A Petri Net approach to consensus in networks with joint-agent interactions

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Abstract

In this paper we consider consensus protocols where an agent might not be influenced by any of his neighbours singularly taken, but could be sensitive to the simultaneous and coherent influence of two or more of them (joint-agent interaction). By abstracting the set of interactions as a Petri Net we provide a graph-theoretical characterization of the ability of the net to attain asymptotic consensus within the considered set-up.

1 Introduction and motivations

The last decade has witnessed a considerable effort of the scientific community in establishing consensus protocols for multi-agent systems (see i.e. [1,2] and references therein). These are distributed algorithms that allow a population of interacting agents to update some internal state-variable or belief so as to converge asymptotically towards a common value which represents a so called consensus equilibrium for the agents' population ([7,9]). The paper by Luc Moreau, ([4]) has investigated the topological nature of interactions that allow for such a consensus configuration to emerge and proposed requirements on the strength of interactions for this to be the case. In particular a fundamental insight of [4] as well as earlier seminal works by Tsitsiklis and Jadbabaie, [5,6], is in proposing a graph theoretical condition that allows the flow of information to spread across the agents population in a way that is necessary and sufficient, within the considered set-up. The original protocol proposed by Moreau is, however, essentially a linear time-varying differential equation. More recently, many authors have focused on extending such results to several nonlinear scenarios (i.e. [16–18,21]), Markovian random ([19]) and time-dependent (i.e.[20,2]) graphs. For instance, interesting work has been developed in the context of extending consensus protocols to nonlinear spaces, i. e. manifolds (see for instance [8,21,3] and references therein). Another recent research direction, has focused on characterizing necessary and sufficient conditions for consensus under asymmetric confidence intervals, ([21]).
Hereby, we are interested in the scenario where an individual would not be influenced by any of his neighbours singularly taken, but might be sensitive to the simultaneous and coherent influence of two or more of them. A similar mechanism may describe complex contagion process (see i.e. [22] and references therein) and common behaviours in diffusion of innovations in social networks (i.e. conservatism [23]), social influence in opinion dynamics (i.e. conformity, social inertia, preservation [24,25]), in economic and financial decision-making (i.e. risk aversion and conformity in herding phenomena [26]). The simplest such possibility is that of an agent A_3 who would not be influenced by neither A_1 or A_2 unless they both express a consistent influence on him. From the mathematical point of view, this interaction can be modeled by introducing in the equations a term of type:

$$f_{\{1,2\}\to 3}(x) := \min\{\max\{x_1 - x_3, 0\}, \max\{x_2 - x_3, 0\}\} + \max\{\min\{x_1 - x_3, 0\}, \min\{x_2 - x_3, 0\}\}, \tag{1}$$

where x_1, x_2 and x_3 are scalar variables encoding the beliefs of agents A_1, A_2 and A_3 respectively. Notice

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that $f_{\{1,2\}\to 3}$ can only be positive provided both terms $x_1 - x_3$ and $x_2 - x_3$ are positive and, in such cases, it will be equal to the smallest of them. Viceversa, it can only be negative when both $x_1 - x_3$ and $x_2 - x_3$ are negative. In all other cases it will be zero. A term of this kind in the equation for $\dot{x}_3$ entails that agent 3, will only upgrade its state provided 1 and 2 exhert a coherent influence on him. This is an interesting notion per se, as it introduces some kind of intrinsically nonlinear dynamics not allowed by standard consensus protocols, and lends itself to the possibility of consensus being achieved more robustly, for instance in the face of exogenous disturbances, faulty or even malicious agents. Adversary Robust Consensus protocols are studied in [12–14] and implemented through the use of *sorting* and *reducing* maps. These maps, suitably deployed, allow to discard influences of 'extreme' opinions and still propagate enough information to let consensus be achieved robustly among non-malicious agents provided suitable topological conditions hold. Joint-agent interactions encompass the effect of *sorting* and *reducing* maps through their ability of filtering out inconsistent influences among neighboors, see [15]. Indeed, the companion paper [15] is devoted to investigating interaction topologies that lend itself to Adversary Robust Consensus protocols in the generalized set-up of joint-agent interaction. At this level it should be enough to remark that, thanks to the joint-agent nature of interactions, information can only propagate when it is shared coherently by a number of agents at least as high as the multiplicity of the interaction, so that isolated malicious agents will not be effective in perturbing the convergence towards consensus of the remaining ones (see also the simulation example presented in Section 7). As a matter of fact, the novelty of joint-agent interactions is not so much that only a single agent within a given group is going to influence their neighbour (this is for instance typical of "gossiping" interactions), but rather that an agent may cross-validate opinions of his neighbours and only if they agree act upon the information received.

Preliminary results along these lines were first discussed in the conference paper [10]. This manuscript complements and extends [10], by including detailed proofs of all results, providing new results (and relative proofs) about unilateral joint agent interactions, and more extensive simulations.

2 Graph theoretical preliminaries

Our aim is to derive suitable graph-theoretical concepts to describe and analyze the ability of general networks with joint-agent interactions to converge towards consensus. Intuitively, the simplest instance of a network that may achieve consensus in the presence of at least one joint-agent interaction is informally described in the previous Section and given by the following list of interactions among three agents A_1, A_2 and A_3:

$$A_1 \to A_2, \quad \{A_1, A_2\} \to A_3. \tag{2}$$

Above, "$\to$" implies the ability of the nodes listed to the left to influence the node on the right. A more complicated network, comprising n agents $A_1, A_2, \dots A_n$ could be the following:

$$\begin{aligned} \{A_1, A_2\} &\to A_3, \\ &\vdots \\ \{A_{n-2}, A_{n-1}\} &\to A_n, \\ \{A_{n-1}, A_n\} &\to A_1, \\ \{A_n, A_1\} &\to A_2. \end{aligned} \tag{3}$$

This is essentially a ring of agents arranged so that each pair of neighboring ones is able to influence the next in line. Our main result will be a necessary and sufficient criterion for understanding if and when networks such as (3), and in fact of arbitrary structure, may always converge asymptotically towards a consensus equilibrium. For the time being it is worth mentioning that the answer, for the specific network (3), will be *yes* provided n is an odd number, and *no* if n is even. In order to state our main criteria we will need to introduce suitable graph theoretical concepts. We borrow, to this end, the language of Petri Nets. Even though we only need Petri Nets as a convenient way to describe *bipartite directed graphs*, and never as discrete event systems (DES), it turns out that some of the structural properties studied in the context of DES are helpful in classifying the flow of information across the network and its ability to induce attainment of consensus among agents.

An (ordinary) Petri Net is a quadruple $\{P, T, E_I, E_O\}$, where P and T are finite sets (with $P \cap T = \emptyset$) referred to as Places and Transitions, respectively. These are regarded as nodes of a directed bipartite graph. Two types of directed edges are allowed: $E_I \subset T \times P$ connecting transitions to places and $E_O \subset P \times T$ connecting places to transitions. The subscripts refer to the fact that, with respect to places, E_I can be seen as input arcs, while E_O are output arcs. In our context Places represent agents, while Transitions are modeling interactions among them. In particular, if agents $A_1, \dots, A_{n+1}$ are represented by places $p_1, \dots, p_{n+1}$ the interaction:

$$\{A_1, A_2, \dots, A_n\} \to A_{n+1} \tag{4}$$

is represented by a single transition $t \in T$, with edges $(p_1, t), (p_2, t), \dots, (p_n, t)$ in E_O and (t, p_{n+1}) in E_I. In particular, every transition can be assumed, within our set-up, to only afford exactly one outgoing edge. This is not the case in general Petri Nets. As an example

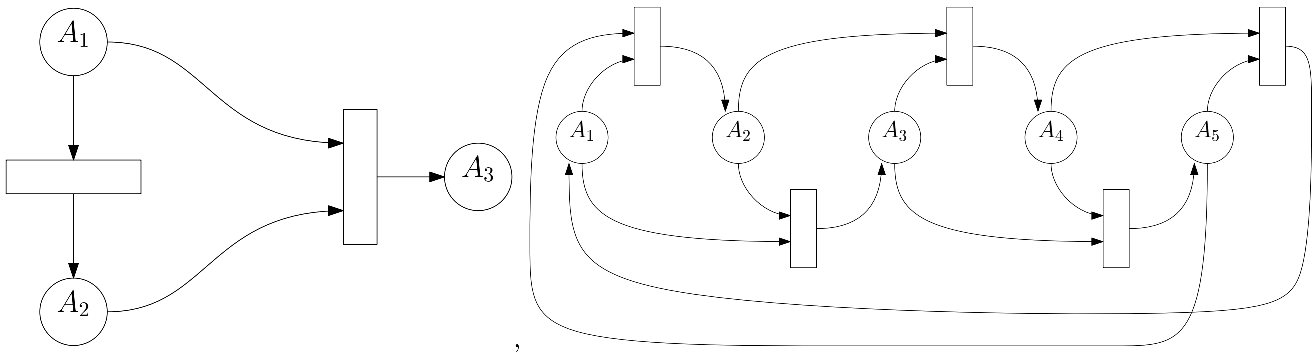

Fig. 1. Petri Nets associated to network of interactions (2) and (3)

we show in Fig. 1 the graphical representation of the Petri Net associated to the list of interactions (2) and (3) (with $n = 5$) respectively. The next concepts will be crucial in characterizing, from the topological point of view, networks that guarantee asymptotic convergence towards consensus. The set of *input transitions* for a place p, is denoted as:

$$I(p) = \{t \in T : (t, p) \in E_I\},$$

and, similarly for a set of places $S \subset P$, its input transitions are $I(S) = \{t \in T : \exists p \in S : (t, p) \in E_I\}$. Symmetrically, output transitions are denoted as: $O(S) = \{t \in T : \exists p \in S : (p, t) \in E_O\}$. Moreover, when dealing with more than one Petri Net, we emphasize the net N currently considered as a subscript, i.e. $I_N(S)$ and $O_N(S)$.

Definition 1 *A non-empty set of places $S \subset P$ is called a* siphon *if $I(S) \subset O(S)$. A siphon is minimal if no proper subset is also a siphon.*

Informally, this means that any influence on such agents needs to come (at least in part) from inside the group.

For intance, in network (2), the set $S = \{p_1\}$ is a siphon. This is trivially true as $I(p_1) = \emptyset$. Notice that S is also a minimal siphon. Moreover, any other siphon includes $\{p_1\}$. Therefore S is the only minimal siphon.

The structure of siphons in example (3) is more complex. Minimal siphons always include 3 places (arranged in a ring) with gaps in between them of at most one place. As an example $\{p_1, p_3, p_5\}$ is a minimal siphon and so is $\{p_2, p_4, p_5\}$ or $\{p_1, p_2, p_4\}$. There are a total of 5 minimal siphons for this network.

Remark 2 *It is worth pointing out that, in traditional consensus protocols, all interactions are of the form $A_i \to A_j$, that are assumed to happen between two agents alone, one acting as a leader and one as a follower. Formulating this situation within the Petri Net framework every transition has exactly one incoming and one outgoing edge. Petri Nets of this special kind are also called* state machines *and, from the topological point of view, are in fact isomorphic to standard directed graphs, with nodes corresponding to places, and arcs (p_i, p_j) replacing each transition t and the two linked edges (p_i, t), (t, p_j).*

The definition of siphon boils down, for standard graphs and thanks to the isomorphism described above, to the notion of a set of nodes without exogenous incoming edges. It is intuitive that sets of agents with this property afford some degree of stubborness that, while not incompatible with consensus, may only happen in specific ways so as to prevent multiple opinions to coexist asymptotically. We will make this precise in Section 5 (see for instance Corollary 12).

3 Problem formulation

We consider, in the following, nonlinear finite dimensional dynamical systems of the following form:

$$\dot{x} = f(x) \tag{5}$$

with state x taking values in $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ a locally Lipschitz map. The unique solutions of (5) corresponding to initial condition $x(0) = x_0$ is denoted as: $x(t) = \varphi(t, x_0)$. This set of differential equations describes the dynamics of n interacting agents. We are going to provide proofs that are based on the machinery of ω-limit sets and time invariance, as this allows for a much more direct analysis. Our goal is to identify conditions under which solutions of (5) asymptotically, converge towards equilibriums of the following form:

$$\lim_{t \to +\infty} \varphi(t, x_0) = \bar{x}\mathbf{1} \tag{6}$$

for some $\bar{x} \in \mathbb{R}$, where $\mathbf{1}$ is the vector of all ones in $\mathbb{R}^n$. When this occurs for all initial conditions we say that system (5) achieves *global asymptotic consensus.* To this end we formulate the following conditions:

Assumption 3 *For any $\bar{x}$ in $\mathbb{R}$ the following holds:*

$$f(\bar{x}\boldsymbol{1}) = 0, \tag{7}$$

moreover for all i, $x_i \mathbf{1} \geq x$ (respectively $x_i \mathbf{1} \leq x$) implies $f_i(x) \leq 0$ (respectively $f_i(x) \geq 0$).

Assumption 3 merely ensures that consensus configurations are equilibria for system (5) and that any agent achieving the maximum (or the minimum) cannot further increase (respectively decrease) its state value. In order to make consensus configurations attractive and stable we need to introduce suitable interactions among the agents. The following notion plays a crucial role in this respect.

Definition 4 *We say that a (non-empty) set of agents $I \subset \{1, 2, \ldots, n\}$ influences agent $j \notin I$, and we denote this by $I \rightarrow j$ if whenever $x_j = \max_i x_i$ (respectively $x_j = \min_i x_i$) and $x_j > x_i$ (respectively $x_j < x_i$) for all $i \in I$ it holds $f_j(x) < 0$ (respectively $f_j(x) > 0$.*

We call this type of interactions joint-agent interactions, as they need, in general, all of the agents in I acting simultaneously and consistently on j in order for the influence to be effective.

Notice that, by definition, if I influences j, any superset $\tilde{I} \supseteq I$ does influence j. Because of this, it is enough to focus our attention on *minimal* joint-agent interactions. We say that the influence between I and $j \notin I$ is minimal whenever no proper subset of I influences j. Notice that single agents interactions are always minimal. We point out that absence of transition $\{i\} \rightarrow j$ between agents i and j does not imply that i does not influence j. A simple example of time-invariant network involving joint-agents interaction is given below:

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= x_1 - x_2 \\ \dot{x}_3 &= f_{\{1,2\} \rightarrow 3}(x_1, x_2, x_3) \end{aligned} \tag{8}$$

Notice that A_1 is a *stubborn* agent as $\dot{x}_1$ is zero and is not influenced by other agents' state values. On the other hand, it is easy to see that 1 influences 2 (according to our definition of joint agent interactions), and also $\{1, 2\}$ influences 3. Moreover, no proper subset of $\{1, 2\}$ has influence on 3. Therefore equation (2) lists all the minimal influences which can be associated to (8). Thanks to Definition 4 we may associate to any system as in (5) a Petri Net $N = \{P, T, E_O, E_I\}$ with the following definition.

Definition 5 *Given a system (5) and its set of minimal joint-agent interactions we construct a Petri Net $\{P, T, E_I, E_O\}$ according to the following rules:*

- *Set of places: $P = \{p_1, p_2, \ldots, p_n\}$, n being the dimension of x (and total number of agents);*
- *Transitions: $T = \{t_{I_1,j_1}, \ldots t_{I_q,j_q}\}$ where influence between I_k and j_k is minimal for all $k \in 1, \ldots, q$;*
- *Input arcs: $E_I = \{(t_{I_1,j_1}, p_{j_1}), \ldots, (t_{I_q,j_q}, p_{j_q})\}$*
- *Output arcs: E_O, of the type (p_i, t_{I_k,j_k}) for all $i \in I_k$ and all $k = 1 \ldots q$*

We are now ready to formulate our main technical tool for assessing the ability of a network with joint-agent interactions to reach consensus.

Definition 6 *We say that a Petri Net fulfils the* siphon overlapping property *if each pair of siphons S_1, S_2 have non-empty intersection.*

Equivalently a Petri Net fulfils the siphon overlapping property if it does not admit two disjoint (minimal) siphons. We show below that, in the case of traditional influence graphs (where all interactions are one to one), the notion of siphon overlapping is equivalent to existence of a spanning-tree. It therefore boils down to the well known connectivity assumptions pioneered by Moreau and widely adopted in the subsequent literature. The following Lemma is fairly straightforward:

Lemma 7 *Consider a Petri Net where for each transition $t \in T$ there exist exactly two distinct places $p_I(t)$ and $p_O(t)$ such that $(t, p_I) \in E_I$ and $(p_O, t) \in E_O$ (such networks are usually referred to as State Machines). Then, the network can be associated to an influence graph $G = \{P, E\}$ where $E = \cup_{t \in T}(p_O(t), p_I(t))$. Moreover a set $\Sigma \subset P$ is a minimal siphon of the Petri Net if and only if it is the set of nodes of a strongly connected component of G which has no incoming edges.*

Proof Let $\Sigma \subset P$ be the set of nodes of a strongly connected component of $\{P, E\}$ without incoming edges. Take any $p \in \Sigma$. For every transition t, such that $(t, p) \in E_I$, there exists a place $p_O(t)$, such that $(p_O(t), t)$ belongs to E_O. Hence $(p_O(t), p)$ belongs to E, and because Σ has no incoming edges, $p_O(t)$ belongs to Σ. This shows that Σ is a siphon in the Petri Net $\{P, T, E_I, E_O\}$. It is minimal because any proper subset $\tilde{\Sigma}$ of Σ admits (by the strong connectivity assumption) incident arcs that come from outside $\tilde{\Sigma}$ and it is therefore not a siphon. Conversely, taken any minimal siphon $\Sigma \subset P$. Let $(q, p) \in E$ be arbitrary and p belong to Σ. Hence there exists a transition t such that $(t, p) \in E_I$ and $(q, t) \in E_O$. Moreover, by definition of siphon and recalling that q is the only transition such that (q, t) belongs to E_O, we see that q belongs to Σ. This implies that Σ has no incoming edges in the graph $\{P, E\}$. Moreover, by minimality of Σ, every proper subset $\tilde{\Sigma}$ is not a siphon and admits a transition t such that $p_I(t) \in \tilde{\Sigma}$ and $p_O(t) \notin \tilde{\Sigma}$. This shows that Σ is strongly connected.

Remark 8 *By virtue of Lemma 7, absence of disjoint minimal siphons in a State-Machine, implies absence of disjoint strongly connected components without incoming edges in the associated graph G. Since, by construction, strongly connected components are always disjoint, whenever distinct, the siphon overlapping property amounts to*

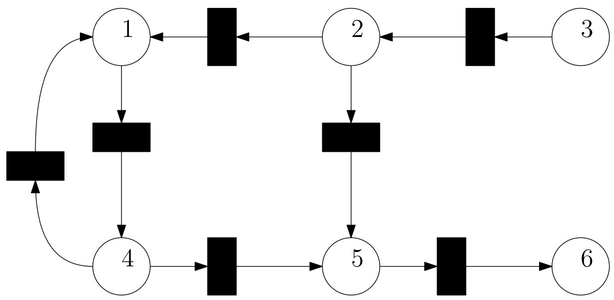

Fig. 2. An example of Petri Net without joint-agent interactions: a State-Machine

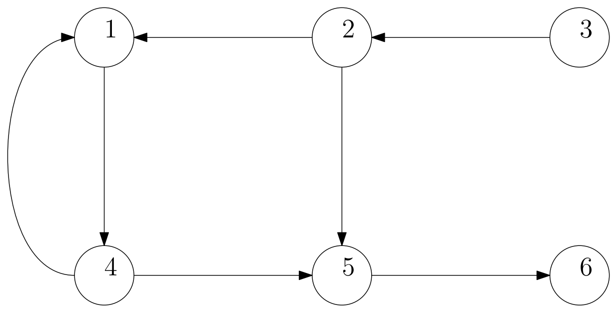

Fig. 3. Graph associated to the State-Machine in Fig 2

the existence of a unique strongly connected component without incoming edges in the influence graph G. This, in turn, is equivalent to existence of a spanning tree in G.

In order to illustrate the above Remark, consider for instance the Petri Net in Fig. 2, which indeed does not exhibit joint-agent interactions. One can easily recast such networks as standard directed graphs, as in Fig. 3. Notice that this graph admits at least one spanning tree (its root node is Agent 3). Moreover, by virtue of Lemma 7, there is a unique minimal siphon in every State-Machine, and this is the unique strongly connected component of the associated graph that has no incoming edges. This is indeed agent $\{3\}$ for the considered example. Every siphon needs to contain a minimal siphon and therefore contains the element 3. For this reason, every pair of distinct siphons intersect non trivially. More in general the unique minimal siphon of a State Machine coincides with the set of all roots of spanning trees for the associated graph. It is worth pointing out that, for a Petri Net N and the associated $\tilde{N}$ which only lists as transitions those corresponding to minimal joint-agent interactions, the siphon overlapping property holds for N if and only if it holds for $\tilde{N}$. This is not obvious, but stated without proof for the sake of space.

4 Main result

We are now ready to state our main result.

Theorem 9 *Consider a cooperative system as in (5) and fulfilling Assumption 3. Then a sufficient condition for global asymptotic consensus is that the associated Petri Net $N = \{P, T, E_I, E_O\}$ fulfils the siphon overlapping property.*

Proof Consider an arbitrary initial condition $x_0 \in \mathbb{R}^n$ and denote by $x(t) := \varphi(t, x_0)$ the corresponding solution at time t. We define the following quantities,

$$x_M(t) := \max_{i \in \{1,2,\dots,n\}} x_i(t)$$

$$x_m(t) := \min_{i \in \{1,2,\dots,n\}} x_i(t).$$

As usual in consensus dynamics, it is relatively straightforward to see that $x_M(t)$ and $x_m(t)$ are, respectively, monotonically non-increasing and non-decreasing. As a consequence, all solutions are uniformly bounded and the limits:

$$\bar{x}_M := \lim_{t \to +\infty} x_M(t)$$

$$\bar{x}_m := \lim_{t \to +\infty} x_m(t)$$

exist. Moreover, by boundedness of solutions, the ω-limit set $\omega(x_0)$ is non-empty, compact and invariant. Pick any state $\hat{x} \in \omega(x_0)$. Clearly $\max_i \hat{x}_i = \bar{x}_M$ and similarly $\min_i \hat{x}_i = \bar{x}_m$. We define the (non-empty) sets

$$M(\hat{x}) := \{i : \hat{x}_i = \bar{x}_M\}$$

$$m(\hat{x}) := \{i : \hat{x}_i = \bar{x}_m\}.$$

Notice that, by invariance of $\omega(x_0)$, $\varphi(t, \hat{x})$ is well defined and belongs to $\omega(x_0)$ for all $t \in \mathbb{R}$. In particular, $M(\varphi(t, \hat{x}))$ is non-empty for all t. Moreover, by Lipschitzianity of f,

$$t_2 \geq t_1 \Rightarrow M(\varphi(t_2, \hat{x})) \subset M(\varphi(t_1, \hat{x})).$$

Given finiteness of the set of agents, for some finite T_M then it holds:

$$M(\varphi(T_M, \hat{x})) = M(\varphi(t, \hat{x})) \qquad \forall\, t \geq T_M. \tag{9}$$

Similarly $m(\varphi(t, \hat{x}))$ is non-empty and fulfils:

$$t_2 \geq t_1 \Rightarrow m(\varphi(t_2, \hat{x})) \subset m(\varphi(t_1, \hat{x})).$$

Hence for some finite time T_m it fulfills

$$m(\varphi(T_m, \hat{x})) = m(\varphi(t, \hat{x})) \qquad \forall\, t \geq T_m.$$

We claim that for all $t \geq T_M$ the set $M(\varphi(t, \hat{x}))$ is a siphon of the associated Petri Net. Similar claim holds for $m(\varphi(t, \hat{x})$. The claim is proved in Lemma 10 given below. We show next the sufficient part of the implication. Assume the siphon overlapping property holds. Then, for $t \geq \max\{T_m, T_M\}$, $M(\varphi(t, \hat{x})) \cap m(\varphi(t, \hat{x}))$ is non-empty. This implies, in particular that $\bar{x}_M = \bar{x}_m := \bar{x}$.

Hence, for all $i \in \{1, 2, \ldots, n\}$

$$\lim_{t \to +\infty} x_i(t) = \bar{x}.$$

This shows global consensus, since x_0 was arbitrary to start with.

The main technical novelty for the proof of Theorem 9 is therefore the following Lemma

Lemma 10 *Let $\hat{x} \in \omega(x_0)$ be arbitrary and $t \geq T_M$ as in equation (9). Then $M(\varphi(t, \hat{x}))$ is a siphon.*

Proof Let $j \in M(\varphi(t, \hat{x}))$ and $x(t) = \varphi(t, \hat{x})$ for simplicity of notation. For any $I \subset P$ such that $I \to j$ we need to show that $I \cap M(\varphi(t, \hat{x}))$ is non-empty. Of course, $x_j(t) = \bar{x}_M$ for all $t \geq T_M$ and therefore $\dot{x}_j(t) = f_j(\varphi(t, \hat{x})) = 0$. By contradiction, assuming $I \cap M(\varphi(t, \hat{x})$ empty and by definition of joint interaction we see that:

$$\bar{x}_M = x_j(t) > x_i(t), \forall\, i \in I \;\Rightarrow\; f_j(x(t)) < 0,$$

which contradicts our previous conclusion. Hence, there exists $i \in I$ that belongs to $M(\varphi(t, \hat{x}))$.

5 Siphons and stubborn sets of agents

We consider next a slightly more specific class of network equations. Restricting the attention to this class will allow to claim necessity of the consensuability conditions stated and also prove additional interesting properties of their dynamics. To start with, for an arbitrary set $I \subset \{1, 2, \ldots, n\}$ and for any $j \notin I$ one might define the following type of influence term:

$$f_{I \to j}(x) = -\frac{\partial}{\partial x_j} |x_j|^2_{\mathrm{co}\{x_i : i \in I\}}.$$

This is an interaction influence from agents in I towards agent j, which grows linearly in proportion to the distance of agent j to the convex hull of all agents in I. It is a generalization, for an arbitrary number of agents, of the function defined in (1). For convenience of notation, given a transition $t \in T$, we let $\bullet t$ denoted the set of agents i (or places) such that (i, t) belongs to E_I. Dually, $t\bullet$ denotes the agent j such that (t, j) belongs to E_O.

In order to define the class of equations considered, we start from a given Petri Net $N_g := \{P, T, E_I, E_O\}$ and build the following nonlinear systems of differential equations:

$$\dot{x} = \sum_{t \in T} f_{\bullet t \to t\bullet}(x) e_{t\bullet}. \tag{10}$$

This is a cooperative network, of the kind considered so far. Its generating Petri net coincides with the net associated according to Definition 5 provided only minimal joint-agent interactions are listed in T. If not, the generating Petri Net N_g includes the network of minimal joint-agent interactions, and is, in this respect, slightly redundant in encoding the information flow as far as consensuability analysis is concerned. The following proposition is remarkable, and may serve as a starting point for designing networks with prescribed global dynamics, in particular for assigning the value of consensus that the network is allowed to reach under suitable siphon overlapping assumptions.

Proposition 11 *Let Σ be a siphon of N_g (or, equivalently, of the associated minimal network of interactions), and $x(t)$ denote $\varphi(t, x_0)$ for an arbitrary initial condition. Then, the functions*

$$\bar{x}_\Sigma(t) := \max_{i \in \Sigma} x_i(t) \qquad \underline{x}_\Sigma(t) := \min_{i \in \Sigma} x_i(t), \tag{11}$$

are, respectively, monotonically non-increasing and non-decreasing.

Proof We prove monotonicity of $\bar{x}_\Sigma$ only, as the proof for $\underline{x}_\Sigma$ follows along the same lines. Let, for each $M \in \mathbb{R}$, C_M denote the closed set given below:

$$C_M := \{x \in \mathbb{R}^n : x_i \leq M, \forall\, i \in \Sigma\}.$$

We remark that C_M is forward invariant with respect to system (10). In fact, adopting the notion of tangent cone we see that

$$TC_x(C_M) = \{v \in \mathbb{R}_n : v_i \leq 0, \forall\, i \in \Sigma : x_i = M\}.$$

Let $x \in C_M$ and $i \in \Sigma$ be such that $x_i = M$. We then compute:

$$\begin{aligned} \dot{x}_i &= e_i' \sum_{t \in T} f_{\bullet t \to t\bullet}(x) e_{t\bullet} \\ &= \sum_{t \in T : i = t\bullet} f_{\bullet t \to t\bullet}(x). \end{aligned}$$

Notice that, for transitions $t \in T$ with $i = t\bullet$, $\bullet t$ contains at least some $j \in \Sigma$, and such that, as a consequence, $x_j \leq M = x_i$. In particular then, $f_{\bullet t \to t\bullet}(x) \leq 0$. This shows that, $\dot{x}_i \leq 0$ and in particular,

$$\sum_{t \in T} f_{\bullet t \to t\bullet}(x) e_{t\bullet} \in TC_x(C_M) \qquad \forall\, x \in C_M.$$

Forward invariance of C_M then follows by Nagumo's Theorem ([11]).

Notice that forward invariance of C_M implies:

$$x(t) \in C_M \Rightarrow x(\tau) \in C_M \quad \forall \tau \geq t.$$

This is equivalent to:

$$\bar{x}_\Sigma(t) \leq M \Rightarrow \bar{x}_\Sigma(\tau) \leq M \quad \forall \tau \geq t.$$

In particular then, forward invariance of C_M for all $M \in \mathbb{R}$ yields letting $M = \bar{x}_\Sigma(t)$,

$$\bar{x}_\Sigma(\tau) \leq \bar{x}_\Sigma(t) \qquad \forall \tau \geq t.$$

This concludes the proof of Proposition 11.

The following is an easy corollary of the previous Proposition.

Corollary 12 *Let $\sigma \in \mathbb{R}$ be arbitrary. Assume that all agents i in a siphon Σ are initialized with $x_i(0) = \sigma$. Then, for all $i \in \Sigma$ and all $t \geq 0$, it holds $x_i(t) = \sigma$.*

In the context of opinion dynamics, agents which preserve their initial opinion value and neglect exogenous influences, are usually defined as *stubborn*. The notion introduced in Corollary 12 is a far-reaching generalization of the concept of stubborn agent, that applies to specific groups of agents depending upon their mutual influence patterns. In particular, siphons of the net, qualify and characterize stubborn groups of agents. Notice that individual agents may not be aware of their being part of a siphon, and therefore of their acting stubbornly, as this is only determined by the global influence patterns of the network, rather than individual predisposition to discard exogenous influences as in the case of traditional stubborn agents.

Corollary 12 allows to prove necessity of the siphon overlapping property for asymptotic consensus. Below we formulate this result explicitly.

Proposition 13 *Consider a Petri Net N, and the associated system of differential equations as in (10). Then a necessary condition for the network to achieve asymptotic consensus regardless of initial conditions $x(0) \in \mathbb{R}^n$, is that N fulfills the siphon overlapping property.*

Proof Assume N does not fulfill the siphon overlapping property. Then, there exist siphons Σ_1 and Σ_2 such that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Take any initial condition with $x(0)$ with $x_i(0) = 1$ for all $i \in \Sigma_1$ and $x_i(0) = 2$ for all $i \in \Sigma_2$. Then, by virtue of Corollary 12 $x_i(t) = 1$ for all $t \geq 0$ and all $i \in \Sigma_1$ and, similarly, $x_i(t) = 2$ for all $i \in \Sigma_2$. This, however, contradicts asymptotic consensus and proves the claim.

Thanks to Proposition 11 and Theorem 9 we can a priori characterize the set of values allowed for asymptotic consensus.

Theorem 14 *Consider a network fulfilling the siphon overlapping property. Then, for all initial conditions $x(0)$, the associated solution $x(t)$ asymptotically converges to some consensus value σ fulfilling:*

$$\sigma \in \bigcap_{\Sigma} [\underline{x}_\Sigma(0), \bar{x}_\Sigma(0)] \tag{12}$$

where the intersection is taken over all (minimal) siphons Σ of the associated Petri Net.

It is worth pointing out that the intersection in (12) is always non-empty for a network fulfilling the siphon overlapping property. In fact, pairwise overlapping intervals of the real line have always non-empty intersection. Notice that, for networks with single-agent interactions, convergence is known to occur within the convex hull of the set of agents which are roots of at least a spanning-tree (this is, in the interaction graph, the unique strongly-connected component without incoming edges). Theorem 14 can be seen as a generalization to the considered set-up of the above result.

6 State-dependent embedding

We consider, next, the situation arising when equation (5) exhibits some form of additive structure, for instance:

$$\dot{x}_i = \sum_{J \in \mathcal{J}_i} k_{J \to i} f_{J \to i}(x), \tag{13}$$

as in most typical examples. In (13), $\mathcal{J}_i \subset 2^{\{1,2,\ldots,n\}}$ denotes the set of joint-agent groups affecting agent i, while $k_{J \to i} > 0$ is a constant quantifying strength of interaction. Under such circumstances, it is possible, with relative ease, to embed the dynamics by using pseudo-linear state-dependent embeddings. In particular, for each permutation $\pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$ of the n agents, we may consider the associated state-space region:

$$X_\pi := \{x \in \mathbb{R}^n : x_{\pi_1} \leq x_{\pi_2} \leq \ldots \leq x_{\pi_n}\}. \tag{14}$$

Denoting by Π the set of all permutations in n elements, we see that:

$$\mathbb{R}^n = \bigcup_{\pi \in \Pi} X_\pi.$$

Hence, we may define the following set valued mapping: $\pi(x) : \mathbb{R}^n \to 2^\Pi$, where

$$\pi(x) := \{\pi \in \Pi : x \in X_\pi\}. \tag{15}$$

Moreover, for each permutation π (and therefore in each of the regions X_π), one may consider a suitably defined matrix A_π so that (13) fulfills standard linear consensus dynamics:

$$\dot{x} = A_{\pi(x)}x. \tag{16}$$

While a priori, equation (16) may seem to have a discontinuous right-hand side, thanks to equivalence with (13) it only admits standard Caratheodory solutions. One may be tempted, in fact, to analyze consensus of equations as (13) according to a family of underlying linear consensus protocols. In particular, by associating to each region X_π, a standard graph G_π, as it would follow by considering equation (16) restricted on X_π. Notice that, for an n-dimensional network there are $n!$ associated regions, and possibly up to $n!$ interaction graphs to consider. As expected our conditions for consensuability *are not equivalent* to asking existence of a spanning tree in G_π for each $\pi \in \Pi$. They are much weaker as seen, for instance, in the example of Section 7. Indeed a spanning-tree in time-varying (and/or state dependent) networks could arise considering the union of interaction graphs experienced along each solution across a shifting time-window (of sufficient length). On the other hand, especially for fairly large values of n, it is normally extremely hard to examine a priori all possible sequence of regions X_π visited along solutions, or how large the considered time-window ought to be. Indeed a combinatorial explosion of this analysis technique ought to be expected except for the most simple instances. Several works have pointed out the convergence time and computation issues related to the agreement problems (see i.e. [30,31]).

In some sense, our analysis technique could be interpreted as a tight necessary and sufficient condition to characterize interaction topologies which, along solutions of (13) and regardless of initial conditions, give rise to sequences of interaction graphs that fulfill existence of a spanning tree in the union $\bigcup_{\tau \geq t} G_{\pi(x(\tau))}$ as in classical consensus protocols.

7 Examples

Next we demonstrate our main result when considering a ring of agents 3 with the following dynamics, including bilateral 2-joint interaction terms:

$$\begin{aligned} \dot{x}_{i+2} = f_{\{i,i+1\} \to i+2}(x) = \min\{&\max\{x_i - x_{i+2}, 0\}, \\ &\max\{x_{i+1} - x_{i+2}, 0\}\} + \max\{\min\{x_i - x_{i+2}, 0\}, \\ &\min\{x_{i+1} - x_{i+2}, 0\}\}. \end{aligned} \tag{17}$$

where indices are all meant modulus n (number of agents). Notice that this expression is only one of many possible alternative functions giving rise to joint-agent interactions. For instance, using sigmoids $\sigma(z) = e^z/(1+e^z)$ one could define

$$f_{1,2\to 3}(x) = \sigma(x_2 - x_3)\sigma(x_1 - x_3) - \sigma(x_3 - x_2)\sigma(x_3 - x_1).$$

We initially consider the case of an odd number of agents $n = 7$. The minimal siphons for the associated Petri Net are listed below:

$$\begin{aligned} &\{1,3,5,7\}, \{1,3,5,6\}, \{1,3,4,6\}, \{1,2,4,6\}, \\ &\{2,3,5,7\}\{2,4,5,7\}, \{7,2,4,6\}. \end{aligned} \tag{18}$$

Notice that they all have 4 elements and obviously, in a net with 7 places, have pairwise non-empty intersections. Therefore, the network fulfills the siphon overlapping property. While finding siphons may be a computationally expensive task, visual inspection is enough in the case of the simple ring topology considered. Moreover, one of the advantages of formulating consensuability conditions by using a well-known concept (siphon) in the field of Petri nets is the possibility of then referring back to the rich literature on characterizations and computation of such objects, see for instance [29] and references therein. We consider the following initial condition, $[20, 1, 18, 2, 16, 3, 14]$. Next, for each minimal siphon Σ, we can compute the intervals $[\underline{x}_\Sigma, \bar{x}_\Sigma]$. This yields the following intervals (corresponding to the siphons listed by (18)):

$$[14, 20],\ [3, 20],\ [2, 20],\ [1, 20],\ [1, 14].$$

Notice that such intervals have, as it should be, non-empty intersection, in particular:

$$[14, 20] \cap [3, 20] \cap [2, 20] \cap [1, 20] \cap [1, 14] = \{14\}.$$

Consistently with equation (12), we see in numerical simulations (see Fig. 5) that consensus is achieved at 14. The latter is a situation where Theorem 9 provides necessary and sufficient conditions for consensuability in the novel setup of joint-agent interactions.

In order to emphasize the merits of our main result we only attempt an analysis using the notion of pseudo-linear embedding. Notice that in this case we would have $7! = 5040$ pernutations and associated regions to consider. Each one of them induces a different graph G_π. For instance, considering $X_\pi = \{x : x_1 \leq x_3 \leq x_5 \leq x_7 \leq x_6 \leq x_4 \leq x_2\}$ we see that the associated graph G_π is as in Fig. 4. Indeed, Agents 7 and 1 are jointly able to influence Agent 2 (as $x_1 \leq x_7 \leq x_2$ in X_π) and do so with an intensity which is proportional to $(x_7 - x_2)$. Hence an edge from 7 to 2 appears in G_π. Similarly Agent 7 and Agent 6 are jointly able to influence Agent 1 as $x_1 \leq x_7 \leq x_6$ and the intensity of interaction is proportional to $(x_7 - x_1)$. On the other hand, the influence $\{1, 2\} \to 3$ is not active in X_π since $x_1 \leq x_3 \leq x_2$. Similarly, for all other interactions $J \to i$, we see that agent i has a state value which is intermediate among those of agents in J. Notice that this graph does not admit a spanning tree, and it is only thanks to the transit towards contiguous regions that overall the connectivity

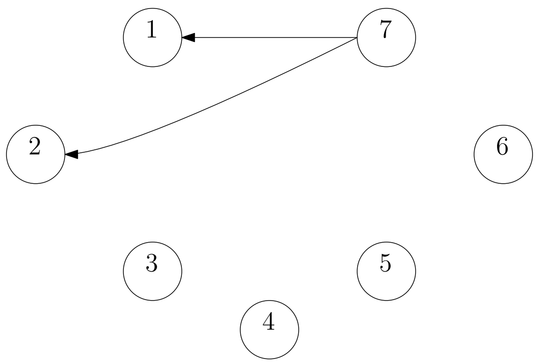

Fig. 4. Example of graph G_π associated to network (17)

conditions guaranteeing sufficient propagation of information can be fulfilled. However, it is not obvious which regions would be visited next, as this is a function of initial conditions. Overall, direct application of standard tools appear to quickly become prohibitive.
While it is true that, for specific networks, symmetry or other ad hoc considerations might reduce the algorithmic complexity of a pseudo-linear embedding (and this might well be the case in the ring topology considered), the example highlights the exponential growth intrinsic in the pseudo-linear approach as well as the difficulty inherent in not having apriori knowledge of the expected switching sequence of interaction graphs.

Next we show, by means of numerical simulations, that consensus is not achieved from every initial condition when a cyclic network is considered in the case of $n = 8$ (Fig. 6 for associated plots). In particular, we picked the following initial condition $[20, 1, 18, 2, 16, 3, 14, 4]'$. Notice that, $\Sigma_1 := \{1, 3, 5, 7\}$ and $\Sigma_2 := \{2, 4, 6, 8\}$ are both siphons of the associated Petri Net. By virtue of Proposition 11 then, $\min_{i\in\Sigma_1} x_i(t)$ is a non-decreasing function. In particular,

$$\min_{i\in\Sigma_1} x_i(t) \geq \min_{i\in\Sigma_1} x_i(0) = 14.$$

Similarly,

$$\max_{i\in\Sigma_2} x_i(t) \leq \max_{i\in\Sigma_2} x_i(0) = 4.$$

The Proposition, in particular, allows to conclude that consensus cannot be expected from the considered initial condition. Notice that, in agreement with the above inequalities, two clusters are achieved asymptotically exactly at the values 4 and 14. This type of bipartite consensus may arise, for instance, when using competitive interactions (i.e. [27,28]). Differently, in the current setup, specific arrangements of initial conditions may result in topologies of active interconnections that are split, and never allow influences to propagate across distinct groups of a certain partition.

Next we validate the proposed conditions over larger networks. Specifically, we consider two random networks of 50 nodes. We initially generate an underlying undirected graph where edges between agents exist with probability 0.25 and 0.12, respectively. Then, we implement joint-agent interactions from each pairs of its neighbours towards every agent in the network. We confirm that relatively high connectivity (i.e. average degree $11.9 \simeq 0.25 \cdot 50$) can yield consensus for random integer initial conditions (Fig. 7), while for lower connectivity values (average degree $5.6 \simeq 0.12 \cdot 50$) we observe a clustered equilibrium with levels related to non-overlapping siphons (Fig. 8). In particular, agents 7 and 29 (with initial conditions 8 and 5 respectively) are a siphon of the network. Similarly the complement, $\{1, \ldots, 50\}\backslash\{7, 29\}$ is a larger, non-overlapped siphon.

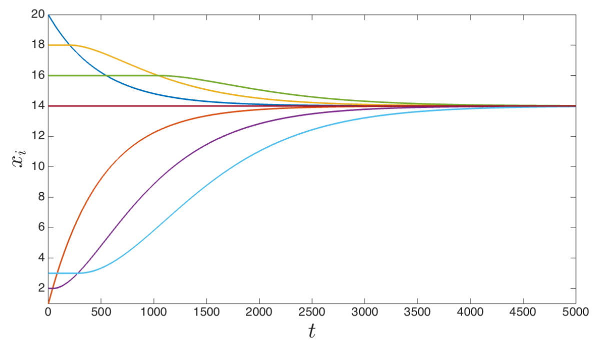

Fig. 5. Convergence to consensus for $n = 7$

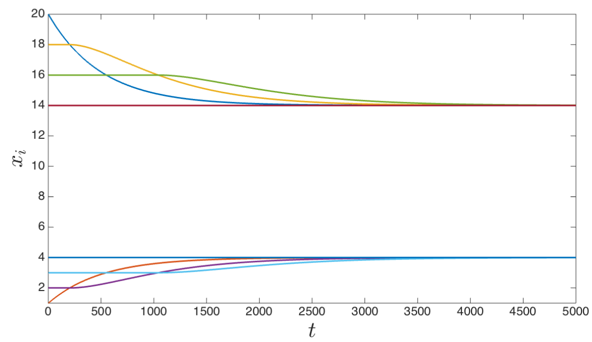

Fig. 6. Convergence to clustered equilibrium for $n = 8$

Finally, we provide numerical evidence of the robustness exhibited by networks with joint-agent interactions by considering again the cyclic network with 7 agents. While the behaviour of the simulation reported in Fig. 9 does not follow from the previous analysis, a companion paper will describe under what topological conditions Adversary Robust Consensus (as proposed in [12–14]) is guaranteed. In particular, we perturb with an additive sinusoidal disturbance the equation of agent 2 and initialize the population with $x_0 = [0, 5, 3, 7, 11, 9, 1]'$. Notice that while the trajectory of agent 2 is disrupted (and possibly disrupting), all remaining agents are nevertheless able to reach a consensus state.

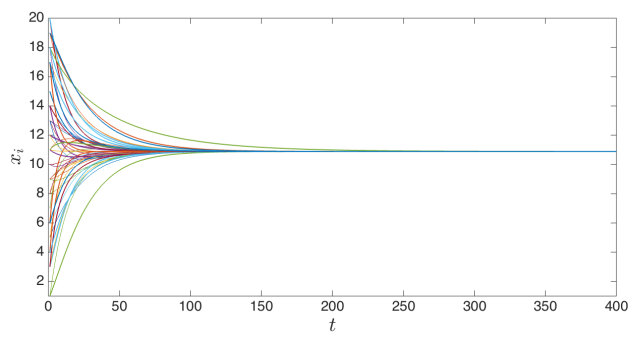

Fig. 7. Random network of 50 nodes with connectivity probability 0.25 and average degree 11.9: convergence to consensus

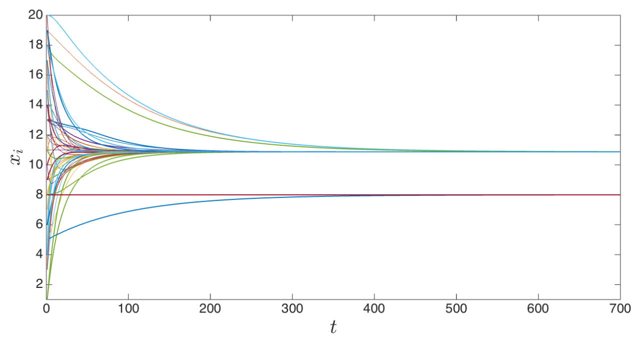

Fig. 8. Random network of 50 nodes with connectivity probability 0.12 and average degree 5.6: convergence to clustered equilibrium

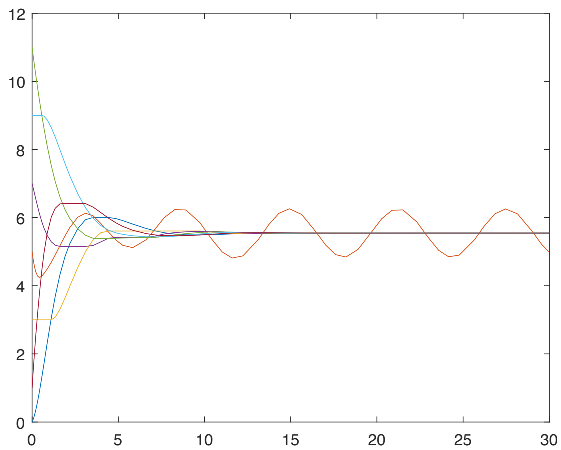

Fig. 9. $n = 7$: robustness under disturbance on agent 2

8 Unilateral joint-agent interactions

Under some circumstances (for instance in synthetic networks designed to converge towards a specific consensus value, or in opinion dynamics) it might be desirable to have separate information flows between agents depending upon the relative ordering of their current state values. In opinion dynamics, for instance, one may allow for asymmetric confidence, that is a situation in which influence from a neighboring agents is felt only within a certain interval of influence that may be asymmetric with respect to the current agents opinion. Even more radically, in the case of unilateral interactions, one accounts for interactions that only occur whenever a neighbor is above (in the case of so called optimistic interactions) or below (for pessimistic ones) the current agents' opinion. This type of intrinsically nonlinear interactions has been introduced in [21].

A similar extension appears very natural also in the context of joing agent interactions.

Definition 15 *We say that a (non-empty) set of agents $I \subset \{1, 2, \dots, n\}$ influences* optimistically *agent $j \notin I$, and we denote this by $I \searrow j$, if for all compact intervals $\mathcal{K} \subset \mathbb{R}$ and all $\bar{x} > x_j \in \mathcal{K}$ there exists a positive definite function ρ, such that*

$$f_j((\bar{x} - x_j)e_I + x_j \mathbf{1}) \geq \rho(|\bar{x} - x_j|) \tag{19}$$

Symmetrically, for pessimistic joint agent interactions:

Definition 16 *We say that a (non-empty) set of agents $I \subset \{1, 2, \dots, n\}$ influences* pessimistically *agent $j \notin I$, and we denote this by $I \nearrow j$, if for all compact intervals $\mathcal{K} \subset \mathbb{R}$ and all $\bar{x} > x_j \in \mathcal{K}$ there exists a positive definite function ρ, such that*

$$f_j((\bar{x} - x_j)e_I + x_j \mathbf{1}) \leq -\rho(|\bar{x} - x_j|) \tag{20}$$

In order to model the graph-theoretic properties of the considered network of influences, we may associate to system (5) a bicolored Petri Net as introduced below.

Definition 17 *A bicolored Petri Net is defined as a quintuple $N = \{P, T_o, T_p, E_I, E_O\}$ where:*

- *P is a (finite) set of places;*
- *T_o, T_p are the (finite) sets of* optimistic *and* pessimistic *transitions respectively, and fulfill $T_o \cap T_p = \emptyset$. In particular $T = T_o \cup T_p$ denotes the set of transitions;*
- *$E_I \subset T \times P$ is the set of Input arcs;*
- *$E_O \subset P \times T$ is the set of Output arcs.*

In particular, thanks to Definitions 15 and 16 we may construct N, associated to (5) according to the following set of rules:

- Set of places: $P = \{p_1, \dots, p_n\}$, n being the dimension of x (and total number of agents);
- Optimistic Transitions: $T_o = \{t^o_{I^o_1, j^o_1}, \dots t^o_{I^o_{q^o}, j^o_{q^o}}\}$ whenever $I^o_k \searrow j^o_k$ is minimal for all $k \in 1, \dots, q^o$;

- Pessimistic Transitions: $T_p = \{t^p_{I^p_1, j^p_1}, \dots t^p_{I^p_{q^o}, j^p_{q^o}}\}$ whenever $I^p_k \nearrow j^p_k$ is minimal for all $k \in 1, \dots, q^p$;
- Input arcs: $E_I = \{(t^o_{I^o_1, j^o_1}, p_{j^o_1}), \dots, (t_{I^o_{q^o}, j^o_{q^o}}, p_{j^o_{q^o}}),$ $(t^p_{I^p_1, j^p_1}, p_{j^p_1}), \dots, (t_{I^p_{q^p}, j^p_{q^p}}, p_{j^p_{q^p}})\}$
- Output arcs: E_O, of the type $(p_i, t_{I^o_k, j^o_k})$ for all $i \in I^o_k$ and all $k = 1 \dots q^o$ and of type $(p_i, t_{I^p_k, j^p_k})$ for all $i \in I^p_k$ and all $k = 1 \dots q^p$.

Notice that the set of input and output arcs E_I and E_O respectively, can, according to the distinction between optimistic and pessimistic transitions, be partitioned in optimistic and pessimistic arcs: $E_I = E^o_I \cup E^p_I$ and $E_O = E^o_O \cup E^p_O$. Accordingly, each bicolored Petri Net, induces two standard Petri Nets, $N_o = \{P, T_o, E^o_I, E^o_O\}$ and $N_p = \{P, T_p, E^p_I, E^p_O\}$ which we denote as the optimistic and pessimistic subnets, respectively. We are now ready to formulate the main technical tool for assessing the ability of a network with unilateral joint-agent interactions to reach consensus.

Definition 18 *We say that a bicolored Petri Net fulfills* siphon overlapping property *if for each pair of siphons S_o and S_p, corresponding to the optimistic and pessimistic subnets respectively, we have $S_o \cap S_p \neq \emptyset$.*

We are now ready to state sufficient conditions for global asymptotic consensus in networks with joint unilateral interactions.

Theorem 19 *Consider a cooperative system as in (5) and fulfilling Assumption 3. Then a sufficient condition for global asymptotic consensus is that the associated bicolored Petri Net $N = \{P, T_o, T_p, E_I, E_O\}$ fulfill* siphon overlapping property*.*

The proof of the result follows along the same lines as the proof of Theorem 9, just remarking that the set of agents achieving the maximum asymptotically are a siphon for the pessimistic subnet, while the set of agents achieving the minimum asymptotically are a siphon for the optimistic subnet.

9 Conclusions

A novel type of interactions between agents within a consensus protocols have been introduced, the so called *joint-agent* interactions. These account for the situation in which an agent is not influenced by any of his neighbours singularly taken, but might be sensitive to coherent influences by two or more of his neighbours. In this respect, graph-theoretical concepts are introduced to analyze the ability of consensus protocols allowing such type of joint agent interactions to converge asymptotically towards an agreement equilibrium. Conditions are written in the language of Petri Nets (treated here as bipartite graphs) and making use of the notion of siphon, a structural invariant which is normally related to deadlock analysis in Discrete Event Systems. A striking feature of the approach is the ability to characterize rather sharply the asymptotic consensus value on the basis of initial conditions and again looking at the maximum and minimum values taken by agents included within each siphon of the network. Simulations examples from a simple ring net (both with an odd and an even number of nodes) have been presented, highlighting their remarkable difference of behaviour exhibited in simulations and correctly predicted by the theory. The original framework and results are extended to the presence of unilateral interaction and can be usefully used to implement distributed estimations of the K-th maximum value of state initial condition. A simulation validation over an All-To-All topology shows the effectiveness of the proposed conditions.

References

[1] R.M. Murray, Recent Research in Cooperative Control of Multivehicle Systems, *Journal of Dynamic Systems, Measurement, and Control, 129, 5, 2007*

[2] J. M. Hendrickx and J. N. Tsitsiklis. Convergence of type-symmetric and cut-balanced consensus seeking systems. *IEEE Transactions on Automatic Control, 58, 1, 2013.*

[3] S. Martin, J. M. Hendrickx, Continuous-time consensus under non-instantaneous reciprocity, in *IEEE Transactions on Automatic Control, 61, 9, 2484 -2495, 2016*

[4] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control, 50, 2, 2005*

[5] D.P. Bertsekas and J.N. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods, *Prentice Hall*, 1989.

[6] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control, 48, 6, :9881001*, 2003.

[7] M. DeGroot, Reaching a consensus, *Journal of the American Statistical Association, vol. 69, pp. 118121, 1974.*

[8] Hegselmann, R., & Krause, U. Opinion dynamics and bounded confidence models, analysis, and simulation, *Journal of Artificial Societies and Social Simulation, 5, 3, 2002.*

[9] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, Mixing beliefs among interacting agents, *Advances in Complex Systems, 3, 87-98, 2000*

[10] D. Angeli and S. Manfredi. On consensus protocols allowing joint-agent interactions. *Proc of* 57^{th} *IEEE Conference on Decision and Control, FL, USA*, 2018

[11] F. Blanchini and S. Miani (2008) *Set Theoretic Methods in Control, Systems and Control Foundations and Applications, Birkhauser.*

[12] H. LeBlanc and X. Koutsoukos, Consensus in Networked Multi-Agent Systems with Adversaries, Proc. of HSCC'2011, Chicago, IL, USA, 2011.

[13] H. LeBlanc and X. Koutsoukos, Low Complexity Resilient Consensus in Networked Multi-Agent Systems with Adversaries, Proc. of HSCC'2012, Beijing, China, 2012.

[14] H. LeBlanc, H. Zhang, S. Sundaram and X. Koutsoukos, Consensus of Multi-Agent Networks in the Presence of Adversaries Using Only Local Information, Proc. of HiCoNS'12, Beijing, China, 2012.

[15] D. Angeli, S. Manfredi On Adversary Robust Consensus protocols through joint-agent interactions. arXiv:1901.02725 [cs.SY]

[16] Z. Lin, B. Francis, and M. Maggiore. State agreement for continuous time coupled nonlinear systems. SIAM J. Contr., 46, 1, 2007

[17] Q. Hui, W. M. Haddad, Distributed nonlinear control algorithms for network consensus, , Automatica 44, 2008

[18] C. Somarakis, J.S. Baras, A simple proof of the continuous time linear consensus problem with applications in non-linear flocking networks. ECC 2015: 1546-1553, 2015

[19] Matei, Baras, Somarakis, Convergence Results for the Linear Consensus Problem under Markovian Random Graphs, SIAM J. Control Optim., 51, 2, 2013.

[20] S. Martin and A. Girard, Continuous-Time Consensus under Persistent Connectivity and Slow Divergence of Reciprocal Interaction Weights SIAM J. Control Optim., 51, 3, 25682584, 2013

[21] S. Manfredi, D. Angeli. Necessary and Sufficient Conditions for Consensus in Nonlinear Monotone Networks with Unilateral Interactions. Automatica, 77, 51 - 60, 2017

[22] Min, Byungjoon and San Miguel, Maxi Competing contagion processes: Complex contagion triggered by simple contagion Scientific Reports, 8, 2018

[23] Y. D. Zhong, V. Srivastava and N. E. Leonard. On the Linear Threshold Model for Diffusion of Innovations in Multiplex Social Networks. To appear in Proc. IEEE Conference on Decision and Control, 2017

[24] M. A. Javarone. Social Influences in Opinion Dynamics: the Role of Conformity. Physica A: Statistical Mechanics and its Applications, 2014

[25] C. Als-Ferrer, S. Hugelschfer and J. Li. Inertia and Decision Making. Frontiers in Psychology, Volume 7, 2016

[26] M. Baddeley Herding, social influence and economic decision-making: socio-psychological and neuroscientific analyses. Phil. Trans. R. Soc. B, 365, 281-290, 2016

[27] C. Altafini, Consensus problems on networks with antagonistic interactions, IEEE Transactions on Automatic Control, Vol. 58, No. 4, 2013

[28] J. Hu, W.X. Zheng, Emergent collective behaviors on coopetition networks, Physics Letters A, 378, 2627, 1787–1796, 2014

[29] G. Liu and K. Barkaoui, A survey of siphons in Petri nets, Information Sciences, Vol. 363, pp. 198-220, 2017

[30] A. Olshevsky, and J. N. Tsitsiklis, Degree Fluctuations and the Convergence Time of Consensus Algorithm, IEEE Transactions on Automatic Control, 58, 10, 2013

[31] A. Bhattacharyya, M. Braverman, B. Chazelle, and H.L. Nguyen, On the convergence of the Hegselmann-Krause system, Proceedings of the 4th conference on Innovations in Theoretical Computer Science, ACM, 2013