Noncommutative gauge theories on D-branes in non-geometric backgrounds

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ABSTRACT: We investigate the noncommutative gauge theories arising on the worldvolumes of D-branes in non-geometric backgrounds obtained by T-duality from twisted tori. We revisit the low-energy effective description of D-branes on three-dimensional T-folds, examining both cases of parabolic and elliptic twists in detail. We give a detailed description of the decoupling limits and explore various physical consequences of the open string non-geometry. The T-duality monodromies of the non-geometric backgrounds lead to Morita duality monodromies of the noncommutative Yang-Mills theories induced on the D-branes. While the parabolic twists recover the well-known examples of noncommutative principal torus bundles from topological T-duality, the elliptic twists give new examples of noncommutative fibrations with non-geometric torus fibres. We extend these considerations to D-branes in backgrounds with R-flux, using the doubled geometry formulation, finding that both the non-geometric background and the D-brane gauge theory necessarily have explicit dependence on the dual coordinates, and so have no conventional formulation in spacetime.

KEYWORDS: D-branes, Flux compactifications, Non-Commutative Geometry, String Duality

Contents

1   Introduction and summary

One of the most striking features of T-duality, which relates different string backgrounds describing the same physics, is that it leads to the possibility of non-geometric backgrounds which do not have a description in terms of conventional Riemannian geometry [1] (see e.g. [2–4] for reviews and further references). Some non-geometric backgrounds arise as T-duals of conventional geometric backgrounds, while others are not geometric in any duality frame. Typical examples start with geometric spaces which admit a torus fibration, with transition functions that are diffeomorphisms of the torus fibres and shifts of the B-field. T-duality transformations along the torus fibres (using the standard Buscher rules [5, 6] on

a covering space) in general lead to T-folds [1]. These are locally geometric — locally they look like a product of the torus with a patch of the base — but the transition functions in general involve T-duality transformations on the torus fibres. The Buscher rules give T-duality in isometric directions. For non-isometric directions, there is a notion of generalised T-duality that can be applied [7]. For a circular direction in which the fields depend explicitly on the coordinate x of that circle, a generalised T-duality transforms this to a configuration in which the fields depend on the coordinate $\widetilde{x}$ of the T-dual circle. This $\widetilde{x}$-dependence means that it cannot be viewed as a conventional background even locally, but has intrinsic dependence on the T-dual coordinates, so a doubled geometry formulation is essential. This concept of generalised T-duality has been checked in asymmetric orbifold limits [7], and is in agreement with the concept of generalised T-duality arising in double field theory [8–10]. We will refer to configurations in which fields and/or transition functions have explicit dependence on the dual coordinates $\widetilde{x}$ as *essentially doubled*.

In this paper we will consider n-dimensional backgrounds obtained by T-dualising the simplest examples of torus bundles, which are fibrations of $n-1$-dimensional tori T^{n-1} over a circle S^1 with vanishing B-field, sometimes referred to as twisted tori [7, 11–15].[1] The monodromy around the base circle is a diffeomorphism of the torus fibres, in the mapping class group $\mathrm{SL}(n-1, \mathbb{Z})$. These and their T-duals give compactifications with a duality twist [11], which are stringy generalisations of Scherk-Schwarz reductions [13]. For definiteness, we will focus on the case of backgrounds in $n = 3$ dimensions, where all of our considerations can be made explicit. Then the simplest case is that of a parabolic monodromy, acting as an integer shift $\tau \mapsto \tau + m$ of the complex structure modulus τ of the two-torus T^2. In this case, the torus bundle is the nilfold of degree m, which is T-dual to a geometric three-torus T^3 with H-flux of the B-field proportional to m [13]. Applying T-duality transformations then results in a much-studied chain of transformations between geometric and non-geometric backgrounds [7, 11, 12, 14, 15]. This is conventionally depicted in a schematic form as [14]

$$ H_{ijk} \;\; \overset{\mathsf{T}_i}{\longleftrightarrow} \;\; f^i{}_{jk} \;\; \overset{\mathsf{T}_j}{\longleftrightarrow} \;\; Q^{ij}{}_k \;\; \overset{\mathsf{T}_k}{\longleftrightarrow} \;\; R^{ijk} \tag{1.1} $$

where T_i denotes a T-duality transformation along the i-th coordinate direction. Successive T-dualities take the three-torus with H-flux to a nilfold with what is sometimes called "geometric flux" f, then to a T-fold with "Q-flux", and finally a generalised T-duality takes this to an essentially doubled space with "R-flux". The cases with f-, H- and Q-flux can be thought of as T^2 conformal field theories fibred over a circle coordinate x, with monodromy in the T-duality group $O(2, 2; \mathbb{Z})$, while the case with R-flux is an essentially doubled space which is a fibration over the T-dual circle with dual coordinate $\widetilde{x}$ and monodromy in the T-duality group. For more general monodromies, such as the elliptic monodromies that we consider in detail below, the results are rather different and do not follow the pattern suggested by (1.1). As we shall see for the elliptic case, acting on the twisted torus 'with f-flux' with either T_i or T_j gives a T-fold, and no dual with only H-flux arises. A further T-duality then gives an essentially doubled space.

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[1]Closely related non-geometric backgrounds involving torus bundles with T-duality monodromy around singular fibres were discussed in [16, 17].

A useful perspective for understanding non-geometry in string theory is to study D-branes in these backgrounds. D-branes can be used as probes to analyse the geometry of a string background and to provide an alternative definition of the background geometry in terms of the moduli space of the probe. D-branes in non-geometric backgrounds were previously discussed from the point of view of doubled (twisted) torus geometry in [1, 18–20], directly in string theory from a target space perspective in [21–23], and from open string worldsheet theory in [24, 25].

In the present paper we focus on an approach based on effective field theory, reinterpreting all of the T-duality transformations in the chain (1.1) for the nilfold and the corresponding chains for other backgrounds in terms of open strings. In this setting it is important to define a low-energy scaling limit which decouples the deformations of geometry due to non-locality of strings from the "genuine" non-geometry due to background fields. In the case of D-branes in flat space with a constant B-field background, the D-brane worldvolume theory is a noncommutative supersymmetric Yang-Mills theory [26], and the decoupling limit was carefully set out in [27]; these considerations were extended to the case of D-branes in curved backgrounds with non-constant B-fields and non-zero H-flux in [28, 29] (see e.g. [30–32] for reviews and further references). This limit is often neglected in the literature on embeddings of worldvolume noncommutative gauge theories in string theory.

To this end, we revisit the problem of formulating effective noncommutative Yang-Mills theories on D-branes in non-geometric tori in the decoupling limit, extending the earlier work of [22, 23] (see also [21]) in various directions. In these works, D3-branes in the simplest T-fold background, originating via T-duality from a flat three-torus with H-flux, are shown to have an effective description as a noncommutative gauge theory on a flat torus. The non-geometry of the background is then interpreted as the dependence of the noncommutativity parameter on the base coordinate $x \in S^1$ of the original torus fibration, with a monodromy around the circle that is a Morita transformation. Morita equivalence was understood in [27] as the open string version of T-duality in the decoupling limit, which is a symmetry of the noncommutative Yang-Mills theory.

In the following we will re-examine D-branes on non-geometric spaces in a more general setting, allowing for more general monodromies beyond the parabolic ones. The elliptic monodromies are of particular interest as they give string theory backgrounds directly, without the need to fibre over some base space [11]. For the particular case of the $\mathbb{Z}_4$ elliptic monodromy, we find that the low-energy effective field theory is defined on a non-geometric torus and that all of its moduli, including the noncommutativity parameter and the Yang-Mills coupling, have a monodromy in the base circle coordinate x in such a way as to render the supersymmetric Yang-Mills theory invariant under Morita duality. To the best of our knowledge, such an example of a noncommutative fibration with non-geometric torus fibres has not appeared before in the literature. The monodromy also interchanges D0-brane and D2-brane charges, which swaps the roles of the rank of the gauge fields and their topological charge in the worldvolume gauge theory.

We also study the effective noncommutative gauge theories in essentially doubled spaces using the doubled twisted torus formalism of [33–36], in which D-branes have been

classified in [19]. Here we find a dependence of the noncommutative gauge theory on the dual base coordinate $\widetilde{x} \in S^1$, thus further exemplifying the need of the doubled formalism in describing such configurations, another point of emphasis which is sometimes neglected in the literature. The general picture of the effective theories on D-branes in non-geometric polarisations of the doubled twisted torus geometry is then that of a parameterised family of noncommutative Yang-Mills theories with monodromies in x or $\widetilde{x}$ that are Morita transformations. These arise as decoupling limits of backgrounds with monodromies in x or $\widetilde{x}$ that are T-duality transformations.

An important feature of our considerations is the role of the doubled geometry. For simple backgrounds, there is a conventional geometry which is seen by particles or momentum modes, while string winding modes will see a T-dual geometry. However, in more complicated settings there is a doubled geometry which cannot be disentangled to give a separate geometry and dual geometry, and the momentum and winding modes see different aspects of the full doubled geometry. For a T-fold, there is a local split, referred to as a polarisation in [1], and local coordinates in a patch can be split into spacetime coordinates and dual coordinates. However, globally this is not possible for a T-fold as the T-duality transition functions mix the two kinds of coordinates so that there is no global polarisation. For essentially doubled spaces, the dependence of the background on the coordinate conjugate to the winding number means that a conventional undoubled formulation is not possible even locally. Configurations that are related to each other by T-dualities all arise as different polarisations of the same doubled geometry. For example, the four configurations in the duality chain (1.1) all arise as different polarisations of the same six-dimensional doubled space [35]. T-duality can be viewed as changing the polarisation [1].

A polarisation splits the doubled coordinates $\mathbb{X}^M$ into "spacetime coordinates" x^m and dual coordinates $\widetilde{x}_m$. For a conventional configuration, the background fields include the closed string metric g, the two-form B-field, and the dilaton ϕ. These background fields depend only on x^m and one obtains the usual spacetime interpretation, at least locally. For an essentially doubled configuration, some of the fields depend explicitly on the dual coordinates $\widetilde{x}_m$. For a conventional configuration with explicit dependence on a coordinate x^ι, a generalised T-duality along the vector field $\partial_\iota = \frac{\partial}{\partial x^\iota}$ will change the dependence of the fields on x^ι to dependence of the fields on the dual coordinate $\widetilde{x}_\iota$, resulting in an essentially doubled background.

The doubled geometry formulation of D-branes has some interesting features [1]. Consider a Dp-brane wrapped on an n-torus T^n with coordinates x^m, where $m = 1, \ldots, n$ and $p \leq n$. Then the ends of open strings will have p coordinates x_{D}^i satisfying Dirichlet boundary conditions on T^n and $n - p$ coordinates x_{N}^a satisfying Neumann boundary conditions. The doubled space is a torus T^{2n} with coordinates $x^m, \widetilde{x}_m$, with $m = 1, \ldots, n$. As T-duality interchanges Dirichlet and Neumann boundary conditions, the coordinates dual to x_{D}^i are p coordinates $\widetilde{x}_{\mathrm{N}\,i}$ satisfying Neumann boundary conditions and the coordinates dual to x_{N}^a are $n - p$ coordinates $\widetilde{x}_{\mathrm{D}\,a}$ with Dirichlet boundary conditions. Then in the doubled torus there are precisely n Dirichlet coordinates $x_{\mathrm{D}}^i, \widetilde{x}_{\mathrm{D}\,a}$, so that whatever the value of p, the doubled picture is that of a Dn-brane wrapping a maximally isotropic (Lagrangian) n-cycle in T^{2n}. As a result, a Dp-brane is secretly a Dn-brane in the doubled space.

The polarisation determines the subset of the n Dirichlet directions which are regarded as physical, and changing the polarisation changes this subset: a T-duality that changes the polarisation from one with p Dirichlet physical coordinates to one with q Dirichlet physical coordinates is interpreted as taking a Dp-brane to a Dq-brane. This picture was developed and extended to more general doubled spaces in [18, 19].

The effective worldvolume theory on a D-brane is a noncommutative Yang-Mills theory coupling to a background open string metric G_{D} with noncommutativity bivector θ and gauge coupling g_{YM}. In general the background fields (G_{D}, θ) as well as the coupling g_{YM} can depend on the coordinates x^m. The action of T-duality on the closed string background (g, B, ϕ) gives rise to Morita transformations of $(G_{\mathrm{D}}, \theta, g_{\mathrm{YM}})$, as we will review in section 2, and in our considerations of D-branes on non-geometric backgrounds we find open string analogues of T-folds in which the dependence $(G_{\mathrm{D}}(x), \theta(x), g_{\mathrm{YM}}(x))$ on a circle coordinate x can have a monodromy that is a Morita transformation. Surprisingly, we also find open string analogues of essentially doubled backgrounds in which $(G_{\mathrm{D}}, \theta, g_{\mathrm{YM}})$ have explicit dependence on a doubled coordinate $\widetilde{x}$, possibly with a Morita monodromy. This suggests that the effective field theory should be defined on the full Dn-brane in the doubled space, and so the fields can depend on all n Dirichlet coordinates $x_{\mathrm{D}}^i, \widetilde{x}_{\mathrm{D}\,a}$.

One of the complications in the case of the flat three-torus with H-flux and its T-duals is that they do not define worldsheet conformal field theories, and so are not solutions of string theory. However, there are string solutions in which these appear as fibres. The simplest case is that in which these are fibred over a line. Taking an NS5-brane with transverse space $\mathbb{R} \times T^3$ and smearing over the T^3 gives a domain wall solution which is the product of six-dimensional Minkowski space with $\mathbb{R} \times T^3$, where there is constant H-flux over T^3 and the remaining fields depend explicitly on the coordinate of the transverse space $\mathbb{R}$ [13]. T-duality then takes this to a metric on the product of $\mathbb{R}$ with the nilfold [13, 21, 22, 37, 38] that is hyperkähler, as was to be expected from the requirement that the background is supersymmetric. Then further T-dualities in the chain (1.1) give T-folds and essentially doubled spaces fibred over a line; the proper incorporation of such spaces in string theory will be discussed further in [39]. This leads to complications in the analysis of D-branes and decoupling limits in such backgrounds [22].

Due to the difficulties arising from such fibrations over a line or other space, we will be particularly interested in examples that do give string theory solutions directly without the need for introducing a fibration. For the cases with elliptic monodromy, at special points in the moduli space the background reduces to an orbifold defining a conformal field theory and so provides a consistent string background. However, we will also be interested in the elliptic monodromy case at general points in the moduli space; these can arise as fibres in which the moduli vary over a line or a higher-dimensional base space.

From the effective field theory point of view, the duality twisted reduction from ten dimensions gives a Scherk-Schwarz reduction of ten-dimensional supergravity to a seven-dimensional gauged supergravity. The fact that in general the product of the internal twisted torus with seven-dimensional Minkowski space does not define a conformal field theory, and so is not a supergravity solution, is reflected in the fact that the seven-dimensional supergravity has a scalar potential. In the parabolic monodromy case, the scalar potential

has no critical points and so there are no Minkowski vacua, but there are domain wall solutions which lift to the ten-dimensional geometry given by the twisted torus fibred over a line. In the elliptic monodromy case, there is a minimum of the potential corresponding to the orbifold compactification to seven dimensions [11], but again there are more general domain wall solutions in which the moduli vary over a line.

We will also consider the dilaton in what follows. For a given background, T-duality will change the dilaton according to the Buscher rules. Defining a conformal field theory requires the metric, B-field and dilaton to satisfy the beta-function equations, but it will be useful to consider general configurations of metric, B-field and dilaton without necessarily requiring them to satisfy the beta-function equations — they then define more general compactifications, as outlined above.

One of our motivations for revisiting these field theory perspectives is to shed some light on the relevance of the noncommutative and nonassociative deformations of closed string geometry which were recently purported to occur in certain non-geometric backgrounds [40–44] (see e.g. [45] for a review and further references). In contrast to these analyses, here we work in a controlled setting with (doubled) twisted tori and quantised fluxes, without any linear approximations and with an exact effective field theory description of the string geometry. Noncommutative and nonassociative geometries were suggested as global (algebraic) descriptions of T-fold and R-flux non-geometries respectively in the mathematical framework of topological T-duality in [46–49], which strictly speaking only applies to the worldvolumes of D-branes, but it was further suggested that such a description should also apply to the closed string background itself. Such a suggestion requires further clarification, insofar that in closed string theory itself there is no immediate evidence for such nonassociative structures. While we reproduce and generalise the non-commutative geometries on D-branes in parabolic T-fold backgrounds, which were shown by [22, 23] to agree with the expectations from topological T-duality, we do not directly find a nonassociative geometry on D-branes in R-folds. Instead, we find that the decoupled noncommutative gauge theory on the D-branes depends explicitly on the transverse doubled coordinate $\widetilde{x}$, and so is essentially doubled and it appears that these cannot be fully understood in an undoubled space.

The organisation of the paper is as follows. In section 2 we briefly review, following [27], the well-known description of the low-energy effective dynamics of D-branes in constant B-fields in terms of noncommutative gauge theory, and in particular its Morita duality on a torus which is inherited from the T-duality symmetry of the closed string background. In section 3 we briefly review some general aspects of string theory compactifications on twisted tori, which are subsequently used to study the worldvolume gauge theories on D-branes on three-dimensional T-folds via T-duality. We treat the cases of parabolic monodromies in section 4 and of elliptic monodromies in section 5. We demonstrate, in both cases of parabolic and $\mathbb{Z}_4$ elliptic twists, that there exist well-defined low-energy scaling limits which completely decouple the open strings from closed strings, and wherein the non-geometry of the T-fold background is manifested in the open string sector as a parameterised family of noncommutative gauge theories which are identified under Morita dualities determined by the particular type of monodromy around $x \in S^1$. We give a

physical interpretation of the scaling limit which reproduces the mathematical description of the field of noncommutative tori probed by the D-branes, and of the Morita equivalence bimodules which implement the Morita duality monodromies. In the case of the elliptic monodromy we examine the theory at the orbifold fixed point where we find that it is equivalent to an ordinary commutative gauge theory on a flat torus for finite area and string slope α'. In section 6 we review the doubled twisted torus formalism and the classification of D-branes therein. This setting allows us to take the final T-duality transformation that describes D-branes in essentially doubled spaces, whose decoupled worldvolume gauge theory is studied in section 7 where we find that in this case the D-branes really probe a noncommutative doubled geometry. For convenience, in appendix A we summarise the Buscher T-duality rules including dimensionful factors.

2 Open string dynamics in B-fields

Consider the standard sigma-model for the embedding of an open string worldsheet Σ into a flat target space with constant metric g, two-form gauge field B and dilaton ϕ. We impose boundary conditions by requiring that the boundary $\partial\Sigma$ is mapped to a submanifold $\mathcal{W}$ of spacetime, which is the worldvolume of a D-brane. At tree-level in open string perturbation theory, Σ is a disk, which can be mapped to the upper complex half-plane by a conformal transformation. The boundary of the upper half-plane with coordinate $t \in \mathbb{R}$ is then mapped to a curve $x^i(t)$ which is the worldline of the end of the string in the D-brane worldvolume $\mathcal{W}$. We are interested in the dynamics of the open string ends located on the D-brane. The two-point function of the x^i on the boundary of the upper complex half-plane is given by [27]

$$\langle x^i(t)\, x^j(t') \rangle = -\alpha'\, G^{ij} \log(t - t')^2 + \tfrac{\mathrm{i}}{2}\, \Theta^{ij}\, \mathrm{sgn}(t - t') \ . \tag{2.1}$$

The metric G and the bivector Θ determine the open string geometry seen by the D-brane, and they are related to the closed string metric g and two-form B by the open-closed string relation

$$G^{-1} + \frac{\Theta}{2\pi\,\alpha'} := \left(g + 2\pi\,\alpha'\,B\right)^{-1}, \tag{2.2}$$

which is equivalent to

$$\begin{aligned} G &= g - (2\pi\,\alpha')^2\, B\, g^{-1}\, B\,, \\ \Theta &= -(2\pi\,\alpha')^2\, (g + 2\pi\,\alpha'\,B)^{-1}\, B\, (g - 2\pi\,\alpha'\,B)^{-1} \ . \end{aligned} \tag{2.3}$$

Of particular interest is the second term in the open string propagator (2.1), which depends only on the ordering of the insertion points of open strings on the boundary of the disk and hence leads to a well-defined target space quantity, independent of the worldsheet coordinates.

In [27] it was shown that there is a consistent decoupling limit where the string slope and closed string metric scale as $\alpha' = O(\epsilon^{1/2})$ and $g_{ij} = O(\epsilon)$, with $\epsilon \to 0$, which decouples the open and closed string modes on the D-brane, and in which the bulk closed string

geometry degenerates to a point. In this limit the first contribution to the propagator (2.1) vanishes, while the open string metric and bivector are finite and are given by

$$G_{\rm D} = -(2\pi\,\alpha')^2\,B\,g^{-1}\,B\,,$$
$$\theta = B^{-1}\,. \tag{2.4}$$

The open string interactions in scattering amplitudes among tachyon vertex operators are captured in this limit by the Moyal-Weyl star-product of fields f, g on the D-brane worldvolume given by

$$f \star g = \cdot \left[\exp\left(\tfrac{\mathrm{i}}{2}\,\theta^{ij}\,\tfrac{\partial}{\partial x^i} \otimes \tfrac{\partial}{\partial x^j}\right) (f \otimes g) \right]\,, \tag{2.5}$$

where $\cdot\,(f \otimes g) = f \cdot g$ is the usual pointwise multiplication of fields. The massless bosonic modes on the D-brane are gauge and scalar fields whose low-energy dynamics in the decoupling limit is described by noncommutative Yang-Mills theory on $\mathcal{W}$. The effective Yang-Mills coupling in the case of a Dp-brane gauge theory can be determined from the Dirac-Born-Infeld action and is generally given by [27]

$$g_{\rm YM}^2 = \frac{(2\pi)^{p-2}}{(\alpha')^{(3-p)/2}}\,g_s\,\mathrm{e}^\phi \left(\frac{\det(g + 2\pi\,\alpha'\,B)}{\det g}\right)^{1/2}\,, \tag{2.6}$$

where g_s is the string coupling. This is finite in the decoupling limit above if $g_s\,\mathrm{e}^\phi = O(\epsilon^{(3-p+r)/4})$, where r is the rank of the antisymmetric matrix B. These considerations can be extended to curved backgrounds with non-constant B-field, including those with non-vanishing H-flux $H = \mathrm{d}B$ [28, 29], in which case the Moyal-Weyl star-product (2.5) is replaced by the more general Kontsevich star-product.

This story becomes particularly interesting in the case when Dp-branes wrap a p-dimensional torus $\mathcal{W} = T^p$. In this case, T-duality of the closed string background translates into open string T-duality which acts on the D-brane charges. The T-duality group $O(p, p; \mathbb{Z})$ acts on the closed string moduli

$$E = \frac{1}{\alpha'}\,\big(g + 2\pi\,\alpha'\,B\big) \tag{2.7}$$

through the fractional linear transformations

$$\widetilde{E} = (a\,E + b)\,(c\,E + d)^{-1} \qquad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(p, p; \mathbb{Z})\,. \tag{2.8}$$

The subgroup $\mathrm{SO}(p, p; \mathbb{Z})$ is a proper symmetry of IIA or IIB string theory; in the decoupling limit, this translates into $\mathrm{SO}(p, p; \mathbb{Z})$ transformations of the open string variables on the Dp-brane given by

$$\widetilde{G}_{\rm D} = (c\,\theta + d)\,G_{\rm D}\,(c\,\theta + d)^\top\,,$$
$$\widetilde{\theta} = (a\,\theta + b)\,(c\,\theta + d)^{-1}\,,$$
$$\widetilde{g}_{\rm YM} = g_{\rm YM}\,\big|\det(c\,\theta + d)\big|^{1/4}\,. \tag{2.9}$$

The remarkable feature is that the noncommutative gauge theory on the Dp-brane inherits this T-duality symmetry. The transformation of the bivector θ on its own is known from topological T-duality to define a Morita equivalence between the corresponding noncommutative tori T^p_θ and $T^p_{\tilde\theta}$, which mathematically preserves their K-theory groups, or more physically the spectrum of D-brane charges on T^p_θ and $T^p_{\tilde\theta}$. Thus open string T-duality in the decoupling limit is a refinement of Morita equivalence, which is referred to as Morita duality of noncommutative gauge theory.

In the mapping of T-duality of the closed string background to Morita equivalence of noncommutative Yang-Mills theory with gauge group U(n), it is generally necessary to introduce a closed two-form [27, 50, 51]

$$\Phi = \frac{1}{2n}\, Q_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \tag{2.10}$$

on the D-brane worldvolume, which can be thought of as an abelian background 't Hooft magnetic flux, where $Q_{ij} \in \mathbb{Z}$ are the Chern numbers of a U(n)-bundle over T^p of constant curvature. The action is then constructed from a shifted form of the noncommutative field strength tensor

$$\mathcal{F} = F_\star + \Phi\, \mathbb{1}_n\,. \tag{2.11}$$

The dependence on Φ simply serves to shift the classical vacuum of the noncommutative gauge theory, giving the fields twisted periodic boundary conditions around the cycles of T^p. Under T-duality, it is required to transform as

$$\widetilde{\Phi} = (c\,\theta + d)\, \Phi\,(c\,\theta + d)^\top + c\,(c\,\theta + d)^\top\,. \tag{2.12}$$

For example, if the components of the noncommutativity bivector θ are rational-valued, then this can be used to provide a Morita equivalence beween noncommutative Yang-Mills theory with periodic gauge fields and ordinary Yang-Mills theory with gauge fields having monodromies in $\mathbb{Z}_n \subset \mathrm{U}(n)$ [51]. The inclusion of Φ also enables one to follow the T-duality orbits of the charges of D-branes wrapping non-contractible cycles of even codimension in the Dp-brane worldvolume, realised as topological charges in the noncommutative gauge theory, which can be suitably arranged into vectors of $\mathrm{SO}(p,p;\mathbb{Z})$ [27, 50].

One purpose of this paper is to investigate this duality in the cases of twisted tori and the non-geometric backgrounds resulting from these under closed string T-duality.

3 Compactification on the twisted torus

In the duality-twisted dimensional reductions of string theory on an n-dimensional twisted torus that we consider here, one first compactifies on an $n-1$-torus T^d, with $d = n-1$. The theory on this internal space is then the conformal field theory with target space T^d, specified by a choice of modulus taking values in the coset space $O(d,d)/O(d) \times O(d)$, which can be represented by a choice of metric g and B-field on the torus T^d. The group $O(d,d)$ then acts naturally on the combination $E = \frac{1}{\alpha'}\,(g + 2\pi\,\alpha'\,B)$ through fractional linear transformations. The automorphism group of the T^d conformal field theory is the

subgroup $O(d, d; \mathbb{Z})$, which is the T-duality group symmetry of the compactified string theory. Configurations related by an $O(d, d; \mathbb{Z})$ transformation are physically equivalent, and so the moduli space $O(d, d)/O(d) \times O(d)$ should be identified under the action of $O(d, d; \mathbb{Z})$.

The next step is to compactify on a further circle S^1 and allow the modulus E of the T^d conformal field theory to depend on the point $x \in S^1$. The x-dependence of $E(x)$ is determined by a map $\gamma : S^1 \to O(d, d)$ given by

$$\gamma(x) = \exp(x\,M)\,, \tag{3.1}$$

for a (dimensionless) mass matrix M in the Lie algebra of $O(d, d)$. This map has monodromy

$$\mathcal{M}(\gamma) = \gamma(0)\,\gamma^{-1}(1) = \exp(M)\,. \tag{3.2}$$

For a consistent string theory background, this monodromy is required to be a symmetry of string theory, and so it must lie in the T-duality group $O(d, d; \mathbb{Z})$ [11, 13]. The condition that this imposes on the mass matrix M can be thought of as a "non-linear quantisation condition".

The map γ is a local section of a principal bundle over S^1 with monodromy $\mathcal{M}(\gamma)$. The moduli of the theory depend on the coordinate x through this section, giving a parameterised family of conformal field theories over S^1 with moduli $E(x)$, so that after a periodic shift $x \mapsto x + 1$ around the base S^1, the conformal field theory returns to itself up to the monodromy $\mathcal{M}(\gamma)$, which is an automorphism of the T^d conformal field theory. Two such bundles are isomorphic, and hence define equivalent theories, if their monodromies lie in the same $O(d, d; \mathbb{Z})$ conjugacy class.

Suppose that the monodromy $\mathcal{M}(\gamma)$ takes values in the geometric subgroup $\mathrm{GL}(d, \mathbb{Z})$ of the duality group consisting of large diffeomorphisms of the torus T^d. In this case we take $\gamma(x)$ in $\mathrm{GL}(d, \mathbb{R}) \subset O(d, d)$, and represent $\gamma(x)$, M and $\mathcal{M}$ by $d \times d$ matrices. Then compactifying on T^d, followed by compactification on S^1 with the duality twist $\mathcal{M}(\gamma)$ amounts to compactification on a torus bundle X over a circle S^1, often referred to as a twisted torus. We denote the local coordinates on this fibration by $(x, y^1, \ldots, y^d)$, where $x \in [0, 1)$ is the coordinate on the base S^1 of radius r and $(y^1, \ldots, y^d) \in [0, 1)^d$ are coordinates on the fibres T^d. The metric is given by

$$\mathrm{d}s_X^2 = (2\pi\,r\,\mathrm{d}x)^2 + h(\tau)_{ab}\,\mathrm{d}y^a\,\mathrm{d}y^b\,, \tag{3.3}$$

where $h(\tau)$ is the metric on the d-torus, which depends on moduli τ taking values in the coset space $\mathrm{GL}(d, \mathbb{R})/\mathrm{SO}(d)$. The x-dependence of τ is defined by

$$h\big(\tau(x)\big)_{ab} = h(\tau^\circ)_{cd}\,\gamma(x)^c{}_a\,\gamma(x)^d{}_b \tag{3.4}$$

for some fixed modulus τ°.

To determine the homologically stable cycles in X which can be wrapped by D-branes, it will prove useful to have another description of these backgrounds. The twisted torus can also be described as the quotient $G_{\mathbb{Z}} \backslash G_{\mathbb{R}}$ of an n-dimensional non-compact Lie group $G_{\mathbb{R}}$ by

a cocompact discrete subgroup $G_{\mathbb{Z}}$, so that much of the local structure of the theory is the same as that for the reduction on the group manifold $G_{\mathbb{R}}$. In particular, the left-invariant Maurer-Cartan forms and the generators of the right action of $G_{\mathbb{R}}$ are well-defined on the compact space $G_{\mathbb{Z}} \backslash G_{\mathbb{R}}$.

The generators $J_1, \ldots, J_d, J_x$ of the Lie algebra of $G_{\mathbb{R}}$ then have brackets

$$[J_a, J_x] = M_a{}^b \, J_b \qquad \text{and} \qquad [J_a, J_b] = 0 \,, \tag{3.5}$$

where $M = (M_a{}^b)$ is the $d \times d$ mass matrix satisfying $\gamma(x) = \exp(x\,M)$, and $G_{\mathbb{R}}$ may be described as a group of $n \times n$ matrices

$$G_{\mathbb{R}} = \left\{ \begin{pmatrix} \gamma^{-1}(x) & y \\ 0 & 1 \end{pmatrix} \,\middle|\, x, y^1, \ldots, y^d \in \mathbb{R} \right\} \,. \tag{3.6}$$

The left action of the discrete subgroup

$$G_{\mathbb{Z}} = \left\{ \begin{pmatrix} \mathcal{M}^{-\alpha} & \beta \\ 0 & 1 \end{pmatrix} \,\middle|\, \alpha, \beta^1, \ldots, \beta^d \in \mathbb{Z} \right\} \tag{3.7}$$

by multiplication on $G_{\mathbb{R}}$ can be expressed in terms of the local coordinates as

$$x \longmapsto x + \alpha \qquad \text{and} \qquad y^a \longmapsto (\mathcal{M}^{-\alpha})^a{}_b \, y^b + \beta^a \,, \tag{3.8}$$

and the resulting quotient

$$X = G_{\mathbb{Z}} \backslash G_{\mathbb{R}} \tag{3.9}$$

is the required twisted torus construction.

The n-manifold X is parallelisable, and the corresponding basis of left-invariant Maurer-Cartan forms is given by

$$\zeta^x = \mathrm{d}x \qquad \text{and} \qquad \zeta^a = \gamma(x)^a{}_b \, \mathrm{d}y^b \,. \tag{3.10}$$

They are globally defined one-forms on the torus bundle which obey the Maurer-Cartan equations

$$\mathrm{d}\zeta^x = 0 \qquad \text{and} \qquad \mathrm{d}\zeta^a + M^a{}_b \, \zeta^x \wedge \zeta^b = 0 \,. \tag{3.11}$$

The metric (3.3) can then be rewritten as the left-invariant metric

$$\mathrm{d}s_X^2 = (2\pi\,r\,\zeta^x)^2 + h(\tau^\circ)_{ab} \, \zeta^a \, \zeta^b \,. \tag{3.12}$$

3.1 SL$(2,\mathbb{Z})$ monodromies

In this paper we will study the examples with $n = 3$ ($d = 2$) in detail. In this case the T-duality group of the string theory compactified on T^2 can be factored as

$$O(2,2;\mathbb{Z}) \simeq \big(\mathrm{SL}(2,\mathbb{Z})_\tau \times \mathrm{SL}(2,\mathbb{Z})_\rho\big) \rtimes \big(\mathbb{Z}_2 \times \mathbb{Z}_2\big) \,. \tag{3.13}$$

The first $\mathrm{SL}(2,\mathbb{Z})$ factor is the mapping class group of T^2 which acts geometrically by fractional linear transformations on the complex structure modulus τ of T^2, while the second acts on the complexified Kähler modulus ρ whose imaginary part is the area of T^2 and whose real part is the restriction of the two-form B-field to T^2. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ factor can be taken to be generated by a reflection in one direction and a T-duality in one direction.

A further compactification on S^1 with the duality twist $\mathcal{M}(\gamma)$ in the geometric subgroup $\mathrm{SL}(2,\mathbb{Z})_\tau$ is equivalent to compactification on a T^2-bundle X over S^1 with monodromy $\mathcal{M}(\gamma)$. The constant metric $h(\tau)_{ab}$ on the T^2 fibers can be written in terms of the complex structure modulus $\tau = \tau_1 + \mathrm{i}\,\tau_2$ and the constant area modulus A of the torus as

$$h(\tau) = \frac{A}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} . \tag{3.14}$$

The torus modulus transforms under $\mathrm{SL}(2,\mathbb{Z})_\tau$ as $\tau \mapsto \mathcal{M}[\tau]$ with

$$\mathcal{M}[\tau] := \frac{a\,\tau + b}{c\,\tau + d} \qquad \text{for} \quad \mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) . \tag{3.15}$$

In the T^2-bundle over S^1, the modulus varies with the circle coordinate x according to the $\mathrm{SL}(2,\mathbb{R})$ transformation

$$\tau(x) = \gamma(x)[\tau^\circ] \tag{3.16}$$

for some fixed modulus τ°, so that $\tau(x + 1) = \mathcal{M}\big[\tau(x)\big]$. The metric on the twisted three-torus X is given by (3.3), which can be rewritten as

$$\mathrm{d}s_X^2 = (2\pi\, r\, \mathrm{d}x)^2 + \frac{A}{\tau_2} \big| \mathrm{d}y^1 + \tau\, \mathrm{d}y^2 \big|^2 . \tag{3.17}$$

In the following we will describe D-branes in non-geometric backgrounds associated with these twisted three-tori. For this, we wrap Dp-branes around suitable p-cycles of X for $p = 1, 2$, which become D$(p+1)$-branes after T-duality to a T-fold background characterised by a monodromy $\mathcal{M}$ in a non-geometric subgroup of the duality group $O(2,2;\mathbb{Z})$. We will study the corresponding noncommutative gauge theory on the D$(p+1)$-branes induced by the metric, B-field and dilaton of the T-fold background, in a scaling limit which decouples open and closed string modes. We shall generally find embeddings into non-geometric string theory of noncommutative Yang-Mills theory whose worldvolume geometry and noncommutativity parameter vary over the base coordinate of a non-geometric "bundle", and hence determine a parameterised family of noncommutative gauge theories which is globally well-defined up to Morita equivalence, the open string avatar of T-duality.

As conjugate monodromies define equivalent backgrounds X, the monodromies leading to physically distinct configurations are classified by $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes [11]. Following [7, 11, 52], the conjugacy classes can be classified into three sets: parabolic ($|\mathrm{Tr}(\mathcal{M})| = 2$), elliptic ($|\mathrm{Tr}(\mathcal{M})| < 2$) and hyperbolic ($|\mathrm{Tr}(\mathcal{M})| > 2$). In this paper we will concentrate on the examples of parabolic monodromies which generate integer translations $\tau \mapsto \tau + m$ of the modular parameter τ, and the $\mathbb{Z}_4$ elliptic monodromies which generate the inversion $\tau \mapsto -\frac{1}{\tau}$. These examples capture many of the essential features of the noncommutative gauge theories on D-branes in non-geometric backgrounds.

4 D3-branes on T-folds: parabolic monodromies

The parabolic conjugacy classes of $\mathrm{SL}(2,\mathbb{Z})$ generate monodromies $\mathcal{M}[\tau] = \tau + m$ of infinite order and are labelled by an integer $m \in \mathbb{Z}$, with mass matrix M and monodromy matrix $\mathcal{M} = \exp(M)$ where

$$\mathcal{M} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad M = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} . \tag{4.1}$$

The local section is given by

$$\gamma(x) = \begin{pmatrix} 1 & m\,x \\ 0 & 1 \end{pmatrix} \qquad \text{with} \quad \tau(x) = \tau^\circ + m\,x , \tag{4.2}$$

where $\tau^\circ = \tau_1^\circ + \mathrm{i}\,\tau_2^\circ$ is some constant modulus, so that

$$\tau(0) = \tau^\circ \qquad \text{and} \qquad \tau(1) = \tau^\circ + m . \tag{4.3}$$

The metric can be brought to the form

$$\mathrm{d}s_X^2 = (2\pi\,r\,\mathrm{d}x)^2 + \frac{A}{\tau_2^\circ} \left(\mathrm{d}y^1 + \omega \right)^2 + A\,\tau_2^\circ \left(\mathrm{d}y^2 \right)^2 , \tag{4.4}$$

where $\omega := (\tau_1^\circ + m\,x)\,\mathrm{d}y^2$. This identifies the twisted torus X in this case as a circle bundle over T^2 of degree m, with fibre coordinate y^1 and base coordinates (x, y^2), while ω is a connection on this bundle with Chern number m. The B-field vanishes and the dilaton is constant in this background.

In this case $G_\mathbb{R}$ is the three-dimensional Heisenberg group whose generators satisfy the Heisenberg algebra

$$[J_1, J_x] = m\,J_2 \qquad \text{and} \qquad [J_1, J_2] = 0 = [J_x, J_2] . \tag{4.5}$$

Then the quotient by the discrete group action

$$x \longmapsto x + \alpha , \quad y^1 \longmapsto y^1 - \alpha\,m\,y^2 + \beta^1 \qquad \text{and} \qquad y^2 \longmapsto y^2 + \beta^2 \tag{4.6}$$

for $\alpha, \beta^1, \beta^2 \in \mathbb{Z}$ is the three-dimensional Heisenberg nilmanifold. A globally defined basis of one-forms on the nilfold is given by

$$\zeta^x = \mathrm{d}x , \quad \zeta^1 = \mathrm{d}y^1 + m\,x\,\mathrm{d}y^2 \qquad \text{and} \qquad \zeta^2 = \mathrm{d}y^2 . \tag{4.7}$$

The Maurer-Cartan equations

$$\mathrm{d}\zeta^x = 0 = \mathrm{d}\zeta^2 \qquad \text{and} \qquad \mathrm{d}\zeta^1 = m\,\zeta^x \wedge \zeta^2 \tag{4.8}$$

imply that $H^1(X, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$ is generated by ζ^x and ζ^2. By Poincaré duality, the second homology $H_2(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by the two-cycles $\xi_{x,1}$ and $\xi_{1,2}$ dual to $\zeta^x \wedge \zeta^1$ and $\zeta^1 \wedge \zeta^2$, and in particular the two-cycle $\xi_{x,2}$ dual to $\zeta^x \wedge \zeta^2$ is homologically trivial [18]. On the other hand, from the Gysin sequence for X viewed as a circle bundle it follows that

$H_1(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_m$, where the $\mathbb{Z}$-valued classes are the one-cycles ξ_x and ξ_2 dual to ζ^x and ζ^2 on the T^2 base, while the $\mathbb{Z}_m$ torsion one-cycle ξ_1 is the class of the y^1 circle fiber.

This background is T-dual to a flat three-torus T^3 with H-flux: Applying the Buscher construction along the abelian isometry generated by the global vector field $\frac{\partial}{\partial y^1}$ on X (see appendix A), T-duality maps the metric (4.4) to the metric and B-field

$$
\begin{aligned}
\mathrm{d}s^2_{T^3} &= (2\pi r \, \mathrm{d}x)^2 + \frac{(2\pi \alpha')^2 \tau_2^\circ}{A} \left(\mathrm{d}y^1\right)^2 + A \, \tau_2^\circ \left(\mathrm{d}y^2\right)^2, \\
B_{T^3} &= (\tau_1^\circ + m \, x) \, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \,.
\end{aligned}
\tag{4.9}
$$

The B-field gives a constant H-flux

$$
H_{T^3} = \mathrm{d}B_{T^3} = m \, \mathrm{d}x \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2
\tag{4.10}
$$

on T^3. This has a monodromy in $\mathrm{SL}(2, \mathbb{Z})_\rho$,

$$
\mathcal{M}[\rho] = \rho + m \,,
\tag{4.11}
$$

giving a shift in B_{T^3} by $m \, \mathrm{d}y^1 \wedge \mathrm{d}y^2$, which represents an integral cohomology class.

4.1 Worldvolume geometry

Let us now wrap a D2-brane around the non-trivial two-cycle $\xi_{x,1}$. T-duality in the y^1-direction then maps the D2-brane to a D1-brane wrapped around the one-cycle dual to ζ^x in the flat three-torus T^3 with metric and B-field given by (4.9), and constant H-flux (4.10). These are both allowed D-brane configurations, according to the doubled torus analysis of [18].

On the other hand, we can consider T-duality along the abelian (covering space) isometry generated by the vector field $\frac{\partial}{\partial y^2}$ which maps the D2-brane to a D3-brane filling the T-fold. The metric and B-field are given by

$$
\begin{aligned}
g &= (2\pi r \, \mathrm{d}x)^2 + \frac{\tau_2(x)}{|\tau(x)|^2} \left(A \left(\mathrm{d}y^1\right)^2 + \frac{(2\pi \alpha')^2}{A} \left(\mathrm{d}y^2\right)^2 \right), \\
B &= \frac{\tau_1(x)}{|\tau(x)|^2} \, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \,,
\end{aligned}
\tag{4.12}
$$

together with the dilaton field

$$
\mathrm{e}^{\phi(x)} = \left(\frac{2\pi \alpha'}{A} \frac{\tau_2(x)}{|\tau(x)|^2} \right)^{1/2} .
\tag{4.13}
$$

where

$$
\tau(x) = \tau_1^\circ + m \, x + \mathrm{i} \, \tau_2^\circ \,.
\tag{4.14}
$$

The area of the T^2 fibres with coordinates (y^1, y^2) is

$$
\mathcal{A} = 2\pi \alpha' \frac{\tau_2(x)}{|\tau(x)|^2} \,.
\tag{4.15}
$$

Then the Kähler modulus of the T^2 fibres is

$$\rho := B_{12} + \frac{\mathrm{i}}{2\pi\,\alpha'}\,\mathcal{A} = \frac{\tau_1(x) + \mathrm{i}\,\tau_2(x)}{|\tau(x)|^2} = \frac{1}{\bar{\tau}(x)} \tag{4.16}$$

so that, as $\tau(x+1) = \tau(x) + m$, this is a T-fold with monodromy

$$\rho(x+1) = \frac{\rho(x)}{1 + m\,\rho(x)} \tag{4.17}$$

in $\mathrm{SL}(2,\mathbb{Z})_\rho$.

Let us now transform to the open string variables seen by the D3-brane [27, 28]. These are the open string metric G and noncommutativity bivector Θ defined from (4.12) through (2.2). Explicit calculation from (4.12) gives a worldvolume $\mathcal{W}_{\mathrm{D3}}$ with the topology of $S^1 \times T^2$ and

$$G = (2\pi\,r\,\mathrm{d}x)^2 + \frac{A}{\tau_2^\circ}\left(\mathrm{d}y^1\right)^2 + \frac{(2\pi\,\alpha')^2}{A\,\tau_2^\circ}\left(\mathrm{d}y^2\right)^2,$$

$$\Theta = (\tau_1^\circ + m\,x)\,\frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2}. \tag{4.18}$$

4.2 Noncommutative Yang-Mills theory

In order to get a low-energy limit with pure gauge theory on $\mathcal{W}_{\mathrm{D3}}$ in which the massive string modes are decoupled and gravity is non-dynamical, we need to take the zero slope limit $\alpha' \to 0$ while keeping G and Θ fixed, which in the present case means keeping the parameters

$$r_1 := \left(\frac{A}{4\pi^2\,\tau_2^\circ}\right)^{1/2} \qquad \text{and} \qquad r_2 := \frac{\alpha'}{(A\,\tau_2^\circ)^{1/2}} \tag{4.19}$$

fixed. This can be achieved by the scaling limit $\alpha' = O(\epsilon^{1/2})$, $A = O(\epsilon^{1/2})$ and $\tau_2^\circ = O(\epsilon^{1/2})$ with $\epsilon \to 0$, and with all other parameters, including the B-field, held fixed. In this limit the closed string metric from (4.12) degenerates along T^2, taking the area to zero while fixing B, whereas the open string parameters on the D3-brane become

$$\mathrm{d}s_{\mathrm{D3}}^2 = (2\pi\,r\,\mathrm{d}x)^2 + \left(2\pi\,r_1\,\mathrm{d}y^1\right)^2 + \left(2\pi\,r_2\,\mathrm{d}y^2\right)^2,$$

$$\theta = (\tau_1^\circ + m\,x)\,\frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2}, \tag{4.20}$$

where in the particular instance of a parabolic twist the open string bivector Θ from (4.18) and its zero slope limit θ happen to coincide.

Finally, the effective Yang-Mills coupling can be determined from (2.6), which in the present case with $p = 3$ is the constant

$$g_{\mathrm{YM}}^2 = \left(\frac{(2\pi)^3\,\alpha'\,g_s^2}{A\,\tau_2^\circ}\right)^{1/2}. \tag{4.21}$$

In order to obtain a well-defined quantum gauge theory, we thus require that

$$\bar{g}_s^2 := \frac{2\pi \, \alpha' \, g_s^2}{A \, \tau_2^\circ} \tag{4.22}$$

remains finite in the zero slope limit, which implies that the string coupling scales as $g_s = O(\epsilon^{1/4})$, which is consistent with the perturbative regime of the string theory that we are working in. Then

$$g_{\mathrm{YM}}^2 = 2\pi \, \bar{g}_s \tag{4.23}$$

is indeed finite in the limit $\alpha' \to 0$.

Since $\frac{\partial}{\partial y^a}\theta = 0$, the supersymmetric noncommutative Yang-Mills theory on the D3-brane is defined by multiplying fields f, g on $S^1 \times T^2$ together with the Kontsevich star-product [21] (see also [53])

$$f \star g = \cdot \left[\exp\left(\tfrac{\mathrm{i}}{2} \, \theta(x) \left(\tfrac{\partial}{\partial y^1} \otimes \tfrac{\partial}{\partial y^2} - \tfrac{\partial}{\partial y^2} \otimes \tfrac{\partial}{\partial y^1} \right) \right) (f \otimes g) \right] . \tag{4.24}$$

Defining $[f, g]_\star := f \star g - g \star f$, this gives a quantisation of the three-dimensional Heisenberg algebra

$$[y^1, y^2]_\star = \mathrm{i}\, \theta(x) \qquad \text{and} \qquad [y^1, x]_\star = 0 = [y^2, x]_\star \, . \tag{4.25}$$

For fixed $x \in S^1$ the star-product (4.24) defines a noncommutative torus $T^2_{\theta(x)}$, which means geometrically that varying $x \in S^1$ determines a field of noncommutative tori in the D3-brane worldvolume $\mathcal{W}_{\mathrm{D3}}$ [46, 53].

The noncommutative torus $T^2_{\theta(x)}$ has Morita equivalence group

$$\mathrm{SO}(2,2;\mathbb{Z}) \simeq \big(\mathrm{SL}(2,\mathbb{Z})_\theta \times \mathrm{SL}(2,\mathbb{Z})_\tau \big)/\mathbb{Z}_2 \, . \tag{4.26}$$

(The $\mathbb{Z}_2$ factor is generated by $(-\mathbb{1}, -\mathbb{1}) \in \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$.) Morita transformations in this group leave invariant the noncommutative gauge theory on the D3-brane filling the T-fold compactification. Here the roles of the two $\mathrm{SL}(2,\mathbb{Z})$ factors in the original duality group (3.13) are interchanged: The $\mathrm{SL}(2,\mathbb{Z})_\rho$ factor in (3.13) now acts on the D3-brane torus with (4.20) as the geometric action of the mapping class group $\mathrm{SL}(2,\mathbb{Z})_\tau$ of the T^2 'fibres' with coordinates (y^1, y^2), leaving θ, the Yang-Mills coupling g_{YM} and the area $V = 4\pi^2 \, r_1 \, r_2$ of the torus T^2 with the metric $\mathrm{d}s_{\mathrm{D3}}^2$ unchanged, while the $\mathrm{SL}(2,\mathbb{Z})_\tau$ factor in (3.13) now acts as the $\mathrm{SL}(2,\mathbb{Z})_\theta$ Morita transformations

$$\begin{aligned} \mathcal{M}[V] &= V \, (c\,\theta + d)^2 \, , \\ \mathcal{M}[\theta] &= \frac{a\,\theta + b}{c\,\theta + d} \, , \\ \mathcal{M}[g_{\mathrm{YM}}] &= g_{\mathrm{YM}} \, |c\,\theta + d|^{1/2} \, . \end{aligned} \tag{4.27}$$

This is the old statement [27] that Morita equivalence is precisely the structure inherited from T-duality in the decoupling limit.

Thus by wrapping a D3-brane we gain an alternative perspective on the non-geometric nature of the T-fold background in terms of noncommutative gauge theory: under a monodromy around the circle coordinate x, the noncommutativity parameter transforms as $\theta(x+1) = \theta(x) + m$, which is precisely an $\mathrm{SL}(2,\mathbb{Z})_\theta$ Morita transformation by the pertinent monodromy matrix (4.1)

$$\theta(x+1) = \mathcal{M}\big[\theta(x)\big] \ . \tag{4.28}$$

The identification of monodromies in x via T-duality in the closed string sector is realised in the open string sector via Morita equivalence in Yang-Mills theory on a noncommutative torus. Under the parabolic monodromy, all other parameters of the gauge theory, including the open string metric $\mathrm{d}s_{\mathrm{D3}}^2$, the area V and the Yang-Mills coupling constant g_{YM}, are invariant.

To summarise, in the case of parabolic twists we have found that although closed strings see non-geometry, open strings see an undeformed conventional geometric torus but the original closed string non-geometry is now reflected in the noncommutativity bivector θ in the dual gauge theory description of D3-branes as a θ-deformed noncommutative supersymmetric Yang-Mills theory. The T-duality monodromy for the geometric moduli of the closed string geometry is mapped to a Morita monodromy for the moduli of noncommutative Yang-Mills theory.

4.3   Interpretation of the decoupling limit

We can give a physical derivation of this noncommutative gauge theory by adapting the description of [26] which considered the case of vanishing fluxes and constant B-field. The essential features can be seen already in the low-energy effective theory of a D1-brane on the twisted torus X wrapping the torsion one-cycle ξ_1, and placed at $y^2 = 0$ and any fixed point $x \in S^1$. We can think of the original torus fibres T^2 of X as the complex plane $\mathbb{C}$, with coordinate $z = y^1 + \mathrm{i}\, y^2$, quotiented by the translations $z \mapsto z + \alpha$ and $z \mapsto z + \beta\, \tau(x)$ for $\alpha, \beta \in \mathbb{Z}$. In the scaling limit $\tau_2^\circ \to 0$ taken above, the torus fibre degenerates to the flat cylinder $S^1 \times \mathbb{R}$ with coordinate $y^1 \in [0, 1)$ quotiented by the additional translations $y^1 \mapsto y^1 + \beta\, \theta(x)$; this is not a conventional Hausdorff space for generic values of $x \in S^1$, but can be precisely interpreted as the noncommutative torus $T_{\theta(x)}^2$, which for irrational values of $\theta(x)$ is sometimes called the 'irrational rotation algebra'. In this geometric picture, the Morita invariance under parabolic monodromies around the base circle is trivially realised as the equality $T_{\theta(x+1)}^2 = T_{\theta(x)+m}^2 = T_{\theta(x)}^2$ under the identification of the periodic coordinate y^1 with $y^1 + m$.

In the gauge theory on the D1-brane, there are additional light states formed by strings winding w^2 times around y^2, viewed as open strings connecting the D1-brane and its images on the covering space over y^2, which have mass proportional to $w^2\, \tau_2^\circ$. The complete low-energy spectrum for $\tau_2^\circ \to 0$ is thus obtained by considering fields $f_{w^2}(y^1)$ with an arbitrary dependence on both $y^1 \in [0, 1)$ and on $w^2 \in \mathbb{Z}$. The open string starting at $(y^1, 0)$ ends at $(y^1, w^2\, \tau_2^\circ)$, which is identified with the point $(y^1 - w^2\, \theta(x), 0)$ on the twisted torus. Since open strings interact via concatenation of paths, in (y^1, w^2) space the interaction of two

fields f_{w^2} and $\tilde{f}_{\tilde{w}^2}$ is given by

$$f_{w^2}(y^1)\,\tilde{f}_{\tilde{w}^2}\big(y^1 - w^2\,\theta(x)\big) = f_{w^2}(y^1)\,\exp\left(-w^2\,\theta(x)\,\frac{\partial}{\partial y^1}\right)\tilde{f}_{\tilde{w}^2}(y^1)\,. \qquad (4.29)$$

By T-duality along the vector field $\frac{\partial}{\partial y^2}$, which maps the winding number w^2 to a momentum mode p_2, followed by the usual Fourier transform of $p_2 = -\mathrm{i}\,\frac{\partial}{\partial y^2}$ to y^2, this interaction is given by the noncommutative star-product $f \star \tilde{f}$ from (4.24) in the gauge theory on the D2-brane in the dual T-fold frame; in particular, this shows that the star-product (4.24) is invariant under the monodromy $\theta(x+1) = \theta(x) + m$. The open string metric on the D2-brane in the scaling limit is obtained from (4.20) with all other parameters as above and is the metric on a flat square torus with radii r_1, r_2. This gives a family of D2-brane gauge theories on $T^2_{\theta(x)}$ parameterised by $x \in S^1$, such that after a monodromy $x \mapsto x + 1$ the noncommutative gauge theory returns to itself up to Morita equivalence, which is a symmetry of the theory; in particular, this leaves the noncommutative Yang-Mills action S^{YM}_x invariant: $S^{\mathrm{YM}}_{x+1} = S^{\mathrm{YM}}_x$. The fibre over x of this parameterised family of noncommutative gauge theories is dual to the low-energy effective theory of a D0-brane placed at $y^1 = y^2 = 0$ and $x \in S^1$ on the three-torus T^3 with constant H-flux (4.10).

5 D2-branes on T-folds: elliptic monodromies

The elliptic conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ are matrices $\mathcal{M} = \exp(M)$ that are conjugate to rotations, so that they are of the form

$$\mathcal{M} = U \begin{pmatrix} \cos(m\,\vartheta) & \sin(m\,\vartheta) \\ -\sin(m\,\vartheta) & \cos(m\,\vartheta) \end{pmatrix} U^{-1} \qquad \text{and} \qquad M = U \begin{pmatrix} 0 & m\,\vartheta \\ -m\,\vartheta & 0 \end{pmatrix} U^{-1}, \quad (5.1)$$

where $U \in \mathrm{SL}(2,\mathbb{R})$, the angle $\vartheta \in (0,\pi]$ and $m \in \mathbb{Z}$. The elliptic conjugacy classes of $\mathrm{SL}(2,\mathbb{Z})$ are matrices of integers that are in elliptic conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$. This is highly restrictive, and the only angles for which there is a U such that $\mathcal{M}$ is integer-valued are $\vartheta = \pi, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}$. These give matrices of finite order, generating the cyclic groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ respectively, which provide the four possible choices of elliptic monodromies.

For $\vartheta = \pi$ (and $m \in 2\mathbb{Z}+1$) and $\vartheta = \frac{\pi}{2}$ (and $m \in 4\mathbb{Z}+1$) the required conjugation is trivial, $U = \mathbb{1}$. These $\mathrm{SL}(2,\mathbb{Z})$ transformations then act on τ by $\mathcal{M}[\tau] = \tau$ and $\mathcal{M}[\tau] = -\frac{1}{\tau}$ respectively. For $\vartheta = \frac{2\pi}{3}$ and $\vartheta = \frac{\pi}{3}$ (with $m = 1$), the conjugation matrix is

$$U = \sqrt{\tfrac{2}{\sqrt{3}}} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}. \qquad (5.2)$$

These generate $\mathcal{M}[\tau] = -\frac{1}{\tau+1}$ and $\mathcal{M}[\tau] = -\frac{\tau+1}{\tau}$ respectively.

The local section is given by

$$\gamma(x) = U \begin{pmatrix} \cos(m\,\vartheta\,x) & \sin(m\,\vartheta\,x) \\ -\sin(m\,\vartheta\,x) & \cos(m\,\vartheta\,x) \end{pmatrix} U^{-1}\,. \qquad (5.3)$$

For definiteness, we now focus our discussion on the case of $\mathbb{Z}_4$ monodromies with $U = \mathbb{1}$, $\vartheta = \frac{\pi}{2}$ and $m \in 4\mathbb{Z} + 1$ so that

$$\mathcal{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{5.4}$$

Then the complex structure modulus is

$$\tau(x) = \frac{\tau^\circ \cos(m\,\vartheta\,x) + \sin(m\,\vartheta\,x)}{-\tau^\circ \sin(m\,\vartheta\,x) + \cos(m\,\vartheta\,x)} , \tag{5.5}$$

with

$$\tau(0) = \tau^\circ \qquad \text{and} \qquad \tau(1) = -\frac{1}{\tau^\circ} . \tag{5.6}$$

In this case $G_{\mathbb{R}} = \mathrm{ISO}(2)$ is the isometry group of the Euclidean plane $\mathbb{R}^2$ whose generators satisfy

$$[J_1, J_x] = m\,\vartheta\,J_2 , \qquad [J_2, J_x] = -m\,\vartheta\,J_1 \qquad \text{and} \qquad [J_1, J_2] = 0 . \tag{5.7}$$

The group manifold of $\mathrm{ISO}(2)$ has the topology of $S^1 \times \mathbb{R}^2$ which is compactified by the discrete group action

$$y^a \longmapsto y^a + \beta^a , \tag{5.8}$$

where $\beta^a \in \mathbb{Z}$ for $a = 1, 2$; then X is topologically $S^1 \times T^2$. For $U = \mathbb{1}$, the Maurer-Cartan equations

$$\mathrm{d}\zeta^x = 0 , \qquad \mathrm{d}\zeta^1 = -m\,\vartheta\,\zeta^x \wedge \zeta^2 \qquad \text{and} \qquad \mathrm{d}\zeta^2 = m\,\vartheta\,\zeta^x \wedge \zeta^1 \tag{5.9}$$

imply that $H^1(X, \mathbb{R}) = \mathbb{R}$ is generated by ζ^x. By Poincaré duality it follows that $H_2(X, \mathbb{Z}) = \mathbb{Z}$ is generated by $\xi_{1,2}$, and in particular now both $\xi_{x,1}$ and $\xi_{x,2}$ are homologically trivial two-cycles. On the other hand, for the $\mathbb{Z}_4$ monodromy, $H_1(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$ is generated by the $\mathbb{Z}$-valued S^1 base one-cycle ξ_x dual to ζ^x, with the $\mathbb{Z}_2$ torsion one-cycle ξ_1 given by the class of the y^1 circle fibre [18].

The metric is given by (3.12). For the parabolic monodromy, T-dualising in y^1 gave a T^3 with H-flux while T-dualising in y^2 gave a T-fold, but for this elliptic case, dualising in either y^1 or y^2 gives the same result, which is a T-fold with H-flux. Starting with the twisted torus metric (3.17), we apply the Buscher construction along the abelian isometry generated by the vector field $\frac{\partial}{\partial y^2}$ to get a non-geometric background with metric and B-field given by

$$g = (2\pi\, r\, \mathrm{d}x)^2 + \frac{\tau_2(x)}{|\tau(x)|^2} \left(A \left(\mathrm{d}y^1\right)^2 + \frac{(2\pi\,\alpha')^2}{A} \left(\mathrm{d}y^2\right)^2 \right) ,$$

$$B = \frac{\tau_1(x)}{|\tau(x)|^2} \,\mathrm{d}y^1 \wedge \mathrm{d}y^2 , \tag{5.10}$$

together with the dilaton field

$$e^{\phi(x)} = \left(\frac{2\pi\,\alpha'}{A} \, \frac{\tau_2(x)}{|\tau(x)|^2} \right)^{1/2} . \tag{5.11}$$

Here the Kähler modulus of the T^2 fibres with coordinates (y^1, y^2) is

$$\rho(x) = \frac{1}{\bar{\tau}(x)} \tag{5.12}$$

so that this is a T-fold with monodromy

$$\rho(x+1) = -\frac{1}{\rho(x)} \tag{5.13}$$

in $\mathrm{SL}(2, \mathbb{Z})_\rho$.

5.1 Worldvolume geometry

Unlike the parabolic case, here we cannot wrap a D2-brane on the base S^1 of the twisted torus. Moreover, unlike the parabolic case, T-dualising the twisted torus with $\mathbb{Z}_4$ elliptic monodromy in either of the torus fibre directions results in something non-geometric. Instead, we wrap a D1-brane around the torsion one-cycle ξ_1 as we did in section 4.3. T-dualising y^2 gives a D2-brane in the T-fold background with metric and B-field in (5.10), and dilaton in (5.11). Transforming now to the open string metric and noncommutativity bivector on the D2-brane using (2.2) we find

$$G = \frac{A}{\tau_2(x)} \left(\mathrm{d}y^1 \right)^2 + \frac{(2\pi\,\alpha')^2}{A\,\tau_2(x)} \left(\mathrm{d}y^2 \right)^2 ,$$
$$\Theta = \tau_1(x) \, \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2} . \tag{5.14}$$

5.2 Noncommutative Yang-Mills theory

In the zero slope limit with the radii (4.19) held fixed, the closed string metric from (5.10) is again degenerate, while the decoupled open string noncommutative geometry is described by the metric and bivector

$$\mathrm{d}s_{\mathrm{D2}}^2 = \left(-\tau_1^\circ \sin(m\,\vartheta\,x) + \cos(m\,\vartheta\,x) \right)^2 \left((2\pi\,r_1)^2 \left(\mathrm{d}y^1 \right)^2 + (2\pi\,r_2)^2 \left(\mathrm{d}y^2 \right)^2 \right) ,$$
$$\theta = \frac{\tau_1^\circ \cos(m\,\vartheta\,x) + \sin(m\,\vartheta\,x)}{-\tau_1^\circ \sin(m\,\vartheta\,x) + \cos(m\,\vartheta\,x)} \, \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2} . \tag{5.15}$$

Again since $\frac{\partial}{\partial y^a}\theta = 0$, the star-product incorporating the dynamics of open strings in this background is given in the same form (4.24) quantising the three-dimensional algebra (4.25), which however is no longer based on a Lie algebra but rather some quantum deformation of the Heisenberg Lie algebra determined by the discrete parameters $m \in 4\,\mathbb{Z}+1$ and $\vartheta = \frac{\pi}{2}$.

As before, the non-geometric nature of the closed string background is captured in the noncommutative gauge theory on the D2-brane via Morita equivalence. Under a monodromy in the circle coordinate x, the noncommutativity parameter transforms in the

expected way from (4.27) under an $\mathrm{SL}(2,\mathbb{Z})_\theta$ Morita transformation corresponding to the elliptic monodromy matrix (5.1):

$$\theta(x+1) = \frac{\cos(m\,\vartheta)\,\theta(x) + \sin(m\,\vartheta)}{-\sin(m\,\vartheta)\,\theta(x) + \cos(m\,\vartheta)} = \mathcal{M}\big[\theta(x)\big]\ . \qquad (5.16)$$

For $m \in 4\,\mathbb{Z}+1$ and $\vartheta = \frac{\pi}{2}$ this reduces to

$$\theta(x+1) = -\frac{1}{\theta(x)}\ . \qquad (5.17)$$

However, in contrast with the case of parabolic twists, here the metric on the D2-brane worldvolume $\mathcal{W}_{\mathrm{D2}}$ is not globally well-defined, so that the open string sector now simultaneously probes both a non-geometric and a noncommutative space. This is exactly what is needed to compensate the Morita equivalence of the corresponding noncommutative fibre tori $T^2_{\theta(x)}$ and render the noncommutative Yang-Mills theory on $\mathcal{W}_{\mathrm{D2}}$ invariant; in particular, the area of the non-geometric worldvolume

$$V(x) = 4\pi^2\,r_1\,r_2\,\big(-\tau_1^\circ \sin(m\,\vartheta\,x) + \cos(m\,\vartheta\,x)\big)^2 \qquad (5.18)$$

transforms under a monodromy $x \mapsto x+1$ in the expected way from (4.27) under the Morita duality corresponding to (5.1):

$$V(x+1) = V(x)\,\big(-\sin(m\,\vartheta)\,\theta(x) + \cos(m\,\vartheta)\big)^2 = \mathcal{M}\big[V(x)\big]\ . \qquad (5.19)$$

For $m \in 4\,\mathbb{Z}+1$ and $\vartheta = \frac{\pi}{2}$ this reduces to

$$V(x+1) = V(x)\,\theta(x)^2\ . \qquad (5.20)$$

Finally, the Yang-Mills coupling of the decoupled noncommutative gauge theory in the non-geometric T^2-bundle over S^1 is x-dependent and is computed from (2.6) with $p=2$ to get

$$g_{\mathrm{YM}}(x)^2 = \left(\frac{2\pi\,g_s^2}{A\,\tau_2(x)}\right)^{1/2}\ . \qquad (5.21)$$

In this case, it is the combination

$$\bar{g}_s^2 := \frac{g_s^2}{2\pi\,A\,\tau_2^\circ} \qquad (5.22)$$

which must be fixed in the zero slope limit, so that now the string coupling scales as $g_s = O(\epsilon^{1/2})$. Then the Yang-Mills coupling in the zero slope limit is still x-dependent and given by

$$g_{\mathrm{YM}}(x)^2 = 2\pi\,\bar{g}_s\,\big|-\tau_1^\circ \sin(m\,\vartheta\,x) + \cos(m\,\vartheta\,x)\big|\ . \qquad (5.23)$$

Hence the Yang-Mills coupling also transforms in the expected way from (4.27) under the Morita duality corresponding to (5.1):

$$g_{\mathrm{YM}}(x+1) = g_{\mathrm{YM}}(x)\,\big|-\sin(m\,\vartheta)\,\theta(x) + \cos(m\,\vartheta)\big|^{1/2} = \mathcal{M}\big[g_{\mathrm{YM}}(x)\big]\ . \qquad (5.24)$$

For $m \in 4\,\mathbb{Z} + 1$ and $\vartheta = \frac{\pi}{2}$ this is

$$g_{\mathrm{YM}}(x + 1) = g_{\mathrm{YM}}(x)\, \big| \theta(x) \big|^{1/2} \ . \tag{5.25}$$

Thus again we obtain a family of noncommutative D2-brane gauge theories on $T^2_{\theta(x)}$ parameterised by $x \in S^1$. We stress that there is a well-defined action of the Morita duality transformations on the gauge theory: all three moduli — the area, the noncommutativity parameter and the gauge coupling — transform in a way that conspires to leave the gauge theory invariant. The underlying noncommutative geometries here are new and generalise the field of noncommutative tori obtained previously in the case of parabolic monodromies.

5.3 Interpretation of the Morita equivalence monodromy

The case of elliptic mondromies exhibits another new feature compared to the case of parabolic twists. Recall that the parabolic monodromy affects only the noncommutativity bivector $\theta(x)$ and is an exact invariance of the noncommutative torus at the topological level, $T^2_{\theta(x+1)} = T^2_{\theta(x)}$; in particular the star-product (4.24) is invariant under integer shifts $\theta(x+1) = \theta(x) + m$. This is no longer true for the $\mathbb{Z}_4$ elliptic twist, which requires the full machinery of Morita equivalence of noncommutative tori to explain the invariance of the Yang-Mills theory on the D2-brane; this necessitates, in particular, the non-trivial actions of the elliptic monodromy on the remaining moduli of the gauge theory described above.

We can give a physical picture for this distinction by including the background magnetic flux modulus Φ which twists the vacuum of the noncommutative gauge theory as discussed in section 2. It shifts the noncommutative field strength tensor $F_\star$ defining the Yang-Mills action by a closed two-form Φ on the D2-brane worldvolume to give

$$\mathcal{F} = F_\star + \Phi \ . \tag{5.26}$$

Under a Morita transformation (4.27) with monodromy $\mathcal{M}$, the magnetic flux Φ transforms as

$$\mathcal{M}[\Phi] = -(c\,\theta + d)^2\, \Phi + c\,(c\,\theta + d) \ . \tag{5.27}$$

This also affects the spectrum of D-brane charges on the T^2 fibres: a generic configuration $(n, q) \in \mathbb{Z}^2$ of n D2-branes wrapping T^2 with q units of D0-brane charge, realised as a background magnetic charge q in $\mathrm{U}(n)$ noncommutative Yang-Mills theory, transforms as a vector under $\mathrm{SL}(2, \mathbb{Z})_\theta$ to

$$\mathcal{M}\left[\begin{pmatrix} n \\ q \end{pmatrix} \right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n \\ q \end{pmatrix} \ . \tag{5.28}$$

For the parabolic monodromies (4.1) this has no effect; in that case $\Phi(x+1) = \Phi(x)$ and the single D2-brane charge with $(n, q) = (1, 0)$ that we have considered is mapped to itself. Thus we can consistently set $\Phi = 0$ for the parabolic case and simply write the standard noncommutative Yang-Mills action in terms of $F_\star$ and single-valued gauge fields on the T^2 fibre.

In contrast, for the $\mathbb{Z}_4$ elliptic monodromy (5.4) the magnetic flux Φ on the D2-brane transforms as

$$\Phi(x+1) = -\theta(x)^2\,\Phi(x) + \theta(x)\,, \tag{5.29}$$

while the single D2-brane charge $(1,0)$ that we started with is mapped to a single D0-brane charge $(0,-1)$; the notation (n,q), as above, denotes n units of D2-brane charge and q units of D0-brane charge. This is solved by

$$\Phi = -\frac{x}{\theta(x)}\;\mathrm{d}y^1 \wedge \mathrm{d}y^2\,, \tag{5.30}$$

which should be properly incorporated in (5.26) in order to obtain the correct compactification of the open string non-geometric background via shifts in x. Choosing the real part of the fixed modular parameter τ° to be $\tau_1^\circ = -1$, the two-form (5.30) then correctly interpolates between the D0-brane charges $\Phi(0) = 0$ and $\Phi(1) = -1$, implying that single-valued gauge fields are mapped to multi-valued gauge fields on the D2-brane under the monodromy in the base coordinate $x \in S^1$ [51].

This effect may be interpreted in terms of open string boundary conditions. By observing that the bivector $\theta(x)$ is constant on the D2-brane worldvolume, we can easily adapt the description of open string ground states given in [27] as Morita equivalence bimodules for a noncommutative torus, as is also done by [21] in a different context. We start with a single D2-brane placed at some fixed point $x \in S^1$, wrapping the T^2 fibre. Since open strings interact by concatenation of paths, the space of open string ground states on the D2-brane forms an algebra $\mathcal{A}_x$ descending from the algebra of open string tachyon vertex operators in the decoupling limit, which is precisely the algebra of functions on the noncommutative torus $T^2_{\theta(x)}$ with the star-product (4.24).

Consider now a string winding once around the x-circle, viewed in the covering space $\mathbb{R}$ over S^1 as an open string stretching with its left end on the $(1,0)$ D2-brane at the point x and with its right end on some other configuration (n,q) of D2–D0-branes placed at the point $x+1$, with the n D2-branes wrapping T^2 and the q D0-branes located at $y^1 = y^2 = 0$ on T^2. The configuration (n,q) comes with its own algebra $\mathcal{A}_{x+1}$ of open string ground states, identified as the algebra of functions on $T^2_{\theta(x+1)}$ for a suitable monodromy (4.27) of the noncommutativity parameter determined by the transformation (5.28) of D-brane charges. Quantisation of the stretched open string with these boundary conditions in the decoupling limit gives a space of states $\mathcal{H}_{n,q}$ which is a left module for the algebra $\mathcal{A}_x$, acting on the left end of the open string, and a right module for $\mathcal{A}_{x+1}$, acting on the right end of the open string. The actions of $\mathcal{A}_x$ and $\mathcal{A}_{x+1}$ commute because they act at opposite ends of the open string, and together they generate the complete algebra of observables on the open string tachyon ground states, acting irreducibly on $\mathcal{H}_{n,q}$. This implies that the algebra $\mathcal{A}_{x+1}$ is the commutant of $\mathcal{A}_x$ in this space (and vice-versa), the maximal algebra of all operators on $\mathcal{H}_{n,q}$ that commute with $\mathcal{A}_x$. In [27] it is shown that the space $\mathcal{H}_{n,q}$ thus defines a Morita equivalence bimodule over $\mathcal{A}_x \times \mathcal{A}_{x+1}$ in this sense, which mathematically implements the Morita equivalence between noncommutative tori whose algebras of functions are $\mathcal{A}_x$ and $\mathcal{A}_{x+1}$; roughly speaking, this implies that there is a bijective mapping between the "gauge bundles" over the noncommutative tori $T^2_{\theta(x)}$ and $T^2_{\theta(x+1)}$.

The parabolic Morita duality above dictates that the right end of the open string should also land on a single $(1,0)$ D2-brane at $x+1$. In this case the Morita equivalence is trivial: The space of open string ground states $\mathcal{H}_{1,0}$ in the decoupling limit is simply a copy of the algebra $\mathcal{A}_x$ itself, or the *free* bimodule over $\mathcal{A}_x$, with the algebra $\mathcal{A}_x$ of functions on the noncommutative torus $T^2_{\theta(x)}$ acting from the left and the algebra $\mathcal{A}_{x+1} = \mathcal{A}_x$ of functions on $T^2_{\theta(x+1)} = T^2_{\theta(x)}$ acting from the right, both via the star-product (4.24); this identifies $\mathcal{H}_{1,0}$ as the space of functions on an ordinary torus T^2. Thus the parabolic monodromies recover the standard free bimodule over $\mathcal{A}_x$.

In contrast, the $\mathbb{Z}_4$ elliptic Morita duality dictates that the right end of the open string should land on a single $(0,-1)$ D0-brane at $x+1$. The space of open string ground states $\mathcal{H}_{0,-1}$ in this case comes from quantising a two-dimensional phase space, which is the cover of the torus T^2 with the Poisson bracket $\{y^1, y^2\} = \theta(x)$. This may be identified as an algebra of functions on $\mathbb{R}$ in a Schrödinger polarisation in which the algebra $\mathcal{A}_x$ acts on $\mathcal{H}_{0,-1}$, regarded as the space of functions of y^2, by representing y^2 as multiplication by y^2 and y^1 as the derivative $\mathrm{i}\,\theta(x)\,\frac{\partial}{\partial y^2}$. The commutant of $\mathcal{A}_x$ in $\mathcal{H}_{0,-1}$ is generated by operators given as multiplication by $y^2/\theta(x)$ together with the derivative $\mathrm{i}\,\frac{\partial}{\partial y^2}$, which quantise the Poisson bracket $\{y^1, y^2\} = \theta(x)^{-1}$. This gives the standard Morita equivalence bimodule over $\mathcal{A}_x \times \mathcal{A}_{x+1}$ [27], with the algebra $\mathcal{A}_x$ of functions on the noncommutative torus $T^2_{\theta(x)}$ acting from the left and the algebra $\mathcal{A}_{x+1}$ of functions on $T^2_{\theta(x+1)} = T^2_{-1/\theta(x)}$ acting from the right.

More general Morita transformations of the D-brane charges, taking the initial configuration of charges $(1,0)$ to a configuration $(1,1)$ with a unit of D0-brane charge inside a D2-brane, are possible with the $\mathbb{Z}_3$ elliptic monodromy, and can be similarly interpreted on the space $\mathcal{H}_{1,1}$ of sections of a line bundle over T^2 with Chern number 1 [27].

5.4 D2-brane theory at the orbifold point

In twisted dimensional reductions, the scaling limits discussed in this section describe D-branes with x-dependent noncommutativity parameters $\theta(x)$ coupled to gauged supergravity. One of the most interesting features of elliptic twists, as compared to parabolic twists, is that they each admit a fixed point in moduli space at which the twisted reduction reduces to an orbifold reduction and so gives an exact string theory realisation [11]. The fixed point for a given elliptic twist is at a minimum of the corresponding Scherk-Schwarz potential at which the potential vanishes, and so gives a stable compactification to Minkowski space [11]. The twist $\gamma(x)$ at the fixed point is independent of x and the monodromy $\mathcal{M}$ generates a cyclic group of order p for some integer p, $\mathcal{M}^p = 1$. The twisted reduction at the fixed point then is realised as a $\mathbb{Z}_p$ orbifold of the theory compactified on T^3. This is given by the compactification on $S^1 \times T^2$ orbifolded by the action of $\mathcal{M}$ on the T^2 conformal field theory together with a shift $x \mapsto x + \frac{1}{p}$ of the coordinate x of the S^1. In particular, from (5.5) it follows that $\tau^\circ = \mathrm{i}$ is a fixed point of the $\mathrm{SL}(2,\mathbb{Z})$ transformation generated by (5.1) for $U = \mathbb{1}$: In that case $\tau(x) = \mathrm{i}$ independently of $x \in S^1$, and the minimum of the potential gives a Minkowski vacuum. The construction is a $\mathbb{Z}_4$ orbifold of the compactification on $S^1 \times T^2_{\tau(x)=\mathrm{i}}$ with the $\mathbb{Z}_4$ twist of the conformal field theory on the T^2 with $\tau^\circ = \mathrm{i}$ accompanied by a shift $x \mapsto x + \frac{1}{4}$.

At this point it is not possible to decouple open and closed string modes on the D2-brane, which would require a scaling limit $\tau_2^\circ \to 0$. In fact, at this point the B-field vanishes and the closed string metric

$$g_{\tau(x)=\mathrm{i}} = (2\pi\,r\,\mathrm{d}x)^2 + \left(2\pi\,r_1\,\mathrm{d}y^1\right)^2 + \left(2\pi\,r_2\,\mathrm{d}y^2\right)^2 \tag{5.31}$$

coincides exactly with the T^3 metric for finite string slope α' and area A. As expected from the open-closed transformation (2.2) with $B = 0$, in this case the open and closed string metrics on the D2-brane coincide while the noncommutativity vanishes, $\Theta = 0$. The dilaton is given by $\mathrm{e}^{\phi_{\tau(x)=\mathrm{i}}} = (2\pi\,\alpha'/A)^{1/2}$ and hence the Yang-Mills coupling at the fixed point is the α'-independent constant

$$\left(g_{\mathrm{YM}}\right)_{\tau(x)=\mathrm{i}} = \left(\frac{2\pi\,g_s^2}{A}\right)^{1/4} . \tag{5.32}$$

Thus in this case the closed string geometry is identical to the open string geometry, and the worldvolume gauge theory on the D2-brane is that of an ordinary *commutative* supersymmetric Yang-Mills theory on a flat torus $T^2 \subset T^3$. The same is expected to be true for the $\mathbb{Z}_3$ (and $\mathbb{Z}_6$) twist at the fixed point $\tau^\circ = \mathrm{e}^{\pi\,\mathrm{i}/3}$, which can be viewed as a $\mathbb{Z}_3$ orbifold and as a toroidal reduction with magnetic flux.

6 D-branes and doubled twisted torus geometry

Having understood the non-geometric T-fold backgrounds, our aim now is to study D-branes in the essentially doubled space obtained by T-duality in the x-direction. However, T-duality along the vector field $\frac{\partial}{\partial x}$ is problematic because the background depends explicitly on $x \in S^1$: The vector field $\frac{\partial}{\partial x}$ does not generate an isometry of the torus bundle and the Buscher construction can no longer be applied. For such cases, we use a generalised T-duality [7] which takes a background with dependence on x to an essentially doubled background in which the fields depend on the coordinate $\widetilde{x}$ of the T-dual circle and so are problematic to interpret in conventional terms.

The reduction with duality twist by an $O(d,d;\mathbb{Z})$ monodromy around the x-circle is generalised to a twisted construction with both a twist along x and along its dual coordinate $\widetilde{x}$, so that the dependence of the moduli $E(x,\widetilde{x})$ is through a local section $\gamma : S^1 \times S^1 \to O(d,d)$ given by

$$\gamma(x,\widetilde{x}) = \exp(x\,M)\,\exp(\widetilde{x}\,\widetilde{M}) , \tag{6.1}$$

with commuting mass matrices $M, \widetilde{M}$ and with the corresponding monodromies

$$\mathcal{M} = \exp(M) \qquad \text{and} \qquad \widetilde{\mathcal{M}} = \exp(\widetilde{M}) \tag{6.2}$$

both valued in $O(d,d;\mathbb{Z})$. A general non-geometric reduction then gives rise to a torus bundle with doubled fibres T^{2d}, and coordinates $y^a, \widetilde{y}_a$, $a = 1,\ldots,d$, over a doubled base $S^1 \times S^1$, with coordinates $x,\widetilde{x}$, such that a generalised T-duality along the vector field $\frac{\partial}{\partial x}$ takes a T^{2d}-bundle over the x-circle to a T^{2d}-bundle over the dual $\widetilde{x}$-circle. In the

remainder of this paper we describe the noncommutative Yang-Mills theories on D-branes in a doubled geometry of this type which is the natural lift of the twisted torus backgrounds considered in section 3.

We will formulate the theory using a doubled geometry with coordinates $(x, \widetilde{x}, y^a, \widetilde{y}_a)$. Such a doubled formulation was first proposed in [33] and developed in [34–36], replacing the doubled torus with a twisted version, the doubled twisted torus. This $2n$-dimensional doubled geometry incorporates all the dual forms of the n-dimensional background, with the different backgrounds arising from different polarisations, which give different n-dimensional 'slices'. This doubled geometry has been recently discussed in [54].

6.1 The doubled twisted torus

Following [35, 36], we extend the twisted torus $X = G_{\mathbb{Z}} \backslash G_{\mathbb{R}}$ of section 3 to the $2n$-dimensional doubled twisted torus

$$\mathscr{X} = \mathscr{G}_{\mathbb{Z}} \backslash \mathscr{G}_{\mathbb{R}} \tag{6.3}$$

where the $2n$-dimensional non-compact Lie group $\mathscr{G}_{\mathbb{R}}$ is the cotangent bundle $\mathscr{G}_{\mathbb{R}} = T^*G_{\mathbb{R}} = G_{\mathbb{R}} \ltimes \mathbb{R}^n$; this is a Drinfeld double, and $\mathscr{G}_{\mathbb{Z}}$ is a discrete cocompact subgroup of $\mathscr{G}_{\mathbb{R}}$. The local structure of $\mathscr{X}$ is given by the Lie algebra of $\mathscr{G}_{\mathbb{R}}$ whose generators $\mathcal{J}_M$, $M = 1, \ldots, 2n$, have brackets

$$[\mathcal{J}_M, \mathcal{J}_N] = t_{MN}{}^P \, \mathcal{J}_P \,, \tag{6.4}$$

with structure constants $t_{MN}{}^P$ satisfying the Jacobi identity $t_{[MN}{}^Q t_{P]Q}{}^T = 0$. The Lie algebra admits an $O(n,n)$-invariant constant symmetric bilinear form η_{MN} of signature (n,n), and so $\mathscr{G}_{\mathbb{R}}$ is a $2n$-dimensional subgroup of $O(n,n)$. The generators $\mathcal{J}_M$ consist of $J_m = \{J_x, J_a\}$, $a = 1, \ldots, d$, $m = 1, \ldots, n$, generating the $G_{\mathbb{R}}$ subgroup, and $\widetilde{J}^m = \{\widetilde{J}^x, \widetilde{J}^a\}$ generating the $\mathbb{R}^n$ subgroup. Then

$$\mathcal{J}_M = \begin{pmatrix} J_m \\ \widetilde{J}^m \end{pmatrix} \tag{6.5}$$

is formally an $O(n,n)$-vector and in this basis the $O(n,n)$-invariant metric is

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \,. \tag{6.6}$$

The Lie algebra is

$$
\begin{aligned}
&[J_a, J_x] = M_a{}^b \, J_b \,, && [J_a, J_b] = 0 && \text{and} && [\widetilde{J}^a, \widetilde{J}^x] = 0 = [\widetilde{J}^a, \widetilde{J}^b] \,, \\
&[J_a, \widetilde{J}^b] = -M_a{}^b \, \widetilde{J}^x \,, && [J_x, \widetilde{J}^a] = M_b{}^a \, \widetilde{J}^b && \text{and} && [J_a, \widetilde{J}^x] = 0 = [J_x, \widetilde{J}^x] \,.
\end{aligned} \tag{6.7}
$$

This has the Drinfeld double form

$$[J_m, J_n] = f_{mn}{}^p J_p \,, \quad [\widetilde{J}^m, \widetilde{J}^n] = 0 \quad \text{and} \quad [J_n, \widetilde{J}^m] = -f_{np}{}^m \, \widetilde{J}^p \tag{6.8}$$

where $f_{mn}{}^p$ are the structure constants for $G_{\mathbb{R}}$.

The group manifold of $\mathscr{G}_\mathbb{R}$ is parameterised by coordinates $(x, y^1, \ldots, y^d)$ on $G_\mathbb{R}$ as in (3.6), and coordinates $(\widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_d)$ on $\widetilde{G}_\mathbb{R} = \mathbb{R}^n$. The quotient by the action of the discrete cocompact subgroup $\mathscr{G}_\mathbb{Z}$ of the $2n$-dimensional group $\mathscr{G}_\mathbb{R}$ results in the compact space $\mathscr{X}$. The T-dual coordinates $(\widetilde{x}, \widetilde{y}_a)$ are all periodic and so parameterise an n-torus T^n, so that $\mathscr{X}$ admits a T^n fibration with fibre coordinates $(\widetilde{x}, \widetilde{y}_a)$ as well as the T^{2d} doubled torus fibration with fibre coordinates $(y^a, \widetilde{y}_a)$.

The action of $\mathscr{G}_\mathbb{Z}$ induces a monodromy in $O(d, d; \mathbb{Z}) \subset \mathrm{GL}(2d, \mathbb{Z})$ acting geometrically by a large diffeomorphism of the doubled torus fibres as

$$\begin{pmatrix} y^a \\ \widetilde{y}_a \end{pmatrix} \xrightarrow{\ x \mapsto x+1\ } \begin{pmatrix} \mathcal{M}^{-1} & 0 \\ 0 & \mathcal{M}^\top \end{pmatrix} \begin{pmatrix} y^a \\ \widetilde{y}_a \end{pmatrix}, \tag{6.9}$$

together with the $\mathrm{GL}(d, \mathbb{Z})$ monodromy

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{y}_1 \\ \vdots \\ \widetilde{y}_{d-1} \end{pmatrix} \xrightarrow{\ y^d \mapsto y^d + 1\ } \mathcal{M}^{-1} \begin{pmatrix} \widetilde{x} \\ \widetilde{y}_1 \\ \vdots \\ \widetilde{y}_{d-1} \end{pmatrix} \tag{6.10}$$

acting geometrically as a large diffeomorphism on the T-dual torus. Here $\mathcal{M}$ is the monodromy matrix of the twisted torus X from section 3, given explicitly for $n = 3$ ($d = 2$) by (4.1) in the case of parabolic twists and by (5.1) for elliptic twists.

The Maurer-Cartan one-forms (3.10) lift to left-invariant forms on $\mathscr{G}_\mathbb{R}$, but $G_\mathbb{R}$ acts non-trivially on $\widetilde{G}_\mathbb{R}$ so one needs to "twist" the left-invariant one-forms $\mathrm{d}\widetilde{x}$, $\mathrm{d}\widetilde{y}_a$ of $\widetilde{G}_\mathbb{R}$ when lifting them to $\mathscr{G}_\mathbb{R}$. A basis of left-invariant one-forms on $\mathscr{G}_\mathbb{R}$ is then given by

$$\begin{aligned} \zeta^x &= \mathrm{d}x && \text{and} && \zeta^a = \gamma(x)^a{}_b\, \mathrm{d}y^b, \\ \widetilde{\zeta}_x &= \mathrm{d}\widetilde{x} - M_a{}^b\, \widetilde{y}_b\, \zeta^a && \text{and} && \widetilde{\zeta}_a = \mathrm{d}\widetilde{y}_a + M_a{}^b\, \widetilde{y}_b\, \zeta^x. \end{aligned} \tag{6.11}$$

The action of $\mathscr{G}_\mathbb{Z}$ is compatible with $\widetilde{G}_\mathbb{R}$, so that the quotient $\widetilde{G}_\mathbb{R} \backslash \mathscr{X}$ is well-defined and corresponds to the n-dimensional twisted torus $X = G_\mathbb{Z} \backslash G_\mathbb{R}$. In this way the conventional spacetime description is obtained for the natural polarisation associated to the coset $\widetilde{G}_\mathbb{R} \backslash \mathscr{G}_\mathbb{R}$, which corresponds to the natural projection on the cotangent bundle $T^* X = X \times \mathbb{R}^n$.

We write coordinates on the quotient $\mathscr{X}$ as $\mathbb{X}^I = (x, y^a, \widetilde{x}, \widetilde{y}_a)$ with $a = 1, \ldots, d$ and $I = 1, \ldots, 2n$, and the one-forms $\zeta^m = \{\zeta^x, \zeta^a\}$ and $\widetilde{\zeta}_m = \{\widetilde{\zeta}_x, \widetilde{\zeta}_a\}$ with $m = 1, \ldots, n$ collectively as

$$\mathcal{P}^M = \mathcal{P}^M{}_I\, \mathrm{d}\mathbb{X}^I. \tag{6.12}$$

We will sometimes denote these coordinates as $\mathbb{X}^M = (x^m, \widetilde{x}_m)$, and write a general group element $g \in \mathscr{G}_\mathbb{R}$ as

$$g = \widetilde{h}\, h \tag{6.13}$$

where

$$h = \exp(x^m J_m) \qquad \text{and} \qquad \widetilde{h} = \exp(\widetilde{x}_m \widetilde{J}^m). \tag{6.14}$$

The left-invariant one-forms $\mathcal{P}^M$, $M = 1, \ldots, 2n$ satisfy the Maurer-Cartan equations

$$\mathrm{d}\mathcal{P}^M + \tfrac{1}{2} t_{NP}{}^M \, \mathcal{P}^N \wedge \mathcal{P}^P = 0 \,, \tag{6.15}$$

so that the $2n$-manifold $\mathscr{X}$ is parallelisable. We further introduce a constant (independent of the coordinates $\mathbb{X}$ on $\mathscr{X}$) metric given by a $2n \times 2n$ symmetric matrix M_{MN} satisfying

$$\mathsf{M}_{MN} = \eta_{MP} \, (\mathsf{M}^{-1})^{PQ} \, \eta_{QN} \,, \tag{6.16}$$

so that it parameterises the left coset $\bigl(O(n) \times O(n)\bigr) \backslash O(n,n)$. The matrix M_{MN} constitutes the moduli of the doubled twisted torus, and becomes a matrix of scalar fields on dimensional reduction, giving a non-linear sigma-model with target space $\bigl(O(n) \times O(n)\bigr) \backslash O(n,n)$.

A natural left-invariant metric and three-form on $\mathscr{X}$ are then defined by

$$\mathrm{d}s_{\mathscr{X}}^2 = \mathsf{M}_{MN} \, \mathcal{P}^M \, \mathcal{P}^N = \mathcal{H}_{IJ} \, \mathrm{d}\mathbb{X}^I \, \mathrm{d}\mathbb{X}^J \tag{6.17}$$

and

$$\mathcal{K} = \tfrac{1}{6} t_{MNP} \, \mathcal{P}^M \wedge \mathcal{P}^N \wedge \mathcal{P}^P = \tfrac{1}{6} T_{IJK} \, \mathrm{d}\mathbb{X}^I \wedge \mathrm{d}\mathbb{X}^J \wedge \mathrm{d}\mathbb{X}^K \,, \tag{6.18}$$

where the doubled metric $\mathcal{H}_{IJ} := \mathsf{M}_{MN} \mathcal{P}^M{}_I \mathcal{P}^N{}_J$ obeys $\mathcal{H}_{IJ} = \eta_{IK} \, (\mathcal{H}^{-1})^{KL} \, \eta_{LJ}$, with $\eta_{IJ} = \eta_{MN} \mathcal{P}^M{}_I \mathcal{P}^N{}_J$, while $T_{IJK} := t_{MNP} \mathcal{P}^M{}_I \mathcal{P}^N{}_J \mathcal{P}^P{}_K$ with $t_{MNP} := \eta_{MQ} t_{NP}{}^Q$ totally antisymmetric. The Wess-Zumino three-form $\mathcal{K}$ is closed, $\mathrm{d}\mathcal{K} = 0$, by virtue of the Jacobi identity $t_{[MN}{}^Q t_{P]Q}{}^T = 0$.

The natural action of $O(n,n)$ on the tangent bundle of $\mathscr{G}_{\mathbb{R}}$ then gives

$$
\begin{aligned}
\mathrm{d}\mathbb{X}^I &\longmapsto (\mathcal{O}^{-1})^I{}_J \, \mathrm{d}\mathbb{X}^J \,, \\
\mathcal{P}^M &\longmapsto (\mathcal{O}^{-1})^M{}_N \, \mathcal{P}^N \,, \\
\mathcal{J}_M &\longmapsto \mathcal{J}_N \, \mathcal{O}^N{}_M \,,
\end{aligned}
\tag{6.19}
$$

for $\mathcal{O} \in O(n,n)$. This is essentially the $O(n,n)$ structure group of generalised geometry, acting on the generalised tangent bundle $TX \oplus T^*X$ of the n-dimensional twisted torus X. Note, however, that $O(n,n;\mathbb{Z})$ is *not* a symmetry in this case. Consider the subgroup $\mathrm{GL}(n,\mathbb{Z}) \subset O(n,n;\mathbb{Z})$. For T^n, the group of large diffeomorphisms is $\mathrm{GL}(n,\mathbb{Z})$, but for the n-dimensional twisted torus X, $\mathrm{GL}(n,\mathbb{Z})$ is not a symmetry, although the subgroup $\mathrm{GL}(d,\mathbb{Z})$ acting on the T^d fibres is. For string theory on X, or any of its T-duals, there is an $O(d,d;\mathbb{Z})$ T-duality symmetry acting in the conformal field theory on the T^d fibres; in the doubled formalism, this acts as a diffeomorphism on the T^{2d} doubled torus fibres of $\mathscr{X}$. For $n = 3$ ($d = 2$), this $O(2,2;\mathbb{Z})$ acting on $(y^a, \widetilde{y}_a)$ recovers the T-duality orbits of the three-dimensional twisted torus from section 4 and section 5. For $\mathcal{O} \in O(d,d;\mathbb{Z}) \subset O(n,n)$, the action (6.19) extends to

$$
\begin{aligned}
\mathbb{X}^I &\longmapsto (\mathcal{O}^{-1})^I{}_J \, \mathbb{X}^J \,, \\
\mathcal{H}_{IJ}(\mathbb{X}) &\longmapsto \mathcal{O}_I{}^K \, \mathcal{H}_{KL}(\mathcal{O}^{-1}\mathbb{X}) \, \mathcal{O}^L{}_J \,, \\
T_{IJK}(\mathbb{X}) &\longmapsto T_{LMN}(\mathcal{O}^{-1}\mathbb{X}) \, \mathcal{O}^L{}_I \, \mathcal{O}^M{}_J \, \mathcal{O}^N{}_K \,.
\end{aligned}
\tag{6.20}
$$

In [35], it was proposed that generalised T-duality acts in the same way, for certain other $\mathcal{O} \in O(n, n)$. In particular, it was proposed that the generalised T-duality in the x-direction is given by an $O(n, n)$-transformation $\mathcal{O}_x$, which for $n = 3$ reads as

$$\mathcal{O}_x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.21}$$

acting on

$$\mathbb{X}^I = \begin{pmatrix} x \\ y^1 \\ y^2 \\ \widetilde{x} \\ \widetilde{y}_1 \\ \widetilde{y}_2 \end{pmatrix} \tag{6.22}$$

corresponding to the exchange $x \leftrightarrow \widetilde{x}$.

The transformation $\mathcal{J}_M \longmapsto \mathcal{J}_N \mathcal{O}^N{}_M$ changes the split of the Lie algebra generators $\mathcal{J}_M$ into $J_n, \widetilde{J}^m$ and so changes the form of the algebra (6.8) to an algebra of the form

$$[J_m, J_n] = f_{mn}{}^p J_p + h_{mnp} \widetilde{J}^p,$$
$$[\widetilde{J}^m, \widetilde{J}^n] = Q_p{}^{mn} \widetilde{J}^p + R^{mnp} J_p,$$
$$[J_n, \widetilde{J}^m] = -f_{np}{}^m \widetilde{J}^p + Q_n{}^{mp} J_p, \tag{6.23}$$

where the various tensors are determined by the choice of $\mathcal{O}$, and are sometimes referred to as fluxes.

In [35], these dualities were interpreted in terms of the choice of polarisation, generalising the picture in [1]. A polarisation splits the tangent bundle $T\mathscr{X}$ of $\mathscr{X}$ at each point into an n-dimensional physical subspace Π and an n-dimensional dual subspace $\widetilde{\Pi}$, and the issue is whether the split of the tangent vectors defines an n-dimensional submanifold (at least locally) which can be viewed as a patch of spacetime. If the spaces Π and $\widetilde{\Pi}$ define an integrable distribution on $\mathscr{X}$, then there is locally a physical subspace of $\mathscr{X}$ and the background is locally geometric. If the distribution is non-integrable, then there is no such local spacetime and the background is not geometric even locally; it is essentially doubled. The polarisation splits the $2n$ Lie algebra generators $\mathcal{J}_M$ into two sets of n generators, the J_m and the $\widetilde{J}^m$. The integrability condition is that the $\widetilde{J}^m$ generate a subgroup $\widetilde{G}_{\mathbb{R}} \subset \mathscr{G}_{\mathbb{R}}$. Then the physical spacetime is defined by the quotient by $\widetilde{G}_{\mathbb{R}}$. The covering space for $\mathscr{X}$ is $\mathscr{G}_{\mathbb{R}}$ and the covering space for the physical subspace is the coset $\mathscr{G}_{\mathbb{R}}/\widetilde{G}_{\mathbb{R}}$. If the action of $\mathscr{G}_{\mathbb{Z}}$ is compatible with the action of $\widetilde{G}_{\mathbb{R}}$, then the background is geometric and given globally by a double quotient of $\mathscr{G}_{\mathbb{R}}$ by $\mathscr{G}_{\mathbb{Z}}$ and $\widetilde{G}_{\mathbb{R}}$. If it is not, then the result is a T-fold, with local n-dimensional patches given by patches of $\mathscr{G}_{\mathbb{R}}/\widetilde{G}_{\mathbb{R}}$. A T-duality transformation is then interpreted as a change of polarisation, changing the physical subspace within the doubled

space, and can be realised as the action of the operator $\mathcal{O}$ on the projectors defining the polarisation [35].

The vielbeins $\mathcal{P}^M{}_I$ are maps $\mathcal{P}: O(n,n) \to O(n) \times O(n)$ and can be brought to lower block-triangular form by an $O(n) \times O(n)$ transformation to get

$$\mathcal{P} = \begin{pmatrix} e & 0 \\ -(2\pi\,\alpha')\,e^{-1}\,B & (2\pi\,\alpha')\,e^{-1} \end{pmatrix}, \tag{6.24}$$

where e is the vielbein for the spacetime metric $g = e^\top e$, and B is the NS-NS two-form potential. By choosing the simple background $\mathsf{M}_{MN} = \delta_{MN}$, the doubled metric can be written as

$$\mathcal{H} = \begin{pmatrix} g - (2\pi\,\alpha')^2\,B\,g^{-1}\,B & (2\pi\,\alpha')^2\,B\,g^{-1} \\ -(2\pi\,\alpha')^2\,g^{-1}\,B & (2\pi\,\alpha')^2\,g^{-1} \end{pmatrix}. \tag{6.25}$$

The expressions for general moduli M_{MN} are given in [35].

If $\widetilde{g}$ denotes the dual metric arising from an $O(n,n)$ transformation (6.20) of (6.25), then the dilaton transforms as

$$\mathrm{e}^\phi \longmapsto \left(\frac{\det \widetilde{g}}{\det g} \right)^{1/4} \mathrm{e}^\phi. \tag{6.26}$$

6.2 D-branes in the doubled twisted torus

D-branes in the doubled picture were discussed for the doubled torus and for doubled torus fibrations in [1, 18], and this was extended to the doubled twisted torus in [19]. Following [19], let us now describe D-branes in the doubled twisted torus geometry.

The starting point is the doubled sigma-model which was introduced in [35] for maps embedding a closed string worldsheet Σ in $\mathscr{X}$. These maps pull back the one-forms $\mathcal{P}^M$ to one-forms $\hat{\mathcal{P}}^M$ on Σ. Introducing a three-dimensional manifold V with boundary $\partial V = \Sigma$ and extending the maps to V, the sigma-model is defined by the action

$$S_{\mathscr{X}} = \frac{1}{4} \oint_\Sigma \mathsf{M}_{MN}\,\hat{\mathcal{P}}^M \wedge *\hat{\mathcal{P}}^N + \frac{1}{2} \int_V \hat{\mathcal{K}}, \tag{6.27}$$

where $\hat{\mathcal{K}}$ is the pullback of the Wess-Zumino three-form $\mathcal{K}$ to V and $*$ is the Hodge duality operator on Σ. To recover the ordinary non-linear sigma-model on a physical target space, this doubled sigma-model is subjected to the self-duality constraint

$$\hat{\mathcal{P}}^M = \eta^{MP}\,\mathsf{M}_{PN} * \hat{\mathcal{P}}^N \tag{6.28}$$

which eliminates half of the $2n$ degrees of freedom by restricting n of them to be right-moving and n of them to be left-moving on Σ. In [35] this constraint was imposed by choosing a polarisation and then gauging the sigma-model.

In the case that the structure constants R^{mnp} in (6.23) vanish, so that the $\widetilde{J}^m$ generate a subgroup $\widetilde{G}_\mathbb{R} \subset \mathscr{G}_\mathbb{R}$, then the reduction to the physical subspace can be achieved by gauging the action of $\widetilde{G}_\mathbb{R}$. On quotienting by the discrete subgroup $\mathscr{G}_\mathbb{Z}$ and eliminating

the worldsheet gauge fields, one obtains a standard non-linear sigma-model on a target space described locally by the coset $\mathscr{G}_{\mathbb{R}}/\widetilde{G}_{\mathbb{R}}$ with coordinates $(x, y^1, \ldots, y^d)$, with metric and B-field given from the generalised metric (6.25), and with physical H-field strength given by

$$H = \mathrm{d}B \ . \tag{6.29}$$

For more general doubled groups that are not Drinfeld doubles, the equation for H has further terms which are given in [35].

On the other hand, if the structure constants R^{mnp} in (6.23) are non-zero, then the sigma-model will depend explicitly on both x and $\widetilde{x}$. In this essentially doubled case, the metric and H-field strength depend on both x and $\widetilde{x}$, and it appears that there is no interpretation of the sigma-model in terms of a conventional n-dimensional spacetime.

In [19] it was shown that the same sigma-model action (6.27) can be used to describe the embedding of an open string worldsheet Σ in the doubled space $\mathscr{X}$. In this case one must specify boundary conditions by demanding that the string maps should send the boundary $\partial\Sigma$ of Σ to a given submanifold $\mathscr{W} \subset \mathscr{X}$, the worldvolume of a D-brane in the doubled space $\mathscr{X}$. This requires that the embedding of the boundary ∂V of the three-dimensional manifold V is the sum of the embedding of Σ with some chain $\mathscr{D} \subseteq \mathscr{W}$, and consistency of the Wess-Zumino term in (6.27) requires that the pullback of the three-form $\mathcal{K}$ to $\mathscr{D}$ vanishes, $\mathcal{K}|_{\mathscr{D}} = 0$. One can then analyse the boundary equations of motion as well as the self-duality constraint (6.28) with these conditions. In [19] it is shown that as a result the worldvolume $\mathscr{W}$ of a D-brane in the doubled twisted torus is a subspace of $\mathscr{X}$ which is maximally isotropic with respect to the $O(n, n)$-invariant metric η. Choosing a polarisation then picks out physical worldvolume coordinates, so that the physical D-brane wraps that part of the physical space which intersects the generalised D-brane subspace $\mathscr{W}$. D-branes in the doubled space are specified by complementary Dirichlet and Neumann projectors that respectively project the tangent bundle of $\mathscr{X}$ at each point into subspaces normal and tangential to the worldvolume wrapped by the D-brane. Both subspaces are null with respect to η, and they are mutually orthogonal to each other with respect to the doubled metric $\mathcal{H}$. The Neumann projector moreover satisfies an integrability condition ensuring that the D-brane worldvolume $\mathscr{W}$ is locally a smooth submanifold of $\mathscr{X}$. The vanishing of the Wess-Zumino three-form $\mathcal{K}$ on $\mathscr{W}$ further constrains the structure constants $t_{MN}{}^P$ of the Lie algebra of $\mathscr{G}_{\mathbb{R}}$ which restricts the orientation of the D-brane in $\mathscr{X}$.

This construction implies, in particular, that for each Neumann condition there is a corresponding Dirichlet condition. Thus there are always n Neumann directions and n Dirichlet directions on the doubled twisted torus $\mathscr{X}$, and these directions each form a null subspace of $\mathscr{X}$. As a consequence, any D-brane in a physical n-dimensional polarisation always arises from a Dn-brane in the extended $2n$-dimensional doubled geometry.

As before, we shall study the cases with $n = 3$ in detail. Starting from the three-dimensional spacetime polarisation above onto the twisted torus X, we will follow the T-duality orbits of D-branes in $\mathscr{X}$. The D-brane projectors transform under the action of the T-duality operator $\mathcal{O}$, and the possible D-branes in the various T-duality frames are classified using the doubled twisted torus formalism by [19]. In particular, some anticipated

Background	Flux	Dp-brane	x	y^1	y^2	$\widetilde{x}$	$\widetilde{y}_1$	$\widetilde{y}_2$
T^3 with H-flux	H-flux	D0-brane	$-$	$-$	$-$	$\times$	$\times$	$\times$
Nilfold	f-flux	D1-brane	$-$	$\times$	$-$	$\times$	$-$	$\times$
T-fold	Q-flux	D2-brane	$-$	$\times$	$\times$	$\times$	$-$	$-$
Essentially doubled	R-flux	D3-brane	$\times$	$\times$	$\times$	$-$	$-$	$-$

Table 1. The Dp-brane configurations considered in the various T-duality frames of the doubled twisted torus geometry for the case of parabolic monodromies. A dash denotes a normal (Dirichlet) direction to the D-brane, while a cross denotes a worldvolume (Neumann) direction. The number of Dirichlet and Neumann directions in each case are both equal to three.

worldsheet classification results are confirmed explicitly in this way; for example, it is known that D3-branes cannot wrap the three-torus T^3 with non-zero H-flux due to the Freed-Witten anomaly [55] (because T^3 is a spin^c-manifold and so anomaly cancellation requires $m = [H] = W_3(T^3) = 0$).

6.3 D2-branes on T-folds

We start by rederiving the results of section 4 and section 5 in the doubled picture, which involve T-duality transformations in the y^a direction corresponding to $\mathcal{O}_{y^a} \in O(2,2;\mathbb{Z})$. Starting from the twisted torus with metric (3.17) and vanishing B-field, we write the corresponding doubled metric from (6.25):

$$
\mathcal{H}_f = \begin{pmatrix}
(2\pi\, r)^2 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{A}{\tau_2(x)} & \frac{A\,\tau_1(x)}{\tau_2(x)} & 0 & 0 & 0 \\
0 & \frac{A\,\tau_1(x)}{\tau_2(x)} & \frac{A\,|\tau(x)|^2}{\tau_2(x)} & 0 & 0 & 0 \\
0 & 0 & 0 & \left(\frac{\alpha'}{r}\right)^2 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{(2\pi\,\alpha')^2\,|\tau(x)|^2}{A\,\tau_2(x)} & -\frac{(2\pi\,\alpha')^2\,\tau_1(x)}{A\,\tau_2(x)} \\
0 & 0 & 0 & 0 & -\frac{(2\pi\,\alpha')^2\,\tau_1(x)}{A\,\tau_2(x)} & \frac{(2\pi\,\alpha')^2}{A\,\tau_2(x)}
\end{pmatrix}, \tag{6.30}
$$

where the complex structure modulus $\tau(x) = \tau_1(x) + \mathrm{i}\,\tau_2(x)$ is given by (4.2) in the case of parabolic twists and by (5.5) (with $m \in 4\,\mathbb{Z}+1$ and $\vartheta = \frac{\pi}{2}$) for the $\mathbb{Z}_4$ elliptic twist. The Wess-Zumino three-form is given by

$$
\mathcal{K}_f = -\tfrac{1}{2}\,M_a{}^b\,\mathrm{d}x \wedge \mathrm{d}\widetilde{y}_b \wedge \mathrm{d}y^a\,, \tag{6.31}
$$

where the components of the mass matrix M can be read off from (4.1) for the case of parabolic twists and by (5.1) (with $U = \mathbb{1}$, $m \in 4\,\mathbb{Z}+1$ and $\vartheta = \frac{\pi}{2}$) for the $\mathbb{Z}_4$ elliptic twist. In this polarisation one thus finds $H = 0$, as expected. As the only non-vanishing structure constants in this case are $f_{ax}{}^b = M_a{}^b$, we can wrap a D1-brane around the torsion one-cycle ξ_1 in the doubled geometry [19], as previously, and follow its orbits under T-duality, which are summarised in table 1.

To dualise along the vector field $\frac{\partial}{\partial y^1}$ of $\mathscr{X}$, we apply (6.20) to (6.30) with

$$
\mathcal{O}_{y^1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , \tag{6.32}
$$

which interchanges y^1 with $\widetilde{y}_1$ in the doubled coordinates $\mathbb{X}$ and all fields, leaving all other components invariant. The transformed doubled metric is given by

$$
\mathcal{H}_H = \mathcal{O}_{y^1}^\top \, \mathcal{H}_f \, \mathcal{O}_{y^1} = \begin{pmatrix} (2\pi\,r)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{(2\pi\,\alpha')^2\,|\tau(x)|^2}{A\,\tau_2(x)} & 0 & 0 & 0 & -\frac{(2\pi\,\alpha')^2\,\tau_1(x)}{A\,\tau_2(x)} \\ 0 & 0 & \frac{A\,|\tau(x)|^2}{\tau_2(x)} & 0 & \frac{A\,\tau_1(x)}{\tau_2(x)} & 0 \\ 0 & 0 & 0 & \left(\frac{\alpha'}{r}\right)^2 & 0 & 0 \\ 0 & 0 & \frac{A\,\tau_1(x)}{\tau_2(x)} & 0 & \frac{A}{\tau_2(x)} & 0 \\ 0 & -\frac{(2\pi\,\alpha')^2\,\tau_1(x)}{A\,\tau_2(x)} & 0 & 0 & 0 & \frac{(2\pi\,\alpha')^2}{A\,\tau_2(x)} \end{pmatrix} . \tag{6.33}
$$

Comparing with (6.25), we can read off the closed string metric and B-field. For the parabolic monodromy (4.2), these agree with (4.9) for the geometric three-torus T^3 with constant H-flux (4.10), and the Wess-Zumino three-form is given by

$$
\mathcal{K}_H = -\tfrac{1}{2}\,m\,\mathrm{d}x \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2 \ . \tag{6.34}
$$

In this polarisation the generators $\widetilde{J}^m$ generate a maximally isotropic subgroup $\widetilde{G}_\mathbb{R} \subset \mathscr{G}_\mathbb{R}$ which is compatible with the action of $\mathscr{G}_\mathbb{Z}$, so that the quotient $\widetilde{G}_\mathbb{R} \setminus \mathscr{X}$ is well-defined and provides a global description of the three-dimensional compactification geometry. Altogether we recover the standard non-linear sigma-model with target space T^3 threaded by a constant H-flux.

On the other hand, dualising along the vector field $\frac{\partial}{\partial y^2}$ implements (6.20) on (6.30) with

$$
\mathcal{O}_{y^2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} , \tag{6.35}
$$

which now interchanges y^2 with $\widetilde{y}_2$ in the doubled coordinates $\mathbb{X}$ and all fields, leaving all other components invariant. The transformed doubled metric is given by

$$
\mathcal{H}_Q = \mathcal{O}_{y^2}^\top \, \mathcal{H}_f \, \mathcal{O}_{y^2} = \begin{pmatrix} (2\pi\,r)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{A}{\tau_2(x)} & 0 & 0 & 0 & \frac{A\,\tau_1(x)}{\tau_2(x)} \\ 0 & 0 & \frac{(2\pi\,\alpha')^2}{A\,\tau_2(x)} & 0 & -\frac{(2\pi\,\alpha')^2\,\tau_1(x)}{A\,\tau_2(x)} & 0 \\ 0 & 0 & 0 & \left(\frac{\alpha'}{r}\right)^2 & 0 & 0 \\ 0 & 0 & -\frac{(2\pi\,\alpha')^2\,\tau_1(x)}{A\,\tau_2(x)} & 0 & \frac{(2\pi\,\alpha')^2\,|\tau(x)|^2}{A\,\tau_2(x)} & 0 \\ 0 & \frac{A\,\tau_1(x)}{\tau_2(x)} & 0 & 0 & 0 & \frac{A\,|\tau(x)|^2}{\tau_2(x)} \end{pmatrix} , \tag{6.36}
$$

while the Wess-Zumino three-form is

$$\mathcal{K}_Q = -\tfrac{1}{2}\,\delta^{ac}\,M_c{}^b\,\mathrm{d}x \wedge \mathrm{d}\widetilde{y}_a \wedge \mathrm{d}\widetilde{y}_b \ . \tag{6.37}$$

Reading off the closed string metric and B-field from (6.25) yields precisely (5.10) for the non-geometric T^2-bundle over S^1, while (6.26), with vanishing dilaton on the twisted torus, yields the anticipated dilaton field (5.11). The H-field strength is given by $H = \mathrm{d}B$. In this case the generators $\widetilde{J}^m$ generate a subgroup $\widetilde{G}_\mathbb{R}$ which is not preserved by $\mathscr{G}_\mathbb{Z}$, so that the quotient $\widetilde{G}_\mathbb{R} \setminus \mathscr{X}$ is locally modelled on the coset $\widetilde{G}_\mathbb{R} \setminus \mathscr{G}_\mathbb{R}$ but is not globally well-defined, and a global description of the background in terms of conventional geometry is not possible. However, the T-fold *is* a submanifold of the doubled twisted torus $\mathscr{X}$, because $O(2,2;\mathbb{Z}) \subset \mathrm{GL}(4,\mathbb{Z})$ acts geometrically on the doubled torus fibres. As the only non-vanishing structure constants are $Q_x{}^{ab} = \delta^{ac}\,M_c{}^b$, we obtain the allowed D2-brane configuration displayed in table 1 [19]; as Morita duality acts entirely within the noncommutative Yang-Mills theory on the D2-brane, and in particular does not mix gauge theory modes with string winding states, the same picture of a parameterised family of D2-brane gauge theories fibred over the x-circle emerges in the doubled geometry, returning to itself under a monodromy $x \mapsto x+1$ up to Morita equivalence, which is a symmetry of the noncommutative gauge theory. On the other hand, the D3-brane considered in section 4 is not a consistent worldvolume when embedded as a three-dimensional subspace of the six-dimensional doubled twisted torus $\mathscr{X}$ [19].

7 D3-branes on essentially doubled spaces

Recall that one motivation for turning to the doubled twisted torus formalism is that it enables us to perform the generalised T-duality transformation along the non-isometric base direction of the original T^2-bundle over S^1. This maps the original D1-brane configuration to a D3-brane wrapping an essentially doubled background. In this final section we will discuss how to make sense of the noncommutative supersymmetric Yang-Mills theory on D3-branes in essentially doubled spaces in the decoupling limit using the doubled twisted torus formalism.

7.1 Worldvolume geometry

To carry out T-duality along the vector field $\frac{\partial}{\partial x}$ of $\mathscr{X}$, we apply (6.20) to (6.36) with the $O(3,3;\mathbb{Z})$ operator (6.21), which interchanges x with $\widetilde{x}$ in the doubled coordinates $\mathbb{X}$ and all fields, leaving all other components invariant. The transformed doubled metric is given by

$$\mathcal{H}_R = \mathcal{O}_x^\top\,\mathcal{H}_Q\,\mathcal{O}_x = \begin{pmatrix} \left(\frac{\alpha'}{r}\right)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{A}{\tau_2(\widetilde{x})} & 0 & 0 & 0 & \frac{A\,\tau_1(\widetilde{x})}{\tau_2(\widetilde{x})} \\ 0 & 0 & \frac{(2\pi\,\alpha')^2}{A\,\tau_2(\widetilde{x})} & 0 & -\frac{(2\pi\,\alpha')^2\,\tau_1(\widetilde{x})}{A\,\tau_2(\widetilde{x})} & 0 \\ 0 & 0 & 0 & (2\pi\,r)^2 & 0 & 0 \\ 0 & 0 & -\frac{(2\pi\,\alpha')^2\,\tau_1(\widetilde{x})}{A\,\tau_2(\widetilde{x})} & 0 & \frac{(2\pi\,\alpha')^2\,|\tau(\widetilde{x})|^2}{A\,\tau_2(\widetilde{x})} & 0 \\ 0 & \frac{A\,\tau_1(\widetilde{x})}{\tau_2(\widetilde{x})} & 0 & 0 & 0 & \frac{A\,|\tau(\widetilde{x})|^2}{\tau_2(\widetilde{x})} \end{pmatrix}. \tag{7.1}$$

Comparing with (6.25) formally gives the closed string metric and B-field

$$g_R = \left(\frac{\alpha'}{r}\,\mathrm{d}x\right)^2 + \frac{\tau_2(\widetilde{x})}{|\tau(\widetilde{x})|^2}\left(A\left(\mathrm{d}y^1\right)^2 + \frac{(2\pi\,\alpha')^2}{A}\left(\mathrm{d}y^2\right)^2\right),$$
$$B_R = \frac{\tau_1(\widetilde{x})}{|\tau(\widetilde{x})|^2}\,\mathrm{d}y^1\wedge\mathrm{d}y^2 \tag{7.2}$$

in the essentially doubled space, while (6.26) yields the dilaton field

$$\mathrm{e}^{\phi_R(\widetilde{x})} = \left(\frac{(\alpha')^2\,\tau_2(\widetilde{x})}{r^2\,A\,|\tau(\widetilde{x})|^2}\right)^{1/2}. \tag{7.3}$$

The explicit dependence on the dual coordinate $\widetilde{x}$ reflects the non-geometric nature of the essentially doubled space: In this polarisation the generators $\widetilde{J}^m$ do not close to a subalgebra and a conventional description of the background cannot be recovered even locally.

In this polarisation the Wess-Zumino three-form

$$\mathcal{K}_R = -\tfrac{1}{2}\,\delta^{ac}\,M_c{}^b\,\mathrm{d}\widetilde{x}\wedge\mathrm{d}\widetilde{y}_a\wedge\mathrm{d}\widetilde{y}_b \tag{7.4}$$

vanishes as required on the worldvolume of the D3-brane, which wraps the directions with coordinates (x, y^1, y^2). It is shown by [35] that it is possible to use the self-duality constraint (6.28) to completely remove the dependence of the doubled worldsheet sigma-model on the pullbacks of $\mathrm{d}\widetilde{x}_m$ and write the doubled theory as a non-linear sigma-model for the metric g_R and B-field B_R in (7.2), depending explicitly on the winding coordinate $\widetilde{x}$, thus rendering the coordinate fields non-dynamical along the dual directions $(\widetilde{x}, \widetilde{y}_1, \widetilde{y}_2)$.

Using (2.2) we now compute the open string metric and noncommutativity bivector on the D3-brane in the R-flux background to find

$$G_R = \left(\frac{\alpha'}{r}\,\mathrm{d}x\right)^2 + \frac{A}{\tau_2(\widetilde{x})}\left(\mathrm{d}y^1\right)^2 + \frac{(2\pi\,\alpha')^2}{A\,\tau_2(\widetilde{x})}\left(\mathrm{d}y^2\right)^2,$$
$$\Theta_R = \tau_1(\widetilde{x})\,\frac{\partial}{\partial y^1}\wedge\frac{\partial}{\partial y^2}. \tag{7.5}$$

Thus even the open string geometry seen by the D3-brane has a non-geometric dependence along the transverse $\widetilde{x}$-direction to its worldvolume in $\mathscr{X}$.

7.2 Noncommutative Yang-Mills theory

To find a decoupling limit with pure gauge theory on the D3-brane worldvolume, we note that the T^2-fibre parts of the open string geometry (7.5) coincide with those of the T-folds, given in (5.14), upon replacing the base S^1 coordinate x with its dual coordinate $\widetilde{x}$. Thus the scaling limit will involve taking $\alpha' = O(\epsilon^{1/2})$, $A = O(\epsilon^{1/2})$ and $\tau_2^\circ = O(\epsilon^{1/2})$, with $\epsilon \to 0$ and the radii (4.19) held fixed exactly as previously, and in addition $r = O(\epsilon^{1/2})$ with the base radius

$$\bar{r}_x := \frac{\alpha'}{2\pi\,r} \tag{7.6}$$

finite in the zero slope limit. Then the open string metric and noncommutativity bivector are finite in this limit and can be written as

$$\mathrm{d}s_R^2 = (2\pi\,\bar{r}_x\,\mathrm{d}x)^2 + \mathrm{d}s_{\mathrm{D2}}^2\big|_{x\to\widetilde{x}}\,,$$
$$\theta_R = \tau_1(\widetilde{x})\big|_{\tau_2^\circ=0}\,\frac{\partial}{\partial y^1}\wedge\frac{\partial}{\partial y^2}\,, \tag{7.7}$$

where $\mathrm{d}s_{\mathrm{D2}}^2\big|_{x\to\widetilde{x}}$ is the decoupled metric of a D2-brane wrapping the T^2-fibre of a T-fold with all x-dependence replaced by $\widetilde{x}$-dependence; these quantities can be read off from (4.20) in the case of parabolic twists and from (5.15) (with $m\in 4\,\mathbb{Z}+1$ and $\vartheta=\frac{\pi}{2}$) in the case of the $\mathbb{Z}_4$ elliptic twist. Note that in this limit, the original twisted torus X and the dual T-fold completely degenerate to a point, even though the bivector Θ_R has no components along the S^1 base.

From (2.6) with $p=3$ we can also compute the Yang-Mills coupling on the D3-brane wrapping the essentially doubled space to get

$$g_{\mathrm{YM}}(\widetilde{x})^2 = \left(\frac{(2\pi\,\alpha')^2\,g_s^2}{r^2\,A\,\tau_2(\widetilde{x})}\right)^{1/2}\,, \tag{7.8}$$

which also generally depends on the dual coordinate $\widetilde{x}$. Thus in this case the relevant parameter to be kept finite in the zero slope limit is given by

$$\bar{g}_s^2 := \frac{(\alpha')^2\,g_s^2}{r^2\,A\,\tau_2^\circ}\,, \tag{7.9}$$

which requires the string coupling to scale as $g_s = O(\epsilon^{1/2})$. Then the finite Yang-Mills coupling in the scaling limit is given by (4.23) in the case of parabolic twists, and by (5.23) (for $m\in 4\,\mathbb{Z}+1$ and $\vartheta=\frac{\pi}{2}$) with $x\to\widetilde{x}$ in the case of the $\mathbb{Z}_4$ elliptic twist.

Since the D3-brane wraps the T^2 fibres over the dual $\widetilde{x}$-circle in this case, we now obtain a parameterised noncommutative worldvolume gauge theory, with noncommutative associative star-products of fields f,g given by the Kontsevich star-product

$$f\,\widetilde{\star}\,g = \cdot\left[\exp\left(\tfrac{\mathrm{i}}{2}\,\theta(\widetilde{x})\left(\tfrac{\partial}{\partial y^1}\otimes\tfrac{\partial}{\partial y^2}-\tfrac{\partial}{\partial y^2}\otimes\tfrac{\partial}{\partial y^1}\right)\right)(f\otimes g)\right]\,, \tag{7.10}$$

which is invariant up to Morita equivalence under monodromies $\widetilde{x}\mapsto\widetilde{x}+1$, in the same sense as explained in section 4 and section 5. This shows that a D3-brane wrapping the essentially doubled space has a sensible low-energy effective description, which can be understood as a noncommutative gauge theory over a compactification of the $\widetilde{x}$-direction transverse to its worldvolume in the doubled twisted torus $\mathscr{X}$ using Morita duality. Following the discussion of section 5.3, in the case of the $\mathbb{Z}_4$ elliptic monodromy the noncommutative Yang-Mills action should be augmented by replacing the noncommutative field strength tensor $F_{\widetilde{\star}}$ with

$$\widetilde{\mathcal{F}} = F_{\widetilde{\star}} + \widetilde{\Phi}\,, \tag{7.11}$$

where

$$\widetilde{\Phi} = -\frac{\widetilde{x}}{\theta(\widetilde{x})}\,\mathrm{d}y^1\wedge\mathrm{d}y^2\,, \tag{7.12}$$

thus exibiting further non-geometric dependence of the noncommutative gauge theory on the winding coordinate $\widetilde{x}$.

This can be interpreted as follows. Consider, for example, a D2-brane wrapping the (y^1, y^2)-directions. As discussed in section 1, this is secretly a D3-brane wrapping the $(\widetilde{x}, y^1, y^2)$-directions in the doubled space (see table 1). In the usual (untwisted) case, there is no $\widetilde{x}$-dependence of either the closed background (g, B, ϕ) or the open string moduli $(G_R, \Theta_R, g_{\mathrm{YM}})$ and one can project to the "physical space" with coordinates (x, y^1, y^2), obtaining a $2 + 1$-dimensional supersymmetric Yang-Mills theory in (y^1, y^2, t)-space as the low-energy effective description of the D2-brane. On the other hand, in the twisted case, the open string background has a genuine $\widetilde{x}$-dependence, and so the theory cannot be interpreted in the physical space. This results in a worldvolume theory in the full $(\widetilde{x}, y^1, y^2, t)$-space. A similar interpretation holds for other D-branes.

Acknowledgments

A   The Buscher construction

The Buscher T-duality rules are given by

$$
\begin{aligned}
\widetilde{g}_{\iota\iota} &= \frac{(4\pi^2\,\alpha')^2}{g_{\iota\iota}}\,, \\
\widetilde{g}_{\iota\alpha} &= -2\pi\,\alpha'^{\,2}\,\frac{B_{\iota\alpha}}{g_{\iota\iota}}\,, \\
\widetilde{g}_{\alpha\beta} &= g_{\alpha\beta} - \frac{1}{g_{\iota\iota}}\left(g_{\iota\alpha}\,g_{\iota\beta} - (2\pi\,\alpha')^2\,B_{\iota\alpha}\,B_{\iota\beta}\right), \\
\widetilde{B}_{\iota\alpha} &= -\frac{g_{\iota\alpha}}{g_{\iota\iota}}\,, \\
\widetilde{B}_{\alpha\beta} &= B_{\alpha\beta} - \frac{1}{g_{\iota\iota}}\left(g_{\iota\alpha}\,B_{\iota\beta} - B_{\iota\alpha}\,g_{\iota\beta}\right), \\
\mathrm{e}^{\widetilde{\phi}} &= \left(\frac{2\pi\,\alpha'}{g_{\iota\iota}}\right)^{1/2}\,\mathrm{e}^{\phi}\,.
\end{aligned}
\tag{A.1}
$$

Here the index ι labels the direction of the Killing vector ∂_ι of an isometry of the initial closed string background (g, B, ϕ). Note that the T-duality relations along multiple isometric directions are formally equivalent to the open-closed string relations (2.2) with $G = \widetilde{g}$ and $\Theta = \widetilde{B}^{-1}$.

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