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Beyond the Low Energy Approximation in Braneworld Cosmology

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Abstract

We develop a four-dimensional effective theory for Randall-Sundrum models which allows us to calculate long wavelength adiabatic perturbations in a regime where the ρ^2 terms characteristic of braneworld cosmology are significant. This extends previous work employing the moduli space approximation. We extend the treatment of the system to include higher derivative corrections present in the context of braneworld cosmology. The developed formalism allows us to study perturbations beyond the general long wavelength, slow-velocity regime to which the usual moduli approximation is restricted. It enables us to extend the study to a wide range of braneworld cosmology models for which the extra terms play a significant role. As an example we discuss high energy inflation on the brane and analyze the key observational features that distinguish braneworlds from ordinary inflation by considering scalar and tensor perturbations as well as non-gaussianities. We also compare inflation and Cyclic models and study how they can be distinguished in terms of these corrections.

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1 Introduction

Braneworlds offer a new approach to the phenomenology of both cosmology and particle physics, and provide an alternative type of low-energy string compactification. From the cosmologist's perspective one of the most interesting aspects of these models is that at high energies, such as those occurring in the early universe, their gravitational behaviour is different from conventional scalar-tensor cosmology. This is shown most clearly in the well known "modified Friedmann equation", which contains additional dark radiation and ρ^2 terms, where ρ is the energy density of matter on the brane. In the simplest models, at low energy and at long wavelengths, braneworlds behave like conventional scalar-tensor theories [1, 2, 3, 4]. However, in the early Universe the low-energy condition may well be violated. Consequently, it makes sense to look beyond this conventional limit. Unfortunately, this usually requires a five-dimensional description, generically making analytic solutions impossible, and requiring numerical methods. To get a better insight into this regime it is useful to use approximation methods that capture the essential physics, if not the precise solution. In this paper we will consider approximation methods that allow us to go beyond the low-energy restriction.

The Gauss-Codacci formalism relates the five-dimensional Riemann tensor to its four-dimensional counterpart induced on the brane. A modified four-dimensional Einstein equation on the brane can be derived but the five-dimensional nature of the model can not be completely ignored. Formally, the system of equations obtained on the brane in terms of four-dimensional quantities is not closed, as the modified Einstein equation contains a new term: the electric part $E_{\mu\nu}$ of the five-dimensional Weyl tensor. This term encodes information about the bulk geometry. Assuming cosmological symmetry, the five-dimensional system can be solved exactly. To start with, we use our knowledge of the background Weyl tensor $E_{\mu\nu}$ to derive its behaviour for adiabatic perturbations and solve the modified four-dimensional Einstein equation. More precisely, for linear perturbations, we will neglect the contribution from the part of the Weyl tensor $E_{\mu\nu}$ which is transverse and vanishes in the background. This procedure is consistent in an adiabatic regime only.

Next, we extend this procedure by including some contribution to the transverse part of the Weyl tensor. We consider the contribution from a specific four-dimensional tensor $A_{\mu\nu}$, that we check is consistent with the proprieties of the Weyl tensor, and thus represents a natural candidate to consider. This tensor represents the term that would be derived if some four-dimensional Weyl-squared terms were included in the four-dimensional effective action. Introducing a Weyl-squared term in the four-dimensional effective action preserves the conformal invariance of the effective theory and is therefore a natural term to consider [5].

Around static branes, the first order perturbations on the brane metrics have been solved exactly [1]. It is well known that in the presence of an extra-dimension, an infinite tower of Kaluza-Klein modes affects the four-dimensional geometry. As long as the extra dimension is of finite size, the modes are discrete. For static branes, we show that including the tensor $A_{\mu\nu}$ to the effective theory correctly reproduces the effects of the first Kaluza-Klein mode in the case where matter is present only on one brane. It seems therefore natural to study the effect of the tensor $A_{\mu\nu}$ in more general setups.

As a first example, we use this formalism within the context of brane inflation for which it reproduces the nearly scale-invariant spectrum for density perturbations of standard inflation.

Recently, the Ekpyrotic and Cyclic models [6, 7, 8, 9, 10] have been introduced as an alternative to inflation, giving a new picture to solve the homogeneity, isotropy and flatness problems. It has been shown that the Ekpyrotic and Cyclic models provide an alternative scenario for the production of a nearly scale-invariant spectrum. While the production of such a spectrum in inflation is based on "slow-roll" conditions in an expanding universe, the Ekpyrotic and Cyclic models, on the other hand, use a "fast-roll" potential in a contracting universe for which the issue of the "beginning of our Universe" is avoided. Further studies [11, 12] have shown an exact duality between the two models in the production of density perturbations making the two models hard to distinguish without bringing in results from the observation of tensor perturbations. In this work we use our prescription to compare the behaviour of the fast-roll and slow-roll models in the case when $A_{\mu\nu}$ contributes to the transverse part of Weyl tensor. We examine how these corrections influence the production of a scale-invariant spectrum, showing how they enable us to distinguish between general models with "slow-roll" and "fast-roll" conditions. Unfortunately, the difference between the standard inflation scenario and the Cyclic model turns out to be negligible.

The paper is organised as follows. In section 2 we review the conventional Randall-Sundrum model. At low-energy, we explain how a four-dimensional effective action can be derived. The extension of the single brane effective theory beyond the low-energy limit is discussed. In section 3 we put this formalism to use: we analyze a brane inflation model in which the inflaton lives on the brane, and the typical energy scale of inflation is not necessarily small compared to the brane tension. The scalar and tensor perturbations are computed and we give an estimation of non-gaussianity. We show how a model of brane inflation may always be reinterpreted as an ordinary inflation model with a redefined potential, or equivalently with redefined slow roll parameters. In section 4 we extend the previous model, taking into account the possibility of the term $A_{\mu\nu}$ to be present in the transverse part of the Weyl tensor. We also compare inflation and Cyclic models and analyze whether they can be distinguished in terms of these corrections. Finally, we discuss the implications of our results in section 5.

In appendix A, we derive the four-dimensional effective theory for a general two brane Randall-Sundrum model which is valid at large energy provided we work in the long wavelength adiabatic regime.

We review the exact behaviour of the Kaluza-Klein modes around static branes in appendix B and show how in this limit the first mode can be reinterpreted in terms of local four-dimensional quantities.

2 Covariant Treatment of Randall-Sundrum Model

2.1 Gauss-Codacci Equations

In what follows, we shall be interested in the two brane Randall-Sundrum model as a specific simple example of braneworld cosmologies. We will see in which limit this model may be extended to the one brane Randall-Sundrum model. To begin with, let us consider the two brane Randall-Sundrum model [13, 14] where the spacetime is five-dimensional, with a compact extra dimension having the topology of an $S_1/\mathbf{Z_2}$ orbifold. The stress energy of the bulk is assumed to be from a pure negative cosmological constant $|\Lambda| = \frac{6}{\kappa L^2}$, $\kappa = \frac{1}{M_5^3} \equiv \frac{L}{M_4^2}$, with M_n the n-dimensional Planck mass. There are two boundary branes (referred as ±-branes) located at the fixed points of the Z_2 symmetry on which gauge and matter fields are confined.

The Ricci tensor on each brane may be expressed in terms of five-dimensional quantities by means of the Gauss-Codacci formalism as in [15, 4]:

$$R^{(4\mathrm{d})}_{\mu\nu} = R^{(5\mathrm{d})}_{\mu\nu} + (K_{\alpha\mu}K^{\alpha}_{\nu} - KK_{\mu\nu}) - E_{\mu\nu}, \tag{1}$$

where $E_{\mu\nu}$ is the electric part of the five-dimensional Weyl tensor. For a Z_2-symmetric brane, the extrinsic curvatures $K_{\mu\nu}$ on each side of the brane are equal and opposite and can be uniquely determined using the Israël matching condition [16]:

$$\Delta K_{\mu\nu} = -\kappa \left(T^{\mathrm{tot}}_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T^{\mathrm{tot}} \right), \tag{2}$$

where $g_{\mu\nu}$ is the induced metric on the brane and $T^{\mathrm{tot}}_{\mu\nu}$ is the total stress-energy on the brane including the gauge and matter fields and the brane tension contribution. We consider the tension on each brane to be fine-tuned to their canonical value: $\mathcal{T}^{\pm} = \pm\frac{6}{\kappa L}$ and we include some matter on each brane with stress-energy tensor $T^{(\pm)}_{\mu\nu}$. Writing the extrinsic curvature in terms of the stress-energy tensor, the projected Ricci tensor on each brane can be expressed as:

$$R^{(\pm)}_{\mu\nu} = \pm\frac{\kappa}{L}\left(T^{(\pm)}_{\mu\nu} - \frac{1}{2} T^{(\pm)} g^{(\pm)}_{\mu\nu} \right) - \frac{\kappa^2}{4}\left(T^{(\pm)\alpha}_{\mu} T^{(\pm)}_{\alpha\nu} - \frac{1}{3} T^{(\pm)} T^{(\pm)}_{\mu\nu} \right) - E^{(\pm)}_{\mu\nu}. \tag{3}$$

We can remark on two features in this modified Friedman equation. The first one is the presence of terms quadratic in the stress-energy tensor. The aim of our study will be to understand the implications of those terms to braneworld cosmology. The second departure from the standard four-dimensional Einstein equation arises from the presence of the Weyl tensor $E_{\mu\nu}$ which is undetermined on the brane. It is worth pointing out that when the cosmological constant in the bulk is important, the length scale L is negligible compared to any other length scale present in the theory. In that case the first term in (3) dominates and we recover standard four-dimensional gravity for the positive tension brane (up to the redefinition of the four-dimensional Planck mass $\kappa \equiv \frac{L}{M_4^2}$).

2.2 Exact Solution for Cosmological Symmetry

Considering the five-dimensional Universe to be homogeneous and isotropic in the three spatial directions, an exact static solution of the five-dimensional Einstein equation can be found: the geometry is of Schwarzschild-Anti-de-Sitter form with parameter $\mathcal{C}$ associated with the black hole mass.

Knowing the bulk geometry exactly, the Weyl tensor can be calculated. For homogeneous and isotropic metrics $E^{(\pm)}_{\mu\nu}$ can be seen to have the form of the stress-energy tensor for radiation [3]:

$$\begin{aligned} E^{(\pm)\ 0}_{\ \ \ 0} &= 3\tfrac{\mathcal{C}}{a_\pm^4}, \\ E^{(\pm)\ i}_{\ \ \ j} &= -\tfrac{\mathcal{C}}{a_\pm^4}\delta^i_j, \end{aligned} \tag{4}$$

where the constant $\mathcal{C}$ is the same as the one associated with the black hole mass.

Following the work done in [3, 17, 18, 19, 20, 21, 22, 23, 24], the expression for the Weyl tensor can be introduced in the modified Einstein equation (3). Each of the two branes satisfy the induced or modified Friedmann equation:

$$H_\pm^2 = -\frac{k}{a_\pm^2} \pm \frac{\kappa}{3L}\rho_\pm + \frac{\kappa^2}{36}\rho_\pm^2 + \frac{\mathcal{C}}{a_\pm^4}, \tag{5}$$

with the $+$ and $-$ indices designating the positive and negative tension brane respectively, H the induced Hubble parameter, a the brane scale factor and ρ the matter and radiation density confined to each brane. In defining ρ we have separated out the part coming from the canonical tension of the brane.

2.3 Four-dimensional Effective Action

In the low-energy limit, assuming the matter and radiation density on the branes to be much smaller than the magnitude of the brane tensions, the $\rho_\pm^2$ terms in (5) may be neglected [2, 3, 25, 26]. In that case, it has been shown that for the purpose of calculation of long wavelength perturbations, the system may be well described by a four-dimensional effective theory derived from the effective action of a scalar field minimally coupled to gravity with non-minimally coupled matter:

$$S = \int d^4x\sqrt{-g}\left(\frac{L}{2\kappa}R - \frac{1}{2}\left(\partial\phi\right)^2\right) + S_m^-[g^-] + S_m^+[g^+], \tag{6}$$

with

$$g^{+}_{\mu\nu} = \left(\cosh\left(\phi/\sqrt{6}\right)\right)^2 g^{4d}_{\mu\nu}, \quad g^{-}_{\mu\nu} = \left(-\sinh\left(\phi/\sqrt{6}\right)\right)^2 g^{4d}_{\mu\nu}, \tag{7}$$

and where $S^{\pm}_m[g^{\pm}]$ are the conventional four-dimensional matter actions on each brane. Here $g^{\pm}_{\mu\nu}$ are the induced geometries on each brane which, in this approximation, are seen to be conformal to each other. These equations are often written in the conventional Brans-Dicke frame, however we have chosen to use the more practical Einstein frame, at the price of having non-minimally coupled matter. Assuming cosmological symmetry, the equations of motion derived from this action are indeed consistent with the modified Friedmann equations (5) in the low-energy limit. Within this limit, the behaviour of long wavelength, adiabatic linear perturbations may therefore be derived from (6), this has been done for instance in [27, 28]. However some braneworld models consider situations where the ρ^2 terms play a significant role [29, 30, 31, 32, 33, 34, 35, 36]. The purpose of the next section is therefore to understand how this covariant formalism may be extended in order to satisfy the correct density dependence in this more general case.

We stress that the extension considered will only be valid for *adiabatic* perturbations where the stress energy of the matter on each brane evolves adiabatically with the scalar field. However we expect that our formalism may be used to model characteristically five-dimensional effects in a more general setting of high energy or velocity without directly dealing with the full five-dimensional formalism as described in [37, 38, 39, 40].

2.4 Treatment of the One Brane High-Energy Regime

In order to focus on the effect of the quadratic terms in $T_{\mu\nu}$, we will consider, in what follows, that the bulk has a pure Anti-de-Sitter (AdS) geometry in the background ($\mathcal{C} = 0$). An extension to the case of matter in a Schwarzschild-AdS bulk is presented in appendix A. In the background, when the bulk is fixed to pure AdS, each brane evolves independently from each other. The negative tension brane could be very close or could be sent to infinity so as to recover the one brane Randall-Sundrum model for the positive tension brane. It is in this limit that we will consider the model in the following.

Since we will be interested in quantities on the positive tension brane only, the superscript $(+)$ will be suppressed.

The Weyl tensor $E_{\mu\nu}$ in the modified Einstein equation (3) is a priori undetermined. This comes from the five-dimensional nature of the theory and the fact that the system of equations is not closed on the brane as the Weyl tensor mediates some information from the bulk to the brane. However $E_{\mu\nu}$ is traceless [17], and we may decompose it into longitudinal and tensor parts:

$$E_{\mu\nu} = \mathcal{E}^{(L)}_{\mu\nu} + E^{TT}_{\mu\nu}, \tag{8}$$

$$\nabla_\mu E^{TT\,\mu}_{\nu} = 0, \quad E^{TT\,\mu}_{\mu} = 0. \tag{9}$$

Furthermore, since the Bianchi identity has to be satisfied for the four-dimensional brane Einstein tensor, the divergence of the Weyl tensor must satisfy:

$$\nabla_\mu E^\mu_\nu = -\frac{\kappa^2}{4} T^\beta_\alpha \left(\nabla_\beta (T^\alpha_\nu - \frac{1}{3} T \delta^\alpha_\nu) - \nabla_\nu (T^\alpha_\beta - \frac{1}{3} T \delta^\alpha_\beta) \right), \tag{10}$$

$$= \nabla_\mu \mathcal{E}^{(L)\,\mu}_{\nu}. \tag{11}$$

The longitudinal part may be determined up to a homogeneous tensor which is absorbed in the transverse and traceless part $E^{TT}_{\mu\nu}$. We may check that for any kind of matter satisfying conservation of energy in a homogeneous and isotropic background, the divergence in (10) vanishes. This is consistent with the fact that the longitudinal part of the Weyl tensor vanishes for the background. (Even when the bulk geometry is not taken to be pure AdS ($\mathcal{C} \neq 0$), the part contributing to the Weyl tensor in (4) is homogeneous.) This four-dimensional tensor part of the Weyl tensor is the only quantity that remains unknown on the brane. For the purpose of the first part of our study we will make the important assumption that $E^{TT}_{\mu\nu}$ *can be neglected*. For pure AdS background, the Weyl tensor vanishes and $E^{TT}_{\mu\nu}$ only comes in at first order in perturbation.

It is a well-known result that for long wavelength *adiabatic* scalar perturbations, the quantity $\frac{\delta\rho}{\dot{\rho}}$ is the same for any fluid with energy density ρ, satisfying conservation of energy, regardless of its equation of state [41]. In particular, this is true for the conserved tensor $E^{TT}_{\mu\nu}$ (with density ρ_E and $\omega_E = \frac{1}{3}$). The transverse condition implies $\dot{\rho}_E = -4\,\frac{\dot{a}}{a}\rho_E$. We consider the fluid with energy density ρ_E and compare it with the energy density ρ of any other fluid present in the theory, or any scalar field Φ present in the theory. The adiabaticity condition at long wavelengths imposes $\frac{\delta\rho_E}{\dot{\rho}_E} = \frac{\delta\rho}{\dot{\rho}} = \frac{\delta\Phi}{\dot{\Phi}}$ which requires that:

$$\delta\rho_E = -4\,\rho_E\,\frac{\dot{a}}{a}\frac{\delta\rho}{\dot{\rho}} = -4\,\rho_E\,\frac{\dot{a}}{a}\frac{\delta\Phi}{\dot{\Phi}}. \tag{12}$$

Since the tensor $E^{TT}_{\mu\nu}$ vanishes in the background, for adiabatic perturbations, its contribution vanishes as well, $\delta\rho_E = 0$. Thus in the adiabatic limit the long wavelength scalar perturbations of $E^{TT}_{\mu\nu}$ can be neglected.
In other words, in an adiabatic regime, the perturbations follow the same evolution as a general background solution. Having a non-vanishing perturbed $E^{TT}_{\mu\nu}$ could be seen as the introduction of a black hole at the perturbed level, changing the background from pure AdS to Schwarzschild-AdS. For the one brane limit, this is not compatible with the bulk boundary conditions at infinity as the bulk geometry would diverge exponentially.

For the three-dimensional-tensor perturbations, the same approximation will be made, although no analogous argument may be given. It is in the context of this approximation that we shall consider inflation on the brane in the next section.

In appendix A a slightly more elaborate version of the treatment of the Weyl tensor is given for the two brane scenario in a general Schwarzschild-AdS background. However as stated earlier we shall for simplicity focus on the one brane limit. In the next section we use this approach to study inflation on the positive tension brane.

3 Inflation on the brane

In this section we shall consider the inflaton to be an additional scalar field living on the positive tension brane. If the energy scale of inflation is well below the brane tension then this system is well described by the Brans-Dicke theory as in section 2.3. However, if the typical energy scale of inflation is comparable to or larger than the brane tension then we may use the formalism of the previous section to get a better insight.

3.1 Background

For simplicity, let us assume that the brane is spatially flat in the background ($k = 0$). The stress-energy tensor of the inflaton is given by:

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \left(\frac{1}{2} (\partial \varphi)^2 + V(\varphi) \right) g_{\mu\nu}. \tag{13}$$

The inflaton φ evolves in a potential $V(\varphi)$ which is assumed to satisfy some slow-roll conditions which will be specified later. The energy density is:

$$\rho = -T^0_0 = V(\varphi_0) + \frac{1}{2a^2} \dot{\varphi}_0^2. \tag{14}$$

where a dot represents derivative with respect to the conformal time.

When the kinetic energy of the scalar field is assumed to be negligible compared to its potential energy, $\frac{1}{a^2}\dot{\varphi}_0^2 \ll V(\varphi_0)$, the modified Friedmann equation (5) reads:

$$H^2 \simeq \frac{2}{L^2 \mathcal{T}} V \left(1 + \frac{V}{2\mathcal{T}} \right) \simeq \text{constant}. \tag{15}$$

It is worth pointing out that an expansion in $\rho/\mathcal{T}$ is equivalent to an expansion in $L^2 H^2$. Usually in the low-energy limit $L^2 H^2 \ll 1$ and $2\rho/\mathcal{T} \sim L^2 H^2$. In what follows we will keep all terms in $L^2 H^2$ so that the theory remains valid at high energy, when $L^2 H^2 \gg 1$. As mentioned before, in the limit where the length scale L is negligible compared to the other length scales of the theory, we should recover the standard four-dimensional results.
Since the inflaton is confined on the four-dimensional brane, the Klein-Gordon field equation of motion coming from the conservation law of the stress-energy tensor $\nabla_\mu T^\mu_\nu = 0$ is the usual four-dimensional equation:

$$V_{,\varphi}(\varphi_0) = -\frac{1}{a^2} \ddot{\varphi}_0 - 2 \frac{\dot{a}}{a^3} \dot{\varphi}_0. \tag{16}$$

This simplifies to the conventional result in the slow-roll regime $\frac{d}{d\tau} \frac{\dot{\varphi}_0}{a} \ll H \dot{\varphi}_0$:

$$\dot{\varphi}_0 \simeq -\frac{a V_{,\varphi}}{3H}. \tag{17}$$

We may now use the formalism of the previous section to consider scalar and tensor perturbations as well as an estimation of non-gaussianity.

3.2 Linear Scalar Perturbations

We consider linear isotropic perturbations around this conformally flat background. In longitudinal gauge,

$$ds^2 = a^2(\tau)\,(-1+2\Phi)\,d\tau^2 + a^2(\tau)\,(1+2\Psi)\,d\mathbf{x}^2, \tag{18}$$
$$\varphi(\tau,\mathbf{x}) = \varphi_0(\tau) + \delta\varphi(\tau,\mathbf{x}). \tag{19}$$

As mentioned previously, in order for the equation of motion to be consistent with the Bianchi identity the Weyl tensor must include a term cancelling the divergence of the quadratic terms in $T_{\mu\nu}$. Using the background equation of motion, we need to have:

$$\nabla_\mu E^\mu_\nu = -\frac{\kappa^2}{4} T^\beta_\alpha \left(\nabla_\beta (T^\alpha_\nu - \frac{1}{3} T \delta^\alpha_\nu) - \nabla_\nu (T^\alpha_\beta - \frac{1}{3} T \delta^\alpha_\beta) \right), \tag{20}$$

$$= -\frac{\kappa^2}{4} \begin{pmatrix} 0 \\ -\frac{2}{3}\frac{\dot\varphi_0^2}{a^4}\left(-\ddot\varphi_0\,\delta\varphi + \frac{\dot a}{a}\dot\varphi_0\,\delta\varphi + \dot\varphi_0^2\,\Phi + \dot\varphi_0\,\delta\dot\varphi\right)_{,i} \end{pmatrix}. \tag{21}$$

We stress that the time-like component of this divergence vanishes. This is a particularity of the stress-energy coming from a scalar field and will not, as far as we know, be true for a general fluid. This allows us to decompose the Weyl tensor in the simple form:

$$E_{\mu\nu} = \mathcal{E}^{(L)}_{\mu\nu} + E^{TT}_{\mu\nu}, \tag{22}$$

$$\text{with} \qquad \mathcal{E}^{(L)}_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & X_{,ij} - \frac{1}{3}\nabla^2 X \gamma_{ij} \end{pmatrix}, \quad \nabla_\mu \mathcal{E}^{(L)\ \mu}_{\nu} = \begin{pmatrix} 0 \\ \frac{2}{3}\nabla^2 X_{,i} \end{pmatrix}. \tag{23}$$

This is remarkable since an expression for the longitudinal part of the Weyl tensor has been found without needing to solve a differential equation involving time derivatives, which would have required us to specify some initial conditions. The expression for X is given by:

$$\nabla^2 X \;=\; \frac{\kappa^2}{4}\frac{\dot\varphi_0^2}{a^2}\left(-\ddot\varphi_0\,\delta\varphi + \frac{\dot a}{a}\dot\varphi_0\,\delta\varphi + \dot\varphi_0^2\,\Phi + \dot\varphi_0\,\delta\dot\varphi\right). \tag{24}$$

Having an expression for the Weyl tensor (23), we need to solve the modified Einstein equation (3):

$$G_{\mu\nu} = \frac{6}{TL^2} T_{\mu\nu} - \frac{9}{T^2L^2}\left(T^\alpha_\mu T_{\alpha\nu} - \frac{1}{3} T T_{\mu\nu} - \frac{1}{2} T_{\alpha\beta} T^{\alpha\beta} g_{\mu\nu} + \frac{1}{6} T^2 g_{\mu\nu}\right) - \mathcal{E}^{(L)}_{\mu\nu} \tag{25}$$

When we derive the (ij) (with $i \neq j$) component of this equation, a second interesting feature arises from the ansatz (23). Indeed the (ij)-equation directly points to the presence of an effective *anisotropic* stress X at high energy:

$$X \quad = \quad \Psi - \Phi. \tag{26}$$

Since only the ρ^2 terms contribute to X, (eq.(24) is quartic in the fields so quadratic in the energy density), we notice that at high-energy, the ρ^2 terms play the role of an effective anisotropic stress.

For linear perturbations, the $(0i)$-component of the modified Einstein equation (25) reads:

$$\delta\varphi = -\frac{2L}{\kappa\,\dot{\varphi}_0\sqrt{1+L^2H^2}}\left(\frac{\dot{a}}{a}\Phi + \dot{\Psi}\right). \tag{27}$$

Using this expression in (24) and (26), we can express $\dot{\Phi}$ in terms of Φ and $\Psi, \dot{\Psi}, \ddot{\Psi}$. Using this, as well as the expression for $\delta\varphi$, in the (00)-component of the modified Einstein equation, we find the following relation between Φ and Ψ:

$$\Phi = \left(1 - \frac{L^2 a\dot{H}}{1+L^2H^2}\right)\Psi = \left(1 - \frac{a}{H}\frac{d}{d\tau}\left(\ln\sqrt{1+L^2H^2}\right)\right)\Psi. \tag{28}$$

Note that the first and second time derivative of Ψ cancel exactly, giving a surprisingly simple expression for the anisotropic stress. In the low-energy regime where $L^2H^2 \ll 1$, the anisotropic part cancels out, and the usual result is recovered.

Substituting this relation between Φ and Ψ into the previous expression we had for $\dot{\Phi}$ in terms of $\Phi, \Psi, \dot{\Psi}, \ddot{\Psi}$ gives the decoupled second order equation for Ψ:

$$\begin{aligned} 0 = {} & \ddot{\Psi} - \nabla^2\Psi + 2\left(\tfrac{\dot{a}}{a} - \tfrac{\ddot{\varphi}_0}{\dot{\varphi}_0}\right)\left(\dot{\Psi} + \tfrac{\dot{a}}{a}\Psi\right) \\ & + 2\,a\dot{H}\left(\Psi - \tfrac{L^2}{1+L^2H^2}\left(H^2\Psi - H\tfrac{\ddot{\varphi}_0}{a\dot{\varphi}_0}\Psi + \tfrac{H}{a}\dot{\Psi}\right)\right) \\ & - L^2\dot{H}^2\tfrac{2-L^2H^2}{1+L^2H^2}\Psi - \tfrac{L^2\ddot{H}H}{1+L^2H^2}\Psi. \end{aligned} \tag{29}$$

We may point out that the only difference with usual four-dimensional scalar perturbations will arise from this second-order equation for Ψ. We have indeed already mentioned that for a given geometry, the Klein-Gordon equation for the scalar field remains the usual four-dimensional one $\nabla_\mu T^\mu_\nu = 0$:

$$\delta\ddot{\varphi} = \nabla^2\,\delta\varphi - 2a^2\Phi V_{,\varphi} + a^2\delta\varphi V_{,\varphi\varphi} - \dot{\varphi}_0\left(3\dot{\Psi} + \dot{\Phi}\right) - 2\frac{\dot{a}}{a}\delta\dot{\varphi}. \tag{30}$$

We can define the gauge invariant variable u related to the metric perturbation by $u = z\Psi$ with $z = \frac{a}{\dot{\varphi}_0\sqrt{1+L^2H^2}}$. It is more significant to express it in terms of the scalar field perturbations:

$$\delta\varphi = \frac{-2L}{a\,\kappa}\left(\dot{u} + \frac{\ddot{\varphi}_0}{\dot{\varphi}_0}u\right), \tag{31}$$

ie. precisely the conventional relation between the Mukhanov variable u and the scalar field perturbations [42] (after defining the four-dimensional Planck mass to be $M_4^2 = \frac{L}{\kappa}$).

Using the equation for Ψ, the Mukhanov variable u satisfies the second order differential equation:

$$\ddot{u}_k + \left(k^2 - \frac{\beta}{\tau^2}\right)u_k = 0, \quad \text{with} \quad \frac{\beta}{\tau^2} \equiv \frac{\ddot{\theta}}{\theta}, \quad \theta = \frac{H}{\dot{\varphi}_0}, \tag{32}$$

which is precisely the standard equation. The braneworld modifications have only altered the expression for the Hubble 'constant' H in the modified Friedmann equation. The corrections to the observable quantities at the linear perturbation level directly result from this background modification alone.

We may now follow the prescription of [11]. Assuming that β may be treated as a constant –the conditions for this assumption will be specified by the slow-roll parameters later on – an analytical solution of (32) for the k-modes u_k can be found. The integration constant may be fixed by requiring the scalar field fluctuations to be in the Minkowski vacuum well inside the Hubble radius, and by normalizing the modes as:

$$k^2\tau^2 \gg \beta \sim 3\epsilon - \eta,$$

$$u_k \sim \frac{ie^{-ik\tau}}{(2k)^{3/2}}, \tag{33}$$

$$\delta\varphi_k \sim -\frac{L}{a\kappa}\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right), \tag{34}$$

which corresponds to the Bunch-Davis vacuum [43].
In the long wavelength regime, the modes satisfy:

$k^2\tau^2 \ll 1,$

$$u_k \approx \frac{\sqrt{\pi}\, k^{-3/2}}{2^{3/2}\sin(n\pi)\Gamma(1-n)} \left(\frac{-k\tau}{2}\right)^{\frac{1}{2}-n} \begin{pmatrix} 1 - e^{-i\pi n}\frac{\Gamma(1-n)}{\Gamma(1+n)}\left(\frac{-k\tau}{2}\right)^{2n} \\ -\frac{\Gamma(1-n)}{\Gamma(2-n)}\left(\frac{-k\tau}{2}\right)^{2} \end{pmatrix}, \qquad (35)$$

with the index $n = \sqrt{\beta + \frac{1}{4}}$.

In order to have a physical interpretation of this result, we can relate it to the gauge invariant curvature perturbation on *comoving* hypersurfaces $\zeta_\varphi = \Psi - \frac{\dot{a}}{a}\frac{\delta\varphi}{\dot{\varphi}}$. To be entirely rigorous, we can also consider the curvature perturbations on uniform-energy-density hypersurfaces ζ_ρ or the curvature perturbations on uniform-*effective*-energy-density hypersurfaces $\zeta_{\rho_{eff}}$:

$$\zeta_\varphi = \Psi - \frac{\dot{a}}{a}\frac{\delta\varphi}{\dot{\varphi}}, \quad \zeta_\rho = \Psi - \frac{\dot{a}}{a}\frac{\delta\rho}{\dot{\rho}} \quad \text{or} \quad \zeta_{\rho_{eff}} = \Psi - \frac{\dot{a}}{a}\frac{\delta\rho_{\rm eff}}{\dot{\rho_{\rm eff}}}, \qquad (36)$$

where $T^{\rm eff}_{\mu\nu}$ is given by the right hand side of (25). All those three quantities are conserved at long wavelengths by conservation of energy. We may indeed check that for energy density coming from an inflationary scalar field,

$$\begin{aligned} \zeta_\rho &= \Psi + \frac{a^2\,\delta\rho_\varphi}{3\dot{\varphi}_0^2}, \\ \dot{\zeta}_\rho &= -\frac{4\frac{\dot{a}}{a} + 2\frac{\ddot{\varphi}_0}{\dot{\varphi}_0}}{3\dot{H}}\,\nabla^2\Psi. \end{aligned} \qquad (37)$$

Since $T^{\rm eff}_{\mu\nu}$ is also conserved (despite not being linearly related to $T_{\mu\nu}$), we may again argue that $\zeta_{\rho_{eff}}$ is conserved at long wavelengths. As pointed out in section 2.4, if the adiabaticity condition holds for linear perturbations, then $\zeta_{\rho_{eff}}$, ζ_ρ and ζ_φ should coincide at long wavelengths,

$$\text{for } k^2\tau^2 \ll 1, \qquad \langle\zeta_\rho^2\rangle \simeq \langle\zeta_{\rho_{eff}}^2\rangle \simeq \langle\zeta_\varphi^2\rangle.$$

In the following we will concentrate on ζ_φ. In the slow-roll regime, at long wavelengths,

$$\delta\varphi_k \simeq -\frac{2LH}{\kappa}u \simeq -\frac{iLH}{\sqrt{2}\kappa k^{3/2}}\,e^{-ik\tau}, \qquad (38)$$

$$\zeta_\varphi \simeq -\frac{\dot{a}}{a}\frac{\delta\varphi_k}{\dot{\varphi}_0} \sim \frac{ia}{\sqrt{2}\,k^{3/2}}\frac{H^2}{\dot{\varphi}_0}\,e^{-ik\tau}. \qquad (39)$$

The power spectrum is therefore given by the standard expression:

$$\mathcal{P} \sim k^3\langle\zeta^2\rangle \sim \frac{L^2}{2\kappa^2}\frac{a^2H^4}{\dot{\varphi}_0^2}\bigg|_{\tau=\tau^*} \sim \frac{H^6}{V_{,\varphi}^2}\bigg|_{\tau=\tau^*} \tag{40}$$

$$\sim \frac{V^3(1+V/2\mathcal{T})^3}{M_4^6\, V_{,\varphi}^2}\bigg|_{\tau=\tau^*}, \tag{41}$$

with τ^* the time of horizon crossing when $k = aH$. Once again the only departure from the standard four-dimensional inflation result only comes from the modification of the Friedmann equation (15) at the background level. Expressed in terms of the potential, the power spectrum will therefore get an overall factor of $\left(1+\frac{V}{2\mathcal{T}}\right)^3$, as mentioned in [29]. For a given potential, it will appear to be redder than for chaotic inflation.

For *small* β (we will study in the following the conditions for β to be small), the spectral index is given by:

$$n_S - 1 = \frac{d\ \ln \mathcal{P}_\zeta}{d\ \ln k} = -2\beta + 2\beta^2 + \mathcal{O}(\beta^3). \tag{42}$$

We define the slow-roll parameters:

$$\epsilon = -\frac{\dot{H}}{aH^2} \quad \text{and} \quad \lambda^{(n)} = \frac{d^n \ln \epsilon}{d \ln a^{\ n}}. \tag{43}$$

Each parameter $\lambda^{(n)}$ may be treated as a constant if $\lambda^{(n+1)} \ll 1$ (writing $\lambda^{(0)} = \epsilon$). In terms of those parameters, β takes the *exact* form:

$$\beta = -\frac{a^2H^2\tau^2}{4}\left[\begin{array}{l} 2\lambda^{(2)} - \left(2+\lambda^{(1)}\right)\left(2\epsilon+\lambda^{(1)}\right) \\ -\frac{L^2H^2}{(1+L^2H^2)^2}\ \epsilon\left(2+6\epsilon+L^2H^2(2+3\epsilon)\right)\end{array}\right]. \tag{44}$$

We have made two assumptions on β: we required it to be small, $\beta \ll 1$, and we treated it as a constant. For β to be small, the parameters ϵ and $\lambda^{(1)}, \lambda^{(2)}$ need to be small. This will be translated into the slow-roll parameter condition. Neglecting the $\lambda^{(n)}$, (ϵ can be considered as a constant), $\epsilon = -\frac{\dot{H}}{aH^2}$ implies that $a \simeq -\frac{1}{H\tau^{\frac{1}{1-\epsilon}}}$. The overall coefficient in the expression (44) of β is therefore $a^2H^2\tau^2 \sim \tau^{-2\frac{\epsilon}{1-\epsilon}}$. Thus it is consistent to treat β as a constant as has been done to get the expression (35) in the regime where $\epsilon \ll 1$, $\lambda^{(1)} \ll 1$ and $\lambda^{(2)} \ll 1$.

Up to first order in the slow-roll parameters ϵ and $\lambda^{(1)}$, (considering all other $\lambda^{(n)}$ to be negligible), the spectral index takes the form:

$$n_S - 1 \;=\; -2\epsilon - \lambda^{(1)} - \frac{L^2 H^2}{1 + L^2 H^2}\,\epsilon. \tag{45}$$

We recover the standard result for the spectral index if we define the second slow-roll parameter η such that:

$$n_S - 1 \;=\; -6\epsilon + 2\eta + \mathcal{O}(\epsilon, \eta). \tag{46}$$

In that case, the parameters ϵ and η may be expressed in terms of the scalar field as pointed out in [29, 44, 45]:

$$\epsilon = -\frac{\dot{H}}{aH^2} = \frac{L}{2\,\kappa}\frac{1 + V/\mathcal{T}}{(1 + V/2\mathcal{T})^2}\frac{V_{,\varphi}^2}{V^2}, \tag{47}$$

$$\eta = -\frac{(HH')'}{H^2 H'} - \frac{1}{2}\frac{L^2 H^2}{1 + L^2 H^2}\,\epsilon = \frac{L}{\kappa}\frac{1}{1 + V/2\mathcal{T}}\frac{V_{,\varphi\varphi}}{V}, \tag{48}$$

where a prime represents the derivative $\frac{d}{dt}$ with respect to the proper time t: $\frac{d}{dt} = \frac{d}{ad\tau}$.

If we worked at higher order in the slow-roll parameters, there would again be some departure from standard four-dimensional gravity in the spectral index that could be eliminated by an adequate redefinition of the third slow-roll parameter ξ^2 (as defined for instance in [41, 46, 45]). We do not wish to show the calculation explicitly here, but it can be seen that if the second order parameter ξ^2 is defined in our case such that $\xi^2 = \frac{1}{1+V/2\mathcal{T}}\xi_{4\mathrm{d}}^2 + \frac{2\,L^2 H^2}{1+L^2H^2}\,\epsilon^2$, the power spectrum to second order in the slow-roll parameters will recover the same form as in the standard four-dimensional chaotic inflation.

At this point, from the knowledge of the amplitude of the scalar perturbations *at long wavelengths* and their scale-dependence, it is not possible to distinguish between a model of standard four-dimensional chaotic inflation with potential $V^{(4\mathrm{d})}$ satisfying the standard slow-roll conditions with parameters $\epsilon_{4\mathrm{d}}$, $\eta_{4\mathrm{d}}$, $\xi_{4\mathrm{d}}$, and a model of brane inflation with a potential V, such that at the beginning of inflation when the first modes exit the horizon,

$$V^{(4\mathrm{d})} \simeq V\left(1+\frac{V}{2\mathcal{T}}\right)\left(1+\frac{V}{\mathcal{T}}\right), \tag{49}$$

$$V^{(4\mathrm{d})}_{,\varphi} \simeq V_{,\varphi}\left(1+\frac{V}{\mathcal{T}}\right), \tag{50}$$

$$V^{(4\mathrm{d})}_{,\varphi\varphi} \simeq V_{,\varphi\varphi}\left(1+\frac{V}{\mathcal{T}}\right), \tag{51}$$

$$V^{(4\mathrm{d})}_{,\varphi\varphi\varphi} \simeq V_{,\varphi\varphi\varphi}\left(1+\frac{V}{\mathcal{T}}\right)+\frac{V^3_{,\varphi}}{V^2}\frac{(1+V/\mathcal{T})^3}{(1+V/2\mathcal{T})}\frac{4V/\mathcal{T}}{1+2V/\mathcal{T}\,(1+V/2\mathcal{T})}\,. \tag{52}$$

We argue that observations of long wavelength scalar perturbations alone are not enough to differentiate between standard chaotic inflation and inflation on a brane with a potential satisfying the modified slow-roll conditions $\epsilon, \eta \ll 1$, η as given in (48). Such observations are not sufficient to distinguish between inflation occurring in a purely four-dimensional universe and on a brane embedded in a fifth dimension.

To extend this study we will consider, in the next section, typical effects that may arise on the brane due to the non-local nature of the theory. We will see how the behaviour of the perturbations may differ in that case from the standard case. However we may first notice that in the limit of large wavelengths comparing scalar and tensor perturbations will give a different signature in steep brane inflation than in standard four-dimensional inflation with one scalar field.

3.3 Tensor Modes

The scalar field φ being the only source of matter of the theory, the effect of the quadratic term in the matter stress-energy tensor on the behaviour of tensor perturbations will strictly be a "background effect" (ie. introducing some $T^0_{\mu\alpha}T^0_{\nu\beta}h^{(t)\,\alpha\beta}$-kind of source terms, with $T^0_{\mu\alpha}$ the background value of the stress-energy tensor). For purely tensor perturbations, the vector part (8) of the Weyl tensor must vanish. Indeed, we can consider the metric perturbation:

$$ds^2 = -a^2(\tau)\,d\tau^2 + a^2(\tau)\,\left(\delta_{ij}+h_{ij}\right)dx^i dx^j, \tag{53}$$

where the three-dimensional tensor h_{ij} is transverse and traceless $h^i_i = 0$, $\partial_i h^i_j = 0$, and indices are raised with δ^{ij}. With respect to this metric, we may indeed check that $\nabla_\mu E^\mu_\nu = 0$. Using the background equation of motion, the tensor modes satisfy the standard equation:

$$\ddot{h}_{ij} + 2\frac{\dot{a}}{a}\dot{h}_{ij} - \nabla^2 h_{ij} = 0, \tag{54}$$

where the only difference to the chaotic inflation case arises in the relation of the scale factor a to the potential through the modified Friedmann equation (14, 15). We can therefore treat the tensor perturbations the same way as is usually done in standard inflation. The power spectrum is given by [29, 46]:

$$\mathcal{P}_g = \frac{72}{M_4^2} H^2|_{\tau=\tau^*} \tag{55}$$

$$\simeq \frac{24}{M_4^4} V \left(1 + \frac{V}{2\mathcal{T}}\right)\Bigg|_{\tau=\tau^*}. \tag{56}$$

Here again, we notice that for a given potential, the power spectrum for gravitational waves will appear to be redder. However the overall factor is less important than for the scalar power spectrum which will be reflected in the ratio.

The tensor spectral index can be derived the usual way:

$$n_T = \frac{d \ln \mathcal{P}_g}{d \ln k} = -\frac{L^2 \mathcal{T}}{6} \frac{V_{,\varphi}^2}{V^2} \frac{1 + \frac{V}{\mathcal{T}}}{(1 + \frac{V}{2\mathcal{T}})^2} = -2\epsilon, \tag{57}$$

and the ratio r of the amplitude of the tensor perturbations at Hubble crossing to the scalar perturbations is modified by a factor:

$$r \simeq \frac{\epsilon}{1 + V/\mathcal{T}}\Bigg|_{\tau=\tau^*}. \tag{58}$$

As mentioned in [29], even though the tensor spectral tilt remains the same as in standard inflation: $n_T = -2\epsilon$, for high-energy $V/\mathcal{T} \gg 1$, the ratio of tensor to scalar perturbations on the brane will in general be smaller than what is expected from ordinary single scalar field inflation. However for slightly more complicated models of four-dimensional inflation, such as hybrid inflation, the ratio r will similarly be reduced in comparison to chaotic inflation and the relation of tensor to scalar perturbations is still not enough to distinguish between such a model and a scenario of brane inflation. The Cyclic-model predicts as well a low amplitude for the tensor modes. In the following section we shall study some more fundamental features of the five-dimensional nature of this model.

3.4 Estimation of the Non-Gaussianity

In order to have a consistency check on the validity of the perturbative approach, we estimate the non-gaussianity corrections to the power spectrum. We consider the non-gaussian part to be strongly dominated by the scalar field perturbations. In general, an estimate can be given by comparing the cubic terms in the Lagrangian to the quadratic ones. Unfortunately this method cannot be used here, so as a first approximation we will compare the quadratic term in the redefined gauge invariant comoving curvature to the previous quantity. Using the result of [47, 48], to second order, the gauge invariant curvature perturbation on uniform-energy-density hypersurfaces $\zeta_\varphi^{(2)}$ is given by:

$$\zeta_\varphi^{(2)} = \frac{\left(\dot{\Psi} + 2\frac{\dot{a}}{a}\Psi + \frac{\dot{a}}{a}\frac{\delta\dot{\varphi}}{\dot{\varphi}}\right)^2}{\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{\dot{a}}{a}\frac{\ddot{\varphi}}{\dot{\varphi}}}. \tag{59}$$

Again we could be interested in the curvature perturbation on uniform-effective-energy-density hypersurfaces or on comoving hypersurfaces instead, but for the purpose of this estimate, those three quantities will give a coinciding result. The ratio of the second term to the first one in this expansion gives an estimation which is the same as in the context of slow-roll inflation:

$$\frac{\sqrt{\zeta_\varphi^{(2)}}}{\zeta_\varphi^{(1)}} \simeq \frac{3}{\sqrt{2}}\,\epsilon. \tag{60}$$

In this estimate, some explicit L^2H^2 corrections will arise at second order in the slow-roll parameter ϵ. However this remains an estimate and it is presented here only to point out that the non-linear terms seem to be damped in comparison to the linear ones by an order of magnitude proportional to the slow-roll parameter which is precisely what is found in the context of standard four-dimensional inflation.

4 Corrections from the Weyl Tensor

In section 2.4, we have pointed out the presence of a transverse and traceless term in the Weyl tensor $E_{\mu\nu}^{TT}$. So far, this term has been neglected. This was motivated by the fact that $E_{\mu\nu}^{TT}$ vanishes for the background and we can check that within this approximation, we recover the long wavelength limit of the exact five-dimensional theory. We gave as well an argument to justify this approximation in the regime of long wavelength adiabatic perturbations for scalar fields. However, there is no reason for the tensor $E^{TT}_{\mu\nu}$ to remain negligible in general. This term actually encodes information about the bulk geometry. At the perturbed level, the bulk geometry is not purely AdS anymore. The fluctuations in the bulk geometry will generate some Kaluza-Klein (KK) corrections on the branes. Those KK modes are mediated by the only term which remains undetermined from a purely four-dimensional point of view: the tensor $E^{TT}_{\mu\nu}$.

In this section, we want to modify the four-dimensional effective theory in order to study the typical kind of corrections that may arise from the perturbed bulk geometry. The only modification consistent with the overall five-dimensional nature of the model is to add a contribution coming from $E^{TT}_{\mu\nu}$. We want to modify the effective theory in order to accommodate terms that are negligible in the long wavelength limit of the five-dimensional theory. We therefore need to consider terms of higher order in derivatives (compared to the other terms already present in the theory). From the properties of the five-dimensional Weyl tensor, together with the Bianchi identity on the brane, $E^{TT}_{\mu\nu}$ satisfies the following properties:

- $E^{TT}_{\mu\nu} = 0$ in the background;
- $E^{TT\ \mu}_{\mu} = 0$;
- $\nabla_\mu E^{TT\ \mu}_{\nu} = 0$.

This has already been mentioned in section 2.4. If we want to consider a non-negligible contribution $E^{corr}_{\mu\nu}$ arising from $E^{TT}_{\mu\nu}$, this correction term $E^{corr}_{\mu\nu}$ has to satisfy the same properties. Since $E^{corr}_{\mu\nu}$ is transverse and traceless, we might think of it as being derived from a conformally invariant action. Furthermore, the correction $E^{corr}_{\mu\nu}$ should vanish for conformally flat spacetimes, (since the effective theory with $E^{TT}_{\mu\nu} = 0$ is exact for that case). We restrict ourselves to a correction $E^{corr}_{\mu\nu}$, which is a functional of the metric only. Then the most straightforward term derived from a conformally invariant action, which vanishes for conformally flat spacetimes and is of higher order in derivative, is:

$$A_{\mu\nu} = \frac{1}{2\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^4x \sqrt{-g}\, C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}, \tag{61}$$

$$\begin{aligned} A_{\mu\nu} = & -\Box R_{\mu\nu} + \frac{1}{3}\nabla_\mu \nabla_\nu R + \frac{1}{6}\Box R g_{\mu\nu} - \frac{1}{6} R^2 g_{\mu\nu} \\ & + \frac{2}{3} R R_{\mu\nu} + \frac{1}{2} R_{\alpha\beta} R^{\alpha\beta} g_{\mu\nu} - 2 R_{\mu\alpha\nu\beta} R^{\alpha\beta}. \end{aligned} \tag{62}$$

From the properties of the four-dimensional Weyl tensor $C_{\alpha\beta\gamma\delta}$, $A_{\mu\nu}$ is indeed traceless and vanishes for conformally flat spacetimes. There is therefore no reason why this term would not be present in the electric part $E_{\mu\nu}$ of the five-dimensional Weyl tensor. The purpose of this section will be to study the effect when such a term is introduced:

$$R_{\mu\nu} = \frac{\kappa}{L}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) - \frac{\kappa^2}{4}\left(T^\alpha_\mu T_{\alpha\nu} - \frac{1}{3} T T_{\mu\nu}\right) - E_{\mu\nu}, \tag{63}$$

$$\begin{aligned} E_{\mu\nu} &= \mathcal{E}^{(L)}_{\mu\nu} + E^{TT}_{\mu\nu}, \\ E^{TT}_{\mu\nu} &= 2\alpha\, L^2 A_{\mu\nu} + O(R^3_{\mu\nu}), \end{aligned} \tag{64}$$

where α is a dimensionless parameter that we assume to be small. All through this study, we will work only up to first order in α. We may emphasize again that $A_{\mu\nu}$ does not alter the background behaviour and will be present only at the perturbative level.

Further motivation for this study is given in appendix B. We show there that this correction correctly reproduces the first KK mode for perturbations around two static branes when matter is introduced only on one brane. Since this term is present in the two brane static limit, it seems natural to study its role within a brane inflation setup.

4.1 Tensor perturbations

We consider the previous scenario of brane inflation from section 3 and keep all terms in L^2H^2 so that the analysis remains consistent at high energy. To start with, we consider tensor perturbations in detail. We denote by h_{ij} the three-dimensional tensor perturbation. By the same argument used previously, the quadratic terms in the modified Einstein equation have only a "background" effect on the tensor perturbations. In particular, they can be derived from the action:

$$S_t^{(2)} = \int d^4x \left(a^2\, \eta^{\mu\nu} \partial_\mu h^i_j\, \partial_\nu h^j_i \right. \\ \left. + \alpha\, L^2\, \eta^{\mu\nu} \eta^{\alpha\beta} \left(2\, \partial_{\mu\alpha} h^i_j\, \partial_{\nu\beta} h^j_i - \partial_{\mu\nu} h^i_j\, \partial_{\alpha\beta} h^j_i \right) \right) . \tag{65}$$

This gives rise to the equation of motion for h_{ij} (omitting the ij-indices for now):

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} - \nabla^2 h + \frac{\alpha L^2}{a^2}\left(\ddot{\ddot{h}} - 2\nabla^2 \ddot{h} + \nabla^4 h \right) = 0. \tag{66}$$

We use here the assumption that the constant α is small and expand h as a series in α: $h = h_0 + \alpha\, h_1 + \mathcal{O}(\alpha^2)$, where $h_{0,1}$ do not depend on α. To lowest order in α, we should recover the results from section 3.3. h_0 therefore satisfies: $\ddot{h}_0 + 2\frac{\dot{a}}{a}\dot{h}_0 - \nabla^2 h_0 = 0$. Keeping only the terms up to first order in α, the last term in eq. (66) can be written as:

$$\frac{\alpha L^2}{a^2}\left(\ddot{\ddot{h}} - 2\nabla^2 \ddot{h} + \nabla^4 h \right) = \frac{\alpha L^2}{a^2}\left(\ddot{\ddot{h}}_0 - 2\nabla^2 \ddot{h}_0 + \nabla^4 h_0 \right) + \mathcal{O}(\alpha^2) \tag{67}$$

$$= \frac{\alpha L^2}{a^2}\Big(-2 \underbrace{\left[\frac{\dddot{a}}{a} - 9\frac{\dot{a}\ddot{a}}{a^2} + 12\frac{\dot{a}^3}{a^3} \right]}_{=2\dot{a}aH^2\epsilon + \epsilon\,\mathcal{O}(\epsilon,\eta)} \dot{h} + 4 \underbrace{\left[\frac{2\dot{a}^2}{a^2} - \frac{\ddot{a}}{a} \right]}_{=a^2H^2\epsilon} \nabla^2 h \Big) + \mathcal{O}(\alpha^2). \tag{68}$$

Using this approximation, eq. (66) simplifies considerably:

$$\ddot{h} + 2\frac{\dot{a}}{a}\left(1 - 2\alpha L^2 H^2\, \epsilon \right) \dot{h} - \left(1 - 4\alpha L^2 H^2\, \epsilon \right) \nabla^2 h = \mathcal{O}(\alpha^2). \tag{69}$$

In this result, the terms beyond first order in the slow-roll parameters have been omitted. Eq. (69) is consistent with the original eq. (66) if we study only the terms up to first order in α in the expression of h. This is an important assumption since the fourth order differential equation reduces to a second order one, allowing us to specify only two parameters on the initial Cauchy surface instead of four. Using this assumption, the requirement that we recover a normalized Minkowski vacuum when the modes are well inside the horizon is then enough to specify h_{ij}. Similarly to the standard case, it is simpler to study the associated variable u instead:

$$z = a\left(1 + \alpha L^2 H^2 (1+\epsilon)\right), \tag{70}$$

$$u = zh, \tag{71}$$

$$c_h^2 = 1 - 4\epsilon\, \alpha L^2 H^2, \tag{72}$$

$$\beta = 2 + 3\epsilon\left(1 - 2\alpha L^2 H^2\right), \tag{73}$$

so that the second order differential equation for the tensor perturbations simplifies to:

$$\ddot{u} + \left(c_h^2\, k^2 - \frac{\beta}{\tau^2}\right) u = 0, \tag{74}$$

up to first order in the slow-roll parameter ϵ and in α.

The main point to notice is that the tensor modes do not propagate at the speed of light anymore but at the speed of sound c_h. The C^2-corrections (as introduced in $A_{\mu\nu}$) will therefore modify the effective speed of linear perturbations on the brane. There is a priori no reason for the constant α to be positive. If α is negative, it will therefore be interesting to study whether or not the speed of sound being effectively larger than the speed of light from a four-dimensional point of view may result in instabilities at higher order.

The amplitude of the tensor perturbations at sound horizon crossing ($aH = c_h k$) is multiplied by an extra mode-dependent factor:

$$\mathcal{P}_g \simeq \frac{24}{M_4^4} V \left(1 + \frac{V}{2\mathcal{T}}\right)\left(1 - 2\alpha L^2 H^2\right)\Big|_{\tau=\tau^*}. \tag{75}$$

The effect of the C^2-corrections is extremely simple and the extra term becomes negligible at long wavelengths as $k^2\tau^2 \ll 1$ and at low-energy, but it is not negligible at high energy. Similarly, the tensor spectral index is modified by an extra mode-dependent factor:

$$n_T = -2\epsilon\left(1 - 2\alpha L^2 H^2\right). \tag{76}$$

Neglecting the slow-roll parameters, H may be treated as constant in this result. We can interpret the previous result as a background redefinition:

$$\bar{\epsilon} = \epsilon\,\left(1 - 2\alpha L^2 H^2\right), \tag{77}$$

even if the C^2-corrections did not perturb the background behaviour.

4.2 Scalar perturbations

The scalar perturbations can be treated within the same philosophy as the tensor perturbations so some of the details will be skipped.

In longitudinal gauge, for scalar isotropic perturbations,

$$A_{\mu\nu} = \frac{2}{3\,a^2} \begin{bmatrix} \nabla^4 & \nabla^2 \partial_i \\ \nabla^2 \partial_i & \left[\frac{1}{2}\left(\partial_\tau^2 - \nabla^2\right)\left(\partial_{ij} - \nabla^2 \delta_{ij}\right) + \partial_{ij}\partial_\tau^2\right] \end{bmatrix} (\Psi + \Phi) \tag{78}$$

Introducing this term in the modified Einstein equation (63), the equation (32) for u gets modified to:

$$u = \tilde{z}\Psi, \tag{79}$$

$$\ddot{u} + \left(c_u^2 k^2 - \frac{\tilde{\beta}}{\tau^2}\right) u = 0, \tag{80}$$

with the modified parameters:

$$\tilde{z} = z\,\left(1 + \alpha\, z_1\right), \tag{81}$$

$$z_1 = \frac{4L^2H^2}{3}\left(1 + \frac{1}{2}\frac{L^2H^2}{1 + L^2H^2}\,\epsilon\right) + 2L^2H^2\left(1 + 2\eta - \frac{1 - \frac{1}{2}L^2H^2}{1 + L^2H^2}\,\epsilon\right), \tag{82}$$

$$c_u^2 = 1 + 8\alpha L^2H^2\left(1 + \frac{2}{3}\frac{1 + 7/4L^2H^2}{1 + L^2H^2}\,\epsilon\right), \tag{83}$$

$$\tilde{\beta} = \beta + 4\alpha L^2H^2\,\epsilon. \tag{84}$$

In the next section, the exact result is developed in the low-energy regime. In particular it is shown that the duality between slow-roll and fast-roll conditions in the production of a scale invariant spectrum remains valid when those kinds of corrections are taken into account. For now, for simplicity, we consider the first order terms in the slow-roll parameter only; in the previous result, terms beyond first order in α or in the slow-roll parameters have been omitted.

We can notice here the same phenomenon as for the tensor perturbations: the scalar perturbations do not propagate at the speed of light any more but at the speed of sound c_u, which is not the same speed as for tensor perturbations.

The rest of the discussion remains the same. Assuming again that $\tilde{\beta}$ may be treated as a constant, an analytical solution can be found for (80) and the constants are chosen so that we obtain Minkowski vacuum $u_k \approx \frac{i\,e^{-ic_u k\tau}}{(2\,c_u k)^{3/2}}$ in the regime where $c_u^2 k^2 \tau^2 \gg \tilde{\beta}$. The expression between the scalar field perturbations $\delta\varphi$ an the variable u is:

$$\delta\varphi = -\frac{2L}{a\kappa}\left[\left(1+2\alpha L^2 H^2 + \mathcal{O}(\epsilon)\right)\dot{u} + \frac{\ddot{\varphi}_0}{\dot{\varphi}_0}\left(1+2\alpha L^2(2\nabla^2 + H^2 + \mathcal{O}(\epsilon))\right)u\right] \tag{85}$$

so that at short distances, the spacetime is locally flat and we recover the Bunch-Davis vacuum for the scalar field perturbations:

$$\begin{aligned}
&\text{for } c_u^2 k^2 \tau^2 \gg \beta^2, \\
&u_k \approx \frac{ie^{-i\,c_u k\tau}}{(2\,c_u k)^{3/2}}, \\
&\delta\varphi_k \approx -\frac{L}{a\kappa}\frac{e^{-ic_\varphi k\tau}}{\sqrt{2c_\varphi k}}\left(1-\frac{i}{c_\varphi k\tau}\right)\left(1+6\alpha L^2 H^2\right), \\
&\text{with } \; c_u = 1+4\alpha L^2 H^2 + \mathcal{O}(\epsilon) \quad \text{and} \quad c_\varphi = 1 + 8\alpha L^2 H^2 + \mathcal{O}(\epsilon).
\end{aligned} \tag{86}$$

We point out that the scalar field perturbations propagate at a speed of sound slightly different from the speed of sound of the Mukhanov scalar u.

Following the same procedure as before, the spectral index for scalar perturbations reads:

$$n_S - 1 = -6\epsilon + 2\eta - 8\alpha L^2 H^2\,\epsilon, \tag{87}$$

which could be again interpreted as a redefinition of the slow roll-parameter:

$$\tilde{\eta} = \eta + 10\alpha L^2 H^2\,\epsilon. \tag{88}$$

The overall amplitude of the scalar perturbations gets a mode-dependent factor:

$$\mathcal{P} \sim \frac{V^3\,\left(1+\frac{V}{2T}\right)^3}{M_4^6\,V_{,\varphi}^2}\left(1-16\,\alpha\,L^2 H^2\right)\Big|_{\tau=\tau^*/c_u}. \tag{89}$$

We obtain the same kind of corrections as for tensor perturbations (75) which again might be significant at high-energy. Since the scalar perturbation propagates at a speed of sound c_u, there is an extra factor when evaluating (89) due to the fact that the modes exit the horizon at a different time: $V\left(1+\frac{V}{2\mathcal{T}}\right)\Big|_{\tau^*/c_u} \sim V\left(1+\frac{V}{2\mathcal{T}}\right)\left(1-8\epsilon\alpha L^2H^2\right)\Big|_{\tau^*}$, but this difference is negligible at long wavelengths.

The ratio between the amplitude of the tensor perturbations to the scalar ones acquires the additional mode-dependent factor (to first order in ϵ):

$$r = \frac{\epsilon}{1+V/\mathcal{T}}\left(1+16\alpha L^2H^2\right)\Big|_{\tau\approx\tau^*}. \tag{90}$$

We may as well check that non-linear corrections remain small in the context of this perturbative approach:

$$\frac{\sqrt{\zeta_\rho^{(2)}}}{\zeta_\rho^{(1)}} \simeq \frac{3}{\sqrt{2}}\,\epsilon - 8\sqrt{2}\alpha L^2H^2\Big|_{\tau=\tau^*}. \tag{91}$$

We can see that the ratio is slightly modified by a mode-dependent term which is *not* damped by the slow-roll parameter. Within this approach the correction term is required to be small since we are making an expansion in α. However, a region can exist for which α might be small but still important compared to ϵ. In this case some deviations from non-gaussianity might be observed. (We can point out that they will not be present at low-energy, when the terms L^2H^2 are negligible.) This is one of the most interesting consequences of the addition of the C^2-corrections within the context of brane inflation. Another result is presented in the next section, where we will study the consequences of those corrections while considering the production of perturbation with a fast-roll potential.

4.3 Slow-Roll, Fast-Roll conditions in the production of scale invariant spectrum

It has been shown [11, 12] that within the low-energy approximation, there is an exact duality between "inflation"-like and "Ekpyrotic or Cyclic"-like potentials which give rise to the same observational features (for scalar perturbations). We intend to study here whether this duality is preserved when the C^2-corrections are taken into account. In the setup of the previous section the scalar field was confined on the brane. We will make the analogy between this case and the case for which the scalar field can be interpreted as the dilaton in a two brane Randall-Sundrum model. In that case the dilaton does not live on a specific brane, however from the four-dimensional effective point of view, the formalism will be the same. In what follows we will work in a low-energy regime and neglect the quadratic terms in the stress-energy tensor. In the Cyclic scenario, the two boundary branes from the Randall-Sundrum model are taken to be empty (at the time when the fluctuations responsible for the structure are produced) and the effective four-dimensional theory as derived in section 2.3 is modified by the introduction of a potential V:

$$S = \int d^4x \sqrt{-g} \left(\frac{L}{2\kappa} R - \frac{1}{2} \left(\partial \phi \right)^2 - V(\phi) \right). \tag{92}$$

V represents an interaction between the two branes which may come from bulk fields such as in the Goldberger-Wise mechanism. The metrics on the branes are conformally related to the effective one by the relations (7). The action (92) gives some equations of motion which are consistent with the background ones. However the addition of the term $\alpha L^2 \, C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}$ in this action will not alter the background behaviour and thus will be a consistent term to consider. When deriving the covariant four-dimensional effective action in section 2.3, there is indeed no reason why such a term could not have been introduced. We will therefore consider the following action:

$$S = \int d^4x \sqrt{-g} \left(\frac{L}{2\kappa} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) + \alpha L^2 \, C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \right). \tag{93}$$

It is important to remember that the Einstein frame has no physical significance here; it is in the sense of the theory translated back into the brane frame that we think of those corrections. But working in terms of (93) we may think of those corrections in the same way as we have done so far and compare them with the case of standard slow-roll inflation on the brane.

Following the work of [11], in the low-energy limit, the differential equation for the scalar modes may be expressed as:

$$\ddot{u} + \left(\hat{c}_s^2 k^2 - \frac{\hat{\beta}}{\tau^2} \right) u = 0, \tag{94}$$

where u is related to the curvature by $u = \frac{a}{\dot{\varphi}_0}\left(1+\alpha\,\hat{z}\right)\Psi$, with notation:

$$\epsilon = -\frac{\dot{H}}{aH^2},\ \lambda^{(n)} = \frac{d^n \ln\epsilon}{d\ln a^{\ n}}, \tag{95}$$

$$\hat{z} = \frac{4}{3}\frac{L^2k^2}{a^2} + 2L^2H^2\left(1+3\epsilon-\lambda^{(1)}\right), \tag{96}$$

$$\hat{c}_s^2 = 1 + \frac{8}{3}\alpha L^2 H^2\left(3+2\epsilon\right), \tag{97}$$

$$\hat{\beta} = -\frac{a^2H^2\tau^2}{4}\left(2\lambda^{(2)} - \left(2+\lambda^{(1)}\right)\left(2\epsilon+\lambda^{(1)}\right)\right), \tag{98}$$

$$+\,2\alpha L^2a^2\tau^2H^4\left(\begin{array}{l}\lambda^{(3)}-\lambda^{(2)}-\lambda^{(1)}\lambda^{(2)}-6\lambda^{(2)}\epsilon,\\ +2\epsilon+4\epsilon^2-6\epsilon^3+3\epsilon\lambda^{(1)}+11\epsilon^2\lambda^{(1)}\end{array}\right).$$

Using the same arguments as in section 3, eq. (94) will admit a nearly scale-invariant solution if the variation of $\hat{\beta}$ is negligible and if $\hat{\beta} \ll 1$. If the parameter ϵ is considered constant as a first approximation (we set all the $\lambda^{(n)}$ to zero to start with), then by integration, $H = \frac{1}{(\epsilon-1)a\tau}$. The expression for $\hat{\beta}$ simplifies to:

$$\hat{\beta} = \frac{\epsilon}{(\epsilon-1)^2}\left[1+4\alpha L^2H^2\left(1+2\epsilon-3\epsilon^2\right)\right], \tag{99}$$

$$H \sim \tau^{-\frac{\epsilon}{\epsilon-1}}. \tag{100}$$

Setting the constant α to zero in this expression gives rise to the standard result: $\hat{\beta}$ remains a small constant $\hat{\beta} \ll 1$ for both $\epsilon \ll 1$ and $\epsilon \gg 1$. However, adding the C^2-corrections in the effective theory gives rise to the slightly more complicated situation: in the standard slow-roll inflation case as studied in the previous section $\epsilon \ll 1$, so that $\hat{\beta} \simeq \epsilon\left(1+4\alpha L^2H^2\right)$, with $H \simeq$ constant during inflation, giving rise to the almost scale-invariant spectrum as discussed previously.

For $\epsilon \gg 1$, on the other hand,

$$\epsilon_{\rm F} = \frac{1}{2\epsilon} \ll 1, \tag{101}$$

$$\hat{\beta} = 2\,\epsilon_{\rm F}\left(1-3\alpha L^2H^2\epsilon_{\rm F}^{-2}\right). \tag{102}$$

$\epsilon_{\rm F}$ denotes the fast-roll parameter as introduced in [11].

Two important problems arise here. First of all, $\alpha L^2 H^2$ should be of the same order of magnitude or smaller than $\epsilon_{\rm F}^2$ to avoid the correction term becoming large. If not, the procedure breaks down as the correction becomes more important than the "leading" term and higher order terms in α have to be introduced as well as other corrections in the Weyl tensor. In the low-energy limit the term $L^2 H^2$ will indeed be small, but not necessarily small compared to $\epsilon_{\rm F}$. The second departure from the standard result arises from the τ-dependence of the correction term. In the context of fast-roll, the Hubble parameter is not constant but varies as $H \sim \tau^{-1}$. In general the corrections to $\hat{\beta}$ will not be constant and it will not be possible to follow the usual derivation from eq.(32) to get the expression (35) and find a scale invariant power spectrum.

In the Cyclic scenario, the large scale structure is produced while the Hubble parameter is still tiny and its variations small enough for the quantity $L^2 H^2 \epsilon_{\rm F}^{-2}$ to remain small and constant during the process. Those corrections will therefore have a negligible effect on the power spectrum of the perturbations. It will be consistent to keep treating $\hat{\beta}$ as a small constant, giving rise to the almost scale-invariant spectrum:

$$n_{S,\,\text{fast-roll}} - 1 = -4\epsilon_{\rm F} - 4\eta_{\rm F} + 12\,\alpha L^2 H^2 \epsilon_{\rm F}^{-1}\Big|_{\tau^*}\,, \tag{103}$$

with the second fast-roll parameter as defined in [11]: $\eta_{\rm F} = 1 - \eta\,\epsilon_{\rm F}$.

However in a more general case of braneworld cosmology, if we consider a general potential satisfying fast-roll conditions, we will generically expect to see some departure from a nearly-scale invariant spectrum when some C^2-corrections are introduced in the Weyl tensor.

5 Conclusion

In the first part of this work we pointed out the presence of a transverse part of the Weyl tensor, that vanishes for the background (but does not necessarily cancel in general). We began with the assumption that this part of the Weyl tensor could be neglected. This was shown to be a valid approximation in the case of adiabatic scalar perturbations.

Using this assumption we solved the modified Einstein equation for brane inflation driven by a scalar field in a slow-roll potential. The corrections coming from the quadratic terms in the stress-energy were tracked throughout.

We showed that the perturbations are anisotropic in longitudinal gauge, in contrast to the case of standard inflation. We showed how to extend the standard inflation variables and parameters to accommodate the behaviour of the quadratic term. The corrections to these terms arise from a purely background effect. Indeed, assuming adiabaticity, the corrections can only influence the background. For a given inflation potential, the power spectrum of both the tensor and the scalar perturbations are redder than in the normal four-dimensional case. Compared to scalar perturbations, the tensor perturbation amplitude is weaker. However for scalar perturbations, this model can be reinterpreted as standard four-dimensional inflation with a redefined potential, giving rise to the same astrophysical observations. These results have already been well-understood in a number of papers [29, 33, 45, 49, 50, 51, 52]. However our study is an important consistency check as it enables us to verify our prescription and to extend it. The relations between the brane inflation variables and the redefined standard inflation variables have been given with precision, and our results are reliable up to second order in the slow-roll parameters. Another important feature of our four-dimensional effective theory is its straightforward extension to more interesting and realistic scenarios where both boundary branes have their own dynamics. This opens the possibility of studying a large range of braneworld scenarios.

To understand the contribution of the Weyl tensor, we have extended what we knew of its background behaviour to knowledge of its behaviour in a quasi-static limit when the extra-dimension is finite. In that limit, when matter is introduced on one brane, the first Kaluza-Klein mode can be modelled by a tensor $A_{\mu\nu}$ which can be expressed in terms of local four-dimensional quantities. Motivated by this result we have studied the contribution of $A_{\mu\nu}$ in two particular examples.

First we analyzed the contribution of the tensor $A_{\mu\nu}$ in a model of brane inflation where again the typical energy scale was important compared to the brane tension. Several new features were observed which, as far as we are aware, extend previous results in chaotic brane inflation. First of all, both tensor and scalar modes propagate at a speed different from the speed of light. The corrections bring a new mode-dependent term in the amplitude of both tensor and scalar perturbations which does not compensate in their ratio and which could be important at high energy. There is as well a new mode-dependent term in the estimation of the non-gaussianity, proportional to the small constant α we have introduced. This term is not damped by a slow-roll parameter. This is a critical feature, addressing the issue of the importance of non-gaussianity when those corrections are introduced. Our prescription only makes sense when the constant α is small, so that we can not address the issue of non-gaussianity outside this regime, however there could be some situations for which the new term in the non-gaussianity estimation might be the leading one. In that case, the cubic terms might not be small compared to the quadratic ones, and a perturbative approach might not be sensible. These are important features that have to be considered seriously in order to distinguish between purely four-dimensional inflation and brane inflation. Comparing these results with observations might give a constraint on the order of magnitude of the constant α.

In the second example, we used the formalism to study how the duality relating density perturbations in expanding and contracting Friedmann cosmologies was affected by the introduction of the tensor $A_{\mu\nu}$. We compared the production of a scale-invariant spectrum in a model of "slow-roll" inflation where the typical energy scale was much smaller than the brane tension, with an Ekpyrotic or Cyclic model for which the scalar field was evolving in a potential satisfying "fast-roll" conditions. The first order corrections, proportional to the constant α, become negligible in the "slow-roll" limit but could be large in a general "fast-roll" limit. We therefore recover the nearly scale-invariant spectrum in the "slow-roll" inflation but the situation becomes more complicated in a general "fast-roll" scenario. In this case, unless some assumptions are made during the production of perturbations responsible for the observed structure (as in the case of Cyclic-model potentials), the departure from a scale-invariant spectrum could become more important when these corrections interfere. This is an important new result enabling us to differentiate between the "fast-roll" and "slow-roll" scenarios. However, when comparing the specific case of the Cyclic model with low-energy inflation, the corrections are negligible.

The main limitation of this study is the assumption made for the behaviour of the homogeneous part of the Weyl tensor. At each step, its behaviour has been imposed by hand. We have tried however to go beyond the usual assumptions and incorporate some effects of the brane nature of our theory by analogy with what is already known from purely four-dimensional theories. To be completely rigorous, one should ultimately try to attack the five-dimensional problem directly. Nonetheless, we hope that our formalism can be used to give greater analytical insight into typical braneworld effects.

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A Covariant formalism including the ρ^2 terms

Our starting point will be the assumption that for long wavelength adiabatic perturbations, the metrics on each brane remain conformal related as in (7),

$$g_{\mu\nu}^{(-)} = \Psi^2 g_{\mu\nu}^{(+)}, \tag{A.1}$$

where the conformal factor Ψ may be expressed in terms of the minimally coupled scalar field ϕ of (6) as $\Psi = -\tanh(\phi/\sqrt{6})$. This assumption is verified for cosmological metrics and as a first step we will assume it will remain valid for long wavelength adiabatic perturbations.

Using this assumption, the Ricci tensor on the negative tension brane can be expressed in terms of the Ricci tensor on the positive one:

$$\Psi^2 R_{\mu\nu}^{(-)} = \Psi^2 R_{\mu\nu}^{(+)} - 2\Psi\nabla_\mu\nabla_\nu\Psi + 4\,\partial_\mu\Psi\,\partial_\nu\Psi - g_{\mu\nu}^{(+)}\left(\Psi\Box\Psi + |\nabla\Psi|^2\right), \tag{A.2}$$

where all derivatives and contractions are taken with respect to the metric on the positive tension brane. This will be our convention throughout this section unless otherwise specified. Taking the trace of eq. (A.2), we get the equation of motion of the scalar field Ψ:

$$\Box\Psi = \frac{1}{6}\left(R^{(+)} - \Psi^2 R^{(-)}\right)\Psi, \tag{A.3}$$

where $R^{(-)} = g^{(-)\mu\nu} R_{\mu\nu}^{(-)}$. We notice that if the negative (resp. positive) tension brane is empty $R^{(-)} = 0$, the scalar field Ψ (resp. $1/\Psi$) is conformally invariant with respect to the positive (resp. negative) tension brane metric.

As already mentioned in section 2.4, the projected Ricci tensor on each brane can be expressed as:

$$R^{(\pm)}_{\mu\nu} = \pm\frac{\kappa}{L}\left(T^{(\pm)}_{\mu\nu} - \frac{1}{2}T^{(\pm)}g^{(\pm)}_{\mu\nu}\right) - \frac{\kappa^2}{4}\left(T^{(\pm)\alpha}_{\mu}T^{(\pm)}_{\alpha\nu} - \frac{1}{3}T^{(\pm)}T^{(\pm)}_{\mu\nu}\right) - E^{(\pm)}_{\mu\nu} \quad \text{(A.4)}$$

with $E^{(\pm)}_{\mu\nu}$ as in (4). For the background, $E^{(-)}_{\mu\nu}$ and $E^{(+)}_{\mu\nu}$ are therefore related:

$$\Psi^2 E^{(-)}_{\mu\nu} = E^{(+)}_{\mu\nu}. \quad \text{(A.5)}$$

A priori, $E^{(\pm)}_{\mu\nu}$ are not determined except for the following properties:

$$g^{(\pm)\mu\nu}E^{(\pm)}_{\mu\nu} = 0, \quad \text{(A.6)}$$

and the fact that their divergences must be consistent with the Bianchi identities on each brane $\nabla^{(\pm)}_{\mu}G^{(\pm)\,\mu}_{\nu} = 0$.

Using eq. (A.4), we can construct the quantity $\left(R^{(+)}_{\mu\nu} - \Psi^2 R^{(-)}_{\mu\nu}\right)$:

$$\left(R^{(+)}_{\mu\nu} - \Psi^2 R^{(-)}_{\mu\nu}\right) = \left(\Pi^{(+)}_{\mu\nu} - \Psi^2\Pi^{(-)}_{\mu\nu}\right) - \Delta E_{\mu\nu}, \quad \text{(A.7)}$$

where

$$\Delta E_{\mu\nu} = \left(E^{(+)}_{\mu\nu} - \Psi^2 E^{(-)}_{\mu\nu}\right), \quad \text{(A.8)}$$

and $\Pi^{(\pm)}_{\mu\nu} = \pm\frac{\kappa}{L}\left(T^{(\pm)}_{\mu\nu} - \frac{1}{2}T^{(\pm)}g^{(\pm)}_{\mu\nu}\right) - \frac{\kappa^2}{4}\left(T^{(\pm)\alpha}_{\mu}T^{(\pm)}_{\alpha\nu} - \frac{1}{3}T^{(\pm)}T^{(\pm)}_{\mu\nu}\right)$. In what follows we will write $\tilde{\Pi}^{(\pm)\mu}_{\ \ \nu} = \Pi^{(\pm)\mu}_{\ \ \nu} - \frac{1}{2}\Pi^{(\pm)}\delta^{\mu}_{\nu}$ with $\Pi^{(\pm)\mu}_{\ \ \nu} \equiv g^{(\pm)\mu\alpha}\,\Pi^{(\pm)}_{\alpha\nu}$.

Using the relation (A.2) between $R^{(-)}_{\mu\nu}$ and $R^{(+)}_{\mu\nu}$, the Ricci tensor on the positive tension brane may be expressed in terms of the scalar field Ψ:

$$R^{(+)}_{\mu\nu} = \Pi^{(+)}_{\mu\nu} + U^{\Psi}_{\mu\nu} - \Delta E_{\mu\nu}, \quad \text{(A.9)}$$

where $U^{\Psi}_{\mu\nu} = -2\Psi\nabla_\mu\nabla_\nu\Psi + 4\,\partial_\mu\Psi\,\partial_\nu\Psi - g^{(+)}_{\mu\nu}\left(\Psi\Box\Psi + |\nabla\Psi|^2\right) + \Psi^2 R^{(+)}_{\mu\nu} - \Psi^2\Pi^{(-)}_{\mu\nu}$ satisfies:

$$g^{(+)\mu\nu}U^{\Psi}_{\mu\nu} = 0, \quad \text{(A.10)}$$

$$\nabla^{(+)}_{\mu}U^{\Psi\,\mu}_{\ \ \ \nu} = -\,\Psi^2\nabla^{(-)}_{\mu}\tilde{\Pi}^{(-)\mu}_{\ \ \ \nu}. \quad \text{(A.11)}$$

The covariant derivatives are taken with respect to $g^{(+)}_{\mu\nu}$, unless otherwise stated.

Since the brane metrics are conformal to each other, the traceless property of $E^{(\pm)}_{\mu\nu}$ can be extended to $\Delta E_{\mu\nu}$. Furthermore, from the Bianchi identity the divergence of $\Delta E_{\mu\nu}$ is:

$$\nabla^{(+)}_{\mu}\left(g^{(+)\mu\alpha}\Delta E_{\alpha\nu}\right) = \nabla^{(+)}_{\mu}\tilde{\Pi}^{(+)\mu}{}_{\nu} - \Psi^2\nabla^{(-)}_{\mu}\tilde{\Pi}^{(-)\mu}{}_{\nu}. \tag{A.12}$$

The traceless tensor $\Delta E_{\mu\nu}$ can be decomposed into a vector part and a traceless, divergenceless tensor:

$$\Delta E_{\mu\nu} = \mathcal{E}_{\mu\nu} + E^{TT}_{\mu\nu}, \tag{A.13}$$

$$\mathcal{E}_{\mu\nu} = \nabla_\mu\mathcal{E}_\nu + \nabla_\nu\mathcal{E}_\mu - \frac{1}{2}g^{(+)}_{\mu\nu}\nabla_\alpha\mathcal{E}^\alpha, \tag{A.14}$$

$$\nabla_\mu E^{TT\,\mu}{}_{\nu} = 0 \text{ and } E^{TT\,\mu}{}_{\mu} = 0. \tag{A.15}$$

The vector $\mathcal{E}_\mu = \mathcal{E}^{(+)}_\mu - \mathcal{E}^{(-)}_\mu$ may be determined using the relation for the divergence of ΔE^μ_ν in eq.(A.12):

$$\mathcal{E}_{\mu\nu} = \mathcal{E}^{(+)}_{\mu\nu} - \mathcal{E}^{(-)}_{\mu\nu}, \tag{A.16}$$

$$\nabla^{(+)}_\mu\mathcal{E}^{(+)\mu}_\nu = \nabla^{(+)}_\mu\tilde{\Pi}^{(+)\mu}{}_{\nu}, \tag{A.17}$$

$$\nabla^{(+)}_\mu\mathcal{E}^{(-)\mu}_\nu = \Psi^2\nabla^{(-)}_\mu\tilde{\Pi}^{(-)\mu}{}_{\nu}. \tag{A.18}$$

We consider that all the divergenceless part of $\Delta E_{\mu\nu}$ is contained in $E^{TT}_{\mu\nu}$. Namely, $\nabla_\mu\mathcal{E}^{(\pm)\mu}_\nu = 0 \Leftrightarrow \mathcal{E}^\pm_\mu = 0$. Adding the second term to the tensor $U^\Psi_{\mu\nu}$, we obtain the conserved and traceless tensor:

$$T^{\Psi\text{ eff}}_{\mu\nu} = U^\Psi_{\mu\nu} + \mathcal{E}^{(-)}_{\mu\nu}, \tag{A.19}$$

$$\begin{aligned} T^{\Psi\text{ eff}}_{\mu\nu} = {} & 4\partial_\mu\Psi\,\partial_\nu\Psi - 2\Psi\nabla_\mu\nabla_\nu\Psi + g^{(+)}_{\mu\nu}\left(2\Psi\Box\Psi - |\nabla\Psi|^2\right) \\ & + G^{(+)}_{\mu\nu}\Psi^2 - \Psi^2\tilde{\Pi}^{(-)}_{\mu\nu} + \mathcal{E}^{(-)}_{\mu\nu}, \end{aligned} \tag{A.20}$$

$$g^{(+)\mu\nu}T^{\Psi\text{ eff}}_{\mu\nu} = 0, \tag{A.21}$$

$$\nabla^{(+)}_\mu T^{\Psi\text{ eff}\,\mu}{}_{\nu} = 0. \tag{A.22}$$

It is interesting to note that when the negative tension brane is empty of matter and radiation, $\Psi^2\tilde{\Pi}^{(-)}_{\mu\nu} = \mathcal{E}^{(-)}_{\mu\nu} = 0$ and $T^{\Psi\text{ eff}}_{\mu\nu}$ is precisely the stress energy tensor of the scalar field Ψ which would then be conformally invariant by (A.3). When matter is present on the negative tension brane, the traceless and divergenceless properties of the pseudo-stress-energy tensor are still satisfied. Namely some $T^{(-)}_{\mu\nu}$ contributions must be added to the stress-energy tensor of Ψ to accommodate the fact that Ψ is no longer conformally invariant with respect to the positive tension brane as mentioned in (A.3).

The only part in (A.9) that remains undetermined is therefore the traceless and divergenceless $E^{TT}_{\mu\nu}$ contribution in $\Delta E_{\mu\nu}$. From the relation (A.5), $\Delta E_{\mu\nu} = E^{(+)}_{\mu\nu} - \Psi^2 E^{(-)}_{\mu\nu} = 0$ for cosmological metrics. So $E^{TT}_{\mu\nu}$ vanishes in the background. Following the argument of section 2.4, for the purpose of long wavelength adiabatic perturbations, it is therefore consistent to set $E^{TT}_{\mu\nu}$ to zero.

Thanks to the Gauss-Codacci formalism and to the conformal assumption (A.1), it is therefore possible to find an expression for the Weyl tensor $E_{\mu\nu}$ up to a conserved traceless tensor that shall be neglected for our regime of interest. Comparing equation (A.4) with (A.9), the expression of the Weyl tensor is:

$$E^{(+)}_{\mu\nu} = -T^{\Psi\ \mathrm{eff}}_{\mu\nu} + \mathcal{E}^{(+)}_{\mu\nu}. \tag{A.23}$$

In this regime, the metric on both branes can be found by solving the set of equations:

$$\begin{aligned} R^{(+)}_{\mu\nu} &= \left(\Pi^{(+)}_{\mu\nu} - \mathcal{E}^{(+)}_{\mu\nu}\right) + T^{\Psi\ \mathrm{eff}}_{\mu\nu}, \\ \Box\Psi &= \tfrac{1}{6}\left(R^{(+)} - \Psi^2 \Pi^{(-)}\right)\Psi, \end{aligned} \tag{A.24}$$

with

$$\left\{ \begin{aligned} & g^{(-)}_{\mu\nu} = \Psi^2 g^{(+)}_{\mu\nu} \\ & \Pi^{(\pm)}_{\mu\nu} = \pm\tfrac{\kappa}{L}\left(T^{(\pm)}_{\mu\nu} - \tfrac{1}{2}T^{(\pm)}g^{(\pm)}_{\mu\nu}\right) - \tfrac{\kappa^2}{4}\left(T^{(\pm)\alpha}_{\mu} T^{(\pm)}_{\alpha\nu} - \tfrac{1}{3}T^{(\pm)}T^{(\pm)}_{\mu\nu}\right) \\ & T^{\Psi\ \mathrm{eff}}_{\mu\nu} = 4\partial_\mu\Psi\,\partial_\nu\Psi - 2\Psi\nabla_\mu\nabla_\nu\Psi + g^{(+)}_{\mu\nu}\left(2\Psi\Box\Psi - |\nabla\Psi|^2\right) \\ & \qquad\quad + \Psi^2 G^{(+)}_{\mu\nu} - \Psi^2\tilde{\Pi}^{(-)}_{\mu\nu} + \mathcal{E}^{(-)}_{\mu\nu} \\ & \mathcal{E}^{(\pm)}_{\mu\nu} = \nabla_\mu\mathcal{E}^{(\pm)}_{\nu} + \nabla_\nu\mathcal{E}^{(\pm)}_{\mu} - \tfrac{1}{2}g_{\mu\nu}\nabla_\alpha\mathcal{E}^{(\pm)\alpha} \\ & \nabla_\mu\mathcal{E}^{(+)\mu}_{\nu} = \nabla^{(+)}_\mu\tilde{\Pi}^{(+)\mu}_{\ \ \nu} \\ & \nabla_\mu\mathcal{E}^{(-)\mu}_{\nu} = \Psi^2\nabla^{(-)}_\mu\tilde{\Pi}^{(-)\mu}_{\ \ \nu} \end{aligned} \right. \tag{A.25}$$

with $\Pi^{(\pm)} = g^{(\pm)\mu\nu}\Pi^{(\pm)}_{\mu\nu}$, $\tilde{\Pi}^{(\pm)\mu}_{\ \ \nu} = \Pi^{(\pm)\mu}_{\ \ \nu} - \frac{1}{2}\Pi^{(\pm)}\delta^\mu_\nu$ and any other contraction and derivatives done with respect to $g^{(+)}_{\mu\nu}$.

These may look formidable, we have an equation of motion for a non-minimally coupled scalar field Ψ and equations for the metric $g^{(+)}_{\mu\nu}$ sourced in a highly non-trivial manner by Ψ and the stress-energy on each brane. However, we argue that the system of equations (A.24) with (A.25) is self-consistent and allows us to calculate the long-wavelength adiabatic perturbations when the ρ^2 terms play a significant role and cannot be neglected. It is perfectly consistent with previous work done in the low-energy limit. Indeed, in the limit where $\rho_\pm \ll \mathcal{T}$, the usual moduli space results are obtained.

The extension of this work to one brane in a non-compact extra-dimension as described in the Randall-Sundrum II model is straightforward by taking the limit $\Psi \to 0$.

This work relies on the important assumption that the branes remain conformal to each other. As we shall see in the following, this assumption is not true when Kaluza-Klein corrections are taken into account; however we may deal with this fact by introducing a different projected Weyl tensor on each brane.

B First Kaluza-Klein Mode in the Effective Theory for Static Branes

B.1 KK Corrections on the RS Model around Static Branes

We consider two static flat branes of positive and negative tension, embedded in a five-dimensional Anti-de-Sitter (AdS) space with cosmological constant $\Lambda = -\frac{6}{\kappa L^2}$. The bulk metric is:

$$ds^2 = dy^2 + e^{-2|y|/L}\eta_{\mu\nu}dx^\mu dx^\nu, \tag{B.1}$$

with the branes located at $y \equiv d^\pm$. Both branes are subject to a Z_2 -reflection symmetry. For this background solution, the induced metric on each brane is flat $\gamma^{(\pm)}_{\mu\nu} = e^{-2\frac{d^\pm}{L}}\eta_{\mu\nu}$. We shall denote by $\Psi_0 = e^{-d/L}$ the constant conformal factor relating the metrics on the two branes $\gamma^{(-)}_{\mu\nu} = \Psi_0^2\,\gamma^{(+)}_{\mu\nu}$.

Following the procedure of [1], we consider the metric perturbations sourced by matter confined on the positive tension brane, with stress-energy tensor $T_{\mu\nu}$. Since matter is introduced as a perturbation, to first order, the matter fields confined on the brane are living on the background flat metric $\gamma^{(+)}_{\mu\nu}$.

As considered in [1], we shall first solve the equations of motion for the perturbed metric in Randall-Sundrum (RS) gauge. In this gauge, the position of the positive tension brane is not fixed. We denote by δy the deviation from its fixed background location. In order to express the perturbed metric induced on each brane, we will change to Gaussian normal coordinates.

When the distance between the two branes is finite, the perturbed metric on the branes can be expanded in momentum space. We shall be interested in the first order correction to the zero mode.

In RS gauge, the perturbed metric is:

$$ds^2 = dy^2 + \left(e^{-2|y|/L}\eta_{\mu\nu} + h_{\mu\nu}\right) dx^\mu dx^\nu \tag{B.2}$$

$$\text{with } h^\mu_\mu = 0 \text{ and } h^\mu_{\nu,\mu} = 0, \tag{B.3}$$

where $h_{\mu\nu}$ is transverse and traceless with respect to the background metric $\gamma_{\mu\nu} = e^{-2|y|/L}\eta_{\mu\nu}$. In this gauge, the five-dimensional Einstein equation and the Israël matching conditions read:

$$\left[e^{2y/L}\Box + \partial_y^2 - \frac{4}{L^2}\right] h_{\mu\nu} = 0 \quad \text{for } d^+ < y < d^-, \tag{B.4}$$

$$\left[\partial_y + \frac{2}{L}\right] h_{\mu\nu} = \begin{cases} -\kappa\, \Sigma_{\mu\nu} & \text{at } y = d^+ \\ 0 & \text{at } y = d^- \end{cases}, \tag{B.5}$$

where $\Box$ is the Laplacian on the four-dimensional Minkowski space. The tensor $\Sigma_{\mu\nu}$ is a functional of the matter on the positive brane: $\Sigma_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}T\gamma^{(+)}_{\mu\nu} - \frac{2}{\kappa}\,\delta y_{,\mu\nu}$, with $T = \gamma^{(+)\,\mu\nu}T_{\mu\nu}$. From eqs. (B.5) and (B.3), the tensor $\Sigma_{\mu\nu}$ must be transverse and traceless with respect to its background flat metric $\gamma^{(+)}_{\mu\nu}$. The deviation δy from the background position is therefore given by $\delta y = -\frac{\kappa}{6}\frac{1}{\Box^{(+)}}T$:

$$\Sigma_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}T\gamma^{(+)}_{\mu\nu} + \frac{1}{3\,\Box^{(+)}}T_{,\mu\nu}. \tag{B.6}$$

Since the background metric is flat, $\Box^{(+)} = e^{2\frac{d^+}{L}}\Box$.

The differential equation (B.4) with boundary conditions (B.5) can be solved, giving the expression for the perturbed metric in RS gauge:

$$h_{\mu\nu}(y, x^\mu) = \kappa\,\hat{F}(y)\Sigma_{\mu\nu}. \tag{B.7}$$

In what follows, for simplicity, we take $d^+ = 0$ and $d^- = d$. In that case, the operator $\hat{F}$ can be expressed as:

$$\hat{F}(y) = \frac{1}{\sqrt{-\Box}} \left[\frac{I_2(e^{y/L} L\sqrt{-\Box}) K_1(e^{d/L} L\sqrt{-\Box}) + I_1(e^{d/L} L\sqrt{-\Box}) K_2(e^{y/L} L\sqrt{-\Box})}{I_1(e^{d/L} L\sqrt{-\Box}) K_1(L\sqrt{-\Box}) - I_1(L\sqrt{-\Box}) K_1(e^{d/L} L\sqrt{-\Box})} \right], \quad \text{(B.8)}$$

with I_n (resp. K_n) the n^{th} Bessel function of first (resp. second) kind. We may notice that when $d^+ = 0$, $\gamma^{(+)}_{\mu\nu} = \eta_{\mu\nu}$ and $\Box^{(+)} = \Box$.

Since in RS gauge, the negative tension brane remains located at $y \equiv d$, $h_{\mu\nu}(x^\mu, y = d)$ is the metric perturbation induced on that brane. However this is not the case for the positive tension brane. In order to find the induced metric on that brane, we need to perform a gauge transformation and work in terms of the Gaussian normal ($\text{GN}^{(+)}$) gauge for this brane:

$$\begin{cases} \bar{y} = y - \delta y \\ \bar{x}^\mu = x^\mu + \zeta^\mu(x^\nu). \end{cases} \quad \text{(B.9)}$$

When working in the $\text{GN}^{(+)}$ gauge, with coordinates $(\bar{y}, \bar{x}^\mu)$, the positive tension brane is located at $\bar{y} \equiv 0$. Performing the gauge transformation (B.9) with $\delta y = -\frac{\kappa}{6}\frac{1}{\Box}T$, the perturbed metric induced on the positive tension brane is given by:

$$\bar{h}^+_{\mu\nu}(\bar{y} \equiv 0) = h_{\mu\nu}(y = 0) + \frac{\kappa}{3L}\eta_{\mu\nu}\frac{1}{\Box}T - \zeta_{(\mu,\nu)}, \quad \text{(B.10)}$$

where the indices are raised and lowered with the metric $\eta^{\mu\nu}$. We may fix the remaining degrees of freedom by imposing the gauge choice:

$$\bar{h}^\mu_{\nu,\mu} = \frac{1}{2}\bar{h}^\mu_{\mu,\nu} \text{ at } \bar{y} = 0. \quad \text{(B.11)}$$

This gauge choice corresponds to the de Donder gauge and is obtained by fixing $\zeta^\mu(x^\nu)$ such that $\zeta_\mu = -\frac{\kappa}{3L\Box^2}T_{,\mu}$.
In de Donder gauge, the metric perturbation induced on the positive tension brane is therefore:

$$\bar{h}^{(+)}_{\mu\nu} = h_{\mu\nu}(y = 0) + \frac{\kappa}{3L}\eta_{\mu\nu}\frac{1}{\Box}T + \frac{2\kappa}{3L}\frac{1}{\Box^2}T_{,\mu\nu}. \quad \text{(B.12)}$$

The perturbed metric on both branes is therefore:

$$\bar{h}^{(+)}_{\mu\nu} = \kappa\left(\hat{F}(0)\Sigma_{\mu\nu} + \frac{1}{3L}\eta_{\mu\nu}\frac{1}{\Box}T + \frac{2}{3L}\frac{1}{\Box^2}T_{,\mu\nu}\right), \quad \text{(B.13)}$$

$$\bar{h}^{(-)}_{\mu\nu} = \kappa\,\hat{F}(d)\Sigma_{\mu\nu}. \quad \text{(B.14)}$$

In what follows we shall be interested in the first order expansion in derivatives (in $L^2\Box$), of the operator $\hat{F}$. The expression (B.8) of $\hat{F}$ has a derivative expansion:

$$\hat{F}(y) = \tfrac{2L\, e^{-2y/L}}{1-e^{-2d/L}} \left[-\tfrac{1}{L^2\Box} + f(y) + \mathcal{O}(L^2\Box)\right], \tag{B.15}$$

$$f(y) = \tfrac{d/2L}{1-e^{-2d/L}} - \tfrac{1}{8}\left(1 - e^{2(2y-d)/L} + 2e^{2y/L}\right). \tag{B.16}$$

The first term in (B.15) can be interpreted as the zero mode and the second one as the first Kaluza-Klein (KK) mode from the infinite KK tower. In the limit where $d \to \infty$, the expansion in (B.15) is ill-defined as the function f in (B.16) diverges and dominates over the zero-mode. In the limit of one positive brane embedded in a non-compactified fifth dimension (RSII model), a derivative expansion is not possible. There is indeed no mass gap between each mode in the KK tower when the fifth dimension is infinite.

To first order in derivatives, the perturbed metric induced on each brane in its respective de Donder gauge is therefore:

$$\bar{h}^{(+)}_{\mu\nu} = -\frac{2L\kappa}{1-\Psi_0^2}\left[\frac{1}{L^2\Box} - f(0) + \mathcal{O}(L^2\Box)\right]\Sigma_{\mu\nu} \tag{B.17}$$

$$+\frac{\kappa}{3L}\eta_{\mu\nu}\frac{1}{\Box}T + \frac{2\kappa}{3L}\frac{1}{\Box^2}T_{,\mu\nu},$$

$$\bar{h}^{(-)}_{\mu\nu} = -\frac{2L\kappa\,\Psi_0^2}{1-\Psi_0^2}\left[\frac{1}{L^2\Box} - f(d) + \mathcal{O}(L^2\Box)\right]\Sigma_{\mu\nu}. \tag{B.18}$$

B.2 Extension of the Standard Four-dimensional Effective Action

B.2.1 The Four-dimensional Effective Theory

In this section, we suggest a possible extension of the four-dimensional effective theory that is capable of recovering the first KK mode as described in the previous section.

We work in a low-energy limit, when the density of the matter confined on the branes is much smaller that the brane tensions. In that case, it has been shown (see for example [2, 3, 26]), that the zero mode of the two brane Randall-Sundrum model can be described by a four-dimensional effective theory.

In this theory, the metrics on both branes are conformally related:

$$g^{(-)}_{\mu\nu} = \Psi^2 g^{(+)}_{\mu\nu}. \tag{B.19}$$

This has already been mentioned in appendix A, and the modified Einstein equation on the positive tension brane is:

$$G^{(+)}_{\mu\nu} = \frac{\kappa}{L}\, T_{\mu\nu} - E^{(+)}_{\mu\nu}, \tag{B.20}$$

with $T_{\mu\nu}$ the stress-tensor of matter on the positive brane. If we introduce matter only on this brane, the expression for the tensor $E^{(+)}_{\mu\nu}$ in this effective theory is:

$$E^{(+)}_{\mu\nu} = -4\partial_\mu\Psi\partial_\nu\Psi + 2\Psi\nabla_\mu\nabla_\nu\Psi - \left(2\Psi\Box\Psi - (\partial\Psi)^2\right) g^{(+)}_{\mu\nu} - \Psi^2 G^{(+)}_{\mu\nu}, \tag{B.21}$$

where all covariant derivatives are taken with respect to $g^{(+)}_{\mu\nu}$ (as will be the case throughout this section unless otherwise specified). This is in complete agreement with the Gauss-Codacci equations in the low-energy limit when the electric part of the induced Weyl tensor $E^{(+)}_{\mu\nu}$ is fixed to the given value (B.21), (cf. appendix A). Using the conformal transformation (B.19), this effective theory predicts the electric part of the Weyl tensor on the negative tension brane to be:

$$E^{(-)}_{\mu\nu} = \Psi^{-2}\, E^{(+)}_{\mu\nu}. \tag{B.22}$$

These relations (B.21,B.22) for $E^{(\pm)}_{\mu\nu}$ can be checked to be true for the background and to model correctly the behaviour of the zero mode (ie. the long wavelength limit of the exact five-dimensional theory). However there is no reason for expressions (B.21,B.22) to remain exact for non-conformally flat spacetimes in general. The electric part of the Weyl tensor $E^{(\pm)}_{\mu\nu}$ is indeed the only quantity which remains unknown from a purely four-dimensional point of view as it encodes information from the bulk geometry. It is through this term that the bulk generates KK corrections on the brane.

From the properties of the five-dimensional Weyl tensor, it can be shown that $E^{(\pm)}_{\mu\nu}$ is traceless. Furthermore, in the low-energy limit, by the Bianchi identity, it is divergenceless. (Outside the low-energy limit, as in the context of appendix A, $E_{\mu\nu}$ is not transverse but it is possible to separate out its transverse part, noted as $E^{TT}_{\mu\nu}$.) If we want to modify the four-dimensional effective theory, the only modification that would be consistent with the five-dimensional nature of the theory is to add a correction to the expression (B.21) of $E^{(+)}_{\mu\nu}$, (or to (B.22) for $E^{(-)}_{\mu\nu}$). Motivated by [54, 55], let us consider a possible term $E^{corr}_{\mu\nu}$ that could be added as a correction to (B.21).

$E^{corr}_{\mu\nu}$ needs to vanish for conformal flat spacetimes since the relation (B.21) is exact in that case. Since $E^{(+)}_{\mu\nu}$ is transverse, so is $E^{corr}_{\mu\nu}$, and we might think of it as being derived from an action S_E (noting however that this is not necessarily the case). Furthermore, since $E^{corr}_{\mu\nu}$ is traceless, S_E must be conformally invariant. Indeed, the variation of this action under a conformal transformation with $\delta g_{\mu\nu} \propto g_{\mu\nu}$ will be:

$$\delta S_E = \int d^4x\sqrt{-g}\frac{1}{2}E^{corr}_{\mu\nu}\delta g^{\mu\nu}, \tag{B.23}$$

and since $E^{corr}_{\mu\nu}$ is traceless, the action S_E must be conformally invariant.

If we want to modify the effective theory in order to accommodate the first KK corrections, we need to add to the action a term of fourth order in derivatives. If we consider a functional of the metric only, the only possible local term at this order that preserves the conformal invariance is:

$$S_{C^2} = \int d^4x\sqrt{-g}\; C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}. \tag{B.24}$$

The stress-tensor derived from the action is:

$$\begin{aligned} A_{\mu\nu}[g] =& \frac{1}{2\sqrt{-g}}\frac{\delta\, S_{C^2}}{\delta g^{\mu\nu}}, \\ A_{\mu\nu}[g] =& -\nabla_\alpha\nabla^\alpha R_{\mu\nu} + \frac{1}{3}\nabla_\mu\nabla_\nu R + \frac{1}{6}\nabla_\alpha\nabla^\alpha R g_{\mu\nu} - \frac{1}{6}R^2 g_{\mu\nu} \\ &+ \frac{2}{3}RR_{\mu\nu} + \frac{1}{2}R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} - 2R_{\mu\alpha\nu\beta}R^{\alpha\beta}. \end{aligned} \tag{B.25}$$

Here all covariant derivatives are taken with respect to the general metric $g_{\mu\nu}$. Since the Weyl tensor vanishes for conformally flat spacetimes, the tensor $A_{\mu\nu}[g]$ has all the requirements it should satisfy: it is indeed traceless, transverse and vanishes for the background. Adding a term $E^{corr}_{\mu\nu} \propto A_{\mu\nu}[g^{(+)}]$ to the expression (B.21) is therefore consistent with the Gauss-Codacci and the Bianchi identities and is consistent with the background results. Following this argument, it seems to us natural to consider the effects that the addition of such terms would have in the theory.

Moreover, from the results of section B.1, we can point out that when the first KK corrections are taken into account, the brane metrics do not remain conformal to each other any longer. We therefore need to modify the relation (B.19) between the brane metrics.

In our procedure we suggest breaking the conformal relation between the brane metrics by including some independent contribution to $E^{(\pm)}_{\mu\nu}$. By independent, we mean that the corrections will not satisfy the conformal relation (B.22), ie. $E^{(-)corr}_{\mu\nu} \neq \Psi^{-2} E^{(+)corr}_{\mu\nu}$.

B.2.2 Ansatz

Our idea is to include a contribution from the stress-tensor (B.25) in the effective theory. More precisely, we will include a term proportional to $A_{\mu\nu}[g^{\pm}]$ to the electric part of the Weyl tensor. We will therefore consider the theory governed by the four-dimensional equations:

$$
\begin{aligned}
G^{(+)}_{\mu\nu} &= \frac{\kappa}{L} T_{\mu\nu} + 4\partial_\mu \Psi \partial_\nu \Psi - 2\Psi \nabla_\mu \nabla_\nu \Psi + \left(2\Psi \Box \Psi - (\partial \Psi)^2\right) g^{(+)}_{\mu\nu} \qquad \text{(B.26)} \\
&\quad + \Psi^2 G^{(+)}_{\mu\nu} - E^{(+)corr}_{\mu\nu}, \\
G^{(-)}_{\mu\nu} &= \frac{1}{\Psi^2}\left(-2\Psi \nabla_\mu \nabla_\nu \Psi + 4\partial_\mu \Psi \partial_\nu \Psi + g^{(+)}_{\mu\nu}\left(2\Psi \Box \Psi - |\nabla \Psi|^2\right)\right) \qquad \text{(B.27)} \\
&\quad + G^{(+)}_{\mu\nu} - E^{(-)corr}_{\mu\nu}, \\
\Box^{(+)} \Psi &= \frac{1}{6} R^{(+)} \Psi, \qquad \text{(B.28)}
\end{aligned}
$$

with the corrections terms:

$$
\begin{aligned}
E^{(+)corr}_{\mu\nu} &= \alpha\, L^2 A_{\mu\nu}[g^+], \qquad \text{(B.29)} \\
E^{(-)corr}_{\mu\nu} &= \beta\, L^2 A_{\mu\nu}[g^-]. \qquad \text{(B.30)}
\end{aligned}
$$

We shall test this modified four-dimensional effective theory against exact results obtained from five-dimensional analysis in what follows.

B.2.3 Verification of the Ansatz for Static Branes

In this last section we shall check the previous ansatz against the exact solution derived in section B.1, in the case of perturbations around static branes. The ansatz we propose has to be considered as a first order correction and we will check if it correctly reproduces the behaviour of the first KK mode from the previous section. We will therefore not solve the system completely but only work out the zero mode and the first order correction generated by the additional terms in (B.29,B.30). We study the same scenario as in section B.1 and we consider metric perturbations around static branes, sourced by the addition of matter on the positive tension brane. In the background, the positive tension brane is located at $y = 0$ and the negative one at $y = d$. To first order in perturbations,

$$g^{(+)}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}^{(+)}_{\mu\nu}, \tag{B.31}$$

$$\Psi = \Psi_0 + \delta\Psi, \quad \Psi_0 = e^{-d/L} = \text{const.}, \tag{B.32}$$

$$g^{(-)}_{\mu\nu} = \Psi_0^2 \ \eta_{\mu\nu} + \bar{h}^{(-)}_{\mu\nu}. \tag{B.33}$$

To first order in perturbations, we can consider the matter fields and the perturbed scalar field $\delta\Psi$ to live on the background metric:

$$R^{(+)} = -\frac{\kappa}{L} T, \tag{B.34}$$

$$\Box\delta\Psi = -\frac{\kappa}{6L} T\Psi_0, \tag{B.35}$$

$$A^{(+)}_{\mu\nu} = -\Box R^{(+)}_{\mu\nu} + \frac{1}{3} R^{(+)}_{,\,\mu\nu} + \frac{1}{6}\Box R^{(+)}\eta_{\mu\nu}, \tag{B.36}$$

giving rise to the modified Einstein equation for the positive tension brane:

$$\begin{aligned} R^{(+)}_{\mu\nu} = \; & \frac{\kappa/L}{1-\Psi_0^2}\left[T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu} + \frac{\Psi_0^2}{6}T\eta_{\mu\nu} + \frac{\Psi_0^2}{3\,\Box}T_{,\mu\nu}\right] \\ & - \frac{\alpha\kappa/L}{(1-\Psi_0^2)^2}\, L^2\Box\left[T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu} + \frac{1}{3\,\Box}T_{,\mu\nu}\right] \\ & + \frac{\kappa}{L}\, \mathcal{O}\left(\alpha^2 L^4 \Box^2 T_{\mu\nu}\right). \end{aligned} \tag{B.37}$$

To get this result, we have only considered the first order corrections generated by the addition of the term (B.24) in the effective four-dimensional action. We notice that up to first order in perturbations and to first order in the correction term $\alpha L^2\Box$, we have the remarkable relation:

$$A^{(+)}_{\mu\nu} = -\frac{\kappa/L}{1-\Psi_0^2}\,\Box\Sigma_{\mu\nu}, \tag{B.38}$$

with the tensor $\Sigma_{\mu\nu}$ as given in (B.6). This is an important result, based on which, our ansatz will be verified.

For the negative tension brane, the modified Einstein equation reads:

$$R^{(-)}_{\mu\nu} = R^{(+)}_{\mu\nu} - 2\frac{\delta\Psi_{,\,\mu\nu}}{\Psi_0} - \frac{\Box\delta\Psi}{\Psi_0}\eta_{\mu\nu} - \beta L^2 A^{(-)}_{\mu\nu}, \tag{B.39}$$

so that:

$$R^{(-)}_{\mu\nu} = \frac{\kappa/L}{1-\Psi_0^2}\left[1 - \left(\beta(1-\frac{1}{\Psi_0^2}) + \frac{\alpha}{1-\Psi_0^2}\right) L^2\Box\right]\Sigma_{\mu\nu}. \tag{B.40}$$

In the de Donder gauge, the Ricci tensor is related to the metric perturbation by: $R^{(\pm)}_{\mu\nu} = -\frac{1}{2}\Box^{(\pm)}\bar{h}^{(\pm)}_{\mu\nu}$. The perturbation of the brane metrics is therefore:

$$\bar{h}^{(+)}_{\mu\nu} = -2\frac{\kappa L}{1-\Psi_0^2}\left[\frac{1}{L^2\Box} - \frac{\alpha}{1-\Psi_0^2} + \mathcal{O}(L^2\Box)\right]\Sigma_{\mu\nu} \tag{B.41}$$

$$+ \frac{2\kappa}{3L\Box^2}T_{,\mu\nu} + \frac{\kappa}{3L\Box}T\eta_{\mu\nu},$$

$$\bar{h}^{(-)}_{\mu\nu} = -2\frac{\kappa L\Psi_0^2}{1-\Psi_0^2}\left[\frac{1}{L^2\Box} - \left(\beta(1-\frac{1}{\Psi_0^2}) + \frac{\alpha}{1-\Psi_0^2}\right)\right]\Sigma_{\mu\nu}. \tag{B.42}$$

We can compare these results with the exact zero and first modes obtained in (B.17) and (B.18). We can check that the zero modes agree perfectly as one should expect. This just confirms the well-known result that the effective theory gives the correct zero mode on the brane.

More remarkable is the result for the first KK mode. Comparing the expression (B.41) with (B.17), the results agree perfectly if the dimensionless constant α is fixed to the value: $\alpha = (1-\Psi_0^2)f(0)$. Similarly, we have a perfect agreement between the ansatz for the negative tension brane (B.42) and the exact result (B.18) if the constant β is fixed to the value: $\beta = \frac{\Psi_0^2}{1-\Psi_0^2}\left(f(0) - f(d)\right)$.

This is an original result. It has been possible to extend the notion of a four-dimensional effective theory in order to accommodate the first KK correction arising from perturbations around static branes.

We should however note that in order to fit to the exact results, the coefficients α and β have to depend on the distance between the branes, or equivalently depend on the scalar field Ψ. If the scalar field is not fixed in the background, $\partial_\mu\Psi_0 \neq 0$, in general the action $\int d^4x\sqrt{-g}\,\alpha(\Psi)C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ is not conformally invariant, and our procedure is not valid. The generalisation of the effective theory to the case for which the branes are not fixed in the background is therefore not straightforward. In that case a more general analysis is needed as in [53]. However, to our knowledge, an exact five-dimensional derivation for the KK modes in a more general scenario has not yet been studied. It is therefore difficult to check any new ansatz for a four-dimensional effective theory in any more elaborate scenario. We hope that our work can be used in order to check the validity of a four-dimensional effective theory and gives insight on how to generalise it to take some typical five-dimensional features into account.

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