A three-phase tessellation: solution and
effective properties

BY R. V. CRASTER† & YU. V. OBNOSOV‡

† Department of Mathematics, Imperial College of Science, Technology and Medicine, London SW7 2BZ, UK
‡ Institute of Mathematics and Mechanics, Kazan State University, University Str., 17, 420008, Kazan, Russia

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Two-dimensional doubly periodic, three-phase, structures are considered in the situation where mean fluxes are applied across the structure. The approach is to utilize complex variables, and to use a mapping that reduces the doubly periodic problem to a much simpler one involving joined sectors.

This is a model composite structure in electrostatics (and mathematically analogous areas such as porous media, anti-plane elasticity, heat conduction), and we find various effective parameters and investigate limiting cases. The structure is also amenable to asymptotic methods in the case of highly varying composition and we provide these solutions, partly as a check upon our analysis, and partly as they are useful in their own right.

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1. Introduction

Multi-phase composites are vital in modern engineering and physics, and as such they have been, and will continue to be, the subject of many studies both theoretical and numerical. The subject is vast, and it is not our aim to review the area here, especially since the emergence of the recent, comprehensive, book by Milton (2002), but one area of interest is in model planar structures that repeat doubly-periodically to fill space. These are pleasant to study both numerically and theoretically as the double periodicity means that one can focus attention upon an elementary cell; this can then be repeated to cover the full plane. Such, apparently simple, model structures are enlightening as they provide benchmarks upon which numerical schemes, theories and bounds can be tested.

Previous work on doubly-periodic structures has tended to focus upon the classical cylindrical inclusion geometry first considered by Rayleigh (1892), or upon square and rectangular checkerboard geometries. Even for these checkerboard geometries, solutions have been slow to emerge; two phase square checkerboards are the simplest cases, and using duality Keller (1964), Dykhne (1971), Mendelson (1975) proved the now classical geometrical mean formula for the effective conductivities, and generalizations thereof. Explicit solutions for the field variables are harder to find, Berdichevski (1985), Emets (1986) considered square two-phased
checkboards and this was later generalized by one of us, Obnosov (1996, 1999),
to rectangular checkerboards and isolated rectangular inclusions. This all relied
upon Riemann-Hilbert, and so-called Markushevich, problems that were hard to
generalize further, and it certainly was not clear that more than two phases could
be treated in this manner. Recently Craster (2000), Craster & Obnosov (2001a,b)
have overcome this difficulty by utilizing a mapping at the outset to a two-sheeted
Riemann surface; the full four-phase rectangular checkerboard can then be dealt
with. Special cases include a proof of the long-standing Mortola and Steffé (1985)
conjecture for square checkerboards, see also Milton (2001). Those solutions are the
only explicit ones known for both the field variables and effective parameters for
more than two phases. Various results involving bounds and approximations can
be deduced, see Milton (2002) for more details.

Our aim here is complementary to those checkerboard analyses, in that we
consider a new non-trivial multi-phased structure that is solvable using a mapping
based upon a similar philosophy, but for a very different geometry; the structure,
shown in Figure 1, has three phases, and a non-trivial geometry constructed from
diamonds. Like the checkerboard structures this should be valuable as effective
parameters can be deduced, as can the dissipation, and limiting cases can also
be investigated. As such it will add to the rather sparse range of exact solutions
available. It is also not unlike three-phase model structures recently considered
using various approximations and bounds by Fel et al. (2000).

The plan of the paper is as follows: We formulate the problem, and set forth
our notation in section 2(a). A vital step is to demonstrate, using some mathem-
atical origami, that the tessellated structure can actually be reduced to a much
simpler, three joined sectors, problem using conformal mappings (section 2(b)).
Once this has been achieved the boundary value problem for the simpler structure
must be solved, this turns out to be rather lengthy and is relegated to Appendix
A. Thereafter, we utilize the solution to deduce effective parameters (section 2(d))
and consider limited cases of two phase structures (section 2(e)), or the asymptotic
situation where one phase is highly conducting, or highly resistive (section 2(f)).
We draw together some concluding remarks in section 3.

2. The three-phased double-periodic structure

We consider three-phase, piecewise continuous, stationary linear media whose
physical fields, electric field, say, can be represented in terms of a vector field that is both
solenoidal and irrotational; this encompasses several physical scenarios in electro-
or magneto-statics, heat flow, hydrology and elasticity. The language of electrostat-
ics is used in later sections. In each phase, distinguished by the subscript \( k \) where
\( k = 1, 2, 3 \), we define a vector field \( \mathbf{v}_k = (v_{kx}, v_{ky}) \) of the horizontal and vertical
components \( v_x, v_y \) such that both

\[
\nabla \cdot \mathbf{v}_k = 0, \quad \nabla \times \mathbf{v}_k = 0.
\]

It is most convenient to utilize complex variables, that is, \( z = x + iy \) and in each
phase, \( \Omega_k, (k = 1, 2, 3) \) piecewise analytic functions \( v_k(z) = v_{kx} - iv_{ky} \) are de-
defined. The continuity boundary conditions between each phase are that the normal
components of \( \mathbf{v}_k \) are continuous across each boundary, and that the tangential
components of $\rho_k v_k$ are similarly continuous; the constant parameters $\rho_k$ correspond to a phase property of each medium. In the terminology of electrostatics we have $v = \sigma E$ with $E$ as the electric field and $v$ as the electric current (often denoted by $J$) and $\sigma$ is the conductivity. The phase property $\rho$ is the resistivity, and it is $1/\sigma$. Each problem naturally requires some boundary conditions, here we apply a constant field at infinity, thus there is a jump in the potential across each elementary cell.

Figure 1. The 3-phased diamond structure.

(a) Formulation

We choose to study the tessellation, in the $z$–plane, shown in Figure 1; although it is not immediately obvious, this precise structure has several features that render it explicitly solvable. Each phase $\Omega_k$, ($k = 1, 2, 3$) consists of an infinite number of rhombii each with side length $l$ and vertex angles $\pi/3$ and $2\pi/3$, and for each phase the orientation of the rhombii in the $z$–plane is identical. Namely, the diagonals of all rhombii of phases $\Omega_1, \Omega_2, \Omega_3$ form three sets of parallel straight lines, one is perpendicular to the real axes, and the others cross it subtending angles $-\pi/6$ and $\pi/6$ with the real axis. This doubly-periodic structure has two primitive periods $2\omega' = \sqrt{3}le^{-i\pi/6}$ and $2\omega = \sqrt{3}le^{i\pi/6}$. For some of the analysis it is more convenient for us to consider as elementary cells either the parallelogram (rhombus) bounded by dashed lines in Figure 1, or the regular hexagon $\Omega$ with vertices at the points $A_k = le^{i(1-k)\pi/3}$, $k = 1, \ldots, 6$ and the centre $O$ at the origin (see Figure 2). This hexagon is composed of three rhombii $\Omega_k$, $k = 1, 2, 3$; we use the same $\Omega$ for each portion of a particular phase. Thus, $\Omega_1, \Omega_2$ and $\Omega_3$ are the rhombii with vertices at the points $O, A_2, A_1, A_6, O, A_4, A_3, A_2$ and $O, A_6, A_5, A_4$.

Using the continuity boundary conditions between each phase, together with
Figure 2. The 3-phased hexagon, and three joined equal sectors.

the $2\omega, 2\omega'$ double periodicity we write the boundary value problem as

\[
\begin{align*}
\text{Re} \left[ \varepsilon (\rho_1 v_1 (\varepsilon x + \delta \varepsilon l) - \rho_2 v_2 (\varepsilon x - \delta l)) \right] &= \text{Im} \left[ \varepsilon (v_1 (\varepsilon x + \delta \varepsilon l) - v_2 (\varepsilon x - \delta l)) \right] = 0, \\
\text{Re} \left[ \varepsilon (\rho_1 v_1 (\varepsilon x + \delta \varepsilon l) - \rho_3 v_3 (\varepsilon x - \delta l)) \right] &= \text{Im} \left[ \varepsilon (v_1 (\varepsilon x + \delta \varepsilon l) - v_3 (\varepsilon x - \delta l)) \right] = 0, \\
\text{Re} \left[ \rho_2 v_2 (-x + \delta l) - \rho_3 v_3 (-x + \delta l) \right] &= \text{Im} \left[ v_2 (-x + \delta l) - v_3 (-x + \delta l) \right] = 0,
\end{align*}
\]

(2.1)

where $0 < x < l$, $\varepsilon = e^{i \pi/3}$ and $\delta$ is either 0 or 1 throughout. The solution of (2.1) can have integrable singularities at the vertices $O, A_1, A_2, \ldots, A_6$. To complete the problem specification, mean fluxes $I_\omega$ and $I_{\omega'}$ are applied through two adjoining sides of the elementary cell ($A_4, A_6$) and ($A_4, A_2$), i.e.

\[
I_\omega = \frac{1}{\sqrt{3}l} \int_{A_4}^{A_6} \mathbf{n} \cdot \mathbf{v}_3(t) ds, \quad I_{\omega'} = \frac{1}{\sqrt{3}l} \int_{A_4}^{A_2} \mathbf{n} \cdot \mathbf{v}_2(t) ds,
\]

(2.2)

where $\mathbf{n}$ is the outward pointing normal to each edge of the triangle.

The feature that renders this particular problem solvable is that we can relate this, via a mapping, to a much simpler problem posed in terms of three joined equal sectors (see figure 2(b)); this reduced problem is more easily solved. This idea of mapping a doubly-periodic problem to a set of joined sectors is similar in philosophy to Craster & Obnosov (2001b) who solved the four-phase checkerboard problems using it.

**Conformal mapping**

To exploit this simpler joined sectors structure we must show that the problem expressed by (2.1) is reduced to it. We begin with the triangle $\Omega_{11}$ with vertices at the points $O, O_1, A_2$, where $O_1$ is the centre of the rhombus $\Omega_1$ (see Figure 3). Using the Schwarz-Christoffel formula, Nehari (1952), the integral

\[
e^{-i \pi/3} \frac{\Gamma(5/6)}{2 \sqrt{3} \Gamma(1/3)} \int_0^\zeta \tau^{-2/3} (1 + \tau)^{-1/2} d\tau,
\]

(2.3)

conformally maps the upper half of the $\zeta$-plane onto the triangle $\Omega_{11}$ with the points $(-1, 0, \infty)$ mapped to $(O_1, O, A_2)$. However, we actually want to map an angular sector, subtending an angle $\pi/3$, to the triangle, and this is achieved by replacing $\zeta$ with $-\zeta^3$ in (2.3); using the definition of the hypergeometric function, the mapping of the sector $\Omega_{11}^* = \{ \zeta : -\pi/3 < \arg \zeta < 0 \}$ onto the triangle $\Omega_{11}$ with...
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Figure 3. The triangle $O, A_2, A_6$ in the $z$-plane and its image in the $\zeta$-plane

points $(1, 0, \infty) \rightarrow (O_1, O, A_2)$ is identified as

$$z(\zeta) = \Lambda \zeta F(1/3, 1/2; 4/3; \zeta^3), \quad \Lambda = \frac{3\Gamma(5/6)}{2\sqrt{\pi}\Gamma(1/3)}. \quad (2.4)$$

Successive analytic continuations, and applications of the symmetry principle, are now used to create the mapped domain corresponding to the hexagonal elementary cell. We begin by generating the mapped domain for a rhombus (see figure 4).

First, the mapping function (2.4) is analytically continued across the line segment $(0, 1)$ onto the sector $\{ \zeta : -\pi/3 < \arg \zeta < \pi/3 \}$ with a branch cut along the ray $\xi > 1$ ($\zeta = \xi + i\eta$). The mapping function, and its analytic continuation across $[0, 1)$, therefore map the sector onto the triangle $O, A_2, A_6$ (figure 3) with the branch cut corresponding to the line $A_2, O_1, A_6$. We now perform one further analytic continuation, this time through both sides of the cut, the resultant function conformally maps the two sheeted sector $\Omega_1^* = \{ \zeta : -\pi/3 < \arg \zeta < \pi/3 \}$ with crosswise glued branches of the cuts onto the rhombus $\Omega_1$. Thus the four triangles $O_1A_6O, O_1A_1A_6, O_1A_2O, O_1A_1A_2$ (see figure 4) are mapped to the two sheeted structure with pairs of triangles overlying each other. It is important to note that $A_1$ is at the origin of the lower sheet, and the images of parallel sides of the rhombus overlie each other in the $\zeta$-plane (see figure 4). This is important as it means later that applying continuity boundary conditions across them will correctly connect fluxes through, say, $A_1A_6$, to those through $A_3A_4$, whilst simultaneously dealing with the conditions across $OA_2$.

Now we actually want to construct the elementary hexagonal cell, figure 2(a), to get this we now invoke the symmetry principle, Nehari (1952), for the rhombus that we have just constructed. By analytical continuation through the sides of the sector $\Omega_1^*$ on the upper sheet we get the function $z(\zeta)$ mapping sectors $\Omega_2^* = \{ \zeta : \exp(2\pi i/3)\zeta \in \Omega_1^* \}$ and $\Omega_3^* = \{ \zeta : \exp(-2\pi i/3)\zeta \in \Omega_1^* \}$ onto rhombii $\Omega_2$ and $\Omega_3$ respectively.

To summarize, the image of the whole hexagon $\Omega$ (figure 2(a)) under the proposed mapping (2.3), and analytic continuations of it, form a two sheeted Riemann surface $\Omega^*$ with branch cuts on the upper sheet ‘criss-cross glued’ along the rays

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Figure 4. The rhombus $O, A_2, A_6, A_1$ in the $z$-plane and its two-sheeted image in the $\zeta$-plane

$|\zeta| > 1$, arg $\zeta = 2k\pi/3$, $k = 0, 1, 2$ and with three cuts on the lower sheet along the rays $|\zeta| > 0$, arg $\zeta = (2k + 1)\pi/3$, $k = 0, 1, 2$.

Having generated, in principle, the mapping required, let us look at its inverse. Often it is convenient to have $\zeta(z)$, that is, the inverse of the function (2.4). Or, more exactly, the function obtained by all possible analytical continuations of the inverse of (2.4). From the elementary cell, figure 1, it is evident that $\zeta(z)$ must be a doubly periodic elliptic function with $2\omega, 2\omega'$ as primitive periods. The elliptic functions are characterized by a parallelogram; we take the parallelogram of interest to be that with vertices at the points $l, l + 2\omega, \lambda + 2\omega', 4l$, that is, the elementary parallelogram shown in figure 1 shifted $2l$ to the right; the function we desire has two simple zeros at the points $\zeta = l$ and $\zeta = 3l$, that is, at the centre of each elementary hexagon, and one pole of second order at the centre of the star-like domain, $\zeta = 2l$ (note that the sum of zeros within the parallelogram equals the sum of poles). It is known (Whittaker & Watson 1927; Lawden 1989) that the residue of $\zeta(z)$ about the single pole $\zeta = 2l$ is zero, and this function is represented as

$$\zeta(z) = \frac{\wp(z - 2l) - \wp(l)}{\wp(\omega + \omega') - \wp(l)}, \quad (2.5)$$

where $\wp(z)$ is a Weierstrass function with primitive periods $2\omega, 2\omega'$. We have also utilized the conditions that $\zeta(l) = \zeta(3l) = 0$, and $\zeta(l/2) = 1$.

It is important to note that the function (2.5) takes one and the same value at any two points symmetric about the centres $O_1, O_2, O_3$ of the diamonds $\Omega_1, \Omega_2, \Omega_3$ (see Figure 2(a)), i.e.

$$\zeta(z) \equiv \zeta(l - z) \equiv \zeta(-l\varepsilon - z) \equiv \zeta(-l\varepsilon' - z). \quad (2.6)$$

Returning to the boundary value problem (2.1), we introduce, using (2.4), the function $v(z(\zeta)) = v(\zeta)$ (we use the same letter for the function after substitution).
Using this function, and (2.1), we now obtain a boundary-value problem that has identical conditions on both sheets of the Riemann surface $\Omega^*$ that we constructed earlier, namely,

\[
\begin{align*}
\text{Re} [z(\rho_1 v_1(\tau \xi) - \rho_2 v_2(\tau \xi))] &= \text{Im} [\varepsilon(v_1(\tau \xi) - v_2(\tau \xi))] = 0, \\
\text{Re} [z(\rho_1 v_1(\tau \xi) - \rho_3 v_3(\tau \xi))] &= \text{Im} [\varepsilon v_1(\tau \xi) - v_3(\tau \xi)] = 0, \\
0 < \xi < \infty. 
\end{align*}
\] (2.7)

It is this remarkable fact that allows this problem to be solved; it, or at least a very similar property, also underlies our earlier solution to the four-phase rectangular checkerboard.

Considering the behaviour of $z(\zeta)$ defined in (2.4) as $\zeta \to 0, \infty$ then a solution of the problem (2.7) has to satisfy the additional conditions that

\[
\begin{align*}
v(\zeta) &= o(1/\zeta) \quad \text{in the vicinity of the origin} \\
v(\zeta) &= o(\zeta^{1/2}) \quad \text{in the vicinity of the infinity}. 
\end{align*}
\] (2.8)

This is simply a statement that there are no sources or sinks at the vertices and any singularities are integrable.

It is important to note that the initial problem (2.1) is reduced to the much simpler problem summarized by (2.7) and (2.8). This latter problem is distinguished as it must be solved in terms of a function taking the same values at points that overlie each other on the upper and lower Riemann sheets.

We now have to solve a plane problem involving three joined equal sectors, the problem is defined by equations (2.7) and (2.8). To get the solution neatly, the procedure is somewhat involved and we relegate it to Appendix A. For the purposes of calculating effective parameters we need the general solution of the problem (2.1) to the tessellated structure, which is that

\[
\begin{align*}
v_1(\zeta) &= c_1 A_1^+ \zeta^{3\alpha^+ - 1} + i c_2 A_1^- \zeta^{3\alpha^- - 1}, \quad \zeta \in \Omega_1, \\
v_2(\zeta) &= c_1 A_2^+ \zeta^{3\alpha^+ - 1} + i c_2 A_2^- \zeta^{3\alpha^- - 1}, \quad \zeta \in \Omega_2, \\
v_3(\zeta) &= c_1 A_3^+ \zeta^{3\alpha^+ - 1} + i c_2 A_3^- \zeta^{3\alpha^- - 1}, \quad \zeta \in \Omega_3, 
\end{align*}
\] (2.9)

and $c_{1,2}$ are, at this stage, arbitrary real parameters $\zeta = \zeta(z)$ is given by the equation (2.5). To ease the notation we introduce

\[
a = \Delta_{12}, \quad b = \Delta_{23}, \quad c = \Delta_{31}, \quad \Delta_{pq} = \frac{\rho_p - \rho_q}{\rho_p + \rho_q}, \quad (2.10)
\]

and $\Delta^2 = -(ab + bc + ac)$, with

\[
\lambda_{\pm} = e^{i \pi \alpha^\pm} = \frac{\sqrt{1 + \Delta}}{2} \pm \frac{i}{2} \sqrt{3 \pm \Delta}. \quad (2.11)
\]

In Appendix A it will be shown that $0 \leq \Delta = \sqrt{-ab - bc - ac} \leq 1$ and this constrains us to have $0 \leq 3\alpha^+ - 1 \leq 1/2$ and $-1/4 \leq 3\alpha^- - 1 \leq 0$. The terms $A_j^+, A_j^-$, $j = 1, 2, 3$ are given by equations

\[
\begin{align*}
A_1^+ &= -(1 - c)(\Delta + a)\lambda_+ - (1 + a)(\Delta - c)\lambda_+, \\
A_2^+ &= (1 + a)(\Delta - c)\lambda_+ + b(1 - c)\lambda_+, \\
A_3^+ &= (1 - c)(\Delta + a)\lambda_+ - b(1 + a)\lambda_+; 
\end{align*}
\] (2.12)
Alternatively, we note that after some effort, see App. (2.16), (2.18) and the identity (A 16).

\[ A_1^- = (1 + c)(\Delta + a)\lambda_+ + (1 - a)(\Delta - c)\lambda_-, \]
\[ A_2^- = -(1 + a)[(\Delta - c)\lambda_+ + b(1 + c)\lambda_-, \]
\[ A_3^- = -(1 - c)[(\Delta + a)\lambda_+ - b(1 - a)\lambda_-. \]

\[(c) The solution for the tessellation\]

The real parameters \( c_1, c_2 \) in the general solution (2.9) should be defined using the flux conditions (2.2), that are transformed to:

\[ I_\omega = -\frac{e^{-i\pi/6}}{\sqrt{3}} \int_{A_4}^{A_4} \text{Re} [e^{-i\pi/3}v_3(t)] dt = \text{Re} \left\{ \frac{i}{\sqrt{3}} \int_{A_4}^{A_4} v_3(t) dt \right\}, \]
\[ I_\omega' = -\frac{e^{i\pi/6}}{\sqrt{3}} \int_{A_4}^{A_4} \text{Re} [e^{-i\pi/3}v_2(t)] dt = \text{Re} \left\{ \frac{-i}{\sqrt{3}} \int_{A_4}^{A_4} v_2(t) dt \right\}. \]

Alternatively, taking into account the relations (2.6), the flux conditions (2.2) take the form

\[ I_\omega = -\frac{2}{\sqrt{3}} \text{Im} \int_{A_4}^{A_4} v_3(t) dt, \quad I_\omega' = \frac{2}{\sqrt{3}} \text{Im} \int_{A_4}^{A_4} v_2(t) dt. \]

After some effort, see Appendix B, one arrives at

\[ I_\omega = c_1\Lambda^+(1 - c)[(\Delta + a - b(1 + a)] - c_2\Lambda^-(1 - c)[(\Delta + a + b(1 - a)], \]

where

\[ \Lambda^+ = \frac{\sqrt{1 - \Delta}}{2\Gamma(1/2 - \alpha^+)}(\Gamma(2/3)\Gamma(1/6)), \quad \Lambda^- = \frac{\sqrt{3 - \Delta}}{2\Gamma(1 - \alpha^-)}(\Gamma(2/3)\Gamma(1/6)). \]

Similarly, for \( I_\omega' \) one derives that

\[ I_\omega' = c_1\Lambda^+(1 + a)[(\Delta - c + b(1 - c)] + c_2\Lambda^-(1 + a)[(\Delta - c - b(1 + c)]. \]

We note that \( \zeta = \zeta(z) \), given by the formula (2.5), satisfies the identities

\[ \zeta \left( e^{\pm 2\pi i/3} \zeta \right) \equiv e^{\pm 2\pi i/3} \zeta, \quad \zeta(\zeta) \equiv \zeta(z). \]

Using these, the general solution (2.9) has the properties (see also Appendix A (b)):

\[ v \left( \zeta \left( e^{\pm 2\pi i/3} \zeta \right) \right) \equiv v \left( e^{\pm 2\pi i/3} \zeta \right), \quad v(\zeta(z)) \equiv v \left( \zeta \right), \]

that allow us to verify equation (2.18), using

\[ I_\omega(a, b, c; c_1, c_2) = I_\omega(-c, -b, -a; c_1, -c_2) \]

which also requires (2.16), (2.18) and the identity (A 16).

The two equations for \( I_\omega \) and \( I_\omega' \) have a unique solution for \( c_1 \) and \( c_2 \)

\[ c_1 = \frac{(\Delta + a)I_\omega + (\Delta - c)I_\omega'}{2\Delta\Lambda^+[(1 + \Delta)(a - c) + 2(\Delta - ac)]}, \]
\[ c_2 = -\frac{(\Delta + a)I_\omega - (\Delta - c)I_\omega' + 2b(1 + a)I_\omega + (1 - c)I_\omega'/(1 + \Delta)}{2\Delta\Lambda^+[(1 + \Delta)(a - c) + 2(\Delta - ac)]}. \]

The term in the denominator, \( D(a, c) = (1 + \Delta)(a - c) + 2(\Delta - ac) \), in (2.21) is zero if \( \rho_1 = 0 \) and \( \rho_2 = \infty \), or if \( \rho_2 = \rho_3 \) and \( \rho_1 < \rho_2 \). These special cases are briefly dealt with separately in sections 2(e), 2(f). It is useful to note that \( b = -(a + c)/(1 + ac) \), and hence \( \Delta(a, c) \) and \( D(a, c) \) are functions of only \( a \) and \( c \).

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(d) Effective properties

Now we are in a position to evaluate some effective parameters of the structure shown in figure 1. We start by evaluating the quantities $E_x, E_y$ defined as

$$E_x = \frac{1}{3l} \int_{-l}^{2l} \text{Re} \left[ \rho_k v_k(x) \right] dx, \quad E_y = \frac{1}{\sqrt{3}l} \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \text{Im} \left[ \rho_1 v_1(l/2 + iy) \right] dy. \quad (2.22)$$

These are integrals along diagonals of the elementary cell $\Omega$. Additionally, we require the fluxes, $I_x, I_y$, defined as

$$I_x = \frac{1}{\sqrt{3}l} \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \text{Re} \left[ v_1(l/2 + iy) \right] dy, \quad I_y = \frac{1}{3l} \int_{-l}^{2l} \text{Im} \left[ v_k(x) \right] dx. \quad (2.23)$$

The paths along which these integrals (and those for $E_x, E_y$) are calculated can be moved horizontally or vertically without affecting their value, it is just convenient to evaluate them along the diagonals.

In accordance with definition (2.2)

$$I_x = -(I_\omega + I_\omega'), \quad I_y = (I_\omega - I_\omega')/\sqrt{3},$$

$$I_\omega = -(I_x - \sqrt{3}I_y)/2, \quad I_\omega' = -(I_x + \sqrt{3}I_y)/2. \quad (2.24)$$

$I_x$ and $I_y$ are the average fluxes through the elementary cell $\Omega$ in the $x$- and $y$-directions. Similarly, if we introduce $E_\omega, E_\omega'$ defined by the formulae

$$E_\omega = \frac{1}{\sqrt{3}l} \int_{A_4} (\rho_3 v_3 \cdot t_\omega) ds = \frac{\rho_3}{\sqrt{3}l} \int_{A_4} (v_3)_x ds,$$

$$E_\omega' = \frac{1}{\sqrt{3}l} \int_{A_4} (\rho_2 v_2 \cdot t_\omega') ds = \frac{\rho_2}{\sqrt{3}l} \int_{A_4} (v_2)_x ds,$$

we have that

$$E_y = E_\omega - E_\omega', \quad E_x = (E_\omega + E_\omega')/\sqrt{3},$$

$$E_\omega = (\sqrt{3}E_x + E_y)/2, \quad E_\omega' = (\sqrt{3}E_x - E_y)/2. \quad (2.25)$$

After some effort one deduces that

$$E_x = \frac{-\rho_1}{\sqrt{3}l} \left\{ c_1 \Lambda^+ m(c, a) \sqrt{(3+\Delta)(1-\Delta)} + c_2 \Lambda^- m(a, c) \frac{1+\Delta}{a+c} M(a, c, \Delta) \right\},$$

$$E_y = \rho_1 \left\{ c_1 \Lambda^+ (a+c) \sqrt{(3+\Delta)(1-\Delta)} - c_2 \Lambda^- (1+\Delta) M(a, c, \Delta) \right\}, \quad (2.26)$$

where $c_1, c_2$ are defined in (2.21), and

$$m(a, c) = 2\Delta - a + c, \quad M(a, c, \Delta) = [2(ac+\Delta) + (a-c)(1-\Delta)]/\sqrt{(3-\Delta)(1+\Delta)}. \quad (2.27)$$

Hence, if one will fix resistivities $\rho_k$, then $E_x, E_y$ are the functions of the external flux components. Namely, they are linear combinations of $I_\omega, I_\omega'$ or alternatively, see (2.24), of $I_x, I_y$, i.e. $E_{x,y} = E_{x,y}(I_x, I_y)$.

We could choose to set up our effective parameters using either integrals of the area of an elementary cell, or in terms of integrals along an edge. The procedure

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is that we identify \( \langle \rho \mathbf{v} \rangle \) \( = \rho_{ij} \langle \mathbf{v} \rangle \) where \( \langle \cdot \rangle \) denotes the average of the quantity, in some sense, and \( \rho_{ij} \) is a tensor, which is identified as the effective resistivity tensor.

First, we consider using integrals over the area of the elementary cell. For brevity we use \( \alpha = \pi / 6 \) and define vectors along, and normal to, the edges of the triangle \( \Omega \) with vertices \( A_2, A_4, A_6 \) as

\[
\mathbf{t}_\omega = (\cos \alpha, \sin \alpha), \quad \mathbf{t}_{\omega'} = (\cos \alpha, -\sin \alpha), \quad \mathbf{n}_y = (0, 1),
\]

\[
\mathbf{n}_\omega = (-\sin \alpha, \cos \alpha), \quad \mathbf{n}_{\omega'} = (-\sin \alpha, -\cos \alpha), \quad \mathbf{n}_y = (1, 0),
\]

then it can be shown that

\[
\mathbf{v} \equiv \frac{2}{3} \left[ \mathbf{n}_y (\mathbf{v} \cdot \mathbf{n}_y) + \mathbf{n}_\omega (\mathbf{v} \cdot \mathbf{n}_\omega) + \mathbf{n}_{\omega'} (\mathbf{v} \cdot \mathbf{n}_{\omega'}) \right],
\]

and

\[
|\Omega| < \mathbf{v} > = \int_{\Omega} \mathbf{v}(z) \, dS = \frac{2}{3} \left[ \int_{\Omega} (\mathbf{n}_y (\mathbf{v} \cdot \mathbf{n}_y) + \mathbf{n}_\omega (\mathbf{v} \cdot \mathbf{n}_\omega) + \mathbf{n}_{\omega'} (\mathbf{v} \cdot \mathbf{n}_{\omega'})) \, dS \right],
\]

where \( |\Omega| = 3\sqrt{3}l^2 / 2 \) is the area of \( \Omega \).

For every summand in the integral above, we have

\[
\int_{\Omega_x} (\mathbf{v} \cdot \mathbf{n}_y) \, dS = \int_{\Omega_x} v_y \, dS = \sqrt{3}l / 2 \int_{A_6} \mathbf{v}_n \, dy = |\Omega| I_x,
\]

where the rectangle \( \Omega_x \), with the vertices at the points \( A_2, A_6 \), and \((-l + i \sqrt{3}l / 2), (-l - i \sqrt{3}l / 2)\) is taken instead of \( \Omega \), this is possible due to the double periodicity of \( \mathbf{v} \).

Analogously,

\[
\int_{\Omega_\omega} (\mathbf{v} \cdot \mathbf{n}_\omega) \, dS = \int_{\Omega_\omega} (\mathbf{v} \cdot \mathbf{n}_\omega) \, dS = \frac{3l}{2} \int_{A_4} \mathbf{v}_n \, dy = |\Omega| I_\omega,
\]

\[
\int_{\Omega_{\omega'}} (\mathbf{v} \cdot \mathbf{n}_{\omega'}) \, dS = \int_{\Omega_{\omega'}} (\mathbf{v} \cdot \mathbf{n}_{\omega'}) \, dS = \frac{3l}{2} \int_{A_4} \mathbf{v}_n \, dy = |\Omega| I_{\omega'},
\]

where \( \Omega_\omega \) and \( \Omega_{\omega'} \) are the rectangles with vertices at the points \( A_4, A_6, -i \sqrt{3}l, \exp(-i \pi / 6) \sqrt{3}l \) and \( A_4, A_2, i \sqrt{3}l, \exp(i \pi / 6) \sqrt{3}l \), respectively.

Thus, using (2.24), we have

\[
\langle \mathbf{v} \rangle = \frac{1}{|\Omega|} \int_{\Omega} \mathbf{v}(z) \, dS = \frac{2}{3} \left[ \mathbf{n}_y I_x + \mathbf{n}_\omega I_\omega + \mathbf{n}_{\omega'} I_{\omega'} \right] = (I_x, I_y).
\]

Similarly, using the identity

\[
\mathbf{v} \equiv \frac{2}{3} \left[ \mathbf{t}_y (\mathbf{v} \cdot \mathbf{t}_y) + \mathbf{t}_\omega (\mathbf{v} \cdot \mathbf{t}_\omega) + \mathbf{t}_{\omega'} (\mathbf{v} \cdot \mathbf{t}_{\omega'}) \right],
\]

we have

\[
|\Omega| < \rho \mathbf{v} > = \int_{\Omega} \rho \mathbf{v}(z) \, dS = \frac{2}{3} \left[ \int_{\Omega} \rho (\mathbf{t}_y (\mathbf{v} \cdot \mathbf{t}_y) + \mathbf{t}_\omega (\mathbf{v} \cdot \mathbf{t}_\omega) + \mathbf{t}_{\omega'} (\mathbf{v} \cdot \mathbf{t}_{\omega'})) \, dS \right].
\]
and, taking into account (2.25), one finds

\[ < \rho \mathbf{v} > = \frac{1}{|\Omega|} \int_\Omega \rho \mathbf{v}(z) \, dS = \frac{2}{3} \left[ t_y E_y + t_\omega E_\omega + t_{\omega'} E_{\omega'} \right] = (E_x, E_y). \]  

(2.29)

Thus, we have \( < \rho \mathbf{v} > = \mathcal{R} < \mathbf{v} > \) with \( \mathcal{R} \) being the effective resistivity tensor. In accordance with (2.28), (2.29) the last equality can be rewritten as follows,

\[ \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} I_x \\ I_y \end{pmatrix}, \]  

(2.30)

The components of the resistivity tensor \( \mathcal{R} \) can be identified from (2.21), (2.26), (2.27), as

\[ \rho_{xx} = E_x(1,0), \quad \rho_{yx} = E_y(1,0), \quad \rho_{xy} = E_x(0,1), \quad \rho_{yy} = E_y(0,1). \]

Utilizing (2.24), (2.26), (2.27), and (2.21), and after some algebra it can be shown that \( \rho_{xy} = \rho_{yx} \), as of course it should on physical grounds. The explicit formulae for the components are:

\[ \begin{align*}
\rho_{xx} &= \rho_1 \left[ M(a,c,\Delta)m^2(a,c)/(a+c)^2 - m^2(c,a)/M(c,a,-\Delta) \right] / (4\Delta \sqrt{3}), \\
\rho_{xy} &= \rho_1 \left[ M(a,c,\Delta)m(a,c)/(a+c) + (a+c)m(c,a)/M(c,a,-\Delta) \right] / (4\Delta), \\
\rho_{yy} &= \rho_1 \sqrt{3} \left[ M(a,c,\Delta) - (a+c)^2/M(c,a,-\Delta) \right] / (4\Delta),
\end{align*} \]

(2.31)

and \( \det \mathcal{R} = -\rho_1^2 M(a,c,\Delta)/M(c,a,-\Delta) \).

The components \( \sigma_{xx}, \sigma_{xy}, \sigma_{yy} \) of the effective conductivity tensor \( \mathcal{S} = \mathcal{R}^{-1} \) are:

\[ \begin{align*}
\sigma_{xx} &= \sigma_1 \sqrt{3} \left[ (a+c)^2/M(a,c,\Delta) - M(c,a,-\Delta) \right] / (4\Delta), \\
\sigma_{xy} &= \sigma_1 \left[ (a+c)m(c,a)/M(a,c,\Delta) + M(c,a,-\Delta)m(a,c)/(a+c) \right] / (4\Delta), \\
\sigma_{yy} &= \sigma_1 \left[ m^2(c,a)/M(a,c,\Delta) - M(c,a,-\Delta)m^2(a,c)/(a+c)^2 \right] / (4\Delta \sqrt{3}).
\end{align*} \]

(2.32)

These can be checked using the interchange of phases, cyclic permutations of the resistivities, this requires the results from Appendix B.

In terms of complex variables, the equality \( < \rho \mathbf{v} > = \mathcal{R} < \mathbf{v} > = \mathcal{S} < \rho \mathbf{v} > \) can be rewritten as \( I_x + iI_y = \rho_{ef}(E_x + iE_y) = \sigma_{ef}(I_x + iI_y) \) with \( \rho_{ef} (\sigma_{ef}) \) being a functional of complex effective resistivity (conductivity):

\[ \rho_{ef} = \frac{E_x + iE_y}{I_x + iI_y}, \quad \sigma_{ef} = \frac{I_x + iI_y}{E_x + iE_y}. \]  

(2.33)

Let us introduce now, as in Obnosov (1996,1999), Craster & Obnosov (2001a,b), real functionals \( \rho_x = E_x/I_x, \quad \rho_y = E_y/I_y \) (\( \sigma_x = 1/\rho_x, \quad \sigma_y = 1/\rho_y \)) of the effective resistivity (conductivity) in \( x \)- and \( y \)-directions, we get

\[ \rho_x = \rho_{xx} + \rho_{xy} \tan \theta, \quad \rho_y = \rho_{yy} + \rho_{xy} \cot \theta, \]

where \( \theta = \tan^{-1}(I_y/I_x) \) is the direction of the given external flux \( I_x + iI_y \). Thus, in general for the present structure, the quantities \( \rho_x, \rho_y \) are dependent on \( \theta \), i.e. \( \rho_x = \rho_x(\theta), \quad \rho_y = \rho_y(\theta) \); this is unlike any of the earlier structures investigated by Obnosov (1996,1999), Craster & Obnosov (2001a,b). Clearly, we have \( \rho_{xx} = \rho_x(0) \),
c = -0.5, a = 0.2j, j = -4, -3, ..., 4
a = -0.5, c = 0.2j, j = -4, -3, ..., 4

Figure 5. Two sets of ellipses of effective conductivity.

This last formula, with all of the $\sigma_j$ fixed, defines an ellipse of effective conductivity in polar coordinates $r, \theta$; this a standard visualization used in the theory of filtration (Bear 1972). Some of these ellipses, for $\rho_1 = 1$ and various values of $a, c$ are presented in figure 5; in the left panel $c = -0.5, a = -0.8$ is the inner ellipse and increasing $j$ moves outwards, in the right panel $a = -0.5, c = -0.8$ is the outer ellipse and increasing $j$ moves the ellipses inwards.

The dissipation $D = <\rho|v|^2>$, deduced as in Craster & Obnosov (2001a), is:

$$D = I_x E_x + I_y E_y = \rho_{xx} I_x^2 + \rho_{yy} I_y^2 + 2 \rho_{xy} I_x I_y.$$  \hspace{1cm} (2.34)

The same result is obtained from the complex functional $\rho_{ef}$ as $D = (I_x^2 + I_y^2) \text{Re} \{\rho_{ef}\}$. Thus, the dissipation is explicitly given by combining the formulæ (2.27), (2.31), (2.34).

\hspace{1cm} (e) Two equal phases

There are considerable simplifications when two phases have identical resistivities, and particularly simple formulæ emerge. This situation is also slightly degenerate in the full analysis, so requires some explanation.

We take two resistivities equal, say, $\rho_2 = \rho_3$, then we have a system of joined diamonds that form parallel, but offset strips. If $\rho_2 = \rho_3$, $\rho_1 > \rho_2$, and correspondingly $b = 0$, $c = -a$, and $\Delta = a$, then the solution of the problem (2.1), (2.2) is given by the formulæ (A 13) with

$$C_1 = \frac{I_\omega + I_{\omega'}}{2\Lambda^+} = -\frac{I_x}{2\Lambda^+}, \hspace{1cm} C_2 = \frac{I_\omega - I_{\omega'}}{2\Lambda^-} = -\frac{\sqrt{3} I_y}{2\Lambda^-}. \hspace{1cm} (2.35)$$

The same formulæ (A 13) and (2.35), but with $a$ replaced by $-a$, give the solution when $\rho_2 = \rho_3$ and $\rho_1 < \rho_2$.

In this case the system of diamonds have evident symmetry and the resistivity tensor is diagonal, the $\rho_{xy}, \rho_{yx}$ terms are zero, and

$$\rho_{xx} = \rho_x = \sqrt{\frac{\rho_2 (2 \rho_1 + \rho_2)}{3}}, \hspace{1cm} \rho_{yy} = \rho_y = \rho_2 \sqrt{\frac{3 \rho_1}{\rho_1 + 2 \rho_2}}. \hspace{1cm} (2.36)$$

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The formulae (2.36) can be derived from the general relations (2.31) with \( \rho_3 \) tending to \( \rho_2 \). From (2.10), (2.27) it follows that

\[
\lim_{\rho_3 \to \rho_2} m(a,c)/(a+c) = 0, \quad \lim_{\rho_3 \to \rho_2} m(c,a) = 4a,
\]

\[
\lim_{\rho_3 \to \rho_2} M(a,c,\Delta) = \frac{4a(1-a)}{\sqrt{(1+a)(3-a)}}, \quad \lim_{\rho_3 \to \rho_2} M(c,a,-\Delta) = -\frac{4a(1+a)}{\sqrt{(1-a)(3+a)}}.
\]

Notably, the final three limits are different from zero and infinity if \( a \neq 0 \) (\( \rho_1 \neq \rho_2 \)) and \( a \neq \pm 1 \).

Remark. If \( \rho_1 = \rho_2 = \rho_3 = \rho \) then, see (2.36), the effective resistivity is \( \rho_{xx} = \rho_{yy} = \rho \), \( \rho_{xy} = 0 \) as it should be. Also we note that \( \rho_{xx}(\rho_1,\rho_2)\rho_{yy}(\rho_1^{-1},\rho_2^{-1}) = 1 \) as it should from Keller’s theorem (Keller 1964, Fel et al 2000).

(f) Asymptotics

We return to the offset strips of diamonds with \( \rho_2 = \rho_3 \), and consider a further limit where either \( \rho_1 \gg \rho_2 \) or \( \rho_1 \ll \rho_2 \), in these cases one phase is highly resistive and the other conducts well; all the current is channelled through a narrow vertex (Keller 1987) and this enables asymptotic progress to be made.

We consider the slightly more general situation of diamonds that subtend an angle of \( \alpha \) at \( A_2,A_6 \) of figure 2, see also figure 6 where the dashed elementary cell has width \( 6l \sin(\alpha/2) \). The diamonds have resistivity \( \rho_1 \) and the surrounding material \( \rho_2 \), the elementary cell is also shown in figure 6. The resistivity of the corner, Keller (1987), is

\[
\rho(\alpha) = \left( \frac{(\pi - \alpha)}{\alpha} \rho_1 \rho_2 \right)^{1/2}.
\]

If \( \rho_1 \ll \rho_2 \) we consider the effective resistivity in the \( y \) direction as the resistance at the corner weighted by the geometry of the elementary cell; we assume the potential has a unit jump at the vertex, then the definition of \( \rho_{yy} \sim E_y/I_y \) yields

\[
\rho_{yy}(\alpha) = \frac{3}{2} \tan \frac{\alpha}{2} \left( \frac{(\pi - \alpha)}{\alpha} \rho_1 \rho_2 \right)^{1/2} \implies \rho_{yy}(\pi/3) = \left( \frac{3\rho_1 \rho_2}{2} \right)^{1/2}.
\]

If we now have the opposite limit, that is, \( \rho_2 \ll \rho_1 \) and consider \( \rho_{xx} \), then the strips of diamonds act a resisting strip and the current is channelled through a vertex of angle \( \pi - \alpha \). A similar argument leads to

\[
\rho_{xx}(\alpha) = \frac{2}{3} \cot \frac{\alpha}{2} \left( \frac{\alpha}{\pi - \alpha} \rho_1 \rho_2 \right)^{1/2} \implies \rho_{xx}(\pi/3) = \left( \frac{2\rho_1 \rho_2}{3} \right)^{1/2}.
\]

In the appropriate limits these results are recovered by the general solution.

Another interesting situation is to consider \( \rho_2 \neq \rho_3 \) and either \( \rho_1 \to 0 \) or \( \rho_1 \to \infty \), that is, one set of diamonds are either perfectly conducting or insulating. In the latter case, \( \rho_1 \to \infty \), and then \( \rho_{xx} \to \infty \) and \( \rho_{yy} \) is given by resistors \( \rho_2,\rho_3 \) acting in series weighted by the vertical distance across each of them, that is,

\[
\rho_{yy}(\alpha) \sim (\rho_2 + \rho_3) \cos \frac{\alpha}{2}, \quad (2.37)
\]
Figure 6. A system of strips of diamonds.

If $\rho_1 \to 0$ then $\rho_{yy} \to 0$ and $\rho_{xx}$ is given by resistors $\rho_2, \rho_3$ acting in parallel, now weighted by the horizontal distance across each cell, thereby taking into account that we are dealing with current densities. Thus

$$\rho_{xx}(\alpha) \sim \left( \frac{\rho_2 \rho_3}{\rho_2 + \rho_3} \right) \sec \frac{\alpha}{2}.$$

If $\alpha = \pi/3$ then these last two results should also emerge from our general analysis.

These super- and non-conducting cases are a bit awkward as the term $D(a,c)$ appearing in (2.21) can be zero. For the non-conducting phase ($\rho_1 = \infty$) we get, via (2.10) and (A 10), that $a = 1, c = -1, \Delta = 1$ and hence $\alpha^+ = 1/2, \alpha^- = 1/4$. That means that the first summmands in (2.9) have non-integrable singularities at all points congruent with $A_2, A_4, A_6$; we have to take $c_1 = 0$ and $A_1^-= 0, A_2^- = -4e^{i\pi/4}, A_3^- = -4e^{-i\pi/4}$. From (2.16), (2.18) it follows that $4\Lambda^- c_2 = -I_\omega = I_\omega'$, which is natural from a physical point of view. Hence, the effective resistivity along the $x$-axis is infinite ($I_x = 0$), and $\rho_{yy}$, due to (2.26), (2.25), is

$$\rho_{yy} = \frac{\sqrt{3}}{2}(\rho_2 + \rho_3).$$

The last result agrees with (2.31) when $\rho_2 = \rho_3$ and $\rho_1 \to \infty$, and with the physical argument presented earlier leading to (2.37).

Now we set $\Omega_1$ to be superconducting, i.e. $\rho_1 \to 0$, so $a = -1, c = 1, \Delta = 1$ and hence $\alpha^+ = 1/2, \alpha^- = 1/4$. Once again we take $c_1 = 0$ and the general solution (2.9) depends on one parameter with coefficients $\lim_{\rho_1 \to 0} A_k^-/(1 - c)$ instead of (2.13). The limits required are

$$A_1^- = -2b \frac{1 - b}{1 + b} e^{i\pi/4} + 2be^{-i\pi/4}, \quad A_2^- = -2b \frac{1 - b}{1 + b} e^{-i\pi/4}, \quad A_3^- = 2be^{i\pi/4}.$$

Correspondingly, the average flux through $A_4A_6$ and $A_4A_2$ can be found as the limits of the ratios $I_\omega/(1 - c)$ and $I_\omega'/(1 - c)$ with $\rho_1 \to 0$. Ultimately, we have

$$I_x = -\frac{2}{1 + b} I_\omega, \quad I_y = \frac{2b/\sqrt{3}}{1 + b} I_\omega.$$

and $c_2 = -I_\omega/(2b\Lambda^-)$. The corresponding limits of expressions (2.26), divided by $1 - c$, are $E_x = -2\rho_3 I_\omega/\sqrt{3}, E_y = 0$. So, the effective resistivities along $x$- and $y$-axes for this limit situation are

$$\rho_{xx} = \frac{2\rho_2 \rho_3}{\sqrt{3}(\rho_2 + \rho_3)}, \quad \rho_{yy} = 0.$$

These final results are in agreement with (2.36) and the earlier physical arguments.
3. Concluding remarks

As a continuation of the approach used for the four-phase rectangular checkerboard considered by the authors (Craster & Obnosov 2001b) we have shown that, at least, one other structure is solvable using the same basic idea. That is, we again use conformal mappings to reduce a doubly periodic problem to one posed on overlying Riemann sheets, these have identical boundary conditions on the upper and lower sheets, and so can be solved directly - although it does require some effort to do so. The structure we consider here, has three distinct phases, and is composed of strips of diamonds. It remains unclear how general our approach is, and whether further non-trivial structures are also explicitly solvable.

The present structure should be of value for several reasons: first, it provides a non-trivial solution leading to a non-diagonal resistivity tensor, and as such it not only increases the number of explicit solutions in the literature, but does so in a useful way, all other solutions lead to diagonal tensors through symmetry. Second, numerical schemes and bounding methods require exact solutions in order to measure how accurate they are, and how robustly they model real structures. When we reduce to two phase structures, or highly contrasting media, it is clear that reductions ensue and the formulae are much simpler; these reduced cases are useful as checks upon the more complex structure, and demonstrate the utility of Keller's (1987) asymptotic approach. Finally, the structure itself is of interest and others have recently embarked upon studies of three phase composites with applications in physics (Fel et al. 2000).

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Appendix A. The solution for three joined equal sectors

We require a solution to the structure of figure 2(b), and the problem is stated in equations (2.7,2.8). The structure is composed of three equal sectors $\Omega_j$ each with solution $v_j$, and a constant resistivity, $\rho_j$, $j = 1, 2, 3$; the sectors are numbered clockwise, beginning from $\Omega_1 = \{z : -\pi/3 < \arg z < \pi/3\}$.

In terms of a function $w(z) = zv(z)$ the boundary conditions (2.7) take the form

\begin{align}
\text{Re} [\rho_1 w_1(\zeta) - \rho_2 w_2(\zeta)] &= \text{Im} [w_1(\zeta) - w_2(\zeta)] = 0, & 0 < \zeta < \infty, \\
\text{Re} [\rho_1 w_1(\zeta) - \rho_3 w_3(\zeta)] &= \text{Im} [w_1(\zeta) - w_3(\zeta)] = 0, & 0 < \zeta < \infty, \\
\text{Re} [\rho_2 w_2(\zeta) - \rho_3 w_3(\zeta)] &= \text{Im} [w_2(\zeta) - w_3(\zeta)] = 0, & -\infty < \zeta < 0,
\end{align}

where, we recall $\zeta = e^{i\pi/3}$. The solutions of this boundary–value problem, $w_j(\zeta)$, should vanish at $\zeta = 0$, and all functions $w_j(1/\zeta)$ should have at $\zeta = 0$ a singularity less than $3/2$.

To place this in a more malleable form we introduce the functions $u_j(\zeta)$, for $j = 1, 2, 3$, as:

\begin{align}
u_1(\zeta) = w_1(\zeta^{1/3}), & \quad u_2(\zeta) = w_2(e^{-12\pi/3}\zeta^{-1/3}), & \quad u_3(\zeta) = w_3(e^{12\pi/3}\zeta^{-1/3}) \quad (A 2)
\end{align}

that are defined and holomorphic in $\zeta$–plane with a cut along the negative part of real axis: $\Omega = \mathbb{C} \setminus \mathbb{R}^\minus$. The branch of the cube root in (A 2) is fixed in $\Omega$ by the condition $-\pi < \arg \zeta < \pi$. We further introduce three functions $u_{4,5,6}$ defined via

\begin{align}
u_{3+k}(\zeta) = \overline{u_k(\zeta)}, & \quad k = 1, 2, 3. \quad (A 3)
\end{align}
Then using equations (A.2) and (2.10) one can show that the boundary–value problem (A.1) is equivalent to the vector Riemann-Hilbert problem

\[ u^+(\xi) = u^-(\xi)G, \quad \text{for } \xi < 0, \quad u(\zeta) = (u_1(\zeta), u_2(\zeta), \ldots, u_6(\zeta)) \]  

(A.4)

where, we recall that \( a, b, c \) are defined in (2.10). The matrix \( G \) is

\[
G = \begin{pmatrix}
0 & 1 + a & 0 & -a & 0 & 0 \\
0 & 0 & 1 + b & 0 & -b & 0 \\
1 + c & 0 & 0 & 0 & 0 & -c \\
c & 0 & 0 & 0 & 0 & 1 - c \\
0 & a & 0 & 1 - a & 0 & 0 \\
0 & 0 & b & 0 & 1 - b & 0
\end{pmatrix}.
\]  

(A.5)

Due to (A.3) the vector \( u \) must satisfy the identity

\[ \overline{u(\zeta)} \equiv u(\zeta)P, \quad \text{where } P = \begin{pmatrix} O & I \\ I & O \end{pmatrix}, \]  

(A.6)

and \( P \) is the 6 \times 6 permutation block matrix involving \( O, I \) that are the 3 \times 3 zero and identity matrices respectively.

In addition, from (A.2) and the behaviour of \( w(\zeta) \) at the origin and infinity,

\[ u_j(0) = 0 \quad \text{and} \quad u_j(\zeta) = o(\zeta^{1/2}) \quad \text{in the vicinity of infinity } (j = 1, \ldots, 6). \]  

(A.7)

Thus, the initial problem (2.7), (2.8) is equivalent to (A.4), (A.6), (A.7) with solution \( u \). Given \( u(\zeta) \) then \( w_1(\zeta) = u_1(\zeta^3), w_2(\zeta) = u_2(e^{2\pi i}\zeta^3), w_3(\zeta) = u_2(e^{-2\pi i}\zeta^3) \) solves (A.1), and

\[ v_1(\zeta) = u_1(\zeta^3)/\zeta, \quad v_2(\zeta) = u_2(e^{2\pi i}\zeta^3)/\zeta, \quad v_3(\zeta) = u_2(e^{-2\pi i}\zeta^3)/\zeta. \]  

(A.8)

is the general solution of (2.7), (2.8).

Thus we require \( u \): Noting that \( G^{-1} = PGP \) and given \( u_0(\zeta) \) as any solution of (A.4) and (A.7), then \( \overline{u_0(\zeta)}P \) is also a solution. Hence,

\[ u(\zeta) = \overline{u_0(\zeta)}P + u_0(\zeta) \]  

(A.9)

is a solution of the Riemann boundary–value problem (A.4), (A.7), satisfying the symmetry condition (A.6).

The matrix (A.5) has two coincident eigenvalues \( \lambda_1 = \lambda_2 = 1 \), and two pairs of complex conjugated eigenvalues \( \lambda_3 = \overline{\lambda_4} = \lambda_5^2 \) and \( \lambda_5 = \overline{\lambda_6} = \lambda_5^2 \), where

\[ \lambda_{\pm} = e^{\pm i2\pi \alpha} = -\frac{1 \pm \Delta}{2} + \frac{i}{2} \sqrt{(3 \pm \Delta)(1 \mp \Delta)} \]  

(A.10)

\[ \Delta = -(ab + bc + ca) = \frac{\rho_1(\rho_2 - \rho_3)^2 + \rho_2(\rho_3 - \rho_1)^2 + \rho_3(\rho_1 - \rho_2)^2}{(\rho_1 + \rho_2)(\rho_2 + \rho_3)(\rho_3 + \rho_1)} \geq 0, \]  

(A.11)

and

\[ 1 - \Delta^2 = \frac{(1 - a^2)(1 - c^2)}{1 + ac} \geq 0, \]

\[ \]
and we take $0 \leq \Delta \leq 1$.

From (A10) it follows that $\lambda_{\pm}^{\pm} e^{i \pi \alpha \pm}$ is given by (2.11). We take the branch of the analytic functions $\zeta^\nu$ ($\nu \in \mathbb{R}$) to be fixed by the condition $|\arg \zeta| \leq \pi$, i.e. the branch in the $\zeta$–plane is chosen to have the cut along the negative part of the real axis. The branch chosen satisfies the condition $(\zeta^\nu) = \zeta^\nu$.

A solution of the problem (A4), vanishing at the origin, is a linear combination of functions $A_{n\pm}^{\pm}$, where $n$ is an arbitrary non-negative integer and $A$ is a constant vector. Due to (2.11) we have $1/3 \leq \alpha^+ \leq 1/2$ and $1/4 \leq \alpha^- \leq 1/3$, and hence only two functions $A_{2,3}^{\pm} \zeta^{\alpha \pm}$ will satisfy both conditions (A7). These functions are linearly independent, hence every one of them, with properly taken vector $A_{\pm}^{\pm}$, will be a solution of the problem (A4). Let us substitute these two functions into the boundary condition (A4), then we get

$$A_{\pm}^{\pm} \lambda_\pm = A_{\pm}^{\pm} \lambda_\pm G, \quad \text{or} \quad A_{\pm}^{\pm} (G - \lambda_\pm^3 I) = 0, \quad \text{or} \quad (G' - \lambda_\pm^3 I) (A_{\pm}^{\pm})' = 0,$$

where prime denotes the transpose and $I$ is the identity matrix. Thus, the vector $A^+$ ($A^-$) is the transposed eigenvector of the matrix $G'$ corresponding to the eigenvalue $\lambda_\pm^3$ ($\lambda_\pm^3$). Additionally, due to the symmetry condition (A6) and the identity $(\zeta^\nu) = \zeta^\nu$, vectors $A_{\pm}^{\pm} = (A_{1}^{\pm}, A_{2}^{\pm}, \ldots, A_{6}^{\pm})$ have to satisfy the condition $A_{\pm}^{\pm} = \tilde{A}_{\pm}^{\pm} P$, where $P$ is the permutation matrix (A6). Hence

$$A_{k}^{\pm} = \tilde{A}_{k+3}^{\pm}, \quad k = 1, 2, 3. \quad (A12)$$

If $A$ is an arbitrary eigenvector of the matrix $G'$ corresponding to an eigenvalue $\lambda$, then both vectors $\tilde{A}P$ and $A + \tilde{A}P$ are its eigenvectors too, and the latter satisfies the condition (A12).

Utilizing symbolic algebra (Mathematica) we get the eigenvectors $A_{\pm}^{\pm}$, but in a complicated form. The last ones can be simplified using the relations (2.10), (A10), (A11) and following useful identities

$$abc + a + b + c = 0, \quad (1 \pm a)(1 \pm b)(1 \pm c) = (1 + ab)(1 + bc)(1 + ca) = 1 - \Delta^2,$$

$$(\Delta + a)(\Delta + b)(\Delta + c) = -(\Delta - a)(\Delta - b)(\Delta - c) = -(a + b)(b + c)(c + a) = abc(1 - \Delta^2),$$

$$1 + \lambda_\pm^3 + \overline{\lambda}_\pm^3 = \mp \Delta, \quad 1 + \lambda_\pm^3 = \sqrt{1 + \Delta \lambda_\pm},$$

$$\pm \Delta + \lambda_\pm^3 = - \sqrt{1 + \Delta \lambda_\pm}, \quad \lambda_\pm^3 = -(1 \pm \Delta) \lambda_\pm - \overline{\lambda}_\pm.$$

Ultimately, we get the components of the vectors $A^+$ and $A^-$:

$$\tilde{A}_{1}^{+} = \sqrt{1 - \frac{b}{a}}(ac - \Delta), \quad \tilde{A}_{2}^{+} = \frac{1 + i \sqrt{\frac{a}{b}}}{2} (\Delta \lambda_+ - b \overline{\lambda}_+), \quad \tilde{A}_{3}^{+} = \Delta \overline{\lambda}_+ - ab \lambda_+,$$

$$\tilde{A}_{4}^{+} = \sqrt{1 - \frac{b}{a}}(a \lambda_+^3 - c \overline{\lambda}_+^2), \quad \tilde{A}_{5}^{+} = \frac{1 + i \sqrt{\frac{a}{b}}}{2} (b \lambda_+ + c \overline{\lambda}_+), \quad \tilde{A}_{6}^{+} = a \lambda_+ - b \overline{\lambda}_+;$$

$$\tilde{A}_{1}^{-} = i \sqrt{1 + \frac{b}{a}}(ac + \Delta), \quad \tilde{A}_{2}^{-} = - i \frac{1 + i \sqrt{\frac{a}{b}}}{2} (\Delta \lambda_- + b \overline{\lambda}_-), \quad \tilde{A}_{3}^{-} = - i (\Delta \overline{\lambda}_- + ab \lambda_-),$$

$$\tilde{A}_{4}^{-} = i \sqrt{1 + \frac{b}{a}}(a \lambda_-^3 + c \overline{\lambda}_-^2), \quad \tilde{A}_{5}^{-} = i \frac{1 + i \sqrt{\frac{a}{b}}}{2} (b \lambda_- - c \overline{\lambda}_-), \quad \tilde{A}_{6}^{-} = i (a \lambda_- - b \overline{\lambda}_-).$$

Naturally, care is required if $\lambda_\pm^3 \neq b/a = -1$, i.e. $\Delta \neq 1$, which takes place for limit situations of $\rho_0 = 0$ or $\rho_0 = \infty$ (the equality $\lambda_\pm^3 = b/a$ is not possible).

Finally, in accordance with $A + \tilde{A}P$ one obtains eigenvectors satisfying the condition (A12). We only need the first three components ($A_1^+, A_2^+, A_3^+$) and
(i \(A_i\), i \(A_j\), i \(A_k\)) of these vectors. After some simple transformations, with the conditions \(c \neq 1\) and \(a \neq -1\), i.e. \(\rho_1 \neq 0\), one, finally, derives \(A_k^\pm\) as in(2.12),(2.13).

Hence, also utilizing (A.8), we obtain the general solution for the \(v_j(\zeta)\) as equation (2.9) of the main text.

\(\text{(a) A special case}\)

We consider the special case, \(\rho_2 = \rho_3\), then due to the representations (2.10), (A 11) \(b = 0\), \(c = -a\), and \(\Delta = a\). From (2.12), (2.13), (2.9) it follows that:

\[v_1(\zeta) = -C_1\sqrt{1-a}^a 3^{a+1-1} + i C_2 \frac{\rho_2}{\rho_1} \sqrt{1+a}^a 3^{a-1}, \quad \zeta \in \Omega_1,\]

\[v_2(\zeta) = C_1 \lambda_2^a 3^{a+1-1} - i C_2 \lambda_2^a 3^{a-1}, \quad \zeta \in \Omega_2, \quad (A 13)\]

\[v_3(\zeta) = C_1^\lambda \lambda_2^a 3^{a+1-1} - i C_2^\lambda \lambda_2^a 3^{a-1}, \quad \zeta \in \Omega_3,\]

where

\[\alpha^+ = \alpha^-(a) = \frac{1}{\pi} \arccos \frac{\sqrt{1+a}}{2}, \quad C_k = 2a(1+a)c_k.\]

If \(\rho_2 = \rho_3\) and \(a < 0\) \((\rho_1 < \rho_2)\) then \(\Delta = -a\) and all the coefficients in (2.12), (2.13) are zero (these coefficients also vanish if \(a = -1\) and \(c = 1\), i.e. \(\rho_1 = 0\) and \(\rho_3 = \infty\)). To overcome this, one has to multiply the \(\hat{A}_k^\pm\) by \(i\) and use

\[\alpha^\pm(a) = \alpha^\mp(-a) = \frac{1}{\pi} \arccos \frac{\sqrt{1+a}}{2}, \quad C_k = 2a(1+a)c_k.\]

The resultant equations are identical to (A 13) to within arbitrary constants; they coincide as \(\alpha^\pm(a) = \alpha^\mp(-a)\) and \(\lambda^\pm(a) = \lambda^\mp(-a)\). Hence, equations (A 13) hold for both \(\rho_1 > \rho_2\) and \(\rho_1 < \rho_2\).

It might appear that there is a contradiction in formulae (A 13) with, apparently, \(v_2(\zeta) \neq v_3(\zeta)\), although physically \(v_2(\zeta) \equiv v_3(\zeta)\). The branches of \(\zeta^{a+1-1}\) and \(\zeta^{a-1}\) are fixed in the \(z\)-plane with the cut along the negative part of the real axis, i.e. by the condition \(-\pi < \arg \zeta < 0\). Hence, \(v_2(\zeta) \equiv v_3(\zeta) = -c_1|\zeta|^{a+1} + i c_2|\zeta|^{a-1}\) for all \(\zeta < 0\); thus \(v_2(\zeta)\) and \(v_3(\zeta)\) in (A 13) are analytical continuations of each other through \(R^\pm\).

\(\text{(b) Useful properties}\)

We use several properties of \(v_\alpha(\zeta)\) (2.9) freely in the main text, these arise from interchanging phases, or cyclic permutation. First, we label \(v(\zeta)\) to explicitly show its dependence on \(a, b, c\) given by (2.10), and on the arbitrary parameters \(c_1, c_2\); the solution is written as \(v(\zeta; a, b, c; c_1, c_2)\).

Interchanging phases \(\Omega_1\) and \(\Omega_2\), and correspondingly altering the solution (2.9), and using (2.10), (A 10), (A 11) gives the solution as \(v(\zeta; -a, -c, -b; c_1, c_2)\). Now the general solution of the initial problem (2.7),(2.8) can be taken in the form

\[\hat{v}(\zeta; a, b, c; c_1, c_2) = \frac{\hat{v}(\zeta; a, b, c; c_1, c_2)}{\zeta}.\]

Comparison of this with (2.9) relates the coefficients \(A_1^+, A_2^-\) and \(\hat{A}_j^+, \hat{A}_j^\pm\), i.e.

\[A_1^+(a, b, c) = \frac{A_2^+(a, b, c)}{A_2^-(a, -c, -b)} = \frac{A_3^+(a, b, c)}{A_3^-(-a, -c, -b)} = k^\pm.\]
It can also be shown that
\[ \frac{\dot{k}^-}{k^+} = \frac{\rho_2}{\rho_1} \dot{k}^- = -\frac{\Delta - c}{\Delta + b} = \frac{a + c}{\Delta - a} \]
and hence
\[ v(\zeta; a, b, c; c_1, c_2) \equiv \varepsilon^2 v(\varepsilon^{-2} \zeta; -a, -c, -b; c_1 k^+, -c_2 k^-). \quad (A 14) \]

In an analogous way, interchanging \( \rho_1, \rho_3 \) \((\rho_1 \leftrightarrow \rho_3)\), the resultant general solution is
\[ \tilde{v}(\zeta; a, b, c; c_1, c_2) = \varepsilon^{-2} v(\varepsilon^{2} \zeta; -b, -a, -c; c_1, -c_2). \]

For the ratios \( A_j^\pm / \tilde{A}_j^\pm \) the equalities
\[ \frac{A_j^\pm (a, b, c)}{A_j^\pm (-b, -a, -c)} = \frac{A_j^\pm (a, b, c)}{A_j^\pm (-b, -a, -c)} = \frac{\Delta}{\Delta - b} = \frac{a + c}{\Delta - a}. \]

Thus, the following identity occurs
\[ v(\zeta; a, b, c; c_1, c_2) = \varepsilon^{-2} v(\varepsilon^{2} \zeta; -b, -a, -c; c_1 k^+, -c_2 k^-). \quad (A 15) \]

Using the substitution \( \rho_2 \leftrightarrow \rho_3 \) one finds that
\[ v(\zeta; a, b, c; c_1, c_2) \equiv \varepsilon^2 v(\zeta; -c, -b, a; -c_1, -c_2). \quad (A 16) \]

Using cyclic permutation \( \rho_1 \to \rho_2 \to \rho_3 \to \rho_1 \), and its inverse \( \rho_1 \to \rho_3 \to \rho_2 \to \rho_1 \), we get a new pair of representations for the solution to (2.7), (2.8):
\[ \tilde{v}(\zeta; a, b, c; c_1, c_2) = \varepsilon^2 v(\zeta; b, c, a; c_1, c_2), \quad \tilde{v}(\zeta; a, b, c; c_1, c_2) = \varepsilon^{-2} v(\varepsilon^{-2} \zeta; c, a, b; c_1, c_2). \]

Furthermore,
\[ \tilde{v}(\zeta; a, b, c; c_1, c_2) \equiv \tilde{v}(\zeta; a, b, c; c_1, -c_2), \quad \tilde{v}(\zeta; a, b, c; c_1, c_2) \equiv \tilde{v}(\zeta; a, b, c; c_1, -c_2), \]
and
\[ \tilde{v}(\zeta; b, c, a; c_1, -c_2) \equiv \tilde{v}(\varepsilon^{-2} \zeta; c, a, b; c_1, c_2), \quad \tilde{v}(\zeta; c, a, b; c_1, c_2) \equiv \tilde{v}(\varepsilon^{-2} \zeta; c, a, b; c_1, -c_2). \]

**Appendix B. Evaluation of flux integrals**

The substitution \( t = \zeta(\tau) \) is used in the flux integrals (2.15), where \( \zeta(z) \) is defined by (2.4). We also analytically continue the hypergeometric function in (2.4) in the vicinity of the infinity (Abramowitz and Stegun 1969 eq.15.3.7), as
\[ z(\zeta) = le^{i \pi/3} \mp i 2 \Lambda \zeta^{-1/2} F(1/6, 1/2; 7/6; \zeta^{-3}), \quad (B 1) \]

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for \( \zeta \in \mathbb{C}^\pm \). Using the substitution (B1) reduces the integral to

\[
\int_{O_3}^A v_3(t) dt = -i2\Lambda \int_{\varepsilon^2}^{\infty} v_3(\tau) d(\tau^{-1/2}F(\tau)),
\]

where \( F(\zeta) = F(1/6,1/2;7/6;\zeta^{-3}) \) and \( \Lambda \) is defined in (2.4). Furthermore (2.9) allows us to transform this to

\[
i2\Lambda \int_{\varepsilon^2}^{\infty} \left(c_1 A_3^+ \text{Re}(3\alpha^+ - 1)\tau^{3\alpha^+ - 5/2} + i c_2 A_3^- \text{Re}(3\alpha^- - 1)\tau^{3\alpha^- - 5/2}\right) F(\tau)d\tau - i2\Lambda(c_1 A_3^+ + i c_2 A_3^-)F(1).
\]

With the assistance of Gradshteyn & Ryzhik (1980) equation 15.1.20 one obtains

\[
I^\pm = \int_{\varepsilon^2}^{\infty} \tau^{3\alpha^\pm - 5/2} F(1/6,1/2;7/6;\tau^{-3})d\tau = -\frac{\lambda_\pm\sqrt{\pi}/6}{(1 - 3\alpha^\pm)} \left[\Gamma(1/6)/\Gamma(2/3) - \Gamma(1/2 - \alpha^\pm)/\Gamma(1 - \alpha^\pm)\right].
\]

(B2)