Abstract

The study of quantum field theories and supersymmetric quantum field theories has thrived in the recent decades. An aspect of this study has a particularly geometrical nature and as such has also started to spearhead progress in related fields of mathematics. This aspect is the study of vacua. Traditional computations in quantum field theory understand particle states as perturbations around a vacuum state. Many theories have a plethora of candidates to be this vacuum state. Different choices of vacuum imply different properties of the particles in the theory. These physical differences can be understood by creating a moduli space: an affine variety in which each point of the variety corresponds to a different vacuum of the theory. The physical properties of the vacua are then realized as geometrical properties of the different points in the variety. In this thesis we analyse these properties and develop new techniques to further their study: the Kraft-Procesi transition and the magnetic quivers. In order to do so we employ a particular fruitful approach to this type of problems: embedding the quantum field theories at hand in a string theoretical background. By this procedure, the low energy dynamics of branes in the string theory set up correspond to motions along the moduli space of vacua of the theory. The new results that we present here aim to be a set of tools that can readily be applied to the analysis of many different supersymmetric quantum field theories. The goal is that the tools are based on manipulations of simple diagrams, which should make them easy to remember and easy to implement.
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Declaration

Originality

All the work in the present thesis is my own and all else is appropriately referenced. The main ideas of this thesis are based on the publications that were created as a result of the PhD endeavour:

- S. Cabrera, A. Hanany and M. Sperling, *Magnetic Quivers, Higgs Branches, and 6d N = (1,0) Theories*, [1904.12293].

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“No bien pues de su luz los horizontes,
que hacían desigual, confusamente,
montes de agua y pílagos de montes,
desorados los siente,
cuando, entregado el mísero extranjero
en lo que ya del mar redimió fiero,
entre espinas crepúsculos pisando,
riscos que aun igualara mal volando
veloz, intrépida ala,
menos cansado que confuso, escala. ”

Góngora
Soledad Primera
Part I

Introduction
Introduction

The current thesis studies a set of new techniques that have been developed during the four years of PhD research. The goal of these techniques is to analyse the geometrical properties of the vacuum of supersymmetric gauge theories. This endeavour is particularly rewarding from the point of view of theoretical physics, since it sits at the boundary between mathematics (algebraic geometry), particle physics (quantum field theory) and string theory (brane dynamics). During these four years, we have learnt that these three areas of study are strongly intertwined and that they can thrive from mutual interaction. In a way, they can all see further in their own disciplines by stepping into each other shoulders.

The two different techniques that are reviewed in this thesis are the Kraft Procesi transition [1] and the magnetic quiver [2]. They form part of the set of results [1, 3, 4, 5, 6, 2, 7, 8] that have been released during the PhD. The structure of the thesis is divided into five parts. Part I is the present introduction. Part II is called Quivers and Branes and it serves as a presentation of all the physical concepts that are needed for the following parts. Within Part II, Chapter 1 is a reminder of the different multiplets that can be encountered in a supersymmetric gauge theory [9] and Chapter 2 surveys a specific Type IIB string theory background [10] in which such type of theories can be realised.

Part III discusses the first of the techniques, the Kraft-Procesi transition [1]. In this case, we took an idea from mathematics (the singularity structure of conjugacy classes of a Lie group) [11, 12, 13, 14] and used it in order to obtain new insights into the vacuum of 3d \( \mathcal{N} = 4 \) gauge theories. In this case, we consider that string theory became the bridge between both areas, since it is via an embedding into a brane system, that the relation between the geometry and the gauge theory was found. Chapter 3 introduces some of the concepts of algebraic geometry that are needed, such as the Hilbert series. Chapter 4 illustrates all the mathematical and physical concepts previously introduced with an example: 3d \( \mathcal{N} = 4 \) SQED with \( N_f \) flavours. Chapters 5 and 6 illustrate some of the details in the computations of the geometry of the vacuum for 3d \( \mathcal{N} = 4 \) quiver gauge theories. Chapter 7 describes the mathematical objects that [11, 12, 13, 14] studied, the nilpotent orbits of \( \mathfrak{sl}(n, \mathbb{C}) \). And finally Chapter 8 analyses the Kraft-Procesi transition and its implication in 3d \( \mathcal{N} = 4 \) gauge theories and the geometry of their vacua.

Part IV studies the second technique, the magnetic quivers [2]. The idea behind this technique is that Higgs branches of theories with 8 supercharges and 3, 4, 5 and 6 space-time dimensions can be characterised via a new set of objects called the magnetic quivers. Our conjecture was that such quivers can be read directly from the brane embedding of the gauge theory. Here we review the case of 5 space-time dimensions, where the conjecture was originally created. Chapter 9 introduces the problem. Chapter 10 studies the set of cases for which the Higgs branch was known and conjectures what should be the corresponding magnetic quivers. Chapter 11 formulates the conjecture on how to obtain the magnetic quivers from the brane embedding of the theories. Chapter 12 collects a set of new results that were obtained for the Higgs branches of 5d \( \mathcal{N} = 1 \) SQCD at infinite coupling. Part V is the conclusion of the thesis.
Part II

Quivers and Branes
Chapter 1

Supersymmetry

1.1 Chiral and Vector Superfields

Our study focuses on quantum field theories with eight supercharges. We pay particular attention to systems with 3 and 5 space-time dimensions. These receive the names 3\textit{d} \(\mathcal{N} = 4\) and 5\textit{d} \(\mathcal{N} = 1\) respectively. In the following pages we introduce the main types of particles that concern our study. The goal of this section is far from being a concise exercise in establishing the main features of supersymmetry \([15, 16, 17]\). Instead, it should be considered as a refreshing recounting of the main concepts that are needed for the thesis, and it is aimed to a reader already familiar with them. A concise explanation from first principles and motivation for the subject of supersymmetry can be found for example in the textbook by Wess and Bagger \([9]\). In said book the reader can also find citations to different papers that contributed to the construction of the topic \([18, 19, 20, 21, 17, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]\).

The different particles that constitute our universe and that high energy physics is devoted to understand are normally classified by the different representations of the inhomogeneous Lorentz group (see for example Section 2 of \([33]\)). In supersymmetry, different representations of the Lorentz group combine to form single multiplets of the supersymmetry algebra. Therefore, the usual scalar, vector or spinor particles are substituted by the chiral supermultiplet or the vector supermultiplet.

In order to be more explicit, let us consider a system of particles in 4 space-time dimensions. Let there be a single set of supercharges \(\mathcal{Q}\) and \(\bar{\mathcal{Q}}\). This system is called 4\textit{d} \(\mathcal{N} = 1\), where \(\mathcal{N}\) counts the number of identical sets of supercharges in the extended Lorentz algebra. In this type of theory, the relevant multiplets can be rewritten as superfields of coordinates \((x, \theta, \bar{\theta})\), were \(x\) is the space-time coordinate and \(\theta\) is the new anticommuting coordinate transforming as a Weyl spinor of the homogeneous Lorentz group in 4\textit{d}. Let us adopt the notation in \([9]\) for the rest of the section and summarise their explanations. The equations that follow are all obtained from \([9]\). A generic superfield \(S(x, \theta, \bar{\theta})\) can be expanded\(^1\) in powers of \(\theta\) and \(\bar{\theta}\):

\[
S(x, \theta, \bar{\theta}) = s(x) + \theta \zeta(x) + \bar{\theta} \bar{\zeta}(x)
+ \theta \theta m(x) + \bar{\theta} \bar{\theta} n(x) + \theta \sigma^\mu \bar{\theta} A_\mu(x)
+ \theta \theta \bar{\lambda}(x) + \bar{\theta} \bar{\theta} \psi(x) + \theta \theta \bar{\theta} d(x).
\]

(1.1)

The fields \(s, m, n, d, A_\mu, \zeta, \bar{\zeta}, \bar{\lambda}\) and \(\psi\) are called the \textit{component} fields, and they do transform as irreducible representations of the Lorentz group. \(s, m, n\) and \(d\) are scalar fields, \(A_\mu\) is a vector field, \(\zeta\) and \(\bar{\psi}\) are Weyl spinors that transform in the representation of \(\mathfrak{so}(4)\) with Dynkin labels \([1, 0]\) and \(\bar{\zeta}\) and \(\bar{\lambda}\) are Weyl spinors in

\(^1\)Spinor indices are omitted and the contractions are as in the appendix of \([9]\).
the representation [0, 1]. The matrices $\sigma^\mu$ are defined as the negative identity and the three Pauli matrices:

$$
\begin{align*}
\sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
$$
(1.2)

The supersymmetric transformation of the superfield $S(x, \theta, \bar{\theta})$ is then defined as:

$$
\delta_\xi S = (\xi Q + \bar{\xi} \bar{Q}) S,
$$
(1.3)

where $Q$ and $\bar{Q}$ are also used to denote the differential operators that implement the group action corresponding to the anticommuting part of the graded Lie-algebra:

$$
\begin{align*}
Q_\alpha &= \frac{\partial}{\partial \theta^\alpha} - i \sigma_\alpha \sigma^m \bar{\theta}^m, \\
\bar{Q}^{\dot{\alpha}} &= \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \sigma^{\dot{\alpha}} \sigma_m \theta^m.
\end{align*}
$$
(1.4)

They transform also as Weyl spinors of the homogeneous Lorentz group. Chiral superfields $\Phi(x, \theta, \bar{\theta})$ are defined as:

$$
\bar{D}\Phi(x, \theta, \bar{\theta}) = 0,
$$
(1.5)

where $D$ and $\bar{D}$ are also differential operators:

$$
\begin{align*}
D_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i \sigma_{\alpha} \sigma^m \bar{\theta}^m, \\
\bar{D}^{\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i \sigma^{\dot{\alpha}} \sigma_m \theta^m.
\end{align*}
$$
(1.6)

Vector superfields $V(x, \theta, \bar{\theta})$ are defined as

$$
V^\dagger(x, \theta, \bar{\theta}) = V(x, \theta, \bar{\theta}).
$$
(1.7)

Both constraints achieve to reduce the components of the superfields to irreducible representations of the superalgebra. Let us show what components remain in each superfield. The chiral superfield is expanded in powers of only $\theta$,

$$
\Phi = \phi(y) + \sqrt{2} \psi(y) + \theta \theta F(y),
$$
(1.8)

if one is to redefine the space coordinate by

$$
y^\mu = x^\mu + i \theta \sigma^\mu \bar{\theta}.
$$
(1.9)

We can see that only three component fields remain. $\phi(y)$ is a complex scalar field with mass dimension 1, $\psi(y)$ is a Weyl spinor with mass dimension 3/2 and $F(y)$ is a complex scalar with mass dimension 2.

Let there be a set of $N$ chiral superfields $\Phi_i$, where $i = 1, 2, \ldots, N$. The Lagrangian for a system with these particles that preserves supersymmetry is given in eq. (5.10) of [9]:

$$
\begin{align*}
\mathcal{L} &= \Phi_i^\dagger \Phi_i \bigg|_{\theta \bar{\theta} \bar{\theta} \text{ component}} + \left[ \frac{1}{2} m_{ij} \Phi_i \Phi_j + \\
&+ \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k + \lambda_i \Phi_i \right] \bigg|_{\theta \bar{\theta} \text{ component}} + \text{h.c.}
\end{align*}
$$
(1.10)

The equations of motion that are obtained from varying this Lagrangian describe the dynamics of one complex
1.1. CHIRAL AND VECTOR SUPERFIELDS

scalar \( \phi_i \) and one Weyl spinor \( \psi_i \) for each chiral superfield \( \Phi_i \). The component fields \( F_\mu \) play the role of auxiliary fields. The degrees of freedom of each superfield before imposing the equations of motion (also referred to as off-shell) are 4 bosonic given by the two complex scalars \( \phi_i \) and \( F_\mu \) and 4 fermionic given by the Weyl spinor \( \psi_i \). The degrees of freedom after imposing the equations of motion (or on-shell) are 2 bosonic carried by the complex scalar \( \phi_i \) and 2 fermionic carried by the Weyl spinor \( \psi_i \).

Let us now review what are the component fields that constitute a vector superfield. The vector superfield can be expanded in powers of \( \theta \) and \( \bar{\theta} \), with some restrictions:

\[
V(x, \theta, \bar{\theta}) = C(x) + i \theta \chi(x) - i \bar{\theta} \bar{\chi}(x) + \frac{i}{2} \bar{\theta} \theta [M(x) + iN(x)] - \frac{i}{2} \bar{\theta} \theta [M(x) - iN(x)] - \theta \sigma^\mu \bar{\theta} A_\mu(x) + i \theta \bar{\theta} \left[ \lambda(x) - \frac{i}{2} \sigma^\mu \partial_\mu \chi(x) \right] + \frac{1}{2} \theta \bar{\theta} \theta \bar{\theta} \left[ D(x) + \frac{1}{2} C(x) \right],
\]

(1.11)

The restrictions are that \( A_\mu, M, N, C \) and \( D \) are all real fields. A Lagrangian can be constructed for a massless vector superfield \( V \), given in eq. (6.15) of [9]:

\[
\mathcal{L} = \frac{1}{4} \left( W^\alpha W_\alpha |_{\theta \theta} + \bar{W}_\alpha \bar{W}^\alpha |_{\bar{\theta} \bar{\theta}} \right),
\]

(1.12)

where the field strength \( W_\alpha \) is a chiral superfield:

\[
W_\alpha = - \frac{i}{4} \bar{D} \bar{D} D_\alpha V, \\
\bar{W}_\alpha = - \frac{i}{4} D \bar{D} D_\alpha V.
\]

(1.13)

The variation of Lagrangian (1.12) produces the equations of motion for a massless vector field \( A_\mu(x) \) and a massless Weyl spinor \( \lambda(x) \), giving 2 bosonic degrees of freedom on-shell and 2 fermionic degrees of freedom on-shell. It can be modified to reproduce the massive vector superfield \( V(x, \theta, \bar{\theta}) \) by adding a term \( m^2 V^2 |_{\theta \theta \theta} \), where the vector field \( A_\mu(x) \) acquires 3 degrees of freedom, and an extra Weyl spinor \( \chi(x) \) and an extra real scalar \( C(x) \) become dynamical. Therefore, the total number of on-shell degrees of freedom for the massive vector superfield are 4 bosonic and 4 fermionic.

As expected, the massless Lagrangian (1.12) is invariant under a \( U(1) \) gauge transformation. Let \( G \) be a gauge group and let \( N \) chiral superfields \( \Phi_i \) with \( i = 1, 2, \ldots, N \) be in an \( N \)-dimensional representation of \( G \). The chiral superfields can be coupled to vector fields \( V \) in the adjoint representation of the gauge group \( G \) via the gauge invariant Lagrangian in equation (7.24) of [9]:

\[
\mathcal{L} = \frac{1}{16k^2} \left( W_\alpha W^\alpha |_{\theta \theta} + \bar{W}_\alpha \bar{W}^\alpha |_{\bar{\theta} \bar{\theta}} \right) + \Phi^i e^V \Phi_i |_{\theta \theta \theta} + \left( \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k \right) |_{\theta \theta} + h.c.
\]

(1.14)

where \( k \) is a normalisation factor, \( V \) now represents a matrix of vector superfields in the adjoint representation of \( G \) and the field strength has been generalised to:

\[
W_\alpha = - \frac{1}{4} DDe^{-V} D_\alpha e^V.
\]

(1.15)

This concludes the introduction of the particles that will be studied in this thesis. In the remaining pages many theories will be discussed that are theories of chiral superfields and vector superfields. They differ from each other in the choice of the gauge group \( G \), and in the choice of the representations of the chiral superfields \( \Phi_i \). In principle they could also differ in the coefficients \( m_{ij} \) and \( g_{ijk} \) that determine the interaction between
the chiral superfields. These terms of the Lagrangian have been traditionally called the superpotential of the theory, and they are normally denoted by $W(\Phi)$. However, the theories that we study all have the same type of superpotential. This is due to the fact that they contain additional supercharges. In the next section we review this set of theories.

### 1.2 Hypermultiplets and Vector Multiplets

The theories we are interested in have extended supersymmetry. This means that different sets of superfields need to be combined in order to produce irreducible multiplets of the extended supersymmetry algebra. This is analogous to the way in which irreducible representations of the Lorentz group join together to create irreducible multiplets of the non-extended supersymmetry algebra (represented by the superfields). In 4 space-time dimensions the minimal way in which supersymmetry can be extended is by adding a new set of supercharges $Q$ and $\bar{Q}$. Therefore, the total set of supercharges becomes $\{Q^1, Q^2, \bar{Q}^1, \bar{Q}^2\}$. There is a new global symmetry of the system called the R-symmetry. The R-symmetry is $SU(2)_R \times U(1)_R$. The supercharges $(Q^1, Q^2)$ form a doublet of $SU(2)_R$, and $(\bar{Q}^1, \bar{Q}^2)$ also form a doublet. The charge with respect to the remaining $U(1)_R$ is $-1$ for the supercharges $Q^i$ and $+1$ for $\bar{Q}^i$. This type of theories are named $4d \ N = 2$ supersymmetry, since the index in $Q^i$ and $\bar{Q}^i$ runs over $i = 1, 2$. These theories have been recently classified in [34]. Let us use the review in Section 2 of [35] in order to refresh what are the new multiplets that transform naturally under the extended supersymmetry algebra.

#### 1.2.1 The Vector Multiplet

The first type of representation is made up of two different superfields: one vector superfield $V(x, \theta, \bar{\theta})$ and one chiral superfield $\Phi(x, \theta, \bar{\theta})$. They combine together to form what is called a vector multiplet:

$$\text{Vector Multiplet} \rightarrow (V, \Phi).$$

(1.16)

In order to retain the language of superfields, we can keep separate the vector superfield and the chiral superfield that make up the $N = 2$ vector multiplet. In this way a theory with only vector multiplets could be a theory with a gauge group $G$ and a vector superfield $V$ together with a chiral superfield $\Phi$ both transforming in the adjoint representation of $G$.

A theory with a single massless vector multiplet has 4 bosonic degrees of freedom, divided into one vector field $A_\mu(x)$ and one complex scalar field $\phi(x)$, and 4 fermionic degrees of freedom divided into two Weyl spinors $\psi(x)$ and $\lambda(x)$. The Weyl spinors form a doublet under the action of $SU(2)_R$, and the bosons transform trivially. Note that each Weyl spinor belongs to a different superfield. Hence, the breaking of the vector multiplet into superfields breaks the global $SU(2)_R$. It is broken to $U(1)_J \subset SU(2)_R$. The chiral superfield $\Phi(x, \theta, \bar{\theta})$ still transforms under the remaining R-symmetry as given by eq. (2.2) of [35]:

$$U(1)_J : \Phi \rightarrow \Phi(e^{-i\alpha})$$

$$U(1)_R : \Phi \rightarrow e^{2i\alpha_\theta}(e^{-i\alpha}).$$

(1.17)

#### 1.2.2 The Hypermultiplet

The other type of representation we are interested in is called the hypermultiplet. The hypermultiplet is made up of two different chiral superfields, that we denote by $Q(x, \theta, \bar{\theta})$ and $\bar{Q}(x, \theta, \bar{\theta})$.

$$\text{Hypermultiplet} \rightarrow (Q, \bar{Q}).$$

(1.18)

For the theory to be consistent $Q$ and $\bar{Q}$ need to transform in complex representations of the gauge group $G$ that are complex conjugated to each other [34], therefore one can refer to the Weyl spinor in $Q$ as the quark,
1.2. HYPERMULTIPLETS AND VECTOR MULTIPLETS

and the Weyl spinor in $\tilde{Q}$ as the anti-quark. In order to write the component fields of the hypermultiplet in a way in which they all share the same representation under the gauge group $G$ we write two complex scalar fields $q(x)$ and $\tilde{q}^\dagger(x)$, and two Weyl spinor fields $\psi_q(x)$ and $\psi_{\tilde{q}}(x)$. Hence, we normally say that the hypermultiplet transforms in the same representation of the gauge group and the flavour group as the chiral superfield $Q$. A massless hypermultiplet carries 4 bosonic degrees of freedom and 4 fermionic degrees of freedom on-shell.

In the case of the R-symmetry, the two complex scalar fields form a doublet $(q, \tilde{q}^\dagger)$ of $SU(2)$. When the component fields are reorganised into chiral superfields the action of the R-symmetry that is still manifest is as in eq. (2.4) of [35]:

$$\begin{align*}
U(1)_J : Q &\to e^{\imath \alpha} Q(e^{-\imath \alpha} \theta) \\
\tilde{Q} &\to e^{\imath \alpha} \tilde{Q}(e^{-\imath \alpha} \theta)
\end{align*}$$

$$\begin{align*}
U(1)_R : Q &\to Q(e^{-\imath \alpha} \theta) \\
\tilde{Q} &\to \tilde{Q}(e^{\imath \alpha} \theta).
\end{align*}$$

1.2.3 The Quiver

Let a 4d $\mathcal{N}=2$ theory have gauge group $G$, massless hypermultiplets $(Q, \tilde{Q})$ transforming in a complex representation of $G$ and massless vector multiplets $(V, \Phi)$ transforming in the adjoint representation of $G$. The superpotential $W(\Phi, Q, \tilde{Q})$ takes the form [35]:

$$W = \sqrt{2} \tilde{Q} \Phi Q,$$

where a single index for the representation of $Q$ under the gauge group $G$ is omitted, same for $\tilde{Q}$, which transforms under the dual representation, and two indices are omitted for $\Phi$, which is in the adjoint. Therefore, in equation (1.20) $Q$ can be also thought of as a column vector, $\tilde{Q}$ as a row vector and $\Phi$ as a matrix.

For the sake of concreteness, let the gauge group be $G = U(N_c)$. $N_c$ is often referred to as the number of colours of the theory. Let there be a number of $N_f$ identical massless hypermultiplets, each of them transforming under the anti-fundamental representation\(^2\) of dimension $N_c$ of $G$. The notation $N_f$ stands for number of flavours. Then, one could write a superpotential with $N_f$ separate terms like the term in (1.20). Alternatively, the notation can be made more compact by letting $Q$ be a $N_c \times N_f$ matrix, letting $\tilde{Q}$ be an $N_f \times N_c$ matrix while $\Phi$ remains an $N_c \times N_c$ matrix. The superpotential is then written as:

$$W = \sqrt{2} \mathrm{Tr} \left( \tilde{Q} \Phi Q \right),$$

This interaction can be represented with the following diagram called quiver $Q$ [36]:

$$Q = \begin{array}{c}
\begin{array}{c}
N_f \\
Q \\
\tilde{Q} \\
\Phi \\
\end{array}
\end{array}$$

The quiver $Q$ has the following elements:

• The gauge node is the circular node. It represents the gauge group $G = U(N_c)$. The node is labelled by the rank of the group $N_c$. Each gauge node always comes with a vector superfield $V$ transforming in the adjoint representation of $G$.

\(^2\)Note that the fundamental representation could have been chosen with equivalent results. The choice of anti-fundamental is made for consistency of the conventions. The hypermultiplets are said to transform under the fundamental representation of the flavour group $SU(N_f)$ and the anti-fundamental representation of the gauge group $U(N_c)$. Note that for $N_c = 1$ this corresponds to the component field $Q$ in the hypermultiplets having negative electric charge, hence their interpretation as electrons, while $\tilde{Q}$ can be interpreted as positrons.
Table 1.1: Examples of the two different ways of writing the quiver of a 4d $\mathcal{N} = 2$ theory with gauge group $G = U(1)^3$. In the left column the arrows represent chiral superfields and the circles represent vector superfields. In the right column the edges represent hypermultiplets and the circles represent vector multiplets.

- The *flavour node* is the square node. It represents a global flavour symmetry of the system $SU(N_f)$. In this case the hypermultiplets transform under the fundamental representation of $SU(N_f)$.

- The *arrows* represent chiral superfields. The arrow indicates the representation of the superfields with respect to the gauge group $G = U(N_c)$ and the flavour group $SU(N_f)$. Each chiral superfield transforms under the fundamental representation of the node towards which the arrow points and under the anti-fundamental representation of the node from which the arrow comes. Note that this is consistent with the statement that $Q$ and $\bar{Q}$ need to transform under complex conjugate representations. Also note that this convention implies that $\Phi$ transforms under the tensor product of fundamental and anti-fundamental of $G = U(N_c)$, which is precisely the adjoint representation of dimension $N_c^2$. In Dynkin labels of $SU(N_c) \subset U(N_c)$:

$$[1, 0, \ldots, 0] \oplus [0, \ldots, 0, 1] = [1, 0, \ldots, 0, 1] \oplus [0, \ldots, 0],$$

where the dots represent zeroes.

The reason why quivers are extremely useful when dealing with 4d $\mathcal{N} = 2$ theories is that they neatly summarise the particle content of the theory under study. Furthermore, some parts of the Lagrangian are also fixed by the quiver, since the superpotential has to be of the type described in (1.21).

Note that, however, the convention with *arrows* allows for any set of superfields in a quiver, not necessarily arranged in 4d $\mathcal{N} = 2$ vector multiplets and hypermultiplets. When dealing with vector multiplets and hypermultiplets the notation can be simplified, assigning a single gauge node to an entire vector multiplet in the adjoint representation and a single *edge* with no arrows to an entire hypermultiplet in the bifundamental representation of the two nodes it connects. The quiver in eq. (1.22) would now look like:

$$Q = \begin{array}{c}
N_f \\
\circ \\
N_c
\end{array}$$

(1.24)

See a more generic example of the two notations in Table 1.1, for a theory with gauge group $G = U(1)^3$. In this case there is a different gauge node for each different $U(1)$ factor of the gauge group.

## 1.3 The Moduli Space

The problems that concern this thesis have to do with the vacuum state of the theories that we want to understand. We aim to study the mechanism that spontaneously breaks the gauge symmetry of the theory, while keeping the supersymmetry unbroken. This study is highly geometrical. The reason for this is that each theory has a set of possible states $|\alpha\rangle$ that can be the vacuum. Let us label them with the letter $\alpha$. They are
1.3. THE MODULI SPACE

distinguished by the expectation values of the scalar fields $s(x)$ of the theory:

$$(\alpha | s(x) | \alpha) = s_\alpha.$$  \hspace{1cm} (1.25)

$s_\alpha$ is called the vacuum expectation value (or VEV) of the scalar field $s(x)$, for the choice of vacuum $|\alpha\rangle$ and it does not depend on $x$. The geometry of the problem becomes manifest with the introduction of the moduli space $\mathcal{M}$. The moduli space is an affine variety such that each point of the variety corresponds to a different value of the VEVs $s_\alpha$ of all the scalar fields in the theory. The geometry of $\mathcal{M}$ and the structure of its singularities has deep implications on the way in which the gauge symmetry is broken and on the particles that remain massless for each different choice of vacuum.

On the theories at hand, with 8 supercharges and in 3, 4 and 5 space-time dimensions, there are two special sets of vacuum states that can be chosen that correspond to two different phases of the theory: a Coulomb phase and a Higgs phase (see for example [37, 38] for $4d \mathcal{N} = 2$). The Coulomb phase is a set of vacuum states with the properties that the only scalar fields that are components of vector multiplets have non-zero VEVs $s_\alpha$. In a Coulomb vacuum the gauge group $G$ of the theory is never fully broken. The maximal breaking that can occur is to the maximal torus of the gauge group $U(1)^{\text{rank}(G)} \subset G$. In the Higgs phase the only scalar fields that have non-zero VEVs $s_\alpha$ are component fields of hypermultiplets. In a Higgs vacuum the gauge group $G$ may be fully broken. These phases are represented by distinct parts of the moduli space $\mathcal{M}$ which are called the Coulomb branch $\mathcal{M}_C \subset \mathcal{M}$ and the Higgs branch $\mathcal{M}_H \subset \mathcal{M}$.

In general there would be other branches $\mathcal{M}'_i$ of the moduli space $\mathcal{M}$ where scalar fields from both vector multiplets and hypermultiplets of the theory have non-zero VEVs. The branches $\mathcal{M}'_i$, $\mathcal{M}_C$ and $\mathcal{M}_H$ have non-empty intersections and the moduli space is the union:

$$\mathcal{M} = \mathcal{M}_C \cup \mathcal{M}_H \cup \mathcal{M}'_i.$$  \hspace{1cm} (1.26)

Exactly how these intersections work can be described very simply using geometry and the mathematical study of singularities in certain type of varieties that we will introduce latter. This is one of the two main goals of this thesis: to introduce the geometric insight from the study of singular varieties into the theory of supersymmetric particles and their different vacuum states [1, 4, 5, 6]. Once this is established it can be used as the base for our second goal: to introduce the magnetic quiver as a new tool to study the vacuum of a theory in any dimension [2, 8].

The tool that we need to achieve both goals is string theory and the dynamics of branes. This is because it is possible to choose a system of branes in 10 space-time dimensions that has a low energy effective description in terms of a supersymmetric quantum field theory with 8 supercharges (see for example the summary in [39]). Furthermore, the values of the scalar fields in the theory acquire a new interpretation as positions of branes. Therefore, this provides a way to depict the different choices of VEVs as different positions of the branes in the string theoretical system. In addition, other parameters of the quantum field theory, like the gauge couplings, the mass parameters or the FI terms also have a realisation in terms of positions of different branes. This is the key feature that we will exploit in this thesis and that will make it possible to create a bridge between the mathematics (algebraic geometry) and the physics (quantum field theory). In the next chapter we introduce the brane systems and the key properties that are needed for our research.
Chapter 2

Brane Construction

2.1 Dynamics of Semi-Infinite D3-branes

Let us review the brane systems [10] for theories in 3 space-time dimensions with 8 supercharges [40, 41]. The description of the connection between the low energy dynamics of these types of brane systems and the effective gauge theories living in their world-volume has been nicely summarised in [39]. We use the same vector multiplets and hypermultiplets as in 4d $\mathcal{N} = 2$ theories, but now in 3d $\mathcal{N} = 4$ there is the following difference:

- The Weyl spinors of $SO(4)$ are now both spin $\frac{1}{2}$ representations of $SU(2)$, the double cover of $SO(3)$. In Dynkin labels the decomposition is:
  \[
  [1,0]_{SO(4)} \rightarrow [1]_{SU(2)},
  [0,1]_{SO(4)} \rightarrow [1]_{SU(2)}.
  \] (2.1)

- The vector field of $SO(4)$ decomposes into a vector field of spin 1 of $SU(2)$ and a real scalar of spin 0 of $SU(2)$. In Dynkin labels:
  \[
  [1,1]_{SO(4)} \rightarrow [2]_{SU(2)} \oplus [0]_{SU(2)}.
  \] (2.2)

With these changes the massless hypermultiplet still carries 4 bosonic degrees of freedom and 4 fermionic, and the massless vector multiplet also carries 4 and 4 with the difference that 3 out of the 4 bosonic are now as scalar fields and the component massless vector field only carries 1 degree of freedom.

These theories live in the world volume of semi-infinite D3-branes (see for example [42]) on a Type IIB string background. The D3-branes end on five-branes along one of their directions, leaving a macroscopic 3d theory on their worldvolume. A special way of arranging D3-branes, D5-branes and NS5-branes is discussed in [10] such that 8 supercharges are preserved. Let us be more explicit. First of all, consider Type IIB string background with space directions $(x^1, x^2, \ldots, x^9)$ and time direction $x^0$. Then, consider an NS5-brane with constant position at the origin of directions $(x_6, x_7, x_8, x_9)$. This system breaks the supercharges of Type IIB from 32 to 16. Let us follow the notation of [10] and use the label $\tilde{w}$ for three of the space directions:

\[
\tilde{w} = (x^7, x^8, x^9).
\] (2.3)

Let us similarly label other three space dimensions (this time all of them spanned by the NS5-brane) with $\tilde{m}$:

\[
\tilde{m} = (x^3, x^4, x^5).
\] (2.4)

One can add any number of additional N5-branes to the system without breaking further supersymmetry, as long as they have constant positions (not necessarily the origin) along directions $\tilde{w}$ and $x^6$. Following [10] we use $\tilde{w}_i$ and $t_i$ for the values of the position of the $i$-th NS5-brane along directions $\tilde{w}$ and $x^6$. A system with
CHAPTER 2. BRANE CONSTRUCTION

Figure 2.1: Two NS5-branes. The horizontal direction is $x^6$. The vertical direction is $\tilde{m}$. The direction perpendicular to the paper is $\tilde{w}$, where the paper is the origin. Both branes are at the origin $\tilde{w}_1 = \tilde{w}_2 = (0, 0, 0)$. This system preserves 16 supercharges.

$$\times$$

NS5

$$\times$$

NS5

Figure 2.2: Two NS5-branes and three D5-branes. The horizontal direction is $x^6$. The vertical direction is $\tilde{m}$. The direction perpendicular to the paper is $\tilde{w}$. The crosses represent D5-branes that have constant positions along $\tilde{m}$ and $x^6$. This system preserves 8 supercharges.

2 such NS5-branes is represented in figure 2.1. In figure 2.1 both NS5-branes have the same position along $\tilde{w}$, this can be set to be the origin without loss of generality:

$$\tilde{w}_1 = \tilde{w}_2 = (0, 0, 0). \quad (2.5)$$

Now one can add additional D5-branes, as long as they have a constant position along directions $\tilde{m}$ and $x^6$. This will break the supersymmetry from 16 down to 8 supercharges [10]. We use the label $\tilde{m}_j$ and $z_j$ for the position of the $j$-th D5-brane along directions $\tilde{m}$ and $x^6$ respectively.

An example of a system with both NS5-branes and D5-branes in this configuration can be seen in figure 2.2. Three extra D5-branes have been added with $z_j$ positions along $x^6$:

$$t_1 < z_1 < t_2 < z_2 < z_3, \quad (2.6)$$

and $\tilde{m}_j$ positions along $\tilde{m}$:

$$\tilde{m}_1 = \tilde{m}_3 \neq \tilde{m}_2. \quad (2.7)$$

Now D3-branes can be added in a way in which the 8 supercharges are still preserved if they have constant position along space directions $\tilde{m}$ and $\tilde{w}$. This means that they stretch along direction $x^6$. Therefore, they can end in either two different NS5-branes, two different D5-branes or an NS5-brane on one end and a D5-brane on the other end. See an example of this configuration in figure 2.3. Let us denote by $\tilde{x}_k$ the position of the $k$-th D3-brane along direction $\tilde{m}$ and let us denote by $\tilde{y}_k$ its position along direction $\tilde{w}$. The three different types of D3-branes are:

- A D3-brane that starts and ends on two different NS5-branes. For this to be possible both NS5-branes must share position $\tilde{w}_1 = \tilde{w}_2$. The three-brane also shares this position $\tilde{y}_1 = \tilde{w}_1 = \tilde{w}_2$ and cannot move in this direction. The D3-brane is free to move along direction $\tilde{m}$. Its position along this direction is labelled $\tilde{x}_1$.

- A D3-brane that starts and ends on two different D5-branes. For this to be possible both D5-branes must share position $\tilde{m}_1 = \tilde{m}_2$. The D3-brane must also share that position $\tilde{x}_1 = \tilde{m}_1 = \tilde{m}_2$ and cannot move along this direction. The D3-brane is free to move along direction $\tilde{w}$. Its position along this direction is labelled $\tilde{y}_1$.

- A D3-brane that starts on an NS5-brane and ends on a D5-brane, or vice-versa. Such D3-brane cannot
move in any direction. Its position along the $\vec{m}$ direction is fixed by the position of the D5-brane: $\vec{x}_1 = \vec{m}_1$.

Let us consider the gauge theory in the worldvolume a single D3-brane that is infinite along all $x^1$, $x^2$ and $x^6$ directions. In the worldvolume theory there is a vector multiplet and a hypermultiplet, as signified by the fact that there are 6 directions in which thebrane can move. This gives a theory with six real scalar components in $4d \mathcal{N} = 4$. In $4d \mathcal{N} = 2$ language, both the vector multiplet and the hypermultiplet are in the adjoint representation of the gauge group $G = U(1)$. When the boundary conditions from the five-branes are imposed the theory becomes 3d. The three directions $\vec{x}_1$ correspond with three scalar fields on a 3d $\mathcal{N} = 4$ supermultiplet. They are related to the vector field $a_\mu$ in 3d via the supersymmetry transformations [10]. The massless field $a_\mu$ can be dualized to a scalar field $a$ in the three space-time dimensional theory. $a$ is called the dual photon. The three directions $\vec{y}_1$ correspond to three component scalar fields of a different 3d $\mathcal{N} = 4$ supermultiplet. They are related to the scalar field $b$ that has its origin as the component of the vector field in four dimensions $A_{\mu}$, with bosonic components $(a, \vec{x}_1)$, and a hypermultiplet, with bosonic components $(b, \vec{y}_1)$.

When the D3-brane ends on an NS5-brane the fields in the vector multiplet $(a, \vec{x}_1)$ receive Newman boundary conditions and are free to change in the boundary, i.e. thebrane can move along direction $\vec{m}$; while the fields in the hypermultiplet $(b, \vec{y}_1)$ receive Dirichlet boundary conditions, i.e. thebrane cannot move along direction $\vec{w}$. On the other hand, when the D3-brane ends on a D5-brane the fields in the vector multiplet $(a, \vec{x}_1)$ receive Dirichlet boundary conditions and the fields in the hypermultiplet $(b, \vec{y}_1)$ receive Newman boundary conditions.

The fact that the vector multiplet contains four scalars $(a, \vec{x}_1)$ makes it a hypermultiplet with scalars $(b, \vec{y}_1)$ possible [41]. This is the reason why we discuss 3d theories in the thesis first. This duality will be crucial in the later parts of the thesis to be able to explain the moduli spaces of theories in 5 (also in 4 and 6) space-time dimensions with 8 supercharges [43, 2, 44, 8].

In the following section we review different 3d $\mathcal{N} = 4$ theories living on D3-branes with different particles, depending on the choice of boundary conditions of the D3-brane. We also demonstrate thebrane interpretation of the choice of vacuum expectation value for the scalar fields in vector multiplets $(a_i, \vec{x}_i)$ and in hypermultiplets $(b_j, \vec{y}_j)$ with the simplest non trivial effective gauge theory, SQED with two flavours.

Please refer to table 2.1 for a summary of the space-time dimensions spanned by the different types of branes in our system.
Figure 2.4: SQED with $N_f = 0$ in 3d $\mathcal{N} = 4$. The vertical lines are NS5-branes. The horizontal line represents a D3-brane, it shares direction $\tilde{y}_1 = \tilde{w}_1 = \tilde{w}_2$ with the NS5-branes. It can move freely along the vertical direction, hence its position $\tilde{x}_1$ is not fixed.

### 2.2 One Vector Multiplet: SQED with $N_f = 0$

Let us first look at a 3d $\mathcal{N} = 4$ theory with a single vector multiplet $(V, \Phi)$ in the adjoint representation of $G = U(1)$ and no hypermultiplets. Let us use the dualised photon $a(x)$ and label the four scalar fields in the vector multiplet as $(a, \tilde{x})$. We say that there are no flavors:

$$N_f = 0. \quad (2.8)$$

This means that the quiver $Q$ in 8 supercharges notation is a single gauge node of rank 1:

$$Q = \begin{array}{c} \circ \\ 1 \end{array} \quad (2.9)$$

In order to build 3d $\mathcal{N} = 4$ SQED let there be two different NS5-branes with the same position along $\tilde{w}$, but different directions along $x^6$:

$$\tilde{w}_1 = \tilde{w}_2, \quad t_1 \neq t_2. \quad (2.10)$$

Now a single D3-brane can be stretched between both NS5-branes, with fixed position $\tilde{y}$:

$$\tilde{y}_1 = \tilde{w}_1 = \tilde{w}_2. \quad (2.11)$$

This is depicted in Figure 2.4.

The inverse gauge coupling squared of $G = U(1)$ is proportional to the distance between the two NS5-branes [10]. Up to a multiplicative constant we have:

$$\frac{1}{g^2} = |t_1 - t_2|. \quad (2.12)$$

In order to find the moduli space, the Lagrangian can be written in terms of component fields, this will produce a potential term for all the scalar component fields $(a(x), \tilde{x}_1(x))$. Note that now $x$ refers to the worldvolume coordinates on the D3-brane at low energy: $x^0, x^1,$ and $x^2$.

These fields are in the adjoint representation of $U(1)$. It can be checked that any constant value of $(a(x), \tilde{x}_1(x))$ that does not depend on $x$ minimises the energy classically and preserves the supersymmetry.

The dual photon $a(x)$ can take vacuum expectation values in the circle $S^1$ [40]. Note that the radius $R$ is proportional to the gauge coupling, therefore at the infrared limit the radius tends to infinity and the moduli space is parametrized by constant values of the component fields $(a, \tilde{x}_1) \in \mathbb{R}^4$. Since all these fields belong to the vector multiplet the entire moduli space is also the Coulomb branch. We write:

$$\mathcal{M}_C = \mathbb{R}^4 = \mathbb{C}^2. \quad (2.13)$$

Since there are not hypermultiplets the Higgs branch is just the trivial variety (a single point).
Figure 2.5: One free hypermultiplet. The horizontal line is a D3-brane. The D5-brane is represented by a cross in (a) and by a vertical dashed line in (b). The vertical direction represents space directions \( \vec{m} \) in (a) and space directions \( \vec{w} \) in (b). Conversely, the direction perpendicular to the paper is identified with \( \vec{w} \) in (a) and with \( \vec{m} \) in (b).

Therefore there are no mixed branches either. In this case the full moduli space is the Coulomb branch:

\[
\mathcal{M} = \mathcal{M}_C = \mathbb{C}^2.
\] (2.14)

Note that there are no singular points in the moduli space. In this case all possible choices of vacuum \( |\alpha\rangle \) are equivalent: the massless matter content of the theory does not change, it is always one vector multiplet that remains massless. The gauge group \( G = U(1) \) therefore always remains unbroken. This is consistent with the fact that the entire moduli space is the Coulomb branch.

### 2.3 One Free Hypermultiplet

Now let us think of a theory where, at low energies, the vector multiplet inside the world volume of the D3-brane is set to zero by the boundary conditions while the hypermultiplet is free. For this system we need two D5-branes with different positions \( z_j \) along \( x^6 \). Both D5-branes need to have the same position along direction \( \vec{m} \):

\[
\vec{m}_1 = \vec{m}_2 \\
z_1 \neq z_2.
\] (2.15)

We can draw a figure like figure 2.5 (a). Note that the D3-brane in figure 2.5 (a) has the same position \( \vec{y}_i \) as the paper, but it can move in the direction \( \vec{w} \) perpendicular to the paper. For this reason one might want to switch perspectives and draw a diagram in which the vertical direction and the direction perpendicular to the paper are swapped. This diagram is drawn in figure 2.5 (b).

In this case the theory has a single hypermultiplet \( (Q, \bar{Q}) \), where the bosonic degrees of freedom are three scalars \( \vec{y}_i(x) \), the position of the D3-brane, and a fourth scalar \( b(x) \). The fact that the brane can move along direction \( \vec{w} \) means that the moduli space only has a Higgs branch, which is non-singular:

\[
\mathcal{M}_H = \mathbb{R}^4 = \mathbb{C}^2.
\] (2.16)

Indeed, in this theory there is no superpotential, just a the kinetic term for a single hypermultiplet. Therefore, any vacuum \( |\alpha\rangle \) is equivalent, and the Higgs mechanism is trivial.

The full moduli space is:

\[
\mathcal{M} = \mathcal{M}_H = \mathbb{C}^2.
\] (2.17)

### 2.4 Mirror Symmetry

Notice that the last two theories have the property that the Higgs branch of one theory is the Coulomb branch of the other and vice-versa. This is the 3d mirror symmetry discovered by [41]. In the brane system this...
mirror symmetry is incarnated by an S-duality transformation \[10\]. The transformation takes the NS5-branes to D5-branes and vice-versa. The D3-branes are mapped to D3-branes. One can see that the brane diagram in figure 2.4 and the brane diagram in 2.5 are related by a combination of S-duality and a rotation that takes \( \tilde{m} \) to \( \tilde{w} \) and \( \tilde{w} \) to \( -\tilde{m} \). This is represented in figure 2.6.

2.5 SQED with \( N_f = 2 \)

Now let us look at our first non-trivial example \[10\]. This example is particularly interesting because the 3d \( \mathcal{N} = 4 \) quiver gauge theory that describes the IR dynamics is self-dual under 3d mirror symmetry. Let us consider the brane system described in figure 2.7. There are two NS5s and two D5s. There is also a single D3-brane that is stretched between the NS5s. Therefore, the two NSs have coincident position along \( \tilde{w} \). The two D5s have set to have the same position along \( \tilde{m} \), but there is no D3-brane ending between them. The world volume theory on the D3-brane is still 3d \( \mathcal{N} = 4 \). There is a single vector multiplet, in the adjoint representation of a gauge group \( G = U(1) \).

There are also two hypermultiplets, charged under the gauge group \( G = U(1) \). This can be seen from the fact that a single F1-string can be stretched between the D3 and one D5-brane. Both hypermultiplets have the same mass parameter in the Lagrangian. This is reflected in the positions \( \tilde{m}_j \) of the D5-branes. The parameter can be set to zero without loss of generality.

The quiver \( Q \) for this theory is:

\[
Q = \begin{array}{c}
2 \\
\circ \\
1
\end{array}
\]  

(2.18)

The moduli space of this theory is more complex than that of the previous two theories: there is a Higgs branch and a Coulomb branch. There is also a special point: the singularity. This point sits precisely at the intersection of the Higgs branch and the Coulomb branch:

\[
\mathcal{M} = \mathcal{M}_C \cup \mathcal{M}_H;
\]  

(2.19)

and the intersection is a single point:

\[
\mathcal{M}_C \cap \mathcal{M}_H = \{0\}.
\]  

(2.20)

Therefore there are three different types of vacua \( |\alpha\rangle \). This can be seen from the point of view of QFT or from the brane embedding. Let us briefly review both.
2.5. $\text{SQED with } N_F = 2$

From the QFT

The singular point can be identified with the origin of both the Higgs and the Coulomb branch. Let us call the corresponding vacuum $|0\rangle$. This is the state for which the VEVs of all scalar fields (from the vector multiplet and from the two hypermultiplets) as they appear in the Lagrangian vanish. Since the Lagrangian describes massless particles, for this choice of vacuum $|0\rangle$ all particles remain massless. The gauge group remains unbroken.

Now there is a choice to go into the Higgs branch or into the Coulomb branch. If we go into the Higgs branch we select a different vacuum $|h_i\rangle$ where the scalars $s(x)$ of one hypermultiplet have non-zero VEV $s_{h_i}$. A new lagrangian can be written in which those fields are redefined as

$$s'(x) = s(x) - s_{h_i}$$

(2.21)

In the new Lagrangian the gauge group is fully broken. This means that the vector multiplet acquires mass and *eats* the remaining hypermultiplet. Therefore, the massless content of the theory for a choice of vacua in the Higgs branch $|h_i\rangle$ is a single hypermultiplet.

On the other hand, one could have chosen a different direction to go from $|0\rangle$, by giving non-zero VEV to the scalars in the vector multiplet. This would keep the vector multiplet massless, but would give mass to the two hypermultiplets. Hence, the gauge group would remain unbroken in the Coulomb branch and the new vacuum is denoted by $|c_i\rangle$.

From the Branes

In here let us discuss how the branes see the VEV of a scalar field and how this VEV can be set to zero.

Let us begin by remembering that motion of lighter branes (D3s) corresponds to the excitations of scalar fields in the effective field theory that describes the low energy dynamics, while motion of heavier branes corresponds to the tuning of the parameters that enter in the Lagrangian, such as the mass terms, the gauge couplings or the FI terms.

Furthermore, the motions where the whole D3-brane has constant position $\tilde{x}_1$ and $\tilde{y}_1$ that do not depend on the space coordinates of the brane itself correspond to configurations of minimum energy. Any other configuration would increase the energy due to the kinetic term for the scalar fields in the effective Lagrangian.

Hence, different positions of D3-branes in the systems where they move as a whole correspond with different choices of vacuum $|\alpha\rangle$ in the quantum field theory. Let us try to identify the brane system which corresponds to a choice of vacuum $|0\rangle$ where all the fields in the effective $3d$ $\mathcal{N} = 4$ SQED with $N_F = 2$ remain massless. This is precisely the point where the D3-brane has position coincident with the NS5s along $\tilde{w}$ and coincident with the D5s along $\tilde{m}$:

$$\tilde{x}_1 = \tilde{m}_1 = \tilde{m}_2$$
$$\tilde{y}_1 = \tilde{w}_1 = \tilde{w}_2.$$  

(2.22)

This is due to the fact that a different position along $\tilde{m}$ would increase the minimum length of a fundamental string between the D3 and either of the D5s, giving a mass to the hypermultiplet fields. This configuration is depicted in figure 2.8 (a).

From this point there are two inequivalent motions of the brane. The choice is the same as choosing to go to the Higgs branch or to the Coulomb branch. On one hand, one could let the D3-brane shift its constant position $\tilde{x}_1$ along $\tilde{m}$:

$$\tilde{x}_1 \neq \tilde{m}_1 = \tilde{m}_2$$
$$\tilde{y}_1 = \tilde{w}_1 = \tilde{w}_2.$$  

(2.23)

This is depicted in figure 2.8 (c). Such position corresponds to a choice of vacuum $|c_i\rangle$ in which the hypermultiplets have acquired a mass spontaneously that is proportional to the difference $|\tilde{x} - \tilde{m}_1|$. 
On the other hand, one could start at $|0\rangle$ and move to the Higgs branch. In order to do so let us go back to figure 2.8 (a). The D3-brane can be split into three distinct segments without losing any extra supersymmetry. The left segment begins in an NS5 and ends on a D5, it is therefore fixed. The right segment begins in a D5 and ends on an NS5. Therefore its position is also fixed. There is the middle segment however which begins on a D5 and ends on a D5. This means that it can now shift freely its position along $\tilde{y}_k$ along $\tilde{w}$. In order to represent this properly let us rotate the diagram. Take figure 2.8 (b). It represents the same brane system as 2.8 (a) where the directions $\tilde{m}$ and $\tilde{w}$ have been rotated into each other. The vertical dashed lines are D5-branes, while the circled crosses are NS5-branes. Now, the motion of the middle segment along $\tilde{w}$ is represented in figure 2.8 (d). This motion corresponds to the choice of vacuum in the Higgs branch $|h\rangle$. This is because a single hypermultiplet can acquire non-zero value in this configuration. This means that the massless spectrum of the theory redefined around this vacuum configuration contains a single hypermultiplet. Hence, the vector multiplet that was massless at $|0\rangle$ has become massive. This is consistent with the disappearance of one of the massless hypermultiplet, which plays the role of the Goldstone particle that has been eaten by the massive vector multiplet.

There are many things that one can learn about the moduli space of the theory $Q$, i.e. SQED with $N_f = 2$, just from the brane system in figure 2.8. For example, the Coulomb branch of this theory has real dimension $\dim_{\mathbb{R}}(\mathcal{M}_C) = 4$, since it describes the motion of a single D3-brane. Similarly, the Higgs branch has also real dimension $\dim_{\mathbb{R}}(\mathcal{M}_H) = 4$.

Furthermore, the Coulomb branch cannot be just $\mathbb{C}^2$, since in this case there is a singular point, signified by the choice or position $\tilde{x}_1$ of the D3 that parametrises the Coulomb branch where it aligns with the D5s and a motion into the Higgs branch directions $\tilde{w}$ becomes possible. A natural guess would be a Kleinian surface singularity of type:

$$\mathcal{M}_C = \mathbb{C}^2/\Gamma,$$

(2.24)

where $\Gamma \subset SU(2)$ is a finite group. An analogous argument on the Higgs branch leads to the conjecture:

$$\mathcal{M}_H = \mathbb{C}^2/\Gamma'$$

(2.25)

with $\Gamma' \subset SU(2)$ the same or a different finite subgroup.

This is indeed correct and there are two ways in which $\Gamma$ and $\Gamma'$ can be determined. They will be described in detail in the following sections but let us present the general lines of reasoning here. The idea is that the Kleinian surface singularities $S$ have distinct Hilbert series $H_S$, that depend on the choice of the finite group. We will describe the Hilbert series in the next chapter, but for now it can be thought of as an infinite series in powers of $t$ that can be expressed as a quotient of polynomials that characterise the variety. For example, for
2.5. SQED WITH $N_f = 2$

$\Gamma = \mathbb{Z}_k$, i.e. the cyclic group of order $k$, the Hilbert series takes the form:

$$H_{\mathbb{C}^2/\mathbb{Z}_k} = \frac{1 - t^{2k}}{(1 - t^2)(1 - t^k)^2}. \quad (2.26)$$

In the case of the Coulomb branch, the Hilbert series of $\mathcal{M}_C$ can be computed employing the monopole formula \[45, 46, 47, 48, 49, 50\]. In the case of the Higgs branch, the Hilbert series can be computed via a hyperkähler quotient \[51, 52, 53, 54, 55\]. Both procedures will be described in deep detail in the next pages of the thesis, but for now we can cite the result of their application to quiver $Q$. The results are:

$$H_{\mathcal{M}_C} = \frac{1 - t^4}{(1 - t^2)(1 - t^2)^2} \quad (2.27)$$

$$H_{\mathcal{M}_H} = \frac{1 - t^4}{(1 - t^2)(1 - t^2)^2}.$$

Which implies that:

$$\mathcal{M}_C = \mathbb{C}^2/\mathbb{Z}_2$$

$$\mathcal{M}_H = \mathbb{C}^2/\mathbb{Z}_2. \quad (2.28)$$

The branes also contain the information about the intersection of both branches. In this case there is a single point, corresponding with the D3-brane aligning with both NS5-branes and both D5-branes. This point can be seen as the origin. Therefore the intersection is trivial:

$$\mathcal{M}_C \cap \mathcal{M}_H = \{0\}. \quad (2.29)$$

Since there are no other possible brane motions that have not been accounted for in this analysis, the brane configuration also tells us that there are no mixed branches $\mathcal{M}_i'$ in the moduli space $\mathcal{M}$. Therefore, this fixes the complete geometry of the moduli space:

$$\mathcal{M} = \mathbb{C}^2/\mathbb{Z}_2 \cup \mathbb{C}^2/\mathbb{Z}_2, \quad (2.30)$$

with trivial intersection at the origin of both cones.

Note how powerful the brane analysis of figure 2.8 is and how much information it can be obtained about $\mathcal{M}$. The only thing we needed to complement the analysis and fully determine $\mathcal{M}$ was a procedure to compute the different Hilbert series $H_{\mathcal{M}_C}$ and $H_{\mathcal{M}_H}$. The results of \[1, 5, 2, 8\] contained in this thesis are simply an extension of the set of information that can be obtained from the branes. This case has a simple moduli space, but as soon as we go to more complicated ones there will be mixed branches $\mathcal{M}_i'$ that will be characterised by the existence of nested singular loci inside the Coulomb branch or inside the Higgs branch or both. One way in which we extended the brane techniques is we explain how to disentangle the information about the nested singularities structure \[1, 4, 5\], and hence about the mixed branches, directly from the brane diagram. This had not been done before and it was a natural extension to the type of analysis with branes for which SQED with $N_f = 2$ has been an example.

The other way in which we enlarged the set of brane techniques is a crucial extension to the reach of this procedure, i.e. of the types of theories that can be analysed in this way. In particular, we took all the lessons learned from the 3d $\mathcal{N} = 4$ brane systems introduced in here and that concern the first part of the thesis and extended them to the analysis of Higgs branches of theories in 4d, 5d and 6d, always with 8 supercharges \[2, 8, 56\]. For example, maybe the reader is not aware that the Higgs branch of SQCD with gauge group $SU(3)$ and $N_f = 4$ had never been fully characterised before, either in 3, 4, 5 or 6 dimensions. Now, one can not only solve this type of unsolved problems \[56\], but also understand the geometrical results in a distinctly pictorial way, which is always a useful way to deal with complicated physical computations.

In the next section we review how to read the quiver of the effective 3d $\mathcal{N} = 4$ gauge theory for a generic
3d Mirror Symmetry

Let us discuss the 3d mirror symmetry [41, 10]. The theory is dual to itself. This is consistent with the fact that the Coulomb branch and the Higgs branch are both the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_2$. Let us see how this can be computed from the branes. Start from the system in figure 2.8 (a). This is the system that describes $3d \mathcal{N} = 4$ SQED with $N_f = 2$. If we perform S-duality we obtain a new brane system, depicted in the right side of figure 2.9. In order to read the quiver for the mirror system, one needs to perform some brane transitions. These are depicted in figure 2.10. After the transitions, the brane system in figure 2.10 (d) is precisely the same brane configuration as the Coulomb branch brane configuration of SQED with $N_f = 2$ depicted in figure 2.8 (c). Hence, the system is self dual, and the theory SQED with $N_f = 2$ is self dual under 3d mirror symmetry.

2.6 A Generic Theory

Let us spend a couple of lines to review how to obtain a $3d \mathcal{N} = 4$ quiver $Q$ from a given brane system. Let a generic brane system be described by figure 2.11.

The low energy dynamics of this brane system is described by an effective $3d \mathcal{N} = 4$ gauge theory, defined by quiver $Q$. In order to read the quiver from the brane system one can follow the steps:

- First make sure that all D3s end on NS5-branes only. This represents a phase in which the Gauge group can only be spontaneously broken down to its maximal torus and therefore allows one to read it directly from the brane system.
2.6. A GENERIC THEORY

Figure 2.11: This is a generic brane system in Type IIB string theory. The vertical lines are NS5s, the crosses represent D5s and the horizontal lines represent D3s.

- The gauge group $G$ consists of the product of a series of unitary factors:

$$G = U(N_1) \times U(N_2) \times \cdots \times U(N_m).$$

Each different interval between NS5s corresponds to a different unitary factor $U(N_i)$ where the number of colours $N_i$ is equal to the number of D3-branes extended in such interval.

- Therefore the quiver $Q$ has one gauge node with label $N_i$ for each such interval.

- There is also one flavour node for each interval with label $K_i$, where $K_i$ is the number of D5s in such interval.

- The flavour symmetry group is of the form

$$S(U(K_1) \times U(K_2) \times \cdots \times U(K_l)),$$

where $S(...)$ denotes that an overall $U(1)$ factor is decoupled, similar to the concept of a decoupling center of mass.

- The gauge nodes of consecutive intervals are connected by a hypermultiplet line in the quiver.

- The flavour node with label $K_i$ is connected by a hypermultiplet line to the gauge node $N_i$.

In the example brane system of figure 2.11 there are four distinct segments between 5 different NS5-branes. The quiver has gauge group:

$$G = U(2) \times U(3) \times U(2) \times U(1)$$

It has flavour group:

$$S(U(1) \times U(2))$$

The quiver $Q$ is:

We can tell from the brane system that the Coulomb branch is given by the motion of $2 + 3 + 2 + 1 = 8$ D3-branes, so it has real dimension $\dim_R(M_C) = 8 \times 4 = 32$. In order to go to the Higgs branch we bring all D3s together to the $|0\rangle$ point of the moduli space, and then re-split them and give non-zero $\tilde{y}_i$ position to the ones that can move along the $\tilde{m}$ direction. This is depicted in figure 2.12 (a-j) (note that Hanany-Witten transitions have been done to obtain a more clear realization of the Higgs branch). The Higgs branch, figure 2.12 (j), has 4 different D3-branes moving along $\tilde{m}$ direction. Therefore the real dimension of the Higgs branch is $\dim_R(M_H) = 4 \times 4 = 16$.

By performing S-duality in the brane system in figure 2.12 (j) we obtain a new brane system, figure 2.13.
Figure 2.12: Generic transition from the Coulomb branch to the Higgs branch, via Hanany-Witten transitions. (a) is the Coulomb branch. (j) is the Higgs branch.

Figure 2.13: Generic 3d mirror theory.

From this system a new quiver can be read:

\[
Q' = \begin{array}{c}
\begin{array}{c}
\bigcirc \\
1
\end{array} & \\
\bigcirc & \\
\begin{array}{c}
\bigcirc \\
3
\end{array}
\end{array}
\begin{array}{c}
\bigcirc \\
5
\end{array}
\end{array}
\]

(2.36)

This is the 3d mirror system of the quiver (2.35).

This concludes Part II of this thesis. In Part III we will look into the moduli space of the theories in more detail and we will use the theory of nilpotent orbits to understand the structure of its singular points.
Part III

The Vacuum and the Nilpotent Orbit
Chapter 3

A Bit of Algebraic Geometry

Before discussing the results about the moduli space of 3d \( \mathcal{N} = 4 \) theories and their realisation as dynamics of branes, let us take a brief detour in order to review some necessary mathematical tools. In particular, we want to employ this chapter to review the concept of Hilbert series. The reader can find a mathematical description of the concept of the Hilbert function in Chapter 13 of the book in algebraic geometry by Harris [57]. Here, we try to convey an intuitive way of thinking about Hilbert series that we have found particularly useful during our research. The Hilbert series of the algebraic variety \( V \) is an infinite power series in one variable, that we will call \( t \), of the form

\[
H_V = \sum_{n=0}^{\infty} h_V(n) t^n,
\]

(3.1)

where \( h_V(n) \) are integer coefficients (these are the coefficients denoted Hilbert functions in [57]). For example the Hilbert series for \( V = \mathbb{C} \) has coefficients \( h_{\mathbb{C}}(n) = 1 \), \( \forall n \):

\[
H_{\mathbb{C}} = \sum_{n=0}^{\infty} t^n.
\]

(3.2)

The infinite sum can be expressed as:

\[
H_{\mathbb{C}} = \frac{1}{1 - t}.
\]

(3.3)

The Hilbert series \( H_{\mathbb{C}} \) can then be understood as a generating function that counts linearly independent holomorphic polynomials that exist in \( \mathbb{C} \), gaging them according to their degree. The coefficient \( h_V(n) \) is the number of such polynomials of degree \( n \). In the case of \( \mathbb{C} \) we can have the complex coordinate \( z \), then there is always one and only one linearly independent holomorphic polynomial at each degree:

- At degree 0: the constant function 1.
- At degree 1: the monomial \( z \).
- At degree 2: the monomial \( z^2 \).
- At degree 3: the monomial \( z^3 \).
- ...
- At degree \( n \): the monomial \( z^n \).

Now let us look at the Hilbert series for \( \mathbb{C}^2 \). The Hilbert series \( H_{\mathbb{C}^2} \) has coefficients:

\[
h_{\mathbb{C}^2}(n) = n + 1.
\]

(3.4)
Therefore, the Hilbert series is:

\[
H_{\mathbb{C}^2} = \sum_{n=0}^{\infty} (n+1)t^n,
\]

or in a quotient form:

\[
H_{\mathbb{C}^2} = \frac{1}{(1-t)^2}.
\]

To explicitly see the different holomorphic polynomials let the complex coordinates of \(\mathbb{C}^2\) be \(\{z_1, z_2\}\). The Hilbert series \(H_{\mathbb{C}^2}\) is counting the number of linearly independent holomorphic polynomials that can be defined on \(\mathbb{C}^2\) at different degrees:

- At degree 0: it is the constant function: 1.
- At degree 1: there are the two generators \(z_1\) and \(z_2\).
- At degree 2: there are three monomials \(z_1^2\), \(z_1 z_2\) and \(z_2^2\).
- \(\ldots\)
- At degree \(n\): there are \(n+1\) monomials: \(z_1^n, z_1^{n-1} z_2, \ldots, z_2^n\).

Now let us look at a non-trivial variety. Let us start with \(\mathbb{C}^2/\mathbb{Z}_2\). In order to define the variety we need to specify the action of the discrete group \(\mathbb{Z}_2\) on the variables of \(\mathbb{C}^2\), \(\{z_1, z_2\}\). Let the two elements of \(\mathbb{Z}_2\) be \(\{e, a\}\), where \(e\) is the identity and \(a\) satisfies \(a^2 = e\). Then the action of \(\mathbb{Z}_2\) is:

\[
\begin{align*}
e(z_1) &= z_1 \\
e(z_2) &= z_2 \\
a(z_1) &= -z_1 \\
a(z_2) &= -z_2.
\end{align*}
\]

The only holomorphic polynomials of \(\mathbb{C}^2/\mathbb{Z}_2\) are those polynomials of \(\mathbb{C}^2\) that are invariant under the action of \(\mathbb{Z}_2\). They are always of even degree:

\[
\{1, z_1^2, z_1 z_2, z_2^2, z_1^4, z_1^3 z_2, z_1^2 z_2^2, z_1 z_2^3, z_2^4, \ldots\}
\]

The corresponding Hilbert series has coefficients \(H_{\mathbb{C}^2/\mathbb{Z}_2}(n)\) that are equal to the coefficients of \(H_{\mathbb{C}^2}\) (eq. (3.4)) when \(n\) is even and are 0 otherwise. This can be expressed as:

\[
H_{\mathbb{C}^2/\mathbb{Z}_2} = \sum_{n=0}^{\infty} (2m+1)t^{2m}.
\]

This series can also be expressed in a quotient form:

\[
H_{\mathbb{C}^2/\mathbb{Z}_2} = \frac{1 - t^4}{(1-t^2)^3}.
\]

Note that the quotient form of the Hilbert series is quite enlightening. The denominator carries information about the monomials that generate the ring of all holomorphic polynomials in \(\mathbb{C}^2/\mathbb{Z}_2\). In this case there are three identical factors \((1-t^2)\). This corresponds to the fact that there are three distinct generators, let us denote them by:

\[
\begin{align*}
g_1 &\equiv z_1^2 \\
g_2 &\equiv z_1 z_2 \\
g_3 &\equiv z_2^2.
\end{align*}
\]
The power of $t$ in the factor $(1 - t^2)$ of each generator $g_i$ is 2 for all $i$. This signifies that the generator $g_i$ is a polynomial of degree 2 on the complex coordinates $\{z_1, z_2\}$. The numerator of the Hilbert series in equation (3.10) relates to relations satisfied by the generators of the polynomial ring. In this case there is a single factor $(1 - t^4)$, where the power of $t$ is 4. We can also show that there is a single relation between generators $g_i$, and that it occurs at degree 4 on the complex coordinates $\{z_1, z_2\}$:

$$g_1g_3 = g_2^2.$$  \hspace{1cm} (3.12)

Not all Hilbert series depict the generators and relations of the polynomial ring in such a clear way, but many of them do. In particular, we could now write the Hilbert series of $\mathbb{C}^k$ very easily. The polynomial ring is freely generated by the complex coordinates of $\mathbb{C}^k$: $\{z_1, z_2, \ldots, z_k\}$. By freely generated we mean that there are no relations between the generators. Therefore, the guess for a Hilbert series would be:

$$H_{\mathbb{C}^k} = \frac{1}{(1 - t)^k}.$$  \hspace{1cm} (3.13)

This is indeed the correct Hilbert series. Now let us think of what could be the Hilbert Series of $\mathbb{C}^2/\mathbb{Z}_k$. In this case the cyclic group $\mathbb{Z}_k$ is generated by a single element $a$, such that $a^k = e$, where $e$ is the identity element. We define the action of $a$ on the coordinates of $\mathbb{C}^2$:

$$a(z_1) = e^{i2\pi/k}z_1,$$
$$a(z_2) = e^{-i2\pi/k}z_2.$$  \hspace{1cm} (3.14)

Then the invariant polynomials that generate the entire holomorphic polynomial ring are:

$$g_1 \equiv z_1^k,$$
$$g_2 \equiv z_1z_2,$$
$$g_3 \equiv z_2^k.$$  \hspace{1cm} (3.15)

and they satisfy a relation of degree $2k$:

$$g_1g_3 = g_2^k.$$  \hspace{1cm} (3.16)

Hence, one could guess a Hilbert series of the form:

$$H_{\mathbb{C}^2/\mathbb{Z}_k} = \frac{1 - t^{2k}}{(1 - t^2)(1 - t^k)^2}.$$  \hspace{1cm} (3.17)

This is indeed the correct Hilbert series. In the following sections we compute the Hilbert series of different branches of the Moduli space to help us identify the geometry of such spaces.
Chapter 4

Interlude: SQED with $N_f$ Flavours

The material that has been covered so far can be used now to present a set of theories that will introduce in more detail the concepts that we are dealing with. This set of $3d$ $\mathcal{N} = 4$ theories is slightly more complicated than the example discussed in the previous chapters, $SQED$ with $N_f = 2$, but not as complex as a generic brane system could be. Actually it occupies a very special place in a complexity scale, since the theories have enough structure to play the role of building blocks. In this sense, any other theory that is more complex, can be decomposed in theories of the type presented here. The objective of this Part III is to illustrate the way in which this is done. Therefore, there are two reasons to study them here: they are the perfect introduction to the problem of computing Higgs and Coulomb branches of complicated theories, and we want to understand them properly since many explanations of the moduli space of more complicated theories (for example those studied in Part IV) will be given in terms of these ones.

4.1 The Quiver

The theory we want to understand is SQED with any number of $N_f$ flavours in $3d$ $\mathcal{N} = 4$. This theory was analysed in [40], its $3d$ mirror theory discovered in [41] and the brane system realisation of such mirror symmetry presented in [10]. This section is devoted to the review of such concepts. The quiver is:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{quiver.png} \\
\end{array}
\]

It has a single vector multiplet, that plays the role of the SQED photon, and $N_f$ hypermultiplets, that play the role of $N_f$ electrons and also their corresponding antiparticles, the $N_f$ positrons. The corresponding brane system is given in figure 4.1.

4.2 The Coulomb Branch

First of all, the brane system is already telling us that the moduli space does not have any mixed branches $\mathcal{M}_i^\prime$, so it is only:

\[
\mathcal{M} = \mathcal{M}_C \cup \mathcal{M}_H.
\]

If this is not evident now, it will hopefully become more clear after this discussion. In the picture 4.1 the Coulomb branch is depicted, since a singe D3-brane is moving along $\vec{m}$ with position $\vec{x}_1$.

In figure 4.1, all D5-branes are set at the same position along $\vec{m}$. Without loss of generality we can set this position to be the origin $\vec{m} = (0,0,0)$. The motion of the D3-brane along direction $\vec{m}$ means that a single
Figure 4.1: Brane configuration for SQED with $N_f$ flavours. The three dots signify that there are a total of $N_f$ D5-branes in the system with the same position $\tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_{N_f}$. They give rise to the $N_f$ different hypermultiplets with zero mass parameter in the lagrangian, charged with negative charge under the gauge group $G = U(1)$.

Figure 4.2: Origin of the moduli space of SQED with $N_f$ flavours. This corresponds to the choice of vacuum state $|0\rangle$, where all the particles in the theory remain massless.

The fact that the Coulomb branch parametrises the position of a single D3-brane signifies that the Coulomb branch has dimension $\dim \mathcal{M}_C = 4$. Again, like for the case of $N_f = 2$, there is only one singular point in the entire Coulomb branch. This point can be identified with the origin of the variety, where all scalars in the vector multiplet $(a, \bar{x}_1)$ have vacuum expectation value of zero. The corresponding vacuum that makes this possible is denoted again as $|0\rangle$. In the brane system this is the vacuum where the D3 aligns with the D5-branes. Let us represent this in figure 4.2.

From the point of view of the branes this is a singular point because at this position $\bar{x}_1 = (0, 0, 0)$ the D3-brane can be split into segments without breaking any further supersymmetry, such that some of the segments will start and end on two D5-branes, and can therefore move along direction $\bar{w}$.

Hence, having dimension $\dim \mathcal{M}_C = 4$ and a singular point at the origin, a Kleinian surface singularity of the type $C^2/\Gamma$ is again a good guess for the Coulomb branch. We leave the computation of the Hilbert series $H_{\mathcal{M}_C}$ for Chapter 6. In here we state the result:

$$H_{\mathcal{M}_C} = \frac{1}{(1-t^2)(1-t^{N_f})^2}. \quad (4.3)$$

As explained before, this confirms our suspicions and identifies the Coulomb branch as the algebraic variety:

$$\mathcal{M}_C = C^2/\mathbb{Z}_{N_f}. \quad (4.4)$$

Note that this nicely generalises the case $N_f = 2$ studied before.

4.3 The Higgs Branch

Let us think now about the Higgs branch. First, the brane diagram can be used to read off the dimension of the Higgs branch. In order to do that let us move to the Higgs branch brane configuration. Let us first rotate the diagram in figure 4.2 so $\bar{m}$ goes to $\bar{w}$ and $\bar{w}$ goes to $-\bar{m}$. This is represented in figure 4.3. Now the D3-brane can be divided into $N_f + 1$ different segments without breaking any supersymmetry. Two of the segments are fixed, while the other $N_f - 1$ are free to move along $\bar{w}$. They have positions $\bar{y}_1, \ldots, \bar{y}_{N_f-1}$. This is depicted in figure 4.4. This means that the real dimension of the Higgs branch, that now parametrizes the position of these brane segments along direction $\bar{w}$, is:

$$\dim \mathcal{M}_H = 4(N_f - 1) \quad (4.5)$$
4.3. THE HIGGS BRANCH

This is correct, since the dimension of the Higgs branch can also be computed with the following formula:

$$\dim_{\mathbb{H}}(M_H) = \#(\text{Hypermultiplets}) - \#(\text{Vector Multiplets}).$$  (4.6)

The subindex $\mathbb{H}$ denotes that this is the quaternionic dimension, which is defined as the real dimension $\dim_{\mathbb{R}}$ divided by 4. This formula represents the fact that in the Higgs branch the gauge group can be fully broken. Therefore the number of $3d \, \mathcal{N} = 4$ super multiplets that are massless in a generic point of the Higgs branch is the total number of hypermultiplets in the theory, these are all massless at the origin $|0\rangle$ or the Higgs branch, minus the number of hypermultiplets that play the role of Goldstone bosons and are eaten up by the now massive vector multiplets. Since the gauge group is fully broken at a generic non-singular point of the Higgs branch, all the vector multiplets spontaneously become massive. In this case there is a single vector multiplet, the photon, that becomes massive, therefore:

$$\dim_{\mathbb{H}}(M_H) = N_f - 1.$$  (4.7)

Now, it is harder to guess what type of variety the Higgs branch is. Looking at the example with $N_f = 2$ it should also be a type of variety that generalises $\mathbb{C}^2/\mathbb{Z}_2$ but that it does so growing in dimension. It should also be a variety with a single singular point at the origin. This is because it is only when all D3-branes that move along directions $\tilde{y}_i$ in the Higgs branch picture align with the two NS5-branes that a transition to the Coulomb branch can be made. This type of variety is what is called the closure of a nilpotent orbit. In the remaining pages of this Part III we explain what those varieties are and the role they play on the study of $3d \, \mathcal{N} = 4$ vacua.

Mirror Symmetry

Let us obtain the $3d$ mirror quiver $[41, 10]$. After applying S-duality to the system in figure 4.4 one obtains figure 4.5. A Hanany-Witten transition takes us to figure 4.6, from where the dual quiver $Q'$ can be read:

$$Q' = \begin{array}{ccc}
1 & \cdots & 1 \\
1 & \cdots & 1 \\
& & 1
\end{array}$$  (4.8)

$$N_f - 1$$
Figure 4.5: SQED brane system after performing S-duality on the Higgs branch brane configuration.

Figure 4.6: SQED 3d mirror.
Chapter 5

The Higgs Branch

5.1 The Higgs Branch as an Affine Variety

Let us spend the next two chapters to comment more explicitly on the computations of the Hilbert series \( H_V \), both in the case of the Higgs branch and in the case of the Coulomb branch. Since the computations are different we devote one chapter to comment on the Higgs branch and another chapter for the Coulomb branch. The idea of these chapters is to serve as an illustration on the computational methods. The full definition on how to compute Hilbert series was developed in [54] for the Higgs branch and in [50] for the Coulomb branch. A nice summary of the techniques can also be found on [58]. The ideas on Highest Weight Generating functions were developed in [59]. Since our present goal is that of illustrating the methods, let us choose the gauge theory discussed just above and focus on its moduli space.

That theory is 3d \( \mathcal{N} = 4 \) SQED with \( N_f \) flavours. Let us discuss its Higgs branch in this chapter. In this particular case we do not need to only compute the Hilbert series, we can obtain all the information about the algebraic variety, as it was done for the surface singularity \( \mathbb{C}^2/\mathbb{Z}_k \) in chapter 3, by specifying the generators of the coordinate ring and the relations between such generators. This can be done following [55].

It is easier to compute the Higgs branch by decomposing the 3d \( \mathcal{N} = 4 \) supermultiplets into chiral superfields and vector superfields. The Higgs branch is parametrised by the VEVs of the scalars in chiral superfields on the hypermultiplets, and in this case the computation can be done directly from the classical Lagrangian [38]. As explained before, the Lagrangian is summarised by the quiver, which in this case we choose to represent with superfields:

\[
Q = \begin{pmatrix} \Phi \\ \tilde{Q} \\ Q \end{pmatrix}
\]

(5.1)

Let us write the superpotential again:

\[
W = \sqrt{2} \text{Tr} \left( \tilde{Q} \Phi Q \right).
\]

(5.2)

This time \( \Phi \) is a 1 \( \times \) 1 matrix, \( Q \) is a row vector of dimension \( N_f \) and \( \tilde{Q} \) is a column vector of dimension \( N_f \). Let us write the elements of the row vector as \( Q_i \). They are chiral superfields with scalar component \( q_i(x) \) and spinor component \( \psi_{q,i}(x) \). The index \( i = 1, 2, \ldots, N_f \) transforms as the fundamental representation of the flavour symmetry \( SU(N_f) \). Let us write the elements of the column vector as \( Q^j \). They are chiral superfields with scalar component \( \tilde{q}^j(x) \) and spinor component \( \psi_{\tilde{q},j}(x) \). The index \( j = 1, 2, \ldots, N_f \) transforms as the anti-fundamental representation of the flavour symmetry \( SU(N_f) \).

The Higgs branch is characterised by VEVs of scalar fields in the hypermultiplets. However, one is only
interested in vacuum expectation values of combination of scalar fields that are gauge invariant. In this case the answer is simple. Remember that the fields in the chiral fields that are represented by arrows in the quiver always transform as fundamental of one group and anti-fundamental of the other group. The fields \( q_i(x) \) transform as fundamentals of \( SU(N_f) \), therefore they have charge \(-1\) under the gauge group \( G = U(1) \). The fields \( \tilde{q}^j(y) \) transform as anti-fundamental of the flavour group \( SU(N_f) \), therefore they have charge \(+1\) under the gauge group \( G = U(1) \). In this case \( Q_i \) are the electrons, while \( \tilde{Q}^j \) are the positrons of the SQED theory.

The simplest gauge invariant combinations that can be made in this case is a product of a field \( q_i(x) \) and a product of a field \( \tilde{q}^j(y) \). However, these operators are inside the chiral ring. This means that their vacuum expectation values, and the vacuum expectation values of their products are not space-time dependent. The fact that their products also have the same property gives this set of operators the structure of the ring. This is a ring under addition and multiplication of operators.

Since we have a ring structure, there is the notion of generators of the ring. All elements that can generate all other elements via linear combinations. The linear combination implies that there are coefficients that can be chosen when two elements of the ring are added. In this case we let any complex number as a coefficient when we add two scalar field operators. In algebraic geometry, the ring \( R \) of all elements generated by products and linear combinations of elements \( \{a, b, c, \ldots\} \) with coefficients in the field \( K \) is written as:

\[
R = K[a, b, c, \ldots].
\] (5.3)

In this case so far there are two rings of scalar operators that we can consider. The simplest one is the ring generated by \( q_i(x) \) and \( \tilde{q}^j(x) \) with coefficients on the field of the complex numbers \( \mathbb{C} \):

\[
R = \mathbb{C}[q_i, \tilde{q}^j].
\] (5.4)

But many operators in this ring are not gauge invariant. Note that we have dropped the space-time dependence of the operators. The other ring is the one generated by gauge invariant combinations \( q_i\tilde{q}^j \):

\[
R' = \mathbb{C}[q_i\tilde{q}^j].
\] (5.5)

\( R \) and \( R' \) are coordinate rings of two different affine varieties \( V_R \) and \( V_{R'} \). None of these varieties are yet the Higgs branch of the theory. The variety associated to \( R \) is simplest to understand, since \( R \) is generated by \( 2N_f \) elements, and they are not related to each other, it is a flat variety:

\[
V_R = \mathbb{C}^{2N_f}.
\] (5.6)

In the case of \( R' \) the generators are related to each other, since they are composed of one operator \( q_i \) and one operator \( \tilde{q}^j \) that can be the same. For example, a product of generators \((q_1\tilde{q}^1)(q_2\tilde{q}^2)\) is equal to the product of generators \((q_1\tilde{q}^2)(q_2\tilde{q}^1)\). These type of relations are evocative of setting the \( 2 \times 2 \) minor of a matrix to zero. In fact, the variety corresponding to \( R' \) can be easily defined as a set of matrices with one extra property. They are \( N_f \times N_f \) matrices with complex elements, and their rank is at most one:

\[
V_{R'} = \{M_{N_f \times N_f} | M_{ij} \in \mathbb{C}, \text{ rank}(M) \leq 1\}.
\] (5.7)

This can be made very explicit by arranging the generators \( q_i\tilde{q}^j \) in a matrix \( M \), such that \( M_{ij} = \tilde{q}^j q_i \). Then, the rank condition can be obtained from the fact that \( M \) can be written as a matrix product of a column vector \( \tilde{q} \) times a row vector \( q \):

\[
M = \tilde{q} q,
\] (5.8)

and from this follows that:

\[
\text{rank}(M) \leq \min(\text{rank}(\tilde{q}), \text{rank}(q)) = \min(1, 1) = 1
\] (5.9)
This variety is not yet the Higgs branch for the reason that we have not fixed the field configurations to have minimum energy, and be therefore in the classical vacuum of the theory. This can be achieved by looking at the scalar potential that comes from the supersymmetric Lagrangian. Once the scalar potential \( V(q_i, \tilde{q}_j) \) is found, if the minimum is achieved at zero \( V = 0 \) this means that the supersymmetry of the system is not spontaneously broken. In this case, it is a standard result that the scalar potential can be obtained by taking the the superpotential \( W(Q, \tilde{Q}, \Phi) \), substituting the chiral superfields with the corresponding complex scalar components, to get:

\[
W(q, \tilde{q}, \phi) = \sqrt{2} \text{Tr}(\tilde{q} \phi q),
\]

and then taking derivative with respect of \( \phi \):

\[
\frac{\partial W(q, \tilde{q}, \phi)}{\partial \phi} = 0,
\]

therefore:

\[
\sum_i q_i \tilde{q}_i = 0,
\]

or in matrix notation:

\[
q \tilde{q} = 0.
\]

The Higgs branch is therefore the variety that results from imposing relation (5.12) to the operators in the variety \( V_{R_{0}} \). In the Language of chiral rings, this defines a new ring \( R'' \) denominated quotient ring:

\[
R'' = R' / I,
\]

where \( I = (\sum_i q_i \tilde{q}_i) \) is the ideal that is set to zero by the relation (5.12). If we want to understand the variety \( V_{R''} \) in terms of sets of matrices it can be done in the following way. It can be checked that relation (5.12) implies that the trace of \( M \) is zero, as well as the square of \( M \) is zero:

\[
\text{Tr}M = \text{Tr}(\tilde{q}q) = \text{Tr}(q\tilde{q}) = 0,
\]

and

\[
M^2 = \tilde{q}qq = 0
\]

This set of relations covers all possible relations among elements of \( M \) that can be derived from relation (5.12). Therefore, the Higgs branch is:

\[
M_{H} = V_{R''} = \{ M_{N_f \times N_f} | M^j_i \in \mathbb{C}, \text{ rank}(M) \leq 1, \text{Tr}(M) = 0, M^2 = 0 \}.
\]

This fully defines the algebraic variety. But what type of variety is this? What are its singular points? From the brane description we know that it has a single singularity at the origin. In this case the singular point corresponds to the matrix with vanishing elements.

We can also check that this is a set that generalises the Higgs branch when \( N_f = 2 \). When \( N_f = 2 \) the variety is \( \mathbb{C}^2 / \mathbb{Z}_2 \). For higher values of \( N_f \) this description grows in dimension, while keeping the single singular point.

Actually, this variety belongs to a larger set of varieties, that can be described employing sets of square matrices and relations among their elements, such as vanishing traces, nilpotent conditions, and restrictions on the rank of the matrix as well as restrictions on the rank of different powers of the matrix or vanishing traces of powers of the matrices. This set of varieties is very important because it can be used to fully characterise the moduli space \( \mathcal{M} \) of a generic 3d \( \mathcal{N} = 4 \) gauge theory, including their Higgs branch, their Coulomb branch, and the mixed branches. We will review a definition and a classification of such varieties in chapter 7. The name for this type of affine varieties is closure of nilpotent orbits of \( \mathfrak{su}(N_f, \mathbb{C}) \).
The other thing that is left to do is to provide a method to compute the Higgs branch in a more systematic fashion. In particular, in this example we presented the relations in \( M \) as a known result, but we did not provide a constructive way of obtaining them directly from the setting the scalar potential to zero. This systematic computation is the computation of the Hilbert series. In the next section we review how it should be computed in general, and also for the current example of 3d \( \mathcal{N} = 4 \) SQED.

### 5.2 The Hilbert Series

In general, for a given quiver, we want a way to systematically implement two calculations: which combinations of scalar fields in the hypermultiplets are gauge invariants, and what are the relations that these combinations of gauge invariant operators satisfy, given that their forming components (which transform under the gauge group) are satisfying relations given by setting the scalar potential to zero.

The answer to these two questions is the computation of the Hilbert series. In the previous example, a Hilbert series can be computed for varieties \( V_R, V_{R^0} \) and \( V_{R^{00}} \). Let us see how this can be done.

The Hilbert series is particularly easy to compute, since it only counts the number or linearly independent operators. For example for \( V_R \), this is precisely the example of \( V = \mathbb{C}^{2N_f} \) covered in Chapter 3, with the difference that now the operators \( q_i \) and \( \tilde{q}_j \) play the role of the complex coordinates \( z_k \). As before, in \( V_R = \mathbb{C}^{2N_f} \), the linearly independent polynomials that can be formed out of the \( 2N_f \) generators \( \{ q_i, \tilde{q}_j \} \) are:

- At degree 0: only one polynomial that does not depend on the generators, we can choose any complex number, for example: 1.
- At degree 1: there are the \( 2N_f \) generators \( q_1, q_2, \ldots, q_{N_f} \) and \( \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_{N_f} \).
- At degree 2: there are \( \binom{2N_f+1}{2} \) linearly independent monomials: \( (q_1)^2, (q_1)(q_2), \ldots, (\tilde{q}_{N_f})^2 \).
- ...
- At degree \( n \): there are \( \binom{2N_f+n-1}{n} \) linearly independent monomials: \( (q_1)^n, (q_1)^{n-1}(q_2), \ldots, (\tilde{q}_{N_f})^n \).

Therefore, the Hilbert series takes the form

\[
H_{\mathbb{C}^{2N_f}} = \sum_{n=0}^{\infty} \binom{2N_f+n-1}{n} t^n \tag{5.18}
\]

This expression can be resumed and written in the form:

\[
H_{\mathbb{C}^{2N_f}} = \prod_{g \in \Lambda} \frac{1}{1 - tl(g)}, \tag{5.19}
\]

where \( \Lambda = \{ q_i, \tilde{q}_j \} \) is the set of generators and \( d(g) \) is the degree of the generators. In this case, there are \( 2N_f \) generators, all of degree 1. Hence the Hilbert series is:

\[
H_{\mathbb{C}^{2N_f}} = \frac{1}{(1 - t)^{2N_f}}. \tag{5.20}
\]

Now let us turn to the gauge invariant ring \( R^0 \). In order to project the counting of operators in \( H_{\mathbb{C}^{2N_f}} \) to the gauge invariant sector we use orthogonality of characters. We give each operator its corresponding character \( y \) for \( \tilde{q}_j \) or \( 1/y \) for \( q_i \) under the gauge group \( U(1) \). In this way we write what is called the **refined Hilbert series** \( \mathcal{H}_{\mathbb{C}^{2N_f}} \):

\[
\mathcal{H}_{\mathbb{C}^{2N_f}} = \frac{1}{(1 - yt)^{N_f}(1 - y^{-1}t)^{N_f}}. \tag{5.21}
\]

Note that the refined Hilbert series is a generating function that counts both the number of operators of degree \( d \), via the power of the fugacity \( t \), and the charge of such operators under the gauge group \( U(1) \), via the power of the fugacity \( y \).
5.3. REFINED HILBERT SERIES

In the same way we introduced characters of the gauge group in the Hilbert series computation, the characters of the flavour group can also be included, to have a Hilbert series with more information. The resulting Hilbert series will be able not only to count the number of linearly independent operators in the chiral ring with each degree, but it also encodes the representation in which they transform under the flavour group. Let us see how this is done. For example, to make it concrete, let fix the number of flavours to \( N_f = 3 \). This means that \( q_i \)

Then we multiply by the complex conjugate of the character of the representation we want to project out, in this case the trivial, and then integrate over the manifold of the group, parametrised by coordinate \( y \), employing the Haar measure of the gauge group, in this case \( dy/2\pi iy \). This produces a new Hilbert series \( H_{V'} \) that only depends on fugacity \( t \).

\[
H_{V'} = \int dy \frac{1}{2\pi iy} H_{C^{2N_f}} = \int dy \frac{1}{2\pi iy} \frac{1}{(1 - yt)(1 - y^{-1}t)^N_f}.
\]

(5.22)

The result of this integration for different values of \( N_f \) is depicted in table 5.1.

Now for the Higgs branch \( M_H = V_{R'} \) we need to impose the condition that the scalar potential vanishes. This condition also transforms under the gauge group. In this case it is given by equation (5.12) and it transforms under the trivial representation of \( G = U(1) \). It has degree \( d = 2 \), therefore, it is represented by fugacities \( y + t^2 \). It can be added to the numerator of the refined Hilbert series \( H_{C^{2N_f}} \), before integrating over \( dy \). This gives a generic way of computing the Hilbert series of the Higgs branch:

\[
H_{M_H} = \int dy \frac{1}{2\pi iy} \frac{1}{(1 - yt)(1 - y^{-1}t)^N_f}.
\]

(5.23)

The result of this integration for different values of \( N_f \) is depicted in table 5.2. Note that the data necessary to build the integrand of (5.23) can be obtained directly from the quiver (5.1). The denominator is given by the representations of the chiral superfields \( Q \) and \( \overline{Q} \), while the numerator is given by the representation of the chiral superfields \( \Phi \).

5.3 Refined Hilbert Series

<table>
<thead>
<tr>
<th>( N_f )</th>
<th>( H_{V'} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{1-t} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1+t}{1-t^2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1+4t^2+t}{1-t^2} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1+9t^2+9t^2+t}{1-t^2} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1+16t^2+16t^2+16t^2+t}{1-t^2} )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{1+25t^2+100t^2+25t^2+25t^2+t}{1-t^2} )</td>
</tr>
</tbody>
</table>

Table 5.1: Results of integrating equation (5.22) for different values of \( N_f \).

<table>
<thead>
<tr>
<th>( N_f )</th>
<th>( H_{M_H} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1+t}{1-t^2} )</td>
</tr>
<tr>
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</tr>
<tr>
<td>6</td>
<td>( \frac{1+25t^2+100t^2+25t^2+25t^2+t}{1-t^2} )</td>
</tr>
</tbody>
</table>

Table 5.2: Results of integrating equation (5.23) for different values of \( N_f \).
transforms in the fundamental representation of $SU(3)$, and $\tilde{q}^i$ in the anti-fundamental, with Dynkin labels:
\begin{align}
q_i &\to [1, 0] \\
\tilde{q}^j &\to [0, 1].
\end{align}

The corresponding characters are:
\begin{align}
[1, 0] &\to a + \frac{b}{a} + \frac{1}{b} \\
[0, 1] &\to b + \frac{a}{b} + \frac{1}{a}.
\end{align}

Now we assign the different monomials in the character to the different operators, $a$ to $q_1$, $b/a$ to $q_2$, etc.
and we write the refined Hilbert series as:
\begin{equation}
\mathcal{H}_{M,H,N,f} = \frac{1 - t^2}{\Phi_1(y)(1 - ayt)(1 - \frac{byt}{a})(1 - \frac{ayt}{b})(1 - \frac{bt}{ay})}.
\end{equation}

The result of the integration is:
\begin{align}
\mathcal{H}_{M,H,N,f} &= 1 + \chi_{[1,1]}(a, b)t + \chi_{[2,2]}(a, b)t^2 + \chi_{[3,3]}(a, b)t^3 + \chi_{[4,4]}(a, b)t^4 + O(t^5)
\end{align}

This notation gives us another way of writing this particular refined Hilbert series as:
\begin{equation}
\mathcal{H}_{M,H,N,f} = \sum_{n=0}^{\infty} \chi_{[n,n]}(a, b)t^n.
\end{equation}

This notation can be useful to see the general pattern for an arbitrary number of flavours. The refined Hilbert series for an arbitrary number of flavours $N_f$ can be computed to be:
\begin{align}
\mathcal{H}_{M,H,N_f} &= \sum_{n=0}^{\infty} \chi_{[n,0,0,\ldots,0,0,n]}t^n
\end{align}

The Highest Weight Generating Function Since there is a general pattern for any number of flavours $N_f$, it is tempting to make it manifest, by creating a new set of fugacities $\mu_a$ where $a = 1, \ldots, N_f - 1$ that encode the Dynkin label of each representation. The resulting generating function is called the highest weight generating function $HWG_{M,H,N_f}$ [59]:
\begin{align}
HWG_{M,H,N_f} &= \sum_n \mu_1^{n_1}\mu_2^{n_2}\cdots\mu_{N_f-1}^{n_{N_f-1}}t^n \\
&= \frac{1}{1 - \mu_1\mu_2\cdots\mu_{N_f-1}t}.
\end{align}

This concludes the example of how the computation of the Hilbert series for SQED with $N_f$ flavours can be carried out. For a more generic quiver, the same computation can be performed. The final Hilbert series will be obtained after integrating over all the gauge group factors an integrand for which the denominator is determined by the representations of the chiral superfields in the quiver that belong to hypermultiplets, while the numerators are determined by the representations of chiral superfields that belong to vector multiplets.
5.3. REFINED HILBERT SERIES

[54, 55, 59, 58].
Chapter 6

The Coulomb Branch

Let us now in this chapter illustrate how the Hilbert series of a 3d $\mathcal{N} = 4$ Coulomb branch $\mathcal{M}_C$ can be computed, utilising the monopole formula [50].

6.1 The Monopole Formula

Let us illustrate the process by studying the Coulomb branch of SQED with $N_f$ flavours. In particular, we have claimed before that the Coulomb branch is the Kleinian surface singularity:

$$\mathcal{M}_C = \mathbb{C}^2/\mathbb{Z}_{N_f}$$

Note that Kleinian singularities of the type $\mathbb{C}^2/\Gamma$ with the finite group $\Gamma \subset SU(2)$ are classified into three families: A, D and E. The family A is the one where the finite subgroup is the cyclic group. For the cyclic group of order $k$, the variety is called $A_{k-1}$, such that $A_0 = \mathbb{C}^2/\mathbb{Z}_1 = \mathbb{C}^2$. Then, we can write, using this notation that is more compact:

$$\mathcal{M}_C = A_{N_f-1}$$

Now, we would like to find the Hilbert series of this Coulomb branch. In this case, it is more tricky than in the Higgs branch, since the Coulomb branch receives quantum corrections. However, we can employ a non-perturbative method that counts operators, analogously to the counting of operators in the chiral ring of the Higgs branch that was described in the previous section. The precise formula that we use was first introduced in [50].

The main idea is that the complex scalar field $\phi(x)$ that comes from the chiral superfield $\Phi(x, \theta, \bar{\theta})$ inside the vector multiplet $(V, \Phi)$ can be treated in the same way as the scalar fields on the hypermultiplets in the computation of the Higgs branch Hilbert series explained in the previous chapter. The only difference is the treatment of the scalar degrees of freedom in the vector superfield $V(x, \theta, \bar{\theta})$ within the vector multiplet. The answer of [50] is that the Coulomb branch is parametrised by the VEVs of gauge invariant combinations of the complex fields in the chiral superfield and by the existence of solitons parametrised by their magnetic charge under the gauge group. Since the work of Mandelstan [60], we can explicitly build quantum operators that create solitonic field configurations. That was in the context of 2d Sine-Gordon theory [61, 62]. These operators where generalised to 3d soliton-creating operators in the works of [45, 46, 47].

The procedure on finding a ring $R$ of operators that characterises the Coulomb branch is then as follows. First of all, one should establish what are all the different soliton creating operators, there will be one for each magnetic charge. In the work of [63], the notion of magnetic charge is generalised from $U(1)$ to a more generic gauge group $G$. For each choice of solitonic field configuration, there is a part of the gauge group that remains unbroken and a part of the gauge group that will be broken. Then, the remaining scalar fields in the chiral superfields $\Phi$ that are part of the adjoint vector multiplets are considered in all configurations that are invariant.
under the unbroken part of the gauge group.

Therefore, there are two types of operators that behave differently. On one hand there are the scalar field operators $\phi(x)$ from the chiral superfields. The Hilbert series of the Coulomb branch should count any product of them that is invariant under the unbroken part of the gauge group. On the other hand, there are the soliton-creating operators $V(x)_m$, that create a soliton with magnetic charge $m$. The Hilbert series should only count product of operators that contain a single soliton-creating operator and an arbitrary number of scalar field operators. Since the soliton-creating operators have been called monopole operators in the literature, the products that the Hilbert series of the Coulomb branch needs to count, with a single monopole operator and an arbitrary number of scalar fields, are called dressed monopole operators.

Let us count all dressed monopole operators. This means that the ring of holomorphic polynomials in the Coulomb branch is in one to one correspondence with the set of all soliton-creating operators, that create solitons with magnetic charge $m$, and the product of such operators with the gauge invariant combinations of the remaining scalar fields in the vector multiplet.

In order to produce the generating function that counts such operators it is necessary to introduce a grading $s$. Since there is a part of the $3d \mathcal{N} = 4$ R-symmetry $SU(2)_C$ that is a global symmetry of $\mathcal{M}_C$, each operator is part of an irreducible representation of $SU(2)_C$ characterised by its spin $s$. The goal is to write a generating function:

$$H_{\mathcal{M}_C} = \sum_{s=0}^{\infty} h_{\mathcal{M}_C}(s)t^{2s},$$

where $h_{\mathcal{M}_C}(s)$ is the number or linearly independent operators with spin $s$. Note that the powers of $t$ are chosen to be twice the spin, $2s$, so the powers of $t$ will always be integer (this differs from the choice of [50], where the powers of $t$ are half compared to this convention and can therefore be half-integers).

The Hilbert series for the Coulomb branch of a $3d \mathcal{N} = 4$ theory defined by a quiver is computed via the monopole formula [50]:

$$H_{\mathcal{M}_C} = \sum_{m \in \Gamma_G^*/W} t^{2\Delta(m)} P_G(m; t),$$

First let us discuss the magnetic charges $m$ of the monopole operators. According to [63], the magnetic charges are elements of a lattice, just like the weights of the group are elements of the weight lattice. Let the gauge group be $G$. Then $\Gamma_G$ denotes the weight lattice of $G$. Given a lattice $\Gamma_G$, the dual lattice $\Gamma_G^*$ can be defined as the set of maps from the lattice to the integer numbers $\mathbb{Z}$. The result of [63] is that magnetic charges $m$ of the group $G$ are elements of the dual weight lattice $\Gamma_G^*$.

Actually, not all elements of $\Gamma_G^*$ define a different magnetic charge, there are a set of reflexions that relate physically equivalent magnetic charges. The group of all such reflexions is a finite group and it is labelled $W$. The reason for this is that this group is the same as the Weyl group of $\Gamma_G^*$. If $\Gamma_G^*$ were to be considered as the weight lattice of some dual magnetic group $G'$.

For each magnetic charge there is a single monopole operator $V_m(x)$, hence the Hilbert series sums over all the magnetic charges. The degree $s = \Delta(m)$ of the operator is given by [46, 48, 64, 49]:

$$\Delta(m) = -\sum_{\alpha \in \Delta_+} |\alpha(m)| + \frac{1}{2} \sum_{i=1}^{n} \sum_{\rho_i \in \mathcal{R}_i} |\rho_i(m)|,$$

where $\Delta_+$ is the set of positive roots of the gauge group $G$ and $\mathcal{R}_i$ are the representations of the different hypermultiplets under the gauge group $G$.

The dressing factor is $P_G(m; t)$. For a given solitonic configuration with magnetic charge $m$, let the unbroken part of the gauge group be $G_m$. Then, let the dimension of the Casimir invariants be $d_i(G_m)$. The dressing factor is the generating function that counts products of gauge invariant scalar fields:

$$P_G(m; t) = \prod_{i=1}^{\text{rank}(G_m)} \frac{1}{1 - t^{2d_i(G_m)}}$$
6.2 An Example, SQED with $N_f$ Flavours

Let us now finally explicitly compute the Hilbert series of $3d \ N = 4$ SQED with $N_f$ flavours and show that it is indeed the Hilbert series of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_{N_f}$, as it was done in [50]. First of all, let us draw the quiver again:

$$Q = \begin{array}{c}
N_f \\
\circ \\
1
\end{array}.$$  \hfill (6.7)

The magnetic lattice is one dimensional:

$$\Gamma_G^* = \mathbb{Z}. \hfill (6.8)$$

The gauge group is always unbroken to $U(1)$:

$$G_m = U(1). \hfill (6.9)$$

Hence, the dressing factor is always computed using the dimension of the single Casimir of $U(1)$, $d_1 = 1$. The dressing factor is:

$$P_G(m; t) = \frac{1}{(1 - t^2)} \quad \forall m. \hfill (6.10)$$

The degree $2s = \Delta(m)$ of the monopole creating operator $V_m$:

$$\Delta(m) = \frac{N_f}{2} |m|. \hfill (6.11)$$

The Hilbert series is:

$$H_{\mathcal{M}_C} = \sum_{m \in \mathbb{Z}} t^{N_f|m|} \frac{1}{(1 - t^2)}$$

$$= \frac{1}{(1 - t^2)} \left( 2 \sum_{m=0}^{\infty} t^{N_f m} - 1 \right)$$

$$= \frac{1}{(1 - t^2)} \left( \frac{2 - (1 - t^{N_f})}{(1 - t^{N_f})} \right)$$

$$= \frac{1}{(1 - t^2)} \left( \frac{1 + t^{N_f}}{(1 - t^{N_f})} \right)$$

$$= \frac{1}{(1 - t^2)(1 - t^{N_f})}$$

This is precisely the Hilbert series of $A_{N_f-1} = \mathbb{C}^2/\mathbb{Z}_{N_f}$. This concludes our illustration of the use of the monopole formula to compute $H_{\mathcal{M}_C}$. Over the years different ways to implement this technique have been tried [65, 66, 67]. Furthermore, mathematicians [68, 69, 70, 71] have taken the monopole formula as the starting point to create a mathematical definition of the Coulomb branch. This is a very exciting direction, since many results concerning Coulomb branches of $3d \ N = 4$ quivers are starting to appear since in the mathematical literature (i.e. [72]).

There is an alternative approach to the computation of the chiral ring of $\mathcal{M}_C$ [73]. This approach has been used by [74] to relate the monopole operators with certain brane insertions in the brane system and find the
relations of the generators in the Coulomb branch directly from the branes. Note that some quivers that have been called *bad quivers* in the literature [48] have problems with the implementation of the monopole formula. A very recent approach on how to compute such Coulomb branches can be found in [75, 76].

The monopole formula and the fact that $3d \mathcal{N} = 4$ Coulomb branches are easy to compute plays a crucial role in the discussion of Part IV of the thesis.

**The Hilbert series and 3d mirror symmetry**  Note that the prediction of 3d mirror symmetry implies that the Hilbert series $H_{M_t}$ of a Higgs branch can be obtained in two very different ways. One by performing the computation described on the previous chapter, the other by applying the monopole formula to the mirror quiver. Both computations are quite different a priori, since one involves integrals over the manifold of the gauge group, while the other contains infinite sums over different points of a lattice. We would like to point out that there as been a recent result [77] in which it is explicitly shown how the integrals can be mathematically transformed into the discrete infinite sums and vice versa. This is the first time to our knowledge that 3d mirror symmetry has been proven at the level of the Hilbert series.
Chapter 7

Nilpotent Orbits

In Chapter 5 we computed the Higgs branch of 3d $\mathcal{N} = 4$ SQED with $N_f$ flavours and stated that it is part of a bigger family of varieties, the closures of nilpotent orbits of $\mathfrak{sl}(n, \mathbb{C})$. This chapter introduces such varieties. In it we review the introduction to nilpotent orbits of [1]. We make the point that due to a recent discovery by Namikawa [78], all moduli spaces that are simple (they have the simplest\footnote{The measure with respect they are simple is the spin of the operator under the global R-symmetry $SU(2)_R$, as it is explained in the following pages.} possible type of generators that are non-trivial) are closures of nilpotent orbits. This is a remarkable result that gives closures of nilpotent orbits a very special status in the study of vacua for quantum theories with 8 supercharges\footnote{Some works in physics in which nilpotent orbits have appeared in the past are [79, 48, 80, 81, 82, 83, 84, 85].}. This status is fully justified in the last part of this thesis, where closures of nilpotent orbits reveal themselves to indeed be the simplest components in the more complicated classification of more complicated vacua of 5d $\mathcal{N} = 1$ theories. The mathematical definition given here of a nilpotent orbit is from the standard book by Collingwood and McGovern [86]. Other mathematical works are [87, 88, 89].

7.1 Classification of Hyperkähler Singularities

The hyperkähler varieties that we are studying have a holomorphic ring and they can be classified with respect of the generators of such ring. In our physical systems the hyperkähler singularities are moduli spaces and there is a global $SU(2)_R$ symmetry that acts non-trivially in such generators. In particular each generator sits in the highest weight of a full multiplet of $SU(2)_R$, where the other members of the multiplet are non-holomorphic polynomials. Therefore, each generator $g_i$ of the variety has a spin $s(g_i) = s_i$ under $SU(2)_R$.

Let us discuss the example of $\mathbb{C}^2$. The generators of the holomorphic ring are the two complex coordinates $z_1$ and $z_2$:

\begin{equation}
\begin{aligned}
g_1 &= z_1 \\
g_2 &= z_2 
\end{aligned}
\end{equation}

They belong to the multiplets $(z_1, \bar{z}_2)$, with weights $(1, -1)$, and $(z_2, \bar{z}_1)$, also with weights $(1, -1)$, under $SU(2)_R$ respectively. These two multiplets have spin $s = 1/2$. Therefore:

\begin{equation}
\begin{aligned}
s(g_1) &= \frac{1}{2} \\
s(g_2) &= \frac{1}{2}
\end{aligned}
\end{equation}

These two representations fix all the multiplets of the remaining holomorphic monomials of the type $z_1^{a_1}z_2^{a_2}$. 
For example, let us take the second symmetrisation of the multiplet $\gamma_1 = (z_1, \bar{z}_2)$:

$$Sym^2(\gamma_1) = (z_1^2, z_1 \bar{z}_2, \bar{z}_2^2)$$  \hfill (7.3)

The weights of the symmetric product can be computed by adding the weights of the original multiplet, one obtains: $(2, 0, -2)$. These are the three weights of a multiplet with spin $s = 1$ under $SU(2)_R$. Hence we say that:

$$s(z_1^2) = 1.$$  \hfill (7.4)

Analogously, the symmetrisation of $\gamma_2 = (z_2, \bar{z}_1)$ can be computed:

$$Sym^2(\gamma_2) = (z_2^2, z_2 \bar{z}_1, \bar{z}_1^2).$$  \hfill (7.5)

The weights are again $(2, 0, -2)$. Therefore

$$s(z_2^2) = 1.$$  \hfill (7.6)

In order to compute the multiplet of $z_1 z_2$ the following tensor product can be computed:

$$\gamma_1 \otimes \gamma_2 = (z_1 z_2, z_1 \bar{z}_1, \bar{z}_2 z_2, \bar{z}_2 \bar{z}_1)$$  \hfill (7.7)

This computation can be written in terms of Dynkin labels of $SU(2)_R$:

$$[1] \otimes [1] = [2] \oplus [0]$$  \hfill (7.8)

When the weights are computed one obtains $(2, 0, 0, -2)$. Since the weight of $z_1z_2$ is 2, it belongs to representation $[2]$ with spin $s = 1$:

$$s(z_1z_2) = 1.$$  \hfill (7.9)

Therefore, the set of maximally linearly independent holomorphic polynomials of degree $d = 2$ in $\mathbb{C}^2$ all have spin $s = 1$.

Now let us analyse the variety $\mathbb{C}^2/\mathbb{Z}_2$. The three generators are:

$$g_1 = z_1^2$$
$$g_2 = z_1 z_2$$
$$g_3 = z_2^2$$  \hfill (7.10)

They all have spin $s(g_i) = 1$ under $SU(2)_R$, since they inherit it from the ring of holomorphic polynomials of $\mathbb{C}^2$.

In general, the varieties of this type can be classified according to the spin of their generators:

- $s = 0$: There is always a constant function with spin $s = 0$, this is not a generator.
- $s = 1/2$: generators with half spin always come in pairs. $2n$ generators $g_i$ with spin $s(g_i) = 1/2$ always generate the variety $\mathbb{C}^{2n}$.
- $s = 1$: generators with spin $s(g_i) = 1$ transform under the adjoint representation of the global symmetry $\mathbb{Z}_2$.
- $s > 1$: These generators's role is to increase the order of the singularity.

**Quantum field theory's view classification**

This classification can now be seen as a classification of moduli spaces of quantum field theories like the ones we are studying, with eight supercharges and branches in the moduli space that are hyperkähler singularities. The generators of the variety correspond to operators in the chiral ring of the theory:
7.2 Nilpotent Orbits

In this part we are explicit about the definition of the closure of a nilpotent orbit. We take the definition from the book [86].

Let \( X \in \mathfrak{g} \) be an element of a complex semisimple Lie algebra \( \mathfrak{g} \). \( X \) is nilpotent if:

\[
\rho(X)^m = \rho(X) \circ \cdots \circ \rho(X) = 0, \tag{7.11}
\]

where \( m > 0 \) is an integer number and

\[
\rho : \mathfrak{g} \mapsto \text{End}(V) \tag{7.12}
\]

is the adjoint representation of the algebra\(^3\) and \( V \) is a complex vector space [86].

7.2.1 The Algebra \( \mathfrak{sl}(n, \mathbb{C}) \)

According to [86], Section 3.1 Type A, let us focus on the algebra \( \mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) \). There is a one to one correspondence between the nilpotent orbits of \( \mathfrak{sl}(n, \mathbb{C}) \) and the partitions of \( n \). A partition \( \lambda \) of \( n \) is a tuple:

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k). \tag{7.13}
\]

It fulfills:

\[
\begin{align*}
\lambda_i & \in \mathbb{N}, \\
\lambda_1 & \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \quad \text{and} \\
\sum_{i=1}^{k} \lambda_i & = n.
\end{align*} \tag{7.14}
\]

A partition can be written in exponential notation. For example, a partition of \( n = 17 \) can be written as:

\[
(5^2, 4, 1^3) = (5, 5, 4, 1, 1, 1). \tag{7.15}
\]

\(^3\)Note that a definition with a different finite representation of the algebra would be equivalent [86].
The set of all partitions of \( n \) is denoted by \( \mathcal{P}(n) \). Then, for \( n = 4 \) we have the example:

\[
\mathcal{P}(4) = \{(4), (3, 1), (2^2), (2, 1^2), (1^4)\}.
\]

(7.16)

A partition can be represented by a diagram. The diagram has squares in positions \((i, j)\) such that \(1 \leq j \geq \lambda_i\) and \(i\) increases from top to bottom while \(j\) increases from left to right [91]. The transpose map \( \lambda^t \) reflects the diagram of the partition along the diagonal. For example, the diagram of \( \lambda = (5^2, 4, 1^3) \) is depicted in figure 7.1 (a). Its transpose is depicted in figure 7.1 (b). It corresponds to:

\[
\lambda^t = (5^2, 4, 1^3)^t = (6, 3^3, 2).
\]

(7.17)

Now let us define an elementary Jordan block \( J_i \) as an \( i \times i \) matrix with zeroes everywhere except on the upper diagonal:

\[
J_i := \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}.
\]

(7.18)

A nilpotent endomorphism of \( \mathbb{C}^n \) can be built from partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}(n) \) with a block matrix \( X_\lambda \):

\[
X_\lambda = \begin{pmatrix}
J_{\lambda_1} & 0 & \ldots & 0 \\
0 & J_{\lambda_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & J_{\lambda_k}
\end{pmatrix}.
\]

(7.19)

Therefore, \( X_\lambda \) is an \( n \times n \) matrix and can be seen as a nilpotent element of the algebra \( \mathfrak{sl}(n, \mathbb{C}) \). The nilpotent orbit \( O_\lambda \) is defined as:

\[
O_\lambda := PSL(n) \cdot X_\lambda.
\]

(7.20)

Where \( PSL(n) = SL(n)/Z \) and \( Z \) is the centre of \( SL(n) \).

Note that given two different partitions \( \lambda, \lambda' \in \mathcal{P}(n) \) they correspond to two disjoint orbits:

\[
O_\lambda \cap O_{\lambda'} = 0 \iff \lambda \neq \lambda',
\]

(7.21)

due to the uniqueness of the Jordan normal form. Note that a nilpotent \( n \times n \) matrix can always be taken to its Jordan normal form via the action of \( PSL(n) \). Hence, any nilpotent element \( X \in \mathfrak{sl}(n, \mathbb{C}) \) belongs to a unique nilpotent orbit \( O_\lambda \subset \mathfrak{sl}(n, \mathbb{C}) \), where \( \lambda \) is determined uniquely by the Jordan normal form of the corresponding
7.3 Example: $\mathfrak{g}(\mathbb{C}) = \mathfrak{sl}(2, \mathbb{C})$

Let us make the previous definitions more concise by studying an example. Let the algebra be

$$\mathfrak{g}(\mathbb{C}) = \mathfrak{sl}(2, \mathbb{C}).$$

(7.22)

The set of all partitions is:

$$\mathcal{P}(n) = \{(2), (1^2)\}$$

(7.23)

Let us compute the orbit with partition $(1^2)$ first. This is called the trivial partition. The Jordan normal matrix $X_\lambda$ is:

$$X_{(1^2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$  

(7.24)

The orbit $O_{(1^2)}$ can be constructed explicitly as a set of matrices:

$$O_{(1^2)} := \left\{ M = S \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot S^{-1} \bigg| S \in SL(2, \mathbb{C}) \right\},$$

(7.25)

where $S \in SL(2, \mathbb{C})$ is an element of the group:

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$ 

(7.26)

Note that since the action of $-S$ gives the same element of the orbit $O_{(1^2)}$ as the action of $S$ the group that generates the orbit is actually $PSL(2, \mathbb{C})$, which is consistent with the definition.

In this case, the orbit contains a single element:

$$O_{(1^2)} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$ 

(7.27)

Let us look at the non trivial orbit given by partition $\lambda = (2)$. The corresponding Jordan matrix is:

$$X_{(2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ 

(7.28)

The nilpotent orbit is defined as:

$$O_{(2)} := \left\{ M = S \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot S^{-1} \bigg| S \in SL(2, \mathbb{C}) \right\}.$$ 

(7.29)

Therefore, we see that elements $M \in O_{(2)}$ can be written as:

$$M = \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix}.$$ 

(7.30)

Then we explicitly have fulfilled the nilpotency condition:

$$M^2 = 0.$$ 

(7.31)

Notice that this orbit is disjoint from the trivial orbit $O_{(1^2)}$, since setting all elements of $M$ to zero would contradict the definition of $S$ with $ad - cd = 1$. Note however that one can get arbitrarily close to it within
The set $\bar{O}_2$ is now an affine variety. It has three generators, which can be graded according to the variables in $S$:

$$g_1 = a^2$$
$$g_2 = c^2$$
$$g_3 = ac.$$  

Hence, we can say that the generators of $\bar{O}_2$ have degree $d = 2$. Notice that they are not fully independent, they satisfy a relation at degree $d = 4$:

$$g_1 g_2 = (g_3)^2. \quad (7.34)$$

We can immediately recognise these properties from Chapter 3. They are the same properties of the coordinate ring for the Kleinian surface singularity $\mathbb{C}^2/\mathbb{Z}_2$. Indeed, the closure of the non-trivial nilpotent orbit of $\mathfrak{sl}(2, \mathbb{C})$ is the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_2$:

$$\bar{O}_2 = \mathbb{C}^2/\mathbb{Z}_2. \quad (7.35)$$

### 7.4 The Hasse Diagram

In the previous example we can form affine varieties by taking the closure of the nilpotent orbits. The closure of the orbits can always be formed by a union of orbits. This is a general result for $\mathfrak{sl}(n, \mathbb{C})$ algebras. It is as follows. Let there be a natural partial ordering between partitions of $P(n)$. Let $\lambda, \lambda' \in P(n)$, then:

$$\lambda > \lambda' \iff \sum_{i=1}^{p} \lambda_i \geq \sum_{i'=1}^{p} \lambda_i' \forall p$$  

(7.36)

This imposes a partial ordering in the partitions, since $\lambda > \lambda'$ and $\lambda' > \lambda''$ implies $\lambda > \lambda''$ In the case of $n = 2$ it is a total ordering, since $(2) > (1^2)$. In the case of $n = 3, n = 4$ and $n = 5$ it is also a total ordering, since:

$$(3) > (2, 1) > (1^3)$$
$$(4) > (3, 1) > (2^2) > (2, 1^2) > (1^4)$$
$$(5) > (4, 1) > (3, 2) > (3, 1^2) > (2^2, 1) > (2, 1^3) > (1^5).$$  

(7.37)

In the case of $n = 6$ the natural ordering is a partial ordering, since $(4, 1^2)$ is neither greater nor lower than $(3^2)$. The best way to represent such partial ordering is to use a Hasse diagram. The Hasse diagram for the partial ordering of $P(4)$, $P(5)$ and $P(6)$ are depicted in figure 7.2.

For $n = 2$ we had the result:

$$\bar{O}_2 = O_{(2)} \cup O_{(1^2)}$$
$$\bar{O}_{(1^2)} = O_{(1^2)}$$  

(7.38)

This result can be generalised to any $\mathfrak{sl}(n, \mathbb{C})$ algebra: the closure of the nilpotent orbit $O_{\lambda}$ is given by the union of all orbits $O_{\lambda'}$ with partition $\lambda' \in P(n)$ such that $\lambda' \leq \lambda$:

$$\bar{O}_{\lambda} = \bigcup_{\lambda' \leq \lambda} O_{\lambda'}.$$  

(7.39)
Such orbits can be read directly from the Hasse diagram, for example, for $n = 5$:

$$
\mathcal{O}_{(2^2, 1)} = \mathcal{O}_{(2^2, 1)} \cup \mathcal{O}_{(2^1)} \cup \mathcal{O}_{(1^3)},
$$

(7.40)

or for $n = 6$:

$$
\mathcal{O}_{(2^3)} = \mathcal{O}_{(2^3)} \cup \mathcal{O}_{(2^2, 1^2)} \cup \mathcal{O}_{(2^1)} \cup \mathcal{O}_{(1^5)}.
$$

(7.41)

This implies that the closures of nilpotent orbits of $\mathfrak{sl}(n, \mathbb{C})$ are contained within each other, according to the partial order structure of the partitions. Let $\lambda, \lambda' \in \mathcal{P}(n)$:

$$
\mathcal{O}_{\lambda'} \subset \mathcal{O}_\lambda \iff \lambda' < \lambda
$$

(7.42)

Therefore, the Hasse diagram also represents a partial order of inclusion of the affine varieties $\mathcal{O}_\lambda \subset \mathfrak{sl}(n, \mathbb{C})$. 
Chapter 8

The Kraft-Procesi Transition

Now we have reached the point in the thesis where we are ready to discuss one of the original results from our research. We have called this result, the Kraft-Procesi transition. We published this work on a series of three papers [1, 4, 5]. These papers followed a natural development that normally occurs in the study of Lie groups. The first one [1] studies the new effect in the case of the unitary algebra, or of type A. The second one [4] extends the results to the remaining classical Lie algebras: the symplectic and the orthogonal, or types B, C and D. The third paper [5] explores a new generalisation of the results that allows an extension to some cases to exceptional Lie algebras, or of type F, G and E.

As mentioned before, these results have to do with the structure of the singularities in the Higgs branches and the Coulomb branches of 3d $\mathcal{N} = 4$ quiver gauge theories. In particular, the different types of singular points in these branches are directly related to the difference in the spectrum of massless particles after spontaneous symmetry breaking. Therefore, it is related to the classification of different mixed branches $\mathcal{M}'_i$ of the moduli space $\mathcal{M}$, where scalar fields from both hypermultiplets and vector multiplets acquire non-zero VEV. Remember that at a generic point in the Coulomb branch only $r$ photons remain massless, where $r$ is the rank of the gauge group $G$:

$$r = \text{rank}(G).$$

(8.1)

By photons we mean vector multiplets that carry the force determined by the maximal torus of the gauge group $U(1)^r \subset G$, which is the gauge subgroup that remains unbroken. At a generic point in the Higgs branch, only $d$ hypermultiplets remain massless, where $d$ is the total number of hypermultiplets in the theory minus the number of vector multiplets:

$$d = \#\text{hypermultiplets} - \#\text{vector multiplets}.$$  

(8.2)

At the generic point in the Higgs branch there are no massless vector multiplets and therefore the gauge group $G$ is fully broken. Remember as well that there is a point in the moduli space that we know is singular: the point where the Coulomb branch and the Higgs branch intersect. This is normally a single point and it is considered to be at the origin of the Higgs branch and at the origin of the Coulomb branch. For this reason, we have denoted the corresponding vacuum state as $|0\rangle$. This is the most singular point in the theory, meaning that at this point a full transition can be made from the Higgs branch to the Coulomb branch, or vice versa.

Now, there will be other types of points in $\mathcal{M}$ which are neither generic (non-singular points) not the origin. Such type of points will be singular loci of $\mathcal{M}$, along some directions and not singular along some other ones. If we were in the Coulomb branch, for example, such points would be points where some hypermultiplets, but not all of them, have also become massless and can acquire non-zero VEV, this will make some, but not all, of the $r$ massless vector multiplets massive. If the transition is made and some of the vector multiplets made massive, one has left the Coulomb branch and finds oneself now in a mixed branch.

Therefore, here there is a nice interplay between the mathematics and the physics. A point in $\mathcal{M}$ that
is singular implies that a transition can be made to a different branch of the moduli space. In other words, massless vectormultiplets are exchanged for massless hypermultiplets or vice versa.

Our results explain how to find and classify such parts of the moduli space of a 3d $\mathcal{N} = 4$ gauge theory. The necessary ingredients are all the concepts already introduced in the previous pages. Without any more ado, let us dive directly into the main idea.

### 8.1 The Closure of a Nilpotent Orbit

In the examples studied so far, we encountered Higgs branches and Coulomb branches that did not have singular points except for the origin. These examples where either the 3d $\mathcal{N} = 4$ theory SQED with $N_f$ flavours, or its 3d mirror dual, with gauge group $U(1)^{N_f-1}$ and bifundamental matter. Now we would like to introduce a case in which there are additional points in the Moduli space that are also singular but are not the origin. For the sake of simplicity in the explanation of the concepts that follow let us focus first on the different points of the Coulomb branch $M_C$.

The simplest case is a 3d $\mathcal{N} = 4$ quiver theory whose Coulomb branch is the closure of a nilpotent orbit of $\mathfrak{sl}(n, \mathbb{C})$. Let us see why. Let us consider the case $\mathfrak{sl}(3, \mathbb{C})$, and let there be a theory with:

$$M_C = \bar{O}(3),$$

(8.3)

i.e. the Coulomb branch is isomorphic to the closure of the maximal nilpotent orbit of $\mathfrak{sl}(3, \mathbb{C})$, $\bar{O}(3) \subset \mathfrak{sl}(3, \mathbb{C})$.

The Hasse diagram of $\mathfrak{sl}(3, \mathbb{C})$, is depicted in figure 8.1 and it illustrates the fact that $\bar{O}(3)$ can be described as the union of three distinct nilpotent orbits:

$$\bar{O}(3) = O(3) \cup O(2,1) \cup O(1^3).$$

(8.4)

Remember that these spaces can be thought of as sets of matrices:

$$O_{(3)} = \left\{ M = S \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot S^{-1} \bigg| S \in SL(3, \mathbb{C}) \right\}$$

$$O_{(2,1)} = \left\{ M = S \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot S^{-1} \bigg| S \in SL(3, \mathbb{C}) \right\}$$

$$O_{(1^3)} = \left\{ M = S \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot S^{-1} \bigg| S \in SL(3, \mathbb{C}) \right\}.$$  

(8.5)

Note that they are all disjoint and that $O_{(1^3)}$ is a single point, since it only contains the matrix with all elements being zero. The work of mathematicians Kraft and Procesi [13, 14] contains precisely the answer.
8.1. THE CLOSURE OF A NILPOTENT ORBIT

<table>
<thead>
<tr>
<th>Orbit</th>
<th>H-plets can acquire VEV ≠ 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathcal{O}(3))</td>
<td>0</td>
</tr>
<tr>
<td>(\mathcal{O}(2,1))</td>
<td>?</td>
</tr>
<tr>
<td>(\mathcal{O}(1^3))</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 8.1: Different singularities in the Coulomb branch \(\mathcal{M}_C = \mathcal{O}(3)\).

that we were looking for. They explain that there are three different types of points in \(\mathcal{O}(3)\) according to their singularities, and they correspond to the three different orbits. Precisely, the orbit \(\mathcal{O}(1^3)\) contains only the most singular point, i.e. the origin. The orbit \(\mathcal{O}(2,1)\) is a set of points that are also singular inside \(\mathcal{O}(3)\), but not along all directions. In particular, they are singular along the directions transverse to the orbit \(\mathcal{O}(2,1)\). Notice that it is the same with \(\mathcal{O}(1^3)\), i.e. it is singular along all directions in \(\mathcal{O}(3)\) that move away from \(\mathcal{O}(1^3)\). However, since the orbit \(\mathcal{O}(1^3)\) is just a point, it is singular along all directions. Kraft and Procesi go even further analysing what type of singularity is there. In this case the structure of a point in \(\mathcal{O}(2,1) \subset \mathcal{O}(3)\) if only the directions transverse to \(\mathcal{O}(2,1)\) are considered is isomorphic to the point situated at the origin of the Kleinian surface singularity of the type:

\[
A_2 = \mathbb{C}^2/\mathbb{Z}_3.
\] (8.6)

Note that this particular result for this particular orbit was already known by Brieskorn [11]. Finally, the orbit \(\mathcal{O}(3)\) corresponds to the set of points that are not singular.

Then, the full classification of the points in the Coulomb branch can be done. There are three types of points, which allow different particles to acquire VEV. In order to be able to talk more explicitly about which are the different multiplets, let us introduce the precise quiver theory with the property \(\mathcal{M}_C = \mathcal{O}(3)\).

The quiver is as follows (see for example [48]):

\[
Q = \begin{array}{c}
3 \\
\circ \\
1 \\
\circ \\
2
\end{array}
\]

(8.7)

Hence, the gauge group \(G\) and the flavour group \(F\) are:

\[
G = U(1) \times U(2),
\]

\[
F = SU(3).
\] (8.8)

Note that we know the particles that are massless, and the particles that can take non-zero VEV away from the corresponding point for two out of the three different set of points. This is depicted in table 8.1. Note that for this theory the number \(r\) of massless photons on the generic point of the Coulomb branch is:

\[
r = \text{rank}(U(1) \times U(2)) = 3.
\] (8.9)

Also, the number \(d\) of massless hypermultiplets in a generic point of the Higgs branch is:

\[
d = (1 \times 2 + 2 \times 3) - (1^2 + 2^2) = 3.
\] (8.10)

Notice that in the generic point of \(\mathcal{M}_C\) there are no massless hypermultiplets that could acquire non-zero VEV. Which means that the generic point is not singular. At the origin of \(\mathcal{M}_C\) all 8 hypermultiplets are massless and there is the possibility of 3 of them acquiring non-zero VEV, since this will give mass to the 5 vectormultiplets of \(G = U(1) \times U(2)\), and they will eat the remaining 5 massless hypermultiplets, that play the role of the Goldstone particles.

We would like to know how the information about the singularity structure in \(\mathcal{O}(2,1)\) can be translated into
information about how many massless hypermultiplets can acquire non-zero VEV at that point in the $\mathcal{M}_C$. In order to find an answer for this question, let us introduce the brane system. The brane system in figure 8.2 represents a generic point in the Coulomb branch. The number of massless vector multiplets is given by the fact that there are three different D3-branes that can move along $\vec{m}$. The number of hypermultiplets that could acquire non-zero VEV is equal to how many D3-brane segments can be broken from existing branes that pass through D5-branes and sent into direction $\vec{w}$ without breaking further supersymmetry. In the case of figure 8.2 it is zero, since no D3-brane aligns with the D5-branes.

From the QFT point of view the argument is simple, if enough hypermultiplets become massless, their scalar fields could acquire non-zero VEV and make some of the previously massless vector multiplets become massive. From the brane perspective it is even simpler, a transition to a mixed branch is possible whenever a D3 brane can be broken into segments, some of which end only in D5-branes, and can be made to move along $\vec{w}$, hence freezing the D3 segments that now end between an NS5 and a D5 and decreasing the total number of D3-branes that move along direction $\vec{m}$.

Now, the brane system can be used to recover the analysis of Kraft and Procesi. The important thing is to be systematic in the analysis. We always start by asking the question: What is the minimal brane motion that can be done that takes the brane system in figure 8.2 from a generic point in $\mathcal{M}_C$ to a singular point? The answer to this question is: to align one of the two rightmost D3-branes with the position of the three D5-branes. The resulting system is depicted in figure 8.3. This brane system corresponds to a point that is singular along some of the directions and non-singular along some of the other. It should then correspond to the orbit $O_{(2,1)}$. In particular, it is singular along 4 directions, represented by the motion of the D3-brane aligned with the D5-branes, and it is non-singular along 8 other directions, represented by the motion of the remaining two branes. This so far matches with the analysis of Kraft-Procesi (and Brieskorn), since the singular directions of the point where predicted to be locally isomorphic to the singularity in:

$$A_2 = \mathbb{C}^2/\mathbb{Z}_3,$$

which has real dimension 4. One can push this analysis a bit further and realise that indeed not only the dimensionality, but all the information about the singularity $A_2$ is encoded in the brane system [1]. The way to do this is to consider the local brane system around that D3-brane. This is depicted in figure 8.4. A local quiver $Q_l$ can be read from this brane system. The local quiver $Q_l$ has a Coulomb branch with a single singular point, at the origin. The singular directions of the system in figure 8.3 will be locally isomorphic to the origin of the Coulomb branch of $Q_l$. 

Figure 8.2: Brane system for $\mathcal{M}_C = \mathcal{O}_{(3)} \subset \mathfrak{sl}(3, \mathbb{C})$. This brane system represents a choice of vacuum inside the Coulomb branch in which only three vector multiplets remain massless.

Figure 8.3: Brane system for $\mathcal{M}_C = \mathcal{O}_{(2,1)} \subset \mathfrak{sl}(3, \mathbb{C})$. 

 CHAPTER 8. THE KRAFT-PROCESI TRANSITION
8.1. THE CLOSURE OF A NILPOTENT ORBIT

![Figure 8.4: Local brane system.](image)

Figure 8.5: Kraft-Procesi transition.

The quiver $Q_l$ read from the local system in figure 8.4 is:

$$Q_l = \begin{array}{c}
3 \\
\circ \\
1 
\end{array} \quad (8.12)$$

We know the Coulomb branch of this theory, it was computed in Chapter 6:

$$\mathcal{M}_C(Q_l) = \mathbb{C}^2/\mathbb{Z}_3 = A_2$$

This is precisely the same singularity that first Brieskorn and then Kraft and Procesi found in their analysis of $\tilde{O}(3)$.

8.1.1 The Kraft-Procesi Transition

The Kraft-Procesi transition comes now. After we have used the branes to establish that there are at least two different types of points in $\mathcal{M}_C = \tilde{O}(3)$: the generic points, that have the geometry of $\tilde{O}(3)$, and a type of points that are partially singular, with the singular part isomorphic to $A_2$. The next question is how many of such points are there, and what is their geometry? Also, are there any other types of singular points? From the mathematical result we know the answers. Let us show how they can be seen from the brane system. Furthermore, by answering this question we will also answer the question of many hypermultiplets can acquire non-zero VEV at $\tilde{O}(2,1) \subset \mathcal{M}_C$.

The way we proceeded was to remove the singularity, to *Higgs it away*. This is what was called the Kraft-Procesi transition [1, 4, 44, 5]. The process consists of breaking the D3-brane that is aligned with the D5-branes into 4 different segments. Two of them can move along their directions $\tilde{y}_1$ and $\tilde{y}_2$, while the other two are fixed. The Kraft-Procesi transition consists of taking the two segments that can move along $\tilde{y}_i$ into this direction and to the limit where they go to infinity and are removed from the brane system. This process is depicted in figure 8.5 (a) and (b). Figure 8.5 (c) is the result of performing two Hanany-Witten transitions [10] in order to annihilate the branes that are fixed. The new brane system, figure 8.5 (c), is the result of the Kraft-Procesi transition. The quiver corresponding to this system can be read from the branes $Q'$:

$$Q' = \begin{array}{c}
1 \\
\circ \\
1
\end{array} \quad (8.14)$$
We know the Coulomb branch of this quiver, since it is 3d mirror to the quiver of SQED with $N_f = 3$ flavours, as it was shown in Chapter 4. Hence, the Coulomb branch of $Q'$ is the same as the Higgs branch of SQED with $N_f = 3$. In Chapter 5 we computed that the variety was the closure of a nilpotent orbit. According to that result, it is isomorphic with the space of $3 \times 3$ complex matrices that are traceless and with rank smaller or equal to 1:

$$M_C(Q') = \{ M_{3\times 3} | M_j^i \in \mathbb{C}, \text{rank}(M) \leq 1, \text{Tr}(M) = 0, M^2 = 0 \}. \quad (8.15)$$

But these are precisely the properties of the closure of the nilpotent orbit of $\mathfrak{sl}(3, \mathbb{C})$ with partition $(2, 1)$, as it can be seen from equation (8.5). Then:

$$M_C(Q') = \bar{O}_{(2,1)}. \quad (8.16)$$

This classifies the geometry of the singular points: it is the same as the geometry of the generic points in $M_C(Q')$, i.e. $O_{(2,1)}$, which is in agreement with the analysis of the singularity structure from the mathematical literature. Now we can also answer the second question, if there are points in $M_C(Q')$ that are not generic, but singular, they will correspond to points in $M_C(Q)$ that are singular not only in 4 directions, and therefore are of a different singular type.

In order to answer this we start the same process again. Trying to be systematic, we ask: What is the minimal brane motion that will take a generic brane configuration like that in figure 8.6 to a singular brane configuration in which some D3-branes can be made to move along direction $\tilde{y}_3$? The only possible answer is to align both D3-branes with both D5-branes (figure 8.7), since then they can be recombined into 3 different segments where two of them are fixed and one can acquire position $\tilde{y}_3$. This point involves all D3-branes, hence it is singular along the 8 directions of $M_C(Q')$. We have found indeed the only singular point in $M_C(Q') = \bar{O}_{(2,1)}$, the origin. But let us pretend we don’t know what is the result and continue with the brane analysis.

In order to continue, we perform the Kraft-Procesi transition on the brane system of figure 8.7. This is depicted on figure 8.8. It takes the D3-brane segment that can move between the two D5s to infinity along its position $\tilde{y}_3$. Notice that after the final Hanany-Witten transitions have been performed (figure 8.8 (c)) there are no D3-branes left in the remaining system. This means that there is a unique singular point in $M_C(Q')$, and that the type of singularity at this point is precisely the full Coulomb branch:

$$a_2 = \bar{O}_{(2,1)}. \quad (8.17)$$

Here, we have used the notation from [13, 14], where the closure of the minimal nilpotent orbit, i.e. the orbit with partition $(2, 1^{n-2})$, of $\mathfrak{sl}(n, \mathbb{C})$ is denoted by $a_{n-1}$.

Since the resulting brane system after the second Kraft-Procesi transition has no D3-branes, the analysis from the point of view of the branes has finished. We can summarise it in the following section.

---

1The fact that two branes have been moved along their directions $\tilde{y}_i$ means that there are 2 massless hypermultiplets that can acquire VEV at $O_{(2,1)} \subset M_C$. Thus one can complete the table 8.1, to table 8.2.
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Figure 8.7: Singular point in the Coulomb branch $M_C(Q') = \tilde{O}_{(2,1)}$.

8.1.2 Summary and Hasse Diagram

The summary of our result can be done in a Hasse diagram like the ones in figure 7.2, with added extra information, in the same fashion as those created by Kraft and Procesi [13, 14]. The Hasse diagram is represented in figure 8.9. The extra information is the labelled links.

Let us review how to read the information about the Coulomb branch $M_C$ from the Hasse diagram. Three different labelled nodes in the Hasse diagram, with labels $\lambda \in \{(3), (2, 1), (1^3)\}$ mean that there are three different types of singular points in $M_C$. Each set of points has the precise geometry of the nilpotent orbit $O_{(\lambda)} \subset sl(3, \mathbb{C})$. The links in the Hasse diagram tell us about the type of singularity. The upper node with $\lambda = (3)$ represents a type of points that are not singular, therefore the subset $O_{(3)}$ of $M_C$ is not singular. The middle node with $\lambda = (2, 1)$ has a link $A_2$ connecting it to the upper node $\lambda = (3)$. This means that the points of $O_{(2,1)}$ sit at the boundary of $O_{(3)}$, they are not singular along directions that take you to other points in $O_{(2,1)}$, but they are singular in transverse directions. These directions are locally isomorphic to the directions that leave the origin of an $A_2$ Kleinian surface singularity. Therefore the points of type $\lambda = (2, 1)$ are singular along 4 real directions and non-singular along the remaining 8 real directions. The bottom node with $\lambda = (1^3)$ has a link $a_2$ connecting it to the middle node with $\lambda = (2, 1)$. This means that the point of $O_{(1^3)}$ (remember this is just the origin of $M_C$) sits at the boundary of $O_{(2,1)}$ and is singular along all 8 real directions within $O_{(2,1)}$. The singular directions are locally isomorphic to the singular directions from the origin of an $a_2$ singularity. But this is a trivial statement, since:

$$a_2 = \tilde{O}_{(2,1)}$$ (8.18)

Finally, let us review once more how the Hasse diagram was obtained from the brane system, via the Kraft-Procesi transitions. In order to build the Hasse diagram from the branes one starts with a brane system in the generic point of $M_C$ (figure 8.2). This is represented by the top node on the Hasse diagram. Then one looks for the minimal way of aligning a D3-brane with D5-branes (figure 8.3). The aligned D3-brane has a local system with a quiver $Q_l$, and Coulomb branch $M_C(Q_l) = A_2$. Therefore we add a link from the top node of the Hasse diagram with label $A_2$. We perform a Kraft-Procesi transition, by breaking the aligned D3-brane into 4 segments and sending two of them to infinite along their positions $\tilde{y}_1$ and $\tilde{y}_2$. The resulting brane system (figure 8.6) corresponds to the middle node in the Hasse diagram. Now again we look for the minimal way of aligning a D3-brane with D5-branes (figure 8.7). The aligned D3-branes have a local system with a quiver $Q'_l$, and Coulomb branch $M_C(Q'_l) = a_2$. Therefore we add a link from the middle node of the Hasse diagram with label $a_2$. We perform a Kraft-Procesi transition, by breaking the aligned D3-branes into 3 segments and sending one of them to infinite along its positions $\tilde{y}_3$. The resulting brane system (figure 8.8 (c)) corresponds to the bottom node in the Hasse diagram, since it has no D3s the procedure is finished.
8.2 Kraft-Procesi Transition: General Definition

This is the generalisation of the procedure described in the previous section [1]. Given a brane system of the type we have described before, with NS5, D5 and D3-branes, the singularity structure of the Coulomb branch (or the Higgs branch using S-duality and 3d mirror symmetry) can be obtained in this way. The initial brane system always corresponds to the top node of a Hasse diagram. Look for minimal ways to align D3-branes with D5-branes (there could be more than one). Choose one. The local brane system around the D3s produces the label of a new link in the Hasse diagram (is always of the type $A_k$, if it is a single D3-brane aligned with $k + 1$ D5-branes, or $a_k$ if $k$ there are D3-branes aligned with 2 D5-branes). Either do the Kraft-Procesi transition, in order to obtain a new brane system corresponding to a new node in the Hasse diagram at the end of the newly added link, or un-align the D3-branes and go back to the brane system corresponding to the node in the Hasse diagram at the top of the newly added link. Iterate this process until all links with their labels and all nodes in the Hasse diagram have been found.

In figure 8.10 we show a generic Kraft-Procesi transition corresponding to a link of type $A_k$, while in figure 8.11 we show a generic Kraft-Procesi transition of type $a_k$.

For more examples of the Kraft-Procesi transition applied step by step to obtain the singularity structure of a quiver $Q$ with $\mathcal{M}_C(Q) = \mathcal{O}_{(4)} \subset \mathfrak{sl}(4, \mathbb{C})$ see Section 5.4 of [1]. In the next section of this chapter we propose a quick algorithm to implement this procedure. The last section of this chapter shows the results of the algorithm when applied to quivers such that their Coulomb branches are the closure of the maximal nilpotent orbit of $\mathfrak{sl}(n, \mathbb{C})$, with $2 \leq n \leq 10$. This is a physical realisation of the mathematical results that Kraft and Procesi found in [13, 14].

8.3 The Matrix Formalism

This section contains an algorithm [1] that analyses the singularities of brane systems with D3, D5 and NS5-branes. Both the Coulomb branch and the Higgs branch different singular points can be obtained with it, and
8.3. THE MATRIX FORMALISM

The nature of the singularities is given by the corresponding label in the obtained Hasse diagrams. In order to make the algorithm clear, let us focus now on a system with Coulomb branch being the closure of the maximal nilpotent orbit of \( \mathfrak{sl}(N, \mathbb{C}) \): \( \mathcal{M}_C = \mathcal{O}_{(N)} \). In order to automate the computation, we encode the D3, D5 and NS5 branes on a \( 2 \times (N + 1) \) matrix \( M \).

8.3.1 Definition

First, in order to encode the brane system into a matrix, take the brane system to a generic point in the Coulomb branch, such that all D3-branes end between NS5-branes.

Then a \( 2 \times (N + 1) \) matrix \( M \) can be created, with matrix elements \( M_{1i} \) counting the number of D5-branes found in the \( i \)th interval between NS5-branes. The intervals should be labelled from left to right, such that \( i = 1 \) is the part to the left of the leftmost NS5 and \( i = N + 1 \) refers to the part to the right of the rightmost NS5. The matrix elements \( M_{2i} \) count the number of D3-branes at the \( i \)th interval between NS5s. For example, this convention would mean that the brane system in figure 8.2 is represented by the following matrix:

\[
M = \begin{pmatrix}
0 & 0 & 3 & 0 \\
0 & 1 & 2 & 0
\end{pmatrix}
\]  

(8.19)

Now let us consider the following matrix:

\[
M = \begin{pmatrix}
0 & N & 0 & \cdots & 0 & 0 & 0 \\
0 & N - 1 & N - 2 & \cdots & 2 & 1 & 0
\end{pmatrix}
\]  

(8.20)

it corresponds to the brane system such that \( \mathcal{M}_C(M) = \mathcal{O}_{(N)} \subseteq \mathfrak{sl}(N, \mathbb{C}) \) [48].

8.3.2 \( A_n \) Kraft-Procesi Transition

Let us discuss how to implement an \( A_n \) Kraft-Procesi transition in this formalism. We look for elements in the first row of the matrix that are different from zero. These represent intervals where there are D5-branes such that the D3-branes can be aligned and the Kraft-Procesi transition can be performed.

The transition \( A_n \) only involves a single D3-brane in a single interval. It has a matrix:
CHAPTER 8. THE KRAFT-PROCESI TRANSITION

\[
M(A_n) = \begin{pmatrix}
0 & (n+1) & 0 \\
0 & 1 & 0 \\
\end{pmatrix}.
\] (8.21)

After the transition is performed, the D3-brane is removed from the Coulomb branch brane system and one D5-brane is moved to the next interval to the left, while another D5-brane is moved to the next interval to the right via Hanany-Witten transitions. The resulting matrix is:

\[
M = \begin{pmatrix}
1 & (n-1) & 1 \\
0 & 0 & 0 \\
\end{pmatrix}
\] (8.22)

Hence, if this system is found as part of any matrix, the \(A_n\) Kraft-Procesi transition can be performed. In a generic brane system with matrix \(M\), a matrix element \(M_{1i} = n + 1 > 1\) always gives rise to a \(A_n\) Kraft-Procesi transition if \(M_{2i} > 0\).

Let us show for example the matrix with \(M_C(M) = \overline{O}(3)\) shown before:

\[
M = \begin{pmatrix}
0 & 3 & 0 & 0 \\
0 & 2 & 1 & 0 \\
\end{pmatrix}.
\] (8.23)

The element \(M_{12} = 3\) indicates that there is an \(A_2\) transition that can be performed. After it is performed, we obtain:

\[
M' = \begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{pmatrix}.
\] (8.24)

8.3.3 \(a_n\) Kraft-Procesi Transition

The branes that are involved have the shape:

\[
M(a_n) = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 1 & 0 \\
0 & 1 & 1 & \ldots & 1 & 1 & 0 \\
\end{pmatrix},
\] (8.25)

where there are \(n\) ones in the bottom row. It is important that all the elements in the top row between the two ones are zeroes. After the transition all D3-branes are removed to obtain:

\[
M = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\end{pmatrix},
\] (8.26)

Let us consider the example of the matrix \(M'\) in equation (8.24):

\[
M' = \begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{pmatrix}.
\] (8.27)

We see that an \(a_2\) Kraft-Procesi transition can be transformed, to obtain:

\[
M'' = \begin{pmatrix}
2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\] (8.28)
8.3. **THE MATRIX FORMALISM**

8.3.4 Example: \( \mathfrak{sl}(4, \mathbb{C}) \)

In this section we review the Kraft-Procesi transitions for \( \mathfrak{sl}(4, \mathbb{C}) \). The starting matrix, is taken from equation (8.20), with \( N = 4 \), let us call this matrix \( M(4) \):

\[
M(4) = \begin{pmatrix}
0 & 4 & 0 & 0 & 0 \\
0 & 3 & 2 & 1 & 0
\end{pmatrix}.
\] (8.29)

Every matrix element in the top row is zero except for \( M_{12} = 4 \). This means that there is a single possible Kraft-Procesi transition: an \( A_3 \) transition involving the second column. After performing it, a new matrix is obtained (let us call this matrix \( M(3,1) \)):

\[
M(3,1) = \begin{pmatrix}
1 & 2 & 1 & 0 & 0 \\
0 & 2 & 2 & 1 & 0
\end{pmatrix}.
\] (8.30)

Now we look for new transitions that could be performed from \( M(3,1) \). The second column has \( M_{12} = 2 \). This means that an \( A_1 \) Kraft-Procesi transition is possible. The third column has \( M_{13} = 1 \). This is not enough for a transition of type \( A_n \) on that interval. On the other hand, another column with \( M_{1i} = 1 \) would be needed for a transition of type \( a_n \). Therefore, there are no other possible transitions that can be performed. The only possible transition is \( A_1 \) involving the second column. After performing it one obtains the following matrix (let us call it \( M(2,2) \)):

\[
M(2,2) = \begin{pmatrix}
2 & 0 & 2 & 0 & 0 \\
0 & 1 & 2 & 1 & 0
\end{pmatrix}.
\] (8.31)

From this matrix the only possible Kraft-Procesi transition is another \( A_1 \) transition this time involving \( M_{13} = 2 \). After performing it, one obtains the matrix (let us call it \( M(2,1,1) \)):

\[
M(2,1,1) = \begin{pmatrix}
2 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0
\end{pmatrix}.
\] (8.32)

In the top row of this matrix there are two elements that are 1, since all elements between them are 0, a Kraft-Procesi transition of type \( a_n \) can be performed. Since there are three intervals involved, the transition is \( a_3 \). After performing it, the resulting matrix (let us call it \( M(1,1,1,1) \)) is obtained:

\[
M(1,1,1,1) = \begin{pmatrix}
3 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\] (8.33)

Now, since all the elements of the bottom row are zero, there are no more Kraft-Procesi transitions that could be performed.

The Hasse diagram has been recovered, figure 8.12. The partitions can be read from the top row of each matrix following [1]. The dimension of the different orbits can be computed from the matrix as:

\[
dim := \sum_i M_{2i}.
\] (8.34)
Figure 8.12: Hasse diagram with Kraft-Procesi transitions for $\mathfrak{sl}(4, \mathbb{C})$.

8.4 Results

This section contains the Hasse diagrams obtained in [1] by applying the Kraft-Procesi partition to the brane system with $\mathcal{M}_C = \tilde{O}(N) \subset \mathfrak{sl}(N, \mathbb{C})$, for $2 \leq N \leq 9$, tables 8.3 - 8.10.

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Partition</th>
<th>$dim$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{sl}(2, \mathbb{C})$</td>
<td>$\begin{pmatrix} 0 &amp; 2 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 1 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$2, 1$</td>
</tr>
<tr>
<td>$\mathfrak{sl}(3, \mathbb{C})$</td>
<td>$\begin{pmatrix} 0 &amp; 3 &amp; 0 &amp; 0 \ 0 &amp; 2 &amp; 1 &amp; 0 \ 1 &amp; 1 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 1 &amp; 0 \ 2 &amp; 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$3, 3, 2, 1, 2, 1, 0$</td>
</tr>
</tbody>
</table>

Table 8.3: Hasse Diagram obtained for $\mathfrak{sl}(2, \mathbb{C})$.

Table 8.4: Hasse Diagram obtained for $\mathfrak{sl}(3, \mathbb{C})$. 
8.4. RESULTS

<table>
<thead>
<tr>
<th>Matrices</th>
<th>$\lambda$</th>
<th>dim</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} 0 &amp; 4 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 2 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>3.1</td>
<td>5</td>
</tr>
<tr>
<td>$\begin{pmatrix} 2 &amp; 0 &amp; 2 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>2.2</td>
<td>4</td>
</tr>
<tr>
<td>$\begin{pmatrix} 2 &amp; 1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 1 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>2.1,1</td>
<td>3</td>
</tr>
<tr>
<td>$\begin{pmatrix} 3 &amp; 0 &amp; 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>1,1,1,1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 8.5: Hasse Diagram obtained for $\mathfrak{sl}(4, \mathbb{C})$.

<table>
<thead>
<tr>
<th>Matrices</th>
<th>$\lambda$</th>
<th>dim</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} 0 &amp; 5 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 4 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>$\begin{pmatrix} 1 &amp; 3 &amp; 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 3 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>4,1</td>
<td>9</td>
</tr>
<tr>
<td>$\begin{pmatrix} 2 &amp; 1 &amp; 2 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 2 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>3.2</td>
<td>8</td>
</tr>
<tr>
<td>$\begin{pmatrix} 2 &amp; 2 &amp; 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 2 &amp; 2 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>3,1,1</td>
<td>7</td>
</tr>
<tr>
<td>$\begin{pmatrix} 3 &amp; 0 &amp; 1 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 2 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>2.2,1</td>
<td>6</td>
</tr>
<tr>
<td>$\begin{pmatrix} 3 &amp; 1 &amp; 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 1 &amp; 1 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>2,1,1,1</td>
<td>4</td>
</tr>
<tr>
<td>$\begin{pmatrix} 4 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>1,1,1,1,1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 8.6: Hasse Diagram obtained for $\mathfrak{sl}(5, \mathbb{C})$. 
Table 8.7: Hasse Diagram obtained for $\mathfrak{sl}(6, \mathbb{C})$. 

<table>
<thead>
<tr>
<th>Matrices</th>
<th>$\lambda$</th>
<th>dim</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} 0 &amp; 6 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 5 &amp; 4 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>6</td>
<td>15</td>
</tr>
<tr>
<td>$\begin{pmatrix} 1 &amp; 4 &amp; 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 4 &amp; 4 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>5,1</td>
<td>14</td>
</tr>
<tr>
<td>$\begin{pmatrix} 2 &amp; 2 &amp; 2 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 3 &amp; 4 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>4,2</td>
<td>13</td>
</tr>
<tr>
<td>$\begin{pmatrix} 2 &amp; 3 &amp; 0 &amp; 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 3 &amp; 3 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>4,1,1</td>
<td>12</td>
</tr>
<tr>
<td>$\begin{pmatrix} 3 &amp; 0 &amp; 3 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 2 &amp; 4 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>3,3</td>
<td>12</td>
</tr>
<tr>
<td>$\begin{pmatrix} 3 &amp; 1 &amp; 1 &amp; 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 2 &amp; 3 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>3,2,1</td>
<td>11</td>
</tr>
<tr>
<td>$\begin{pmatrix} 3 &amp; 2 &amp; 0 &amp; 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 2 &amp; 2 &amp; 2 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>3,1,1,1</td>
<td>9</td>
</tr>
<tr>
<td>$\begin{pmatrix} 4 &amp; 0 &amp; 0 &amp; 2 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 2 &amp; 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>2,2,2</td>
<td>9</td>
</tr>
<tr>
<td>$\begin{pmatrix} 4 &amp; 0 &amp; 1 &amp; 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 2 &amp; 2 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>2,2,1,1</td>
<td>8</td>
</tr>
<tr>
<td>$\begin{pmatrix} 4 &amp; 1 &amp; 0 &amp; 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 1 &amp; 1 &amp; 1 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>2,1,1,1</td>
<td>5</td>
</tr>
<tr>
<td>$\begin{pmatrix} 5 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>1,1,1,1,1,1</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 8.8: Hasse Diagram obtained for $\mathfrak{sl}(7, \mathbb{C})$. 
Table 8.9: Hasse Diagram obtained for $\mathfrak{sl}(8, \mathbb{C})$.
8.4. RESULTS

\[ \mathfrak{sl}(9, \mathbb{C}) \]

Table 8.10: Hasse Diagram obtained for \( \mathfrak{sl}(9, \mathbb{C}) \).

<table>
<thead>
<tr>
<th>Matrices</th>
<th>( \lambda )</th>
<th>dem</th>
</tr>
</thead>
</table>
| \begin{pmatrix} 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 2 & 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 2 & 6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 6 & 6 & 5 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 6 & 6 & 5 & 4 & 3 & 2 & 1 \\ 4 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 4 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 6 & 6 & 5 & 4 & 3 & 2 & 1 \\ 3 & 5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 5 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 6 & 5 & 4 & 3 & 2 & 1 \\ 4 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & 1 \\ 4 & 3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 5 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 5 & 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \\ 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 4 & 4 & 3 & 2 & 1 & 0 \\ 5 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 5 & 5 & 4 & 3 & 2 & 1 \\ 6 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 5 & 5 & 4 & 3 & 2 & 1 \\ 5 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 4 & 4 & 3 & 2 & 1 & 0 \\ 6 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 4 & 4 & 3 & 2 & 1 & 0 \\ 6 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ 5 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 6 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 4 & 4 & 3 & 2 & 1 \\ 7 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 \\ 6 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 7 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 6 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \\ 7 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \\ 7 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 36 \\ 8,1 & 35 \\ 7,2 & 34 \\ 7,1,1 & 33 \\ 6,3 & 33 \\ 6,2,1 & 32 \\ 5,4 & 32 \\ 5,3,1 & 31 \\ 6,1,1,1 & 30 \\ 4,4,1 & 30 \\ 5,2,2 & 30 \\ 5,2,1,1 & 29 \\ 4,3,2 & 29 \\ 3,3,3 & 27 \\ 5,1,1,1,1 & 26 \\ 4,2,2,1 & 27 \\ 3,3,2,1 & 26 \\ 4,2,1,1,1 & 25 \\ 3,3,1,1,1 & 24 \\ 3,2,2,2 & 24 \\ 4,1,1,1,1,1 & 21 \\ 3,2,1,1,1,1 & 23 \\ 2,2,2,2,2 & 20 \\ 3,2,1,1,1,1,1 & 20 \\ 2,2,2,1,1,1 & 18 \\ 3,1,1,1,1,1,1 & 15 \\ 2,2,1,1,1,1,1,1 & 14 \\ 2,1,1,1,1,1,1,1,1 & 8 \\ 1,1,1,1,1,1,1,1,1,1,1 & 0
This concludes the Part III of the thesis and the discussion about the Kraft-Procesi transition. Note that any brane system of the type described in Chapter 2 can be written in a matrix form and the study of its singularities can be carried out. Hence, this is a generic process that can be readily applied to analyse other 3d $\mathcal{N} = 4$ quiver gauge theories different from those discussed here. The work on [4] extended the notion of Kraft-Procesi transition beyond this type of brane systems to brane systems with O3-planes. This allowed us to recover the physical realisation of Kraft and Procesi results for the algebras $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$ [14]. Our work in [5] extended the Kraft-Procesi results to 3d $\mathcal{N} = 4$ quiver gauge theories that do not necessarily have an embedding on Type IIB string theory, by introducing the concept of quiver subtractions. This allowed us to find the physical realisation of some of the results that mathematicians Fu, Juteau, Levy and Sommers have carried out in the spirit of Kraft and Procesi, but for the exceptional algebras $\mathfrak{e}(6, \mathbb{C})$, $\mathfrak{e}(7, \mathbb{C})$, $\mathfrak{e}(8, \mathbb{C})$, $\mathfrak{f}(4, \mathbb{C})$ and $\mathfrak{g}(2, \mathbb{C})$ [92].

Let us finish this part stressing once more that we are convinced that these results are important due to two different reasons. On one hand, they constitute a physical realisation of ideas that were in the mathematical literature since the 70s [11, 12, 13, 14], but that are still a current matter of research [92]. On the other hand, they are methods that are simple to implement and can reveal a wealth of information about the singularity structure of the vacuum in 3d $\mathcal{N} = 4$ gauge theories, and hence about the massless spectrum of each choice of VEV, and about the mixed branches of the moduli space. In the next part of the thesis we turn to a different aspect of the analysis of moduli spaces of quiver gauge theories with eight supercharges.
Part IV

The Magnetic Quiver
Chapter 9

The Problem

9.1 The Higgs Branch of SQCD with $G = SU(N_c)$

There is a question about the structure of vacua with 8 supercharges that has been unsolved for a long time. The question is: What is the Higgs branch of SQCD with 8 supercharges, gauge group $SU(N_c)$ and $N_f$ flavors? For example, is the Higgs branch the union of two cones or just a single cone? And if there are two cones, what is their geometry? What is their intersection? These Higgs branches were studied in [37, 38, 93, 94]. The magnetic conjecture that we proposed in [2] helps to tackle this question in a future work [56].

The conjecture is useful because it makes manifest in a simple way the components of the Higgs branch of theories with more complicated gauge groups and matter content. And it does not only explicitly shows the number of cones, but it fully characterises each cone. In particular the Hilbert series of the cone can be readily extracted. It also provides a simple way in which the intersection between the cones can be specified, in particular, the intersections are also hyperkähler cones whose Hilbert series can be computed. The conjecture goes further, since it proposes a new way of understanding brane systems and the information they contain about the vacua, that can be applied to theories with 8 supercharges and 3, 4, 5 or 6 space-time dimensions.

Interestingly enough, the first part of this problem that called our attention was not for example the fact that generic cases, like the Higgs branch of $G = SU(3)$ with $N_f = 4$ was not known as a variety, i.e. what are the precise cones and the precise intersections. And maybe this is because there was indeed a computation of the chiral ring of that theory for a finite value of the gauge coupling. The operators were known. Instead, we first turned our eye towards a set of theories for which not even the chiral ring of the Higgs branch was known. The fact that was known is that when SQCD is placed in 5d with 8 supercharges its Higgs branch changes at the UV limit where the gauge coupling becomes infinite [95, 96, 97, 98, 99, 100, 101, 102, 103, 43]. Outside of this regime, for finite value of the gauge coupling, the Higgs branch is the same in 3, 4, 5 or 6 dimensions, and its chiral ring can be computed using the hyperkähler quotient described in Chapter 5. In other words, for finite gauge coupling the chiral ring of the Higgs branch can be computed and only depends on the values $N_c$ and $N_f$. But for 5d and infinite gauge coupling, the Higgs branch changes, its chiral ring cannot be computed using hyperkähler quotient, and it also depends on the absolute value of the Chern-Simons level $|k|$.

So the first task that the magnetic conjecture solved was how to obtain the chiral ring of the Higgs branch of 5d SQCD at infinite coupling (or at least its Hilbert series). So, what was known about these theories? First, at finite coupling the global symmetry acting on the Higgs branch is $U(N_f)$. At infinite coupling that symmetry gets enhanced. The following table 9.1 summarises the enhancement [99, 100, 101, 102].

The other thing we can do is the following: we do have a full description of the Higgs branch at infinite coupling for the cases where the gauge group is $SU(2)$. These are special cases where the Global symmetry gets enhanced to the exceptional group $E_{N_f+1}$ [95]. The Higgs branch in that case is the closure of the minimal nilpotent orbit of the corresponding algebra $e(N_f+1, \mathbb{C})$ (also known as the reduced moduli space of one $E_{N_f+1}$ instanton in $\mathbb{C}^2$) [95].

In this and the remaining chapters of this Part IV we show the results of [2]. In that paper the known Higgs
<table>
<thead>
<tr>
<th>Parameter region</th>
<th>Global symmetry at UV</th>
</tr>
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<tbody>
<tr>
<td>$N_c - \frac{1}{2} N_f &gt;</td>
<td>k</td>
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<td>$N_c - \frac{1}{2} N_f &gt;</td>
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Table 9.1: Symmetry groups that act as global symmetries on the Higgs branch of 5d SQCD with gauge group $G = SU(N_c)$, $N_f$ flavours and Chern-Simons level $k$. At finite coupling the global symmetry is $SU(N_f) \times U(1)$ times an instanton group $U(1)$ that acts trivially on the Higgs branch.

branches of 5d $\mathcal{N} = 1$ SQED with $G = SU(2)$ and $N_f$ flavours at infinite coupling were used to find a pattern in the corresponding brane system and predict what are the Higgs branches for different values of $N_c$, $N_f$ and CS level $k$.

### 9.2 The First Case

Let us discuss the theory $SU(2)$, with $N_f = 2$. We want a 5d theory with 8 supercharges, which is denoted as 5d $\mathcal{N} = 1$.

At infinite coupling the Higgs branch becomes the union of two cones $[95, 96]$:

\[
\mathcal{M}_{H, \infty} \left( \frac{2}{SU(2)} \right) = \min_{E_2} = \min_{A_2} \cup \min_{A_1}.
\]  

(9.1)

Here we have adopted a new notation for the closure of nilpotent orbits of $\mathfrak{sl}(n, \mathbb{C})$ $[44, 5]$. The reason for this is that there are no partitions that can be used to label the orbits of exceptional algebras, and this notation can be used for any Lie group as long as we keep to the minimal orbit and the next to minimal orbit (these are the two orbits of smaller dimension that are not the trivial orbit of zero dimension).

We can describe both cones employing a 3d $\mathcal{N} = 4$ quiver for which the cone is the Coulomb branch. We can recognise the quivers from Chapter 4 as the mirror quivers for 3d $\mathcal{N} = 4$ SQED with $N_f = 3$ and $N_f = 2$. Precisely the theories that we suggested would act as building blocks of more complicated moduli spaces in 5d $\mathcal{N} = 1$. This is the first case where that happens. Let us write them as:

\[
\min_{A_2} = \mathcal{M}_{C,3d} \left( \begin{array}{c} 1 \\ \circ \end{array} \right).
\]

(9.2)

\[
\min_{A_1} = \mathcal{M}_{C,3d} \left( \begin{array}{c} 1 \\ \circ \\ 1 \end{array} \right).
\]

(9.3)
9.2. THE FIRST CASE

Figure 9.1: Toric diagram from where the brane web of $5d \mathcal{N} = 1$ with $G = SU(2)$ with $N_f = 2$ flavours can be recovered.

Figure 9.2: Brane web of $5d \mathcal{N} = 1$ with $G = SU(2)$ with $N_f = 2$ flavours. The left diagram represents a system with non-infinite gauge coupling. In the right diagram the limit of infinite gauge coupling and zero mass parameters has been taken. The lines represent $(p, q)5$-branes. The horizontal lines are $(1,0)5$-branes (or D5-branes), the vertical lines are $(0,1)5$-branes (or NS5-branes) and the diagonal lines are $(1,−1)5$-branes. The circles represent $(P,Q)7$-branes where $P$ and $Q$ of a 7-brane are of the same type as the $p$ and $q$ of the 5-brane ending on it.

Where the double edge in (9.3) means that there are two identical sets of $3d \mathcal{N} = 4$ hypermultiplets. Note that since the quivers in equations (9.2) and (9.3) do not have flavour nodes one of their nodes of rank 1 can be turned into a flavour, breaking the necklace shape and recovering the quivers for closures of minimal nilpotent orbits discussed in Chapter 4, equation (4.8).

It was already known that the Higgs branch at infinite coupling decomposes into two cones and that both cones only intersect at their respective origins can be seen directly from the brane web. In order to draw the brane web one can use the method of toric diagrams [98]. Note that since we are dealing with $5d$ effective gauge theories, the brane system that we use is different from that of the previous chapters. In this case, we substitute the D3-branes for D5-branes and the D5-branes for D7-branes. Note that now the D5-branes and the NS5-branes form a web [98, 104, 105]. The toric diagram (it can be taken from [106], figure 2) is depicted in figure 9.1. The resulting brane system before and after taking the limit of zero mass and infinite gauge coupling is represented in figure 9.2. The brane system after the limit can be subdivided into sub-webs. There are two inequivalent maximal subdivisions that are represented in figure 9.3 that do not break the supersymmetry of the system. Any other subdivision of the brane system would break the supersymmetry. The motion of the sub-webs in the directions perpendicular to the paper correspond to a different choice of vacuum in the Higgs branch. The fact that there are two inequivalent maximal subdivisions corresponds to the fact that the Higgs branch is the union of two cones.

The new result is the magnetic conjecture, in which we claimed [2] that not only the fact that there are two cones can be seen in the brane system, but also the exact geometry of the cones can be computed. The way to obtain this information was a procedure from which the quivers in equations (9.3) and (9.2) can be read directly from the brane webs. The final conjecture was called Conjecture 1 in [2] and it is presented here in Chapter 11.

In the present chapter and the following chapter we aim to develop an intuition, as well as exposing the reader to the different steps that took us to the final discovery. We try to do this by presenting the cases of $5d \mathcal{N} = 1$ theories with $G = SU(2)$ and $1 < N_f < 6$, from which the geometry of the Higgs branch at infinite coupling was already known to be the closure of the minimal nilpotent orbit [95]:

$$\mathcal{M}_{H,\infty} \left( \begin{array}{c} N_f \\ SU(2) \end{array} \right) = \min E_{N_f+1}. \quad (9.4)$$

We compare each brane system at the infinite coupling limit with the corresponding $3d \mathcal{N} = 4$ quiver or quivers
for which the Coulomb branch is isometric to the Higgs branch of the 5d theory.

Let us now go back to the case with $G = SU(2)$ and $N_f = 2$ and look in more detail at how the branes could be related to the two different quivers (9.2) and (9.3). From figure 9.3 one can compute the quaternionic dimension of the different cones of the Higgs branch. The subdivision in figure 9.3 (a) has three different sub-webs, coloured red, blue and green. Each contributes to the quaternionic dimension by 1. The total dimension is $\dim_{\mathbb{H}} = 3 - 1 = 2$, where the $-1$ accounts for the choice of the origin along the perpendicular directions to the paper. Therefore, the system in figure 9.3 (a) is the candidate to be the cone with quaternionic dimension 2, i.e. $\min_{A_2}$. The system in figure 9.3 (b) has two sub-webs, hence its quaternionic dimension is $\dim_{\mathbb{H}} = 2 - 1 = 1$. Therefore, it is the candidate to be the cone $\min_{A_1}$.

From now on, we refer to the quivers of equations (9.2) and (9.3) as the magnetic quivers. From this example we can induce some facts that were needed [2] to formulate a general conjecture on how to read the magnetic quivers from the brane webs sub-divisions:

1. In the magnetic quiver, there is a separate gauge node with rank 1 for each different sub-web in the brane system.

2. In the magnetic quiver, the hypermultiplets between the gauge nodes are determined by the intersections of sub-webs.

Let us elaborate on point number 2. For figure 9.3 (a), the intersection between the sub-webs is nothing else than the intersection number between $(p, q)$ fivebranes. Let $(p_1, q_1)$ and $(p_2, q_2)$ be two different fivebranes, their intersection number is:

$$I\{(p_1, q_1), (p_2, q_2)\} = \text{Abs} \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}.$$  \hfill (9.5)

Since the branes in figure 9.3 (a) are $(1, 0)$, $(0, 1)$ and $(1, -1)$ they all have intersection 1 with each other, giving rise to the magnetic quiver in equation 9.2.

Note that in the system of figure 9.3 (b) the sub-webs are not simple $(p, q)$ fivebranes. If we want to define a notion of intersection for them we can turn to the mathematical field known as tropical geometry [108]. In tropical geometry the brane webs can be understood as tropical curves. Then, given two tropical curves there is a notion defined between them called stable intersection. The full definition can be found in [108]. Here let us illustrate the concept with the example at hand. The stable intersection between the two sub-webs in figure 9.3 (b) is computed in figure 9.4. In the diagram on the left we depict the two brane sub-webs as tropical curves. In order to compute the stable intersection we need to shift their respective position slightly. This is depicted in the central diagram. Then, the brane sub-webs intersect at points that locally look like the intersection of two $(p, q)$ fivebranes. The intersection number $I\{(p_i, q_i), (p_j, q_j)\}$ for such local intersections can be computed and the stable intersection $SI$ is the sum of all such intersection numbers. The term stable means that if a different shift is chosen the separate values for $I\{(p_i, q_i), (p_j, q_j)\}$ might change, but the total sum $SI$ will not.

Note that the number $SI$ can also be computed as the total area of multicoloured polygons in the dual toric diagram, figure 9.4, right. In the case at hand the stable intersection is:

$$SI = 1 + 1 = 2.$$  \hfill (9.6)
9.2. THE FIRST CASE

Figure 9.4: Computation of the stable intersection between two tropical curves. In the left there are the two curves. In the center both curves have been shifted from their original position. The right diagram depicts the toric diagram corresponding to the figure in the center. The stable intersection is the area of polygons with two colored sides. In this case there are two squares of area 1: \( SI = 1 + 1 = 2 \).

Therefore the quiver in equation (9.3) is related as the magnetic quiver in this way to the brane web in figure 9.3 (b), and the value \( SI = 2 \) corresponds to the two identical hypermultiplets in the magnetic quiver.
Chapter 10

The 5d Higgs Branch at Infinite Coupling for $G = SU(2)$

10.1 $G = SU(2)$ with $N_f = 3$: $E_4$

The next system that was discussed in [2] was the 5d $\mathcal{N} = 1$ theory with $G = SU(2)$ and $N_f = 3$. According to the discussion before, the Higgs branch at infinite coupling is expected to be:

$$\mathcal{M}_{H,\infty}\left(\begin{array}{c}
\begin{array}{c}
\text{3} \\
\text{SU(2)}
\end{array}
\end{array}\right) = \min_{E_4} = \min_{A_4}$$

(10.1)

For this we already have a 3d $\mathcal{N} = 4$ quiver that has the precise Coulomb branch [41]:

$$\min_{A_4} = \mathcal{M}_{C,3d}\left(\begin{array}{cccc}
\text{1} \\
\text{0} \\
\text{1} \\
\text{1} \\
\text{1} \\
\text{1} \\
\end{array}\right)$$

(10.2)

Let us examine the brane webs. The toric diagram is depicted in figure 10.1. From there a brane system can be read, using the same procedure as in the chapter before, it is depicted in figure 10.2. In figure 10.2 $(p, q)$5-branes end on their respective $(p, q)$7-branes, represented by circles. That picture also shows the brane system when the limit to infinite gauge coupling and the origin of the Coulomb branch is taken. The question now is can we see the magnetic quiver of equation (10.2) in the brane system of figure 10.2?

This case is different from the $E_3$ case discussed in the previous chapter since there is a unique maximal subdivision of the brane web. Such subdivision is depicted in figure 10.3. The different colours represent different sub-webs that can move in the direction perpendicular to the paper without any supersymmetry being broken. If we are to recover the magnetic quiver in equation 10.2 from the brane system in figure 10.3 we need that each of the brane sub-webs corresponds to a different gauge node of rank 1. There are five sub-webs: green, blue, red, orange and purple. Then, each gauge node of rank 1 is connected via a hypermultiplet to two and only two other nodes. From the brane system we can see that orange can only connect with blue and with red. Similarly, pink can only connect with green and with red. This means that if the magnetic quiver is to be recovered green needs to be connected with blue and no more connections should be made. To see how this could be possible one would like to first compute the stable intersections $SI$ between red, blue and green sub-webs. Note that when the sub-webs are of the form of a single $(p, q)$ segment, like the case of blue and green in the present example, their stable intersection $SI$ can be computed and it reduces to their intersection.
number. All the stable intersections are computed in table 10.1.

From table 10.1 one can see that the stable intersection between red and blue, between red and green, and between green and blue is always \( SI = 1 \). For the green and blue this would correspond to the hypermultiplet that we expect to find in the magnetic quiver. Let us point out now that the other main difference between this example and the \( E_3 \) example in the previous chapter is that in this one different sub-webs are found to end on the same sevenbrane. This also affects the magnetic quiver. There are two possibilities, if the different sub-webs end in the same sevenbrane from opposite directions the number of hypermultiplets in the magnetic quiver between the gauge nodes is increased by +1. This is the case of the connection of the orange sub-web with the red sub-web, the orange with the blue, the pink with the red, and the pink with the green. On the other hand, if the two different sub-webs end on the same sevenbrane from the same direction the number of hypermultiplets in the magnetic quiver that connects their respective gauge nodes gets a correction of -1. In this way there are no links between the red sub-web and the blue sub-web or the red sub-web and the green sub-web in the magnetic quiver, since the contribution from the stable intersection gets countered by the contribution of the sevenbranes. With this prescription the magnetic quiver read from the branes takes the expected form:

\[
\begin{array}{c}
\text{1} \\
\text{0}
\end{array}
\]

(10.3)

The facts found about the magnetic quivers so far were summarised as follows [2]:

**Summary 1 (Sevenbranes and Stable Intersection for the Magnetic Quiver)** The number of hypermultiplets between two gauge nodes of the magnetic quiver is equal to the stable intersection \( SI \) between the corresponding sub-webs modified by the presence of common sevenbranes. If the two sub-webs end on the same sevenbrane from opposite directions the contribution to the number of hypermultiplets is positive, otherwise the contribution is negative.

**10.2 \( G = SU(2) \) with \( N_f = 4 \) : \( E_5 \)**

Let us discuss now the next example that was analysed in [2], the 5d \( \mathcal{N} = 1 \) theory with \( G = SU(2) \) and \( N_f = 4 \). The Higgs branch at infinite coupling is the reduced moduli space of one \( E_5 \) instanton on \( \mathbb{C}^2 \), or equivalently the closure of the minimal nilpotent orbit of \( \mathfrak{e}(5, \mathbb{C}) \) (note that the algebra of type \( E_5 \) is the same as the one of
10.2. $G = SU(2)$ WITH $N_F = 4$: $E_5$

We have not discussed in the previous chapters nilpotent orbits of $D$-type. In this case, the closure of minimal nilpotent orbit of the Lie algebra $\mathfrak{g}(\mathbb{C})$ can always be identified with the reduced moduli space of 1 $\mathfrak{g}$ instanton on $\mathbb{C}^2$. Therefore, the magnetic quiver can be written following the results of [41]:

$$\min_{D_5} = \mathcal{M}_{C,3d} \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right).$$

(10.5)

The toric diagram for $E_5$ is depicted in figure 10.4. The corresponding brane system is depicted in figure 10.5. In this case the brane web at infinite coupling admits a unique maximal decomposition, figure 10.6. This brane decomposition has a new feature that was not present in the case of $E_3$ and $E_4$. The feature is the presence of identical subwebs. In this case there are two identical $(1, 0)$ fivebranes that have been coloured green and two identical $(0, 1)$ fivebranes that have been coloured blue. The interpretation of this phenomenon is simple, they correspond to gauge nodes of rank 2 in the magnetic quiver. We can see that the magnetic quiver of equation 10.5 has indeed two nodes of rank 2. Therefore, using the previous rules a magnetic quiver can be read:

$$\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \begin{array}{c} 1 \\ 2 \\ 2 \end{array} \begin{array}{c} 1 \end{array}$$

(10.6)

This is precisely the expected magnetic quiver.
10.3 $G = SU(2)$ with $N_f = 5$: $E_6$

The set of examples from the $E_n$ cases was finished in [2] with $E_6$. The theory with $G = SU(2)$ and $N_f = 5$ at infinite coupling it is predicted to have the Higgs branch:

$$M_{H, \infty} \left( \begin{array}{c} 5 \\ SU(2) \end{array} \right) = \min_{E_6}.$$

In this case the magnetic quiver is the Dynkin diagram of affine $E_6$ where the rank of the nodes are the corresponding indices [41]:

$$\min_{E_6} = M_{C,3d} \left( \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{array} \right).$$

In order to see this from the branes we can depict the toric diagram in figure 10.7. The brane system and its infinite coupling limit is depicted in figure 10.8. There is a unique maximal sub-web subdivision for the infinite coupling system, figure 10.9. Notice that there are several copies of identical sub-webs. The gauge nodes of the magnetic quiver will have ranks from 1 to 3. The hypermultiplets that link the gauge nodes are given uniquely via contributions from sub-webs ending on the same sevenbrane from opposite directions. The quiver that one obtains from applying the same rules in Summary 1 is the expected one:

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{array}.$$

10.4 $G = SU(N_c)$ with $N_f = 0$: Super Yang-Mills

The final example investigated in [2] was a different type of 5d $\mathcal{N} = 1$ quiver gauge theory for which the Higgs branch at infinite coupling was also known. This is Super Yang-Mills, with gauge group $G = SU(N_c)$ and no flavours $N_f = 0$. For $N_c > 2$ the Chern-Simons level $k$ affects the Higgs branch. In this example let us set it to
10.4. $G = SU(N_C)$ WITH $N_F = 0$: SUPER YANG-MILLS

$G = SU(N_C)$ WITH $N_F = 0$.

Figure 10.6: Brane web maximally subdivided for $E_5$.

Figure 10.7: Toric diagram corresponding to the 5d SQCD theory with $SU(2)$ gauge group and $N_f = 5$.

The predicted Higgs branch is [103] (note that in [95] the case of $N_c = 2$ was already discussed and its brane system appeared in [98]):

$$M_{H,\infty} \left( \frac{0}{SU(N_c)_0} \right) = \mathbb{C}^2/Z_{N_c}. \quad (10.11)$$

This is a Kleinian surface singularity of type A. Its magnetic quiver has been discussed before and it can be written as:

$$\mathbb{C}^2/Z_{N_c} = M_{C,3d} \left( \begin{array}{c} \circ \hspace{1cm} \circ \hspace{1cm} N_c \\ 1 \end{array} \right) \quad (10.12)$$

where the label in the hypermultiplets indicate that there are $N_c$ hypermultiplets in total. Note that this quiver is equivalent to the 3d $\mathcal{N} = 4$ quiver of SQED with $N_c$ flavours, since a $U(1)$ factor of the gauge group $G = U(1) \times U(1)$ needs to decouple.

The brane system is depicted in figure 10.10 and the maximal subdivision in figure 10.11. From here there are only two sub-webs, corresponding to the two gauge nodes of rank 1 in the magnetic quiver. The number of hypermultiplets is given by the $SI$ which in this case reduces to the intersection number:

$$I\{(-N_c,1),(0,1)\} = \text{Abs} \left( \begin{array}{c} -N_c \\ 0 \end{array} \right) = N_c. \quad (10.13)$$

The magnetic quiver is precisely the expected one:

$$\begin{array}{c} \circ \hspace{1cm} \circ \hspace{1cm} N_c \\ 1 \end{array} \quad (10.14)$$

This concludes the set of examples whose Higgs branch at infinite coupling was known and that were used to develop the ideas of a magnetic quiver that could be extracted directly from the branes.
CHAPTER 10. THE 5D HIGGS BRANCH AT INFINITE COUPLING FOR \( G = SU(2) \)

Figure 10.8: Brane webs for the 5d \( \mathcal{N} = 1 \) theory with \( G = SU(2) \) and \( N_f = 5 \). The web has been depicted before and after taking the gauge coupling to infinity.

Figure 10.9: Brane web maximally subdivided for \( E_6 \).

Figure 10.10: Brane webs for the 5d \( \mathcal{N} = 1 \) theory with \( G = SU(N_c) \) and \( N_f = 0 \). The web has been depicted before and after taking the gauge coupling to infinity.

Figure 10.11: Brane web maximally subdivided for SYM with \( G = SU(N_c) \).
Chapter 11

Magnetic 5d Conjecture

The present chapter contains the magnetic conjecture as it was presented in [2]. According to this conjecture the Higgs branch of a 5d $\mathcal{N} = 1$ gauge theory can be obtained from the brane web in Type IIB realisation of said theory. The conjecture provides a simple procedure to read a magnetic quiver for each component of the Higgs branch (baryonic branch, mesonic branch, instantonic branch). The same procedure can be applied to the intersections between the different components. Once the magnetic quiver is obtained, the Hilbert series can be computed applying the monopole formula to the magnetic quiver.

In the next chapter this conjecture is applied to the set of theories with gauge group $G = SU(N_c)$, number of flavours $N_f$ and CS level $k$. In the case where the Higgs branch has a single component the obtained magnetic quiver is in full agreement with the prediction of [43]. In the case of several components the magnetic quivers, and therefore the geometric description of the different components of the Higgs branch, was presented in [2] for the first time.

11.1 Relevant Quantities

The first thing that we need to state the conjecture is to define some quantities that can be computed from two different sub-webs in a brane web system. The first one is the stable intersection $SI$. This has been explained in the previous chapter and a mathematical definition can be found in [108]. Please, refer to figure 9.4 for an example. The idea is that both sub-webs can be shifted until they only intersect at points that are locally like $(p_a, q_a)$ and $(p'_a, q'_a)$ fivebranes intersections. The intersection numbers of such local intersection points can be computed via the determinant:

$$I\{(p_a, q_a), (p'_a, q'_a)\} = \text{Abs} \begin{pmatrix} p_a & q_a \\ p'_a & q'_a \end{pmatrix},$$  \hspace{1cm} (11.1)

and then added:

$$SI = \sum_a I\{(p_a, q_a), (p'_a, q'_a)\}. \hspace{1cm} (11.2)$$

Next, one needs to quantify the contributions of the sevenbranes that are shared by both sub-webs. Let us denote by $A_i$ ($i = 1, 2, 3, ...$) all the sevenbranes that are shared by both sub-webs. There are two quantities that can be computed for sevenbrane $A_i$:

- $X_i$ is the total number of pairs of two fivebranes, one from each sub-web, that end on sevenbrane $A_i$ from opposite directions.
- $Y_i$ is the total number of pairs of two fivebranes, one from each sub-web, that end on sevenbrane $A_i$ from the same direction.

Now we illustrate the computation of these quantities with an example from [2].
11.1.1 Example of Computation

Let us consider the brane system in figure 11.1. There are three sevenbranes that are shared between the two sub-webs. They are labelled as $A_1$, $A_2$ and $A_3$. Let us compute the complete intersection. The sub-webs can be shifted, figure 11.2. Then, the dual toric diagram can be obtained, figure 11.3, giving a complete intersection of:

$$SI = 2 + 2 = 4.$$  \hfill (11.3)

The quantities related to the sevenbranes $A_1$, $A_2$ and $A_3$ are:

$$X_1 = 0$$ \hfill (11.4)
$$X_2 = 0$$ \hfill (11.5)
$$X_3 = 1 \times 2 = 2,$$ \hfill (11.6)

and

$$Y_1 = 1 \times 1 = 1$$ \hfill (11.7)
$$Y_2 = 1 \times 1 = 1$$ \hfill (11.8)
$$Y_3 = 2 \times 2 = 4.$$ \hfill (11.9)

11.2 The Magnetic Conjecture

Now that we have defined the $SI$, and the contributions from the sevenbranes $X_i$ and $Y_i$ we can state the conjecture:

**Conjecture 1 (The Magnetic Quiver)** Let there be a fivebrane web in Type IIB, where sevenbranes are placed as boundary conditions of the fivebranes. The moduli space that represents the motion of a particular subdivision of the web in the directions spanned by the sevenbranes can be described as the 3d $\mathcal{N} = 4$ Coulomb branch of a magnetic quiver. The procedure to read the magnetic quiver is as follows. For every set of $m$ identical sub-webs there is a $U(m)$ gauge node in the magnetic quiver. For every pair of gauge nodes, there are $E$ edges in the quiver that are obtained as follows. Take two different sub-webs, one for each of the nodes.
11.3 Consequences of the Conjecture

Figure 11.3: This diagram is dual to the brane system in figure 11.2.

Figure 11.4: Generalised stable intersection. Figure created by Futoshi Yagi for [2].

Compute their stable intersection $SI$ and the contributions $X_i$ and $Y_i$ from the sevenbranes that both sub-webs have in common. The number of edges in the quiver is:

$$E = SI + \sum_i X_i - \sum_i Y_i.$$  \hspace{1cm} (11.10)

Let us make one comment on the quantity $E$. It could be understood as a generalisation of the stable intersection $SI$ from tropical geometry. The main difference we introduced with respect the tropical geometry description is that the fivebranes of the sub-webs do not extend to infinity, but end on their respective sevenbranes. This fact means that the brane system can be deformed by moving the sevenbranes in a fashion that changes the $SI$ of two different sub-webs. For example consider in figure 11.4 (a) the stable intersection between the a $(0,1)$ fivebrane and a $(-N_c,1)$ fivebrane. Their stable intersection is:

$$SI = N_c.$$  \hspace{1cm} (11.11)

However, if one were to move the bottom $(0,1)$ sevenbrane upwards until it crosses the $(-N_c,1)$ fibrane one would obtain the brane system in figure 11.4 (b). In this system the $SI$ has changed from $N_c$ to zero. However the quantity $E$ between the two brane systems has remained constant.

In this way, $E$ is the stable quantity that generalises the stable intersection when one passes from a brane web with fivebranes stretching to infinity to a brane web where the fivebranes end on sevenbranes.

11.3 Consequences of the Conjecture

In the next chapter we present a systematic application of Conjecture 1 to the family of 5$dN = 1$ SQCD theories with gauge group $G = SU(N_c)$, number of colours $N_f$ and CS level $k$. Before we do that we present some new physical concepts that have raised during that analysis [2] and that are a direct consequence of applying the conjecture.
One of the new possibilities is the fact that the Higgs branch is the union of three cones. One of the simpler examples is the 5d $\mathcal{N} = 1$ theory with $G = SU(5)$, $N_f = 6$. The toric diagram is represented in figure 11.5 and the corresponding brane system in figure 11.6. The brane system at infinite coupling has three inequivalent maximal subdivisions. They are represented in figure 11.7. This means that the Higgs branch is the union of three cones:

$$\mathcal{M}_{H,\infty} \left( \begin{array}{c} 6 \\ \frac{1}{5} \\ SU(5) \end{array} \right) = C_1 \cup C_2 \cup C_3. \quad (11.12)$$

The geometry of the three cones can be described using the Conjecture 1. Each subdivision is assigned with a magnetic quiver. The magnetic quivers are depicted in figure 11.7. Each cone is the 3d $\mathcal{N} = 4$ Coulomb branch of the corresponding magnetic quiver:

$$C_1 = \mathcal{M}_{C,3d} \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \right). \quad (11.13)$$

$$C_2 = \mathcal{M}_{C,3d} \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \right). \quad (11.14)$$

$$C_3 = \mathcal{M}_{C,3d} \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \right). \quad (11.15)$$

To our knowledge this is the first time that it was discussed that the Higgs branch at infinite coupling can
11.3. CONSEQUENCES OF THE CONJECTURE

Figure 11.7: Three inequivalent maximal subdivisions for the brane system for the 5d $\mathcal{N} = 1$ theory with $G = SU(5)$, $N_f = 6$ and level $k = 1$. The Magnetic Conjecture 1 can be applied to each subdivision to obtain a magnetic quiver that has been displayed underneath each subdivision.

be the union of three cones.

11.3.2 The Intersection of Several Cones

Note that the Conjecture 1 can be applied to any brane web subdivision, not only a maximal subdivision. In this way, it can be utilised to describe the geometry of the double or triple intersections of the cones that make up the Higgs branch at infinite coupling. For example, let us illustrate this concept with the computation of the double intersection of cones $C_1$ and $C_2$ of the previous sections. In order to compute the intersection we find the maximal subdivision $S$ such that both subdivisions corresponding to $C_1$ and $C_2$ are subdivisions of $S$, see figure 11.8. The Conjecture 1 can be applied to $S$ to produce a magnetic quiver. The intersection of both cones is given by the 3d $\mathcal{N} = 4$ Coulomb branch of the magnetic quiver:

$$C_1 \cap C_2 = M_{C_3, 3d} \left( \begin{array}{c}
1 \\
\circ \circ \circ \\
\circ \circ \circ \\
\circ \circ \circ \\
\end{array} \right).$$

(11.16)

This quiver is equivalent to:

$$M_{C_3, 3d} \left( \begin{array}{c}
1 \\
\circ \circ \circ \\
\circ \circ \circ \\
\circ \circ \circ \\
\end{array} \right) = M_{C_3, 3d} \left( \begin{array}{c}
1 \\
\circ \circ \circ \\
\circ \circ \circ \\
\circ \circ \circ \\
\end{array} \right) = n_{\min A_5}. \quad (11.17)$$

Hence, Conjecture 1 produces the result:

$$C_1 \cap C_2 = n_{\min A_5}, \quad (11.18)$$

where $n_{\min A_5}$ is the closure of the next to minimal nilpotent orbit of $\mathfrak{sl}(6, \mathbb{C})$ (with the partition notation, we have $n_{\min A_5} = O_{(22,1^{n-3})}$). This is one more case in which the closures of nilpotent orbits appear as the fundamental objects to understand the geometry of complicated moduli spaces.

Similarly, the union of three cones $C_1$, $C_2$ and $C_3$ can be computed. The relevant subdivision $S'$ that gives the intersection is represented in figure 11.9. The magnetic quiver obtained via Conjecture 1 is:

$$C_1 \cap C_2 \cap C_3 = M_{C_3, 3d} \left( \begin{array}{c}
1 \\
\circ \circ \circ \\
\circ \circ \circ \\
\circ \circ \circ \\
\end{array} \right).$$

(11.19)

Hence:

$$C_1 \cap C_2 \cap C_3 = \overline{n_{\min A_5}}. \quad (11.20)$$
Figure 11.8: Computation of an intersection between two of the components of the Higgs branch.

Figure 11.9: Intersection of the three components of the Higgs branch with brane system depicted in figure 11.7.

Note how once more the closure of a nilpotent orbit plays a fundamental role in the analysis of the moduli space.
Chapter 12

Applications

12.1 Computation of $\mathcal{M}_{H,\infty}$ for SQCD

Let us review in here all the results that were obtained in [2] by applying the Conjecture 1 to the set of bane webs corresponding to $5d \mathcal{N} = 1$ theories with $G = SU(N_c)$, $N_f$ flavours and Chern-Simons level $k$. The quiver is:

$$
\begin{array}{c}
\includegraphics[width=0.25\textwidth]{quiver.png}
\end{array}
$$

There is a condition imposed by the brane web [109]:

$$ |k| \leq N_c - \frac{N_f}{2} + 2. $$

(12.2)

According to our analysis there are four different classes with different types of magnetic quivers:

1) $|k| < N_c - \frac{N_f}{2}$  
2) $|k| = N_c - \frac{N_f}{2}$  
3) $|k| = N_c - \frac{N_f}{2} + 1$  
4) $|k| = N_c - \frac{N_f}{2} + 2$

(12.3) (12.4) (12.5) (12.6)

Let us present the magnetic quivers that describe the Higgs branch of each different class as they were presented in [2].

12.2 Class 1: $|k| < N_c - \frac{N_f}{2}$

12.2.1 $|k| > \frac{1}{2}$

The first class has two degenerate cases, for values of the CS level $|k| = 0, |k| = \frac{1}{2}$. The non-degenerate case is:

$$ |k| > \frac{1}{2}. $$

(12.7)

The brane system is represented in figure 12.1. See the corresponding toric diagram in figure 12.2. Note that the triangulation of the toric diagram does not affect the Higgs branch at infinite coupling. There are three inequivalent maximal decompositions of the brane system. In figure 12.3 the three subdivisions are labelled I,
Figure 12.1: Brane system for $0 \leq k < N_c - \frac{N_f}{2}$. The label $x$, see (12.8), differentiates between an even or an odd number of flavours $N_f$. Figure created by Futoshi Yagi for [2].

Figure 12.2: Diagram of case $0 \leq k < N_c - \frac{N_f}{2}$. Figure created by Futoshi Yagi for [2].

II and III and the different sub-webs are depicted. The variable $x$ is:

\[
x = 0 \quad \text{(for } N_f: \text{even}), \quad x = \frac{1}{2} \quad \text{(for } N_f: \text{odd}). \quad (12.8)
\]

The sub-webs of each maximal sub-division are:

Phase I: \((Ia) \times 1 + (Ib) \times 1 + \sum_{i=1}^{N_f-k} (c_i) \times i + \sum_{i=N_f-k+1}^{N_f-1} (c_i) \times \left( \frac{N_f}{2} - k \right)
\]
\[+ \sum_{i=1}^{N_f-k} (d_i) \times i + \sum_{i=N_f-k+1}^{N_f-1} (d_i) \times \left( \frac{N_f}{2} - k \right) + \left( e \right) \times \left( \frac{N_f}{2} - k \right).\]

Phase II: \((Iia) \times 1 + (Iib) \times 1 + \sum_{i=1}^{N_f-N_c} (c_i) \times i + \sum_{i=N_f-N_c+1}^{N_f-1} (c_i) \times (N_f - N_c)
\]
\[+ \sum_{i=1}^{N_f-N_c} (d_i) \times i + \sum_{i=N_f-N_c+1}^{N_f-1} (d_i) \times (N_f - N_c) + \left( e \right) \times (N_f - N_c).\]

Phase III: \((Iia) \times 1 + \sum_{i=1}^{N_f-1} (c_i) \times i + \sum_{i=1}^{N_f-1} (d_i) \times i + \left( e \right) \times \left( \frac{N_f}{2} - x \right). \quad (12.9)

To each subdivision we apply Conjecture 1 and obtain the corresponding magnetic quiver. They are in table 12.1. At infinite gauge coupling the global symmetry of the Higgs branch gets enhanced to:

\[SU(N_f) \times U(1) \times U(1). \quad (12.10)\]

Note that in each component of the Higgs branch might only see the action of a subgroup of the full global symmetry group. This subgroup can be computed by looking at the gauge nodes that are balanced. They form
12.2. CLASSE 1: $|K| < N_C - \frac{N_T}{2}$

the Dynkin diagram of the semisimple factor of the subgroup. The abelian factor $U(1)^k$ appears when there are $k + 1$ unbalanced gauge nodes. These groups are depicted also in table 12.1.

Note that not in all cases the Higgs branch at infinite coupling is the union of three cones. The cone labelled by I for example is only present if:

$$\frac{N_f}{2} \geq |k|. \quad (12.11)$$

The cone labelled by II appears if:

$$N_f \geq N_c. \quad (12.12)$$

The cone labelled by III appears if:

$$N_f \geq 2. \quad (12.13)$$

**Intersections between components** The intersection between each component of the Higgs branch at infinite coupling can be computed following the same procedure explained in Section 11.3.2. The results are depicted in table 12.2.

12.2.2 $|k| \leq \frac{1}{2}$

Let us look at the special case with:

$$|k| = \frac{1}{2}. \quad (12.14)$$

In this case there are not three cones but only two since the subdivision labelled with III is no longer maximal but it can be further subdivided to subdivision I. The two remaining cones have magnetic quivers depicted in table 12.3.

Now, let us consider case:

$$|k| = 0. \quad (12.15)$$

This case has the same features as $|k| = \frac{1}{2}$, and the two different magnetic quivers are in table 12.4.
Table 12.1: Magnetic quivers for $\frac{1}{2} |k| < N_c - \frac{N_f}{2}$. Note that the labels in the quivers need to be non-negative integers. This puts constrains on the theories that will have all of the three components as part of their Higgs branch at infinite coupling. The gauge nodes that have been omitted with the ... are increasing rank one by one, from 1 to the rank of the next depicted node.

12.3 Class 2: $|k| = N_c - \frac{N_f}{2}$

12.3.1 $|k| > 1$

For this class there are exceptional cases when $|k| \leq 1$. Let us first consider the generic case:

$$|k| > 1.$$  \hfill (12.16)

The enhancement of the global symmetry is:

$$SU(N_f) \times SU(2) \times U(1).$$  \hfill (12.17)

The brane web is represented in figure 12.4. Figure 12.5 depicts the corresponding toric diagram. In this case the Higgs branch at infinite coupling is the union of two cones, labelled I and III. The labels are given due to the fact that there are two maximal subdivisions that follow the same pattern as the subdivisions I and III in the previous Class 1. The sub-webs that make up the two different subdivisions are represented in figure 12.6.
12.3. CLASS 2: $|K| = N_C - \frac{N_f}{2}$

<table>
<thead>
<tr>
<th>Intersection</th>
<th>Quiver</th>
<th>Global Symmetry</th>
<th>Dimension</th>
</tr>
</thead>
</table>
| $I \cap II, II \cap III$ | $\begin{array}{c}
\circ \quad 1 \\
\circ \quad N_f - N_c \\
\circ \quad N_f - N_c \\
\circ \quad 1 \\
\end{array}
\begin{array}{c}
\circ \quad 1 \\
\circ \quad N_f - N_c \\
\circ \quad N_f - N_c \\
\circ \quad 1 \\
\end{array}$ | $SU(N_f)$ | $N_c(N_f - N_c)$ |
| $I \cap III$ | $\begin{array}{c}
\circ \quad 1 \\
\circ \quad N_f - |k| \\
\circ \quad N_f - |k| \\
\circ \quad 1 \\
\end{array}$ | $SU(N_f)$ | $\frac{N_f^2}{4} - k^2$ |
| $I \cap II \cap III$ | $\begin{array}{c}
\circ \quad 1 \\
\circ \quad N_f - N_c \\
\circ \quad N_f - N_c \\
\circ \quad 1 \\
\end{array}$ | $SU(N_f)$ | $N_c(N_f - N_c)$ |

Table 12.2: Intersection of several cones in the Higgs branch of Class 1 with $\frac{1}{2} < |k| < N_c - \frac{N_f}{2}$.

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
</table>
| I     | $\begin{array}{c}
\circ \quad 1 \\
\circ \quad N_f - \frac{N_f + \frac{1}{2}}{2} \\
\circ \quad 1 \\
\end{array}$ | $SU(N_f) \times U(1)$ |
| II    | $\begin{array}{c}
\circ \quad 1 \\
\circ \quad 2N_c - N_f \\
\circ \quad 1 \\
\end{array}$ | $SU(N_f) \times U(1)$ |
| I \cap II | $\begin{array}{c}
\circ \quad 1 \\
\circ \quad N_f - N_c \\
\circ \quad N_f - N_c \\
\circ \quad 1 \\
\end{array}$ | $SU(N_f)$ |

Table 12.3: Magnetic quivers for $\frac{1}{2} = |k| < N_c - \frac{N_f}{2}$.

Phase I: $(Ia) \times 1 + (Ib) \times 1 + \sum_{i=1}^{N_f-k} (c_i) \times i + \sum_{i=N_f-k+1}^{N_f-1-x} (c_i) \times \left( \frac{N_f}{2} - k \right)$

$$+ \sum_{i=1}^{N_f-k} (d_i) \times i + \sum_{i=N_f-k+1}^{N_f-1+x} (d_i) \times \left( \frac{N_f}{2} - k \right) + (e) \times \left( \frac{N_f}{2} - k \right) + (f) \times 1$$

Phase III: $(IIIa) \times 1 + \sum_{i=1}^{N_f-1-x} (c_i) \times i + \sum_{i=1}^{N_f-1+x} (d_i) \times i + (e) \times \left( \frac{N_f}{2} - x \right) + (f) \times 1 \quad (12.18)$

The Conjecture 1 can be used in every subdivision to obtain the magnetic quiver that corresponds. Such quivers are collected in table 12.5. This table also includes the intersection of both cones.

12.3.2 $|k| \leq 1$

The first exceptional case of Class 2 is for:

$$|k| = 1. \quad (12.19)$$
In this case the subdivision III would break supersymmetry. In order to avoid it, it should be modified such that sub-web (IIIa) and sub-web (f) combine onto a single sub-web. The new magnetic quivers are given in table 12.6.

Now let us consider the exceptional case:

$$|k| = \frac{1}{2}. \quad (12.20)$$

In this case subdivision III is not maximal, since sub-web (IIIa) can be broken into sub-webs (Ia) and sub-web (Ib) without breaking of the supersymmetry. This means that the cone III is fully included in cone I. Therefore, there Higgs branch at infinite coupling is a single cone. This is consistent with the prediction of [43]. The magnetic quiver is represented in table 12.7.

The last exceptional case is:

$$|k| = 0. \quad (12.21)$$

This case has a different pattern of global symmetry enhancement, to:

$$SU(N_f) \times SU(2) \times SU(2). \quad (12.22)$$

The brane system is not as in the generic case, but rather as in figure figure 12.7 (with toric diagram in figure 12.8).

This brane system has a single maximal subdivision, labelled by I'. The magnetic quiver is in table 12.8 and it is consistent with the prediction of [43].
12.4. Class 3: \( |K| = N_C - \frac{N_f}{2} + 1 \)

12.4.1 \( |k| > \frac{3}{2} \)

There is a general case, given by absolute value of the CS level:

\[
|k| > \frac{3}{2} \tag{12.23}
\]

The enhancement of the global symmetry is:

\[
SU(N_f+1) \times U(1) \tag{12.24}
\]

The corresponding brane web for finite coupling is represented in figure 12.9, see [109, 110]. Figure 12.10 depicts the toric diagram. Note that actually this is the generalised toric diagram introduced in [105] and called dot diagram there. The white dots represent several fivebranes that end on the same sevenbrane. The brane system at infinite coupling is in figure 12.11. The sub-webs of the maximal subdivisions are depicted in figure 12.12. Labels I and III are used for the two cones, since they follow the same pattern as the cones with the same label in Class 1:
CHAPTER 12. APPLICATIONS

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td><img src="image" alt="Quiver" /></td>
<td>$SU(N_f) \times SU(2) \times U(1)$</td>
</tr>
<tr>
<td>III $(N_f \text{ even})$</td>
<td><img src="image" alt="Quiver" /></td>
<td>$SU(N_f) \times SU(2)$</td>
</tr>
<tr>
<td>III $(N_f \text{ odd})$</td>
<td><img src="image" alt="Quiver" /></td>
<td>$SU(N_f) \times SU(2)$</td>
</tr>
<tr>
<td>I $\cap$ III</td>
<td><img src="image" alt="Quiver" /></td>
<td>$SU(N_f) \times SU(2)$</td>
</tr>
</tbody>
</table>

Table 12.5: Magnetic quivers for $1 < |k| = N_c - \frac{N_f}{2}$.

Phase I: $(Ia) \times 1 + (Ib) \times 1 + \sum_{i=1}^{N_f-2-k+1} (c_i) \times i + \sum_{i=N_f-2-k+2}^{N_f-2-x} (c_i) \times \left( \frac{N_f}{2} - k + 1 \right)$

$$+ \sum_{i=1}^{N_f-2-k+1} (d_i) \times i + \sum_{i=N_f-2-k+2}^{N_f-2-x} (d_i) \times \left( \frac{N_f}{2} - k + 1 \right) + (e) \times \left( \frac{N_f}{2} - k + 1 \right)$$

Phase III: $(IIIa) \times 1 + \sum_{i=1}^{N_f-2-x} (c_i) \times i + \sum_{i=1}^{N_f-2-x} (d_i) \times i + (e) \times \left( \frac{N_f}{2} + x \right)$

Table 12.9 contains the corresponding magnetic quivers.

12.4.2 $|k| \leq \frac{3}{2}$

First, let us discuss:

$$|k| = \frac{3}{2}.$$  \hspace{1cm} (12.26)

For this value of the CS level the sub-web labelled (IIIa) breaks supersymmetry. Therefore the only maximal subdivision is I. Hence there is a single cone in the Higgs branch at infinite coupling. This theory falls in the family of theories studied by [43] and it is consistent with their results. Table 12.10 contains the magnetic quiver.

Let us now study:

$$|k| = 1.$$  \hspace{1cm} (12.27)

In this case sub-web (IIIa) can be subdivided into (Ia) and (Ib) without breaking supersymmetry. Therefore,
12.4. CLASS 3: $|K| = N_C - \frac{N_f}{2} + 1$

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td><img src="image" alt="Quiver" /></td>
<td>$SU(N_f) \times SU(2) \times U(1)$</td>
</tr>
<tr>
<td>III ((N_f) even)</td>
<td><img src="image" alt="Quiver" /></td>
<td>$SU(N_f)$</td>
</tr>
<tr>
<td>I (\cap) III</td>
<td><img src="image" alt="Quiver" /></td>
<td>$SU(N_f)$</td>
</tr>
</tbody>
</table>

Table 12.6: Magnetic quivers for \(1 = |k| = N_c - \frac{N_f}{2}\).

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td><img src="image" alt="Quiver" /></td>
<td>$SU(N_f) \times SU(2) \times U(1)$</td>
</tr>
</tbody>
</table>

Table 12.7: Magnetic quiver \(\frac{1}{2} = |k| = N_c - \frac{N_f}{2}\).

the unique maximal subdivision is I. This theory has a single cone as the Higgs branch, it is consistent with the result of [43]. Table 12.11 shows the magnetic quiver.

Now let us look at

$$|k| = \frac{1}{2}. \quad (12.28)$$

The pattern of global symmetry enhancement changes to:

$$SU(N_f + 1) \times SU(2). \quad (12.29)$$

The brane system at infinite coupling is depicted in figure 12.13 (the toric diagram is in figure 12.14). In this case we label the unique maximal subdivision $I'$. This theory was also studied in [43] and our results are consistent. Table 12.12 shows the magnetic quiver.

Let us see the last case:

$$|k| = 0 \quad (12.30)$$

The pattern of global symmetry enhancement changes to:

$$SU(N_f + 2). \quad (12.31)$$

The corresponding brane system is depicted in figure 12.15 (toric diagram in figure 12.16). The unique maximal subdivision is labelled $I'$. The magnetic quiver given in table 12.13 is also consistent with consistent with [43].
### 12.5 Class 4: \(|k| = N_c - \frac{N_f}{2} + 2\)

The last class of theories has parameters \(|k| = N_c - \frac{N_f}{2} + 2\). Figure 12.17 represents the brane system at finite coupling. Hanany-Witten transitions take us to the brane system in figure 12.18 when \(|k| > 2\) or to figure 12.19 when \(|k| \leq 2\).

#### 12.5.1 \(|k| > 2\)

Let us discuss the generic case, with:

\[
|k| > 2. \tag{12.32}
\]

The pattern of global symmetry enhancement is:

\[
SO(2N_f) \times U(1). \tag{12.33}
\]

Figure 12.20 shows the toric diagram and figure 12.21 shows the brane system at infinite coupling.

The CS level \(k\) can be divided into four cases:

\[
|k| = 2n + \alpha \quad \text{with} \quad \alpha = 0, \frac{1}{2}, 1, \frac{3}{2}. \tag{12.34}
\]

Variables \(y, z, y', z'\) can be introduced:

\[
\begin{align*}
    y &= n - 2, \quad z = 0, \quad y' = n - 2, \quad z' = 0 \quad \text{for} \quad \alpha = 0, \\
    y &= n - 1, \quad z = 1, \quad y' = n - 2, \quad z' = 0 \quad \text{for} \quad \alpha = \frac{1}{2}, \\
    y &= n - 1, \quad z = 1, \quad y' = n - 1, \quad z' = 1 \quad \text{for} \quad \alpha = 1, \\
    y &= n - 1, \quad z = 0, \quad y' = n - 1, \quad z' = 1 \quad \text{for} \quad \alpha = \frac{3}{2}.
\end{align*} \tag{12.35}
\]
12.5. CLASS 4: \(|K| = N_C - \frac{N_f}{2} + 2\)

\[
\begin{align*}
C : (-k + 1 - x, 1) \\
D : (k - x, 1)
\end{align*}
\]

Figure 12.11: Brane system for \(0 \leq k = N_c - \frac{N_f}{2} + 1\), infinite coupling limit. Figure created by Futoshi Yagi for [2].

Also, for integer \(k\) variables \(v\) and \(w\) are defined as:

\[
v = 2, \quad w = 2n - 2 \quad \text{for} \quad \alpha = 0 \\
v = 0, \quad w = 2n - 2 \quad \text{for} \quad \alpha = 1 \tag{12.36}
\]

Figure 12.22 shows the maximal subdivisions and the different sub-webs. There are two maximal subdivisions, labelled IV and V (note that subdivision IV is only there for even \(N_f\)):\(\geq 2\)):

\[
\begin{align*}
\text{Phase VI: } (IVa) \times 1 + (c) \times \left( \frac{N_f - v}{2} \right) + (d_{N_f-1}) \times \left( \frac{N_f - 2 + w}{2} \right) + \sum_{i=1}^{N_f-2} (d_i) \times i \\
\text{Phase V: } (Va) \times 1 + (Vb) \times 1 + (c) \times \left( \frac{N_f - z - z'}{2} \right) + (d_{N_f-1}) \times \left( \frac{N_f + z + z' - 2}{2} \right) + \sum_{i=1}^{N_f-2} (d_i) \times i \tag{12.37}
\end{align*}
\]

Note that in the case \(N_f = 0\), there is no cone V for odd \(k\) (\(\alpha = 1\)), while for even \(k\) (\(\alpha = 0\)) maximal subdivision V is just \((Va) \times 1 + (Vb) \times 1\). Note also that in the case \(N_f = 1\), for any half integer \(k\), the subdivision V is \((Va) \times 1 + (Vb) \times 1\).

Table 12.14 shows the corresponding magnetic quivers.

12.5.2 \(|k| \leq 2\)

Let us now focus on the exceptional cases, with CS level such that:

\[
|k| \leq 2 \tag{12.38}
\]

Figure 12.23 displays the brane web at infinite coupling for the different values of \(k\) (see figure 12.24 for the corresponding toric diagrams). Note that there is a single maximal subdivision, V. The sub-webs \((Va)\) and
<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td><img src="image" alt="Quiver I" /></td>
<td>$SU(N_f + 1) \times U(1)$</td>
</tr>
<tr>
<td>III (N_f even)</td>
<td><img src="image" alt="Quiver III (even)" /></td>
<td>$SU(N_f + 1)$</td>
</tr>
<tr>
<td>III (N_f odd)</td>
<td><img src="image" alt="Quiver III (odd)" /></td>
<td>$SU(N_f + 1)$</td>
</tr>
<tr>
<td>I ∩ III</td>
<td><img src="image" alt="Quiver I ∩ III" /></td>
<td>$SU(N_f + 1)$</td>
</tr>
</tbody>
</table>

Table 12.9: Magnetic quivers for $\frac{3}{2} < |k| = N_c - \frac{N_f}{2} + 1$.

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td><img src="image" alt="Quiver I" /></td>
<td>$SU(N_f + 1) \times U(1)$</td>
</tr>
</tbody>
</table>

Table 12.10: Magnetic quiver for $\frac{3}{2} = |k| = N_c - \frac{N_f}{2} + 1$.

(Vb) that differ in each exceptional case are given in figure 12.25. The other sub-webs are the same with the exception that sevenbranes $E$ and $A_{N_f}$ replace $A_{N_f}$ and $A_{N_f, -1}$ respectively for the value $|k| = 1/2$.

First, let us look at the case:

$$|k| = 2.$$  \hfill (12.39)

As we said before the Higgs branch is a single cone, labelled V. The maximal subdivision is as in (12.37). Table 12.15 shows the magnetic quiver.

Now let us look at value:

$$|k| = \frac{3}{2}.$$  \hfill (12.40)

There is a single cone V, (12.37). The magnetic quiver is shown in table 12.16.

Let us look at case:

$$|k| = 1.$$  \hfill (12.41)

The pattern of enhancement of the global symmetry is changed to:

$$SO(2N_f) \times SU(2).$$  \hfill (12.42)

There is a unique maximal subdivision, which in this case we label V':

$$\text{Phase V': } (Va) \times 1 + (Vb) \times 2 + (c) \times \left( \frac{N_f - 2}{2} \right) + (d_{N_f - 1}) \times \left( \frac{N_f}{2} \right) + \sum_{i=1}^{N_f - 2} (d_i) \times i.$$  \hfill (12.43)
12.5. CLASS 4: $|K| = N_c - \frac{N_f}{2} + 2$

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
</table>
| $1$   | $\begin{array}{c}
1 \\
\cdots
\end{array}$ | $SU(N_f + 1) \times U(1)$ |

Table 12.11: Magnetic quiver for $1 = |k| = N_c - \frac{N_f}{2} + 1$.

Note that it differs from (12.37). The magnetic quiver is given in Table 12.17.

Let us look at the next case:

$$|k| = \frac{1}{2}.$$  \hspace{1cm} (12.44)

The pattern of enhancement of the global symmetry is changed to:

$$SO(2N_f + 2).$$  \hspace{1cm} (12.45)

There is a unique maximal subdivision, which in this case we label $V'$:

$$\text{Phase } V': (Va) \times 2 + (Vb) \times \left( \frac{N_f + 1}{2} \right) + (c) \times \left( \frac{N_f - 1}{2} \right) + \sum_{i=1}^{N_f-1} (d_i) \times i$$  \hspace{1cm} (12.46)

Table 12.18 shows the magnetic quiver.

The final case is for the value:

$$|k| = 0.$$  \hspace{1cm} (12.47)

The theory in this case does not have a 5d, but a 6d fixed point at infinite coupling.

**Exception to the four classes:** $|k| = N_c - \frac{N_f}{2} + 3$ and $N_c = 3$ See [2] for a discussion of theories that fall out of the four classes discussed in here. There is a finite set of theories and they all can be analysed with Conjecture 1, obtaining the respective magnetic quivers.
Table 12.12: Magnetic quiver for $\frac{1}{2} = |k| = N_c - \frac{N_f}{2} + 1$.

Figure 12.15: Brane system for $0 = k = N_c - \frac{N_f}{2} + 1$. Figure created by Futoshi Yagi for [2].

Table 12.13: Magnetic quiver for $0 = |k| = N_c - \frac{N_f}{2} + 1$.

Figure 12.16: Toric diagram for $0 = k = N_c - \frac{N_f}{2} + 1$. Figure created by Futoshi Yagi for [2].

Figure 12.17: Brane system for $0 \leq k = N_c - \frac{N_f}{2} + 2$. Figure created by Futoshi Yagi for [2].

Figure 12.18: A Hanany-Witten transition has been performed for $k = N_c - \frac{N_f}{2} + 2 > 2$. Figure created by Futoshi Yagi for [2].

Figure 12.19: A Hanany-Witten transition has been performed for $0 \leq k = N_c - \frac{N_f}{2} + 2 \leq 2$. Figure created by Futoshi Yagi for [2].
Figure 12.20: Diagram for $k = N_c - N_f/2 + 2 > 2$.
Figure created by Futoshi Yagi for [2].

Figure 12.21: Brane system at infinite coupling for $k = N_c - N_f/2 + 2 > 2$. Figure created by Futoshi Yagi for [2].

Figure 12.22: Maximal subdivision for $k = N_c - N_f/2 + 2 > 2$. Only a value of integer $|k| (\geq 3)$ has sub-web (IVa). Also there is an exception for the value $|k| = \frac{5}{2}$, since $y' = -1$ and $z' = 0$. $E$ is a $(1,1)$ sevenbrane, the sub-web (Vb) becomes a single $(1,1)$ fivebrane between $E$ and $C$. Figure created by Futoshi Yagi for [2].

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>IV ($N_f$ even)</td>
<td>$\begin{array}{c} N_f - 2 \ \downarrow \vdots \ 1 \end{array}$</td>
<td>$SO(2N_f)$</td>
</tr>
<tr>
<td>V ($N_f$ even)</td>
<td>$\begin{array}{c} N_f - 2 \ \downarrow \vdots \ 1 \end{array}$</td>
<td>$SO(2N_f) \times U(1)$</td>
</tr>
<tr>
<td>V ($N_f$ odd)</td>
<td>$\begin{array}{c} N_f - 1 \ \downarrow \vdots \ 1 \end{array}$</td>
<td>$SO(2N_f) \times U(1)$</td>
</tr>
<tr>
<td>IV $\cap$ V ($N_f$ even)</td>
<td>$\begin{array}{c} N_f - 2 \ \downarrow \vdots \ 1 \end{array}$</td>
<td>$SO(2N_f)$</td>
</tr>
</tbody>
</table>

Table 12.14: Magnetic quivers for $2 < |k| = N_c - N_f/2 + 2$. 

12.5. CLASS 4: $|K| = N_C - \frac{N_f}{2} + 2$
CHAPTER 12. APPLICATIONS

+ 2. \( N = N + 2 \)

Global Symmetry = \( \text{Quiver} \)

2. Figure created by Futoshi Yagi for [2].

\[
\begin{align*}
&k = 2 & k = \frac{3}{2} & k = 1 & k = \frac{1}{2} \\
&N_f \quad N_f - 1 \quad N_f - 2 \quad N_f - 1 \quad N_f + 1 \\
&k = 2 & k = \frac{3}{2} & k = 1 & k = \frac{1}{2}
\end{align*}
\]

3. Figure created by Futoshi Yagi for [2].

\[
D \quad E
\]

(Va): \( k = 2, \frac{3}{2} \) \quad (Va): \( k = 1, \frac{1}{2} \) \quad (Vb): \( k = 2 \) \quad (Vb): \( k = \frac{3}{2}, 1, \frac{1}{2} \)

Figure 12.25: Sub-webs that change for \( |k| = N_c - \frac{N_f}{2} + 2 \leq 2 \). Figure created by Futoshi Yagi for [2].

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V (N_f \text{ even}) )</td>
<td>( \circ - \circ - \cdots - \circ - \circ - \circ - \circ - \circ - \circ )</td>
<td>( SO(2N_f) \times U(1) )</td>
</tr>
</tbody>
</table>

Table 12.15: Magnetic quiver for \( 2 = |k| = N_c - \frac{N_f}{2} + 2 \).

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V (N_f \text{ odd}) )</td>
<td>( \circ - \circ - \cdots - \circ - \circ - \circ - \circ - \circ - \circ )</td>
<td>( SO(2N_f) \times U(1) )</td>
</tr>
</tbody>
</table>

Table 12.16: Magnetic quiver for \( \frac{3}{2} = |k| = N_c - \frac{N_f}{2} + 2 \).

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V' (N_f \text{ even}) )</td>
<td>( \circ - \circ - \cdots - \circ - \circ - \circ - \circ - \circ - \circ )</td>
<td>( SO(2N_f) \times SU(2) )</td>
</tr>
</tbody>
</table>

Table 12.17: Magnetic quiver for \( 1 = |k| = N_c - \frac{N_f}{2} + 2 \).
### Table 12.18: Magnetic quiver for $\frac{1}{2} = |k| = N_c - \frac{N_f}{2} + 2$

<table>
<thead>
<tr>
<th>Phase</th>
<th>Quiver</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V'$ (Nf odd)</td>
<td>$\circ \circ \cdots \circ \frac{N_f-1}{2} \circ \frac{1}{2} \circ \circ \circ_{N_f-2} \circ \circ \circ_{N_f-1} \circ \circ \circ_{N_f+1} \frac{1}{2}$</td>
<td>$SO(2N_f + 2)$</td>
</tr>
</tbody>
</table>

Table 12.18: Magnetic quiver for $\frac{1}{2} = |k| = N_c - \frac{N_f}{2} + 2$. 

12.5. CLASS 4: $|K| = N_C - \frac{N_f}{2} + 2$
Part V

Conclusion
Conclusion

In the present thesis two of the main results from the PhD research effort have been reviewed. They are the Kraft-Procesi transition [1] and the magnetic quivers [2]. The first one is a physical realisation of the mathematical results of [11, 12, 13, 14]. Those results studied the structure of singularities of affine varieties known as closures of nilpotent orbits. In [1] we showed how these results can be applied to study the physical properties of the vacuum of 3d $\mathcal{N} = 4$ quantum field theories. In particular, we used a Type IIB string theory embedding for such theories [10] and interpreted the mathematical results of Kraft and Procesi in terms of motions of branes. We believe that these results are simple enough such that they can be used in the future to analyse the singularity structure of more complicated moduli spaces of quantum field theories with eight supercharges. The work reviewed here has already been extended into several directions. In [4] the brane setting was modified with the introduction of O3-planes. This allowed to extend the Kraft-Procesi transitions to 3d $\mathcal{N} = 4$ theories with symplectic and orthogonal factors in the gauge group [111, 48, 112, 83]. In [5] the notion of Kraft-Procesi transitions was generalised beyond the brane setting. It was conjectured as an operation between quivers that was named quiver subtraction. This allowed to study the singularity structure of moduli spaces of 3d $\mathcal{N} = 4$ theories that did not have a known embedding in Type IIB string theory. Also the works of [113, 114] have extended the notion of Kraft-Procesi transitions to circular quivers and D-type quivers respectively.

Another reason for why the work in the Kraft-Procesi transition was important is because it helped to better understand the geometry of closures of nilpotent orbits. Two of the other works that resulted from this PhD also took this direction [3, 6]. Since the results of Namikawa [78], closures of nilpotent orbits are believed to be the building blocks that are needed in order to understand the geometry of more complicated spaces that appear as moduli spaces of quantum field theories with 8 supercharges.

This fact became particularly apparent with the introduction of our second result, the magnetic quiver [2]. The idea behind this result is that Higgs branches of theories with eight supercharges and 6, 5, 4 or 3 space-time dimensions that cannot be built as hyperkähler quotients can be constructed instead by giving a new set of diagrammatic data called the magnetic quiver. In particular, the Hilbert series of the Higgs branch in question can be obtained by applying the monopole formula [50] to the magnetic quiver (remember that the monopole formula was initially introduced to compute Hilbert series of Coulomb branches for 3d $\mathcal{N} = 4$ quiver gauge theories). In the result that we review in this thesis we focus on the case of 5d $\mathcal{N} = 1$ and propose a conjecture that allows to read the magnetic quiver corresponding to each component of the Higgs branch directly from the brane embedding of the theory (in this case a fivebrane web). We think that the main advantage of our conjecture is that it is relatively simple to implement. It is also very interesting that in order to formulate it we needed to borrow some ideas from the mathematical field of tropical geometry [108]. The results reviewed here have already been extended to the case of 6d $\mathcal{N} = (1, 0)$ in [8]. That work is particularly relevant since it analyses Higgs branches of 6d theories with 8 supercharges by combining the ideas of magnetic quivers and Kraft-Procesi transitions. In a future work [56] the conjecture reviewed here is used in order to understand the geometry of the Higgs branch of SQCD with 8 supercharges at finite coupling in 3, 4, 5 and 6 space-time dimensions.

We would like to conclude by saying that we hope to have contributed to the study of the supersymmetric vacua of quiver gauge theories in different dimensions by providing a set of techniques that help analyse different
aspects of the geometry of the moduli spaces. We formulated the techniques in the language of diagrams (brane diagrams and quiver diagrams) since we find this useful and simple to remember and to implement. Finally, we would like to say that we had have a lot of fun researching this topic and we also hope that some of that joy has stayed in these pages and has managed to find its way to the reader, along with the physical and mathematical ideas. It has been an incredible journey and we consider ourselves exceptionally lucky to have had the chance to take part in it.
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