EIGENVALUES OF THE BILAYER GRAPHENE OPERATOR WITH A COMPLEX VALUED POTENTIAL

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Abstract. We study the spectrum of a system of second order differential operator D_m perturbed by a non-selfadjoint matrix valued potential V. We prove that eigenvalues of $D_m + V$ are located near the edges of the spectrum of the unperturbed operator D_m.

1. Statement of the main results

Spectral properties of non-selfadjoint operators have been recently a subject of interest of many papers. A particular interest was related to the location of eigenvalues of differential operators in the complex plane $\mathbb{C}$. The corresponding results for Schrödinger operators can be found in [1], [3]-[4] and in [5]. Some other problems were studied in the papers [6]-[10] and [12].

The operator we study is related to the quantum theory of a material consisting of two layers of graphene. Namely, we consider the operator $D = D_m + V$, where

$$D_m = \begin{pmatrix} m & 4\partial_{\bar z}^2 \\ 4\partial_z^2 & -m \end{pmatrix}, \qquad \partial_{\bar z} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + \frac{1}{i}\frac{\partial}{\partial x_2}\Big), \qquad \partial_z = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - \frac{1}{i}\frac{\partial}{\partial x_2}\Big), \quad m \geq 0.$$

This operator acts in the Hilbert space $L^2(\mathbb{R}^2;\mathbb{C}^2)$. The domain of D is the Sobolev space $\mathcal{H}^2(\mathbb{R}^2;\mathbb{C}^2)$. The potential V is a not necessary self-adjoint matrix-valued function

$$V(x) = \begin{pmatrix} V_{1,1}(x) & V_{1,2}(x) \\ V_{2,1}(x) & V_{2,2}(x), \end{pmatrix}$$

where the matrix elements are allowed to take complex values. For the matrix V we denote

$$|V(x)| = \sqrt{\sum_{i,j=1,2} |V_{i,j}(x)|^2}.$$

Assuming that V decays at the infinity in some integral sense we would like to answer the question: "Where are the eigenvalues of D located?"

Note that since $D_m^2 = \Delta^2 + m^2$, the spectrum $\sigma(D_m)$ of D_m is the set $(-\infty, m] \cup [m, \infty)$. Our results show that the eigenvalues of D are located near the edges of the absolutely continuous spectrum, i.e. near the points $\pm m$. Since the spectrum of the unperturbed operator has two edges, our results resemble some of the theorems of the paper [2] related to the Dirac operator. However, the main difference between the two papers is that we study a differential operator on a plane, while the article [2] deals with operators on a line.

Theorem 1.1. *Let $k \notin \sigma(D_m)$ be an eigenvalue of the operator D. Let $1 < p < 4/3$. Then*

$$\frac{C_p \int_{\mathbb{R}^2} |V(x)|^p dx}{|\mu|^{p-1}} \Big(\sqrt{\Big|\frac{k-m}{k+m}\Big|} + \sqrt{\Big|\frac{k+m}{k-m}\Big|} + 1 \Big)^p \geq 1, \qquad \mu^2 = k^2 - m^2,$$

with $C_p > 0$ independent of V, k and m. In particular, if $m = 0$, then

$$|k|^{p-1} \le 3^p C_p \int_{\mathbb{R}^2} |V(x)|^p dx, \qquad 1 < p < 4/3.$$

The next statement tells us about what happens when $p \to 1$.

Theorem 1.2. *Let $k \notin \sigma(D_m)$ be an eigenvalue of the operator D. Let $\mu^2 = k^2 - m^2$. Then*

$$C\Big(|\ln|\mu|| \sup_{x\in\mathbb{R}^2} \int_{|x-y|<(2|\mu|)^{-1}} |V(y)|\, dy + \sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} \Big(1 + |\ln|x-y||\Big)|V(y)|dy\Big) +$$

$$+C \int_{\mathbb{R}^2} |V(x)|dx \Big(\sqrt{\Big|\frac{k-m}{k+m}\Big|} + \sqrt{\Big|\frac{k+m}{k-m}\Big|} + 1\Big) \ge 1,$$

where the constant $C > 0$ is independent of V, k and m.

Note that this statement also holds true for $m = 0$.

Corollary 1.1. *Let $m = 0$ and let $k \notin \mathbb{R}$ be an eigenvalue of the operator D. Then*

$$C\Big(|\ln|k|| \sup_{x\in\mathbb{R}^2} \int_{|x-y|<(2|k|)^{-1}} |V(y)|\, dy + \sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} \Big(1 + |\ln|x-y||\Big)|V(y)|dy\Big) +$$

$$+3C \int_{\mathbb{R}^2} |V(x)|dx \ge 1,$$

where the constant $C > 0$ is independent of V and k.

In particular, we see that if $m = 0$, then for small V, the eigenvalues of D are situated in the circle $\{k \in \mathbb{C} : \ |k| < r\}$ of radius r which has the following asymptotical behavior

$$r \asymp \exp\Big(-\frac{C}{\int |V|dx}\Big), \qquad \text{as} \qquad \int |V|dx \to 0.$$

The proof of Theorems 1.1 and 1.2 are given in Section 2. In Section 3 we consider a special case where $V = iW^2$, $W = W^*$, In this case we can get a more precise information about location of the complex eigenvalues, see Theorem 3.1. It is interesting to note that if $m = 0$ (no gap in the continuous spectrum), then perturbations by such matrix-functions do not create any complex eigenvalues. Here we have similarities with the result obtained for the one dimensional Dirac operators in [2].

2. Proofs of the main results

In order to prove our main results we need the Birman-Schwinger principle formulated below.

Proposition 2.1. *Let $V = W_2 W_1$, where W_1 and W_2 are two matrix-valued decaying functions. A point $k \in \mathbb{C} \setminus \sigma(D_m)$ is an eigenvalue of D if and only if -1 is an eigenvalue of the operator*

$$X(k) := W_1(D_m - k)^{-1} W_2.$$

In particular, if $k \in \mathbb{C} \setminus \sigma(D_m)$ is an eigenvalue of D then $||X(k)|| \ge 1$.

The proof of this statement is standard and it is left to the reader as an exercise.

Below we always denote

$$W = \sqrt{V^*V}$$

and use the Birman-Schwinger principle with $W_1 = W$ and $W_2 = VW^{-1/2}$.

Proof of Theorem 1.1. Since

$$(D_m - k)^{-1} = (D_m + k)(D_m - k)^{-1}(D_m + k)^{-1} = (D_m + k)(D_m^2 - k^2)^{-1},$$

it is easy to see that

$$(D_m - k)^{-1} = (m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} + (D_0 - \mu)^{-1}, \tag{1}$$

where

$$D_0 = \begin{pmatrix} 0 & 4\partial_{\bar z}^2 \\ 4\partial_z^2 & 0 \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One can also note that the last term in the right hand side of (1) can be rewritten in the form

$$(D_0 - \mu)^{-1} = (D_0 + \mu)(\Delta^2 - \mu^2)^{-1}. \tag{2}$$

The operator $(\Delta^2 - \mu^2)^{-1}$ is an integral operator with the kernel

$$g_k(x,y) = \frac{i}{8\mu}\Big(H(\sqrt{\mu}r) - H(i\sqrt{\mu}r)\Big),$$

where $H(z) = H_0^{(1)}(z)$ is the Hankel function of first kind and $r = |x-y|$. It is a simple consequence of the fact that

$$(\Delta^2 - \mu^2)^{-1} = \frac{1}{2\mu}\Big((-\Delta - \mu)^{-1} - (-\Delta + \mu)^{-1}\Big).$$

The kernel of $(-\Delta - \mu)^{-1}$ is $4^{-1}iH(\sqrt{\mu}r)$. Another useful representation of $g_k(x,y)$ follows from the fact that the kernel of $(-\Delta - \mu)^{-1}$ equals (see [11])

$$(2\pi)^{-1}K_0(-i\sqrt{\mu}|x-y|),$$

where

$$K_0(z) = \frac{e^{-z}}{\Gamma(1/2)}\sqrt{\frac{\pi}{2z}}\int_0^\infty e^{-t}t^{-1/2}\big(1 + \frac{t}{2z}\big)^{-1/2}dt, \qquad |\arg z| < \pi.$$

Let us define

$$G(z) = H(z) - H(iz).$$

We need to know the behaviour of the function G only in the region $0 < \arg z < \pi/2$, where we have

$$|G(z)| + |G'(z)| + |G''(z)| \le \frac{C}{\sqrt{|z|}}, \qquad \text{if} \qquad |z| > 1/2.$$

The behaviour of the function G near zero is determined by the expansion of the Hankel function in the neighbourhood of $z = 0$. It turns out that

$$|G(z)| \le C, \qquad |G'(z)| \le C_1|z|\ln|z|^{-1}, \qquad |G''(z)| \le C_1 \ln|z|^{-1}, \qquad \text{if} \qquad |z| < 1/2.$$

Let $\rho_\mu(|x-y|)$ be the kernel of the integral operator $(D_0 - \mu)^{-1}$

$$\rho_\mu(|x-y|) = \frac{i}{8\mu}\begin{pmatrix} \mu G(\sqrt{\mu}|x-y|) & \partial_{\bar z}^2 G(\sqrt{\mu}|x-y|) \\ \partial_z^2 G(\sqrt{\mu}|x-y|) & \mu G(\sqrt{\mu}|x-y|) \end{pmatrix}$$

Therefore

$$|\rho_\mu(|x-y|)| = \frac{1}{8|\mu|}\sqrt{2|\mu|^2|G(\sqrt{\mu}|x-y|)|^2 + |\partial_{\bar z}^2 G(\sqrt{\mu}|x-y|)|^2 + |\partial_z^2 G(\sqrt{\mu}|x-y|)|^2}.$$

As a consequence, if we denote by $\rho_\theta(|x-y|)$ the kernel of the operator $(D_0 - e^{i\theta})^{-1}$ then

$$|\rho_\theta(r)| \le C \ln r^{-1}, \qquad \text{if} \qquad r < 1/2, \tag{3}$$

and

$$|\rho_\theta(r)| \le C r^{-1/2}, \qquad \text{if} \qquad r > 1/2. \tag{4}$$

In order to prove the latter relations, one has to differentiate the integral kernel of $(\Delta^2 - \mu^2)^{-1}$, using the formulas

$$\frac{\partial r}{\partial z} = \frac{1}{2}\frac{\bar z}{r}, \qquad \frac{\partial^2 r}{\partial z^2} = -\frac{1}{4}\frac{\bar z^2}{r^3}$$

and

$$\frac{\partial r}{\partial \bar z} = \frac{1}{2}\frac{z}{r}, \qquad \frac{\partial^2 r}{\partial \bar z^2} = -\frac{1}{4}\frac{z^2}{r^3}.$$

Since the integral kernel of $(\Delta^2 - \mu^2)^{-1}$ is $\frac{i}{8\mu} G(\sqrt{\mu} r)$, we obtain from (2) that

$$8|\rho_\theta(r)| \le \Big(\Big| \frac{\partial^2 G(e^{i\theta/2} r)}{\partial z^2}\Big|^2 + \Big| \frac{\partial^2 G(e^{i\theta/2} r)}{\partial \bar z^2}\Big|^2 + 2|G(e^{i\theta/2} r)|^2\Big)^{1/2}$$
$$\le C(r^{-1}|G'(e^{i\theta/2} r)| + |G''(e^{i\theta/2} r)| + |G(e^{i\theta/2} r)|).$$

The positive constants in the inequalities (3) and (4) do not depend on $\theta \in [0, \pi/2]$. In particular,

$$M := \sup_\theta \sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} |\rho_\theta(|x-y|)|^q dy < \infty, \qquad q > 4.$$

Let us estimate now the norm of the operator $T = W(D_0 - e^{i\theta})^{-1} W$ with the kernel

$$\tau(x,y) = W(x)\rho_\theta(|x-y|)W(y).$$

For that purpose, we estimate the sesquie-linear form of this operator :

$$(Tu, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \bar v(x) W(x) \rho_\theta(|x-y|) W(y) u(y)\, dxdy.$$

Obviously,

$$|(Tu,v)|^2 = \Big|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \bar v(x) W(x) \rho_\theta(|x-y|) W(y) u(y)\, dxdy\Big|^2 \le$$
$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v(x)|^2 |\rho_\theta(|x-y|)||W(y)|^2\, dxdy \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |W(x)|^2|\rho_\theta(|x-y|)||u(y)|^2\, dxdy \le$$
$$\Big(\sup_x \int_{\mathbb{R}^2} |\rho_\theta(|x-y|)||W(y)|^2\, dy\Big)^2 \|u\|^2\, \|v\|^2 \le$$
$$\Big(\int_{\mathbb{R}^2} |\rho_\theta(|x-y|)|^q\, dy\Big)^{2/q} \|V\|_p^2\, \|u\|^2\, \|v\|^2, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad q > 4.$$

Therefore,

$$\|T\| \le C\|V\|_p, \qquad 1 < p < 4/3.$$

We are now able to estimate the norm of the operator $T_k = W(D_0 - k)^{-1}W$ for $k \notin \sigma(D_0)$. Indeed,

$$|(T_k u, v)| = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x) W(x) \rho_\theta(\sqrt{|k|}|x-y|) W(y) u(y)\, dx dy \right| =$$

$$\frac{1}{|k|^2} \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x/\sqrt{|k|}) W(x/\sqrt{|k|}) \rho_\theta(|x-y|) W(y/\sqrt{|k|}) u(y/\sqrt{|k|})\, dx dy \right| \leq$$

$$\frac{C}{|k|^2} \|V(\cdot/\sqrt{|k|})\|_p \|u(\cdot/\sqrt{|k|})\| \; \|v(\cdot/\sqrt{|k|})\| = \frac{C\|V\|_p}{|k|^{(p-1)/p}} \|u\| \; \|v\|.$$

Consequently,

$$\|T_k\| \leq \frac{C\|V\|_p}{|k|^{(p-1)/p}}.$$

Observe now that the kernel of the operator $(\Delta^2 - \mu^2)^{-1}$ is the function $iG(\sqrt{\mu}|x-y|)/(8\mu)$. The function $G(\sqrt{\mu}|x-y|)$ has the same properties as $\rho_\theta(\sqrt{|\mu|}|x-y|)$. Moreover it is bounded. Therefore, by mimicking the above arguments, one proves that

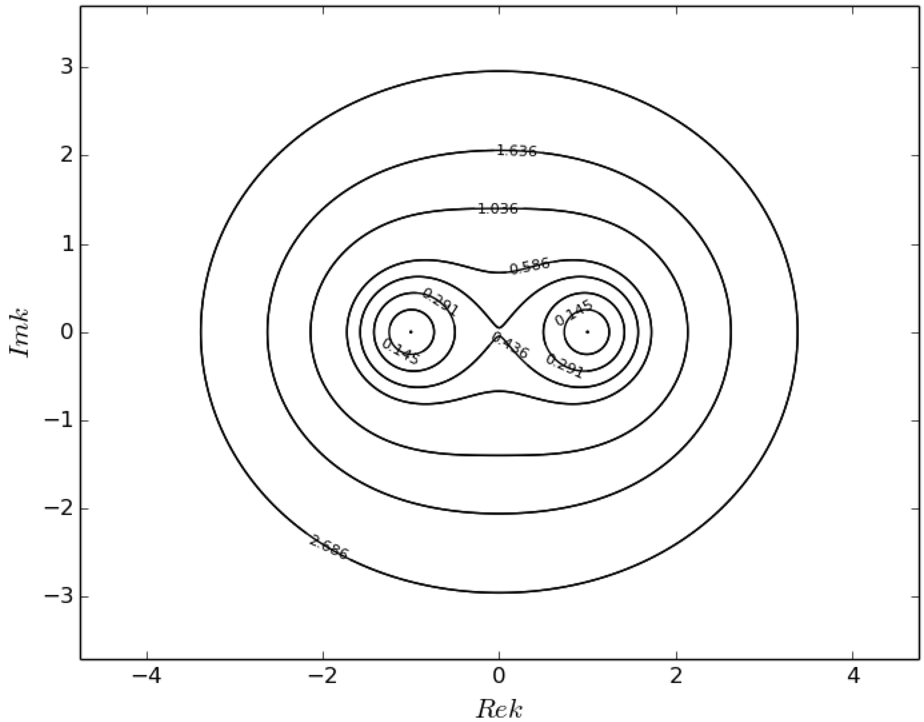
$$\|W(\Delta^2 - \mu^2)^{-1}W\| \leq \frac{C\|V\|_p}{|\mu|^{(2p-1)/p}}, \qquad 1 \leq p < 4/3. \tag{5}$$

This leads to the estimate

$$\|W(D_m - k)^{-1}W\|^p \leq \frac{C\int_{\mathbb{R}^2} |V(x)|^p dx}{|\mu|^{p-1}} \left(\sqrt{\left|\frac{k+m}{k-m}\right|} + \sqrt{\left|\frac{k-m}{k+m}\right|} + 1 \right)^p, \qquad \mu^2 = k^2 - m^2.$$

Now the statement of our theorem follows from the fact that if k is an eigenvalue of $D = D_m + V$, then $\|W(D_m - k)^{-1}W\| \geq 1$. The proof is complete.

In the picture below we describe the areas of possible location of complex eigenvalues depending on the value of $C\int_{\mathbb{R}^2} |V(x)|^p\, dx$, where $m = 1$ and $p = 1.2$.



Proof of Theorem 1.2. As before we use the representation

$$(D_m - k)^{-1} = (m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} + (D_0 - \mu)^{-1}.$$

The operator $(\Delta^2 - \mu^2)^{-1}$ is an integral operator with the kernel

$$g_k(x, y) = \frac{i}{8\mu}\Big(H(\sqrt{\mu}r) - H(i\sqrt{\mu}r)\Big),$$

where $H(z) = H_0^{(1)}(z)$ is the Hankel function of first kind. Again we denote

$$G(z) = H(z) - H(iz).$$

We need to know the behavior of the function G only in the region $0 < \arg z < \pi/2$, where this function is bounded. The boundedness of G implies the estimate (5) with $p = 1$.

It remains to estimate the norm of the operator $T_\mu = W(D_0 - \mu)^{-1}W$ for $\operatorname{Im}\mu > 0$. We already know that if $\mu = |\mu|e^{i\theta}$ the operator $(D_0 - \mu)^{-1}$ is an integral operator with the kernel $\rho_\theta(\sqrt{|\mu|}|x - y|)$, where ρ_θ is a function having the properties

$$|\rho_\theta(r)| \le C \ln r^{-1}, \qquad \text{if} \qquad r < 1/2, \tag{6}$$

and

$$|\rho_\theta(r)| \le C r^{-1/2}, \qquad \text{if} \qquad r > 1/2. \tag{7}$$

The positive constants in these inequalities do not depend on $\theta \in [0, \pi]$. As before, we estimate the sesquie-linear form of this operator :

$$(T_\mu u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \bar{v}(x)W(x)\rho_\theta(\sqrt{|\mu|}|x - y|)W(y)u(y)\,dxdy.$$

Obviously,

$$|(T_\mu u,v)|^2 = \left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \bar v(x)W(x)\rho_\theta(\sqrt{|\mu|}|x-y|)W(y)u(y)\,dxdy\right|^2 \le$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v(x)|^2|\rho_\theta(\sqrt{|\mu|}|x-y|)||W(y)|^2\,dxdy \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |W(x)|^2|\rho_\theta(\sqrt{|\mu|}|x-y|)||u(y)|^2\,dxdy \le$$

$$\left(\sup_x \int_{\mathbb{R}^2} |\rho_\theta(\sqrt{|\mu|}|x-y|)||V(y)|\,dy\right)^2 \|u\|^2\,\|v\|^2.$$

Therefore,

$$\|T_\mu\| \le \sup_x \int_{\mathbb{R}^2} |\rho_\theta(\sqrt{|\mu|}|x-y|)||V(y)|\,dy.$$

The bounds (6) and (7) imply

$$\sup_x \int_{\mathbb{R}^2} |\rho_\theta(\sqrt{|\mu|}|x-y|)||V(y)|\,dy$$
$$\le C\left(|\ln|\mu||\sup_{x\in\mathbb{R}^2}\int_{|x-y|<(2|\mu|)^{-1}} |V(y)|\,dy + \sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}(1+|\ln|x-y||)\,|V(y)|dy\right),$$

which leads to

$$\|T_\mu\| \le C\left(|\ln|\mu||\sup_{x\in\mathbb{R}^2}\int_{|x-y|<(2|\mu|)^{-1}} |V(y)|\,dy + \sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}(1+|\ln|x-y||)\,|V(y)|dy\right).$$

Since

$$\|W(D_m-k)^{-1}W\| \le \|W(m\gamma_0+k-\mu)(\Delta^2-\mu^2)^{-1}W\| + \|T_\mu\|$$

and since (5) holds with $p=1$, we obtain

$$\|W(D_m-k)^{-1}W\|$$
$$\le C\left(|\ln|\mu||\sup_{x\in\mathbb{R}^2}\int_{|x-y|<(2|\mu|)^{-1}} |V(y)|\,dy + \sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}(1+|\ln|x-y||)\,|V(y)|dy\right)$$
$$+\,C\int_{\mathbb{R}^2}|V(x)|dx\left(\sqrt{\left|\frac{kt-m}{k+m}\right|}+\sqrt{\left|\frac{k+m}{k-m}\right|}+1\right).$$

The statement of Theorem 1.2 follows from the fact that if k is an eigenvalue of $D = D_m + V$, then $\|W(D_m-k)^{-1}W\| \ge 1$.

3. A special case

Consider now a special case, when $V(x) = iW^2(x)$, where $W(x) = W^*(x)$ is a matrix valued function. It turns out, that in this case we can get a more precise information about the spectral properties of the operator D.

Theorem 3.1. *Let $k \notin \sigma(D_m)$ be an eigenvalue of the operator $D = D_m + V$, where $V = iW^2$. Let μ be the number in the upper half-plane defined by $\mu^2 = k^2 - m^2$. Then*

$$\left(C\left(\left|\frac{k+m}{\mu}-1\right|+\left|\frac{k-m}{\mu}-1\right|\right)+1\right)\frac{1}{4}\int_{\mathbb{R}^2}\mathrm{tr}|V|dx \ge 1, \tag{8}$$

where the constant C is independent of V, m and k.

Proof. According to the Birman-Schwinger principle, k is an eigenvalue of the operator D if and only if 1 is an eigenvalue of the operator $X = -iW(D_m - k)^{-1}W$. On the other hand, if 1 is an eigenvalue of X then $\|\mathrm{Re}X\| \geq 1$. Since,

$$\mathrm{Re}X = W\,\mathrm{Im}(D_m - k)^{-1}W,$$

we would like to have the explicit expression for the operator $\mathrm{Im}(D_m - k)^{-1}$. Let us first obtain this representation for the case $m = 0$. Observe that D_0 is the operator with the symbol

$$\begin{pmatrix} 0 & -(\xi_1 + i\xi_2)^2 \\ -(\xi_1 + i\xi_2)^2 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\pm|\xi_1 + i\xi_2|^2$. The orthogonal projections $P_1(\xi)$ and $P_2(\xi)$, $\xi = \xi_1 + i\xi_2$, onto the eigenvectors depend only on $\arg(\xi)$. Therefore, the symbol of D_0 is

$$|\xi|^2 P_1(\xi) - |\xi|^2 P_2(\xi),$$

which implies that the integral kernel of $\mathrm{Im}\,(D_0 - k)^{-1}$ is

$$(2\pi)^{-2}\int_{\mathbb{R}^2} \exp(i\xi(x-y))\left(\frac{(\mathrm{Im}\,k)\,P_1(\xi)}{(|\xi|^2 - \mathrm{Re}\,k)^2 + (\mathrm{Im}\,k)^2} + \frac{(\mathrm{Im}\,k)\,P_2(\xi)}{(-|\xi|^2 - \mathrm{Re}\,k)^2 + (\mathrm{Im}\,k)^2}\right)\,d\xi.$$

It follows from this representation that the kernel of the operator $\mathrm{Im}\,(D_0 - k)^{-1}$ is bounded by $1/4$ as using polar coordinates and changing variables $|\xi|^2 = t$ we obtain

$$\|\mathrm{Im}\,(D_0 - k)^{-1}\| \leq (2\pi)^{-2}\int_{\mathbb{S}^1}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\,k}{t^2 + (\mathrm{Im}\,k)^2}\,dt = \frac{1}{4}, \qquad \mathrm{Im}\,k > 0.$$

Consequently,

$$\left\|W\mathrm{Im}\,(D_0 - k)^{-1}W\right\| \leq \mathrm{tr}\,\left(W\mathrm{Im}\,(D_0 - k)^{-1}W\right) \leq \frac{1}{4}\int_{\mathbb{R}^2} \mathrm{tr}\,W^2(x)dx.$$

If $m > 0$, then we have

$$\|\mathrm{Re}X\| \leq \left\|\frac{1}{2\mu}W(m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1}W\right\| + \left\|W\,\mathrm{Im}(D_0 - \mu)^{-1}W\right\|$$

and that

$$\left\|\frac{m\gamma_0 + k - \mu}{2\mu}\right\| \leq \frac{1}{2}\left(\left|\frac{k+m}{\mu} - 1\right| + \left|\frac{k-m}{\mu} - 1\right|\right).$$

It remains to note that according to (5) with $p = 1$,

$$\|W(\Delta^2 - \mu^2)^{-1}W\| \leq C\int_{\mathbb{R}^2} \mathrm{tr}|\mathrm{V}|\,dx$$

The proof is completed. $\square$

The next result says that the spectrum of the operator D_0 is stable with respect to small perturbations of the form $V = iW^2$.

Corollary 3.1. *Let $m = 0$ and let $V = iW^2$ with $W^* = W$. Assume that*

$$\frac{1}{4}\int_{\mathbb{R}^2} \mathrm{tr}|V|dx < 1. \tag{9}$$

Then the operator $D = D_0 + V$ does not have eigenvalues outside of the real line $\mathbb{R}$, i.e. the spectrum of D is real.

References

[1] A. A. Abramov, A. Aslanyan, E. B. Davies, *Bounds on complex eigenvalues and resonances.* J. Phys. A **34** (2001), 57–72.

[2] J-C. Cuenin, A. Laptev and Ch. Tretter, *Eigenvalue estimates for non-selfadjoint Dirac operators on the real line*, Annales Henri Poincare **15** (2014), 707-736

[3] E. B. Davies, *Non-self-adjoint differential operators.* Bull. London Math. Soc. **34** (2002), no. 5, 513–532.

[4] E. B. Davies, J. Nath, *Schrödinger operators with slowly decaying potentials.* J. Comput. Appl. Math. **148** (2002), 1–28.

[5] R. L. Frank, *Eigenvalue bounds for Schrödinger operators with complex potentials.* Bull. Lond. Math. Soc. **43** (2011), no. 4, 745–750.

[6] R. L. Frank, A. Laptev, E. H. Lieb, R. Seiringer, *Lieb–Thirring inequalities for Schrödinger operators with complex-valued potentials.* Lett. Math. Phys. **77** (2006), 309–316.

[7] R.L. Frank, A. Laptev and O. Safronov, *On the number of eigenvalues of Schrödinger operators with a complex potentials* J. of LMS to appear

[8] R. L. Frank, A. Laptev, R. Seiringer, *A sharp bound on eigenvalues of Schrödinger operators on the half-line with complex-valued potentials.* Spectral theory and analysis, 39–44, Oper. Theory Adv. Appl. **214**, Birkhäuser/Springer Basel AG, Basel, 2011.

[9] R. L. Frank, J. Sabin, *Restriction theorems for orthonormal functions, Strichartz inequalities and uniform Sobolev estimates.* Preprint (2014), http://arxiv.org/pdf/1404.2817.pdf

[10] R. L. Frank, B. Simon, *Eigenvalue bounds for Schrödinger operators with complex potentials. III.* J. Spectr. Theory, to appear.

[11] I. M. Gel'fand, G. E. Shilov, *Generalized functions. Vol. 1. Properties and operations.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1964 [1977].

[12] A. Laptev, O. Safronov, *Eigenvalue estimates for Schrödinger operators with complex potentials.* Comm. Math. Phys. **292** (2009), 29–54.

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