ON THE DIAMETERS OF MCKAY GRAPHS FOR FINITE SIMPLE GROUPS

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Abstract. Let G be a finite group, and α a nontrivial character of G. The McKay graph $\mathcal{M}(G,\alpha)$ has the irreducible characters of G as vertices, with an edge from χ_1 to χ_2 if χ_2 is a constituent of $\alpha\chi_1$. We study the diameters of McKay graphs for simple groups G of Lie type. We show that for any α, the diameter is bounded by a quadratic function of the rank, and obtain much stronger bounds for $G = \mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$.

1. Introduction

For a finite group G, and a (complex) character α of G, the *McKay graph* $\mathcal{M}(G,\alpha)$ is defined to be the directed graph with vertex set $\mathrm{Irr}(G)$, there being an edge from χ_1 to χ_2 if and only if χ_2 is a constituent of $\alpha\chi_1$. The famous McKay correspondence [11] shows that if G is a finite subgroup of $\mathrm{SU}_2(\mathbb{C})$ and α is the corresponding 2-dimensional character of G, then $\mathcal{M}(G,\alpha)$ is an affine Dynkin diagram of type A, D or E. The purpose of this paper is to initiate the study of McKay graphs for simple groups, focussing particularly on their diameters.

By a classical result of Burnside and Brauer [3], $\mathcal{M}(G,\alpha)$ is connected if and only if α is faithful, and moreover in this case an upper bound for the diameter $\mathsf{diam}\mathcal{M}(G,\alpha)$ is given by $N-1$, where N is the number of distinct values of α. (Indeed, in this case $\sum_{j=0}^{N-1}\alpha^j$ contains every irreducible character of G. Taking β to be an irreducible constituent of $\bar{\chi}_1\chi_2$, we can find $0 \le j \le N-1$ such that

$$0 < [\alpha^j,\beta]_G \le [\alpha^j,\bar{\chi}_1\chi_2]_G = [\alpha^j\chi_1,\chi_2]_G,$$

i.e. a directed path of length j connects χ_1 to χ_2.)

An obvious lower bound for $\mathsf{diam}\mathcal{M}(G,\alpha)$ (when $\alpha(1) > 1$) is given by $\frac{\log \mathsf{b}(G)}{\log \alpha(1)}$, where $\mathsf{b}(G)$ is the largest degree of an irreducible character of G. One can do slightly better, by observing that if $d := \mathsf{diam}(M,\alpha)$, then

$$2\alpha(1)^d > \sum_{i=0}^{d}\alpha(1)^i \ge \sum_{\chi\in\mathrm{Irr}(G)}\chi(1) > \Big(\sum_{\chi\in\mathrm{Irr}(G)}\chi(1)^2\Big)^{1/2} = |G|^{1/2}.$$

It follows that

$$\mathsf{diam}\mathcal{M}(G,\alpha) \ge \tfrac{1}{2}\frac{\log(|G|/4)}{\log\alpha(1)}.$$

This bound is far from tight for many groups G. However, for finite simple groups we make the following conjecture.

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Conjecture 1. *There is an absolute constant C such that for any finite non-abelian simple group G of Lie type, and any nontrivial irreducible character α of G,*

$$\mathsf{diam}\mathcal{M}(G,\alpha) \le C\frac{\log|G|}{\log\alpha(1)}.$$

Note that [9] gives the analogous bound for conjugacy classes: namely, for a nontrivial conjugacy class S of a finite (non-abelian) simple group G, we have $\mathsf{diam}\Gamma(G,S) \le C\frac{\log|G|}{\log|S|}$, where $\Gamma(G,S)$ is the Cayley graph of G with respect to S.

In general, Conjecture 1 cannot hold for arbitrary faithful character of G. However, once it holds for faithful irreducible characters, then it also holds for all faithful *multiplicity-free* characters, albeit with a different constant C. To see this, note that Theorem 1.1 of [10] implies that the number $r_m(G)$ of irreducible characters of degree m of a non-abelian finite simple group G satisfies $r_m(G) = o(m^{1+\epsilon})$ for any fixed $\epsilon > 0$. This implies that G has at most m^c irreducible characters of degree at most m, where c is an absolute constant. Now let β be a faithful multiplicity-free character of G, and let α be an irreducible constituent of β of maximal degree. Then $\beta(1) \le \alpha(1)^{c+1}$, so assuming Conjecture 1 we obtain

$$\mathsf{diam}\mathcal{M}(G,\beta) \le \mathsf{diam}\mathcal{M}(G,\alpha) \le C\frac{\log|G|}{\log\alpha(1)} \le C(c+1)\frac{\log|G|}{\log\beta(1)}.$$

In this paper we prove Conjecture 1 for many families of simple groups of Lie type.

Theorem 2. *There is an absolute constant C such that $\mathsf{diam}\mathcal{M}(G,\alpha) \le Cr^2$ for any finite simple group G of Lie type of rank r and any nontrivial irreducible character α of G. Hence Conjecture 1 holds for simple groups of Lie type of bounded rank.*

Our proof of Theorem 2 shows that in fact one can take $C = 489$.

Note that the character covering number $\mathsf{ccn}(G)$ of a finite simple group G was defined by Arad, Chillag and Herzog [1] as the minimal positive integer m such that, for any non-trivial irreducible character α of G, α^m contains all irreducible characters of G as constituents. It is proved in [1] that $\mathsf{ccn}(G)$ is bounded above by an explicit quadratic function of $k(G)$, the number of conjugacy classes of G. For G of Lie type of rank r over the field with q elements, $k(G)$ is roughly q^r [5], yielding $\mathsf{ccn}(G) = O(q^{2r})$.

Note that for any finite non-abelian simple group G we have

$$D(G) := \max_{1_G \neq \alpha \in \mathrm{Irr}(G)} \mathsf{diam}\mathcal{M}(G,\alpha) \le \mathsf{ccn}(G) \le 2D(G)(k(G)-1).$$

Indeed, if $\mathsf{ccn}(G) = N$, then, for any nontrivial $\alpha \in \mathrm{Irr}(G)$, α^N contains all $\chi \in \mathrm{Irr}(G)$, and so, as explained above, $\mathsf{diam}\mathcal{M}(G,\alpha) \le N$. Conversely, suppose $\mathsf{diam}\mathcal{M}(G,\alpha) = D$. By Burnside's lemma, the number of real-valued irreducible characters of G is equal to the number of real conjugacy classes of G, which is at least 2 since $|G|$ is even. Hence we can find a nontrivial real-valued character $\beta \in \mathrm{Irr}(G)$. Now, from 1_G we can get to β by a path of length $1 \le l \le D$ in $\mathcal{M}(G,\alpha)$, i.e. α^l contains β, whence α^{2l} contains 1_G. By [1, Corollary 1.4(b)], $\alpha^{2l(k(G)-1)}$ contains every irreducible character of G. Hence, $\alpha^{2D(G)(k(G)-1)}$ contains every irreducible character of G by [1, Lemma 1.3(a)].

Moreover, if G is of Lie type, then, choosing β to be the Steinberg character St of G, we then have that β^3 contains every irreducible character of G by Proposition 2.1 (below), whence the same holds for α^{3l}. This shows that

$$D(G) \le \mathsf{ccn}(G) \le 3D(G)$$

in this case. As a consequence, our bound in Theorem 2 on the diameters of all McKay graphs $\mathcal{M}(G,\alpha)$ yields a much stronger bound $\mathsf{ccn}(G) \le 1467r^2$ for any simple group of

Lie type of rank r. We will return to the problem of bounding $\mathsf{ccn}(G)$ in a forthcoming paper.

For classical groups of unbounded rank, we are able to handle the projective special linear and unitary groups $\mathrm{PSL}_n^\epsilon(q)$ (with $\mathrm{PSL}^+ = \mathrm{PSL}$ and $\mathrm{PSL}^- = \mathrm{PSU}$), where q is large compared to n. The proof uses major new advances in the theory of character ratios, taken from [2, 12].

Theorem 3. *There exist an absolute constant C and a function $g : \mathbb{N} \to \mathbb{N}$ such that the following holds. If $G = \mathrm{PSL}_n^\epsilon(q)$ with $n \geq 2$, $\epsilon = \pm$, and $q > g(n)$, then*

$$\mathsf{diam}\mathcal{M}(G, \alpha) < C\frac{\log|G|}{\log \alpha(1)},$$

for all nontrivial irreducible characters α of G.

As one can see from our proof, C can be taken to be 15, and g can also be made explicit.

Theorem 3 gives rise to the following extension.

Corollary 4. *With the function $g(n)$ and C as in Theorem 3, we have*

$$\mathsf{diam}\mathcal{M}(G, \alpha) < 2.5C\frac{\log|G|}{\log \alpha(1)},$$

for any simple group $G = \mathrm{PSL}_n^\epsilon(q)$ with $n \geq 2$, $\epsilon = \pm$, $q > \max\{g(n), 11\}$, and for any faithful multiplicity-free character α of G.

For other types of classical groups of unbounded ranks, we have not yet been able to prove the conjecture, but we do have results bounding the diameter by a linear function of the rank. These results, which require much more work, as well as new character bounds, will be discussed in a forthcoming paper.

As for alternating groups $G = \mathsf{A}_n$, a theorem of Zisser [15] shows that $\mathsf{ccn}(\mathsf{A}_n) = n - \lceil\sqrt{n}\rceil$ for every integer $n \geq 6$. This obviously implies $\mathsf{diam}\mathcal{M}(\mathsf{A}_n, \alpha) \leq n - \lceil\sqrt{n}\rceil$. Nevertheless, we offer (in §4) a short proof of Theorem 5 giving a weaker upper bound $4n - 4$, which is still of the right magnitude; moreover, various ideas of the proof can and will be applied in a forthcoming paper to bound $\mathsf{diam}\mathcal{M}(G, \alpha)$ for several further families of simple groups.

This paper is organized as follows. In Section 2 we prove Theorem 2. Section 3 is devoted to the proof of Theorem 3 and Corollary 4, using new developments in character bounds (see [2, 12]). In Section 5 we briefly discuss the diameter of McKay graphs for quasi-simple groups.

2. Preliminaries and groups of bounded rank

Our proof uses the following results, taken from [7] and [6].

Proposition 2.1. *Let G be a finite simple group of Lie type, and let St denote the Steinberg character of G. Then provided G is not a unitary group in odd dimension, St^2 contains every irreducible character of G as a constituent. In all cases, St^3 contains every irreducible character.*

Proof. The first statement is [7, Theorem 1.2]. Consider the exceptional case $G = \mathrm{PSU}_n(q)$ with $2 \nmid n \geq 3$. Then, again by [7, Theorem 1.2], St^2 contains all $\chi \in \mathrm{Irr}(G)$ but the unique unipotent character α of degree $(q^n - q)/(q + 1)$. Let $\chi \in \mathrm{Irr}(G)$ and suppose that $\mathsf{St} \cdot \bar{\chi}$ is a multiple of α:

$$\mathsf{St} \cdot \bar{\chi} = k\alpha$$

for some $k \in \mathbb{Z}$. Then $k = \mathsf{St}(1)\chi(1)/\alpha(1) \neq 0$, and so $\alpha(t) = \mathsf{St}(t) \cdot \bar{\chi}(t)/k = 0$ for any transvection $t \in G$. However, $\alpha(t) = -(q^n - q(-1)^n)/(q+1) \neq 0$ by [14, Lemma 4.1], a contradiction. Hence $\mathsf{St} \cdot \bar{\chi}$ contains some character $\beta \in \mathrm{Irr}(G) \smallsetminus \{\alpha\}$. It follows that

$$0 < [\mathsf{St} \cdot \bar{\chi}, \beta]_G \leq [\mathsf{St} \cdot \bar{\chi}, \mathsf{St}^2]_G = [\mathsf{St}^3, \chi]_G,$$

i.e. χ is an irreducible constituent of St^3.

A similar argument as above shows that St^3 contains all irreducible characters of G, for any simple group G of Lie type. ■

Proposition 2.2. [6] *Let G be a finite simple group of Lie type over $\mathbb{F}_q$, and let $1 \neq g \in G$. Then for any $\chi \in \mathrm{Irr}(G)$,*

$$\frac{|\chi(g)|}{\chi(1)} \leq \min\left(\frac{3}{\sqrt{q}}, \frac{19}{20}\right).$$

We can now prove Theorem 2. Let G be a simple group of Lie type over a field $\mathbb{F}_q$ (of characteristic p) of rank r, and let $G_{\rm ss}$ denote the set of semisimple elements of G. Recall (see [4, 6.4.7]) that the values of the Steinberg character St are

$$\mathsf{St}(g) = \begin{cases} \epsilon_g |\mathbf{C}_G(g)|_p, & \text{if } g \in G_{\rm ss}, \\ 0, & \text{if } g \notin G_{\rm ss}, \end{cases} \tag{2.1}$$

where $\epsilon_g = \pm 1$.

Lemma 2.3. *There is an absolute constant D such that for any $l \geq Dr^2$ and any $\chi \in \mathrm{Irr}(G)$, we have $[\chi^l, \mathsf{St}]_G \neq 0$. Indeed, $D = 163$ suffices.*

Proof. By (2.1),

$$\begin{aligned} [\chi^l, \mathsf{St}]_G &= \frac{1}{|G|} \sum_{g \in G_{\rm ss}} \epsilon_g \chi^l(g) |\mathbf{C}_G(g)|_p \\ &= \frac{\chi^l(1)}{|G|} \left(|G|_p + \sum_{1 \neq g \in G_{\rm ss}} \epsilon_g \left(\frac{\chi(g)}{\chi(1)}\right)^l |\mathbf{C}_G(g)|_p \right). \end{aligned} \tag{2.2}$$

Hence $[\chi^l, \mathsf{St}]_G \neq 0$ provided $\Sigma_l < |G|_p$, where

$$\Sigma_l := \sum_{1 \neq g \in G_{\rm ss}} \left| \frac{\chi(g)}{\chi(1)} \right|^l |\mathbf{C}_G(g)|_p.$$

Note that $|G| < q^{4r^2}$. Assume first that $q > 9$. Then Proposition 2.2 implies that $\Sigma_l < |G|_p$ provided $q^{4r^2} \cdot (3/q^{1/2})^l < 1$, which holds if $l \geq 96r^2$. For $q \leq 9$ we need $q^{4r^2} \cdot (19/20)^l < 1$, and this holds when $l \geq 163r^2$. □

Now let $1 \neq \alpha \in \mathrm{Irr}(G)$. It follows from Lemma 2.3 and Proposition 2.1 that α^{3Dr^2} contains all irreducible characters of G. Hence, given any two $\chi_1, \chi_2 \in \mathrm{Irr}(G)$,

$$0 \neq \left[\alpha^{3Dr^2}, \bar{\chi}_1\chi_2\right]_G = \left[\alpha^{3Dr^2}\chi_1, \chi_2\right]_G,$$

i.e. a directed path of length $\leq 3Dr^2$ connects χ_1 to χ_2 in $\mathcal{M}(G, \alpha)$. We conclude that $\mathsf{diam}\mathcal{M}(G, \alpha) \leq 3Dr^2$, completing the proof of Theorem 2.

3. Projective special linear and unitary groups

Throughout this section, which is devoted to prove Theorem 3, let $G = \mathrm{PSL}_n^\epsilon(q)$ with $\epsilon = \pm$. For a semisimple element $g \in G_{\rm ss}$, let $\hat{g}$ be a preimage of g in $\mathrm{SL}_n^\epsilon(q)$, and define $\nu(g) = \mathrm{supp}(\hat{g})$, the codimension of the largest eigenspace of $\hat{g}$ over $\bar{\mathbb{F}}_q$.

We shall need the following bound for character ratios of semisimple elements, which follows from the deep results in [2, 12].

Theorem 3.1. *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for any $g \in G_{\rm ss}$ with $s = \nu(g)$, and any $\chi \in {\rm Irr}(G)$, we have*

$$|\chi(g)| < f(n)\chi(1)^{1-\frac{s}{n}}.$$

Proof. Let $\mathcal{G} = {\rm SL}_n(K)$, $K = \bar{\mathbb{F}}_q$ be the ambient algebraic group with $G = \mathcal{G}^F/\mathbf{Z}(\mathcal{G}^F)$, where F is a Frobenius endomorphism. Then $\mathbf{C}_{\mathcal{G}}(\hat{g}) = \mathcal{L} := \tilde{\mathcal{L}} \cap \mathcal{G}$, where $\tilde{\mathcal{L}} = \prod_{i=1}^m {\rm GL}_{n_i}(K)$, $1 \le n_1 \le \cdots \le n_m$, and $\sum_{i=1}^m n_i = n$. Note that $\nu(g) = s = n - n_m$; and that $\mathcal{L}$ is F-stable (but not necessarily split).

We now apply [12, Cor. 1.11(c)] (which is an extension of [2, Thm. 1.1]). That gives a function $f : \mathbb{N} \to \mathbb{N}$ such that for any $\chi \in {\rm Irr}(G)$,

$$|\chi(g)| < f(n)\chi(1)^{\alpha(\mathcal{L})},$$

where $\alpha(\mathcal{L})$ is the maximum value of $\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}}$ over nontrivial unipotent elements $u \in \mathcal{L}$ (and $\alpha(\mathcal{L}) = 0$ if $\mathcal{L}$ is a torus). Note that the function $f(n)$ can be chosen to be explicit; an explicit choice for $f(n)$ is given in [12, 1.28] with the main term of $(n!)^{5/2}$. Although this choice may seem to be inflated, it is noted in [2, Remark 1.2(iii)] that any choice of $f(n)$ should be at least $\mathsf{b}(\mathsf{S}_n) > e^{-1.3\sqrt{n}}\sqrt{n!}$.

Let $\tilde{\mathcal{G}} = {\rm GL}_n(K)$ and let $\alpha(\tilde{\mathcal{L}})$ be the maximum value of $\frac{\dim u^{\tilde{\mathcal{L}}}}{\dim u^{\tilde{\mathcal{G}}}}$ over nontrivial unipotent elements $u \in \tilde{\mathcal{L}}$ (and $\alpha(\tilde{\mathcal{L}}) = 0$ if $\tilde{\mathcal{L}}$ is a torus). It is easy to see that $\alpha(\mathcal{L}) = \alpha(\tilde{\mathcal{L}})$. Furthermore, $\alpha(\tilde{\mathcal{L}}) \le \frac{n_m}{n}$ by [2, Thm. 1.10]. (Note that this bound is only stated for ${\rm GL}_n(q)$ in [2, Theorem 1.10], but its proof applies to bound $\alpha(\tilde{\mathcal{L}})$ for any proper Levi subgroup $\tilde{\mathcal{L}}$ of the algebraic group $\tilde{\mathcal{G}}$.) Hence $\alpha(L) \le \frac{n-s}{n}$, and the conclusion follows. □

The next lemma gives some properties of elements of G of support s.

Lemma 3.2. *For $1 \le s < n$, define $N_s(G) = \{g \in G_{\rm ss} : \nu(g) = s\}$ and let $n_s(G) := |N_s(G)|$.*

- (i) *If $g \in N_s(g)$ and $s < \frac{n}{2}$ then $|\mathbf{C}_G(g)|_p < q^{\frac{1}{2}n^2+s^2-ns}$.*
- (ii) *If $g \in N_s(g)$ and $s \ge \frac{n}{2}$ then $|\mathbf{C}_G(g)|_p < q^{\frac{1}{2}(n^2-ns)}$.*
- (iii) *$\sum_{n-1 \ge s \ge n/2} n_s(G) < |G| < q^{n^2-1}$.*
- (iv) *If $s < n/2$, then $n_s(G) < cq^{s(2n-s)+n-1}$, where c is an absolute constant that can be taken to be 44.1.*

Proof. (i) Let $g \in N_s(G)$ with $s < \frac{n}{2}$. Then $\hat{g} = {\rm diag}(\lambda I_{n-s}, X)$ for some $\lambda \in \mathbb{F}_{q^u}^*$ (where $u = 1$ if $\epsilon = +$ and $u = 2$ if $\epsilon = -$) and a suitable $s \times s$-matrix X, and one can see that

$${\rm GL}^{\epsilon}_{n-s}(q) \le \mathbf{C}_{{\rm GL}_n(q)}(\hat{g}) \le {\rm GL}^{\epsilon}_{n-s}(q) \times {\rm GL}^{\epsilon}_{s}(q). \qquad (3.1)$$

Now the statement follows, since $|\mathbf{C}_G(g)|_p \le |\mathbf{C}_{{\rm GL}^{\epsilon}_n(q)}(\hat{g})|_p$.

(ii) Let $g \in N_s(G)$ with $s \ge \frac{n}{2}$. Then

$$\mathbf{C}_{{\rm GL}^{\epsilon}_n(q)}(\hat{g}) = \prod_{i=1}^{t} {\rm GL}^{\epsilon_i}_{d_i}(q^{k_i}),$$

where $n - s = d_1 \ge d_2 \ge \ldots \ge d_t \ge 1$ and $\sum_{i=1}^t d_i k_i = n$. Hence, $|\mathbf{C}_{{\rm GL}^{\epsilon}_n(q)}(\hat{g})|_p = q^D$, where

$$D := \sum_{i=1}^{t} k_i d_i(d_i - 1)/2 = \big(\sum_{i=1}^{t} k_i d_i^2 - n\big)/2.$$

Using the obvious inequality $x^2 + y^2 < (x+1)^2 + (y-1)^2$ when $x \ge y$, we observe that, over all m-tuples $(x_1 \ge x_2 \ge \ldots \ge x_m)$ of integers $0 \le x_i \le d_1$ and with fixed $\sum_{i=1}^m x_i$,

$\sum_{i=1}^m x_i^2$ is maximized when $(x_1, x_2, \dots, x_m)$ is $(d_1, d_1, \dots, d_1, e, 0, \dots, 0)$ with $0 \le e < d_1$. Applying this observation to

$$(x_1, x_2, \dots, x_m) = \big(\underbrace{d_1, \dots, d_1}_{k_1 \text{ times}}, \underbrace{d_2, \dots, d_2}_{k_2 \text{ times}}, \dots, \underbrace{d_t, \dots, d_t}_{k_t \text{ times}}\big)$$

(and $m = \sum_{i=1}^t k_i$), we see that

$$\sum_{i=1}^t k_i d_i^2 \le a d_1^2 + b,$$

where $n = ad_1 + b$ with $0 \le b < d - 1$. It follows that

$$2D \le ad_1(d_1 - 1) < ad_1^2 \le nd_1 = n(n-s),$$

and we are done as in (i).

(iii) This is obvious, since $|G| \le |\mathrm{SL}_n^\epsilon(q)| < q^{n^2-1}$.

(iv) By [8, Lemma 4.1],

$$\frac{9}{32} q^{n^2} < |\mathrm{GL}_n(q)| < |\mathrm{GU}_n(q)| \le \frac{3}{2} q^{n^2}.$$

It now follows from (3.1) that

$$\begin{aligned} |g^G| &\le |\hat{g}^{\mathrm{GL}_n^\epsilon(q)}| = [\mathrm{GL}_n^\epsilon(q) : \mathbf{C}_{\mathrm{GL}_n^\epsilon(q)}(\hat{g})] \\ &\le [\mathrm{GL}_n^\epsilon(q) : \mathrm{GL}_{n-s}^\pm(q)] < \frac{(3/2)q^{n^2}}{(9/32)q^{(n-s)^2}} = \tfrac{16}{3} q^{s(2n-s)} \end{aligned}$$

for any $g \in N_s(G)$. Since the total number of conjugacy classes in G is at most $8.26q^{n-1}$ by Propositions 3.6 and 3.10 of [5], the statement follows. □

Lemma 3.3. *Let $1 \ne \chi \in \mathrm{Irr}(G)$, and for $1 \le s < n$, let $g_s \in N_s(G)$ be such that $|\chi(g_s)|$ is maximal. For $l \ge 1$, define*

$$\Delta_l := \sum_{1 \le s < n/2} c q^{ns + \frac{3n}{2} - 1} \left| \frac{\chi(g_s)}{\chi(1)} \right|^l + \sum_{n/2 \le s < n}^{n-1} q^{n^2 - \frac{1}{2}n(s-1) - 1} \left| \frac{\chi(g_s)}{\chi(1)} \right|^l,$$

with c as in Lemma 3.2. If $\Delta_l < 1$, then $[\chi^l, \mathsf{St}]_G \ne 0$.

Proof. As in the proof of Lemma 2.3, we have $[\chi^l, \mathsf{St}]_G \ne 0$ as long as $\Sigma_l < |G|_p$, where

$$\Sigma_l := \sum_{1 \ne g \in G_{\mathrm{ss}}} \left| \frac{\chi(g)}{\chi(1)} \right|^l |\mathbf{C}_G(g)|_p.$$

Using Lemma 3.2, we have

$$\begin{aligned} \Sigma_l &\le \sum_{s=1}^{n-1} n_s(G) \left| \frac{\chi(g_s)}{\chi(1)} \right|^l |\mathbf{C}_G(g)|_p \\ &\le \sum_{1 \le s < \frac{n}{2}} c q^{s(2n-s)+n-1} \left| \frac{\chi(g_s)}{\chi(1)} \right|^l q^{\frac{1}{2}n^2 + s^2 - ns} + \sum_{n/2 \le s < n} q^{n^2-1} \left| \frac{\chi(g_s)}{\chi(1)} \right|^l q^{\frac{1}{2}(n^2 - ns)} \\ &= |G|_p \Delta_l, \end{aligned}$$

where Δ_l is as in the statement of the lemma. The conclusion follows. □

Proof of Theorem 3

Adopt the notation of Lemma 3.3. By Theorem 3.1, for any $\chi \in \mathrm{Irr}(G)$,

$$|\chi(g_s)| < f(n)\chi(1)^{1-\frac{s}{n}}.$$

Hence for $l \geq 1$,

$$\Delta_l < f(n)^l \left(\sum_{1\leq s<n/2} cq^{ns+\frac{3n}{2}-1}\chi(1)^{-sl/n} + \sum_{n/2\leq s<n} q^{n^2-\frac{1}{2}n(s-1)-1}\chi(1)^{-sl/n} \right). \quad (3.2)$$

Now we choose

$$l = 5\frac{\log|G|}{\log\chi(1)} = 5\frac{\log_q|G|}{\log_q\chi(1)}.$$

We claim that

$$8(n+2) \geq l > \frac{3n^2}{\log_q\chi(1)}. \quad (3.3)$$

This is obvious if $G = \mathrm{PSL}_2(q)$ is simple. If $G = \mathrm{PSL}_n(q)$ with $n \geq 3$, then by [13, Theorem 1.1],

$$\chi(1) > q^{n-1}, \text{ on the other hand, } q^{n^2-1} > |G| > q^{n^2-2}$$

(where the last inequality follows from [8, Lemma 4.1(ii)]), and so (3.3) holds. If $G = \mathrm{PSU}_n(q)$ with $n \geq 3$, then again by [13, Theorem 1.1],

$$\chi(1) > q^{n-2}, \text{ on the other hand, } q^{n^2-1} > |G| > q^{n^2-3},$$

(where the last two inequalities can be checked using the proof of [8, Lemma 4.1(iv)]), and so (3.3) holds.

Now (3.3) implies that $\chi(1)^{-sl/n} < q^{-3ns}$. Hence the first summand inside the parenthesized part of (3.2) is at most

$$c\sum_{1\leq s<n/2} q^{3n/2-1-2ns} < cq^{-n/2-1}\sum_{j=0}^{\infty}\frac{1}{q^{2nj}} < \frac{16c}{15}q^{-n/2-1}.$$

The second summand inside the parenthesized part of (3.2) is at most

$$\sum_{n/2\leq s<n} q^{n^2-7ns/2+n/2-1} < \frac{n}{2}q^{-3n^2/4+n/2-1} \leq \frac{n}{2}q^{-n-1} < q^{-n/2-1}.$$

Since $c \leq 44.1$, it follows that

$$\Delta_l < f(n)^l\left(\frac{16c}{15}+1\right)q^{-n/2-1} < f(n)^l\left(\frac{49}{q}\right)^{n/2+1}.$$

Taking

$$q \geq (49f(n))^{16},$$

we obtain by (3.3) that $\Delta_l < 1$. Hence $[\chi^l, \mathsf{St}]_G \neq 0$ by Lemma 3.3.

Now Theorem 3 follows, using exactly the same argument as in the last paragraph of Section 2.

Proof of Corollary 4

Write $\alpha = \alpha_1 + \ldots + \alpha_k$, with $\alpha_i \in \mathrm{Irr}(G)$ and $\alpha_1(1) \leq \alpha_2(1) \ldots \leq \alpha_k(1)$. Since α is faithful, $\alpha_k(1) \geq \mathsf{d}(G) > 1$, where $\mathsf{d}(G)$ is the smallest degree of nontrivial irreducible characters of G; furthermore, $k \leq k(G) := |\mathrm{Irr}(G)|$ as α is multiplicity-free. It is easy to check that $\mathsf{d}(G)^{1.5} > k(G)$ for $G = \mathrm{PSL}_2(q)$ with $q \geq 11$. For $n \geq 3$ and $G = \mathrm{PSL}_n^{\epsilon}(q)$, it follows from [13, Theorem 1.1] and [5, Propositions 3.6, 3.10] that

$$\mathsf{d}(G)^{3/2} \geq \left(\frac{q^n - q}{q+1}\right)^{3/2} \geq \left(\frac{5}{6}q^{n-1}\right)^{3/2} > 8.3q^{n-1} > k(G).$$

It follows that $\alpha_k(1)^{5/2} > k(G)\alpha_k(1) \geq k\alpha_k(1) \geq \alpha(1)$. By Theorem 3, for some

$$N \leq C\frac{\log|G|}{\log \alpha_k(1)} < 2.5C\frac{\log|G|}{\log \alpha(1)},$$

we have $\sum_{i=0}^{N} \alpha_k^i$ contains all irreducible characters of G, whence $\sum_{i=0}^{N} \alpha^i$ also contains all irreducible characters of G, i.e. $\mathsf{diam}\mathcal{M}(G, a) \leq N$.

4. Symmetric and alternating groups

Theorem 5. *Let $n \geq 5$ and let $G = \mathsf{A}_n$ or S_n. Then for any faithful irreducible character α of G, we have $\mathsf{diam}\mathcal{M}(G, \alpha) \leq 4n - 4$.*

Proof of Theorem 5 As explained in the Introduction, $\mathsf{diam}\mathcal{M}(G, \alpha)$ is at most $N = N(\alpha)$, if N is the smallest positive integer such that $\sum_{i=0}^{N} \alpha^i$ contains $\mathrm{Irr}(G)$. Let $G := \mathsf{S}_n$, $S := \mathsf{A}_n$, and let $H := \mathsf{S}_{n-1}$, $K := \mathsf{S}_{n-2} \times \mathsf{S}_2$, $K' = \mathsf{S}_{n-2} < K$, and $L := \mathsf{S}_{n-3} \times \mathsf{S}_3$ be Young subgroups of G. If $\lambda \vdash n$ is a partition of n, let χ^{λ} denote the irreducible character of S_n labeled by λ.

Given a faithful irreducible character α of G or H, we will now bound $N(\alpha)$ in a sequence of steps.

Step 1. If $\alpha \in \mathrm{Irr}(G)$ and $\alpha = \chi^{(n-1,1)}$, then $N(\alpha) \leq n - 1$.

Indeed, α takes n distinct values $-1, 0, 1, \ldots, n-3, n-1$, hence $N(\alpha) \leq n - 1$ by [3].

Step 2. If $\alpha \in \mathrm{Irr}(G)$ and $\alpha|_H$ is reducible, then $N(\alpha) \leq 2n - 2$. In particular, $N(\chi^{(n-2,2)}) \leq 2n - 2$. Likewise, if $n \geq 7$ and $\mu = (mu_1 \geq \mu_2 \geq \ldots \geq \mu_s \geq 1) \vdash n$ and $n - 1 \geq \mu_1 \geq n - 3$, then $N(\chi^{\mu}) \leq 2n - 2$.

Indeed, by assumption we have that

$$2 \leq [\alpha|_H, \alpha|_H]_H = [\alpha^2|_H, 1_H]_H = [\alpha^2, \mathrm{Ind}_H^G(1_H)]_G.$$

Recall that $\mathrm{Ind}_H^G(1_H) = 1_G + \chi^{(n-1,1)}$ and $[\alpha^2, 1_G]_G = 1$. It follows that α^2 contains $\chi^{(n-1,1)}$, and so $N(\alpha) \leq 2n - 2$ by Step 1.

The branching rule for complex representations of S_n implies that

$$\chi^{(n-2,2)}|_H = \chi^{(n-2,1)} + \chi^{(n-3,2)},$$

i.e. $\chi^{(n-2,2)}$ is reducible over H. Similarly, $\chi^{\mu}|_H$ is reducible for the μ listed above when $n \geq 7$, whence we are done.

Step 3. If $\alpha \in \mathrm{Irr}(G)$, then $N(\alpha) \leq 4n - 4$.

Consider $K = \mathsf{S}_{n-2} \times \mathsf{S}_2$, where $\mathsf{S}_2 = \langle s \rangle$ is generated by a transposition s. If $\alpha|_K$ is irreducible, then by Schur's Lemma s acts as a scalar, and so $\alpha = 1_G$ or the sign character, contradicting the faithfulness of α. Thus $\alpha|_K$ is reducible, and so

$$2 \leq [\alpha|_K, \alpha|_K]_K = [\alpha^2|_K, 1_K]_K = [\alpha^2, \mathrm{Ind}_K^G(1_K)]_G.$$

Recall that $\mathrm{Ind}_K^G(1_K) = 1_G + \chi^{(n-1,1)} + \chi^{(n-2,2)}$ and $[\alpha^2, 1_G]_G = 1$. If α^2 contains $\chi^{(n-1,1)}$, then $N(\alpha) \leq 2n - 2$ by Step 1. Otherwise we must have that α^2 contains $\chi^{(n-2,2)}$, and so $N(\alpha) \leq 4n - 4$ by Step 2.

From now on we will assume that $\alpha \in \mathrm{Irr}(S)$ and that α is an irreducible constituent of the restriction of $\chi = \chi^{\lambda} \in \mathrm{Irr}(G)$ to S.

STEP 4. If α extends to G, then $N(\alpha) \leq 4n-4$.

Indeed, in this case $\alpha = \chi|_S$. By Step 3, $\sum_{i=0}^{4n-4} \chi^i$ contains χ^μ for all $\mu \vdash n$. It follows that $\sum_{i=0}^{4n-4} \alpha^i = (\sum_{i=0}^{4n-4} \chi^i)|_S$ contains $\chi^\mu|_S$ for all $\mu \vdash n$, whence it contains all $\mathrm{Irr}(S)$.

From now on, we will assume that α does not extend to G; equivalently, λ is self-associated: $\lambda = \lambda^*$. For $n = 5, 6$, the character α takes at most 5 different values on S, and so $N(\alpha) \leq 4$ by [3]. We will therefore assume $n \geq 7$.

STEP 5. If α is real-valued then $N(\alpha) \leq 4n-4$.

The assumption implies that $[\alpha^2|_S, 1_S] = 1$. Next, by inspecting the character table of A_5, we see that any nontrivial complex irreducible representation Φ of A_5 affords all three distinct eigenvalues $1, \omega, \omega^2$ for the 3-cycle $t = (1,2,3)$ ($\omega \neq 1$ being a cubic root of unity in $\mathbb{C}$). We prove by induction that the same statement holds for any $n \geq 5$. For the induction step $n \geq 6$, suppose $\Phi(t)$ affords at most two distinct eigenvalues. By induction hypothesis, all composition factors of $\Phi|_{\mathsf{A}_{n-1}}$ are trivial. By Frobenius' reciprocity, the character φ of Φ is a constituent of

$$\mathrm{Ind}_{S\cap H}^S(1_{S\cap H}) = (\mathrm{Ind}_H^G(1_H))|_S = (\chi^n + \chi^{(n-1,1)})|_H,$$

and so $\varphi = \chi^{(n-1,1)}|_S$. But clearly in this case $\Phi(t)$ affords all three eigenvalues $1, \omega, \omega^2$, a contradiction.

Applying the established assertion to a complex representation Φ affording α, we see that $\Phi(t)$ affords all three eigenvalues $1, \omega, \omega^2$. We can choose the Young subgroup $L = \mathsf{S}_{n-3} \times \mathsf{S}_3$ such that $t \in \mathsf{S}_3 \cap L$, in which case $\langle t \rangle \lhd L \cap S$. It follows that $\alpha|_{L\cap S}$ is reducible, and so

$$2 \leq [\alpha|_{S\cap L}, \alpha|_{S\cap L}]_{S\cap L} = [\alpha^2|_{S\cap L}, 1_{S\cap L}]_{S\cap L} = [\alpha^2, \mathrm{Ind}_{S\cap L}^S(1_{S\cap L})]_S.$$

Observe that

$$\mathrm{Ind}_{S\cap L}^S(1_{S\cap L}) = (\mathrm{Ind}_L^G(1_L))|_S = 1_S + \sum_{i=1}^{3} \chi^{(n-i,i)}|_S,$$

and $\chi^{(n-i,i)}|_S$ is irreducible for $i \leq 3$. It follows that α^2 contains $\chi^{(n-j,j)}|_S$ for some $1 \leq j \leq 3$. As $N(\chi^{(n-j,j)}) \leq 2n-2$ by Step 2, we have that $N(\alpha) \leq 4n-4$.

STEP 6. If $\alpha \neq \bar{\alpha}$ and $\lambda \neq (a^a)$ with $a \in \mathbb{Z}_{\geq 1}$, then $N(\alpha) \leq 2n-2$.

Since we are assuming that α does not extend to G and χ^λ is real-valued, we have that $\chi^\lambda|_S = \alpha + \bar{\alpha}$ and that $\lambda = \lambda^*$. Let μ be obtained from λ by removing the last node of the shortest row of (the Young diagram of) λ. As $\lambda \neq (a^a)$, observe that $\mu \neq \mu^*$. But $\lambda = \lambda^*$, so by symmetry we see that $\chi^\langle|_H$ contains $\chi^\mu + \chi^{\mu^*}$. The condition $\mu \neq \mu^*$ also implies that $\beta := \chi^\mu|_{\mathsf{A}_{n-1}} = \chi^{\mu^*}|_{\mathsf{A}_{n-1}}$ is irreducible. It follows that $\alpha|_{S\cap H}$ contains the real-valued irreducible character β, and so $\alpha^2|_{S\cap H}$ contains β^2, which in turns contains $1_{S\cap H}$. Thus we have

$$1 \leq [\alpha^2|_{S\cap H}, 1_{S\cap H}]_{S\cap H} = [\alpha^2, \mathrm{Ind}_{S\cap H}^S(1_{S\cap H})]_S.$$

Now

$$\mathrm{Ind}_{S\cap H}^S(1_{S\cap H}) = (\mathrm{Ind}_H^G(1_H))|_S = 1_S + \chi^{(n-1,1)}|_S,$$

and $[\alpha^2, 1_S]_S = 0$ since $\alpha \neq \bar{\alpha}$. Hence α^2 contains $\chi^{(n-1,1)}|_S$, and so $N(\alpha) \leq 2n-2$ by Step 1.

FINAL STEP. If $\alpha \neq \bar{\alpha}$ and $\lambda = (a^a)$ with $a \in \mathbb{Z}_{\geq 3}$, then $N(\alpha) \leq 4n-4$.

As in Step 6, since we are assuming that α does not extend to G and χ^λ is real-valued, we have that $\chi^\lambda|_S = \alpha + \bar{\alpha}$. Let ν be obtained from λ by removing the last two nodes of the last row of (the Young diagram of) λ, so that $\nu \neq \nu^*$. But $\lambda = \lambda^*$, so by symmetry we see that $\chi^\lambda|_K$ contains $\chi^\nu + \chi^{\nu^*}$. The condition $\nu \neq \nu^*$ also implies that $\gamma := \chi^\nu|_{\mathsf{A}_{n-2}} = \chi^{\nu^*}|_{\mathsf{A}_{n-2}}$

is irreducible. It follows that $\alpha|_{S\cap K'}$ contains the real-valued irreducible character γ, and so $\alpha^2|_{S\cap K'}$ contains γ^2, which in turns contains $1_{S\cap K'}$. Thus we have

$$1 \le [\alpha^2|_{S\cap K'}, 1_{S\cap K'}]_{S\cap K'} = [\alpha^2, \mathrm{Ind}_{S\cap K'}^S(1_{S\cap K'})]_S.$$

Now

$$\mathrm{Ind}_{S\cap K'}^S(1_{S\cap K'}) = (\mathrm{Ind}_{K'}^G(1_{K'}))|_S = 1_S + \chi^{(n-1,1)}|_S + \chi^{(n-2,2)}|_S + \chi^{(n-2,1^2)}|_S,$$

and $[\alpha^2, 1_S]_S = 0$ since $\alpha \ne \bar{\alpha}$. Hence α^2 contains at least one of (irreducible characters) $\chi^{(n-1,1)}|_S$, $\chi^{(n-2,2)}|_S$, $\chi^{(n-2,1^2)}|_S$, and we conclude that $N(\alpha) \le 4n-4$ by Step 2. ∎

5. McKay graphs for quasi-simple groups

McKay graphs $\mathcal{M}(G,\alpha)$ are usually considered for any finite group G possessing a faithful character α (to guarantee connectedness). In this section, we show that the diameters of McKay graphs for faithful irreducible characters of quasi-simple groups (with cyclic center) can be bounded by the diameters of McKay graphs for simple groups.

Theorem 5.1. *Let G be a finite quasi-simple group with cyclic center $\mathbf{Z}(G)$, and let χ be a faithful irreducible character of G. Then there is a nontrivial irreducible character β of the simple group $S := G/\mathbf{Z}(G)$ such that*

$$\mathsf{diam}\mathcal{M}(G,\chi) \le |\mathbf{Z}(G)| \cdot \mathsf{diam}\mathcal{M}(S,\beta) + |\mathbf{Z}(G)| - 1.$$

In particular

$$\max_{\alpha\in\mathrm{Irr}(G),\ \alpha\ \mathrm{faithful}} \mathsf{diam}\mathcal{M}(G,\alpha) \le |\mathbf{Z}(G)| \cdot \left(\max_{1_S\ne\gamma\in\mathrm{Irr}(S)} \mathsf{diam}\mathcal{M}(S,\gamma) + 1\right) - 1.$$

Proof. Let $e := |\mathbf{Z}(G)|$. Since $\mathrm{Ker}(\chi^e)$ contains $\mathbf{Z}(G)$ but not G, we can find a nontrivial $\beta \in \mathrm{Irr}(S)$ such that β inflated to G is an irreducible constituent of χ^e. Now consider arbitrary $\varphi, \psi \in \mathrm{Irr}(G)$. Then there is $0 \le i \le e-1$ such that the nontrivial character $\varphi\chi^i\overline{\psi}$ is trivial at $\mathbf{Z}(G)$ and so contains a nontrivial $\delta \in \mathrm{Irr}(S)$. Thus

$$[\varphi\chi^i\overline{\delta}, \psi]_G = [\varphi\chi^i\overline{\psi}, \delta]_G > 0.$$

Next, we can find some $d \le \mathsf{diam}\mathcal{M}(S,\beta)$ such that β^d contains $\overline{\delta}$. It follows that

$$[\varphi\chi^{i+de}, \psi]_G \ge [\varphi\chi^i\beta^d, \psi]_G \ge [\varphi\chi^i\overline{\delta}, \psi]_G > 0,$$

i.e. a directed path of length $i + de$ connects φ to ψ in $\mathcal{M}(G,\alpha)$. ∎

As a final remark, we note that one cannot remove the term $|\mathbf{Z}(G)|$ from the upper bound in Theorem 5.1. Indeed, any directed path connecting 1_G to any other $1_S \ne \psi \in \mathrm{Irr}(S)$ in $\mathcal{M}(G,\alpha)$ must have length divisible by $|\mathbf{Z}(G)|$.

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