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$U(1)$–Extended Gauge Algebras in p-Loop Space

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ABSTRACT

We consider, for p odd, a p–brane coupled to a $(p+1)$th rank background antisymmetric tensor field and to background Yang-Mills (YM) fields *via* a Wess-Zumino term. We obtain the generators of antisymmetric tensor and Yang-Mills gauge transformations acting on p–brane wavefunctionals (functions on 'p-loop space'). The Yang-Mills generators do not form a closed algebra by themselves; instead, the algebra of Yang-Mills and antisymmetric tensor generators is a $U(1)$ extension of the usual algebra of Yang-Mills gauge transformations. We construct the p-brane's Hamiltonian and thereby find gauge-covariant functional derivatives acting on p–brane wavefunctionals that commute with the YM and $U(1)$ generators.

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1. Introduction

In many supergravity theories the graviton supermultiplet includes a $(p+1)$-form gauge potential that couples naturally to a p-dimensional extended object, *i.e.* a p–brane. Two examples of interest are a string coupled to ten-dimensional ($d = 10$) supergravity in the 2-form formulation and a fivebrane coupled to $d = 10$ supergravity in the 6-form formulation. A feature of the two-form formulation is that the two-form potential acquires a non-trivial Yang-Mills (YM) transformation when YM fields are included [1]. Although there is no analogous 'anomalous' YM variation of the six-form in the six-form formulation of *classical* supergravity/YM theory, such a variation is required for anomaly cancellation in the quantum theory [2]. In both formulations, therefore, one finds that the YM algebra is modified in the sense that a commutator of two YM transformations yields not only another YM transformation but also an antisymmetric tensor gauge transformation. One would expect that, upon quantization of the p–brane in the YM and antisymmetric tensor background, this modified algebra should be realized in terms of functional differential operators acting on the p–brane wavefunctional. This is a function on p-loop space, *i.e.* the space of maps of the p–brane to spacetime. Since the antisymmetric tensor transformation of the $(p+1)$-form on spacetime is equivalent to a $U(1)$ transformation of an associated one-form on p-loop space [3], one expects the modified algebra to be a $U(1)$ extension of the algebra of YM gauge transformations.

To investigate this point one first needs an action for the p–brane coupled to these background fields. For p *odd*, which includes the $p = 1$ and $p = 5$ cases under discussion, such an action has been proposed for the bosonic p–brane [4]. This action describes a p–brane propagating in a curved background locally diffeomorphic to $M \times G$, where M is spacetime and G is a group manifold, and includes a Wess-Zumino (WZ) term. (The coupling to target spacetime gauge fields considered here is analogous to, but should not be directly confused with, the gauging of Wess-Zumino terms *via* world-volume gauge fields as considered in [5].) For $p = 1$ this action reduces to the bosonic sector of an action for the heterotic string appearing in earlier work [6]. More recently, a set of YM generators acting on string wavefunctionals was deduced from this action and shown to satisfy an affine Kač-Moody algebra [7]. The central charge appearing in this algebra can be interpreted as the eigenvalue of the $U(1)$ generator associated with the antisymmetric tensor gauge transformation [8]. One purpose of this paper is to obtain the analogous results for all odd p (partial results have appeared [9] during the course of writing this paper). Our method is also novel; from a path integral representation of the p–brane wavefunctional Ψ, together with a careful treatment of boundary terms, we find that Ψ satisfies the conditions

$$G_\epsilon \Psi = 0 \qquad G_\Lambda \Psi = 0, \tag{1.1}$$

with specific operators G_ϵ and G_Λ. These operators are not purely operators on p-loop space because they also contain functional derivatives with respect to the background gauge fields; consequently, eqs. (1.1) do not constrain Ψ for given fixed background fields but simply determine the response of the wavefunctional to a gauge transformation (if the background fields were treated as dynamical variables, eqs. (1.1) could be interpreted as continuity equations.). The consistency of these equations follows from the fact that the operators G_ϵ and G_Λ form a *closed* algebra, for which the only non-zero commutator is

$$[G_{\epsilon_1}, G_{\epsilon_2}] = G_{[\epsilon_1,\epsilon_2]} - G_\Lambda \tag{1.2}$$

where

$$\Lambda = k_p \omega_p^2(A, \epsilon_1, \epsilon_2) \tag{1.3}$$

is a p–form 2-cocycle of the algebra of YM gauge transformations and k_p is a normalisation constant. For $p = 1, 3, 5$, which are the principal cases of interest (see section 3 for details of the notation),

$$\begin{aligned}
\omega_1^2(A, \epsilon_1, \epsilon_2) &= -\,2\mathrm{tr}\,\epsilon_1 d\epsilon_2 \\
\omega_3^2(A, \epsilon_1, \epsilon_2) &= -\,\mathrm{tr}\,\{d\epsilon_1, d\epsilon_2\}A \\
\omega_5^2(A, \epsilon_1, \epsilon_2) &= \frac{1}{15}\mathrm{tr}\,(5F - 3A^2)\,[2A\{d\epsilon_1, d\epsilon_2\} - d\epsilon_1 A d\epsilon_2 + d\epsilon_2 A d\epsilon_1]\ .
\end{aligned} \tag{1.4}$$

A special feature of the string is that in this case the cocycle (1.3) is background-field independent, so the algebra defined by (1.2) is a Lie algebra. For $p > 1$ the structure 'constants' of the algebra are background-field dependent.

Note that these results are similar to those found from the analysis of chiral anomalies in the Hamiltonian formalism of $(p+1)$-dimensional gauge theories [10, 11], but there are several differences. One is that here the gauge fields are not dynamical so the anomalies in question are of 'sigma-model' type. A more significant, but related, difference is the presence of the $(p+1)^{\mathrm{th}}$ rank antisymmetric tensor and its 'anomalous' YM transformation (without which there would be no background gauge invariance of the string action; *cf.* the sigma-model anomaly cancellation in the fermionic formulation of the heterotic string [12]). With the antisymmetric tensor field, the algebra is anomaly-free, despite the central extension, in the sense that equations (1.1) do not imply a vanishing wavefunctional.

Since our p-brane action is worldvolume reparametrization invariant, its canonical Hamiltonian is a sum of constraints (*cf.* General Relativity). The 'Hamiltonian' constraint function associated with time reparametrizations is a quadratic function of the momenta conjugate to the worldvolume fields x^μ and y^m (these being maps from the worldvolume to M and G respectively). The invariance of the Hamiltonian under background field transformations is a consequence of the invariance of particular linear combinations of the momenta that become, upon quantization, functional YM and $U(1)$ 'covariant' derivatives acting on p-brane wavefunctionals. There are *two* covariant derivatives, $\mathcal{D}_\mu$ and $\mathcal{D}_a \equiv L_a{}^m \mathcal{D}_m$, corresponding to covariant differentiation with respect to x^μ or to y^m, respectively ($L_a{}^m$ are the components of the left-invariant Killing vectors on G). The first of these was introduced for the string in Ref. [7], where it was used to derive dynamical equations for the background fields *via* the principle of lightlike integrability. Existence arguments for $\mathcal{D}_\mu$ in the general case have also been given [13]. From the point of view of this paper the covariant derivatives are functional differential operators on p–loop space satisfying

$$\begin{aligned}
&[\mathcal{D}_\mu, G_\Lambda] = 0 \qquad [\mathcal{D}_\mu, G_\epsilon] = 0 \\
&[\mathcal{D}_a, G_\Lambda] = 0 \qquad [\mathcal{D}_a, G_\epsilon] = \epsilon^b f^d{}_{ab} \mathcal{D}_d\ .
\end{aligned} \tag{1.5}$$

Another result of this paper is the construction of these covariant derivatives for all odd p *via* the construction of the Hamiltonian for a p-brane in a YM and antisymmetric tensor background.

In section 2 we shall begin our presentation with an explanation of the method used. For this purpose we shall consider in detail the special cases of the string (recovering previous results) and the three-brane (since this is the simplest illustration of the new features that occur beyond $p = 1$). We then proceed, in section 3, to a discussion of the general case and in section 4 we construct the p-brane Hamiltonian and find the covariant derivatives.

2. Strings and Three-branes

We begin with the $p = 1$ case, *i.e.* a closed string moving on $M \times G$. We first introduce some notation. Let x^μ and y^m be local coordinates on M and G respectively and let $A_\mu(x) = A^a_\mu T_a$ and $B_{\mu\nu}(x)$ be background YM and tensor gauge fields on M. Let us denote by $L_a = L_a{}^m(y)\partial_m$ the left-invariant vector fields on G; they satisfy $[L_a, L_b] = L_c f^c{}_{ab}$ where $f^c{}_{ab}$ are the structure constants of G. We shall also need a background Riemannian metric $g_{\mu\nu}$ on M and an invariant Riemannian metric $g_{mn} = L_m{}^a L_n{}^b d_{ab}$ on G, where $d_{ab} = \mathrm{tr}(T_a T_b)$ is a multiple of the Cartan-Killing inner product on the Lie algebra of G. We also introduce a potential b_{mn} on G chosen to satisfy the relation $3\partial_{[m} b_{np]} = f_{abc} L_m{}^a L_n{}^b L_p{}^c$, where $f_{abc} = d_{ad} f^d{}_{bc}$ and it is to be understood in what follows that all lowering and raising of the group indices will be done with d_{ab} and its inverse.

Introducing local worldsheet coordinates $\xi^i = (\tau, \sigma)$ and an (independent) worldsheet metric $\gamma_{ij}(\xi)$ (with inverse γ^{ij}), we can now write the string action as

$$S = \int d^2\xi \Big[-\tfrac{1}{2}\sqrt{-\gamma}\gamma^{ij} \Big(\partial_i x^\mu \partial_j x^\nu g_{\mu\nu} + (D_i y)^m (D_j y)^n g_{mn}\Big) + \epsilon^{ij}\left(\tfrac{1}{2}B_{ij} - k_1 L^a_i A_{ja} - \tfrac{1}{2}k_1 b_{ij}\right)\Big] \tag{2.1}$$

where B_{ij}, b_{ij}, L^a_i and A^a_i are the pullbacks to the worldsheet of $B_{\mu\nu}$, b_{mn}, L^a_m and A^a_μ, respectively, $\gamma = \det \gamma_{ij}$, and $(D_i y)^m \equiv \partial_i y^m - \partial_i x^\mu A^a_\mu L^m_a$ is a YM-covariant derivative. The coefficient k_1 is a normalisation constant; in order that $\exp(-\frac{1}{2} i k_1 \int \epsilon^{ij} b_{ij})$ be well-defined in the quantum theory, the coefficient k_1 is restricted to be an integer multiple of some numerical factor, whose precise value will not concern us here. The last two terms in (2.1) are not YM invariant by themselves, but their variation can be cancelled, *up to a total derivative*, by an anomalous variation of $B_{\mu\nu}$. Specifically, under YM and antisymmetric tensor gauge transformations the fields transform as

$$\begin{aligned} \delta x^\mu &= 0 \ , \\ \delta y^m &= \epsilon^a(x) L_a{}^m(y) \ , \\ \delta A^a_\mu &= \partial_\mu \epsilon^a + f^a{}_{bc} A^b_\mu \epsilon^c = (D_\mu \epsilon)^a \ , \\ \delta B_{\mu\nu} &= -2k_1 A^a_{[\mu}\partial_{\nu]}\epsilon_a + 2\partial_{[\mu}\Lambda_{\nu]} \ . \end{aligned} \tag{2.2}$$

The variation of the action (2.1) under these transformations is

$$
\begin{aligned}
\delta S &= \int d^2\xi\, \varepsilon^{ij}\partial_i\big[k_1\epsilon^a\partial_j y^m L_m{}^b(d_{ab}-b_{ab}) + \partial_j x^\mu \Lambda_\mu\big] \\
&= \int d\tau\partial_\tau \oint d\sigma\big[k_1\epsilon^a\partial_\sigma y^m L_m{}^b(d_{ab}-b_{ab}) + \partial_\sigma x^\mu \Lambda_\mu\big] \ ,
\end{aligned}
\tag{2.3}
$$

where the second line follows from the fact that a closed string has no boundary. We remark that the inclusion of the WZ term in the action, with the consequent complications, is not obligatory from a worldsheet point of view, but it *is* required for the background gauge field transformations needed for invariance of the action to coincide with those known from the D-dimensional supergravity/YM theory, at least for the particular cases (D=10, p=1,5) of most interest to us here.

Our aim now is to determine the transformation properties of the string wave-functional. Let us suppose the world-sheet to be a two-manifold with an S^1 boundary component representing a closed string at a given time, and consider the string wave-functional

$$
\Psi[x,y;A,B] = \int^{x,y} [dX][dY] e^{iS[X,Y;A,B]} \ , \tag{2.4}
$$

where the arguments $\big(x^\mu(\sigma), y^m(\sigma)\big)$ are the boundary values of the integration variables $\big(X^\mu(\tau,\sigma), Y^m(\tau,\sigma)\big)$. In the spirit of the 'no boundary' proposal [14] of quantum cosmology, we can avoid having to deal with boundary conditions at an earlier time by supposing that there is no such boundary. The consistency of this viewpoint requires that iS be replaced by minus the Euclidean action, obtained by analytic continuation of the worldsheet metric from Lorentzian to Euclidean signature, but the relevant terms in the action are metric-independent and therefore unaffected by the difference of signature. Now, assuming an invariant path-integral measure and keeping only first-order variations,

$$
\begin{aligned}
\Psi[x,y+\delta y;A+\delta A,B+\delta B] &= \int^{x,y+\delta y} [dX][dY] e^{iS[X,Y;A+\delta A,B+\delta B]} \\
&= \int^{x,y} [dX][dY] e^{iS[X,Y+\delta Y;A+\delta A,B+\delta B]} \\
&= \int^{x,y} [dX][dY] e^{i(S+\delta S)} \\
&= (1+i\delta S)\Psi[x,y;A,B] \ ,
\end{aligned}
\tag{2.5}
$$

where the last line follows from the asssumption that δS is a surface term and therefore independent of the integration variables. The assumption of the invariance of the measure is justified for the bosonic string considered here. If we were dealing with the formulation in which the YM fields couple to the worldsheet *via* heterotic fermions then the surface term would arise not from the variation of the classical action but from the non-invariance of the measure; however, the final result would be the same. Expanding the left-hand-side of (2.5) to first order in variations we deduce that Ψ satisfies the functional differential equation

$$\begin{aligned}\Big\{ \int d^D x' \Big[\delta A^a_\mu(x') \frac{\delta}{\delta A^a_\mu(x')} + \delta B_{\mu\nu}(x') \frac{\delta}{\delta B_{\mu\nu}(x')}\Big] \\ + \oint d\sigma \delta y^m(\sigma) \frac{\delta}{\delta y^m(\sigma)} \ - i\ \delta S(x,y) \Big\} \Psi[x,y;A,B] = 0\ .\end{aligned} \tag{2.6}$$

where D is the dimension of M. The functional derivatives here, and henceforth, are defined to be densities; for example, $\delta y^n(\sigma')/\delta y^m(\sigma) = \delta^n_m \delta(\sigma - \sigma')$ where the delta function is a density. Substituting the particular variations of (2.2) into (2.6) and taking into account the independence of ϵ^a and Λ, we find that

$$G_\epsilon \Psi(x,y;A,B) = 0 \qquad G_\Lambda \Psi(x,y;A,B) = 0 \tag{2.7}$$

where

$$\begin{aligned} G_\epsilon = \int d^D x' \Big\{ \big(D_\mu \epsilon(x')\big)^a \frac{\delta}{\delta A^a_\mu(x')} - 2k_1 \big(\partial_\nu \epsilon(x')\big)_a A^a_\mu(x') \frac{\delta}{\delta B_{\mu\nu}(x')} \Big\} \\ + \oint d\sigma \epsilon^a\big(x(\sigma)\big) D_a(\sigma) \end{aligned} \tag{2.8a}$$

$$G_\Lambda = \int d^D x'\ 2\partial_\mu \Lambda_\nu(x') \frac{\delta}{\delta B_{\mu\nu}(x')} \ - i \oint d\sigma\ \partial_\sigma x^\mu(\sigma) \Lambda_\mu\big(x(\sigma)\big) \tag{2.8b}$$

with $D_a(\sigma)$ given by

$$D_a(\sigma) = L_a{}^m \frac{\delta}{\delta y^m(\sigma)} - i k_1 \partial_\sigma y^m L_m{}^b (d_{ab} - b_{ab})\ . \tag{2.9}$$

Eqs. (2.7) state that the wavefunctional (2.4) is invariant under the transformations (2.2) up to a phase. Although this result was found for a particular wavefunctional, its general validity is clearly required for physical quantities to be gauge-independent.

The computation of the algebra of the generators of eqs. (2.8) is greatly simplified by the fact that the operators $D_a(\sigma)$ are background-field independent. As shown in [7], these operators generate an affine Kač-Moody algebra with a central extension. Using this result, we find that all commutators of the complete generators vanish except for (1.2) with $[\epsilon_1, \epsilon_2] \equiv f^a{}_{bc} \epsilon^b_1 \epsilon^c_2 T_a$ and $\Lambda_\mu = k_1 \epsilon^a_2 \overleftrightarrow{\partial}_\mu \epsilon_{1a}$.

Before proceeding to the general odd-p case, we shall discuss the closed three-brane as this provides the simplest illustration of the complications that arise beyond $p = 1$. The action of [4] describing the coupling of a three-brane to YM fields exists only for those groups for which there is a third order symmetric invariant tensor d_{abc} (which satisfies the identity $f^d{}_{e(a} d_{bc)d} = 0$); a simple example would be $SU(3)$. Introducing worldvolume coordinates $\xi = (\tau; \sigma^r, r = 1,2,3)$, we can write the action as

$$S = \int d^4\xi \Big\{ -\tfrac{1}{2}\sqrt{-\gamma}\gamma^{ij}\left[\partial_i x^\mu \partial_j x^\nu g_{\mu\nu} + d_{ab}(D_i y)^a (D_j y)^b\right] + \sqrt{-\gamma}$$
$$- \frac{k_3}{8}\varepsilon^{ijkl} d_{abc}\Big[f^a{}_{de}\Big(L_i{}^b A_j^c A_k^d A_l^e - L_i{}^c L_j{}^d A_k^b A_l^e - L_i{}^c L_j{}^d L_k{}^e A_l^b \Big) + 4 L_i{}^a \partial_j A_k^b A_l^c \Big]$$
$$+ \frac{1}{24}\varepsilon^{ijkl}\Big(B_{ijkl} - k_3 b_{ijkl}\Big)\Big\} \ , \tag{2.15}$$

where k_3 is a normalisation constant, and B_{ijkl} and b_{ijkl} are the pull-backs of $B_{\mu\nu\rho\sigma}$ and b_{mnpq} respectively, the latter being chosen to satisfy

$$5\partial_{[m} b_{npqr]} = -\frac{3}{2} d_{st[m} f^s{}_{np} f^t{}_{qr]} \ . \tag{2.16}$$

The YM and antisymmetric tensor gauge transformations are now

$$\begin{aligned} \delta y^m &= \epsilon^a(x) L_a{}^m(y) \\ \delta A_\mu^a &= (D_\mu \epsilon)^a \\ \delta B_{\mu\nu\rho\sigma} &= -12 k_3 d_{abc}\Big(A^a_{[\mu}\partial_\nu A^b_\rho + \frac{1}{4} f^a{}_{de} A^d_{[\mu} A^e_\nu A^b_\rho \Big)\partial_{\sigma]}\epsilon^c + 4\partial_{[\mu}\Lambda_{\nu\rho\sigma]} \ . \end{aligned} \tag{2.17}$$

The variation of the action under these transformations is

$$\begin{aligned} \delta S = \int d^4\xi\, \varepsilon^{ijkl}\partial_i \Big\{ &-\frac{1}{6} k_3 \epsilon^a \Big[L_j{}^b L_k{}^c L_l{}^d (b_{abcd} + \frac{3}{4} d_{eab} f^e{}_{cd}) + 3 d_{abc}\partial_l (L_j{}^b A_k^c)\Big] \\ &+ \frac{1}{6}\partial_j x^\mu \partial_k x^\nu \partial_l x^\rho \Lambda_{\mu\nu\rho}\Big\} \ . \end{aligned} \tag{2.18}$$

Since the three-brane has no boundary, this surface term can be written as

$$\begin{aligned} \delta S = \int d\tau \partial_\tau \oint d^3\sigma\, \varepsilon^{rst}\Big\{ &-\frac{1}{6} k_3 \epsilon^a \big[L_r{}^b L_s{}^c L_t{}^d (b_{abcd} + \frac{3}{4} d_{eab} f^e{}_{cd}) + 3 d_{abc}\partial_t (L_r{}^b A_s^c)\big] \\ &+ \frac{1}{6}\partial_r x^\mu \partial_s x^\nu \partial_t x^\rho \Lambda_{\mu\nu\rho}\Big\} \ . \end{aligned} \tag{2.19}$$

This result leads, by the same reasoning as before, to the generators

$$G_\epsilon = \int d^D x' \Big\{ \big(D_\mu \epsilon(x')\big)^a \frac{\delta}{\delta A_\mu^a(x')} - 12 k_3 d_{abc}\big(A^a_\mu \partial_\nu A^b_\rho + \frac{1}{4} f^a{}_{de} A^d_\mu A^e_\nu A^b_\rho \big)\big(\partial_\sigma \epsilon^c(x')\big)\frac{\delta}{\delta B_{\mu\nu\rho\sigma}(x')}\Big\}$$
$$+ \oint d^3\sigma\ \epsilon^a\big(x(\boldsymbol{\sigma})\big) D_a(\boldsymbol{\sigma}) \tag{2.20a}$$

$$G_\Lambda = \int d^D x'\, 4\partial_\mu \Lambda_{\nu\rho\sigma}(x')\frac{\delta}{\delta B_{\mu\nu\rho\sigma}(x')} - i\oint d^3\sigma\ \varepsilon^{rst}\partial_r x^\mu \partial_s x^\nu \partial_t x^\rho \Lambda_{\mu\nu\rho}\big(x(\boldsymbol{\sigma})\big) \tag{2.20b}$$

where $\boldsymbol{\sigma} = \{\sigma^r\}$, and

$$D_a(\boldsymbol{\sigma}) = L_a{}^m \frac{\delta}{\delta y^m(\boldsymbol{\sigma})} + \frac{i k_3}{6}\varepsilon^{rst}\big[L_r{}^b L_s{}^c L_t{}^d (b_{abcd} + \frac{3}{4} d_{eab} f^e{}_{cd}) + 3 d_{abc}\partial_t (L_r{}^b A_s^c)\big] \tag{2.21}$$

Observe that, in contradistinction to the string case, the operators $D_a(\boldsymbol{\sigma})$ *depend on the background fields* so that their algebra differs from the algebra of the complete generators, and therefore has no obvious significance. However, the background-field dependence of $D_a(\boldsymbol{\sigma})$ is such that the algebra of the complete generators, G_ϵ and G_Λ, is the same as (1.2) with $[\epsilon_1, \epsilon_2] \equiv f^a{}_{bc}\epsilon_1^b\epsilon_2^c T_a$ and $\Lambda_{\mu\nu\rho} = k_3 d_{abc}\partial_{[\mu}\epsilon_1^a\partial_\nu\epsilon_2^b A^c_{\rho]}$. For a related discussion of the algebra of gauge transformations in the context of field-theoretic gauged Wess-Zumino terms, see Ref. [11].

3. The general odd-p case

We now turn to the case of a closed p–brane propagating in $M \times G$ for p odd but otherwise arbitrary. The background fields on M are the metric $g_{\mu\nu}(x)$, the antisymmetric tensor field $B_{\mu_1\dots\mu_{p+1}}(x)$, and the YM field $A_\mu = A_\mu^a(x)T_a$ valued in some representation of the Lie algebra of G. The background fields on G are now the left-invariant metric $g_{mn}(y)$ and the potential $b_{m_1\dots m_{p+1}}(y)$ satisfying

$$\partial_{[m_1}b_{m_2\dots m_{p+2}]} = -c_p(p+1)!\mathrm{tr}L_{[m_1}\dots L_{m_{p+2}]}\ , \tag{3.1}$$

where $L_m = L_m{}^aT_a$, and c_p is the constant $c_p = (-1)^{\frac{p+1}{2}}\left(\frac{p+3}{2}\right)\Gamma(\frac{p+3}{2})^2/\Gamma(p+3)$.

We may now write the action (for unit p-volume tension) as [4]

$$\begin{aligned} S = \int d^{p+1}\xi \Big\{ &-\tfrac{1}{2}\sqrt{-\gamma}\gamma^{ij}\left(\partial_i x^\mu\partial_j x^\nu g_{\mu\nu} + D_i y^m D_j y^n g_{mn}\right) + \frac{(p-1)}{2}\sqrt{-\gamma} \\ &+ \frac{1}{(p+1)!}\varepsilon^{i_1\dots i_{p+1}}\left[B_{i_1\dots i_{p+1}} + k_pC_{i_1\dots i_{p+1}} - k_pb_{i_1\dots i_{p+1}}\right]\Big\}\ , \end{aligned} \tag{3.2}$$

where k_p is a normalisation constant, $B_{i_1\dots i_{p+1}}$ and $b_{i_1\dots i_{p+1}}$ are the pullbacks to the worldvolume of the corresponding antisymmetric tensors on M and G, and $C_{i_1\dots i_{p+1}}$ are the components of the pullback of a $(p+1)$-form C_{p+1} on $M\times G$ that is constructed as follows. Let $A_t = tA + (1-t)L$ and $F_t = dA_t + A_t^2$, where $A = dx^\mu A_\mu$ and $L = dy^m L_m$. Defining the operator

$$\ell_t = dt(A^a - L^a)\frac{\partial}{\partial F_t^a}\ , \tag{3.3}$$

we have

$$C_{p+1}(A,F,L) = \int_0^1 \ell_t\ \omega^0_{p+2}(A_t,F_t)\ , \tag{3.4}$$

where ω^0_{p+2} is the Chern-Simons form defined by the relation

$$d\omega^0_{p+2} = \mathrm{tr}F^{\frac{p+3}{2}}. \tag{3.5}$$

The cases of most interest are $p = 1,3,5$, for which

$$\begin{aligned} \omega^0_3(A,F) &= \mathrm{tr}\Big(FA - \frac{1}{3}A^3\Big) \\ \omega^0_5(A,F) &= \mathrm{tr}\Big(F^2A - \frac{1}{2}FA^3 + \frac{1}{10}A^5\Big) \\ \omega^0_7(A,F) &= \mathrm{tr}\Big(F^3A - \frac{2}{5}F^2A^3 - \frac{1}{5}FAFA^2 + \frac{1}{5}FA^5 - \frac{1}{35}A^7\Big)\ , \end{aligned} \tag{3.6}$$

and substituting these expressions into (3.4) we find that

$$\begin{aligned}
C_2 =& \mathrm{tr}(AL)\\
C_4 =& \frac{1}{4}\mathrm{tr}\left[2(FA+AF-A^3)L+ALAL-2AL^3\right]\\
C_6 =& \frac{1}{30}\mathrm{tr}\big[(10F^2A+10FAF+10AF^2-8FA^3-8A^3F-4AFA^2-4A^2FA+6A^5)L\\
&+2F(A^2L^2-L^2A^2+3ALAL-3LALA)-6A^3LAL\\
&+3F(LAL^2-L^2AL+2L^3A-2AL^3)+6A^3L^3\\
&-3L^2A^2LA+3A^2L^2AL+2ALALAL+6L^3ALA+6AL^5\big]\ .
\end{aligned} \tag{3.7}$$

The three-brane action of (2.15) agrees with (3.2) if the identification $d_{abc} = \mathrm{tr}\big(T_a\{T_b, T_c\}\big)$ is made. The YM gauge variation of the Chern–Simons forms defines the $(p+1)$-form ω^1_{p+1}:

$$\delta_\epsilon \omega^0_{p+2}(A,F) = d\omega^1_{p+1}(A,F,\epsilon)\ . \tag{3.8}$$

As explained in [15], ω^1_{p+1} can be written in the form

$$\omega^1_{p+1}(A,F,\epsilon) = \mathrm{tr}\, d\epsilon\, \phi_p(A,F) \tag{3.9}$$

where the p-form $\phi_p = \phi^a_p T_a$ is a Lie algebra-valued polynomial in A and F, given, for $p = 1, 3, 5$, by

$$\begin{aligned}
\phi_1 =& -A\\
\phi_3 =& -\tfrac{1}{2}(FA+AF-A^3)\\
\phi_5 =& -\tfrac{1}{3}\left[(F^2A+FAF+AF^2)-\tfrac{4}{5}(A^3F+FA^3)-\tfrac{2}{5}(A^2FA+AFA^2)+\tfrac{3}{5}A^5\right]
\end{aligned} \tag{3.10}$$

The components of the form ω^1_{p+1} appear in the YM transformation of the antisymmetric tensor field; the full YM and antisymmetric tensor gauge transformations are

$$\begin{aligned}
\delta x^\mu =& 0\\
\delta y^m =& \epsilon^a(x)L^m_a(y)\\
\delta A^a_\mu =& \partial_\mu\epsilon^a + f^a{}_{bc}A^b_\mu\epsilon^c\\
\delta B_{\mu_1\ldots\mu_{p+1}} =& -k_p\omega^1_{\mu_1\ldots\mu_{p+1}}(A,F,\epsilon)+(p+1)\partial_{[\mu_1}\Lambda_{\mu_2\ldots\mu_{p+1}]}
\end{aligned} \tag{3.11}$$

The variation of C_{p+1} is determined as follows. Note first that ℓ_t commutes with δ_ϵ since $A-L$ and F both transform homogeneously. Then, from the definition (3.4) of C_{p+1}, and (3.8) it follows that

$$\delta_\epsilon C_{p+1} = \int_0^1 \ell_t\ d\omega^1_{p+1}(A_t, F_t, \epsilon)\ . \tag{3.12}$$

Now apply the homotopy formula [15]

$$d_t \equiv dt\frac{d}{dt} = \ell_t d - d\ell_t\ ; \tag{3.13}$$

this leads to the formula

$$\delta_\epsilon C_{p+1} = \omega^1_{p+1}(A,F,\epsilon) - \omega^1_{p+1}(L,\epsilon) + d\int_0^1 \ell_t \mathrm{tr}\, d\epsilon\phi_p(A_t,F_t) \tag{3.14}$$

where $\omega^1_{p+1}(L,\epsilon) = \omega^1_{p+1}(L,0,\epsilon)$, and we have rewritten the last term using (3.9). Finally, defining a Lie algebra-valued $(p-1)$-form $\chi_{p-1} = \chi^a_{p-1}T_a$ by

$$\chi_{p-1} = \int_0^1 \ell_t\ \phi_p(A_t,F_t)\ , \tag{3.15}$$

we obtain

$$\delta_\epsilon k_p C_{p+1} = k_p\Big(\omega^1_{p+1}(A,F,\epsilon) - \omega^1_{p+1}(L,\epsilon) + d\big(\mathrm{tr}\,[\epsilon d\chi_{p-1}]\big)\Big)\ . \tag{3.16}$$

The variation of the last term in (3.2) is given by

$$\begin{aligned} &-\frac{k_p}{p!}\epsilon^{i_1\dots i_{p+1}}\partial_{i_1}\Big\{\epsilon^a L_a^{m_1}\partial_{i_2}y^{m_2}\dots\partial_{i_{p+1}}y^{m_{p+1}}b_{m_1m_2\dots m_{p+1}}\Big\} \\ &-\frac{(p+2)k_p}{(p+1)!}\epsilon^{i_1\dots i_{p+1}}\epsilon^a L_a^n\partial_{i_1}y^{m_1}\dots\partial_{i_{p+1}}y^{m_{p+1}}\partial_{[n}b_{m_1\dots m_{p+1}]}\ . \end{aligned} \tag{3.17}$$

The first term here is a surface term; using (3.1), the second term is seen to be equal to

$$k_pc_p(p+2)\mathrm{tr}\ \epsilon L^{p+1} = k_p\mathrm{tr}\ \epsilon d\phi_p(L) = k_p\Big(\omega^1_{p+1}(L,\epsilon) - d(\mathrm{tr}\ \epsilon\phi_p(L))\Big)\ , \tag{3.18}$$

where we have used the fact that $\phi_p(L)\equiv\phi_p(L,0) = c_p(p+2)L^p$ and then (3.9). One can now easily prove the invariance of the action *up to a surface term*. The terms in the first line of (3.2) are manifestly invariant. The variation of B cancels the first term on the r.h.s. of (3.16). The first term on the r.h.s. of (3.18) cancels the second term on the r.h.s. of (3.16).

All that remains are the first term in (3.17), the third term on the r.h.s. of (3.16) and the second term on the r.h.s. of (3.18). Also taking into account the surface terms coming from the tensor gauge transformations, we find

$$\begin{aligned} \delta S = \int d\tau\partial_\tau \oint d^p\sigma\frac{1}{p!}\epsilon^{r_1\dots r_p}\Big\{&\Lambda_{r_1\dots r_p} \\ &- k_p\epsilon^a\Big[L_a{}^m b_{mr_1\dots r_p} + d_{ab}\big(\phi^b_p(L)\big)_{r_1\dots r_p} - pd_{ab}\partial_{[r_1}\big(\chi^b_{p-1}\big)_{r_2\dots r_p]}\Big]\Big\}\ , \end{aligned} \tag{3.19}$$

where we have decomposed the i-index into a time and a space part according to $i=(0,r_1\dots r_p)$. In this formula, appropriate pull-backs with $\partial_r x^\mu$ and $\partial_r y^m$ are understood. (For comparison with the string and three-brane formulae of section 3 we note here that $\chi^a_0 = 0$, $\chi^a_2 = -\frac{1}{2}d^a{}_{bc}A^bL^c$; also note $\chi^a_4 = d^a{}_{bcd}L^bA^c[F^d - \frac{1}{10}f^d{}_{fg}(3A^fA^g + 3L^fL^g - 2L^fA^g)]$, where $d_{abcd} = \mathrm{tr}(T_{(a}T_bT_cT_{d)})$).

Following the reasoning of the string and three-brane examples we deduce from (3.19) that the generators acting on the p–brane wavefunctional are given by

$$\begin{aligned} G_\epsilon = \int d^Dx'\,\Big\{\big(D_\mu\epsilon^a(x')\big)\frac{\delta}{\delta A^a_\mu(x')} - (p+1)k_p\big(\partial_{\mu_1}\epsilon^a(x')\big)d_{ab}(\phi^b_p)_{\mu_2\dots\mu_{p+1}}(A,F)\frac{\delta}{\delta B_{\mu_1\dots\mu_{p+1}}(x')}\Big\} \\ + \oint d^p\sigma\ \epsilon^a D_a(\boldsymbol{\sigma}) \end{aligned} \tag{3.20a}$$

$$G_\Lambda = \int d^D x' \left\{ (p+1)\partial_{\mu_1}\Lambda_{\mu_2\dots\mu_{p+1}}(x') \frac{\delta}{\delta B_{\mu_1\dots\mu_{p+1}}(x')} \right\} \\ - i \oint d^p\sigma\, \varepsilon^{r_1\dots r_p}\partial_{r_1}x^{\mu_1}\cdots\partial_{r_p}x^{\mu_p}\Lambda_{\mu_1\dots\mu_p} \;, \tag{3.20b}$$

where $D_a(\boldsymbol{\sigma})$ is given by

$$D_a(\boldsymbol{\sigma}) = L_a{}^m \left[\frac{\delta}{\delta y^m(\boldsymbol{\sigma})} + i\frac{k_p}{p!}\varepsilon^{r_1\dots r_p}\left\{ b_{mr_1\dots r_p} + L_{mb}\left[\left(\phi_p^b(L)\right)_{r_1\dots r_p} - p\partial_{[r_1}\left(\chi_{p-1}^b\right)_{r_2\dots r_p]}\right]\right\}\right] \tag{3.21}$$

We refer to [16] for a computation of the algebra of these generators, which is that given in (1.2). We note here that once it is known that they form a *closed* algebra, (1.2) follows from the transformations (3.11) because G_ϵ and G_Λ reproduce these transformations when acting on the individual fields.

4. Covariant Derivatives in p–Loop Space

The coupling to the antisymmetric tensor $B_{\mu\nu}$ and the WZ term in the action (3.2) are linear in time derivatives and can therefore be expressed as

$$\int d\tau \oint d^p\sigma \left[\dot{x}^\mu \mathbf{B}_\mu - k_p \dot{y}^m \mathbf{b}_m \right] \;, \tag{4.1}$$

where the overdot indicates differentiation with respect to τ. Using the chain rule for $C(A,F,L)$ we have that

$$\begin{aligned} \mathbf{B}_\mu &= \frac{1}{p!}\varepsilon^{r_1\dots r_p}\partial_{r_1}x^{\nu_1}\cdots\partial_{r_p}x^{\nu_p}B_{\mu\nu_1\dots\nu_p} + k_p A_\mu^a \frac{\partial C}{\partial A_0^a} + k_p F_{\mu\nu}^a \partial_r x^\nu \frac{\partial C}{\partial F_{0r}^a} \\ \mathbf{b}_m &= \frac{1}{p!}\varepsilon^{r_1\dots r_p}\partial_{r_1}y^{n_1}\cdots\partial_{r_p}y^{n_p}b_{mn_1\dots n_p} - L_m{}^a \frac{\partial C}{\partial L_0{}^a} \;. \end{aligned} \tag{4.2}$$

Introducing the momenta $p_\mu(\tau,\boldsymbol{\sigma})$ and $p_m(\tau,\boldsymbol{\sigma})$, conjugate to the worldvolume fields x^μ and y^m respectively, and the Lagrange multipliers $\ell(\tau,\boldsymbol{\sigma})$ and $s^r(\tau,\boldsymbol{\sigma})$ for, respectively, the time and space reparametrization constraints, we can rewrite the action (3.2) in the equivalent 'first-order' form

$$S = \int d\tau \oint d^p\sigma \left[\dot{x}^\mu p_\mu + \dot{y}^m p_m - \ell\, \mathcal{H}_0 - s^r\, \mathcal{H}_r \right] \;. \tag{4.3}$$

The constraint functions are

$$\begin{aligned} \mathcal{H}_0 &= g^{\mu\nu}\mathcal{P}_\mu\mathcal{P}_\nu + d^{ab}\mathcal{P}_a\mathcal{P}_b + \det\left(\partial_r x^\mu \partial_s x^\nu g_{\mu\nu} + (D_r y)^m (D_s y)^n g_{mn}\right) \\ \mathcal{H}_r &= \partial_r x^\mu \mathcal{P}_\mu + (D_r y)^m L_m{}^a \mathcal{P}_a \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \mathcal{P}_\mu &= p_\mu - \mathbf{B}_\mu + A_\mu^a \mathcal{P}_a \\ \mathcal{P}_a &= L_a{}^m (p_m + k_p \mathbf{b}_m) \end{aligned} \tag{4.5}$$

The (classical) equivalence of (4.3) to (3.2) can be proved by eliminating all auxiliary variables from both actions.

Observe that the action (4.3) can be rewritten as

$$
\begin{aligned}
S = \int d\tau \oint d^p\sigma \big[\dot{x}^\mu \mathcal{P}_\mu + (D_0 y)^m L_m{}^a \mathcal{P}_a - \ell\, \mathcal{H}_0 - s^r\, \mathcal{H}_r \big] \\
+ \int d\tau \oint d^p\sigma \big[\dot{x}^\mu \mathbf{B}_\mu - k_p \dot{y}^m \mathbf{b}_m \big] \ .
\end{aligned}
\tag{4.6}
$$

The second integral is just (4.1) which was shown in section 3 to be invariant, up to a surface term, under the gauge transformations (3.11). The first integral is invariant if these transformations are supplemented by transformations of p_μ and p_m chosen such that the functions $\mathcal{P}_\mu$ and $\mathcal{P}_a$ are covariant, *i.e.* such that

$$
\delta \mathcal{P}_\mu = 0 \qquad\qquad \delta \mathcal{P}_a = -\mathcal{P}_c f^c{}_{ab} \epsilon^b \ . \tag{4.7}
$$

The corresponding statement in the quantum theory, obtained by the replacements

$$
p_\mu(\boldsymbol{\sigma}) \to \hat{p}_\mu(\boldsymbol{\sigma}) = -i \frac{\delta}{\delta x^\mu(\boldsymbol{\sigma})} \qquad p_m(\boldsymbol{\sigma}) \to \hat{p}_m(\boldsymbol{\sigma}) = -i \frac{\delta}{\delta y^m(\boldsymbol{\sigma})} \ , \tag{4.8}
$$

is that the operators $\mathcal{D}_\mu \equiv i\hat{\mathcal{P}}_\mu$ and $\mathcal{D}_a \equiv i\hat{\mathcal{P}}_a$ obey (1.5).

Of chief importance is the derivative $\mathcal{D}_\mu$. Using (4.2) and (4.5) we find that

$$
\begin{aligned}
\mathcal{D}_\mu = \frac{\delta}{\delta x^\mu(\boldsymbol{\sigma})} - \frac{i}{p!} \varepsilon^{r_1 \dots r_p} \partial_{r_1} x^{\nu_1} \cdots \partial_{r_p} x^{\nu_p} B_{\mu\nu_1\dots\nu_p} - i k_p F^a_{\mu\nu} \partial_r x^\nu \frac{\partial C}{\partial F^a_{0r}} \\
+ A^a_\mu L_a{}^m \Big[\frac{\delta}{\delta y^m(\boldsymbol{\sigma})} + \frac{i k_p}{p!} \varepsilon^{r_1 \dots r_p} \partial_{r_1} y^{n_1} \cdots \partial_{r_p} y^{n_p} b_{m n_1 \dots n_p} - k_p L_m{}^b \Big(\frac{\partial C}{\partial L_0{}^b} + \frac{\partial C}{\partial A^b_0} \Big) \Big] \ .
\end{aligned}
\tag{4.9}
$$

It can be shown [16] that

$$
\frac{\partial C}{\partial A^a_0} + \frac{\partial C}{\partial L_0{}^a} = d_{ab} \Big(\phi^b_p(A, F) - \phi^b_p(L) + d\chi^b_{p-1} \Big) \ . \tag{4.10}
$$

from which it follows that

$$
\mathcal{D}_\mu = \frac{\delta}{\delta x^\mu(\boldsymbol{\sigma})} - i\mathcal{C}_\mu + A^a_\mu D_a(\boldsymbol{\sigma}) \tag{4.11}
$$

where $D_a(\boldsymbol{\sigma})$ is the p–loop space differential operator given previously in (3.21) and

$$
\mathcal{C}_\mu = \frac{1}{p!} \varepsilon^{r_1 \dots r_p} \partial_{r_1} x^{\nu_1} \cdots \partial_{r_p} x^{\nu_p} B_{\mu\nu_1\dots\nu_p} + k_p A^a_\mu \phi^b_p(A, F) d_{ab} + k_p F^a_{\mu\nu} \partial_r x^\nu \frac{\partial C}{\partial F^a_{0r}} \ , \tag{4.12}
$$

which we identify as the $U(1)$ gauge potential. For $p = 1$ this reproduces the results of [7,8].

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