $d$, the derivative of the extrinsic curvature on the positive-tension brane is then:

$$K''_\nu^\mu(y = 0) = \left[ \frac{\sqrt{O}}{L} \tanh \left( \frac{\sqrt{O}}{2} \right) \right] \delta_\nu^\mu + \frac{1}{L} O \left( \frac{d}{L} \right)$$

(63)

with

$$F_\nu^\mu \equiv \hat{O} \delta_\nu^\mu = 2\partial_\nu d_\alpha d - (\partial d)^2 \delta_\nu^\mu$$

(64)

$$\hat{O} F_\nu^\mu = (\partial d)^4 \delta_\nu^\mu.$$  

(65)

Therefore to leading order in $d/L$, the Weyl tensor on the positive-tension brane is:

$$E^{(+)}_\nu^\mu = -\frac{1}{d} K''_\nu^\mu(y = 0) - D_\mu D_\nu d +$$

$$- \frac{D_\mu D_\nu d}{d}$$

(66)

$$- \frac{1}{2dL} \left[ \partial d \left( \tanh \frac{\partial d}{2} - \tanh \frac{\partial d}{2} \right) \delta_\nu^\mu + \frac{1}{\partial d} \left( \tanh \frac{\partial d}{2} + \tanh \frac{\partial d}{2} \right) F_\nu^\mu \right] + \frac{1}{dL} O \left( \frac{d}{L} \right),$$

with $|\partial d| \equiv \sqrt{-g^{(+)}_\alpha^\mu \partial_\mu \partial_\alpha d}$ and $F_\nu^\mu = 2\partial_\nu d_\alpha d - (\partial d)^2 \delta_\nu^\mu$.

Here the covariant derivative and the raising of indices are performed with respect to $g^{(+)}_\alpha^\mu = q_{\mu\nu}(y = 0)$. Note that, because of the modulus signs, we have

$$(\partial d)^2 \equiv g^{(+)}_\alpha^\mu \partial_\mu \partial_\alpha d = -|\partial d|^2. $$

(67)

We may note that these functions are well defined only for $|\partial d| < \pi$. For $|\partial d| \geq \pi$, the inverse of the power expansion on the left hand side of (61) is ill defined. For the
purpose of this study we shall therefore restrict ourselves
to the case where $|\partial d| < \pi$ for which the expressions \[66\] is well defined. Whether or not the theory possesses ac-
tual velocity-dependent divergences as $|\partial d| \rightarrow \pi$ would be
interesting to investigate with numerical simulations.

All this analysis could be repeated using the negative-
tension brane as reference brane instead. This would be
equivalent to taking $-d$ instead $d$ all the way through.
We would have then obtained the induced Weyl tensor
on the negative-tension brane $E^{(-)}_{\nu}^\mu$ as

\[ E^{(-)}_{\nu}^\mu = -\frac{D^\mu D_\nu d}{d} + \frac{1}{2dL} \left[ |\partial d| \left( \tanh \frac{|\partial d|}{2} - \tan \frac{|\partial d|}{2} \right) \delta^\nu_\mu \right. \]
\[ + \left. \frac{1}{|\partial d|} \left( \tanh \frac{|\partial d|}{2} + \tan \frac{|\partial d|}{2} \right) F^\nu_\mu \right] + \frac{1}{dL} \mathcal{O}\left( \frac{d}{L} \right), \]

describing gravity coupled to a scalar field in a non-trivial
way. This theory has some higher-derivative corrections as
expected but, remarkably, they are both simple in form and
entirely first order, involving only powers of first derivatives.
This leads to the important result that the theory remains second order in derivatives even when
correct to all orders in velocity. The initial data on a
Cauchy surface that needs to be specified for the theory
to be solved is the same as that needed in the low-energy
effective theory. There will be no need to specify extra
information or to consider the corrections to be small,
a feature which will make the theory straightforward to
solve.

In order to solve this theory, we need to specify the
equation of motion of the scalar field. So far the traceless
property of the Weyl tensor has not been used, which
gives rise to the modified Klein Gordon equation for the
radion:

\[ \Box d = \frac{|\partial d|}{L} \left[ \tan \frac{|\partial d|}{2} \right. \]
\[ \left. - 3 \tanh \frac{|\partial d|}{2} \right] + \frac{1}{L} \mathcal{O}\left( \frac{d}{L} \right). \]

Using this equation of motion for the scalar field and the
modified Einstein equation \[69\], the geometry on the
brane can be found in the close-brane limit. We may
point out that only four-dimensional quantities are in-
volved. Although this theory does not seem to be derivable
from a four-dimensional action, it may be solved
using standard four-dimensional methods.

**B. Low-energy limit**

In the low-energy limit, the four-dimensional effective
theory presented in Section 1.1.1 is exact. For consist-
tency we may check that, in the small velocity limit, we
recover the small distance limit of this theory. To leading
order in velocity, we may use $\tan \frac{d}{2L} \approx \tanh \frac{|\partial d|}{2} \approx \frac{1}{2} |\partial d|$ in \[69\] and \[70\], giving rise to the modified Klein Gordon equation and the equation of motion for the scalar field

\[ G^{(+)}_{\mu \nu} = \frac{D_\mu D_\nu d}{d} + \frac{1}{dL} \left( \partial_\mu \partial_\nu d - \frac{1}{2} (\partial d)^2 g^{(+)}_{\mu \nu} \right) \]
\[ \Box d = \frac{(d d)^2}{L}. \]

which is precisely the leading order in $d/L$ of the effective
theory \[24\] \[26\]. The equation of motion for the scalar field is actually exact to all orders in $d/L$.

We might think that the low-energy effective theory
\[26\] could still give a correct answer provided that the relation between $\Psi$ and $d$ was modified. If that was the
case, the relation between these two variables should in-
clude some higher order velocity terms: $\Psi = \Psi(d, \partial d)$ such that, in the low-velocity limit, $\Psi(d, \partial d) \sim e^{-d/L}$. The
theory should then include some terms of higher order
in derivative of the form $\partial^\alpha d D_\mu \partial_\nu d \Psi \partial d$. This is
not compatible with the theory in \[69\] which only con-
tains terms up to second order in derivatives. The close
limit theory in \[69\] and the low-energy effective theory
\[26\] are therefore genuinely different.
C. Conservation of Energy

In order for the theory to be consistent, the Bianchi identity on the brane needs to be preserved. This means that, to leading order in $d/L$, the traceless condition must imply the right hand side of $G_{\mu\nu}$ to be transverse. In other words, if we consider the right hand side of $\bar{E}_{\mu\nu}$ to be the stress-energy tensor of the scalar field $d$, the Klein Gordon condition should impose that it is conserved, i.e. divergenceless. Using the equation of motion for the scalar field, we may rewrite $\bar{E}_{\mu\nu}$ and $(\bar{E}_{\mu\nu})_{\text{leading order}} \equiv \bar{E}_{\mu\nu}$ in the form:

$$G_{\mu\nu}^{(+)} = T_{\mu\nu} - \bar{E}_{\mu\nu}$$

where for simplicity we used the notation $z = |\partial d|$, $f \equiv f(z^4) = \frac{2}{3} (\tanh \frac{z}{2} - \tan \frac{z}{2})$ and $g \equiv g(z^4) = \frac{1}{2} (\tanh \frac{z}{2} + \tan \frac{z}{2})$. $f$ and $g$ are viewed as functions of $z^4 = (\partial d)^4$ since their Taylor expansions only contain powers of $z^4$. Therefore, for example,

$$\partial_\mu f(z) = 2z^2 f'(z) \partial_\mu (z^2).$$

We have as well added the next-order correction $d\bar{E}_{\mu\nu}$ where $\bar{E}_{\mu\nu} \sim d^9$. As we shall see, although $\bar{E}_{\mu\nu}$ does not contribute to the leading order of the Einstein equation, it does contribute to the leading order of the divergence of $T_{\mu\nu}$. This is due to the fact that, although $d\bar{E}_{\mu\nu} \sim d$ is negligible, its derivative is not $(\partial_\mu d) \bar{E}_{\nu\nu} \sim d^3$. We therefore need to consider its contribution as well. The transverse requirement will therefore impose a condition on the next to leading order in $d$. Using the result of Appendix B, the divergence of the stress-energy tensor as defined in $(21)$ is given by:

$$dD_\nu T^\nu_\mu = -\frac{12}{L^2} \left[ (g' + z^2 g) (f' + z^2 g') \right] \partial_\nu d$$

$$- \left[ \bar{E}^\nu_\nu - \bar{E} \delta^\nu_\nu \right] \partial_\nu d + \frac{1}{L^2} \mathcal{O}(d/L) \tag{75}$$

For the stress-energy to be transverse to leading order in $d$ we therefore have a constraint equation on the next-to-leading order contribution $\bar{E}_{\mu\nu}$. Furthermore, from the low-energy effective theory (using $(25)$), to leading order in velocities and in $d/L$, $\bar{E}_{\mu\nu}$ should be:

$$\bar{E}_{\mu\nu} = \frac{1}{L^2} \left[ D_\mu D_\nu d + \frac{1}{L} \left( \partial_\mu d \partial_\nu d - \frac{1}{2} (\partial d)^2 g_{\mu\nu}^{(+)} \right) \right] + \frac{1}{L^2} \mathcal{O}(d/L). \tag{76}$$

A natural ansatz for $\bar{E}_{\mu\nu}$ is therefore:

$$\bar{E}_{\mu\nu} = \frac{1}{L^2} \left[ D_\mu D_\nu d + \frac{1}{L} \left( \bar{f}(z^4) g_{\mu\nu}^{(+)} + \bar{g}(z^4) F_{\mu\nu} \right) \right]. \tag{77}$$

Using this ansatz in the constraint $(75)$, we find the equations for $\bar{f}$ and $\bar{g}$:

$$\bar{f}(z^4) = f + 4g f + 4z^4 (f g f' - g f') \tag{78}$$

$$\bar{g}(z^4) = g + 4f f' - 4z^4 g g'. \tag{80}$$

In the slow-velocity limit, we may check that $\bar{f} = \mathcal{O}(\partial d)^4$ and $\bar{g} = \frac{1}{2} + \mathcal{O}((\partial d)^4)$ so we recover the result from the low-energy effective theory $(76)$. Furthermore, using the expression $(77)$, the next to leading order $\bar{E}_{\mu\nu}$ will be conformally related to the leading order $E_{\mu\nu} = \frac{1}{L^2} \bar{E}^{\mu\nu}_{\nu\nu}$ only if $\bar{f} = f$ and $\bar{g} = g$. But from the expressions $(79)$ $(80)$, this can only be the case in the low-velocity limit where $\bar{f} = f = -\frac{1}{8} g + \mathcal{O}(z^8)$ and $\bar{g} = g = \frac{1}{2} + \mathcal{O}(z^8)$. The higher order terms in velocities do not match and the conformal relation $\bar{E}_{\mu\nu} = \frac{d}{d \bar{E}_{\mu\nu}}$ does not hold.

This expression for $\bar{E}_{\mu\nu}$ with the relations $(79)$ $(80)$ is consistent with the close limit theory since it ensures that the stress-energy tensor is conserved to leading order in $d/L$. However we may emphasise that this is not the only possible answer and relies strongly on the ansatz $(77)$. Since we only want to focus in the leading order in $d/L$ of the theory, the exact expression of $\bar{E}_{\mu\nu}$ will not be relevant, it is enough to show that it is possible to introduce its contribution in such a way that the transverse condition is preserved. If we wanted to go further in the study of the next to leading order term of the theory, we should check that this expression of $\bar{E}_{\mu\nu}$ is consistent with its exact expression that can be derived in the background using the SADS five-dimensional geometry. This is beyond the purpose of this work which aims to focus on the close brane limit and to neglect any terms which are not of leading order in the expansion.

D. Consistency check for the background

Having a theory which should be exact in the close brane limit, we may now check that it correctly reproduces the exact background behaviour obtained in $(10)$
Taking the positive-tension brane to be endowed with a flat FRW induced metric with scale factor $a_+(t)$, the Einstein equations \[69\] reduce to
\[
3H_+^2 = \frac{d}{d} + \frac{d}{L} \tan \frac{d}{2} \quad (81)
\]
and the Klein Gordon equation \[70\] gives
\[
\ddot{d} + 3H_+ \dot{d} = \frac{d}{L} \left( 3 \tanh \frac{d}{2} - \tan \frac{d}{2} \right). \quad (83)
\]
Combining \[81\] and \[83\] gives
\[
H_+^2 = \frac{d}{d} \left( \frac{1}{L} \tanh \frac{d}{2} - H_+ \right). \quad (84)
\]
Also we have immediately that $a_+(t)$ is linear in conformal time (which is merely a consequence of the induced metric being FRW with empty branes). As with the analogous low-energy result \[20\], \[84\] has two solutions, one of which has $\dot{d} = 0$ at the collision and for which both $d$ and $H_+$ are finite and non-vanishing. Again, it is the latter we consider since this is the one corresponding to branes moving in opposite directions. Therefore for this study both $H_+$ and $a_+$ are finite in the limit $d \to 0$, giving the leading order behaviour of $H_+$ from \[84\] as
\[
H_+ = \frac{1}{L} \tanh \frac{d}{2} + O \left( \frac{d}{L^2} \right). \quad (85)
\]
Replacing $d$ with $-d$ will give the corresponding result for the negative-tension brane:
\[
H_- = - \frac{1}{L} \tanh \frac{d}{2} + O \left( \frac{d}{L^2} \right). \quad (86)
\]
The modified Friedmann equations on both branes \[84\] are precisely the exact result \[20\] we obtained from solving the five-dimensional equations in Section 11A. This is a non-trivial check on the validity of this close-brane four-dimensional theory as it reproduces the right result to all order in velocities in the small distance limit and not only to leading order in $d$ as was the case for the low-energy effective theory.

V. EXTENSIONS

A. Introduction of a potential to low-energy effective theory

The low-energy presented in Section 11A was expressed in the positive-tension brane frame. In order to add a potential to this theory, it is more natural to work in the Einstein frame. The Einstein frame is related to the brane frame by the conformal transformation:
\[
\bar{g}_{\mu\nu}^{(+)} = \left( \cosh \left( \phi/\sqrt{6} \right) \right)^2 \bar{g}_{\mu\nu}.
\]
\[
\bar{g}_{\mu\nu}^{(-)} = \left( - \sinh \left( \phi/\sqrt{6} \right) \right)^2 \bar{g}_{\mu\nu}.
\]
where any “bar” quantity designates a quantity with respect to the Einstein frame. The conformal scale factor in \[20\] is related to the scalar field $\phi < 0$ by $\Psi = e^{-d/L} = - \tanh \left( \phi/\sqrt{6} \right)$. In that frame, the four-dimensional effective theory can be derived from the effective action of a scalar field minimally coupled to gravity:
\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{2} \bar{R} - \frac{1}{2} \left( \partial \phi \right)^2 \right), \quad (89)
\]
giving rise to the equation of motion for the scalar field:
\[
\Box \phi = 0, \text{ Cf. } [9, 48].
\]
It is possible to modify this effective theory by adding a potential by hand in \[89\]:
\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{2} \bar{R} - \frac{1}{2} \left( \partial \phi \right)^2 - U(\phi) \right). \quad (90)
\]
In that case the equation of motion for the scalar field is modified to:
\[
\Box \phi = U'(\phi), \quad (91)
\]
and the Einstein equation in Einstein frame becomes:
\[
\bar{G}_{\mu\nu} = \partial_{\mu} \phi \partial_{\nu} \phi - \left( \frac{1}{2} \left( \partial \phi \right)^2 + U \right) \bar{g}_{\mu\nu} \quad (92)
\]
We can now go back to the positive-tension brane frame by performing the conformal transformation \[84\] with $\left( \cosh \left( \phi/\sqrt{6} \right) \right)^2 \simeq \frac{L}{d}$. To linear order in $d/L$, the theory on the brane is modified to:
\[
G_{\mu\nu}^{(+)} = \frac{1}{d} \left[ \partial_\mu \partial_\nu d - \Box d \bar{g}_{\mu\nu}^{(+)} \right]
\]
\[
+ \frac{1}{L} \left( \partial_\mu d \partial_\nu d + \frac{1}{2} \left( \partial d \right)^2 \bar{g}_{\mu
u}^{(+)} \right) - V(d) \bar{g}_{\mu\nu}^{(+)} \quad (93)
\]
\[
\Box d = \frac{\left( \partial d \right)^2}{L} + \frac{2}{3} \left( d V'(d) - 2 V(d) \right), \quad (94)
\]
where all covariant derivatives and index raising is taken with respect to $g_{\mu\nu}^{(+)}$ and for simplicity we used the potential $V(d)$ defined as:
\[
V(d) = d \left( 1 - \Psi^2 \right) U \simeq \frac{d^2}{L} U. \quad (95)
\]
The modification of the equation of motion of the scalar field \[95\] ensures the right hand side of the Einstein equation \[89\] to stay conserved even after the addition of a potential. The same procedure can now be applied to the theory in the close brane limit.
B. Addition of a potential in the close brane theory

We consider the addition of a potential in the close brane theory by modifying the stress-energy tensor in the Einstein equation \( T_{\mu \nu} \).

\[
T_{\mu \nu} = \frac{1}{d} \left[ D_\mu D_\nu d + \frac{2}{L} g \partial_\mu d \partial_\nu d - d \hat{E}_{\mu \nu} \right] - \left[ \frac{1}{d} z^2 g + \frac{3f}{L} - d \hat{E} + V(d) \right] g_{\mu \nu}^{(+)} .
\]

(96)

For this stress-energy to remain conserved after the addition of the potential, the equation of motion of the scalar field \( \phi \) has to be modified as well:

\[
\Box \phi = - \frac{1}{L} \left( 2z^2 g + 4f \right) + d \hat{E} + W ,
\]

(97)

where the correction term \( W \) is a functional of the potential \( V(d) \) to be determined. The modified Klein Gordon equation \( \Box \phi = 0 \) should be consistent with the conservation of the modified stress-energy tensor in (96). The divergence of \( T_{\mu \nu} \) is calculated in the appendix B using the constraint (13) for \( \hat{E}_{\mu \nu} \):

\[
d D_\mu T^\mu_\nu = \partial_\mu \left[ - V' \right] + \partial_\nu \left[ 3W + 4V \right] + \frac{2d}{L} \left( 3W + 2V \right) + \frac{12}{L} z^2 \left( f' + z^2 g' \right) \left( V + W \right) + \frac{1}{L^2} O(d/L) .
\]

(98)

To lowest order in \( d/L \) we therefore need to modify the equation for the \( d \) with a term \( W \):

\[
W = \frac{2}{3} \left( 2z^2 g + 4f \right) - 2V(d) .
\]

(99)

This is precisely the same term which had to be introduced in the equation (13) for \( d \) in the low-energy theory. This procedure is therefore completely consistent with the low-energy theory and will give the same result to leading order in \( \partial d \). It is possible to modify the close brane theory by introducing a potential, in such a way that the conservation of energy and thus the Bianchi identity remain unaffected.

C. Detuned tensions

A full analysis of how the argument of Section IV would run in the presence of matter on the branes is beyond the scope of this paper. We can, however, consider the simple example of brane cosmological constants where the stress-energy tensors are proportional to the induced metric to get an idea for how the coupling to matter might look. Specifically, we allow now for a detuning of the tensions to

\[
\lambda_+ = \frac{6}{L} + \sigma_+ \quad \lambda_- = -\frac{6}{L} + \sigma_- .
\]

(100)

The analysis is only slightly modified from the above and we shall only sketch it here for brevity. Using (13), (92) is modified to

\[
\sum_{n \geq 1} \frac{1}{n!} \delta K^{(+)(n)}_{\mu \nu} = \frac{1}{6} \left( \sigma_+ + \sigma_- \right) \delta \|_\nu ,
\]

(101)

which gives rise to a more complicated version of (66):

\[
K^{(+)}_{\mu \nu} \left( y = 0 \right) = \left( \frac{1}{L} + \frac{\sigma_+}{6} \right) \left[ \sqrt{O} \tanh \left( \frac{\sqrt{O}}{2} \right) \right] \delta \|_\nu \\
+ \frac{\sigma_+ + \sigma_-}{6} \frac{\sqrt{O}}{\sinh \sqrt{O}} \delta \|_\nu .
\]

(102)

The analogous result to (66) is then

\[
E^{(+)}_{\mu} = -\frac{1}{d} K^{(+)}_{\nu} \left( y = 0 \right) - \frac{D^\mu D_\nu d}{d} - \frac{\delta \|_\mu}{3} \left( \frac{\sigma_+}{L} + \frac{\sigma_-}{12} \right) ,
\]

(103)

\[
= - \frac{D^\mu D_\nu d}{d} \\
- \frac{1}{2d} \left[ \frac{1}{L} + \frac{\sigma_+}{6} \right] \left[ \left| \partial d \right| \left( \tanh \left( \frac{\partial d}{2} \right) - \tanh \left( \frac{\partial d}{2} \right) \right) \delta \|_\mu \\
+ \frac{1}{|\partial d|} \left( \tanh \left( \frac{|\partial d|}{2} \right) + \tanh \left( \frac{|\partial d|}{2} \right) \right) F^\mu_\nu \right] \\
+ \frac{1}{3} \left[ \left| \partial d \right| \left( \cosh \left| \partial d \right| + \cosh \left| \partial d \right| \right) \delta \|_\mu \\
+ \frac{1}{|\partial d|} \left( \cosh \left| \partial d \right| - \cosh \left| \partial d \right| \right) F^\mu_\nu \right] \\
- \frac{\delta \|_\mu}{3} \left( \frac{\sigma_+}{L} + \frac{\sigma_-}{12} \right) + \frac{1}{dL} O \left( \frac{d}{L} \right) .
\]

(102)

The Einstein and Klein-Gordon equations are then modified to include complex couplings of the tensions to both the radion \( d(x) \) and powers of its first (but not higher) derivative.

\[
G^{(+)}_{\mu \nu} = -E^{(+)}_{\mu \nu} - \frac{1}{d} g^{(+)}_{\mu \nu} \left( \frac{1}{L} \sigma_+ + \frac{1}{12} \sigma_-^2 \right) \\
= \frac{D_\mu D_\nu d}{d} + \frac{1}{d} \left( \frac{1}{L} + \frac{\sigma_+}{6} \right) \left[ \left| \partial d \right| \tanh \left( \frac{\partial d}{2} \right) g^{(+)}_{\mu \nu} \right] \\
+ \frac{1}{\left| \partial d \right|} \left( \tanh \left( \frac{|\partial d|}{2} \right) + \tanh \left( \frac{|\partial d|}{2} \right) \right) \partial_\mu d \partial_\nu d \\
+ \Sigma_{\mu \nu} + \frac{1}{d} O \left( \frac{d}{L} \right) 
\]

(104)

\[
\Sigma_{\mu \nu} = \frac{\left( \sigma_+ + \sigma_- \right)}{6d} \left[ \left| \partial d \right| \cosh \left| \partial d \right| g^{(+)}_{\mu \nu} \right] \\
+ \frac{1}{\left| \partial d \right|} \left( \cosh \left| \partial d \right| - \cosh \left| \partial d \right| \right) \partial_\mu d \partial_\nu d \\
- \frac{2}{3} g^{(+)}_{\mu \nu} \left( \frac{\sigma_+}{L} + \frac{\sigma_-}{12} \right) 
\]

(105)
In the low-energy limit, these reduce to
\[
G_{\mu\nu}^{(+)} = \frac{D_{\mu}D_{\nu}\partial d}{d} + \frac{1}{dL} \left( \partial_{\nu}d\partial_{\nu}d - \frac{1}{2} (\partial d)^2 g_{\mu\nu}^{(+)} \right) + \frac{\sigma_+ + \sigma_-}{6d} g_{\mu\nu}^{(+)} + \frac{1}{d} \mathcal{O}(d/L, \partial d^4, \sigma_{\pm} \partial d^2) \tag{107}
\]
\[
\Box d = \frac{(\partial d)^2}{L} - \frac{2}{3} (\sigma_+ + \sigma_-). \tag{108}
\]
which agree with the standard results of the low-energy effective theory in the presence of detuned tensions \[27\].

**VI. CONCLUSION**

In the first part of this work we pointed out the discrepancy between predictions from the effective four-dimensional low-energy theory and the exact five-dimensional results. The difference has been established in the regime where the branes were close and all through this paper we worked to leading order in $d/L$.

In order to go beyond the low-energy effective theory, we established a formalism to find the electric part $E_{\mu\nu}$ of the five-dimensional Weyl tensor on both branes. This tensor represents the only quantity which is unknown from a four-dimensional point of view as it encodes information about the bulk geometry. Finding its expression on the brane is therefore the key element in order to study the geometry of a brane within an extra dimension.

Using this formalism in the small distance limit, we found an exact expression for the Weyl tensor on each brane, valid at leading order in $d/L$ but at all orders in velocities, or for any energy scale. We were therefore able to modify the regime of validity of the effective theory from a low-energy regime valid independently of the branes distance to a regime valid at all energies for close branes. This regime of validity is relevant for cosmology since it represents one of the main focus of present braneworld models. Understanding the behaviour of branes just before or just after a collision is indeed crucial if our Universe emerged from a brane collision. If the Big Bang were initiated by such a collision, it is consistent to assume that the large scale structure was produced in a regime where both branes were close. Even if this regime is not valid after a while, effects that originated during this period are unlikely to be eliminated once the branes are far apart and the fifth dimension has opened up, leaving the possibility of exciting the Kaluza Klein modes. We therefore believe that understanding the consequences of cosmologies in this limit will allow us to study the viability of such scenarios by comparing them with observational data \[17, 21, 33, 49, 50, 51\].

In this paper we have established a theory that allows us to study such scenarios and have checked the consistency of its predictions for homogeneous and isotropic geometries. We argue that this theory will be remarkably straight-forward to use since it includes only four-dimensional quantities and is effectively second order in derivatives, the only higher-order corrections coming as powers of first derivatives. This feature will facilitate any comparison with other four-dimensional theories. It is different from other higher-derivative theories \[16, 52, 53\] in that it is not derived using the assumption that it can be derived from an action, and it is different from a theory relying on purely string effects \[54, 55, 56\]. Already for the background, an interesting result can be pointed out: when the branes are empty the Hubble parameter on each brane is bounded by $L^2H^2 < 1$, which could not have been derived from the low-energy effective theory. This does not hold anymore when some matter is introduced on the branes or if a potential is added for the radion. Another interesting result of the theory is that the expansion seems to break down when the velocities are of order $|\partial d| \sim \pi$. This is a feature which has never been pointed out before and we believe it suggests the presence of interesting physics which might be worth studying.

To extend the theory, we considered the addition of a potential for the radion. This will be interesting if we want to study curvature perturbations as the origin of the large scale structure. In order to begin to understand the way matter on the brane would be coupled in this theory we have extended the model to the case where the brane tensions are detuned. Both these extensions will be useful for further progress in the understanding of braneworld cosmology after or before a collision.

**Acknowledgements**

We would like to thank Anne Davis for supervising this work and for her comments on the manuscript. SLW is supported by PPARC and would like to thank James O’Dwyer and David Richards for pointing out the breakdown of the expansion as $|\partial d| \rightarrow \pi$. CdR is supported by DAMTP and would like to thank Andrew Tolley for useful discussions throughout the development of this paper.

**APPENDIX A: LEADING ORDER DERIVATIVE OF THE EXTRINSIC CURVATURE**

1. **Gauge where $A$ is $y$-independent**

The purpose of this Section is to formulate the recursive expression for the derivatives of the extrinsic cur-
vature on the brane in the regime where the branes are close $d \ll L$.

We recall from Section [1] that on the positive tension brane, for $y = 0$, we have $K^{(m)} \sim d^0$ for $m = 0, 1, 2$. Furthermore, from [2], at $y = 0$, we have $\Gamma' \sim \partial d \sim d^0$. We may as well calculate its second derivative:

$$
\Gamma_{\mu\nu}''(y) = (dK_{\nu}^{(\alpha')})_{\mu} + (dK_{\mu}^{(\alpha')})_{\nu} - (dK_{\mu\nu})^{(\alpha)} (A1) \\
+ d \left( \Gamma^{\alpha}_{\mu\nu} K^{\rho}_{\nu} + \Gamma^{\alpha}_{\nu\rho} K^{\mu}_{\rho} - 2 \Gamma^{\alpha}_{\mu\rho} K^{\rho}_{\nu} \right) \\
- d q^{\alpha\beta} q_{\mu\nu} \left( \Gamma^{\sigma}_{\beta\rho} K^{\rho}_{\nu} - \Gamma^{\rho}_{\beta\nu} K^{\sigma}_{\rho} \right) \\
- d \left( q^{\alpha\beta} q_{\mu\nu} \right)' \left( \Gamma^{\sigma}_{\beta\rho} K^{\rho}_{\nu} - \Gamma^{\rho}_{\beta\nu} K^{\sigma}_{\rho} \right). 
$$

On the brane the last line is of order $(d/L)^2$ and the two middle ones are of order $d/L$. They are therefore negligible. The first line gives a contribution of order $\Gamma^{(m)}(0) \sim \partial dK^{(m)}(0) \sim d^0$:

$$
\Gamma^{(m)}(0) = \partial_{\mu} dK_{\nu}^{(m)}(0) + \partial_{\nu} dK_{\mu}^{(m)}(0) - \partial^\alpha d K_{\mu\nu}^{(m)}(0)(A2)
$$

where terms of order $\frac{d}{L}$ have been omitted.

In what follows we will use a symbolic notation omitting any indices or coefficients. We write symbolically, $q_{\alpha\beta} = q$, $K^{(m)} = K$, $\partial_{\alpha} = \partial$ and $\partial_{\nu} = \gamma$. In particular, we have: $q' = dq$. Using this notation and expressing $E_{\nu}^{(m)}$ in terms of $K_{\nu}^{(m)}$ in (B8), we may symbolically write an equation for $K_{\nu}^{(m)}$:

$$
K'' = d \left( \partial^2 + \partial \Gamma + \Gamma^2 \right) q' + d K^3 + dK \\
+ dK K' + q \partial d \left( \Gamma' + d K + d \partial K \right) + \partial^2 d q', 
$$

where again by $\partial$ we designate a normal derivative (as opposed to covariant derivative) with respect to any coordinate $x^\mu$ along the four-dimensional $y = \text{const}$ hypersurface. This expression is true for any $y$. The leading order in $d/L$ in this expression comes from the term $q \partial d \Gamma' \sim d^0$. All the other terms are of order $d$.

For $n = 2$, we therefore have on the brane:

$$
K^{(m)}(0) \sim \Gamma^{(m)}(0) \sim d^0 \quad \forall m \leq n. 
$$

Now let us assume that this relation holds for a given $n = l + 1$. In particular, $\Gamma^{(l+1)}(0) \sim d^0$ and

$$
q^{(m+1)}(0) \sim d K^{(m)} d \quad \forall 0 \leq m \leq l + 1, 
$$

$$
q(0) \sim d^0. 
$$

For the next order in $n + 1$, we have:

$$
K^{(n+1)} = \partial_y (I) K'' \\
= \partial_y \left[ \left( \partial^2 + \partial \Gamma + \Gamma^2 \right) q' + d K^3 + dK K' + \partial^2 d q' \right] \\
+ q \partial d \left( \Gamma' + d K + d \partial K \right), 
$$

with

$$
\partial_y (I) d \partial d q' \bigg|_{y=0} = d \partial^2 \left[ q^{(l+1)}(0) \right] \sim d. 
$$

Similarly we may check that

$$
\partial_y (I) \left[ d \left( \Gamma' + \partial \Gamma + \Gamma^2 \right) q' + d K^3 + dK \right] \bigg|_{y=0} \sim d 
$$

and

$$
\partial_y (I) \left[ d K K' + q \partial d \left( d K + \partial K \right) + \partial^2 d q' \right] \bigg|_{y=0} \sim d, 
$$

and

$$
\partial_y (I) \left[ q \Gamma' \partial d \right] \bigg|_{y=0} = \partial d \sum_{n=0}^{l} C^n q^{l-n} \Gamma^{(l+1)}(0), 
$$

with $q^{(l-n)}(0)\Gamma^{(l+1)}(0) \sim d^1$ if $m < l$ and $q\Gamma^{(l+1)}(0) \sim d^0$ so that:

$$
\partial_y (I) \left[ q \Gamma' \partial d \right] \bigg|_{y=0} \sim d^0. 
$$

We have therefore shown that $K^{(n+1)}(0) \sim d^0$ and its leading contribution comes from the derivative of the Christoffel symbol uniquely.

We want now to show that the same is true for the Christoffel symbol. We have $K^{(m)}(0) \sim d^0$ for any $m \leq n + 1$ and $\Gamma^{(m)}(0) \sim d^0$ for any $m \leq n$. Using the relation (A7) for $\Gamma^{(\alpha)}_{\mu\nu}$, we have:

$$
\Gamma^{(\alpha)}_{\mu\nu}(y) = (dK^{(\alpha)})_{\nu} + (dK^{(\alpha)})_{\mu} - q_{\mu\rho}(dK^{(\alpha)})_{\rho} \\
+ d \partial_y \left[ \Gamma^{(\alpha)}_{\mu\nu} + \Gamma^{(\alpha)}_{\nu\rho} K^{\rho}_{\mu} - 2 \Gamma^{(\alpha)}_{\mu\rho} K^{\rho}_{\nu} \right] \\
- q^{\alpha\beta} q_{\mu\nu} \left( \Gamma^{(\sigma)}_{\beta\rho} K^{\rho}_{\nu} - \Gamma^{(\rho)}_{\beta\nu} K^{(\alpha)}_{\rho} \right) \\
- \sum_{m=1}^{n-1} \frac{C^m}{C_{m-1}} \left( dK_{\nu}^{(m)} \right) \partial_y \left( q^{(m-n)}(0) \Gamma^{(n)} \right) (A14) \\
- \sum_{m=0}^{n-1} C_{m-1} \left( dK_{\nu}^{(m)} \right) \partial_y \left( q^{(n-m)}(0) \Gamma^{(m)} \right) \\
$$

where $d \partial_y (\Gamma) K \sim d$, $d \partial_y (\Gamma) \sim d$ for $n > m$ so the last sum goes as $d$. Finally the first term goes as: $(dK_{\nu}^{(m)}),_{\mu} = d_{\mu} K^{(m)}_{\nu} + O(d)$. The Christoffel symbol therefore goes as $\Gamma^{(n+1)} \sim d^0$ for all $n$ and its leading contribution is:

$$
\Gamma^{(\alpha)}_{\mu\nu}(0) = d_{\mu} K_{\nu}^{(n)} + d_{\nu} K_{\mu}^{(n)} - d_{\alpha} K_{\mu\nu}^{(n)}(A15) \\
$$

where terms of order $O(d^1)$ have been omitted. We can therefore finally conclude that

$$
K^{(n)} \sim \Gamma^{(n)} \sim d^0 \quad \text{for any } n \geq 0, 
$$

and the leading contribution for the extrinsic curvature comes from the derivative of the Christoffel symbol only. Indeed for $n = 2$, we have:

$$
K_{\nu}^{(\mu)}(0) = q^{\beta\nu} \Gamma_{\beta\lambda} K^{\lambda}_{\omega} \partial_\omega d + O \left( \frac{d}{L} \right), 
$$

and similarly for any $n \geq 2$, the leading term in the extrinsic curvature comes uniquely from the derivative of the Christoffel symbol:

$$
K_{\nu}^{(\mu)} \bigg|_{y=0} = q^{\beta\nu} \Gamma_{\beta\lambda}^{(n-1)} \partial_\lambda d + O \left( \frac{d}{L} \right) \\
= \left( d_{\mu} K_{\nu}^{(n)} + d_{\nu} K_{\mu}^{(n)} - d_{\alpha} K_{\mu\nu}^{(n)} \right) \left( \frac{d}{L} \right), 
$$

where terms of order $O(d^1)$ have been omitted. We can therefore finally conclude that

$$
K^{(n)} \sim \Gamma^{(n)} \sim d^0 \quad \text{for any } n \geq 0, 
$$

and the leading contribution for the extrinsic curvature comes from the derivative of the Christoffel symbol only. Indeed for $n = 2$, we have:

$$
K_{\nu}^{(\mu)}(0) = q^{\beta\nu} \Gamma_{\beta\lambda} K^{\lambda}_{\omega} \partial_\omega d + O \left( \frac{d}{L} \right), 
$$

and similarly for any $n \geq 2$, the leading term in the extrinsic curvature comes uniquely from the derivative of the Christoffel symbol:

$$
K_{\nu}^{(\mu)} \bigg|_{y=0} = q^{\beta\nu} \Gamma_{\beta\lambda}^{(n-1)} \partial_\lambda d + O \left( \frac{d}{L} \right) \\
= \left( d_{\mu} K_{\nu}^{(n)} + d_{\nu} K_{\mu}^{(n)} - d_{\alpha} K_{\mu\nu}^{(n)} \right) \partial_\alpha d + O \left( \frac{d}{L} \right). 
$$
We therefore have a recursive expression for the leading order in \( d/L \) of the normal derivatives of the extrinsic curvature that we can use in order to formally sum the Taylor expansion (14) or (22).

2. Gauge where \( A \) depends on \( y \)

The previous result has been derived assuming that \( A \) in (39) was independent of \( y \). It is possible to show that such a dependence does not affect the final result. In order to show this we will dissociate the derivatives of \( A \) from the other derivatives. Then by summing over the derivatives of \( A \), we will recover the quantity \( d \). We denote by \( \delta_A = A'(y)\partial_A + \cdots \) the \( y \)-derivative which acts exclusively on \( A \).

In order to differentiate any other quantity, we can apply the operator \( \partial_y = \partial_y - \delta_A \) which has no effect on \( A \). In particular we write:

\[
\delta_A \partial_y Q(y) = \partial_y Q(y) = Q'(y), \quad (A20)
\]

for any quantity \( Q = K^\mu_\nu, E^\mu_\nu, q_{\mu\nu}, \Gamma^\mu_{\nu\rho} \). In the previous Section, \( A \) was \( y \)-independent and so the operator \( \delta_A \) had no effect, we simply had \( \partial_y = \partial_y \). Now this does not hold anymore and we need to include the effect of \( \delta_A \) in the Taylor expansion (16) bearing in mind that \( \delta_A \) and \( \partial_y \) do not commute. Indeed, a new \( A \) appears each time the operator \( \partial_y \) is applied on a quantity \( Q \) since \( \partial_y Q = AL_n Q \), where \( L_n = \partial_z \) is the Lie derivative along the normal direction. So \( \delta_A Q = 0 \) whereas \( \delta_A \partial_y Q = \bar{O}A'(y) \) where \( \bar{O} \) is some four-dimensional operator. If we consider for instance the \( y \)-derivative of the extrinsic curvature in (18), we have:

\[
\delta_A \partial_y K^\mu_\nu(y) = \delta_A K^\mu_\nu'(y)
= -A'(y)E^\mu_\nu(y) - D^\mu D_\nu A'(y)
- \frac{1}{L^2} K^\mu_\nu(y).
\quad (A21)
\]

In the Taylor expansion, (15), there will be a \( y \)-term, which we denote by \( K^\mu_\nu(0) \), that has only the first derivative along \( \partial_y \) and all the other along \( \delta_A \):

\[
K^\mu_\nu(y) = \sum_{m \geq 1} \frac{1}{m!} \delta_A^{(m-1)} \partial_y^{m-1} K^\mu_\nu(y). \quad (A22)
\]

We may write

\[
\delta_A^{(m-1)} \partial_y K^\mu_\nu(y) = \hat{O}^\mu_\nu A^{(m-1)}(y), \quad (A23)
\]

where he operator \( \hat{O}^\mu_\nu \) can be read off from (A21). Using this relation, we have:

\[
K^\mu_\nu(y) = \hat{O}^\mu_\nu \sum_{m \geq 1} \frac{1}{m!} A^{(m-1)}(y)
= \hat{O}^\mu_\nu \int_y^{y+1} A(y') dy'. \quad (A24)
\]

In particular the term \( K^\mu_\nu(0) \) that contributes to the Taylor expansion (15) is:

\[
K^\mu_\nu(0) = \hat{O}^\mu_\nu(y = 0) \int_0^1 A(y) dy = -d E^\mu_\nu - D^\mu D_\nu, \quad (A25)
\]

which is precisely the first term (63) that was contributing in the Taylor expansion when \( A \) was assumed to be \( y \)-independent.

It what follow we shall see that the Taylor expansion can be expressed as a sum of \( K \) which expression is precisely the same as \( K^{(n)} \) when \( A \) is assumed to be \( y \)-independent.

We shall work in a symbolic way, omitting any indices. First we notice that the Taylor expansion (16) (22) can be written in the form:

\[
\sum_{n \geq 1} \frac{1}{n!} K_n(0) = 0, \quad (A27)
\]

with

\[
\frac{1}{n!} K_n(y) = \sum_{m_1, \ldots, m_n \geq 0} \frac{1}{n!} F_{m_1, \ldots, m_n}, \quad (A28)
\]

where we wrote \( l_n = (m_1 + \cdots + m_n + n) \) and

\[
F_{m_1, \ldots, m_n} = \delta_A^{(m_n)} \partial_y^{(m_n-1)} \cdots \partial_y \partial_y A^{m_1} \partial_y K. \quad (A29)
\]

Recalling that each time \( K \) is differentiated with respect to \( y \), a new power of \( A \) appears, so that

\[
\partial_y^{(n)} K = \hat{O}_n \left[ A\hat{O}_{n-1} \left( \cdots \hat{O}_1 A \right) \right], \quad (A30)
\]

where each operator \( \hat{O}_i \) depends on \( y \) but includes only derivatives along the \( y = \text{const} \) hypersurface and quantities \( Q = K, E, q, \Gamma \). Using this notation, \( F_n \) is:

\[
F_{m_1, \ldots, m_n} = \hat{O}_n \left[ A\hat{O}_{n-1} \left( \cdots \hat{O}_1 A \right) \right]^{m_1} \cdots \left( \hat{O}_{m_n-1} \right)^{m_n},
\]

where the operators \( \hat{O}_i \) should be treated as independent variable of \( y \) so that for instance

\[
\left( A\hat{O}_1 A^{(m_1)} \right)^{m_2} = \sum C_{m_2}^{k} A^{(k)} \hat{O}_1 A^{(m_1+m_2-k)}. \quad (A32)
\]

When all the sums are performed, we obtain:

\[
\frac{1}{n!} K_n(y) = \hat{O}_1 \left[ A\hat{O}_2 \left( A \int \cdots \int \left( \hat{O}_n \int A dy \right) dy \right) dy \right] dy.
\]

So on the brane, we can express \( K_n \) in the form:

\[
K_n(0) = n! U_1(1) \quad (A34)
\]

\[
U_m(y) = \hat{O}_m \int_0^y A(y) U_{m+1}(y) dy, \quad m < n \quad (A35)
\]

\[
U_n(y) = \hat{O}_n \int_0^y A(y) dy = \hat{O}_n \hat{d}(y). \quad (A36)
\]
So far this result is exact. When the branes are close, this result will simplify remarkably. First we point out that \(0 < d(y) < d \ll L\) for any \(0 < y < 1\). This follows from the fact that the bulk geometry is completely regular, so between the branes, \(A(y, x) > 0\), which implies \(0 < \int_0^y A(y, x) dy < d(x) \ll L\) for any \(0 < y < 1\). This holds similarly for any multiple integral. The same simplifications as in the previous Section will therefore be valid for. For instance if in the previous Section we had \(\hat{O}_1 d \simeq Dd\), then this will remain true in this case as well: \(\hat{O}_1 \int_0^y A(y, x) dy \simeq D \int_0^y A(y, x)\).

In the following we shall label with a “bar” any quantity which was derived in the previous Section assuming \(A\) was independent of \(y\). In the previous Section, we had \(\hat{A} = d\) and so \(U_m\) was simply:

\[
U_m(y) = \frac{y^{n+1-m}}{(n + 1 - m)!} \hat{O}_m \left[ d\hat{O}_{m+1} \left( d \cdots d\hat{O}_n d \right) \right], \tag{A37}
\]

so we had:

\[
\hat{K}^{(n)}(0) = \hat{O}_1 \left[ d\hat{O}_2 \left( d \cdots d\hat{O}_1 d \right) \right], \tag{A38}
\]

using the fact that at \(d \ll L\), the action of the operators \(\hat{O}_i\) was considerably simplified and at leading order in \(d/L\), they were equivalent to the overall operator \(\hat{O}^{(n)}\):

\[
\hat{K}^{(n)}(0) = \hat{O}^{(n)} d^n \tag{A39}
\]

with \(\hat{O}^{(2n+1)} = \frac{1}{(2n + 1)!} \left( -E(0) - D^2 \right) D^{(2n)} \tag{A40}\)

\[
\hat{O}^{(2n)} = \frac{1}{2n!} K(0) D^{(2n)} \tag{A41},
\]

where \(D\) is a derivative along the \(y = \text{const}\) hypersurface. The leading contribution was indeed:

\[
\hat{K}^{(2n+1)}(0) = K'(0) (\partial d)^{2n} \tag{A42}
\]

\[
\hat{K}^{(2n)}(0) = K(0) (\partial d)^{2n}. \tag{A43}
\]

Now we may go back to the situation where \(A\) has some \(y\)-dependence. Because \(0 < \int_0^y A(y, x) dy < d(x) \ll L\) for any \(0 < y < 1\) and similarly for any multiple integral in \(U_m\), when the operators \(\hat{O}_i\) are applied on these integrals we can reproduce step by step exactly the same procedure as we followed in the previous Section in order to keep only the leading order in \(d/L\). In particular, the repeated action of each \(\hat{O}_i\) on each multiple integral can be substituted, in the small \(d\) limit (which implies a small \(\int_0^y \cdots \int_0^y A d y\) limit) by the action of an overall operator \(\hat{O}^{(n)}\). If we do so, the leading contribution can be expressed in the same way as in \(\text{A39}\):

\[
\frac{1}{n!} K_n(0) = \hat{O}^{(n)} \left[ \int_0^1 A \left( \int_0^y \cdots A \int_0^y A d y \right) \right], \tag{A44}
\]

with the operator \(\hat{O}^{(n)}\) as given in \(\text{A40 A41}\). Now the multiple integral is simply:

\[
\int_0^1 A \left( \int_0^y \cdots A \int_0^y A d y \right) = \frac{1}{n!} \left( \int_0^1 A d y \right)^n = \frac{d^n}{n!}, \tag{A45}
\]

So the rest follows exactly as in the previous case. For any \(n\), the leading contribution to \(K_n\) is:

\[
K_n(0) = \hat{O}^{(n)} d^n, \tag{A46}
\]

exactly as in \(\text{A39}\). So the leading order in \(d/L\) of the Taylor expansion \(\text{A27}\) is independent of the \(y\)-dependence of \(A\), and in particular we obtain the same result whether \(\hat{A} = d\) or any other function of \(y\) such that \(\int_0^y A d y = d\). The close brane limit theory derived in Section \(\text{IIC}\) is therefore valid independently of the precise expression of \(A\), as this should be since our result is gauge invariant.

### APPENDIX B: DIVERGENCE OF THE STRESS-ENERGY TENSOR

In order to work out in detail the divergence of the stress-energy tensor, we will consider directly the situation with a potential in section \(\text{V B}\). We therefore consider the stress-energy tensor as given in \(96\) with the equation of motion for \(d\) given in \(97\). We can work in the situation where no potential is present as in section \(\text{IV C}\) by setting \(V = W = 0\) in the following.

Using \(\text{B6}\), the divergence of the stress-energy tensor may be written as:

\[
d D_{\nu} T^\mu_{\nu} = -V'(d) \partial_\nu d - T^\mu_{\nu} \frac{d}{\partial \mu} d + R^\mu_{\nu} \partial_\mu d + \frac{1}{L} (6 z^2 f' + 4 z^2 g' + 2 g) \partial_\nu z^2 + \frac{2}{L} z^2 g' F^\mu_{\nu} \partial_\mu z^2
\]

\[
+ \frac{1}{L} g D_{\mu} F^\mu_{\nu} - \partial_\mu d \left[ \bar{E}^\mu_{\nu} - \bar{E} \delta^\mu_{\nu} \right] - d D_{\mu} \left[ \bar{E}^\mu_{\nu} - \bar{E} \delta^\mu_{\nu} \right],
\]

where again \(f\) and \(g\) are taken as function of \(z^4\). We stress that \(\bar{E}^\mu_{\nu} \sim d^0\) so \(d D_{\mu} \bar{E}^\mu_{\nu} \) will be neglected in what follows.

The first line simplifies, using the relation

\[
T^\mu_{\nu} = R^\mu_{\nu} + \frac{1}{2} T \delta^\mu_{\nu}
\]

\[
= R^\mu_{\nu} - \frac{1}{2d} (3 W + 4 V) \delta^\mu_{\nu}. \tag{B2}
\]

If from the Einstein equation and the equation for \(d\) \(\text{B7}\), we have:

\[
\partial_\mu z^2 = -2 D_{\mu} D_{\nu} d \partial^\nu d
\]

\[
= -2 \left( \frac{1}{L} (f g_{\mu \nu} + g F_{\mu \nu}) + (V + W) g_{\mu \nu} \right) \partial^\nu d
\]

\[
= \frac{2}{L} (f - z^2 g) \partial_\mu d - 2 (V + W) \partial_\mu d, \tag{B3}
\]

where terms of order \(d/L^2\) have been omitted.

Using this relation and

\[
D_{\mu} F^\mu_{\nu} = 2 \Box d \partial_\mu d = -\frac{2}{L} (2 z^2 g + 4 f - LW) \partial_\mu d, \tag{B4}
\]
the expression \( [E^\nu_\nu - \hat{E}^\nu_\nu] \partial_\nu d \) simplifies to:
\[
d D_\mu T_{\nu}^\mu = - \left[ E^\nu_\nu - \hat{E}^\nu_\nu \right] \partial_\nu d \tag{B5}
\]
\[
- \frac{12}{L^2} \left[ g f + z^2 \left( f - z^2 g \right) (f' + z^2 g') \right] \partial_\nu d + \left[ - V'(d) + \frac{3W + 4V}{2d} + \frac{2g}{L} (3W + 2V) \right] \partial_\nu d + \frac{1}{L^2} \mathcal{O} \left( \frac{d}{L} \right).
\]

In the absence of a potential, the divergence of \( T_{\nu}^\mu \) simplifies to the first two lines. Both
\[
\left[ E^\nu_\nu - \hat{E}^\nu_\nu \right] \partial_\nu d = - \frac{12}{L^2} \left[ g f \right.
\]
\[
+ z^2 \left( f - z^2 g \right) (f' + z^2 g') \left] \partial_\nu d \right.
\]
and, in the presence of a potential,
\[
V'(d) = \frac{3W + 4V}{2d} + \frac{2g}{L} (3W + 2V) \tag{B7}
\]
\[
+ \frac{12}{L^2} \left[ z^2 \left( f' + z^2 g' \right) (V + W) \right]
\]
need to hold independently, in the close brane limit.

**APPENDIX C: EVOLUTION OF THE WEYL TENSOR**

In this Section we derive the evolution equation for \( E^\nu_\nu \). Essentially the same results are quoted in \[ ]^{10, 54, 57} but we repeat them here for completeness. We assume the general form \([33]\) for the metric. \( n = A^{-1} \partial / \partial y \) is the unit normal to hypersurfaces of constant \( y \) and, for a purely four-dimensional tensor, the Lie derivative with respect to \( n \) is given by
\[
L_n = n = A^{-1} \partial / \partial y
\]
The projection tensor \( h_{ab} = g_{ab} - n_a n_b \) then satisfies \( h_5^5 = h_5^\mu = h_5^\nu = 0 \), \( h_\mu^\nu = \delta_\mu^\nu \) and so can be left implicit in the index convention. The four-dimensional and five-dimensional Riemann tensors are related by
\[
R_{\mu \nu \alpha \beta} = (5) R_{\mu \nu \alpha \beta} + 2K_{\mu \alpha} K_{\beta \nu} \tag{C1}
\]
The electric and magnetic parts of the bulk Riemann tensor are defined to be
\[
E_{\mu \nu} \equiv (5) R_{\alpha \beta \mu \nu} n^\alpha n^\beta \quad \tilde{B}_{\mu \nu} = h_\mu^c h_\nu^d \left[ (5) R_{\alpha \beta \mu \nu} \right] \tag{C2}
\]
and are purely four-dimensional. \[ ]^{11} then follows identically from a direct evaluation of the Riemann tensor for the metric \([33]\). Denoting by \( \nabla_\alpha \) the five-dimensional covariant derivative, we find that
\[
\frac{1}{A} \nabla_5 \left[ (5) R_{\mu \nu \alpha \beta} n^\nu \right] = L_n \tilde{B}_{\mu \nu} + \frac{2}{A} \partial_{[\mu} A \tilde{E}_{\nu]} \tag{C3}
\]
\[
+ 2K_{\mu \beta} \tilde{B}_{\nu \beta} - K_{\mu \beta} \tilde{B}_{\nu \beta} \tag{C4}
\]
As a consequence of the five-dimensional Bianchi identities, we find
\[
\nabla_\beta \left[ (5) R_{\mu \nu \alpha \beta} n^\nu \right] \equiv \left( \nabla_\beta n^\nu \right) (5) R_{\mu \nu \alpha \beta}
\]
which, from \[ ]^{33} and \[ ]^{34}, gives
\[
L_n \tilde{B}_{\mu \nu} = K_{\mu \alpha} \tilde{B}_{\rho \nu} - \frac{2}{A} \partial_{[\mu} A \tilde{E}_{\nu]} \alpha - 2D_{[\mu} \tilde{E}_{\nu]} \alpha
\]
tensor (which will vanish when contracted with $K^{\alpha \beta}$):

$$L_n E_{\mu \nu} = 2K^{\alpha}_{\mu} E_{\alpha \mu} - \frac{3}{2} K E_{\mu \nu} - \frac{1}{2} K^{\alpha}_{\mu} E_{\alpha \mu} q_{\mu \nu}$$

which reduces to [18] using $B_{\mu \nu \rho} = 2D_{[\mu \nu} K_{\rho]}$.


