Computational Investigations of Viscous Fingering in Enhanced Oil Recovery

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Declaration

I declare that the work presented in this thesis is my own. Where other sources of information have been used they have been indicated and acknowledged.

Shahrizal Anuar Abdul Hamid

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Abstract

Viscous fingering is important in many applications and industries and can occur over a wide range of time and length scales. Examples include fluid separation using chromatography and carbon dioxide (CO$_2$) sequestration. The particular type of viscous fingering discussed occurs in first contact miscible (FCM) displacements, which are often encountered in enhanced oil recovery applications such as CO$_2$ injection.

Viscous fingering is a highly non-linear phenomenon with a strong dependence on viscosity contrast, diffusion, dispersion and permeability variation. Its dynamics cannot be described analytically except at early time when the fingers grow independently of their neighbours. Predicting the growth of viscous fingering is challenging as it requires numerical simulation with high resolution to ensure physical diffusion and dispersion dominates over numerical errors.

We begin our investigation with a quantitative assessment on the inherent errors in numerical simulations resulting from truncation of Taylor’s expansion when approximating the governing partial differential equations describing the flow in porous media. We study the effects of grid orientation and different numerical schemes. We show how the truncation error reduces the growth rate of immiscible viscous fingers for wavenumbers greater than 1 in all cases but does not affect the growth rate of higher wavenumber fingers as much as would be seen if capillary pressure were present.

We then proceed to study the life-cycle of miscible fingering, from the early to late time regime. We use simple but effective methods to precisely detect nonlinear phenomena such as finger merging, coalescence and tip-splitting. By investigating the impacts of viscosity ratio and anisotropic dispersion on the fingering patterns, we propose how the scaling of non-linear growth can be made using the results from linear stability analysis. This subsequently allows us to predict when the late time regime will occur.
Next we investigate how spatial variations in the permeability including channelling and highly correlated reservoirs influence the viscous fingering. We study how macroscopic efficiency is reduced due to bypassed-oil in channelised reservoirs, and attempt to capture this using a modified 1D Koval model. We use several metrics to provide demarcations between the type of heterogeneity and resulting flow regimes.

Finally, we perform numerical simulations of polymer slug fingering using FCM simulator and black-oil simulator with Todd-Longstaff model. We show how the latter may suffer excessive numerical dispersion and we present a simple method to overcome this. Finally, we show that the miscible fingering at the back of the slug may be simplified by using a semi-analytical model that allows a rapid estimation of slug size required to maintain the integrity of the polymer slug.
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Dedication

I dedicate this thesis to my beloved late parents, Abdul Hamid Abdul Wahab and Saayah Zainon.
Nomenclature

\( A \) Reservoir cross-sectional area, \([L^2]\)

\( c \) Dimensionless solvent concentration

\( C_p \) Normalized polymer concentration

\( c_p \) Polymer concentration in mass of polymer per volume of water

\( c_{p,\text{inj}} \) Injected polymer concentration in mass of polymer per volume of water

\( C_{p,\text{max}} \) Maximum value of normalized polymer concentration in each of the grid blocks

\( CF \) Flow conductivity indicator

\( CFL \) Courant number

\( D \) Diffusion, \([L^2t^{-1}]\)

\( f \) Fractional flow

\( f_p \) Fractional flow of polymer

\( F_s \) Fractional flow of solvent

\( F_w \) Fractional flow of aqueous phase

\( H \) Reservoir width, \([L]\)

\( H_k \) Koval heterogeneity factor
$K$ Permeability, [$L^2$]

$k$ Wavenumber

$L$ Reservoir length, [L]

$L_{mix}$ Mixing length, [L]

$M$ Mobility ratio

$M_e$ Effective mobility

$N_p$ Oil production, [PV]

$p$ Pressure, [mL$^{-1}$t$^{-2}$]

$Pe$ Peclet number

$PV$ Pore volumes

$PVI$ Pore volumes injected

$Q$ Total injection flow rate, [$L^3t^{-1}$]

$Q_p$ Polymer injection flow rate, [$L^3t^{-1}$]

$Q_w$ Water injection flow rate, [$L^3t^{-1}$]

$R$ ln($M$)

$S_1$ Connate water bank saturation

$S_2$ Slug front water saturation

$S_3$ Slug trailing edge water saturation

$S_4$ Chase water front saturation

$S_o$ Oil phase saturation

$S_w$ Water saturation

$S_{F1}$ Fluid 1 saturation
$S_{F2}$  Fluid 2 saturation

$S_{or}$  Irreducible oil saturation

$S_{wc}$  Connate water saturation

$t$  Time, $[t]$  

$t_{D}$  Transverse diffusive dimensionless time

$t_{d}$  Dimensionless time, $[PVI]$  

$t_{bt}$  Breakthrough time, $[PVI]$  

$t_{D,A}$  Advective dimensionless time

$t_{d,slug}$  Dimensionless slug size

$v_{1}$  Connate water bank speed

$v_{2}$  Slug front water speed

$v_{3}$  Slug trailing edge water speed

$v_{4}$  Chase water front speed

$v_{d}$  Characteristic speed

$v_{h}$  Hydrocarbon phase velocity

$v_{s}$  Spreading wave speed

$v_{t}$  Total velocity

$v_{w}$  Aqueous phase velocity

$v_{fingertips}$  Approximated finger tips velocity

$x_{1}$  Connate water front position

$x_{2}$  Slug front position

$x_{3}$  Slug trailing edge position
\( x_d \)  Dimensionless distance in direction of flow

\( DX \)  Number of grid blocks in x direction

\( DY \)  Number of grid blocks in y direction

\( DZ \)  Number of grid blocks in z direction

**Greek Letters**

\( \mu_m \)  Mixture viscosity, \([\text{mL}^{-1}\text{t}^{-1}]\)

\( \mu_o \)  Oil viscosity, \([\text{mL}^{-1}\text{t}^{-1}]\)

\( \mu_p \)  Polymer viscosity, \([\text{mL}^{-1}\text{t}^{-1}]\)

\( \mu_{p,\text{eff}} \)  Polymer effective viscosity, \([\text{mL}^{-1}\text{t}^{-1}]\)

\( \mu_{w,\text{eff}} \)  Aqueous effective viscosity, \([\text{mL}^{-1}\text{t}^{-1}]\)

\( \mu_{w,e} \)  Water effective viscosity, \([\text{mL}^{-1}\text{t}^{-1}]\)

\( \omega \)  Todd and Longstaff mixing parameter

\( \omega' \)  Modified Todd and Longstaff mixing parameter

\( \omega_{gr} \)  Growth rate, \([\text{t}^{-1}]\)

\( \phi \)  Fractional rock pore volume

\( \varepsilon \)  Transverse-to-longitudinal diffusion ratio

**Subscripts**

\( inj \)  Injected

\( L \)  Longitudinal

\( m \)  Maximum

\( num \)  Numerical

\( o \)  Oileic phase
oo    Oil component in oleic phase
os    Oil solvent in oleic phase
p     Polymer
s     Solvent
T     Transverse
w     Aqueous phase
wp    Polymer component in aqueous phase
ww    Water component in aqueous phase
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Chapter 1

Introduction

1.1 Background and motivation

Fluid and hydrodynamic instability can be observed everywhere ranging from the formation of swirls when milk poured into a cup of tea or the mushroom cloud that accompanies an atomic explosion. In these two examples, the instability occurs due to the fact that pressure and density gradients are acting in the opposite direction which causes any initial disturbances at the fluid interface to grow into what are perceived to be highly chaotic patterns.

There is also a particular type of instability that is driven by viscosity difference between two fluids flowing in porous media called Saffman-Taylor instability, named after two prominent British scientists who published the paper in this subject in the mid-20th century (Saffman and Taylor, 1958). In this instance, a less viscous fluid displacing a more viscous fluid may give rise to the finger-like patterns, which is why this phenomenon is also commonly referred to as viscous fingering.

The study of viscous fingering is important in many different areas. Some cases of industrial interest involving porous medium include
1. Separation of fluid mixtures in the laboratory using chromatography techniques
2. Transport of contaminant or pollutant plumes in groundwater systems
3. Carbon sequestration into deep geological formations
4. Injection of fluids such as water or CO$_2$ into oil reservoirs

We are particularly interested in the last example where viscous fingering may significantly affect the efficiency of oil recovery.

1.1.1 EOR and viscous fingering

Although in general the efficiency of oil recovery is relatively low at around 30% to 50%, it has increased quite significantly over the past decades. This is due to several factors, such as the emergence of new production technologies such as hydraulic fracturing (fracking), horizontal drilling, as well as the implementation of enhanced oil recovery (EOR). In the Norwegian continental shelf (NCS), the efficiency was reported to be approximately 40% in 1995, and today this figure has increased to 46% (Norwegian Petroleum Directorate, 2014). A certain portion of this increment can be attributed to EOR, which has been successfully applied to 12 fields in the NCS, including large fields such as Ekofisk and Statfjord.

Oil is recovered from porous and permeable rocks (reservoirs) through production wells. It is possible for the oil to flow naturally from underground to the surface as the reservoir pressure is much higher than atmospheric pressure – this is known as primary recovery method. However, the reservoir pressure will decrease over time (except when the pressure is sustained by an adjacent water aquifer). Hence in almost all cases worldwide, oil recovery requires water or gas flooding whereby water is injected into the reservoirs to physically displace the oil as well as for pressure maintenance. This is referred to as secondary recovery.
1.1. Background and motivation

EOR, on the other hand, are the techniques to increase oil production (relative to water/gas flooding) often by modifying the transport properties of the oil in porous rocks. For example, heavy oil can be made to flow more easily by reducing its viscosity using steam injection. This method is used extensively in Alberta, Canada and the Duri field in Indonesia. Although EOR is typically used as a tertiary recovery method, it is possible to deploy these methods earlier in the secondary phase. EOR techniques can be broadly divided as follows:

1. Miscible gas (CO$_2$, nitrogen)

2. Chemical (polymer, surfactant, alkaline)

3. Thermal (steam, steam-assisted gravity drainage, fire-lift)

4. Others (low salinity, microbial)

These techniques have different levels of acceptance and success in the industry. Two of these techniques, namely miscible gas and polymer flooding, share a common characteristic. Although they work differently in increasing the amount of oil recovered, both techniques are susceptible to miscible viscous fingering.

1.1.1.1 Miscible gas flooding

Miscible gas flooding is one of the most popular EOR techniques. It works by lowering or eliminating the interfacial tension between the displaced oil and the displacing gas, which in turn reduces the amount residual oil that would be trapped in the rock if the displacing fluid was immiscible. Two fluids are said to be miscible if they mix in all proportions to form a single phase. Examples of gases that are miscible with oil (also known as solvents) are short-chain hydrocarbon gases or high-pressure CO$_2$ (above its minimum miscibility pressure). In general, fluid miscibility is controlled by the pressure, temperature and fluid compositions.
In the United States, following its peak in the mid-1980s, the utilisation of EOR had begun to decline primarily due to the oil price collapse. This trend was then reversed in the early 2000s, as miscible flooding started to overtake thermal as the main EOR method in 2002 for the first time in history, as shown in Figure 1.1. In 2014, miscible gas injection accounted for more than 50% of the total EOR projects. A similar trend is observed in the North Sea, where miscible gas flooding is equally popular. Out of 19 EOR field projects reported from 1975 to 2005, six were miscible gas injection and three were miscible water-alternating-gas (WAG), which represent nearly half the total of 19 EOR field projects (Awan et al., 2008).

The appeal of miscible gas injection is expected to continue in the future as it may also be executed as part of carbon sequestration, utilisation and storage (CCUS) projects. CCUS refers to technologies or activities around capturing CO$_2$ (usually from the fuel combustion emissions in power plants), its transportation through pipelines as well as its utilisation for EOR or sequestration into underground geological systems. This is a promising way forward to avoid the release of CO$_2$ to the
atmosphere on an industrial scale. Currently, the Petra Nova project is the world’s biggest CCUS project (Miyamoto et al., 2017). Around 90% of CO$_2$ released from a coal-fired power plant in Texas, USA, is captured and sent to nearby oil fields for EOR. It is worth noting that CCUS was previously known as CCS – this change reflects the importance of CO$_2$ utilisation (in the form of injection for EOR) as part of global efforts in reducing CO$_2$ emission to the atmosphere.

Compared to other EOR techniques, the economics of miscible gas injection are often marginal. Large capital and operating expenditures (CAPEX and OPEX) are required including

1. Cost of facilities – dedicated infrastructures may need to be built such as pipelines to deliver CO$_2$ (or other solvents) and compression systems for solvent injection. CO$_2$, in particular, is highly corrosive hence necessitates investments in higher grade materials to resist corrosion. It also requires high injection pressure to achieve miscibility, all of which contribute to the high cost of the facility.

2. Cost of solvent – based on the size of the reservoir, several tens of billions of cubic feet to more than a trillion cubic feet of solvent may be required (Stalkup, 1983). However, in some cases, solvent is readily available due to lack of production gas export capacities. Jarell et al. (2002) reported that CO$_2$ purchases are the single largest expense in CO$_2$ EOR flooding for the first 10 years of field operation, which represent up to 68% of total costs.

One issue that may hamper such projects being economically successful is viscous fingering. This is not uncommon as the solvents are typically less viscous than oil and is manifested in an early solvent breakthrough at production wells, reduced oil production and/or larger amounts of expensive solvent being required to sweep the oil.
1.1.1.2 Polymer flooding

In water flooding, immiscible viscous fingering may also occur if the viscosity ratio between oil and water is unfavourable. Mixing polymer such as hydrolyzed polyacrylamide (HPAM) into the injected water increases the viscosity of the aqueous phase and in turn reduces its mobility relative to the oil. This technique is called polymer flooding and it is typically used in more viscous oil reservoirs \((1 < \mu_o < 100 \text{ centipoise})\). It is sometimes referred to as augmented waterflooding or water-based flooding, as the physical properties of the injected water are modified by adding the polymer.

Many successful implementations of polymer flooding have been observed in many fields particularly in China. In fact, it is the most widely used chemical EOR technique in the world, with more incremental oil recovery attributed to this method than all other types of chemical EOR combined (Pope, 2011). It is particularly relevant and attractive today, because oil companies are increasingly looking to develop more viscous oil fields as the fields with lighter crudes become mature.

As for other EOR techniques such as miscible gas flooding, the additional costs associated with polymer injection are typically higher than the water flooding. The main extra cost is the polymer itself, incurred continuously as OPEX. Sheng et al. (2015) reported that the average cost of polymer (excluding related costs such as processing cost) is around USD 4.00 per barrel of incremental oil achieved, which can be a substantial addition to the oil lifting cost.

Costs can be reduced by injecting a fixed volume (or slug) of polymer solution, followed by water (usually referred to as chase water). However, as the viscosity of the polymer slug is higher than that of chase water, miscible fingering of the water into the polymer solution is expected. Hence, care has to be taken to ensure that the polymer slug is sufficiently large that it maintains its integrity from the injection to
1.1. Background and motivation

the production well. This often requires a good understanding of the flow behaviour associated with the viscous fingering phenomenon.

1.1.2 Other Applications

Apart from EOR, the study of viscous fingering is also very important in other applications. For example, in protein purification using size exclusion chromatography, salts are removed from proteins by injecting a plug of sample mixture into a vertical column. This column is packed with porous resin beads and initially saturated with a mobile liquid that acts as a carrier. Due to the difference in their molecular size, the salts will travel through the porous beads while the proteins will pass around them. This causes the salts and proteins to flow at different rates across the column (resulting in different retention times), allowing the purified proteins to be collected separately at the column outlet.

Viscous fingering may occur at the leading or trailing edge of the mixture plug depending on its viscosity relative to that of the carrier liquid. The presence of fingering will modify the protein retention time and broaden the purified protein plug and ultimately will reduce the efficiency of protein purification. Figure 1.2 shows examples of leading and trailing edge fingers in a chromatography column (Shalliker et al., 2007).

In CO₂ sequestration, CO₂ has to be isolated within its injection site to avoid catastrophic environmental consequences. Potential sites include structural and stratigraphic traps which inhibit upward migration of CO₂ (Szulczewski et al., 2013). When the CO₂ is injected (below a trap), it dissolves into the groundwater. Consequently, the denser CO₂ saturated water tends to sink and minimise the risk of CO₂ leakage via the trap.

The CO₂ dissolution rate is governed by diffusion. The density contrast between CO₂
saturated water and the ambient groundwater causes the downward flow of the CO$_2$ saturated water to be unstable which changes the rate of CO$_2$ mixing or dissolution. This unstable process leads to downward fingering that has many similarities to the viscous fingering. Hence a fundamental qualitative understanding of how the mixing of unstable miscible flow may be applied to CO$_2$ dissolution during sequestration.

![Figure 1.2: Examples of viscous fingering in a chromatography column with a viscosity ratio of 7.8 (taken from Shalliker et al., 2007). The flow is in downward direction. (a) Leading edge fingering; (b) Trailing edge fingering.](image)

1.1.3 Challenges

Embarking on oil and gas projects requires very high capital investments especially for offshore fields. This is particularly true for EOR projects as the returns from the incremental increase in the oil recovered must be higher than the capital spent for the expansive EOR implementations.

Viscous fingering is a highly nonlinear phenomenon making its analytical analysis non-trivial. The number of fingers that initially form and their growth are governed by the viscosity contrast between the fluids and the amount of physical diffusion (or, in the case of immiscible displacements, capillary pressure). Another important
aspect influencing viscous fingering is the presence of spatial variability of the rock properties as encountered in the geological systems.

Many researchers (for e.g. Saffman and Taylor, 1958; Chuoke et al., 1959; Tan and Homsy, 1986; Bacri et al., 1991) rely on linear stability analysis to provide the underlying scientific understanding of viscous fingering. In this analysis, a base steady state describing the initial fluid interface is established. Small perturbations, typically in the form of sinusoidal signals, then imposed on the base steady state. The resulting non-linear equations are linearised and solved. The solution will indicate whether the perturbations will be damped (hence stabilised) or amplified further leading to an unstable flow. While this analysis provides valuable information, the assumption of linearity means that the solution is restricted only to the regime in which the amplitude of perturbations remains very small i.e. the early time period. Because of this, such analysis cannot tell us about the fate of fingers at the time scales we are interested in.

Unsurprisingly, there has been an extensive use of numerical computations to understand the physics of fluids experiencing viscous fingering. In petroleum engineering, this is known as reservoir simulation and it is a routine tool used by the engineers in their work. The most widely used methods are finite difference, finite volume, finite element and combinations thereof, in which the spatial domain of interest is discretised using a mesh of elements or grid-blocks and the partial derivative in time is expressed as the change in the property or properties of interest over a finite time interval. This reduces the partial differential equations into a set of algebraic equations that are amenable to solution on a computer.

These numerical methods have provided many valuable insights into the nature of the fingers such as spreading, shielding and tip splitting. These dynamics are influenced by physical diffusion and dispersion. Numerical simulations have to be sufficiently resolved to ensure this physical diffusion and dispersion dominates over
numerical errors. As physical diffusion and dispersion are quite small on the reservoir scale, this can mean that very fine meshes are needed involving high computational cost.

Despite the development of increasingly powerful computers over the last several decades, the simulation of viscous fingering remains a challenge. Very high resolution simulations are required to properly capture the growth and development of viscous fingers. Using the required number of grid cells is impractical, if not impossible, especially in field-scale studies. There is a big trade-off between accuracy and computational costs that needs to be considered when performing numerical simulation.

It is now standard to approximate the average behaviour of fingering in miscible gas and polymer slug injection by incorporating the Todd and Longstaff model (Todd and Longstaff, 1972) into reservoir simulators. It was conceived based on the previous empirical 1D model by Koval (1963). The main advantage the Todd and Longstaff model method is that it allows coarser grid blocks to be used and hence speeds up the simulations. One drawback of these fingering models, however, is that the fitting parameters in their formulation may need to be calibrated by comparison with detailed simulation. Furthermore, most of these models assume that, on average, the spreading of the fingered region as a function of time is linear which may not be true especially for heterogeneous reservoirs.

1.2 Aims and objectives

The main aim of this thesis is to improve our understanding of the effects of various key parameters such as viscosity contrast, diffusion/dispersion, reservoir properties and numerical methods on viscous fingering in the context of miscible displacements.
Our objectives are to

1. Quantify numerical errors due truncation of the discretised governing equations in different numerical solvers and grid orientations, as well as to understand their impacts on the growth of viscous fingering.

2. Investigate the influences of diffusion and viscosity ratio on the growth of the fingers from early to late time regimes.

3. Identify the different flow regimes in heterogeneous reservoirs (advective, diffusive, channelling) and propose the suitable 1D models that can be used to represent these flow regimes.

4. Develop an analytical model to estimate the slug size required in the polymer injection in the presence of viscous fingering and validate it against high-resolution simulations.

For objectives (1) and (2), we will use an idealised system (first contact miscible flow in a homogeneous 2D rectilinear domain) that will allow the results to be applicable to EOR as well as other related cases such as chromatographic separation.

1.3 Outline of the thesis

In this thesis we address these objectives by means of numerical experiments and mathematical analyses. Chapter 2 reviews the history and current state of research in viscous fingering, mainly from the petroleum engineering perspective.

Chapter 3 walks through the methods used in this thesis. In particular, we describe the governing equations used in the numerical simulators used in this thesis. This is then followed by the non-dimensionalisation of the variables which form the basis of the studies presented in the subsequent chapters.
Chapter 1. Introduction

In Chapter 4, we compare the level of numerical errors quantitatively in three different numerical schemes (finite-volume first order, higher-order finite difference, and control volume finite-element methods) and Cartesian grid orientation (parallel and diagonal). This provides a basis for choosing a suitable grid resolution in subsequent studies such that the numerical error is not dominant. We conclude the chapter by highlighting the impact of truncation error on the growth at early time for miscible and immiscible fingering.

Chapter 5 begins with a brief review of linear stability theory. Equipped with the physics of fingering at early time, we investigate how diffusion and mobility ratio affect the life cycle of the fingers. We use several methods to quantify the growth of the fingers as a function of time which leads to a more detailed understanding of the complex nonlinear viscous fingering mechanisms.

Chapter 6 briefly explores the impact of permeability heterogeneity on viscous fingering. Here, we use the standard example of a heterogeneous reservoir from the SPE10 model 2 (Christie and Blunt, 2001) as well synthetic reservoirs. In lieu of rigorous analytical models, a pragmatic approach is adopted whereby we modify and calibrate the Todd and Longstaff model to include the bypassed oil that occurs due to channelling.

In Chapter 7, we turn our attention to miscible fingering in polymer slug injection. We first review the 1D analytical analyses of polymer injection with and without fingering at the rear of the slug. Their validity are evaluated with numerical simulations. We also show how the industry-standard black-oil type simulator used to model polymer injection can suffer from excessive artificial smearing at the polymer front and how this problem can be addressed easily.

We conclude our study and present some of the potential future work in Chapter 8.
1.4 Publications

This thesis contains previously published studies as follows:


The ideas in this thesis were presented at the following conferences:


2. 9th International Conference on Porous Media, Rotterdam, Netherlands (2017).

3. 15th European Conference on the Mathematics of Oil Recovery (ECMOR XV), Amsterdam, Netherlands (2016).
Chapter 2

Literature Review

2.1 Introduction

The study of instability phenomena in fluids can be traced back to Kelvin-Helmholtz theory in the mid-19th century (Thomson, 1871), from which the science behind everyday observations such as the formation of swirly cloud and ripples on water surface can be explained. Since then, other closely related formalisms of fluid instabilities by influential scientists emerged including the instability that occurs when the inertia dominate over the viscous force (i.e. turbulent flow), the Rayleigh-Taylor instability (due to a density difference in two fluids; Lord Rayleigh, 1883; Taylor, 1950), and Rayleigh-Bernard convection (due to temperature gradients).

In this chapter, we will review some of the past research on the instabilities driven by viscosity differences between fluids flowing in porous media i.e. the Saffman-Taylor instability (Saffman and Taylor, 1958), more commonly known as viscous fingering.
2.2 Physics of viscous fingering

2.2.1 Governing equations

Single phase, two-component transport in porous media can be described by the mass balance and equation and Darcy’s law

\[ \frac{\partial c}{\partial t} + \mathbf{v} \cdot \nabla c = D \nabla^2 c \]  

(2.1)

\[ \mathbf{v} = -\frac{K}{\mu} \nabla p \]  

(2.2)

where \( c \) is the concentration (mass fraction) of the displacing fluid, \( K \) is the permeability, \( p \) is the pressure, \( \mu \) is the viscosity, which depends upon concentration and \( D \) is the dispersion tensor, given by

\[ D = \begin{pmatrix} D_o + D'_L & 0 \\ 0 & D_o + D'_T \end{pmatrix} \]  

(2.3)

Here \( D_o \) is the molecular diffusion in the porous medium, \( D'_L \) is the longitudinal dispersivity and \( D'_T \) is the transverse dispersivity.

Molecular diffusion refers to the transport between two miscible components due to Brownian motion. It occurs regardless of whether there is flow or not and it is driven by a concentration gradient as given by Fick’s law with typical values of \( D_o \) for liquids are in the order of \( 10^{-9} \text{ m}^2/\text{s} \). On the other hand, dispersion commonly refers to the additional mixing due to velocity fluctuations in the bulk flow caused by the constant changes in direction as the fluid traverses the pores. Unlike the molecular diffusion coefficient which is always isotropic, the dispersion in the direction of flow will be different to that transverse to the principal flow direction. Longitudinal
dispersion is typically 10 to 100 times than the transverse dispersion. Following Bear (1972), we can model \( D'_L \) and \( D'_T \) as velocity-dependent

\[
D'_L = \alpha_L |v|, \quad D'_T = \alpha_T |v| \tag{2.4}
\]

Here, \( \alpha_L \) and \( \alpha_T \) are the dispersivity coefficients.

The governing equations (Eqs. 2.1 and 2.3) represent a limiting case where we work under several tacit assumptions including first contact miscibility (FCM), incompressible fluids, Newtonian flow and no gravity. There is also a myriad of studies that look into cases that depart from this rather idealised case. For example, Azaiez and Singh (2002) investigated non-Newtonian miscible flow in a Hele-Shaw cell. By looking into the vorticity disturbances, they found that a flow with a shear thinning displacing fluid has a higher instability compared to its Newtonian counterpart. More recently, Fu et al. (2017) investigated the case when two fluids are only partially miscible. Unlike in the FCM case where molecular diffusion is driven by concentration gradient as per Eq. 2.1, they showed that diffusion occurs mainly in the direction of chemical potential gradient which ultimately may alter the growth of the fingers. A number of studies looking into other effects such as non-isothermal systems (Islam et al., 2007), chemical reactions (Hejazi and Azaiez, 2013) and adsorption (Rana et al., 2014; Mishra et al., 2007) have also been carried out.

In the 1D case, with appropriate boundary conditions, diffusion will smear the travelling front. This can be calculated using the error function (Crank, 1975)

\[
c = 0.5 \left( \text{erf} \left( \frac{x - t}{2\sqrt{Dx}} \right) \right) \tag{2.5}
\]

where \( x \) is the distance in \( x \) direction and \( t \) is time. Eq. 2.5 may also be used to
2.2. Physics of viscous fingering

characterise a stable flow in 2D or 3D cases. Viscous fingering, on the other hand, is a manifestation of fluid instabilities and it cannot be sufficiently described by Eq. 2.5.

2.2.2 Linear stability analysis

One of the earliest analytical studies in viscous fingering can be traced back to the work by Hill (1952), who was looking into the purification of sugar in a refinery which involves displacement of sugar liquor with water in a vertical column packed with granular bone charcoal. Interestingly, this study pre-dated the work by Saffman and Taylor (1958) (in fact Homsy, 1987 argued that the instability phenomenon should be referred to as the 'Hill instability') and it provides an intuitive way to understand how a small perturbation in the interface of two fluids can trigger instability as illustrated in Figure 2.1(a). Here, the flow is moving vertically upwards with a velocity of $v_0$ and a pressure of $p_0$ at the interface. When a small perturbation of $\delta y$ occurs at the interface, we denote the pressure just above and below the perturbation to be $p_1$ and $p_2$ respectively.

(a) $\mu_1, \rho_1$

(b) $\mu_1, \rho_1$

Figure 2.1: Illustrations of stability analysis of two miscible fluids with upward flow and a sharp interface (indicated by the dotted line). (a) A simple small perturbation analysis based on the work by Hill (1952); (b) Linear stability analysis taken from Saffman and Taylor (1958).
where $g$ is the gravity acceleration and $k$ is the permeability. If $p_1 > p_2$, the perturbation will be damped and the flow will be stable. Hence we have

$$\frac{v_0}{k} (\mu_2 - \mu_1) + g(\rho_2 - \rho_1) > 0$$

(2.7)

If the influence of gravity is made insignificant, we obtain the required condition of $M = \mu_1 / \mu_2 < 1$ for a stable flow.

In the linear stability analysis presented by Saffman and Taylor (1958), the perturbation is expressed as sinusoidal signal with a wavenumber of $k$ as illustrated in Figure 2.1(b). In this case, the growth rate of the perturbation is given by

$$\omega_{gr} = \frac{k}{\mu_1 + \mu_2} \left[ v_0(\mu_1 - \mu_2) + (\rho_1 - \rho_2) \right]$$

(2.8)

Again, if the effect of gravity can be neglected

$$\omega_{gr} = k v_0 \frac{M - 1}{M + 1}$$

(2.9)

Eq. 2.9 is unphysical as it implies that the growth rate increases linearly with no bound as a function of wavenumber. A more realistic analysis would require the inclusion of diffusion that will give a cutoff wavenumber as illustrated in Figure 2.2.

The earliest such analysis for miscible flow was by Chuoke et al. (1959) (available in the appendix of Gardner and Ypma, 1984) who assumed that the initial concentration profile in the direction of flow was a step and that viscosity varied
2.2. Physics of viscous fingering

Figure 2.2: Schematic plot of initial growth rate of viscous fingers as a function of wavenumber, showing the growth rate in the absence of diffusion (or capillary pressure) and how diffusion (or capillary pressure) reduces the growth rate of higher wavenumber fingers.

linearly with concentration. Tan and Homsy (1986) later extended this by incorporating a time-dependent diffusive base state concentration. This is more realistic and results in there being a short period of time between the beginning of the displacement and the formation of the fingers as observed in many laboratory experiments (e.g. Bacri et al., 1991).

There is a large body of literature on linear stability analysis starting with the aforementioned work of Tan and Homsy (1986). These include Zimmerman and Homsy (1991); Bacri et al. (1992); De Wit and Homsy (1997); De Wit (2004). We will review this analysis further in Chapter 5.

2.2.3 Nonlinear growth

The dynamics of viscous fingers are highly non-linear and cannot be described analytically except at early time when the fingers grow independently of their neighbours. In this case, as we have discussed earlier, the growth rate and number of fingers can be estimated using perturbation analysis (Chuoke et al., 1959; Perrine,
1961; Gardner and Ypma, 1984; Tan and Homsy, 1986). At intermediate times numerical simulations and/or physical experiments are required to determine the complex interactions between the fingers that occur as they merge and split in a non-linear fashion until at late times only a single finger remains (Zimmerman and Homsy, 1991, 1992; Nijjer et al., 2018).

It is challenging to model viscous fingering numerically, especially in miscible displacements, as the pattern and growth of the fingers depend upon physical diffusion and dispersion as well as the viscosity ratio between the displacing and displaced fluids (Tan and Homsy, 1986, 1988; Zimmerman and Homsy, 1992; De Wit and Homsy, 1997; De Wit, 2004). The larger the transverse dispersion then the smaller the number of fingers that grow initially (Zimmerman and Homsy, 1992). Simulations typically require very fine meshes and/or higher order methods to ensure physical diffusion and dispersion dominate over truncation errors and predict the correct number of fingers and their growth rates (Christie and Bond, 1987; Meiburg and Homsy, 1988). As a result, many authors have investigated viscous fingering as a means for demonstrating the utility of their numerical schemes (Islam and Azaiez, 2005; Scovazzi et al., 2017; Adam et al., 2016).

Although there is a wide literature on viscous fingering in porous media, most investigations have focused on the early time regime (when the system is amenable to mathematical analysis). Some studies have investigated fingering at intermediate times (when there are multiple fingers interacting) but few have quantified the change in the number of fingers and identified which of the various mechanisms that occur during these regimes (decay, merging and splitting) as a function of viscosity ratio and dispersion. Only Tan and Homsy (1988); Zimmerman and Homsy (1991, 1992); Nijjer et al. (2018) have studied the full life cycle of viscous fingering and these studies considered systems with periodic boundary conditions perpendicular to the average flow direction. Most real systems (chromatography columns, aquifers being
used for carbon dioxide sequestration and oil reservoirs) have no flow boundaries perpendicular to the average flow direction.

The average behaviour of the flow in the different fingering regimes, and the times at which the different regimes start, are of particular interest to engineers as they typically use empirical models such as those of Koval (1963) or Todd and Longstaff (1972) to predict the impact of fingering on the trapping of carbon dioxide or oil recovery. These empirical models assume that the average concentration profile of the displacing fluid grows linearly with time. Various studies (Fayers et al., 1992; Sorbie et al., 1995; Yang et al., 2002) have shown that these empirical models provide a good match to both experimental data and detailed simulations of viscous fingering for the intermediate regime at modest viscosity ratios. There is, however, some evidence that the average concentration of the displacing fluid is not self-similar with respect to distance/time, as assumed by these models, for larger mobility ratios (Malhotra et al., 2015), for high Peclet numbers (Tardy and Pearson, 2006) and in the late time, single finger regime (Abdul Hamid and Muggeridge, 2018).

2.2.4 Heterogeneous reservoirs

The advective-dominated flow observed as a result of viscous fingering starts to deviate when it comes to heterogeneous reservoirs. In the absence of gravity, miscible flow in heterogeneous reservoirs can be dominated by one of the following flow regimes (Fayers et al., 1992; Rashid et al., 2012)

1. Diffusive

2. Advective (with viscous fingering)

3. Channelling (large correlation length heterogeneity)
As the viscosity ratio or the correlation length and standard deviation of the permeability distribution increase, the flow becomes advective. If the viscosity ratio is greater than 1 then there will also be viscous fingering. In both these cases the spread of the front (which will appear as mixing when the fluids are commingled in the production well) is proportional to $t$.

Fayers et al. (1992) suggested that diffusive or advective flow can be identified from the product of standard deviation of log-permeability, $\sigma$, and correlation length, $\lambda$

$$N_{AD} = \sigma \lambda$$

(2.10)

The same term has been used by other authors to investigate flow dispersivity (Gelhar and Axness, 1983; Kempers, 1990).

The third flow regime, channelling, occurs when the correlation length of the permeability distribution approaches the interwell spacing. This results in there being a preferential route for the solvent to flow from the injector to the producer. Channellised flow is also characterised by low sweep efficiency due to the bypassing of oil in the low permeability regions. It can potentially override the effect of viscous fingering, even at adverse viscosity ratios. For this reason, some researchers downplay the importance of viscous fingering in real reservoirs. Nevertheless, the displacement efficiency may still be affected by the small scale fingers (Luo et al., 2018).

This is best illustrated by the observations made Brock and Orr Jr (1991). In their experiment, they used a glass beadpack with checkerboard pattern as shown in Figure 2.3(a). Even with a relatively low permeability contrast of 4, the displacements of $M = 1$ (Figure 2.3b) and $M = 80$ (Figure 2.3c) follow the same preferential route suggesting channelised flows. However, it was observed that $M = 80$ gives more smaller scale fingers with a faster breakthrough time of
2.2. Physics of viscous fingering

\[ t_d = 0.2 \text{ PVI} \] (dimensionless time in pore volume injected unit) (as opposed to \( t_d = 0.66 \) when \( M = 1 \)) and a much larger unswept area. This highlights the interplay between channelling due to heterogeneity and viscosity contrast.

![Figure 2.3: The channelling flow due to permeability heterogeneity as observed in the experiment by Brock and Orr Jr (1991) using a 7.6 cm × 30.5 cm glass beadpack. (a) The checkerboard pattern with permeability contrast of 4; (b) The concentration field at \( t_d = 0.4 \text{ PVI} \) when \( M = 1 \); (c) The concentration field at \( t_d = 0.25 \text{ PVI} \) when \( M = 80 \).](image)

Many workers have proposed different methods for quantifying the impact of heterogeneity on flow with varying success. Knudby and Carrera (2005) and Renard and Allard (2013) proposed several metrics to determine how well the reservoirs are connected, including \( K_{\text{eff}}/K_{\text{geo}} \) (the ratio of effective permeability and geometric mean of the permeability). This metric was able to flag channelling in reservoirs for stable displacements but it has not been tested for miscible gas injection in which the flow is unstable.
2.2.5 Miscible viscous fingering in polymer slug injection

Injecting polymer solution rather than water results in a higher shock front saturation compared to an ordinary waterflood, reducing watercut until the polymer front breaks through, as well as improving the overall macroscopic sweep efficiency in heterogeneous reservoirs (Chang, 1978). It can also help stabilize displacements for which the oil-water shock front mobility ratio is greater than 1 (Blunt and Christie, 1994), thus reducing or preventing the degree of viscous fingering. An example of this is shown in Figure 2.4. Here, the oil is 50 times more viscous than water, so the injected water front is unstable and immiscible fingering between the injected water and resident oil is observed, Figure 2.4(a). Dissolving polymer in the injected water, so that the viscosity of the aqueous phase matches that of the oil, makes the polymer front stable although the leading shock front between the connate water bank and the oil may still be unstable (Figure 2.4b). Figure 2.4(c) shows the average saturation profiles between the injection and production wells for the two cases.

![Figure 2.4](image)

Figure 2.4: Illustration of the stabilization of viscous fingering by polymer injection in a line drive case with an oil-water viscosity ratio of 50 and a polymer-oil viscosity ratio of 1. (a) Saturation distribution seen in water injection. (b) Saturation distribution seen in polymer injection. (c) Average water saturation between the injector and the producer from (a) and (b). (d) Oil recovery curves. All shown at 0.1 PVI. All the parameters used in this simulation are as per Table 7.1.
Various factors including adsorption and viscous fingering can potentially destroy the slug. Adsorption means that polymer is lost progressively from the slug as it moves through the reservoir. If the slug is too small then all the polymer may be adsorbed before it reaches the production well. Viscous fingering occurs at the trailing edge of the polymer slug because the chase water is less viscous and thus more mobile than the polymer solution. Miscible viscous fingers of water tend to form and grow into the polymer slug. If these fingers reach the leading edge of the slug then the slug integrity is destroyed and the benefits of polymer injection are lost (Figure 2.5). A number of authors have presented analytical solutions that can be used to estimate the the impact of adsorption on a polymer slug including Bedrikovetsky (1993), Ribeiro and Pires (2008), Sheng et al. (2015), de Paula and Pires (2015), and Borazjani et al. (2016) but to date no one has derived a solution to describe the impact of viscous fingering.
2.3 Modelling approaches

2.3.1 Direct simulation

The dependence of viscous fingering on diffusion means that particular care has to be taken to ensure levels of numerical diffusion are much less than the physical diffusion in numerical simulations of these instabilities. For example, one of the earliest attempts in simulating viscous fingering was made by Peaceman and Rachford (1962) using finite difference method. Although they were able to reproduce the oil recovery curves observed in the experiments by Blackwell et al. (1959), they did not however manage to capture short wavelength fingers at early time due to the relatively coarse grid of $40 \times 20$ grid and a numerical scheme with first order accuracy.

In general numerical methods only give an approximate solution to the system of partial differential equations of interest as the discretisation results in a truncation error (Lantz, 1970; Fanchi, 1983). Lantz (1970) showed that the dominant truncation error in first order methods can be described by a diffusion-like term, hence the term numerical diffusion (Lantz, 1970; Mattax and Dalton, 1990). In fact, numerical diffusion combines with physical diffusion (Peaceman, 1977). The magnitude of the numerical diffusion depends on the spatial and temporal discretisation techniques employed, the resolution of the mesh as well as the size of the time step. For example, when using fully implicit single point upstream weighting, the numerical diffusion $D_{num,im}$ is given by Lantz (1971)

$$D_{num,im} = \frac{1}{2} v f' \Delta x (1 + CFL) \quad (2.11)$$

where $v$ is the fluid bulk velocity, $f'$ is the front velocity, $\Delta x$ is the grid size, and
2.3. Modelling approaches

\(CFL\) is the Courant number defined as (Lantz, 1971)

\[
CFL = v_{max} \frac{\Delta t}{\Delta x}
\]  

(2.12)

where \(v_{max}\) is the maximum fluid velocity, \(\Delta t\) is the time step and \(\Delta x\) is the grid size. Similarly, the numerical diffusion \(D_{num}\) for implicit pressure explicit saturation (IMPES) is given by

\[
D_{num,ex} = \frac{1}{2} v f' \Delta x (1 - CFL)
\]  

(2.13)

For explicit numerical solutions to be stable, Eq. 2.13 shows that \(CFL < 1\) so that \(D_{num,ex}\) is not an unphysical negative value.

For fully implicit simulation, numerical diffusion can therefore be reduced by refining the grid or mesh and/or reducing the time-step but this significantly increases the computational expense in terms of both storage of data and operations performed to solve the equations. This has driven the development of higher order methods that aim to reduce the truncation error associated with a given discretisation. Todd et al. (1972) demonstrated increased accuracy could be obtained by using two-point upstream. However, as with other higher order methods, it is susceptible to spurious oscillations at the shock front, which led Christie and Bond (1985) to couple it with a flux limiter called Flux Corrected Transport, FCT (Boris and Book, 1973). The same strategy was found to be successful in capturing the fingering pattern observed in physical experiments (Christie and Bond, 1987; Christie et al., 1990; Davies et al., 1991).

Apart from spatial and temporal resolutions, grid orientation (Figure 2.6) also has significant effects on the level of numerical error observed (Todd et al., 1972). These are particularly evident in miscible displacement simulations and are exacerbated by two-point point upstream weighting. Traditionally in finite difference or finite volume methods, a 5-point stencil is used where a grid block is not able to
'communicate' with its diagonal neighbours. Figure 2.7 shows an example of the error in oil recovery observed in 5-spot simulations using 5-point stencil taken from Todd et al. (1972). Here, the error from using a parallel grid is larger than that of diagonal grid. Yanosik and McCracken (1979) presented a version of a 9-point discretisation scheme to address this problem. This was later reformulated by Shiralkar and Stephenson (1991) for general applications including cases with gravity and capillary forces.

(a) (b)

Figure 2.6: An illustration of grid orientation relative to principal flow direction. (a) Parallel (0°); (b) Diagonal (45°).

In reservoir simulation, the salient features of the flow such as moving fronts and fingering patterns are where the computational effort should be spent most. However, using a fixed fine grid is rather inefficient because the computational effort is distributed unevenly across the domain. This motivates the use of dynamic adaptive gridding (Edwards and Christie, 1993; Mulder and Meyling, 1993; Adam et al., 2016). In this method, first proposed by Berger and Oliger (1984), refinement or amalgamation (or both) of the base grid is done by actively looking into the gradient of a defined variable such as flux or saturation. There are however computational overheads associated with adaptive gridding which should also be taken into account.
2.3. Modelling approaches

Figure 2.7: The oil recovery curve obtained from simulations with parallel and diagonal grid as compared to analytical solution (the upper most curve) from Todd et al. (1972).

Other numerical studies have resorted to using spectral methods to minimise the impact of numerical errors (Tan and Homsy, 1988; Zimmerman and Homsy, 1991; De Wit and Homsy, 1997; Nijjer et al., 2018). Spectral methods make use of high-order polynomials or Fourier series to achieve 'spectral accuracy' (Trefethen, 2000), where the rate of accuracy can be made to converge exponentially. They are, however, not feasible for general reservoir engineering problems - their accuracy superiority is unfortunately limited by the fact that only relatively simple geometries can be modelled and limitation in handling more complex domain boundaries.

Mostaghimi et al. (2016) have already shown that different numerical schemes and meshes can result in different viscous fingering patterns even when using the same fluid properties, but there appear to be few attempts to quantify and compare the truncation error between different schemes and approaches to meshing. Papers describing new algorithms for various higher order methods tend to compare the
accuracy of their proposed method with a standard first order scheme (e.g. Blunt and Rubin, 1992) but, to date, there have been very few studies that compare the truncation error resulting from different numerical schemes and discretizations.

### 2.3.2 Averaged models

Very high resolution simulations are required to properly capture the growth and development of viscous fingers (e.g. Christie, 1989; Ewing, 1991). Using the required number of grid cells is prohibitively expensive especially in field-scale studies. This has driven the development of empirical fingering models which capture the average behaviour of a fingered front.

![Figure 2.8: An illustration of the transversely averaged concentration in the fingering region.](image)

Figure 2.8: An illustration of the transversely averaged concentration in the fingering region.
Koval (1963) was the first to exploit the observation that, although the unstable flow gives finger sizes spanning many orders of magnitude, the length of the fingered zone (illustrated in Figure 2.8 as $L_{mix}$) grows approximately linearly ($L_{mix} \sim t$). They proposed an averaged model to capture the growth of the mixing zone using a 1D first order hyperbolic transport equation in the form of

$$\frac{\partial c}{\partial t} + \frac{\partial F_s(c)}{\partial x} = 0 \quad (2.14)$$

where $F_s$ is a flux function or commonly referred to as fractional flow. $F_s$ is analogous to the fractional flow of water in the Buckley-Leverett formulation (Buckley and Leverett, 1942) except that the former is a monotonic function defined by

$$F_s(c) = \frac{1}{1 + \frac{1}{M_e} \frac{1 - c}{c}} \quad (2.15)$$

where the fractional flow model is parameterised by an effective viscosity ratio, $M_e$. Koval proposed an empirical value of $M_e$ in the form of

$$M_e = H_k(0.78 + 0.22M_1^{0.25})^4 \quad (2.16)$$

where $H_k$ is the heterogeneity index. The constant values in Eq. 2.16 correspond to an empirical oil-solvent mixing ratio of 78% to 22%, which was found to give an excellent prediction of the concentration profile in 1D compared with experiments by Blackwell et al. (1959). However, one of the shortcomings of the model is that it is not readily scalable from 1D to higher dimensions. As demonstrated by Sorbie et al. (1995), the assumption of constant oil-solvent ratio gives a constant effective viscosity in the fingered region and subsequently, the model failed to predict the pressure drop observed in 2D laboratory experiments. Other averaged models were proposed by Todd and Longstaff (1972) and Fayers (1988). Like the Koval model, these other empirical models also rely on an adjustable fractional flow formulation.
which then appears in a hyperbolic transport equation.

The Todd and Longstaff model modifies the fractional flow curve by introducing a parameter $\omega$. Physically, this represents the degree of mixing induced in the fingered region between the two components (i.e. $L_{\text{mix}}$ in Figure 2.8). Its value can vary between 0 (no fingering) and 1 (complete mixing) but is typically set to $\omega = 2/3 = 0.67$ based on the calibration made against the experimental results of Blackwell et al. (1959). The fractional flow takes the same form as the Koval model (Eq. 2.15) but $M_e$ is now given by

$$M_e = \frac{\mu_{o,e}}{\mu_{s,e}}$$

where $\mu_{o,e}$ is the effective oil viscosity and $\mu_{s,e}$ is the effective solvent viscosity defined as

$$\mu_{o,e} = \mu_m \mu_o^{1-\omega}$$

$$\mu_{s,e} = \mu_m \mu_s^{1-\omega}$$

In Eqs. 2.18 and 2.19, $\mu_m$ is the oil-solvent mixture viscosity defined using the quarter-power rule (Lohrenz et al., 1964)

$$\mu_m = \left( \frac{c}{\mu_s^{\frac{1}{4}}} + \frac{1-c}{\mu_o^{\frac{1}{4}}} \right)^4$$

One drawback of these fingering models, however, is that the fitting parameters in their formulation may need to be calibrated by comparison with detailed simulation. A survey of field applications by Klins (1984) found a wide range of $\omega$ values from 0.5 to 0.93. Consequently, using the default value of $\omega=2/3$ may lead to erroneous results.

Using such empirical models to describe fingering has also enabled various authors
to subsequently derive analytical solutions to describe various miscible gas injection processes. For example, Blunt and Christie (1993) and Juanes and Blunt (2006) have shown how it is possible to predict the behaviour of Water Alternating Gas (WAG) displacements using the Todd and Longstaff model. Such analytical models enable rapid estimation of the best ratio of water to gas to inject to maximize recovery as well as providing a means for validating numerical models.

Although it was initially developed for miscible flooding, the Todd and Longstaff model is also used in polymer flooding (Bondor et al., 1972) which involves two phase three component flow simulation. There are several analyses which look into the analytical solutions of polymer slug flow (e.g. Bedrikovetsky, 1993) but most of them assumed stable flows. To date, there has been no analytical study incorporating the fingering at the rear of the slug.

2.3.3 Other methods

While it is obvious that the modelling of viscous fingering should be done by numerically solving the governing equations derived from first principles, other theories have also been employed to give more insights into this phenomenon. The random nature of viscous fingering leads to development of several stochastic models. For e.g., Nittman et al. (1985) observed some degree of self-similarity in the Hele-Shaw experiments that characterises fractals (Mandelbrot, 1982).

Unsurprisingly, the fractal-like fingers were only observed when there was low diffusion or from the fractal/chaos theory perspective – the stabilising effect is low compared to the level of randomness. Diffusion-limited aggregation (DLA) which involves random walk simulation first postulated by Witten and Sander (1983) and later, Paterson (1984) showed that direct analogy between DLA and viscous fingering if \( M \sim \infty \) and \( D \sim 0 \). Subsequent researchers (King, 1987; Sideiqui and Sahimi,
Chapter 2. Literature Review

1990) showed that the adaptation of DLA for viscous fingering is equally applicable for finite values of $M$.

2.4 Conclusions

Although a great deal of work has been carried out especially from the petroleum engineering perspective, viscous fingering remains a subject that is not well understood. This is due to its non-linear behaviour involving a complex interplay between viscosity contrast, diffusion and dispersion, as well as permeability heterogeneity in porous media. While there is a large body of work dedicated to linear stability analyses, laboratory or numerical studies of the intermediate to late time regimes are rather limited. There are also only very few studies that have been carried out to understand the propagation of miscible fingers in a polymer slug flow. We highlighted some of the areas that we believe deserve further attention including

1. How does numerical dispersion affect instability in the numerical simulations?
2. How well are grid orientation effects understood?
3. The complete morphology of fingering from onset to late time
4. The effects of anisotropic diffusion/dispersion on the growth of the fingers
5. The effects of reservoir heterogeneity on apparent flow regime (advective, diffusive, channelised)
6. How well do the 1D Koval or Todd and Longstaff models represent the fingering in cases such as channelised flow or chase water when injecting polymer slugs?

We shall limit our investigation to that of first contact miscible flow (single phase, two component), except for in Chapter 4 where we consider the unstable growth
2.4. Conclusions

during immiscible displacement, and Chapter 7 where we deal with two phase, three component flow to investigate polymer slug injection.
Chapter 3

Methods

In this chapter, we summarise the general methods employed throughout this study. The numerical simulators and any assumptions made to arrive at their governing equations are discussed. This includes the casting of variables into their dimensionless form.

3.1 Numerical simulation

We used a higher order, IMPES (implicit pressure, explicit saturation), finite difference simulator developed to model the details of first contact miscible viscous fingering in three component, two phase flows. Its ability to predict viscous fingering in miscible displacements has previously been validated by comparison with experimental results by Christie and Bond (1987), Christie et al. (1990), and Al-Shuraiqi et al. (2003), among others.

The commercial simulator ECLIPSE (Schlumberger, 2015) is also used in this study. Specifically, we used its black oil module (ECLIPSE 100) to investigate the truncation error in a first order accurate solver in Chapter 4.
3.1 Numerical simulation

While ECLIPSE 100 is a fast and robust way to model black oil problems, it does not have enough flexibility for any modification in its source code. Because of that, we also utilised an open-source simulator called MATLAB Reservoir Simulation Tools, MRST (Lie, 2016). As presented in Chapter 7, we were able to implement a more numerically accurate model of polymer slug injection using MRST.

3.1.1 Formulation

3.1.1.1 First contact miscible simulation

We now describe the formulation of the FCM simulator, based on the presentations by Christie and Bond (1985) and Barley (1992).

Conservation of mass

The FCM simulator models two immiscible phases – oileic and aqueous. The conservation of mass for each phase are given by

\[
\phi \frac{\partial}{\partial t} (\rho_w S_w) + \nabla \cdot (\rho_w v_w) = q_w \rho_w
\]

(3.1)

\[
\phi \frac{\partial}{\partial t} (\rho_o S_o) + \nabla \cdot (\rho_o v_o) = q_o \rho_o
\]

(3.2)

The oileic phase consists of oil and solvent components that are assumed to be fully miscible on the first contact. The mass transfer between them are modelled as diffusion and dispersion giving rise to the advection-diffusion equations as follows

\[
\phi \frac{\partial}{\partial t} (\rho_{oo}(1 - c)S_o) + \nabla \cdot (\rho_{oo}(1 - c)v_o) + \rho_{oo} \nabla \cdot (S_o \mathbf{D} : \nabla c) = q_o (1 - c) \rho_{oo}
\]

(3.3)

\[
\phi \frac{\partial}{\partial t} (\rho_{os}cS_o) + \nabla \cdot (\rho_{os}cv_o) + \rho_{os} \nabla \cdot (S_o \mathbf{D} : \nabla c) = q_o c \rho_{os}
\]

(3.4)

where subscripts \( oo \) and \( os \) correspond to the oil and solvent components respectively.
Chapter 3. Methods

\( c \) is the dimensionless solvent concentration defined as

\[
c = \frac{c_s}{c_{s,\text{inj}}}
\]  

(3.5)

where \( c_s \) is the solvent concentration in mass of solvent per volume of oleic phase and \( c_{s,\text{inj}} \) is the injected concentration of solvent in the same units.

The diffusion tensor \( \mathbf{D} \) is assumed to be diagonal and characterised by molecular diffusion and velocity-dependent dispersion

\[
\mathbf{D} = \begin{pmatrix}
D_o + \alpha_x |v_t| & 0 \\
0 & D_o + \alpha_y |v_t|
\end{pmatrix}
\]  

(3.6)

where \( D_o \) is molecular diffusivity, and \( \alpha_x \) and \( \alpha_y \) are dispersion constant in the \( x \) and \( y \) direction respectively. \( v_t \) is the total interstitial velocity given by the sum of the velocities of both phases

\[
v_t = v_o + v_w
\]  

(3.7)

The fluid velocities \( v_o \) and \( v_w \) are assumed to follow Darcy velocities, hence we have

\[
v_w = -\lambda_w \left( \nabla p_w - \rho_w g \right)
\]  

(3.8)

where \( g \) is the acceleration due to gravity and \( \lambda_w \) is the mobility of water phase defined as

\[
\lambda_w = K \frac{k_{rw}}{\mu_w}
\]  

(3.9)

Similarly, the velocity and mobility of the oleic phase are given by

\[
v_o = -\lambda_o \left( \nabla p_o - \rho_o g \right)
\]  

(3.10)

\[
\lambda_o = K \frac{k_{ro}}{\mu_o}
\]  

(3.11)
$\mu_o$ in Eq. 3.11 is the oil-solvent mixture viscosity which is a function of solvent concentration and is assumed to follow the empirical quarter-power mixing rule.

$$
\mu_o = \left( \frac{c}{\mu_{os}} + \frac{1-c}{\mu_{oo}} \right)^{-\frac{1}{4}}
$$

(3.12)

The oleic phase density due to mixing between oil and solvent components is assumed to be linearly dependent on solvent concentration

$$
\rho_o = \rho_{os}c + \rho_{oo}(1 - c)
$$

(3.13)

Additionally, the following constraints also apply

$$
p_c(S_w) = p_o - p_w
$$

(3.14)

$$
S_w + S_o = 1
$$

(3.15)

**Pressure equation**

To define a unique representative pressure describing oleic and aqueous phases, the simulator uses the (arithmetic) average pressure as proposed by Peaceman (1977)

$$
p = \frac{1}{2}(p_o + p_w)
$$

(3.16)

The Darcy velocities are expressed in terms of $q_w$ and $q_o$ by summing Eqs. 3.1 and 3.2 after dividing them with their respective densities to yield

$$
\nabla \cdot (v_w + v_o) = q_w + q_o
$$

(3.17)

$$
\nabla \cdot ((\lambda_o + \lambda_w)\nabla p) = \nabla \cdot (\lambda_o \rho_o + \lambda_w \rho_w)g - \frac{1}{2} \nabla \cdot ((\lambda_o - \lambda_w)\nabla p_c) - (q_w + q_o)
$$

(3.18)

**Fractional flow**
The velocities $v_w$ and $v_o$ are expressed in terms of fractional flow $f_w$. To do this, $v_w$ and $v_o$ are first defined in terms of the average pressure $p$ and capillary pressure $p_c$

\[ v_w = -\lambda_w (\nabla p - \frac{1}{2} \nabla p_c - \rho_w g) \]  
\[ (3.19) \]

\[ v_o = v_t - v_w = -\lambda_o (\nabla p + \frac{1}{2} \nabla p_c - \rho_o g) \]  
\[ (3.20) \]

Then by taking (3.19)–(3.20)

\[ \frac{v_w}{\lambda_w} - \frac{v_t - v_w}{\lambda_o} = \nabla p_c + (\rho_w - \rho_o)g \]  
\[ (3.21) \]

\[ (\lambda_w + \lambda_o)v_w = \lambda_w v_t + \lambda_w \lambda_o \nabla p_c + \lambda_w \lambda_o (\rho_w - \rho_o)g \]  
\[ (3.22) \]

\[ v_w = \frac{\lambda_w}{(\lambda_w + \lambda_o)} v_t + \frac{\lambda_w \lambda_o}{(\lambda_w + \lambda_o)} \nabla p_c + \frac{\lambda_w \lambda_o}{(\lambda_w + \lambda_o)} (\rho_w - \rho_o)g \]  
\[ (3.23) \]

\[ v_w = f_w v_t - \psi \nabla S + \lambda_o f_w (\rho_w - \rho_o)g \]  
\[ (3.24) \]

where $f_w$ is the fractional flow of aqueous phase and $\psi$ is the diffusion due to capillary forces

\[ f_w = \frac{\lambda_w}{\lambda_w + \lambda_o} \]  
\[ (3.25) \]

\[ \psi = \frac{\lambda_w \lambda_o}{\lambda_w + \lambda_o} \frac{dp_c}{ds} \]  
\[ (3.26) \]

**Non-dimensionalisation of the variables**

Internally, the governing equations in the FCM simulator are further simplified by expressing all the variables in dimensionless form. This is achieved by scaling the variables with characterestic parameters (identified with subscript 0). For example, the dimensionless viscosity of oil, $\mu_o$, is made dimensionless by taking the ratio between the oil viscosity $\mu_o$ to a characteristic viscosity $\mu_0$. $\mu_0$ is chosen be equal to $\mu_w$, hence we have

\[ \hat{\mu}_o = \frac{\mu_o}{\mu_0} = \frac{\mu_o}{\mu_w} \]  
\[ (3.27) \]
3.1. Numerical simulation

The full list of characteristic variables are shown in Table 3.1. By using the variables in their dimensionless forms and expressing the gradient operator as

$$\hat{\nabla} = \frac{1}{x_0} \nabla$$  \hspace{1cm} (3.28)

the pressure equation (3.18) can be expressed as

$$\hat{\nabla} \cdot ((\hat{\lambda}_w + \hat{\lambda}_o)\hat{\nabla}p) + \frac{1}{2} \hat{\nabla} \cdot ((\hat{\lambda}_o - \hat{\lambda}_w)\hat{\nabla}p_c) + \frac{\mu_0 x_0^2}{K_0 p_0} (q_o + q_w) = \frac{x_0 \rho_0}{\rho_o} \hat{\nabla} \cdot (\hat{\lambda}_w \rho_w + \hat{\lambda}_o \rho_o) \dot{g}$$  \hspace{1cm} (3.29)

which can be further simplified as

$$\hat{\nabla} \cdot ((\hat{\lambda}_w + \hat{\lambda}_o)\hat{\nabla}p) + \frac{1}{2} \hat{\nabla} \cdot ((\hat{\lambda}_o - \hat{\lambda}_w)\hat{\nabla}p_c) + \dot{Q} = \hat{\nabla} \cdot (\hat{\lambda}_w \rho_w + \hat{\lambda}_o \rho_o) \dot{g}$$  \hspace{1cm} (3.30)

### Triggering of fingers

Throughout this study, unless stated otherwise, we explicitly induce the formation of fingering by perturbing the velocity field. This is achieved by specifying a random solvent concentration initialisation (uniformly distributed between 0 and 1) at the inlet prior to injection.

3.1.1.2 Black oil model

ECLIPSE 100 and MRST (Schlumberger, 2015; Lie, 2016) use similar standard black oil formulations, which has been discussed extensively by various authors including Peaceman (1977) and Aziz and Settari (1979). The formulation models three phases (oil, gas, and water) with no diffusion between the phases. At reservoir conditions, gas can be partially or completely dissolved in the oilic phase and this is defined through gas-oil ratio, $R_g$. Similarly, the oil-gas ratio, $R_v$, defines the volume of oil vapourised into gaseous phase. To take into account the difference in volume
### Scaling value

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Relationship</th>
<th>Dimension</th>
<th>Dimensionless variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>$x_0$</td>
<td>chosen</td>
<td>[L]</td>
<td>$\hat{x} = \frac{x}{x_0}, \hat{y} = \frac{y}{x_0}, \hat{z} = \frac{z}{x_0}$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$v_0$</td>
<td>chosen</td>
<td>[LT$^{-1}$]</td>
<td>$\hat{v} = \frac{v}{v_0}$</td>
</tr>
<tr>
<td>Permeability</td>
<td>$K_0$</td>
<td>chosen</td>
<td>[L$^2$]</td>
<td>$\hat{K} = \frac{K}{K_0}$</td>
</tr>
<tr>
<td>Viscosity</td>
<td>$\mu_0$</td>
<td>$\mu_0 = \mu_w$</td>
<td>[ML$^1$T$^{-1}$]</td>
<td>$\hat{\mu} = \frac{\mu}{\mu_0}$</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho_0$</td>
<td>$\rho_0 = \rho_w$</td>
<td>[ML$^3$]</td>
<td>$\hat{\rho} = \frac{\rho}{\rho_0}$</td>
</tr>
<tr>
<td>Time</td>
<td>$t_0$</td>
<td>$t_0 = \frac{v_0}{v_0}$</td>
<td>[T]</td>
<td>$\hat{t} = \frac{t}{t_0}$</td>
</tr>
<tr>
<td>Pressure</td>
<td>$p_0$</td>
<td>$p_0 = \frac{x_0v_0\mu_0}{K_0}$</td>
<td>[ML$^1$T$^{-2}$]</td>
<td>$\hat{p} = \frac{p}{p_0}$</td>
</tr>
<tr>
<td>Gravity</td>
<td>$g_0$</td>
<td>$g_0 = \frac{v_0}{x_0\rho_0}$</td>
<td>[LT$^{-2}$]</td>
<td>$\hat{g} = \frac{g}{g_0}$</td>
</tr>
<tr>
<td>Diffusivity</td>
<td>$D_0$</td>
<td>$D_0 = x_0v_0$</td>
<td>[L$^2$T$^{-1}$]</td>
<td>$\hat{D} = \frac{D}{D_0} = \frac{1}{Pe}$</td>
</tr>
<tr>
<td>Source term</td>
<td>$q_0$</td>
<td>$q_0 = \frac{v_0}{x_0} = \frac{p_0K_0}{\mu_0x_0^2}$</td>
<td>[T$^{-1}$]</td>
<td>$\hat{q}_w = \frac{q_w}{q_0}$</td>
</tr>
<tr>
<td>Mobility</td>
<td>$\lambda_0$</td>
<td>$\lambda_0 = \frac{K_0}{\mu_0}$</td>
<td>[M$^{-1}$L$^3$T]</td>
<td>$\hat{\lambda} = \frac{\lambda}{\lambda_0}$</td>
</tr>
</tbody>
</table>

Table 3.1: Characteristic variables used in the FCM simulator
occupied by phase $\alpha$ at reservoir and surface conditions, the formation-volume factor $B_\alpha$ is used. The conservation of mass for water, oil, and gas are given by

$$\frac{\partial}{\partial t} \left( \frac{\phi S_w}{B_w} \right) + \nabla \cdot \left( \frac{v_w}{B_w} \right) + \frac{q_w}{B_w} = 0$$ (3.31)

$$\frac{\partial}{\partial t} \left( \frac{\phi S_o}{B_o} + \frac{R_v S_g}{B_g} \right) + \nabla \cdot \left( \frac{v_o}{B_o} + \frac{R_v v_g}{B_g} \right) + \left( \frac{q_o}{B_o} + \frac{R_v q_g}{B_g} \right) = 0$$ (3.32)

$$\frac{\partial}{\partial t} \left( \frac{\phi S_g}{B_g} + \frac{R_s S_o}{B_o} \right) + \nabla \cdot \left( \frac{v_g}{B_g} + \frac{R_s v_o}{B_o} \right) + \left( \frac{q_g}{B_g} + \frac{R_s q_o}{B_o} \right) = 0$$ (3.33)

### 3.2 Reservoir models

Homogeneous reservoirs were used throughout the study, except in Chapter 6 where the miscible displacement in realistic heterogenous reservoirs was investigated using SPE 10 Model 2 (Christie and Blunt, 2001) and synthetic normally distributed log-permeability generated from Sequential Gaussian Simulation (Deutsch and Journel, 1997). We will discuss this further in Chapter 6.

### 3.3 Reservoir configuration

Previous studies (e.g. Zimmerman and Homsy, 1992; Christie et al., 1993) showed that many of the important features of viscous fingering in a 3D linear displacement can be replicated in 2D domain. Hence, to reduce the computational effort, a 2D line-drive injection was used throughout the study. Using a linear displacement also allows us to compare the simulations with analytical solution such as those given by Tan and Homsy (1986).
The injection well (inlet) and production well (outlet) are located at the left and right of the reservoir respectively as illustrated in Figure 3.1. A no flow boundary condition is imposed at the top and bottom of the reservoir.

### 3.4 Dimensionless variables

As we have seen in Section 3.1.1.1, the use of dimensionless variables greatly simplifies the governing equations in the FCM simulator. Additionally, dimensionless variables also allows direct comparison between the results from the FCM simulator and other reservoir simulators (which typically use field, metric or laboratory units).

Consider a 2D reservoir with line-drive injection illustrated in Figure 3.1. Dimensionless distances in $x$ and $y$ directions are

\[ x_d = \frac{x}{L} \quad (3.34) \]

\[ y_d = \frac{y}{H} \quad (3.35) \]

In presenting our results, we will mostly use the dimensionless time in terms of pore volumes injected (PVI), $t_d$ that is commonly used in petroleum engineering. This is
defined as
\[
t_d = \frac{\text{volumetric injected rate} \times \text{time}}{\text{reservoir pore volume}} = \frac{Qt}{\phi AL} \tag{3.36}
\]
where \(Q\) is the constant injection rate and \(A\) is the reservoir cross-sectional area.

In Chapter 5, however, we will cast the time in diffusive and advective time scales, \(t_D\) and \(t_{D,A}\) respectively. We will discuss this further in Chapter 5.

Mobility ratio \(M\) is one of the most important variables in determining the degree of miscible viscous fingering. It is defined as the ratio of displacing fluid mobility to that of the displaced fluid
\[
M = \frac{\lambda_{\text{displacing}}}{\lambda_{\text{displaced}}} = \frac{K_{\text{displacing}}/\mu_{\text{displacing}}}{K_{\text{displaced}}/\mu_{\text{displaced}}} \tag{3.37}
\]
In FCM displacements this reduces to the viscosity ratio \(\mu_{\text{displaced}}/\mu_{\text{displacing}}\).

We use dimensionless concentration \(c\) throughout the thesis. As presented in Section 3.1.1.1, dimensionless solvent concentration \(c\) is given by
\[
c = \frac{c_s}{c_{s,\text{inj}}} \tag{3.38}
\]
where \(c_s\) is the solvent concentration in mass of solvent per volume of oleic phase and \(c_{s,\text{inj}}\) is the injected concentration of solvent in the same units. Hence we have \(0 \leq c \leq 1\).

### 3.5 Fluid properties

A typical miscible flow in porous media exhibits a higher dispersion longitudinally compared to those in transverse direction due to its dependency on fluid velocity. The impact of this dispersion anisotropy will be investigated in Chapter 5. Although highly unlikely, we will also explore cases when the transverse dispersion is higher
than longitudinal dispersion. This is to allow us to investigate the interplay between longitudinal and transverse dispersion as well as to access the validity of the linear stability theory when $D_T > D_L$.

To ensure that the longitudinal and transverse dispersion remain constant throughout the simulation, we use velocity-independent dispersion coefficients in both directions.

Finally, to exclude the influence of gravity on the viscous fingers, we use fluids with equal density.

### 3.6 Conclusions

We have presented the general methodology used throughout this study, including the overview of the numerical simulators, reservoir model and well configuration, fluid properties as well as the non-dimensionalisation of the relevant parameters.

As the nature of viscous fingering requires high resolution numerical simulation, a higher order solution was used to simulate first contact miscible fingering in two-phase, three-component flows. A discussion on this solution and its impact on numerical dispersion and modelling of viscous fingering will be presented in the next chapter (Chapter 4).

In order to simplify the modelling and reduce the computational effort, several assumptions were made in our simulation including

1. Non-periodic 2D domain
2. Linear displacement
3. No gravity effect
4. First contact miscible fluids with quarter-power mixing rule for mixture viscosity

5. Velocity independent dispersion

While these assumptions may limit the applicability of our study to other cases, the results should provide a qualitative understanding and a starting point for future studies.

Although originally written for water-oil-solvent system, the FCM simulator can also be used to study polymer injection system (water-polymer-oil) as the governing equations for both systems are analogous (provided certain assumptions are made). Similarly, the standard black oil simulators are also capable of modelling fingering during polymer slug injection by implementing an empirical model to describe the mixing between two miscible components. We will present the modelling strategies specific to polymer injection in Chapter 7.
Chapter 4

Truncation Error

In this study, we perform various numerical experiments to quantify the level of numerical dispersion in different numerical schemes and meshes and its impact on the growth of both miscible and immiscible viscous fingering. The numerical schemes considered are

1. First order finite volume (FOFV)
2. Higher order finite difference (HOFD)
3. Control volume finite element (CVFE)

The first two methods use structured meshes with rectangular grid blocks and we examine the impact of aligning flow with the grid as well as using a grid that is diagonal to flow. The CVFE approach can be applied using either a structured or an unstructured mesh, but in both cases the meshes are formed of triangular elements.

The CVFE simulation in this chapter was carried out by Dr. Alexander Adam at the request of the author of this thesis. This is the continuation of his previous work on immiscible viscous fingering that can be found in Adam et al. (2016).
4.1 Theory

This section presents the results from a perturbation analysis using the flow equations described in the previous chapter that can be used to estimate the impact of diffusion (numerical or physical) on the initial growth rate of viscous fingers.

4.1.1 Linear stability analysis

Miscible viscous fingering is a function of the viscosity ratio \( M = \mu_2/\mu_1 \) between the fluids and the diffusion and dispersion whereas immiscible fingering is a function of the shock front mobility ratio \( M_{sf} = \lambda_{t,f}/\lambda_{t,0} \) and capillary pressure. \( \lambda_{t,f} \) is the total mobility at shock front saturation and \( \lambda_{t,0} \) is the total mobility at the displacing fluid saturation ahead of the shock front where

\[
\lambda_t = \frac{K_{r1}/\mu_1}{K_{r1}/\mu_1 + K_{r2}/\mu_2}
\]

and the shock front saturation can be determined using the Welge construction (Welge, 1952) and Buckley-Leverett analysis (Buckley and Leverett, 1942).

In the linear stability analysis, the growth rate refers to the rate at which a perturbation grows. This perturbation is typically expressed as a sinusoidal wave at the fluid interface (see Figure 2.1b), where the frequency of the wave is referred to as the wavenumber. In the absence of diffusion (miscible) or capillary pressure (immiscible) then the growth rate of a finger \( \omega_{gr} \) increases with wavenumber \( k \) for miscible displacements according to (Christie and Bond, 1987)

\[
\omega_{gr} = kv_0 \frac{M - 1}{M + 1}
\]

and for immiscible displacements according to (Yortsos and Huang, 1986; Riaz and
Chapter 4. Truncation Error

Tchelepi, 2004)

\[ \omega_{gr} = kv_0 \frac{f_{F1,f}}{S_{F1,f} - S_{F1,c}} \frac{M_{sf} - 1}{M_{sf} + 1} \]  \( (4.2) \)

where \( v_0 \) is the mean interstitial velocity, \( S_{F1,f} \) is the displacing fluid saturation at the shock front that forms between displacing and displaced fluids in an immiscible displacement (Buckley and Leverett, 1942), \( S_{F1,c} \) is the displacing fluid saturation ahead of the shock front, \( f_{F1,f} \) is the fractional flow at shock front saturation given by

\[ f_{F1,f} = \frac{K_{r1}(S_{F1,f})/\mu_1}{K_{r1}(S_{F1,f})/\mu_1 + K_{r2}(S_{F1,f})/\mu_2} \]

The effect of physical diffusion (or capillary pressure) is to reduce the growth rate of higher wavenumber fingers to below the values given by Eqs. 4.1 or 4.2. This leads to the phenomenon of the ‘most dangerous’ wavenumber (the wavenumber with the highest growth rate) and the critical wavenumber (the wavenumber above which growth rate is negative) as shown schematically in Figure 2.2. In real physical systems no fingers can form with a wavenumber higher than this critical wavenumber. Numerical diffusion may mean that the growth rate of higher wavenumber fingers in numerical simulations is lower than expected physically or, in the worst case, may prevent any fingers growing at all.

4.2 Numerical schemes

We aim to compare the truncation errors associated with three different numerical schemes by first quantifying the amount of transverse and longitudinal numerical diffusion found in those simulations and then comparing the early time growth rate of immiscible viscous fingers with the predictions of linear stability analysis (Eq. 4.2).
4.2. Numerical schemes

4.2.1 Numerical methods

The three numerical schemes are a first order finite volume scheme, a second order finite difference scheme and a first order control volume finite element scheme. We also compare the impact of using a 5 point and a 9 point stencil (Yanosik and McCracken, 1979) in the first order finite volume simulator. We will use a structured mesh with rectangular grid blocks in both the finite difference and finite volume schemes and compare the outcomes when the grid is aligned with the principal flow direction and when it is diagonal (i.e. at 45°) to the principal flow direction. The control volume finite element simulation uses a mesh with triangular elements and we shall compare the outcomes when using a fixed structured mesh, a fixed unstructured mesh and dynamically adapting unstructured mesh (that is more refined where saturation changes most quickly, Adam et al., 2016).

The commercial black-oil simulator ECLIPSE 100 (Schlumberger, 2015) was used for our first order finite volume (FOFV) simulations. It employs a standard black-oil formulation using either a 5 point or 9 point stencil (Yanosik and McCracken, 1979) with an option to use either an IMPES (implicit pressure, explicit saturation) or a fully implicit solver. In this study we used the fully implicit solver option.

The second simulator we used was the second order, finite difference (HOFD), IMPES simulator described in Chapter 3. Only the 5 point stencil is available for this simulator.

The third simulator used in this study is the control volume finite element (CVFE) simulator described by Jackson et al. (2015) and Gomes et al. (2017). In this formulation pressure, permeability, porosity and viscosity are represented using finite elements but saturation and relative permeability are described on control volumes. The use of control volumes for saturation helps ensure mass conservation. We used the \( P_1(DG) - P_2 \) element in this study (first order discontinuous finite element...
representation for velocity, second-order continuous finite element representation for pressure). All equations were solved implicitly.

4.3 Method

We evaluated the effect of truncation error in the different simulators first by performing a series of unit mobility ratio, first contact miscible numerical experiments (without modelling physical diffusion) and then calculating the level of longitudinal and transverse diffusion in each case for different mesh resolutions. We then illustrate how these levels of numerical error influence the pattern of viscous fingering that occurs for an adverse mobility ratio, first contact miscible displacement. Finally, we consider an unstable immiscible displacement and compare the early time growth rate predicted as a function of grid resolution, mesh type and numerical scheme.

4.3.1 Test cases

All simulations were performed in the square, 2D model system illustrated in Figure 3.1. The system was initially filled with fluid 2. Fluid 1 was then injected at constant rate $Q$ along the left-hand face of the model. Both fluids have the same viscosity ($M = 1$). Fluid was produced at the same constant rate along the right-hand face. The top and bottom edges of the model were no-flow boundaries. Gravity was eliminated by giving the injected fluid and the displaced fluid equal densities. The permeability was constant and homogeneous in all cases. No physical diffusion (or capillary pressure in immiscible displacements) was input into any simulation so all apparent diffusion was due to truncation errors.

Numerical diffusion (longitudinal and transverse) was evaluated as a function of
grid resolution by performing a series of runs using different mesh resolutions, mesh types and orientations. The finite volume and finite difference mesh simulations used Cartesian meshes with square grid block and the grid resolutions listed in Table 4.2. We considered both diagonal and parallel meshes as well as comparing the impact of using a 5 point versus a 9 point stencil to calculate the pressure differences in the FOFV simulations. The simulations using the CVFE simulator used triangular mesh elements. We compared the outcomes from using a fixed structured mesh, a fixed unstructured mesh and a dynamic adaptive mesh that used the method described by Adam et al. (2016). Figure 4.1 shows examples of these meshes. The total number of elements used are also listed in Table 4.2.

We used a CFL number of 1.0 for the first order finite volume simulations and a CFL number of 0.4 for the second order finite difference simulations. The CVFE simulations used a CFL number of 0.05.

Truncation error is a function of time step size as well as mesh resolution: in a fully implicit scheme then the larger the timestep the larger the numerical diffusion (Lantz, 1971) whereas numerical diffusion increases the further the timestep is from the CFL limit in a scheme that is explicit in saturation.

The FOFV and CVFE simulators used here were originally developed for immiscible applications. To enable these to simulate miscible displacements, we modified the input relative permeabilities as described by Lantz (1970). In this case the relative permeability curves become

\[ K_{r1} = \frac{\mu_1 S_{F1}}{\mu(S_{F1})} \]  

\[ K_{r2} = \frac{\mu_2 S_{F2}}{\mu(S_{F2})} \]  

where

\[ \mu(S_{F1}) = \left( \frac{S_{F1}}{\mu_1^2} + \frac{1 - S_{F1}}{\mu_1^2} \right)^{-4} \]
Chapter 4. Truncation Error

$K_{r1}$ and $K_{r2}$ are straight lines if $\mu_1 = \mu_2$. For the immiscible displacement simulations, we used the relative permeability curves taken from Jaure et al. (2014) as shown in Figure 4.2. These represent a water-wet system, and are described using the Corey correlation (Corey, 1954)

$$K_{r1} = 0.06 \left( \frac{S_{F1} - S_{or}}{1 - S_{wc} - S_{or}} \right)^2$$

$$K_{r2} = 0.74 \left( \frac{1 - S_{F1} - S_{or}}{1 - S_{wc} - S_{or}} \right)$$

$S_{wc}$ was set to be 0.3 and $S_{or}$ was 0.4. Capillary pressure was not input as we were not modelling physical diffusion and dispersion.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
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<td>Fluid viscosity</td>
<td></td>
</tr>
<tr>
<td>Fluid 2</td>
<td>$\mu_2 = 303$</td>
</tr>
<tr>
<td>Fluid 1</td>
<td>$\mu_1 = 1$</td>
</tr>
<tr>
<td>Inital condition, $t_d = 0$</td>
<td></td>
</tr>
<tr>
<td>$x_d &lt; 0.1$</td>
<td>$S_{F1} = 1 - S_{F2}$</td>
</tr>
<tr>
<td>$x_d &gt; 0.1$</td>
<td>$S_{F1} = S_{wc} + \frac{A_s}{2}(1 - S_{wc} - S_{or})(1 + \cos(2\pi k y_d))$</td>
</tr>
<tr>
<td>$t_d &gt; 0$</td>
<td>$S_{F1} = 1$</td>
</tr>
</tbody>
</table>

Table 4.1: Fluid properties for the immiscible displacement simulations

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Max. no. of grid blocks/ no. of elements</th>
<th>Range of element lengths (in flow direction)</th>
<th>Courant number</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOFV</td>
<td>$400 \times 400$</td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>HOHD</td>
<td>$400 \times 400$</td>
<td>$0.0025 - 0.005$</td>
<td>0.4</td>
</tr>
<tr>
<td>CVFE</td>
<td>$50000$</td>
<td></td>
<td>$\sim 0.05$</td>
</tr>
</tbody>
</table>

Table 4.2: Summary of data used for the immiscible displacement simulations
4.3. Method

Figure 4.1: Illustrations of the different meshes used in the CVFE simulations. (a) fixed structured mesh; (b) fixed unstructured mesh; (c) unstructured; dynamically adapting mesh (adapting to changes in displacing fluid saturation).

Figure 4.2: Relative permeability curves used in the immiscible viscous fingering simulations (from Jaure et al., 2014).
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4.3.2 Calculation of numerical diffusivity

We first performed a series of numerical experiments to quantify the amount of longitudinal and transverse numerical diffusion as a function of numerical scheme, mesh resolution and mesh orientation. These experiments are identical to laboratory and numerical experiments performed by many researchers including Taylor (1953), Ogata and Banks (1961), and Perkins and Johnston (1963) to measure longitudinal dispersion; and Hiby (1962) and Blackwell (1962) to measure transverse dispersion.

4.3.2.1 Longitudinal dispersion

The model was initially fully saturated with oil, and at $t_d > 0$, we injected miscible solvent of equal viscosity into the left-hand boundary. This results in a stable but dispersed front as shown in Figure 4.3(b).

The analytical solution to Eq. 2.1 for a 1D displacement in the presence of diffusion is given by

$$
c = 0.5 \left( \text{erf} \left( \frac{x_d - t_d}{2 \sqrt{D_{L,num} x_d}} \right) \right)$$

(4.6)

where we have replaced $D$ with $D_{L,num}$ to emphasise that the diffusivity in our simulations is solely due to the longitudinal numerical dispersion.

Inspection of Eq. 4.6 indicates that length of the dispersed front grows proportionally to $\sqrt{t_d}$, which is the main characteristic of dispersion dominated flow. The numerical dispersivity $D_{L,num}$ can be found by fitting the solvent concentration $c$ from simulation with Eq. 4.6 or alternatively, as demonstrated by Taylor (1953), by calculating the dispersivity directly from the displacement concentration contours

$$
D_{L,num} = \frac{1}{t_d} \left( \frac{x_{c=0.9} - x_{c=0.1}}{3.625} \right)^2
$$

(4.7)
where $x_{c=0.1}$ and $x_{c=0.9}$ are the locations where the concentration values are $c = 0.1$ and $c = 0.9$ respectively. In this study we fitted the simulated concentration profile $(0 \leq c \leq 1)$ with Eq. 4.6 to find $D_{L,\text{num}}$ such that the sum of squared error was minimised.

### 4.3.2.2 Transverse dispersion

To calculate the numerical transverse diffusivity, we continuously injected oil across the upper half of the left-hand face of the model whilst injecting solvent of equal viscosity at the same rate across the bottom half of the left-hand face, as shown in Figure 4.3(c). The solvent and oil were injected until all the initial reservoir fluid was displaced and a steady state distribution of oil and solvent was reached. Numerical transverse diffusion, $D_{T,\text{num}}$, mixes oil and solvent across the midline of the model and the width of this mixing zone increases from left to right.

The vertical concentration profile taken at distance $x_d = 0.5$ from the inlet is plotted in Figure 4.3(d) which shows a smooth transition from $c = 1$ to $c = 0$ rather than a step change because of the transverse mixing between the two fluids. This curve can be used to find $D_{T,\text{num}}$ by fitting it to the following analytical solution (Hiby, 1962)

$$
    c = 0.5 \left( \text{erf}\left(\frac{y_d + 0.5}{2\sqrt{D_{T,\text{num}}x_d}}\right) + \text{erf}\left(\frac{y_d - 0.5}{2\sqrt{D_{T,\text{num}}x_d}}\right) \right) 
$$

(4.8)

### 4.3.3 Calculation of growth rates

To calculate the initial growth rates of immiscible viscous fingers the initial saturation field was perturbed as follows. The first 10% of the model to the right of the inlet was fully saturated with water i.e.

$$
    S_{F1} = 1 - S_{or}; \quad x_d < 0.1 
$$

(4.9)
Figure 4.3: Numerical experiments used to calculate the values of longitudinal and transverse numerical dispersions. (a) An example of concentration map used to determine longitudinal numerical dispersion. (b) Transversely averaged concentration obtained from (a). (c) An example of concentration map to used determine transverse numerical dispersion. (d) Concentration $c$ at $x_d = 0.5$, as a function of vertical distance $y_d$ obtained from (c).
The water saturation in the rest of the model was initialised according to the chosen wavenumber $k$ using

$$S_{F1} = S_{wc} + \frac{A_s}{2} (1 - S_{wc} - S_{or})(1 + \cos(2\pi ky_d)); \quad x_d \geq 0.1 \quad (4.10)$$

where $A_s$ is a scaling factor that changes the amplitude of the cosine wave (chosen to be 0.1 in this study). The saturation curve for a wavenumber $k$ of 2 is shown in Figure 4.4(a) as an example, while Figure 4.4(b) illustrates the corresponding water saturation map. Here, the wavenumber is measured based on the distance perpendicular to the flow between the fingertips. We chose to initiate the fingering part way into the model so that the early time growth of the fingers was not influenced by the inlet boundary.

![Figure 4.4: The initial conditions used to investigate the growth rate of immiscible viscous fingering. (a) Initial saturation perturbation, shown here for wavenumber $k = 2$; (b) Initial saturation map for wavenumber $k = 2$.](image)

We were only interested in the early time growth rate, so the simulations were only run for 0.05 PVI. To calculate the growth rate, we extracted the contour line of what would be the shock front saturation ($S_{F1,f} = 0.36$) if there were no fingers. We then found the finger amplitude $L_{mix}$ by determining the amplitude of the contour line as illustrated in Figure 4.5(b) at different times. The growth rate $\omega_{gr}$ was then
calculated using
\[ \omega_{gr} = \frac{dL_{\text{mix}}}{dt_d} \left( (1 - S_{or}) - S_{w,p} \right) \] (4.11)

where \( dL_{\text{mix}}/dt_d \) is the gradient of the \( L_{\text{mix}} \) plotted against \( t_d \), and \( S_{w,p} \) is the average value of the perturbation which is 0.315 (see Figure 4.4b).

Figure 4.5: Illustration of growth rate calculation in miscible displacement for wavenumber of 2. (a) Saturation map at \( t_d = 0.05 \). (b) The mixing length \( L_{\text{mix}} \) is found from the contour line of \( S_w \) of 0.36 at various times. (c) The growth rate is taken as the gradient of \( L_{\text{mix}} \) vs. \( t_d \) plot.

4.4 Results and Discussion

4.4.1 Numerical diffusion

Figure 4.6 compares the concentration profiles obtained from the FOFV and the HOFD simulations with their best fit profiles obtained from Eq. 4.6. There is a good match between the profile obtained from the fine grid FOFV simulation and the analytical curve, (Figure 4.6a, 200×200 mesh with \( \Delta x_d = 1/200 = 0.005 \)), because the truncation error is dominated by the second order terms, as expected from the analysis of Lantz (1971). We notice, however, that the agreement between simulation and fitted analytical is less good for the FOFV simulation with a coarse mesh (Figure 4.6b, \( \Delta x_d = \Delta t_d = 0.025 \)) and the HOFD simulation with \( \Delta x_d = 0.01 \). This is because the contribution from the higher order truncation errors are more significant in these cases.
4.4. Results and Discussion

Figure 4.6: Concentration profiles from numerical simulations fitted with the analytical solution by minimizing the sum of squared error. (a) FOFV with relatively high resolution ($\Delta x_d = 0.005$) is virtually identical to (4.6) indicating higher order terms are negligible. Both (b) FOFV with relatively large grid ($\Delta x_d = 0.025$) and; (c) HOFD ($\Delta x_d = 0.01$) deviate from the error function as the contribution from high order terms is more significant.

Figure 4.7 shows the longitudinal and transverse dispersivities calculated from FOFV, HOFD and CVFE simulations as a function of grid block width and height, $\Delta x_d$ and $\Delta y_d$. For the CVFE simulator, $\Delta x_d$ and $\Delta y_d$ are the minimum element edge lengths in the horizontal and vertical direction respectively.

Both the FOFV method and the CVFE give longitudinal numerical dispersivities that vary linearly with mesh element size, regardless of the grid orientation and stencil (5 or 9). The gradient is 1 for the FOFV method and 0.525 for the CVFE method. These values are consistent with the analysis based on a Taylor series expansion given by Lantz (1971) that predicts that in a first-order, implicit scheme

$$D_{L,\text{num}} = \frac{\Delta x_d + \Delta t_d}{2} = \frac{\Delta x_d}{2} (1 + \text{CFL}) \quad (4.12)$$

This is true for all mesh orientations and discretisation schemes except for simulations using both the diagonal grid with a 9 point scheme, which seem to be slightly more dispersive than expected. The numerical longitudinal dispersivities are lower in the HOFD simulations, as would be expected for a second order
Figure 4.7: Longitudinal and transverse numerical dispersion values, $D_{L,num}$ and $D_{T,num}$, as a function of spatial resolution measured from various numerical schemes. (a) $D_{L,num}$ for FOFV are in agreement with the analytical solution, although the dispersion in the diagonal grid simulations with a 9 point stencil are slightly higher. The results for the CVFE simulations with different mesh configurations are almost identical at $D_{L,num} = 0.5 \Delta y_d$. (b) For FOFV, $D_{T,num}$ also varies linearly with element length and is around 1/6 to 1/3 of the corresponding $D_{L,num}$ values. The CVFE simulations produce a non-linear relationship between $D_{T,num}$ and $\Delta y_d$. 
4.4. Results and Discussion

A best fit to the data from the HOFD simulations gives a relationship $D_{L,\text{num}} = 1.16\Delta x_d^2$.

The transverse dispersivities, $D_{T,\text{num}}$, are systematically smaller than the longitudinal dispersivities in all cases. In the HOFD and the FOFV simulations with a parallel grid and a 5 point stencil $D_{T,\text{num}} = 0$ for all grid sizes as the vertical velocity component is zero everywhere. In the CFVE simulations and the other FOFV simulations the numerical transverse diffusivity varies linearly with mesh size and is between 1/6 and 1/3 of the corresponding $D_{L,\text{num}}$. As for the numerical longitudinal dispersivity, the numerical transverse dispersivity is highest in the FOFV scheme when a diagonal grid with a 5 point stencil is used. The CVFE simulations using the dynamic adaptive unstructured mesh gave transverse dispersivities that were only slightly higher than those obtained from the fixed mesh although these simulations were significantly less expensive to run because the dynamic adaptivity reduced the total number of cells used.

The CVFE solution uses an unstructured mesh and thus it is easier for flow to move at an angle to the principal flow direction as cell boundaries are not either parallel to or perpendicular to this principal flow direction. There is then truncation error as concentration is immediately spread across the element.

These results suggest that miscible viscous fingering in a line drive may be best modelled using a finite volume or finite difference method using a grid parallel to flow as this minimises the level of numerical transverse diffusivity. Tan and Homsy (1986) have shown that lower levels of physical transverse diffusivity (compared to longitudinal diffusivity) result in more fingers being formed initially and higher growth rates. Scientists and engineers simulating viscous fingering will typically aim to choose a mesh resolution such that the input physical diffusion dominates over the numerical diffusion. They should also choose the smallest possible CFL number if their simulator is fully implicit. Reducing the amount of numerical transverse...
Chapter 4. Truncation Error

diffusivity in a simulation by choosing a parallel grid may mean that a coarser mesh resolution may be needed with a consequent saving in computational effort. We note, however, that in simulations of non-linear viscous fingering the flow field will not always be parallel to the mesh and thus there will be non-zero transverse numerical diffusion even when using a parallel grid. It may thus be sensible and more conservative to assume that the level of transverse numerical diffusion is more similar to that seen when using a diagonal grid.

Using the results from Figure 4.7, we can calculate the number of grid blocks required in our simulation so that the physical diffusion dominates. For example, consider a typical longitudinal physical diffusivity of $D_L = 1 \times 10^{-8} \text{ m}^2/\text{s}$ (an order of magnitude approximation). We first need to convert this into a dimensionless value. From Table 3.1 in Chapter 3, we use

$$D_L[\text{dimensionless}] = \frac{D_L[L^2/T]}{x_0 v_0[L^2/T]}$$  \hspace{2cm} (4.13)

where $x_0$ is the characteristic length (in this study, we chose this to be the reservoir length) and $v_0$ is the characteristic velocity (injection velocity). Hence, for a reservoir length of 1 m and injection velocity of $1.2 \times 10^{-5} \text{ m/s} \sim 1 \text{ m/d}$, we then have a dimensionless $D_L = (1 \times 10^{-8} \text{ m}^2/\text{s})/(1 \text{ m} \times 1.2 \times 10^{-5} \text{ m/s}) = 8.33 \times 10^{-4}$.

For the FOFV simulation ($D_{L,num} = \Delta x_d$, from Figure 4.7), if we want to keep the numerical diffusion to be 10% or smaller than the physical diffusion ($D_L/D_{L,num} > 10$), then the minimum dimensionless grid length required is $\Delta x_d = D_{L,num} < 8.33 \times 10^{-5}$. This gives a minimum number grid blocks of $1/(8.64 \times 10^{-5}) \sim 12,000$.

For the HOFD simulation ($D_{L,num} = 1.16 \Delta x_d^2$), the minimum required grid blocks, in contrast, is only 118. Figure 4.8 shows the number of grid blocks required (in the $x$ direction) as a function of physical longitudinal diffusion $D_L$ for the FOFV and HOFD simulations in order to obtain $D_L/D_{L,num} = 1, 10$. Unsurprisingly, we can see here that a smaller physical diffusion requires a higher number of grid blocks.
Figure 4.8: Grid block requirement (in $x$ direction) as a function of physical longitudinal diffusion in order to obtain $D_L/D_{L,num} = 1, 10$. Here we assume that the reservoir length is 1 m and injection velocity is 1.2 m/s.

Figure 4.9 shows how the different levels of numerical dispersivity associated with different mesh orientations and stencils in the FOFV scheme affect the non-linear fingering pattern that is predicted. These simulations used a mesh resolution such that $\Delta x_d \approx \Delta y_d \approx 0.005$. The fluids had a mobility ratio of $M = 10$. Fingers were triggered using random permeability field at the inlet. The mixing lengths $L_{mix}$ here were taken as the distance $x_{c=0.1} - x_{c=0.9}$.

The simulations confirm the analyses of Tan and Homsy (1986) that transverse dispersivity controls the growth of viscous fingering as we see different wavelengths of fingering corresponding to the levels of transverse numerical dispersivity measured (Figure 4.7). The simulation using the parallel grid with the 5 point stencil predicted the largest number of fingers with the highest growth rate. This is consistent with this simulation having the lowest level of transverse numerical dispersion. The simulations with the diagonal grid (which has the highest transverse dispersivity) had the fewest fingers with the lowest growth rate as these have the highest levels of numerical transverse dispersion. It is interesting to note the growth of an extra, large finger adjacent to the top and bottom boundaries in these simulations. These
Figure 4.9: Miscible fingering for various schemes using a FOFV scheme with mobility ratio of $M = 10$ and $\Delta x_d \approx \Delta y_d \approx 0.005$ at $t_d = 0.5$. (a) Parallel grid with 5 point stencil; (b) Parallel grid with 9 point stencil; (c) Diagonal grid with 5 point stencil; (d) Diagonal grid with 9 point stencil; (e) Mixing lengths $L_{mix}$ ($x_c=0.1 - x_c=0.9$).
4.4. Results and Discussion

grow due to the no-flow boundaries (see Yang and Yortsos, 1998). The simulation using the parallel grid and the 9 point scheme predicted slightly more fingers than when a diagonal grid is used, consistent with the results shown in Figure 4.7.

4.4.2 Immiscible viscous fingering

Having shown how transverse numerical dispersion influences the miscible fingering pattern predicted by numerical simulations we now investigate its influence on the initial growth rate of immiscible viscous fingers.

Figure 4.10 shows the growth rate observed in FOFV, HOFD and CVFE simulations as a function of finger wavenumber initiated in the simulations. We compare the growth rates against the analytical solution for the case without capillarity (Eq. 4.2). In our simulations $M_{sf} = 1.8$, $v_0 = 1$ and $f_{F1,f}/(S_{F1,f} - S_{F1,c}) = 9.2$. As expected there is good agreement between the analytical growth rates and those predicted by all the simulations for low wavenumbers, however the early time growth rate seen in the simulations is lower than that predicted for wavenumbers greater than 1. The wavenumber at which the simulated growth rates diverge from the analytical solution is similar for all numerical methods, meshes and stencils. It is interesting to see that the initial growth rate is then approximately constant for wavenumbers greater than 4. Based on linear stability analysis we would expect the growth rate to reduce at higher wavenumbers if numerical dispersivity behaves in the same way as capillary pressure. This apparent stabilisation may be because we did not consider wavenumbers higher than 10 in our simulations as these could not be properly resolved by the meshes we used for this study or alternatively this may be because, unlike capillary pressure, the truncation error does not vary with saturation. The FOFV simulations, using a $200 \times 200$ mesh, predict the lowest growth rate at larger wavenumbers, as we would expect as these have the highest levels of transverse numerical dispersivity. Doubling the mesh resolution results in higher predicted
Chapter 4. Truncation Error

growth rates similar to those predicted by the HOFD simulations using a 200×200 mesh. This is consistent with the second order accuracy of this simulator.

![Graph showing growth rates](image)

Figure 4.10: The growth rates $\omega_{gr}$ in FOFV, HOFD and CVFE for various wavenumbers $k$. The dotted line shows the analytical solution calculated using Eq. 4.2 taken from Yortsos and Huang (1986).

These results confirm that very fine mesh resolutions are needed to avoid results being adversely affected by truncation error. Alternatively the use of dynamic adaptive meshing produce similar results at a fraction of the computational cost. These results further suggest that numerical diffusion cannot be used to approximate the effects of capillary pressure on viscous fingering. The initial growth rate predicted by all the numerical schemes tested deviated from the analytical solution at wavenumber of $\sim 1$, despite the different levels of truncation error quantified in the previous subsection. Even the growth rate predicted by the coarser grid,
4.5 Conclusions

We have compared the numerical error associated with three different numerical schemes (first order finite volume, second order finite difference and control volume finite element) and different meshes (structured Cartesian, structured triangular, unstructured triangular and dynamically adapting, unstructured triangular). This has been quantified as a function of mesh resolution in terms of the apparent longitudinal and transverse diffusivity seen in unit mobility ratio miscible displacements as well as in terms of the early time growth rate of immiscible viscous fingers.

Overall these results indicate that, for the numerical schemes and meshes tested, truncation error can, to first order, be characterised as a numerical diffusivity. Furthermore, these suggest that these schemes can be used to predict viscous fingering behaviour provided that the input physical diffusivity or capillary pressure is larger than the underlying numerical diffusivity. For miscible displacements a Taylor series analysis of truncation error can provide a good estimate of the longitudinal numerical diffusivity in finite difference and finite volume schemes. It seems that the transverse numerical diffusivity is typically much smaller that the longitudinal numerical diffusivity. It is also interesting to see that the level of numerical error associated with dynamic adaptive meshing is very similar to that seen in a fixed unstructured mesh with the same average element size but involving
significantly more computational effort.
Chapter 5

Fingering Regimes in Unstable Miscible Displacements

In this chapter, we study the different fingering regimes observed in unstable, miscible displacement in homogeneous rectilinear systems with no-flow boundaries perpendicular to the average direction of flow. The impacts of mobility ratio $M$, transverse diffusion $D_T$ and longitudinal diffusion $D_L$ on the flow regimes are investigated. We aim to understand how the non-linear growth of the fingers at intermediate and late times. We also investigate the average behaviour of the fingers in these time regimes, which is important for oil recovery or CO$_2$ sequestration applications.

5.1 Theory

We shall analyse finger growth, the number of fingers observed in a displacement as a function of time and their average behaviour in a first contact miscible displacement through a semi-infinite, two-dimensional rectilinear, porous medium (Figure 5.1). As in previous chapter, the system is initially filled with a fluid of viscosity $\mu_1$. A
lower viscosity fluid (viscosity $\mu_2$) that is first contact miscible with the ambient fluid is injected continuously, at constant rate, into the system along the left-hand boundary. There are no-flow boundaries top and bottom. The system is horizontal, so gravity can be neglected. It is assumed that both the fluids and the porous medium are incompressible.

\[ \mu = \mu_2, \ c = 1 \]
\[ \mu = \mu_1, \ c = 0 \]

Figure 5.1: 2D semi-infinite model with line-drive injection.

### 5.1.1 Dimensionless scaling

Previous workers have suggested a range of different ways of scaling the system. In particular we note that Tan and Homsy (1986) and Zimmerman and Homsy (1992) scale all lengths by the longitudinal dispersive length $\left( \frac{D_L}{|v|} \right)$ as their system has periodic boundaries and hence no natural physical length scale. In contrast, Nijjer et al. (2018) choose the thickness of the domain as the reference physical length scale. As the system is semi-infinite and bounded perpendicular to flow we follow Nijjer et al. (2018) in choosing the reference length scale to be the system width, $H$, so the dimensionless distances are

\[ x_D = \frac{x}{H} \]  \hspace{1cm} (5.1)  
\[ y_D = \frac{y}{H} \]  \hspace{1cm} (5.2)  

where $x$ is the physical distance measured parallel to the average direction of flow and $y$ is the physical distance measured transverse to the average direction of flow.
Choosing the dimensionless time to be

\[ t_D = \frac{t D_T}{H^2} \]  

(5.3)

allows us to rewrite Eq. 2.1 in Chapter 2 as

\[ \phi \frac{\partial c}{\partial t_D} + P e_T \varepsilon v_D \nabla c = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \nabla^2 c \] 

(5.4)

where \( \varepsilon = D_T/D_L \), \( P e_T = \frac{H v_T}{\phi D_T} \) is the transverse Peclet number and \( v_D = v/v_T \). \( v_T \) is the average Darcy velocity, determined from the injection rate and the cross-sectional area of the system perpendicular to the average flow direction. This choice of time scale is motivated by the analysis of Peters et al. (1984) and Tan and Homsy (1988) that showed that transverse dispersion is the main control on initial finger number and growth.

### 5.1.2 Linear perturbation theory

As discussed previously in Chapter 2, a number of authors have used perturbation analysis to analyse the early time behaviour of viscous fingers. They all show that when there is diffusion, there is a dimensionless finger wavenumber \( k_m \) with maximum dimensionless growth rate \( \omega_{gr,m} \) and a cutoff wavenumber \( k_c \) above which the perturbation will not grow. This cutoff wavenumber depends upon dispersion: if there were no dispersion then growth rate would grow linearly with wavenumber and there would be no cutoff wavenumber (e.g. see analysis of Christie and Bond, 1987). If fingers grow from an initial concentration profile that is itself changing in time (due to longitudinal diffusion) then a similar behaviour is observed but \( k_c \), \( k_m \) and \( \omega_{gr,m} \) reduce with time (Figure 5.2). This is because the concentration gradient in the direction of flow is reducing due to longitudinal diffusion and is thus stabilising
the displacement. If the viscosity varies exponentially with concentration

\[ \mu_m = \mu_1 e^{-clnM} \]  

(5.5)

then Tan and Homsy (1986) showed that the initial growth rate (at \( t_D=0 \)) as a function of wavenumber, mobility ratio and dispersion is given by

\[ \omega_{gr} = \frac{1}{2} Pe_T^2 \varepsilon k \left[ \frac{R}{Pe_T} - k - \sqrt{k^2 + 2 R \frac{Pe_T}{Pe_T} k + 2(1 - \varepsilon)k} \right] \]  

(5.6)

where \( R = \ln(M) \) and we have converted their equation into our system of dimensionless units. Solving this equation for the cutoff wavenumber when \( \omega_{gr} = 0 \) gives

\[ k_c = \frac{R}{2 Pe_T} \left( \frac{1}{\varepsilon + \sqrt{\varepsilon}} \right) \]  

(5.7)

The most dangerous wavenumber \( k_m \), occurs when \( d\omega_{gr}/dk = 0 \). Eq. 5.6 can only be solved exactly when diffusion is isotropic in which case

\[ k_m = \frac{(2\sqrt{5} - 4) R Pe_T}{4} \simeq 0.118 R Pe_T \]  

(5.8)

and

\[ \omega_{gr,m} = \frac{(-2 + \sqrt{5})(7 - 3\sqrt{5}) R^2 Pe_T^2}{3 - \sqrt{5}} \frac{4}{4} \simeq 0.0225 R^2 Pe_T^2 \]  

(5.9)

Tan and Homsy (1986) provide two approximate solutions for when \( \varepsilon >> 1 \) (transverse diffusion is much greater than longitudinal)

\[ k_m \simeq \frac{R Pe_T}{4} \left( 1 - \frac{3}{2\sqrt{2}\varepsilon} \right) \]  

(5.10)

\[ \omega_{gr,m} \simeq \frac{R^2 Pe_T^2}{8\varepsilon} \left( \frac{1}{2} - \frac{1}{\sqrt{2}\varepsilon} \right) \]  

(5.11)
and when $\varepsilon << 1$ (the more usual case when transverse diffusion is much less than longitudinal)

\[
k_m \simeq \frac{RP_e T \varepsilon}{2} \left( \varepsilon^{\frac{1}{3}} - \frac{5}{3} \right) \tag{5.12}
\]

\[
\omega_{gr,m} \simeq \frac{R^2 P_e T^2}{4} \left( 1 - 3\varepsilon^{\frac{1}{3}} \right) \tag{5.13}
\]

We see that in all cases the wavenumber scales proportionally to $RP_e T$ and the growth rate of the fingers scales proportionally to $R^2 P_e T^2$. Note that in our system of dimensionless parameters the dimensionless wavenumber is $2\pi n_f$ where $n_f$ is the number of fingers.

Manickam and Homsy (1993) have shown that the preceding analysis can be extended to any function relating mixture viscosity to concentration by replacing $R$ with $\Lambda$, where $\Lambda$ is defined as

\[
\Lambda = -\frac{d\mu}{dc}_{c=0} + \frac{d\mu}{dc}_{c=1} \frac{\mu_1}{\mu_1 + \mu_2} \tag{5.14}
\]
Overall we see from this analysis that the initial fingering pattern (number of fingers and growth rate) is controlled by the transverse dispersion as originally suggested by Slobod and Thomas, 1963.

### 5.1.3 Number of fingers at early time

Let us now investigate further quantitatively the formation of fingers at early time.

Assuming that $k_m$ is representative of the number of fingers $n_f$, we have

$$n_f = \frac{k_m}{2\pi} \quad (5.15)$$

For simplicity we only consider $k_m$ at $t = 0$ as given in Eq. 5.6. First we divide both sides of Eq. 5.6 by $R^2$ to give

$$\frac{\omega_{gr}}{R^2} = 1 + \frac{k}{R} \left[ \frac{1}{Pe_T} - \left( \frac{k}{R} \right) - \sqrt{\left( \frac{k}{R} \right)^2 + 2 \frac{1}{Pe_T} \left( \frac{k}{R} \right)} + 2(1 - \varepsilon) \left( \frac{k}{R} \right) \right] \quad (5.16)$$

Eq. 5.16 indicates that $k$ scales with $R$, i.e. $\frac{k}{R}$, and this ratio is a function of $Pe_T$ and $\varepsilon$ only. To find the scaled wavenumber with maximum growth rate $\frac{k_m}{R}$, we solve $\frac{d(\omega_{gr}/R^2)}{d(k/R)} = 0$ numerically. The results are shown in Figure 5.3 where we plot $\frac{k_m}{2\pi R} = \frac{n_f}{R}$ as a function of $Pe_T$ for $\varepsilon = 1 \times 10^{-5}, 0.2, 1, 5, 1 \times 10^5$. Here we also plot the analytical solutions for $\varepsilon = 1$ (Eq. 5.8) and $\varepsilon >> 1$ (Eq. 5.10), both of which match the numerical solutions.

For a specific $\varepsilon$, there is a linear relationship between $\frac{n_f}{R}$ and $Pe_T$. We see clearly the influence of $\varepsilon$: increasing $\varepsilon$ gives a steeper gradient, which translates into a higher number of fingers. However, there is an upper limit of this gradient as given by Eq. 5.8 which is $\frac{n_f}{R} = \frac{1}{4 \times 2\pi}$. In contrast, a very small $\varepsilon$ gives virtually no fingers, which is expected as per Eq. 5.12.
5.2 Fingering regimes

There are four fingering regimes that can be observed as miscible viscous fingers develop over time, i.e.

1. Very early time, during which no fingers are observed because the growth rate of the fingers is less than the spreading of the interface due to longitudinal diffusion and dispersion. This seems to have been first described by Perkins and Johnston (1963) although it was subsequently also described by Bacri et al. (1992) and Loggia et al. (1999).

2. Early time, when the growth rate of the fingers outstrips the rate of longitudinal mixing. The number and growth rate of the fingers can be calculated using linear perturbation theory such as that of Tan and Homsy (1986).
3. Intermediate time when the fingers begin to interact non-linearly. Depending on the viscosity ratio, the anisotropy in diffusion and the Peclet number the fingers may branch, merge or fade, although ultimately it seems that merging and fading dominate so the total number of fingers reduces.

4. Late time, when a single finger is left and propagates through the system. This was first described by Saffman and Taylor (1958) and analysed by Outmans (1962) who showed that the finger width occupies approximately half of the overall system width. The transition and duration of this regime has been analysed more recently by Zimmerman and Homsy (1992) and Nijjer et al. (2018).

The intermediate time regime and the transition to the late time regime are of particular interest as these are the regimes most often encountered in physical systems of engineering interest such as CO$_2$ sequestration and enhanced oil recovery. Although the number of fingers may increase at specific times through splitting, it seems that on average, in miscible displacements in porous media, the number of fingers decreases with time. This splitting tends to occur at the finger tips for lower viscosity ratios (e.g. $M = 47.5$, Suekane et al., 2017) and along the finger for higher viscosity ratios (e.g. $M = 383$, Blackwell et al., 1959). It seems that the Peclet number in laboratory scale experiments is such that diffusion and dispersion tend to limit the ability of fingers to split. Perkins and Johnston (1963) described this reduction in the number of fingers as being due to the mechanisms of merging and fading, ascribing the fading of trailing fingers as being due to the growth of leading fingers which grow slightly more quickly and thus take more of the lower viscosity solvent. In contrast Zimmerman and Homsy (1991) explained the fading of fingers as being due to spreading of the leading finger and subsequent fading of the trailing finger.

At late times, when only one single finger remains, there remains some uncertainty
as to the ultimate fate of the finger. Tan and Homsy (1988) observed, in numerical simulations, that the single finger would split then one of the two fingers would fade forming a single finger and that this process would repeat. Zimmerman and Homsy (1992) found that this was less likely to occur with anisotropic dispersion (when the transverse dispersion was weak). They speculated that the behaviour of this finger at late times would be independent of Peclet number and the ratio of transverse to longitudinal dispersion for large Peclet numbers. Interestingly Nijjer et al. (2018) do not see any tip splitting in the late time, single finger regime despite using isotropic diffusion. Furthermore, their analysis suggests that the once the single finger forms it decays exponentially due to transverse diffusion.

### 5.3 Numerical simulation

In this chapter we have used numerical simulation to explore the growth and decay of fingers in a semi-infinite system with no flow boundaries perpendicular to flow. This was achieved using the FCM simulator discussed in Chapter 3. Note that this simulator uses quarter power mixing rule.

### 5.4 Simulation input parameters

A range of system aspect ratios were investigated from square to $L/H=10$, together with a range of viscosity ratios, transverse Peclet numbers and diffusion anisotropies (Table 5.1).

The investigations used anisotropic diffusion rather than velocity dependent dispersion to ensure that $\epsilon$ was constant across the model and thus simplify analysis of the results. Further work is needed to compare these results with those that would be obtained if only longitudinal dispersion is velocity dependent (as assumed by
Zimmerman and Homsy, 1991) and if both transverse and longitudinal dispersion are velocity dependent, as is usually assumed to be the case on the field scale. Velocity dependent dispersion would lead to higher dispersion within a finger compared to the bypassed oil between fingers.

The simulation domain was discretised into rectangular grid blocks and the number of grid blocks chosen using a grid refinement study to ensure that physical diffusion dominated over numerical diffusion. The FCM simulator calculates each time-step using a CFL criterion to minimise numerical diffusion (Lantz, 1971). For a second order scheme $\text{CFL} = \frac{v\Delta t}{\Delta x}$ (where $\Delta t$ is the time-step, $\Delta x$ is the grid block size and $v$ is the maximum velocity in any grid cell) should be 0.5 but in this study we chose CFL=0.4 to allow for a small margin for error in the simulator estimates of velocity.

Miscible solvent was injected at a constant rate across the inlet face. This was matched by the rate of production from the outlet face. The fingers were explicitly triggered by using uniformly distributed random concentration ($0 < c < 1$) at the inlet when $t = 0$.

We performed two sets of simulations, one to investigate the very early to intermediate time regime and the second to investigate the intermediate to late time regime. The first set used a square domain and a $600 \times 200$ grid so $\Delta x_D = \frac{1}{600} = 0.00167$ and $\Delta y_D = \frac{1}{200} = 0.005$. The higher resolution was used in $x$-direction to ensure the simulation captured the changes in $L_{mix}$ and $n_f$ with time more accurately. The second set of simulations required a longer domain hence we used $L/H = 10$ and a grid of $2000 \times 200$, giving $\Delta x_D = \Delta y_D = \frac{1}{200} = 0.005$. The simulation input parameters used are summarised in Table 5.1.
5.4. Simulation input parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Very early to intermediate time</th>
<th>Intermediate to late time</th>
</tr>
</thead>
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<tr>
<td>Domain dimension</td>
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<td>(L = 10, H = 1)</td>
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<tr>
<td>No. of grid blocks, (DX \times DY)</td>
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<td>2000×200</td>
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<td>Viscosity ratio, (M)</td>
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<td>20 ((R = 3))</td>
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<tr>
<td>Transverse Peclet number, (Pe_T)</td>
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<td>125, 250, 500</td>
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<tr>
<td>Ratio of transverse-to-longitudinal dispersion, (\varepsilon)</td>
<td>0.2, 1, 5</td>
<td>0.25, 0.5, 1, 2</td>
</tr>
</tbody>
</table>

Table 5.1: Summary of data used in the investigation of flow fingering regimes

5.4.1 Analysis of results

The simulation outputs were analysed in terms of average concentration profiles (averaged transverse to the principle flow direction), growth of fingers and the number of fingers versus dimensionless advective time, \(t_{D,A}\) given by

\[
t_{D,A} = \frac{t | v_T |}{H \phi}
\]

as well as dimensionless time based on transverse diffusion (Eq. 5.3).

To quantify the growth of the fingers we express their spreading in the direction of flow in terms of the dimensionless mixing length \(L_{mix}\)

\[
L_{mix} = x_{D,c=0.02} - x_{D,c=0.98}
\]

where \(x_{D,c=0.02}\) and \(x_{D,c=0.98}\) are the dimensionless distances in the direction of flow where the average concentration equals \(c = 0.02\) and \(c = 0.98\) respectively.

We used contour lines to keep track of the evolution of the fingers as a function of time. This was necessary because of the finite difference simulation approach adopted. In spectral methods, this can be done by directly taking the wavenumber...
with highest power, e.g. Zimmerman and Homsy (1991).

At every $5^{th}$ timestep, we computed the contour of the concentration at a specified concentration level $c_{\text{level}}$ using the marching squares method (also known as marching cubes method for 3D contouring, see Lorensen and Cline, 1987). Due to coalescence of the fingers and blob formation due to pinchoff of the fingers, there could be multiple unconnected contour lines for a given $c_{\text{level}}$ as illustrated in Figure 5.4(a). In this instance, only the longest contour line would be selected (and the rest of the lines ignored). We then found the local minima and maxima along this selected contour line which correspond to the peaks and troughs. We excluded any peak if it was located less than two grid blocks ($\Delta x = 2/DX$) further towards the producer than its adjacent trough, as shown in Figure 5.4(b). This was particularly important at very early times to exclude the initial perturbation introduced at $t = 0$. $c_{\text{level}} = 0.5$ was used in this study. This value was found to capture the number of fingers correctly, as shown in the results section.

Figure 5.4: Illustration of computing the number of fingers $n_f$ using the contour lines. (a) Only the longest contour line is selected for computation of $n_f$; (b) $n_f$ is given by the number of peaks. We exclude any peak that is located less than two grid blocks ($\Delta x = 2/DX$) further towards the producer than its adjacent trough.
5.5 Results

We now present the simulation results from very early to late time regimes. For the very early and early times, comparison against the analytical solutions from linear stability theory will be made. However, it should be noted that there are two main assumptions made in this study that will lead to differences when comparing the simulation results against the linear stability theory.

1. Finding the analytical solution of the linear stability analysis for $t_D > 0$ is complex and requires a numerical solver as discussed by Tan and Homsy (1986). For simplicity, we only use the solution at $t_D = 0$. As illustrated in Figure 5.2, the main ramification of this assumption is that the analytically predicted growth rate and number of fingers at $t_D = 0$ will be higher than those observed in the simulations.

2. The simulator uses the quarter power rule for the mixture viscosity whereas the linear stability results in Figure 5.3 is based on the exponential mixing rule. We will discuss the ramification of this assumption further in Section 5.5.2.

5.5.1 Very early time regime

At very early time the instability induced by a sharp front is diffused by longitudinal dispersion $D_L$. This initially spreads more quickly than the fingers grow, due to the very high concentration gradient across the front. Figure 5.5(a) shows the crossover between diffusive flow at very early time to the viscous-dominated unstable flow for $R = 3, 4$ and $Pe_T = 250, 500$ when $\varepsilon = 1$. For a given $Pe_T$, $L_{mix}$ at very early time for different $R$ is the same but the crossover from diffusion to fingering is earlier for higher $R$ as expected. Kempers (1990) argued heuristically that this crossover
can be roughly estimated by comparing the mixing length induced by longitudinal dispersion against the linear spreading based on the Koval model

\[ 3.625 \sqrt{D_L t_{D,A}} \sim (M_e - \frac{1}{M_e}) t_{D,A} \] (5.19)

The left hand side of Eq. 5.19 is the length of diffusive spreading from \( c = 0.1 \) to \( c = 0.9 \) (Perkins and Johnston, 1963) and the right hand side is the linear spreading from \( c = 0 \) to \( c = 1 \) predicted using the Koval model where \( M_e \) is the effective viscosity ratio defined as

\[ M_e = (0.78 + 0.22M^4) \] (5.20)

Similarly, Bacri et al. (1992) proposed that this crossover can be estimated using Eq. 5.19, but instead of using the linear mixing length estimated by Koval model, they suggested that the growth of the initial perturbation should be estimated from linear stability theory using

\[ L_{\text{mix}} = l_D e^{\omega_m t_{D,A}} \] (5.21)

where \( l_D \) is the initial perturbation length in the \( x \) direction.

We plot the growth of the initial perturbation using Eq. (5.21) in Figure 5.5(b) together with the diffusive length \( (3.625 \sqrt{D_L t_{D,A}}) \) for \( R = 3, Pe_T = 500 \) and \( \varepsilon = 1 \). Here, \( l_D \) was taken as the dimensionless length of 1 grid block in the \( x \) direction \((l_D = \Delta x = 1/DX)\) as this is the length of the initial random perturbation made in the simulation. The intersection of these two lengths in Figure 5.5(b) indicates the crossover from the very early time to early time regime.

It can be seen that this approach predicts a much higher growth rate than is observed in the simulation: the simulation transitions to early time fingering at around 0.12 PVI (compared with \(~0.04\) PVI suggested by the Bacri et al. (1991) approach).
5.5. Results

This is probably because the growth rate of the fingers is slower than predicted by the perturbation analysis (which is calculated at $t_{D,A} = 0$), as the dispersed front is less unstable (Figure 5.2).

Also shown in Figure 5.5(b) is the linear mixing length computed using Koval model $((M_e - 1/M_e)t_{D,A})$. This predicts a much earlier transition to fingering than is seen in the simulations. This poor prediction is not unexpected as apart from viscosity ratio, the crossover time is a function of $Pe_T$ and $\varepsilon$ – neither of which are captured by the effective viscosity in Eq. (6.2).

![Figure 5.5: The crossover between the very early time to early time regime. (a) Mixing lengths for $R = 3, 4$ and $Pe_T = 250, 500$ calculated from detailed simulations; (b) The crossover times estimated by comparing the longitudinal spreading due to $D_L$ at very early time ($= 3.635\sqrt{D_L t_{D,A}}$) with: (i) the growth of disturbance using Eq. 5.21 (here we used the initial longitudinal disturbance length $l_D$ of $1/DX = 1/600$), (ii) Koval model ($=(M_e - 1/M_e)t_{D,A}$).](image)

5.5.2 Early time regime

We next investigated the fingering behaviour predicted by the simulator at early time for a range of viscosity ratios, Peclet numbers, and $\varepsilon$. Figure 5.6 shows the fingers and their development at $0 < t_{D,A} < 0.5$ for $R = 3, 4, 5$ and $Pe_T = 250, 500$.
with isotropic dispersion. As expected more fingers start growing sooner and grow more quickly for larger $Pe_T$ or $M$. The fingering dynamics become non-linear sooner with more splitting/branching occurring at higher viscosity ratios and $Pe_T$ (we will discuss this further in the next section). Figure 5.7 shows the effects of increasing and reducing longitudinal dispersion ($\varepsilon = 0.2, 5$) when $Pe_T = 250$. As expected from Figure 5.3, more fingers form and grow more rapidly for $\varepsilon = 5$ compared to the isotopic case, whilst fewer fingers form that grow more slowly for $\varepsilon = 0.2$.

Figure 5.6: The early to intermediate time growth of the fingers ($t_{D,A} = 0.5$) with isotropic dispersion for $Pe_T = 250, 500$ and $R = 3, 4, 5, (M = 20, 55, 148)$. The number of fingers and the interaction between the fingers increase as we increase $R$ and $Pe_T$. We observe various events including spreading (SP), shielding (SH), coalescence (CO), double coalescence (DC), tip splitting (TS), and side branching (SB). We discuss the non-linear events further in Section 5.5.3.

Figure 5.8 shows that the number of fingers obtained at early time from the simulation follow the same linear trend as the maximum number of fingers predicted by the analytical solution as shown in Figure 5.3 (which assumed that mixture
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Figure 5.7: The effects of anisotropic dispersion ($\varepsilon = 0.2, 5$) on the early time growth of the fingers ($t_{D,A} = 0.5$) for $Pe_T=250$ and $R = 3, 4, 5$. The corresponding cases with isotropic dispersion are shown in Figure 5.6.
viscosity varies exponentially with concentration). We see that the number of fingers increases with the ratio of transverse to longitudinal dispersion for a given \( Pe_T \) as seen in Figure 5.7. This is because there is less smoothing of the initial step concentration profile by longitudinal dispersion at early time. There are fewer fingers for lower \( \epsilon \) because there is more smearing of the front by longitudinal dispersion at early time and hence the front is less unstable.

It can be seen, however, that the gradients of \( n_f \) vs. \( Pe_T \) in Figure 5.8 are only half of those predicted in Figure 5.3 i.e. our simulations only give half the number of fingers predicted by linear stability theory.

![Graph showing maximum number of fingers vs. PeT](image)

**Figure 5.8:** Maximum number of fingers obtained from the simulation (see Figure 5.6) for \( R = 3, 4, 5 \) compared to the analytical solutions for the maximum number of fingers predicted by linear stability analysis (Tan and Homsy, 1986; Eq. 5.6) assuming an exponential mixing rule for viscosity. The maximum number of fingers obtained in the simulations is approximately half of the predictions.

Figure 5.9 shows that there is a significant increase in the maximum number of fingers predicted by linear stability analysis if the quarter power or exponential mixing rules are used. The simulator actually uses a quarter power mixing rule for viscosity so comparison of Figures 5.8 and 5.9 suggests it is predicting significantly
fewer fingers than the maximum number obtained from linear stability analysis. This could be due to the numerical method using a linear interpolation of concentration between grid blocks or due to Eq. 5.6 having been derived by assuming that the maximum number of fingers is controlled by the initial state of the system. The instability is reduced because the initial step concentration profile is smoothed by longitudinal dispersion before the fingers grow. This reduces the viscosity contrast across the front and thus results in fewer fingers growing more slowly (Figure 5.2). Suekane et al. (2017) also found the maximum number of fingers measured in their laboratory experiments were approximately two orders of magnitude fewer than the number predict by linear stability analysis. Zimmerman and Homsy (1991) too found that the number of fingers formed initially in their simulations was often half that predicted by linear stability theory. They explain this by a pairwise interaction mechanism. We cannot find any other comparisons of numbers of fingers predicted by linear stability analysis with those obtained either by numerical simulation or in the laboratory. Tchelepi et al. (1993) compared the growth rates observed experimentally with those predicted by linear stability analysis and found the experimental growth rates were larger than those predicted analytically. They suggested that this was because the finger dynamics in their experiments were non-linear even at the early times observed.

5.5.3 Intermediate time regime

In the intermediate time regime, the fingers no longer grow independent of each other. Many laboratory experiments (Saffman and Taylor, 1958; Wooding, 1969; Park and Homsy, 1985; Malhotra et al., 2015) and numerical simulations (Bensimon, 1986; Tan and Homsy, 1988; Zimmerman and Homsy, 1992; Islam and Azaiez, 2005) have shown several well-identified mechanisms during this non-linear growth of viscous fingering. These include shielding, spreading, coalescence and tip splitting.
We now discuss some of these mechanisms by drawing examples from our earlier observations in Figure 5.6 where $R = 3, 4, 5$ and $Pe_T = 250, 500$.

1. Shielding (abbreviated as SH in Figure 5.6) occurs when a finger that is slightly ahead of its neighbours becomes more dominant and consequently halts the growth of its neighbouring fingers.

2. Spreading (SP) is the lateral growth of the fingers which reduces the average wavelength of the fingers. This can either be purely due to transverse dispersion or as a result of finger coalescence.

3. Coalescence (CO) occurs when the fingers merge to become a single larger finger. This seems to result when a finger makes a sharp turn to coalesce with its neighbour. There are also instances of double coalescence (DC) as discussed by Islam and Azaiez (2005) where a pair of fingers adjacent to a dominant, shielding finger bend and merge into its base.

4. Tip splitting (TS) refers to the instability at the finger tip when a finger tip
5.5. Results

splits into two. Tan and Homsy (1988) argue that for this to happen, the finger tip needs to spread so that it is wider than two fingers of the most dangerous wavelength for the $Pe_T$ and $\varepsilon$ of the system and the longitudinal concentration gradient at the tip.

5. Side branching (SB) is also observed particularly at high mobility ratio.

In general, we observe that the frequency of splitting events increases when we have a higher mobility ratio and $Pe_T$. For a given $Pe_T$, increasing longitudinal dispersion stabilises the flow with less fingers formed as shown in Figure 5.7 for $\varepsilon = 0.2$. In the same figure, we also show the case when $\varepsilon = 5$. In particular, there are more side branching events as the flow is influenced more heavily by $D_T$. Although this case is unrealistic as the longitudinal dispersion $D_L$ is typically higher than $D_T$ especially for flow in porous media, it provides further evidence that smaller $D_L$ will increase the flow instability consistent with the early time linear stability analysis.

Figure 5.10 shows the number of fingers $n_f$ as a function of $t_{D,A}$ for $Pe_T = 500$, $\varepsilon = 1$, and $R = 3, 4, 5$ (the corresponding concentration maps are shown in Figure 5.6). The onset of fingering occurs earlier for $R = 5$ as we have seen before. After a maximum number of fingers is attained, the non-linear mechanisms such as shielding and coalescence reduce the number of fingers relatively quickly. The laboratory experiments by Wooding (1969) showed that the number of fingers reduce diffusively as a function of time i.e, $n_f \sim t^{-0.5}$, when $D_T$ is the main mechanism responsible for this behaviour. We observed here similar behaviour as shown in Figure 5.7(b) especially for $R = 3$. However, for higher mobility ratios, we observed the following

1. Initially, the number of fingers very approximately as $n_f \sim t^{-1}$ (Figure 5.7b).

Nijjer et al. (2018) also observed this in their simulations and argued that there is a small period when advection dominates shortly after the early time regime. They proposed that $n_f \sim (Rt^{-1})$. 
2. After this period, \( n_f \) starts to increase as a result of branching and tip splitting, before there is further merging and coalescence.

No tip splitting was observed in \( R = 3 \) and \( Pe_T = 500 \) for \( t_{D,A} \) up to 0.5. This is similar to the observations by Zimmerman and Homsy (1991) where they had to increase \( Pe_T > 500 \) before the tip splitting occurred in their spectral method simulations.

We now attempt to scale \( n_f \) with our system of dimensionless (transverse diffusive) time \( t_D \) for \( Pe_T = 500, \epsilon = 1, 0.2, 5, \) and \( R = 3, 4, 5 \) in two ways

1. \( n_f \sim Pe_T R, \quad t_D \sim 1/Pe_T^2 R^2 \)

2. \( n_f \sim k_m/2\pi, \quad t_D \sim 1/\omega_{gr,m} \)

The first scaling variables are based on Eqs. 5.8 and 5.9 (i.e. for isotropic dispersion). Nijjer et al. (2018) found the same scaling, but based on heuristic arguments. The second scaling variables are based directly on the linear stability results i.e. by solving Eq. 5.6 to find \( k_m \) and \( \omega_{gr,m} \) which has to be done numerically for \( \epsilon \neq 1 \).

The results are shown in Figure 5.11. The first method of scaling (Figure 5.11b) does collapse the curves, but only into groups according to their \( \epsilon \) values. This is because the scaling variables (Eqs. 5.8 and 5.9) did not take into account the anisotropy of dispersion. Interestingly the second method of scaling (Figure 5.11c) gives a good result suggesting that the non-linear dynamics are related to the initially linear behaviour of the fingers at early time.

### 5.5.4 Late time regime

We now investigate the dynamics and scaling of the intermediate to late time fingering regimes. Figure 5.12 illustrates a typical set of results for the case \( R = 3, \)
Figure 5.10: (a) The number of fingers from early to intermediate time for $Pe_T = 500$ and $\varepsilon = 1$. The corresponding concentration maps are shown in Figure 5.6; (b) Same as (a) using logarithmic scales.
Figure 5.11: Number of fingers from very early to intermediate time regimes. (a) $n_f$ as a function of $t_{D,A}$ for $P_e T = 500$, $R = 3, 4, 5$ and $\varepsilon = 0.2, 1, 5$; (b) Scaling of $n_f$ using $k_m$ and $\omega_{gr,m}$ assuming isotropic diffusion (Eqs. 5.8 and 5.9); (c) Scaling of $n_f$ using the early time analysis (Eq. 5.6).
$Pe_T = 500, \varepsilon = 1$. The aspect ratio of the simulation model is 10. Figure 5.12(a) shows how the 50% concentration contour evolves over time. The finger tips are marked with different colours to indicate the dimensionless advective time $t_{D,A}$.

![Figure 5.12: Results for $R = 3, Pe_T = 500, \varepsilon = 1$. (a) The tracking of the finger tips using the method presented in Section 5.4.1. The fingers tips are marked with different colour to represent time; (b) Concentration map at $t_{D,A} = 5$; (c) Number of fingers as a function of time; (d) Mixing length as a function of time.](image)

The corresponding concentration distribution shown in Figure 5.12(b) shows the concentration map at $t_{D,A} = 5$ for visual reference. Figure 5.12(a) shows that the displacement starts with 12 fingers but very quickly (by $t_{D,A} = 1.8$) 9 of those fingers have faded due to shielding and coalescence. Two of the remaining fingers experience tip splitting at approximately $t_{D,A} = 2$ and $t_{D,A} = 2.5$ but there are only 2 fingers by $t_{D,A} = 3$. We also observed an effect of the no flow boundary in the lower part of the domain in Figure 5.12(a) where one of the dominant fingers tends to flow along the boundary, before making a turn towards the centre.

Figures 5.12(c) and (d) show the number of fingers and mixing length as a function of $t_{D,A}$, respectively. Note that the time scale in Figure 5.12(d) is logarithmic. In
Figure 5.12(c) there is a short interval at the beginning of the displacement when no fingers are observed because the initial step concentration profile initially spreads due to longitudinal dispersion, as discussed in the very early time section. This can also be seen in Figure 5.12(d) where the mixing length grows proportionally to \( t_{D,A}^{0.5} \) until \( t_{D,A} \sim 0.1 \). After this time the number of fingers decays monotonically (except for a brief period when there are tip splitting event between \( t_{D,A} = 2 \) and \( t_{D,A} = 3 \)) to 2 fingers at \( t_{D,A} = 3.3 \). The mixing length grows linearly with \( t_{D,A} \) over this period.

We highlight how longitudinal and transverse diffusion influence the flow from very early to late time regimes in Figure 5.13. Plotting \( n_f \) and \( L_{mix} \) of the simulations with the same longitudinal dispersion \( (D_L=0.004) \) but different \( Pe_T = 125, 250, 500 \) in Figure 5.13(b) gives identical \( L_{mix} \) at very early time, as expected. However, at intermediate to late time regimes, \( n_f \) and \( L_{mix} \) are dominated by \( Pe_T \) as shown in Figure 5.13(c) where we plot the cases with constant transverse Peclet number \( (Pe_T = 500) \) and different \( \varepsilon = 0.25, 0.5, 1 \). In this instance, both \( n_f \) and \( L_{mix} \) curves are, to a certain degree, superposed.

### 5.5.4.1 Tip-splitting

Tip-splitting is important as it may alter the time at which the flow transitions to the late time regime (i.e. when the fingers are reduced to one). Interestingly, unlike Nijjer et al. (2018), we observed tip-splitting of the single remaining finger at intermediate to late times as previously shown in Figures 5.12. This was also observed by Tan and Homsy (1988).

It is unclear why Nijjer et al. (2018) did not observe the tip-splitting, but one of the possible reasons is due to their method of capturing the fingers where they averaged the number of fingers (as a function of time) obtained from two different simulation runs to reduce the noise.
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Figure 5.13: The plots of $n_f$ and $L_{mix}$ (inset plots) for (a) Constant $\epsilon$ (=1); (b) Constant longitudinal dispersion $D_L$ (=0.004); (c) Constant $Pe_T$ (=500).
Figure 5.14 shows tip-splitting was observed in cases with $Pe_T = 250$ but different longitudinal diffusivities ($\varepsilon = 2, 1, 0.5$). In this regime the remaining dominant finger splits at its tip (Figures 5.14a and 5.14b) then one of the fingers fades whilst the other grows. Tan and Homsy noticed that the finger can only split when it is wide enough to sustain two fingers according to linear stability analysis. As discussed in the theory section, the most dangerous number of fingers in a given system depends upon the viscosity ratio and the transverse Peclet number. A finger-tip can only split when it is wide enough to sustain two fingers based on the mobility ratio between the finger and the displaced fluid, and $Pe_T$. However, the finger is formed of a mixture of displacing and displaced fluid and so there is a lower viscosity ratio between the finger and the displacing fluid (see Figure 5.14) than is the case initially. This means that the maximum number of fingers predicted by linear stability theory will be lower and they will have a greater wavelength than at the beginning of the displacement. The maximum finger concentration is controlled by the interplay of transverse dispersion, longitudinal dispersion and viscosity ratio. A smaller $D_L$ results in a steeper concentration gradient in the direction of flow at the front of the finger but $D_T$ will tend to spread the finger transverse to flow. There is thus an optimum combination of transverse dispersion and mobility ratio for tip-splitting. In particular, if $D_L$ is too large then there will be no tip splitting (see Figure 5.14c). This also means that tip-splitting is less likely to occur if $\varepsilon < 1$ as was also found by Zimmerman and Homsy (1991).

Tan and Homsy (1988) speculated that repeated tip splitting could prevent the formation of a single dominant finger in some cases. Our simulations suggest that in fact tip-splitting will delay the time at which a single finger forms but not prevent this. Figure 5.15(b) shows the change in the concentration along the path of the dominant finger over time. The concentration in the finger tip is gradually reducing, so will eventually decay to the point that the viscosity ratio between the tip and the displacing fluid is too low for tip-splitting to occur. Figure 5.15(c) shows that the
5.5. Results

Figure 5.14: The average concentration profiles for $0 < t_{D,A} < 5$ and the concentration maps at $t_{D,A} = 5$. In all cases $Pe_T = 250$. (a) $\epsilon = 2$; (b) $\epsilon = 1$; (c) $\epsilon = 0.5$. Increasing $D_L$ reduces the tendency of the flow to exhibit tip splitting at intermediate to late time regime.
finger tip concentration scales approximately proportionally to $t_{D,A}^{-0.5}$

![Diagram](image)

**Figure 5.15:** Change in the concentration along the path of the dominant finger over time for $R = 3$, $Pe_T = 250$ and $\varepsilon = 1$. (a) Illustration of the maximum concentration $c_{\text{max},x}$ (shaded black) along the dominant finger; (b) $c_{\text{max},x}$ (red lines) across $x$ axis over time ($0 < t_{D,A} < 5$). We track the front of the finger tip (marked as blue circles) by finding the concentration corresponding to a constant $dc_{\text{max},x}/dx$ value (chosen to be -10); (c) The concentration in the finger tip from (b) is gradually reducing suggesting there will be no tip splitting at late time.

### 5.5.4.2 Late time scaling

Figure 5.16 shows the scaling of the number of fingers $n_f$ against dimensionless diffusive time $t_D$ as previously carried out in Figure 5.11 using $k_m/2\pi$ and $\omega_{gr,m}$. These variables still give a satisfactory scaling for $n_f$ even in the late time regime. It appears that $n_f/(k_m/2\pi) \sim (t_D\omega_{gr,m})^{-0.5}$ or

$$tn_f^2 \sim \frac{(k_m/2\pi)^2 H^2}{\omega_{gr,m} D_T} \quad (5.22)$$

This suggests that using this scaling we can predict the start of the late time regime (when there is only one remaining finger left, $n_f = 1$). By assuming isotropic
diffusion, and substituting $k_m$ (Eq. 5.8) and $\omega_{gr,m}$ (Eq. 5.9) in Eq. 5.22, we then have

$$t = 0.016 \frac{H^2}{D_T}$$

(5.23)

Figure 5.16: Scaling of $n_f$ curves from Figure 5.13 using the early time analysis (Eq. 5.6).

This is consistent with the heuristic discussion in Nijjer et al. (2018) who suggested that for isotropic dispersion, the scaling of late time regime is given by $t \sim O(Pe_T)$ (they did not investigate anisotropic diffusion/dispersion).

In Eq. 5.23, if we express $t$ in terms of velocity and the domain length $L$ (i.e.
t = L/v_T), the number of fingers will reduce to 1 when

$$\frac{LD_T}{v_T H^2} = 0.016$$  \hspace{1cm} (5.24)

A similar expression to the left hand side of Eq. 5.24 was used by Gardner and Ypma (1984) to argue that there is a transition between the viscous-dominated unstable regime to stable flow (with one finger) as shown in Figure 5.17(a), where they quantified this transition in terms of oil recovery. This corresponds to the crossover from the intermediate to the late time regime (n_f = 1) in our study. Figure 5.17(b) shows the unrecovered oil measured in laboratory experiments performed by Gardner and Ypma (1984). The unrecovered oil for different core lengths scales well with (LD_T)/(v_T H^2). These experiments were performed using long cores with a range of aspect ratios up to 22. There is a sharp decrease in unrecovered oil (suggesting there is only one finger left) at approximately (LD_T)/(v_T H^2) = 0.02, which is consistent with Eq. 5.24.

### 5.5.5 Conclusions

This study has investigated the dynamics and scaling of miscible viscous fingering in a system with no flow boundaries perpendicular to the principle flow direction. We have studied the influence of the transverse Peclet number, viscosity ratio and the diffusion anisotropy on the behaviour. Overall the behaviour is similar to that observed in previous studies (Zimmerman and Homsy, 1991; Nijjer et al., 2018) that investigated fingering in systems with periodic boundary conditions.

Four fingering regimes have been identified and studied namely

1. Very early time, when longitudinal diffusion dominates over finger growth.

2. Early time, when fingers start to grow. This is the flow regime typically
Figure 5.17: Late time scaling based on Eq. 5.24. (a) The transition between unstable displacement regime (many fingers) and stable displacement (one finger) taken from Gardner and Ypma (1984). Here they used core diameter $d$ which is roughly equivalent to our domain width $H$; (b) Unrecovered oil at 1.1 PVI in miscible core floodings by Gardner and Ypma (1984). The upper most curve is their attempt to simulate viscous fingering.
investigated using perturbation analysis.

3. Intermediate time, when multiple fingers grow and interact. This regime can only be investigated using numerical simulation or physical experiments because of the non-linear behaviour.

4. Late time, when all the fingers but one have faded or coalesced.

The number of fingers predicted by the simulations at early time were approximately half the number expected from perturbation analysis. This was probably because the analysis assumes an initial step change in concentration from which the fingers grow, whereas in reality this sharp interface is smeared by longitudinal diffusion before the fingers start growing, reducing the instability and hence the number of fingers. Nonetheless the linear dependence of the number of fingers on $Pe_T$ and their variation with diffusion anisotropy agreed with the predictions of the perturbation analysis of Tan and Homsy (1986).

At intermediate and late times we have shown that the number of fingers over time scale with the most dangerous wavenumber and the growth rate of that wavenumber determined from early time growth analysis. Initially the number of fingers reduces as $1/t$ but at later times they decay according to $1/\sqrt{t}$. The along flow spreading of the fingers scales linearly with time in these regimes. This suggests that the empirical models based on modified fractional flow, such as the Koval (1963) and Todd and Longstaff (1972) models, can be used to model the average flow behaviour in field scale simulation of enhanced oil recovery of CO$_2$ sequestration. We have also shown that the time at which all the fingers merge or fade into one depends only on the aspect ratio of the system and the transverse Peclet number.

As our analysis was confined to idealised cases (constant dispersion, linear displacement in homogeneous reservoirs), further work is needed to investigate the implications of the results for modelling physical systems such as
5.5. Results

- Velocity-dependent dispersion is expected to cause deviations from the existing results especially at very early and early times when the growth of the fingers is very sensitive to the amount of dispersion.

- Radial flow requires the use of a different solution of linear stability theory for the fingering scaling.

- Heterogenous reservoirs with small correlation lengths may increase the frequency of tip-splitting events.

Nevertheless, the current results should provide a qualitative understanding of the fingering scaling and dynamics for physical systems.
Chapter 6

Flow Regimes in Heterogeneous Reservoirs

In Chapter 4, we showed that modelling viscous fingering requires a very high resolution grid to reduce the numerical dispersion to an acceptably low level. Such a requirement means that explicit modelling of viscous fingering is not always feasible for studies of physical systems. Many studies have demonstrated the ability of empirical models to capture the average behaviour of miscible flows. This has also been demonstrated in Chapter 5 where we observed that, in homogeneous domains, the mixing length during early and intermediate times grows linearly from early to late times. This allows empirical advective flow models such as Koval or Todd-Longstaff models to be used.

However, as noted in Chapter 2, Section 2.2.4, realistic reservoirs are heterogeneous which may lead to different dominant flow regimes including

1. Advective – The flow instability is either caused by the adverse viscosity contrast as previously studied in Chapter 5 or induced by the heterogeneous reservoirs with large correlation lengths.
2. Diffusive – A stable miscible displacement in reservoirs with relatively small degree of heterogeneity variations. In this case $L_{mix} \sim \sqrt{t}$.

3. Channelling – This is typically identified by the presence of bypassed regions in the reservoirs during flooding due to well-defined channels with high permeability contrast.

In this chapter, we investigate how the different flow regimes in heterogeneous reservoirs can be identified and what are the suitable 1D models that can be used to represent these flow regimes. We first establish a workflow to predict different flow regimes in miscible gas injection. Then, we present suitable averaged models that can be used to predict average flow performance in the dominant flow regimes encountered in heterogeneous reservoirs using an empirical model. Specifically, we propose a modification to Koval (1963) model to allow it to capture the bypassed oil from channelling. This is easier than using other empirical models (e.g. Fayers, 1988) as there are fewer adjustable parameters. We shall limit the discussion by excluding gravity-dominated flow and neglecting physical diffusion.

6.1 Overview

6.1.1 Koval model

As presented in Chapter 2, the Koval (1963) model gives an empirical description of solvent flow represented by the fractional flow

$$F_s(c) = \frac{1}{1 + \frac{1}{M_e} \frac{1 - c}{c}}$$  \hspace{1cm} (6.1)
Permeability heterogeneity and viscosity contrast are captured in Eq. 6.1 using the effective mobility ratio $M_e$ defined as

$$M_e = H_k(0.78 + 0.22M^{rac{1}{4}})^4$$ (6.2)

$H_k = 1$ means the reservoir is homogeneous and the solvent flow only depends viscosity ratio $M$. A high $H_k$ indicates a high degree of variation in permeability that adversely affects the flow. The usual way to find $H_k$ is by running a numerical simulation with a unit-mobility displacement, but this can be time-consuming.

Figure 6.1: (a) Fractional flow $F_s$ and; (b) dimensionless velocity $v_d$ modelled using Koval model (Eq. 6.1) for $M_e = 1, 2, 3$.

Figure 6.1(a) shows the fractional flow modelled using Koval model for $M_e = 1, 2, 3$. Similar to Buckley-Leverett method, we obtain the dimensionless solvent velocity $v_d$ (Figure 6.1b) by differentiating the fractional flow curve

$$\frac{dF_s}{ds} = \frac{M_e}{(M_ec - c + 1)^2}$$ (6.3)

The velocities of the leading edge of the flow (i.e. the finger tip) and the trailing
The solvent breaktrough occurs when the leading edge reaches the production well ($x_d = 1$) i.e.

$$t_{bt} = \frac{1}{M_e}$$

(6.5)

### 6.1.1.1 Modified Koval model

In its original form, the empirical fractional flow formulation (Eq. 6.1) will result in 100% oil recovery. However, in a channelized flow there will be areas with low permeabilities in which the oil is bypassed and not recovered for a finite amount of solvent injection. This consequently reduces the sweep efficiency. To take this into account, an additional term, $S_{orb}$, is added to the original fractional flow equation as given in Eq. 6.6.

$$F_s = \frac{1}{1 + \frac{1}{M_e} \left( \frac{1 - c - S_{orb}}{c} \right)}$$

(6.6)

This results in various changes to the average flow behaviour predicted by the Koval model, as summarised in Table 6.1. Figure 6.2 compares the average concentration profiles and oil recovery profiles ($N_p$) for the original and modified Koval models with $M_e = 2$ and $S_{orb} = 0.5$. In this study, the value of $S_{orb}$ will be simply taken as
Chapter 6. Flow Regimes in Heterogeneous Reservoirs

Figure 6.2: Comparison between the original and modified Koval models (see Table 6.1). Here, $M_e = 2$ for both models and $S_{orb} = 0.5$ in the modified model.

the fractional volume occupied by low permeabilities below a selected threshold in the reservoir.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Original Koval</th>
<th>Modified Koval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fractional flow</td>
<td>$F_s = \frac{1}{1 + \frac{1}{M_e} \left(1 - c \cdot \frac{1}{c}\right)}$</td>
<td>$F_s = \frac{1}{1 + \frac{1}{M_e} \left(1 - c - S_{orb} \cdot \frac{1}{c}\right)}$</td>
</tr>
<tr>
<td>Dimensionless velocity</td>
<td>$M_e$</td>
<td>$\frac{M_e}{(1 - S_{orb})}$</td>
</tr>
<tr>
<td>Leading edge</td>
<td></td>
<td>$\frac{M_e}{M_e(1 - S_{orb})}$</td>
</tr>
<tr>
<td>Trailing edge</td>
<td>$\frac{1}{M_e}$</td>
<td></td>
</tr>
<tr>
<td>Maximum recovery</td>
<td>1 PV</td>
<td>$(1 - S_{orb})$ PV</td>
</tr>
</tbody>
</table>

Table 6.1: Comparison between the key parameters in the original and modified Koval models.
6.1.2 Stiles method

The Stiles method was originally developed to estimate the oil recovery for waterflooding in stratified reservoirs. Consider a layered 2D reservoir with varying permeability and layer thickness as shown in Figure 6.3(a). Here each layer is defined by layer thickness $\Delta y_d$ and permeability $K$. Assuming a piston-like displacement with no communication between the layers, uniform porosity, and a constant pressure difference between injection and production wells, the displacement front velocity in each layer will be proportional to the permeability of the layer.

The Stiles method involves rearranging the layers based on their permeability values into ascending (or descending) order as shown in Figure 6.3(b). Next, by dividing the permeability in each layer with the area under the curve in Figure 6.3(b), a dimensionless permeability curve $K'$ shown Figure 6.3(c) is obtained. This gives the area under the $K'$ curve equal to unity. If the reservoir is homogeneous, the $K'$ curve is a straight line ($K'=1$), as shown by the dashed line.

Then a ‘capacity’ curve is constructed by integrating the $K'$ curve. This is shown in Figure 6.3(d). The capacity curve provides the information on the amount of oil recovered and the relative breakthrough time (when injected water reaches the production well) for each layer. It also provides a visual representation of the heterogeneity level in stratified reservoirs – a highly stratified reservoir will give a highly concave capacity curve while a homogeneous reservoir is represented by a $45^\circ$ line shown by the dashed line in Figure 6.3(d). To obtain the fractional flow and recovery curves for immiscible displacement, the Stiles method requires the capacity curve to be used together with the fluid properties (viscosities, formation volume factor).
6.1.2.1 Application for miscible displacement

For a stable miscible displacement, we can show that the $K'$ curve is representative of the solvent velocity profile (Figure 6.1a) while the capacity curve is analogous to the fractional flow (Figure 6.1b). To illustrate this, we performed a simulation with $M = 1$ by using the permeability distribution from Figure 6.3(a). Figure 6.4(a) shows the concentration distribution in the reservoir at $t_d = 0.3$ PVI. The average concentration as a function of dimensionless velocity is shown in Figure 6.4(b). Here we see that the velocity profile can be solely predicted using $K'$ from Figure 6.3(c) as the assumption that the velocity is proportional to permeability is valid in this case.

Recall that the Koval model gives the leading edge velocity of $H_k$ when $M = 1$ (Eq. 6.4a). Since the leading edge velocity also equals to the maximum dimensionless permeability $K'_{max}$, we then have

$$H_k = K'_{max}$$ (6.7)

This allows a quick determination of $H_k$ in layered reservoirs without the need for high-resolution simulation.

Using the same idea for heterogeneous non-layered reservoirs, we propose the estimation of $H_k$ as follows

1. Compute the harmonic average permeability $K_{har}$ of each row of the grid blocks perpendicular to the flow direction (Figure 6.5a).

2. Arrange the permeabilities in ascending order (Figure 6.5a).

3. Find the dimensionless permeability $K'_{har}$ by dividing (2) with the area under its curve (Figure 6.5b).
From the $K'_{har}$ curve found in step (3), there are two ways to proceed. First, we could assign the largest value of dimensionless permeability $K'_{har,max}$ as $H_k$ and use it in Koval’s formulation to represent flow at various $M$ as given in Eq. 6.2. Alternatively, one could directly use the $dF_s/dc = K'_{har}$ curve found in step (3) for the recovery calculation similar to Buckley-Leverett formulation. In this case, the oil recovered $N_p$ at time $t_d$ is given by

$$N_p = \int_{x_d=0}^{x_d=1} \frac{dF_s}{dc} t_d \, dx_d$$  \hspace{1cm} (6.8)$$

This, however, only valid for $M = 1$.

### 6.1.3 Flow conductivity indicator, CF

The flow conductivity indicator $CF$ is simply the ratio of effective permeability to its geometric mean (Renard and Allard, 2013)

$$CF = \log \frac{K_{eff}}{K_{geo}}$$  \hspace{1cm} (6.9)$$

If $CF > 0$ then flow is channelised. If $CF < 0$ then flow is not channelised.

In this study, the effective permeability, $K_{eff}$, for each 2D model was obtained from single phase flow simulations, run using the Matlab Reservoir Simulation Toolbox, MRST (Lie, 2016). Constant pressures at the injection well $P_{inj}$ and the production well were used. The resultant flux was used to calculate the effective permeability using

$$K_{eff} = \frac{QL}{H(P_{inj} - P_{prod})}$$  \hspace{1cm} (6.10)$$

where $L$ and $H$ are the reservoir length and width, respectively.
Figure 6.3: Illustration of Stiles method. (a) It is assumed that the front velocity in each layer is proportional to its permeability; (b) The layers are rearranged according to their permeability values; (c) The dimensionless permeability curve $K'$ is obtained by dividing each of the layer permeability with the area under the curve in (b); (d) The cumulative capacity curve is constructed by integrating $K'$. 
6.1. Overview

Figure 6.4: Simulation of a stable miscible flow \((M = 1)\) in a layered reservoir identical to Figure 6.3(a). (a) Concentration map at \(t_d = 0.3\) PVI; (b) Average concentration vs. dimensionless velocity at \(t_d = 0.3\) PVI and the dimensionless permeability curve from Figure 6.3(c).

Figure 6.5: Computation of \(H_k\) based on Stiles method. (a) Rearrange the harmonic-averaged horizontal permeability in increasing order; (b) Find the dimensionless permeability \(K_{har}^{'}\). \(H_k\) equals to the maximum value of the dimensionless permeability curve \(K_{har,max}^{'}\).
Chapter 6. Flow Regimes in Heterogeneous Reservoirs

6.2 Method

In this study, we ran a series of flow simulations in three different types of geological models to test the different methods for quantifying the impact of heterogeneity of unstable miscible displacements.

6.2.1 Permeability fields

The three types of permeability heterogeneities considered in this study were (a) highly heterogeneous reservoirs (SPE 10 model 2, layers 1-35), (b) channelised reservoirs (SPE 10 model 2, layers 36-85), and (c) geostatistically generated synthetic reservoirs.

The SPE 10 model 2 (Christie and Blunt, 2001) was originally developed as part of the PUNQ project investigating history matching approaches (Bos, 2000) but was adapted for the SPE comparative study into upscaling. It has a grid of $220 \times 60 \times 85$, and physical dimensions of $1200 \times 2200 \times 170$ feet. The top part of the model (layers 1-35) represents the Tarbert formation, a prograding near shore environment. The lower part (layers 36-85) represents the Upper Ness formation, a fluvial environment. Statistically, the former has a correlated log-normal distribution, while the latter exhibits channelling with a bimodal log-normal distribution. In this study each of the 85 layers taken to form a series of 85 2D models. We examine the flow behaviour in each of these models.

Additional synthetic permeability fields were generated to investigate the crossover from dispersive flow to advective flow. These were generated using Sequential Gaussian Simulation (SGS) which is a popular algorithm due to “its relative simplicity and effectiveness at creating numerical models with correct spatial statistics” (Deutsch, 2002). The algorithm used in this study was taken from
6.2. Method

Figure 6.6: The permeability field for SPE 10 model from Christie and Blunt (2001).

Figure 6.7: The histogram plots of the permeability field for SPE 10 model 2 from Christie and Blunt (2001).
the Geostatistical Software Library, GSLIB (Deutsch and Journel, 1997), accessed through a graphical user interface called Stanford Geostatistical Modeling Software, SGeMS (Remy et al., 2009). Spatial variation was characterized using a spherical variogram model, with a range value set to the desired correlation length and sill set to zero. Isotropic permeability variation was assumed, i.e. the correlation length in the $x$ direction is the same as in the $y$ direction. $200 \times 100$ grid blocks with an aspect ratio of 1:0.5 were used for all cases (to roughly follow the dimension and grid resolution of the SPE 10 model 2).

### 6.2.2 Miscible flow simulation

The reservoirs were initially saturated with 100% oil. Injection and production wells were modelled along the left and right hand edges of each 2D reservoir model, respectively, spanning the entire reservoir width, $H$. Constant injection and production rates were used. Simulations were run for viscosity ratios of 1, 10 and 100. Viscous fingering was explicitly triggered by a small random concentration seed in the row of grid blocks containing the injection well. No physical dispersion was modelled as we assumed that the level of numerical diffusion was similar to the physical diffusion that would be encountered on these scales.

Two metrics were used to monitor the performance of each layer: solvent breakthrough time, $t_{bt}$ and oil recovery, $N_p$, at $t_d = 1$ PVI.
6.3. Results

6.3.1 Channelised vs. advective flow

In Figure 6.8(a), we show the values of $K_{eff}$ and $K_{geo}$ for each layer in the SPE10 model. The fairly homogeneous layers (layers 1-35) tend to have roughly similar values of $K_{eff}$ and $K_{geo}$, whereas in channellised layers $K_{eff}$ is much higher compared to $K_{geo}$. A plot of $K_{eff}$ vs. $K_{geo}$ as shown in Figure 6.8(b) shows a clear demarcation between the channellised vs. non-channellised layers.

The CF indicator was found to be able to discriminate the channellised layers in SPE 10 model successfully (Figure 6.9). All the non-channellised layers have CF values less than 0. Interestingly, some of the layers that appear to have channels, when we examine the permeability maps, give rather low CF values, ranging from -0.51 to 0.18 (layers 36, 42, 47, and 68). Upon inspection, this occurs because either the channel area is very small or the orientation of the channel relative to the well placement does not create a “short-circuit” path during production.

This is best illustrated by considering layer 36 as shown in Figure 6.10. Although
Figure 6.8: Homogeneous layers tend to have roughly similar values of $K_{\text{eff}}$ and $K_{\text{geo}}$, whereas in channelised layers $K_{\text{eff}}$ is much higher compared to $K_{\text{geo}}$. (a) $K_{\text{eff}}$ and $K_{\text{geo}}$ values; (b) $K_{\text{eff}}$ vs. $K_{\text{geo}}$.

Figure 6.9: CF values for SPE10 model 2 calculated using Eq. 6.9.
there is a high permeability channel in this layer, its dimension and orientation do not create a preferential path from injection well to production well, giving a rather efficient sweep similar to non-channeled reservoir, as seen in Figure 6.10(b). This suggests that these layers can potentially be modelled using the original Koval model without the need to include the bypassed oil volume.

![Figure 6.10](image)

Figure 6.10: (a) Permeability field for layer 36; (b) concentration map at $t_d = 1$ PVI.

Figure 6.11 shows the distributions of breakthrough time $t_{bd}$ and oil recovery $N_p$ (at $t_d = 1$ PVI) for $M = 1$ obtained from FCM simulations as a function of $K_{geo}$, $K_{eff}$ and (c) $CF$. We see here that $K_{geo}$ and $K_{eff}$ on their own are not sufficient to differentiate the channeled vs. non-channeled layers.

![Figure 6.11](image)

Figure 6.11: The distributions of breakthrough time $t_{bd}$ and oil recovery $N_p$ (at $t_d = 1$ PVI) for $M = 1$ obtained from FCM simulations as a function of (a) $K_{geo}$; (b) $K_{eff}$; and (c) $CF$. 
6.3.1.1 Tuning of the Koval and modified Koval models

Renard and Allard (2013) suggested that flow would be channelled when CF > 0. However, in this study, CF = 1 was chosen as the cut-off point, i.e.

- CF ≤ 1: Advective flow modelled using the original Koval model
- CF > 1: Channelised flow modelled using the modified Koval model

\( CF \leq 1 \)

In the Koval model, we can show that for a unit-mobility displacement \( (M = 1) \), the breakthrough time and heterogeneity index have the following relationship

\[ t_{bt} = \frac{1}{H_k} \]  \hspace{1cm} (6.11)

Hence, a way of tuning \( H_k \) is by running a simulation with \( M = 1 \) to obtain the breakthrough time and assigning \( H_k \) using Eq. 6.11. In this study, we attempted to tune \( H_k \) by using the Stiles method to predict the breakthrough time as discussed in Section 6.1.2.1.

Figure 6.12 compares the breakthrough time and oil recovery at \( t_d = 1 \) PVI obtained by the detailed simulation and the Koval formulation, tuned using the Stiles method for all the layers with CF values of less than 1. The Stiles method gives satisfactory results for non-channelled layers (layers 1-35). The line of best fit for the breakthrough times for these layers (Figure 6.12a) suggests that the Stiles method slightly underpredicts and moderately correlates with the simulation results (with the coefficient of determination, \( r^2 = 0.30 \)).

However, it performs badly in channelled layers with low CF. This is expected, since Stiles method just takes the average permeability values parallel to the flow direction – any long high permeability streaks that are at some angle to the direction
of the flow will not be captured properly. In this case, further tuning of $H_k$ may be needed.

![Diagram](image)

Figure 6.12: Comparison between the breakthrough times (blue) and oil recovery at $t_d = 1$ PVI (red) from detailed simulation and the Koval model for reservoir layers with low CF values ($< 1$). The Stiles method was used to obtain $H_k$. Square markers correspond to the channelized layers with low CF (layers 36, 42, 47, and 68). The Stiles method does not give satisfactory results for these layers. (a) $M = 1$; (b) $M = 10$; and (c) $M = 100$.

**CF > 1**

As discussed in Section 6.1.1.1, the modified Koval model requires two parameters to be defined, namely $S_{orb}$ and $H_k$. These two parameters control the breakthrough time while the maximum recovery is controlled by $S_{orb}$ only.

From Figure 6.7, the bimodal distribution for channelized layers shows that the upper tail of the low permeability ends at approximately at $K_{log} = 1$. Hence we used this value as the upper threshold for low permeability. For each layer, the fraction of its area that is lower than $K_{log} = 1$ is taken as its $S_{orb}$ value. $H_k$ has to be tuned once $S_{orb}$ is determined. However, we found that using an empirical $H_k$ value of 2 is sufficient to obtain satisfactory results for all of the channelized layers.
without any further tuning.

Figure 6.13 shows its performance in predicting the breakthrough time and oil recovery observed in the detailed simulation for viscosity ratios of $M = 1$, 10 and 100. A data point on the solid 45° line indicates a perfect agreement between detailed simulation and the modified Koval model.

![Figure 6.13: Comparison between the breakthrough times (blue) and oil recovery at $t_d=1$ PVI (red) from simulation and modified Koval model for reservoir layers with high CF values (>1); (a) $M = 1$; (b) $M = 10$; and (c) $M = 100$.](image)

In general, the predictions made by the modified Koval model were found to be good for both breakthrough time and oil recovery. However, notice that at viscosity ratios of 10 and 100, the model seems to slightly under-predict the breakthrough time and over-predict the oil recovery. Figure 6.14 shows the oil recovery curve and concentration profile for layer 50 with $M = 10$. The inclusion of $S_{orb}$ term clearly allows for better approximation of the average concentration profile, but as per the original Koval model, no details of spatially varying concentration in $x$ direction is captured.
Figure 6.14: A better representation can be made by modifying the Koval model to include a bypassed oil (quantified by $S_{orb}$ term) in channelling cases, shown here for layer 50, $M = 10$. (a) Permeability field; (b) concentration at $t_d = 0.5$; (c) oil recovery; (d) concentration profile across $x$ direction.
6.3.2 Sensitivity studies

Recall that viscous fingering was explicitly triggered by random concentration values at the injection well at the start of simulations. The details of fingering formation will depend on these random values, which were generated by means of a seed number. Changing the seed to give different random concentration values at the inlet did not alter the results, as the growth of the fingers in these cases with correlated permeability distributions is mainly dictated by the spatial permeability distribution. Figures 6.15(b) and 6.15(c) show the fingering patterns generated from two different seed numbers in layer 1 at $M = 100$.

The effect of numerical dispersion was also investigated by doubling the number of grid blocks in the $x$ and $y$ directions. The more refined grid blocks caused the fingers to be narrower but longer (Figure 6.15d) as might be expected from the previous analysis in Chapter 5. However, the general fingering patterns are still the same, and the results in term of breakthrough time and production are within 0.02 PV, as shown in Figure 6.15(e).

6.3.3 Dispersive vs. advective flow

We also investigated whether the criterion discussed by Fayers et al. (1992) (Eq. 2.10) can be used to determine when flow is dispersion dominated and when it is advection dominated. In this case we used synthetic permeability fields generated using Sequential Gaussian Simulation with different standard deviations and correlation lengths. Detailed flow simulations for displacements with a viscosity ratio of 1 were performed and transversely-averaged concentration vs. $x$ distance profiles were calculated at $t_d =$ 0.2 intervals up until $t_d =$ 1.0 PVI. These curves were then normalized in by shifting them using $(x - t)/\sqrt{t}$. If the flow is diffusive, the curves should collapse onto a single curve. Figure 6.16 shows the normalized
Figure 6.15: The effects of using different initial concentration and increasing grid blocks on the fingering at $t_d = 0.1$ in layer 1 when $M = 100$. (a) Permeability field; (b) base case concentration profile; (c) different seed number; (d) double grid blocks in $x$ and $y$ directions; (e) oil recovery profiles.
concentration curves for standard deviations of 1 and 2. In both cases, the normalised curves start to deviate from each other when $\sigma \lambda = 0.1$ (the threshold between dispersive and advective flow suggested by Fayers et al. (1992)), and deviate much further when $\sigma \lambda = 0.2$, which suggests that diffusive model is no longer suitable.

The CF values for reservoirs that exhibit diffusive flow are found to be around 0.9, which is of the same order of magnitude as the non-channellised layers in SPE 10 model. This means that one can use $N_{AD}$ to differentiate between diffusive and advective flows when $CF < 0$.

![Figure 6.16: Normalised concentration profiles at $t_d = 0.2, 0.4, 0.6, 0.8$ and 1.0 PVI for different values of correlation lengths $\lambda$. (a) $\sigma = 1.0$; (b) $\sigma = 2.0$.](image)

6.4 Conclusions

We have used detailed simulation to investigate the different flow regimes that occur during first contact miscible displacements in heterogeneous reservoirs when gravity is neglected. We have shown that the diffusivity number proposed by Gelhar
and Axness (1983) for determining when there is macrodispersion in aquifers and later adopted by Fayers et al. (1992) can be used to determine when tracer flow behaves dispersively. We have also shown that the connectivity factor proposed by Knudby and Carrera (2005) and Renard and Allard (2013) can be used to determine when first contact miscible displacements are dominated by flow channels. Using these dimensionless numbers we propose that the average effects of heterogeneity on first contact miscible displacements on the field scale may be characterised by the following relationships

**Dispersive** \( M \leq 1, N_{AD} < 0.1 \)

\[
c(x,t) = \frac{1}{2} \left( 1 - \text{erf} \left( \frac{x - vt}{2\sqrt{Dt}} \right) \right)
\]

**Advective** \( M > 1, CF < 1 \)

\[
F_s = \frac{1}{1 + \frac{1}{M_e} \left( \frac{1-c}{c} \right)}
\]

**Channeling** \( M > 1, CF > 1 \)

\[
F_s = \frac{1}{1 + \frac{1}{M_e} \left( \frac{1-c - S_{orb}}{c} \right)}
\]

where \( F_s \) is the fractional flow of the solvent (miscible gas), \( c \) is the concentration of the solvent and \( S_{orb} \) is the fraction of bypassed oil. The effective viscosity \( M_e \) is given by \( M_e = H_k(0.78 + 0.22M^\frac{1}{4})^4 \), where \( H_k \) is the heterogeneity index.

The range of validity of these different relationships can be visualised by the phase or regimes diagram sketched in Figure 6.17. Further work is needed to confirm this suggested phase diagram, especially the location and width of the boundaries between the different regimes, as only a small number of example cases were studied here that investigated the difference between channeling and advective flow regimes and the difference between advective and dispersive regimes.

The fractional volume occupied by the low permeability zones was taken as the \( S_{orb} \) value in the channelled flow. Alternatively it may be calculated using the methods to compute sweep efficiency as proposed by Gago et al. (2018). For the SPE10
model 2, we also found that $H_k = 2$ is sufficient to provide reasonable results for the channellised flows although we have no physical interpretation behind this value.

For advective flow, we showed that $H_k$ can be estimated using the Stiles method. However, caution needs to be taken since the naive Stiles method just takes the average permeability values parallel to the flow direction – any long high permeability streaks that are at some angle to the direction of the flow will not be captured properly as observed in some of the SPE 10 layers. In this case, further tuning of may be needed. These ideas can be easily extended to the more widely used Todd and Longstaff (1972) empirical model.

![Diagram](image)

Figure 6.17: Approximate map of flow regimes for miscible flow in heterogeneous reservoirs using the flow conductivity indicator $CF$ (Renard and Allard, 2013) to identify channelised flow and the Fayers et al. (1992) criterion ($\sigma \lambda$) to identify when flow is likely to be dispersion dominated. Flow is influenced by multiple effects in the white regions.
This chapter presents an analytical expression to estimate the minimum polymer slug size needed to maintain the effectiveness of the polymer flood in the event of viscous fingering of the chase water case. This is obtained from a new semi-analytical solution that describes the average effects of the fingering of chase water into a polymer slug in 1D. We first briefly review the analytical solution of continuous polymer injection, following which the derivation describing the dynamics of a stable chase water front position as a function of time is made. These solutions are then combined and extended to capture the effect of miscible viscous fingering of the chase water into the polymer slug. This is achieved by assuming that the average effect of fingering can be represented by the empirical Todd and Longstaff model. Finally, the validity of the analytical technique is demonstrated by comparing it against its numerical equivalent as well as high resolution first contact miscibility simulation.


Chapter 7. Polymer Slug Injection

7.1 Analytical solution for continuous polymer injection

Let us first consider continuous polymer injection into a homogeneous 1D model so that we can understand the key features of the displacement. We assume that the system is incompressible, there is no adsorption of polymer onto the rock, and physical diffusion and dispersion are negligible. We define a normalised concentration of polymer solution in the water phase,

\[ C_p = \frac{c_p}{c_{p,\text{inj}}} \] (7.1)

where \( c_p \) is the polymer concentration in mass of polymer per volume of water and \( c_{p,\text{inj}} \) is the injected concentration of polymer in the same units. Using this definition \( \mu_w(C_p = 0) = \mu_w \) and \( \mu_w(C_p = 1) = \mu_p \), where \( \mu_w \) is the water viscosity and \( \mu_p \) is the polymer viscosity at the injection well.

The conservation of aqueous phase saturation \( S_w \) and \( C_p \) can be expressed as two hyperbolic, first order equations

\[ \frac{\partial S_w}{\partial t_d} + \frac{\partial F_w}{\partial x_d} = 0 \] (7.2)

\[ \frac{\partial C_p S_w}{\partial t_d} + \frac{\partial f_p F_w}{\partial x_d} = 0 \] (7.3)

where \( F_w \) is the fractional flow of the water phase and \( f_p \) is the fractional flow of the polymer component in the water phase.

If we assume the effects of gravity and capillarity are negligible, the fractional flow
of the aqueous phase $F_w$ is given by

$$F_w = \frac{Q_w}{Q} = \frac{1}{1 + \frac{\mu_w k_{rw}}{k_{ro} \mu_o}}$$  \hspace{1cm} (7.4)$$

and the fractional flow of the polymer component in the water phase is given by

$$f_p = \frac{Q_p}{Q_w}$$  \hspace{1cm} (7.5)$$

where $Q_w$ is the flow rate of the water phase, $Q_p$ is the flow rate of the polymer component in the water phase, $k_{rw}$ is the relative permeability of water, $\mu_w$ is the water viscosity, $k_{ro}$ is the relative permeability to oil and $\mu_o$ is the viscosity of oil. In this chapter, we assume $k_{rw}$ and $k_{ro}$ curves depend only on water phase saturation (shown in Figure 7.4) and any changes in the fractional flow function (Eq. 7.4) are due only to changes in $\mu_w$ when the polymer is added to the water phase. In the absence of viscous fingering, $f_p = C_p$ as the polymer solution and the chase water are first contact miscible.

For a line drive in which polymer is injected continuously into a reservoir containing oil and immobile connate water (at saturation $S_{wc}$), the initial conditions are

$$S_w = S_{wc}, C_p = 0; \text{ for all } x_d$$

For constant water phase injection at $x_d = 0$, with constant polymer concentration $C_p$, this is a Riemann problem in the half-plane which can be solved by using the method of characteristics (MOC), with $x_d/t_d$ as the self-similar variable. Following Pope (1980), the solution to this problem can be obtained graphically using two fractional flow curves, for water-oil ($F_w(S_w, C_p = 0)$) and polymer-oil ($F_w(S_w, C_p = 1)$) as shown in Figure 7.1(a). Note that we use $C_p = 1$ here for simplicity. The following solutions are valid for any values of $0 \leq C_p \leq 1$ as long as the $F_w$ curves
are correct for the selected concentration values.

The resulting solution for $S_w$ against $x_d$ is shown in Figure 7.1(b). Unlike the Buckley-Leverett solution for waterflooding, the profile of $S_w$ against $x_d$ is characterised by the existence of two shocks and a spreading wave. The first shock with saturation $S_1$ exists due to the formation of a connate water bank ($C_p = 0$) that has been displaced by the more viscous polymer. This is followed by another shock, at which $C_p$ increases from 0 to 1, and water saturation increases from $S_1$ to $S_2$. $S_2$ is found by the tangent to the polymer fractional flow curve that goes through origin as illustrated in Figure 7.1(a), while the “jump” from water to polymer curves ($S_1$ to $S_2$) must satisfy the following condition which imposes the conservation of mass at the vicinity of the shock

$$\frac{F_w(S_2) - F_w(S_1)}{S_2 - S_1} = \frac{F_w(S_2)}{S_2}$$  

(7.6)

7.2 Analytical solution for polymer slug injection

Having understood the key features of continuous polymer injection into an oil reservoir with no mobile water initially, we now examine how this is modified when we inject a slug of polymer followed by chase water. This analytical solution is complicated by the fact that there is a third discontinuity that occurs between the chase water and the trailing edge of the polymer slug, in addition to the two shocks described in the previous section. The analytical solutions to this problem have been presented by Bedrikovetsky (1993), Ribeiro and Pires (2008), de Paula and Pires (2015), Borazjani et al. (2016), and Vicente et al. (2014). However these solutions are more mathematically complex that those presented here, moreover they focussed on assessing the effects of adsorption and did not consider the effect of viscous instability.
7.2. Analytical solution for polymer slug injection

7.2.1 No fingering

We first consider the case when there is no fingering of the chase water into the polymer slug. Polymer injection is stopped at time $t_{d,\text{slug}}$ and followed by chase water. The injection rate of the chase water is the same as the injection rate used to inject the polymer solution.

In the case of chase water injection, a further discontinuity forms at the trailing edge of the polymer slug, in addition to the shock at the front of the connate water bank and the shock at the leading edge of the polymer slug. This continuity does not travel at constant speed, unlike the first two shocks. This was first analysed by Bedrikovetsky (1993) using Green’s theorem, however we shall explain the dynamics of this discontinuity more simply here using a geometric construction in the graph of fractional flow against water saturation. This will enable us to describe the late
time behaviour of the discontinuity and subsequently derive an analytical expression for estimating the minimum slug size in the presence of viscous fingering in the next sub-section.

The boundary conditions corresponding to the injection of the chase water are

\[ S_w = 1, C_p = 1; \quad 0 < t_d < t_{d,\text{slug}} \]

\[ S_w = 1, C_p = 0; \quad t_d > t_{d,\text{slug}} \]

The discontinuity between the chase water and the polymer slug is found at distance \( x_3 \) from the inlet. The water saturation immediately downstream of the discontinuity is \( S_3 \) and the water saturation immediately upstream is \( S_4 \).

Let us now find \( x_3 \) and and saturation \( S_3 \). First, select a value \( S_3 \) on the polymer-oil fractional flow curve remembering that \( S_2 < S_3 < 1 - S_{\sigma} \). We then draw a line tangent to this fractional flow curve with a gradient of \( v_3 = \frac{dF_w(S_3)}{dS_w} \). This line passes through the y-axis and x-axis at points A and -B respectively as illustrated in Figure 7.2(a).

Now consider Figure 7.2(b). The area under the curve representing the slug is given as the sum of \( A_1 \) and \( A_2 \). For the polymer volume to be conserved (assuming no adsorption), this area must equal the injected volume of \( t_{d,\text{slug}} \). Hence, we have

\[ t_{d,\text{slug}} = S_2(v_2t_d - x_3) + \int_{S_2}^{S_3} x_d \, dS_w - (S_3 - S_2)x_3 \]

where \( v_2 \) is the characteristic velocity of the leading edge of the polymer slug.

Since for \( x_d \geq x_3, x_d = \frac{(dF_w(S_w,C_p = 1))}{dS_w} t_d \), we then have

\[ t_{d,\text{slug}} = S_2v_2t_d - S_2x_3 + (F_3 - F_2)t_d - (S_3 - S_2)x_3 \]
where \( F_2 = F_w(S_w = S_2, C_p = 1) \) and \( F_3 = F_w(S_w = S_3, C_p = 1) \).

From Buckley-Leverett theory and the Welge construction (Figure 7.1a), \( F_2 = S_2 v_2 \).

Subsequently,

\[
t_{d,\text{slug}} = S_2 v_2 t_d + F_3 t_d - S_2 v_2 t_d - S_3 x_3
\]

\[
t_{d,\text{slug}} = F_3 t_d - S_3 x_3
\]

(7.7)

Since \( x_3 = v_3 t_d \), 7.7 can be rearranged to give

\[
\frac{t_{d,\text{slug}}}{t_d} = F_3 - S_3 v_3 = A
\]

(7.8)

\[
x_3 = v_3 t_d = \frac{A t_d}{B}
\]

(7.9)

Note that when \( t_d = t_{d,\text{slug}} \), \( x_3 = 0 \) and so \( F_3 = 1 \). This indicates that the trailing edge of the slug is at the injection well and the water saturation at the injection well is 1, as expected. As \( t_d \) increases, \( F_3 - S_3 v_3 \) decreases reflecting a decreasing value of \( S_3 \) which causes \( F_3 \) to decrease and \( v_3 \) to increase (remember that \( S_2 \) the leading shock front saturation of the polymer slug remains constant). To find the limiting value of \( S_3 \) (the saturation at the trailing edge of the polymer slug) and \( v_3 \), we let \( t_d \) go to infinity in Eq. 7.8. This gives \( v_3 = F_3 / S_3 \). This can only be true if \( S_3 \) has decreased to \( S_2 \) so the trailing edge of the polymer slug is travelling at the same speed as the leading front i.e.

\[
v_3 = \frac{F_3}{S_3} = \frac{F_2}{S_2} = v_2
\]

(7.10)

This means that at late times, we have a rectangular saturation profile for the slug. The length of the slug, \( x_2 - x_3 \), is simply given by

\[
x_2 - x_3 = \frac{t_{d,\text{slug}}}{S_2}
\]

(7.11)
At the chase water front, the value of $S_4$ can be determined graphically from $S_3$ at any given time by performing a jump from polymer-oil curve to water-oil curve in Figure 7.2(a) similar to Eq. 7.6

$$\frac{F_w(S_4) - F_w(S_3)}{S_4 - S_3} = \frac{F_w(S_4)}{S_4}$$

(7.12)

![Figure 7.2](image.png)

Figure 7.2: 1D solution for polymer slug injection. (a) Fractional flow curves for water-oil and polymer-oil. The blue dashed line with a gradient of $v_3$ that goes through points A and B can be used to calculate $x_3$ using 7.9. The change in saturation from $S_3$ to $S_4$ can be determined by taking a jump between the two curves using a line that goes through the origin, $S_3$ and $S_4$. (b) The position of slug trailing edge $x_3$ is determined by considering the area $A_1 + A_2$ which should be equal to $t_{d,slug}$

### 7.2.2 Empirical model of miscible viscous fingering

The discussion thus far assumes that the interface between chase water and slug tail is stable without any occurrence of viscous instability. In reality the interface is unstable and viscous fingering of the chase water into the polymer slug is expected as the chase water is less viscous than the polymer slug and the water and polymer solution are first contact miscible.
In principle, high resolution simulation is required to model the evolution of each of the fingers, however, as noted previously, Bondor et al. (1972) proposed using the Todd and Longstaff (1972) model to describe the average behaviour of the water and polymer components when this fingering occurs. This is now the standard way of representing the effects of viscous fingering in commercial simulations because (a) it can be implemented relatively easily in the existing black oil simulator framework and (b) it provides a fast way of approximating the likely impact of viscous fingering of the chase water on a polymer slug.

Bondor et al. (1972) proposed that fingering of the chase water into the polymer slug could be treated as the fractional flow of two components of the water phase: the chase water and the polymer slug. The effective polymer slug and chase water viscosities in each grid block, $\mu_{p,\text{eff}}$ and $\mu_{w,e}$, should be calculated using

$$\mu_{p,\text{eff}} = \mu_m(C_p)^\omega \mu_p^{(1-\omega)}$$  \hspace{1cm} (7.13)

$$\mu_{w,e} = \mu_m(C_p)^\omega \mu_w^{(1-\omega)}$$  \hspace{1cm} (7.14)

where $\mu_p$ is the viscosity of the polymerized water at maximum polymer concentration ($C_p = 1$), $\mu_w$ is the pure water viscosity and $\mu_m(C_p)$ is the viscosity of a mixture of pure water and polymer solution as a function of the polymer concentration. $\omega$ is a mixing parameter that can vary between 0 (no fingering) and 1 (complete mixing) but is typically set to 0.67 when there is fingering mainly based on the calibration made against experimental results made earlier by Blackwell et al. (1959). In principle, this needs to be calibrated by comparison with detailed fingering simulations. The effective aqueous phase viscosity, $\mu_{w,\text{eff}}$, used in the 2-phase black oil model is then calculated using

$$\frac{1}{\mu_{w,\text{eff}}} = \frac{1 - C_p}{\mu_{w,e}} + \frac{C_p}{\mu_{p,\text{eff}}}$$  \hspace{1cm} (7.15)
7.2.3 Analytical solution with fingering

As noted above, existing analytical solutions describing the dynamics of injecting a polymer slug followed by chase water ignore the impact of the chase water fingering into the trailing edge of the polymer slug. We now develop an approximate analytical solution that captures this effect and show that the resulting solution can be used to estimate the minimum polymer slug size needed to ensure that it is not destroyed by fingering before it reaches the injection well. We achieve this by using the Todd and Longstaff (1972) model to represent the average effects of the fingering.

To incorporate miscible fingering between the polymer slug and the chase water using the Todd and Longstaff model, we take the mass balance equation (7.3) and replace the fractional flow \( f_p \) with

\[
f_p = \frac{1}{1 + \frac{\mu_{p,eff}}{\mu_{w,e}} \frac{1 - C_p}{C_p}} = \frac{1}{1 + \frac{\mu_p (1 - \omega) 1 - C_p}{\mu_w C_p}}
\]  

(7.16)

using Eqs. 7.13 and 7.14. An analogous formulation for the water-oil-solvent system was used by Blunt and Christie (1993) and more recently by Juanes and Blunt (2006) to derive an analytical solution for WAG injection, although in WAG injection the fingering takes place at the front between the solvent (gas) and the displaced oil. They found exact solutions by simultaneously solving the conservation Eqs. 7.2 and 7.3.

From Eqs. 7.10 and 7.11, we assume that the water phase saturation in the polymer slug is constant and given by the saturation at the leading edge of the polymer slug. We further justify this assumption with the observation that this saturation is typically high and the change in saturation across the slug, from trailing to leading
7.2. Analytical solution for polymer slug injection

edge, is relatively small. This assumption means that, we can reduce (7.3) to

\[ S_2 \frac{\partial C_p}{\partial t_d} + F_2 \frac{\partial f_p}{\partial x_d} = 0 \]

\[ S_2 \frac{\partial C_p}{\partial t_d} + F_2 \frac{df_p}{dC_p} \frac{\partial C_p}{\partial x_d} = 0 \]  

(7.17)

In this case, we can obtain the dimensionless position of a given mean concentration of polymer \( C_p \), from the injection well, using

\[ x_{d,C_p} \simeq v_2(t_d - t_{d,slug}) \frac{df_p}{dC_p} \bigg|_{C_p} \] 

(7.18)

For example, by using \( \omega = 0.67 \) in Eq. 7.18 allows visualization of the spreading wave at the back of the slug as shown in Figure 7.3. The front of the spreading wave \( (C_p = 1) \) characterises the finger tips, which travel at a faster speed than the slug front. Its approximated velocity is given by

\[ v_{fingertips} = v_2 \frac{df_p}{dC_p} \bigg|_{C_p=1} \] 

(7.19)

Eventually, the finger tips will reach the slug front and cause the slug to break down. The time at this occurs when the blue line (giving the approximate characteristic speed of the leading fingers) intersects the red line (giving the characteristic speed of the leading edge of the polymer slug).

The minimum slug size needed to ensure that the chase water fingers only just cross the polymer slug by the time the polymer breaks through (i.e. at \( t_d = 1/v_2 \)) is given by

\[ t_{d,slug,min} = \frac{1}{v_2} - \frac{1}{v_{fingertips}} \] 

(7.20)

Substituting \( v_2 = S_2/F_2 \) (Eq. 7.10) and \( v_{fingertips} = v_2 \frac{df_p}{dC_p} \bigg|_{C_p=1} \) (Eq. 7.19) in
Eq. 7.20, we then have

$$t_{d,\text{slug,min}} = \frac{S_2}{F_2} \left( 1 - \frac{1}{\left. \frac{df_p}{dC_p} \right|_{C_p=1}} \right)$$  \hspace{1cm} (7.21)$$

Eq. 7.21 can also be derived graphically as presented in Section 7.4.

Figure 7.3: Incorporation of Todd and Longstaff model into the slug trailing edge using Eq. 7.14, assuming $\omega = 0.67$ and using a slug size of $t_{d,\text{slug}} = 0.5$ PVI. The red line shows the evolution of the leading edge of the polymer slug whilst the solid black line shows the evolution of the trailing edge using the analytical solution derived in Section 7.2.1. The dashed black line shows the motion of the trailing edge if the water saturation in the polymer slug is constant and equal to the leading shock front saturation. The upper blue line shows the motion of the finger tips using the Todd and Longstaff model, which indicates that the fingers will reach the leading edge of the polymer slug at approximately $t_d = 0.69$ PVI. The lower blue line shows the motion of the trailing edge of the fingering. All data used in this calculation is from Table 7.1.

### 7.3 Evaluation of the analytical approach

We now evaluate the above analytical solution by comparing its predictions with results from detailed simulation of the viscous fingering of chase water into a polymer
7.3. Evaluation of the analytical approach

slug and black oil simulation in which the Todd and Longstaff (1972) model has been implemented in the polymer options.

7.3.1 Method

To model miscible viscous fingering into the trailing edge of the slug, we used the higher order, IMPES (implicit pressure, explicit saturation), finite difference FCM simulator described in Chapter 3. ECLIPSE (Schlumberger, 2015) and MRST (Lie, 2016) were used for the black oil simulations using the Todd and Longstaff model to capture the average effects of fingering. We used the fully implicit scheme in both these simulators throughout the study.

All simulations used the data summarized in Table 7.1, unless stated otherwise. Grid dimensions were chosen following a grid refinement study.

Figure 7.4: Relative permeabilities used the polymer slug injection study (as defined in Table 7.1)
Chapter 7. Polymer Slug Injection

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluid viscosities</td>
<td></td>
</tr>
<tr>
<td>Oil</td>
<td>$\mu_o = 50$</td>
</tr>
<tr>
<td>Water</td>
<td>$\mu_w = 1$</td>
</tr>
<tr>
<td>Polymer</td>
<td>$\mu_p = 50$</td>
</tr>
<tr>
<td>Water-polymer mixture viscosity</td>
<td>$\mu_m = \left( \frac{1 - C_p}{\mu_w^{1/4}} + \frac{C_p}{\mu_p^{1/4}} \right)^{-4}$</td>
</tr>
<tr>
<td>Relative permeabilities</td>
<td></td>
</tr>
<tr>
<td>Oil</td>
<td>$\left( \frac{1 - S_w - S_{or}}{1 - S_{wc} - S_{or}} \right)^2, S_{or} = 0.2$</td>
</tr>
<tr>
<td>Water, polymer</td>
<td>$\left( \frac{S_w - S_{wc}}{1 - S_{wc} - S_{or}} \right)^2, S_{wc} = 0.2$</td>
</tr>
<tr>
<td>Reservoir initial state</td>
<td>$S_w = 0.2, S_o = 0.8$</td>
</tr>
<tr>
<td>Injection state</td>
<td></td>
</tr>
<tr>
<td>$0 &lt; t_d &lt; t_{d,slug}$</td>
<td>$S_w = 1, C_p = 1$</td>
</tr>
<tr>
<td>$t_d &gt; t_{d,slug}$</td>
<td>$S_w = 1, C_p = 0$</td>
</tr>
<tr>
<td>Grid blocks</td>
<td></td>
</tr>
<tr>
<td>MRST (1D model)</td>
<td>$1000 \times 1$</td>
</tr>
<tr>
<td>ECLIPSE (1D model)</td>
<td>$10000 \times 1$</td>
</tr>
<tr>
<td>FCM (2D model)</td>
<td>$300 \times 150$ (aspect ratio=1:0.5)</td>
</tr>
</tbody>
</table>

Table 7.1: Summary of data used in the polymer slug injection study
7.3. Evaluation of the analytical approach

7.3.2 The modified Todd and Longstaff model

Standard implementations of polymer flooding models in commercial simulators, such as ECLIPSE, include the Todd and Longstaff (1972) model to describe the effect of mixing between polymer solution and water and its effect on water phase viscosity. This is presumably intended to allow engineers to model the fingering of chase water into polymer as proposed by Bondor et al. (1972), however using $\omega > 0$ can also result in unphysical spreading of the shock front at the leading edge of the polymer slug. In the absence of adsorption, the front at the leading edge of the polymer slug is no longer self-sharpening and considerably more grid blocks are then required to resolve it.

This effect is illustrated for a continuous polymer flood, in Figure 7.5. Figure 7.5(a) shows the polymer front obtained from 1D ECLIPSE and MRST simulations with the analytical solution for $\omega = 0$ (no mixing between polymer and water) and $\omega = 1$ (complete mixing). 100 grid blocks were used in both cases. Both simulators give similar results in term of numerical accuracy (first order) and in both cases the polymer front is smeared when $\omega = 1$. Figure 7.5(b), shows that 10000 grid blocks are needed in this case to obtain a shock front close to the analytical solution. Clearly this is impractical for field scale simulations. One solution is to use higher order solvers (Liu et al., 1995; Mykkeltvedt et al., 2016), but these are more computationally expensive. Alsofi and Blunt (2013) used a simpler method in streamline-based simulator whereby the weighted average fractional flow is used to segregate the flow between the regions with and without polymer.

In this study, we adapted MRST using an approach comparable to the method proposed by Alsofi and Blunt (2013). At every time step, we calculate the maximum value of polymer concentration, $C_{p,max}$ reached in each of the grid blocks since the beginning of injection. We then modify the Todd and Longstaff mixing parameter
Figure 7.5: Polymer front simulation in ECLIPSE and MRST. (a) Comparison between MRST, ECLIPSE and analytical solution. (b) Grid refinement in ECLIPSE for $\omega = 1$. It can be seen that many more grid blocks ($\sim 10000$) are needed to resolve the leading front of the polymer slug when $\omega = 1$. Note that the connate water bank has already broken through in this case.

Using

$$\omega' = \omega C_{p,\text{max}}$$

(7.22)

where $\omega$ is the input value of the Todd and Longstaff mixing parameter. This means that the mixing parameter is zero in cells that have never seen polymer and reduced in cells that see polymer increasing (as the polymer slug advances) but is equal to the input value at the trailing edge of the polymer slug. This strategy was found to give better resolution of the leading polymer front in MRST as shown in Figure 7.6.

### 7.3.3 Results - no viscous fingering

We first verified that the black oil simulators can reproduce the analytical solution derived by Bedrikovetsky (1993) for polymer slug injection for the case without viscous fingering. Figure 7.7 shows the comparison between the analytical solution
7.3. Evaluation of the analytical approach

Figure 7.6: Comparison of the concentration and saturation profiles obtained with ECLIPSE and the modified MRST model, when simulating a polymer flood using 100 grid blocks. The modification to the Todd and Longstaff parameter reduces smearing of the leading polymer front.

and the predictions of the modified MRST using $\omega = 1$ for a slug volume of 0.4 PVI. Other input data is as per Table 7.1. A very high resolution ECLIPSE model (DX=10000) is also shown here.

The simulators correctly capture the location of the trailing edge $x_3$ and the relatively small jump in saturation from $S_3$ to $S_4$ at $x_3$. The immobile oil region, formed by the locus of chase water front saturation $S_4$, is also observed.

7.3.4 Results - viscous fingering

We now evaluate the ability of the new analytical model to predict viscous fingering of the chase water into the trailing edge of the polymer slug by comparing its predictions with those obtained when we model the fingering explicitly. Here we consider two cases, $\mu_p/\mu_w$, of 10 and 50. We keep $\mu_p = \mu_o$ in all cases so that the potential impact of immiscible fingering at the slug front can be ruled out.
Figure 7.7: Comparison of the concentration and saturation profiles obtained with ECLIPSE and the modified MRST model, when simulating a polymer flood using 100 grid blocks. The modification to the Todd and Longstaff parameter reduces smearing of the leading polymer front.

The polymer slug size is 0.5 PVI. We also compare these two predictions with those obtained from the modified MRST using the Todd and Longstaff model with $\omega = 0.67$.

Figure 7.8 shows that there is very good agreement between the three approaches suggesting that the assumptions made in the derivation of the analytical model are appropriate. In particular, the time when the finger tips reach the polymer front is successfully predicted analytically.

### 7.4 Minimum slug size computation

We now show how the analytical model can be used to calculate the minimum polymer slug size that will maintain its integrity between injection and production wells even if there is fingering of the chase water into the trailing edge of the slug.
7.4. Minimum slug size computation

Figure 7.8: Comparison between FCM simulator, MRST and analytical model with $\omega = 0.67$, showing that the new analytical model can predict the development of the viscous fingers into the trailing edge of the slug.
Figure 7.9 shows the evolution of the fingering for $t_{d,\text{slug}} = 0.3$ PVI on an $x_d - t_d$ plot. The red line shows the evolution of the leading edge of the polymer slug whilst the solid black line shows the evolution of the trailing edge using the analytical solution derived in Section 7.2.1. The dashed black line shows the motion of the trailing edge if the water saturation in the polymer slug is constant and equal to the leading shock front saturation. The upper blue line shows the motion of the finger tips using the Todd and Longstaff model, assuming $\omega = 0.67$. The lower blue line shows the motion of the trailing edge of the fingering. Clearly, with $t_{d,\text{slug}} = 0.3$ PVI, we expect the slug will start to breakdown in the middle of the reservoir at around $t_d = 0.41$ PVI.

![Figure 7.9: Optimal slug size estimation for polymer-water viscosity ratio of 50.](image)

The smallest slug size to minimize the breakdown of the slug by fingering should be selected such that the finger tips reach the production well ($x_d = 1$) at the same
Minimum slug size computation

Time as the polymer front. This can be graphically determined by drawing a line with a gradient of \( v_{\text{finger tips}} \) given in (7.19) that meets \( x_2 \) at \( x_d = 1 \). The green line in Figure 7.9 illustrates this. Then, \( t_{d,\text{slug,min}} \) can be directly determined from the value of the \( x \)-intercept. Assuming \( \omega = 0.67 \), we found that \( t_{d,\text{slug,min}} = 0.52 \) PVI.

To verify this, we ran the FCM simulation for \( t_{d,\text{slug}} \) of 0.3 PVI, 0.52 PVI and 0.65 PVI. The results are shown in Figure 7.10. As expected, \( t_{d,\text{slug}} = 0.3 \) PVI is too small and viscous fingering of the chase water causes the slug to break down in the middle of the reservoir. This is reflected in the oil recovery plot in Figure 7.10(d) which indicates that the polymer breakthrough at the production well occurs rather early, at around \( t_d = 0.5 \) PVI. We also observe that the recovery is less efficient as the increase in oil is very gradual post polymer breakthrough. The recoveries obtained for \( t_{d,\text{slug}} = 0.65 \) PVI and \( t_{d,\text{slug}} = 0.52 \) PVI are found to be almost identical, which means that, in this case, the minimum slug size needed to ensure fingering of the chase water does not destroy the slug is \( t_{d,\text{slug}} = 0.52 \) PVI.

The minimum slug size needed to ensure fingering of the chase water does not completely break down the polymer slug for various values of \( \omega \) and viscosity ratios are shown in Figure 7.11. We can see here the expected trend on how high oil viscosities or low \( \omega \) values necessitate the use of larger slug size. Generally in polymer flooding, polymer viscosity needs to be high enough so that the oil can be swept more efficiently. However, interestingly, too high a polymer-oil viscosity ratio, \( \mu_p/\mu_o \), has an adverse effect instead – a large slug size is required as the chase water tends to finger more through the polymer slug as we can see in Figure 7.11(b).

Although we need to consider many parameters to calculate the required slug size, for moderate values of \( \omega \) we typically encountered say around 0.4 to 0.8, Figure 7.11 suggests that \( t_{d,\text{slug,min}} \) of around 0.5 to 0.6 PVI is a good first approximation that may be used in the field.
Figure 7.10: Oil recovery for different slug sizes. (a) A slug size of 0.3 PVI is too small as fingering of the chase results in the breakdown of the polymer slug before the slug reaches the production well. (b) The fingering pattern seen at the same time for a 0.52 PVI slug. The fingers have just reached the leading edge of the slug as the polymer front reaches the production well. (c) Very little fingering is seen for a slug size of 0.65 PVI. (d) The recovery curves obtained from the 0.52 PVI and 0.65 PVI slugs are virtually identical whereas recovery is reduced for the 0.3 PVI slug. The minimum slug size that will not be destroyed by viscous fingering at polymer breakthrough is 0.52 PVI.

Figure 7.11: Optimum slug size as a function $\omega$ and viscosity ratios. All data used in this calculation is from Table 7.1, using Eq. 7.14.
7.5 Conclusions

We have investigated the fingering of chase water into a polymer slug during secondary polymer flooding in the absence of adsorption using a mixture of numerical simulations and analytical approaches. Both detailed numerical simulations, describing the fingers explicitly, and black oil simulations using a Todd and Longstaff model to represent the average effects of the fingering were performed. We have reviewed the existing analytical solutions that predict both continuous polymer flooding and a slug of polymer followed by chase water in the absence of fingering.

Existing solutions describing the injection of chase water following a polymer slug assume a stable interface between chase water and polymer slug. The predictions of the analytical solution first presented by Bedrikovetsky (1993) were compared successfully against numerical simulation. We showed analytically that at late times the water saturation throughout the polymer slug tends to the saturation of the leading shock of the polymer slug.

We have extended this analysis to obtain an approximate method for predicting the growth of the fingering of chase water into the polymer slug due to fingering, in the absence of adsorption. This is an original work where we proposed a simple analytical expression that can be used to estimate the minimum polymer slug size needed to ensure that it is not destroyed by fingering of the chase water before polymer breakthrough.

Comparison of detailed numerical simulations of the fingering of chase water with black oil simulations using the Todd and Longstaff model to represent the average effects of the fingering have shown that a value of $\omega = 0.67$ can be used, for oil-water viscosity ratios of 10 and 50. However, due to the empirical nature of the Todd and Longstaff model, this value may not work for cases with heterogeneous reservoirs, very high viscosity ratios or polymer-oil viscosity ratios that are less than 1. We
expect calibration of $\omega$ is required for more general problems. When $\omega$ is provided, the optimum slug size can be rapidly determined using the approximate analytical solution. Such results can potentially be used during EOR screening or feasibility studies, during which only order of magnitude estimations are required and more accurate but highly expensive computational study may not be necessary.

We expect that the analytical solution presented can be extended to include the effects of adsorption and possibly to estimate flow behaviour when polymer is injected subsequent to a waterflood.
Chapter 8

Conclusions and future work

Perhaps it is not surprising that even after more than 70 years since it was formally studied by Hill (1952) and Saffman and Taylor (1958), our understanding of viscous fingering is still not complete. We have seen that miscible viscous fingering exhibits a highly non-linear behaviour with a strong dependence on viscosity contrast and dispersion. This is further complicate by the fact that reservoirs are not homogeneous – the presence of permeability heterogeneity may significantly alter the growth of the fingers.

This thesis pushes the research of miscible viscous fingering in three directions, (a) a better fundamental understanding of the non-linear growth of viscous fingering, (b) the influences imposed by various numerical solvers, and (c) the application of empirical averaged models to provide fast estimations of unstable flows in heterogeneous reservoirs and fingering in polymer slug injection.

As viscous fingering is highly dependent on dispersion, we started our investigations in the first half of Chapter 4 by quantifying the apparent longitudinal and transverse diffusion due to truncation errors in various numerical schemes, the influence of types of mesh, and grid orientation. This was followed by a brief look on the impacts of
these numerical schemes and meshes on the growth of the immiscible fingers.

We focused on the physics of miscible fingering in Chapter 5 by exploring the very early time to late time regimes. We singled out and examined the influence of longitudinal and transverse components of diffusion as well as the viscosity contrast: first by examining the linear stability analysis, then using numerical simulations. We tracked the life cycle of the fingers from their initial inception, growth and merging and proposed how the scaling of the fingers can be made for any given values of mobility contrast and anisotropic dispersion. In this chapter we used a homogeneous reservoir domain with a standard first contact miscible formulation which means the results are directly applicable to other applications such as chromatography.

Chapter 6 dealt with the modelling of viscous fingering in heterogeneous reservoirs for engineering purposes. We were able to identify channelled reservoirs using the flow conductivity indicator and proposed a simple modification and tuning of the Koval model to capture the unswept oil in the low permeability regions.

In Chapter 7, we incorporated the Todd and Longstaff model to the existing analytical solution of polymer slug flow to capture the fingering at the rear of the slug. This allowed us to track the average movement of the fingers and rapidly estimate the slug size required to avoid the polymer front from being invaded by the fingers.

8.1 Summary of key findings

1. For miscible displacements a Taylor series analysis of truncation error can provide a good estimate of the longitudinal numerical diffusivity in finite difference and finite volume schemes. It seems that the transverse numerical diffusivity is typically much smaller than the longitudinal numerical diffusivity.
2. The comparison between three simulators (first order finite volume, high order finite difference, control-volume finite-element methods) used to simulate immiscible displacements showed similar unstable growth at early time period. Apart from the computational speeds (which we did not compare) and the number of meshes required, there seems to be no clear reason to choose one simulator over another.

3. Using transverse Peclet number $\text{Pe}_T$ in the linear stability analysis, we highlighted the importance of transverse diffusion in controlling the number of fingers at early time as well as other non-linear events at intermediate to late time regimes such as coalescence, tip-splitting and side branching.

4. The number of fingers and the time scale over which they interact, split and merge scale with the most dangerous wavenumber $k_m$ and the growth rate of that wavenumber $\omega_{gr,m}$ determined from early time growth analysis (Tan and Homsy, 1986)

$$\omega_{gr} = \frac{1}{2} \text{Pe}_T^2 k \left[ \frac{R}{\text{Pe}_T \varepsilon} - k - \sqrt{k^2 + 2 \frac{R}{\text{Pe}_T \varepsilon} k + 2(1 - \varepsilon)k} \right]$$

This scaling was found to work well even for anisotropic diffusion.

5. Using this scaling, we propose that the occurrence of late time regime can be estimated as

$$t \sim \frac{(k_m/2\pi)^2 H^2}{\omega_{gr,m} D_T}$$

For isotropic diffusion, this is reduced to

$$t \sim 0.016 \frac{H^2}{D_T}$$

This estimation can be used to predict the flow regime in laboratory experiments (e.g. the results from Gardner and Ypma, 1984 shown in Chapter
5) or the flow regime in EOR miscible displacements.

6. The mixing length or spreading of the fingered region is controlled by longitudinal diffusion at very early time, and transverse diffusion at intermediate to late time regimes. This suggests that tuning or calibration of the empirical models (such as Todd and Longstaff, 1972) would depend on the magnitude of transverse diffusion.

7. The analytical method originally proposed by Stiles can be used to roughly estimate the solvent breakthrough time in non-channelised reservoirs. This allows the heterogeneity index $H_k$ for the Koval model to be tuned rapidly. However, its prediction of the breakthrough time was found to be less accurate compared to other methods that rely on numerical simulation (e.g. the vorticity method by Rashid et al., 2012). Nevertheless, it is useful in providing a quick estimation that may be used as a baseline result or diagnostic tool.

8. Using $K_{\text{eff}}/K_{\text{geo}}$ allows us to quickly identify channelised reservoirs, which can then be modelled using a modified Koval model to take into account the low permeability regions. We showed that miscible displacements in channelised reservoirs are governed mainly by the high permeability channels.

9. The introduction of the Todd and Longstaff mixing parameter $\omega > 0$ to modify the water-polymer mixture viscosity results in highly numerically dispersed fronts. We showed that by selectively applying $\omega$, we were able to maintain sharp fronts while still capturing the fingering effects at the back of the slug.

10. A good agreement was obtained between our analytical model of polymer slug fingering, black oil simulation and high resolution FCM simulation. This analytical solution can potentially be used during EOR screening or feasibility studies to estimate the polymer slug size required.
8.2. Recommendations for further research

The scope of this thesis was limited to viscous fingering in a density-matched, first contact miscible flow with a 2D linear displacement. There are many potential extensions of this thesis including:

1. In Chapter 4, we saw that the grid orientation has a significant impact not only on the value of numerical dispersion produced, but also on the fingering patterns. In a quarter five spot simulation, the flow is not parallel to the grid or mesh unlike in the linear displacements studied in this thesis. It would be interesting to see how the triggering of the fingers and their subsequent evolution will vary in numerical simulations in a quarter five spot geometry and whether the scaling can be approximated using the linear stability theory as found in Chapter 5. It is worth noting that the linear stability analysis we used in this thesis will no longer be valid as they were derived based on a linear displacement.

2. To simply our analyses, we assumed in Chapter 5 that the velocity-induced dispersion can be represented by molecular diffusion. A natural extension is to include the former in the simulations. This is likely to have further implications on the growth of the fingers, but we do not anticipate it to be significantly different from our current approach of using diffusion as a proxy for dispersion.

3. The inclusion of fluids with a density contrast will further complicated the growth of the fingers as the viscous force will now compete with gravity. In a gravity-dominated flow, a gravity tongue may form and it may alter or completely override the fingering patterns we have observed in this thesis. There is a possibility of axial movements of the fingers due gravity, and such an effect requires a 3D simulation.
4. Throughout the thesis we only used a 2D domain. Some researchers (Zimmerman and Homsy, 1992; Christie et al., 1993) have argued that there is little difference especially for linear case, between 2D and 3D domain. However, as we have mentioned, the transition between gravity to viscous fingering dominated flow would require 3D simulation (Tchelipi and Orr, 1994). Riaz and Tchelipi (2004) also showed that the number of fingers observed at early time are different in 2D and 3D cases, hence we suggest to extend our study into 3D domain.

5. One of the challenges moving from 2D to 3D simulation is the significant increase in computational power required. Some of the simulations runs in this study took several days to complete on a modern workstation. Scalibility of the existing FCM simulator is weak as it was not written to be solved on multi-core central processing units (CPU). To extend similar studies performed in this thesis for 3D will require a more efficient approach. Apart from parallelisation on multi-core CPU, one can choose to port the current code to allow it to be solved in a graphical processing unit (GPU) which has been shown to be able to handle very large matrix operations more efficiently compared to normal CPUs.

6. We studied the impact of permeability channelling on macroscopic recovery efficiency, but did not explore the combined impact of heterogeneity and dispersion on the formation and growth of small scale fingers and possible crossflow between and low permeability regions. This may have further implications on the recovery which may require further analysis.

7. Immiscible flows with a very high viscosity ratio are becoming increasingly important as more heavy oil reservoirs are being exploited. Many studies of immiscible fingering rely on Hele-Shaw cells which result in a sharp interface between the fluids. This is, to a certain degree, representative of many linear...
8.2. **Recommendations for further research**

stability analyses including that of Saffman and Taylor (1958). While there is a large body of work in miscible displacement, fingering regimes in immiscible displacements remain relatively under-researched especially for realistic porous medium with shock fronts and rarefactions between the two fluids. The tools and workflow we developed in this thesis (such as the algorithm to track the number of fingers) can be directly used in immiscible fingering with little or no modifications required.

8. A good understanding of the growth of immiscible fingering is also useful in polymer displacement. As the viscosity of the injected polymer may be less than that of oil, we expect immiscible-type fingering between polymer front and the resident oil, which may be also be modelled using empirical models.

9. Further analytical studies of fingering in polymer slug flow are suggested. The analysis presented in Chapter 7 could be extended to include other physical effects in polymer displacements such as adsorption. Such analysis would not only beneficial as it would allow us to quickly predict the displacements, it would also be important to understand and verify similar cases solved using numerical solvers.
Bibliography


