Perturbative and non-perturbative approaches to string sigma-models in AdS/CFT

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Supervisor’s foreword

This thesis, which encompasses results from the six peer-reviewed publications that Dr. Vescovi co-authored in his three years at Humboldt University Berlin, deals with the most relevant string model in the framework of the gauge/gravity duality – one of the most far-reaching recent ideas in theoretical physics. This work is in my opinion remarkable for three aspects. First, it contains some of the technically hardest calculations in this framework (next-to-leading order results for string excitations which are highly coupled, as well as next-to-next-to-leading order results, the last one setting the current limit of perturbative string world-sheet analysis). Second, it contains the results of a new, highly interdisciplinary project – involving various themes in mathematical and high-energy physics – which initiates an entirely new way to analyze worldsheet string theory models: the use of Monte Carlo simulations for their lattice-discretized version. Finally, the thesis is very well written and highly pedagogical, thus enabling the reader to follow easily (and I believe with pleasure) the complex subjects treated and the analytic and numerical results reported.

Prof. Dr. Valentina Forini

Berlin, Germany
April 2017
Abstract

This thesis discusses perturbative and non-perturbative aspects of type II superstring theories in $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ backgrounds relevant for the AdS/CFT correspondence. We present different approaches to compute observables in the context of the duality and we test the quantum properties of these two superstring actions. Our methods of investigation span from the traditional perturbative techniques for the worldsheet sigma-model at large string tension, consisting in expanding around minimal-area surfaces, to the development of a novel non-perturbative analysis, based upon numerical methods borrowed from lattice field theory.

We review the construction of the supercoset sigma-model for strings propagating in the $AdS_5 \times S^5$ background. When applied to the $AdS_4 \times \mathbb{CP}^3$ case, this procedure returns an action that cannot consistently describe the general quantum dynamics of the superstring. This can be attained instead by an alternative formulation based on the double dimensional reduction of the supercoset action for a supermembrane moving in $AdS_4 \times S^7$.

We then discuss a general and manifestly covariant formalism for the quantization of string solutions in $AdS_5 \times S^5$ in semiclassical approximation, by expanding the relevant sigma-model around surfaces of least area associated to BPS and non-BPS observables amenable to a dual description within the gauge/gravity duality. The novelty of our construction is to express the bosonic and fermionic semiclassical fluctuation operators in terms of intrinsic and extrinsic invariants of the background geometry for given arbitrary classical configuration.

We proceed with two examples in the more general class of quantum small fluctuations, governed by non-trivial matrix-valued differential operators and so far explored only in simplifying limits. Our results stem from the exact solution of the spectral problem for a generalization of the Lamé differential equation, which falls under a special class of fourth-order operators with coefficients being doubly periodic in a complex variable. Our exact semiclassical analysis applies to two-spin folded closed strings: the $(J_1, J_2)$-string in the $SU(2)$ sector in the limit described by a quantum Landau-Lifshitz model and the bosonic sector of the $(S, J)$-string rotating in $AdS_5$ and $S^5$. In both situations, we write the one-loop contribution to the string energy in an analytically closed integral expression that involves non-trivial nested combinations of Jacobi elliptic functions.

Similar techniques allow to address the strong-coupling behaviour of 1/4-BPS latitude Wilson loops in planar $SU(N)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. These operators are holographically mapped to fundamental strings in $AdS_5 \times S^5$. To compute the first correction to their classical values, we apply a corollary of the Gel’fand-Yaglom method for the functional determinants to the matrix-valued operators of the relevant semiclassical fluctuations. To avoid ambiguities due to the absolute normalization of the string partition
function, we consider the ratio between the generic latitude and the maximal 1/2-BPS circular loop. Our regularization procedure reproduces the next-to-leading order predicted by supersymmetric localization in the dual gauge theory, up to a certain remainder function that we comment upon and that was later confirmed in a different setup by other authors. We also study the AdS light-cone gauge-fixed string action in $AdS_4 \times \mathbb{C}P^3$ expanded around the null cusp background, which is dual to a light-like Wilson cusp in the planar $\mathcal{N} = 6$ Chern-Simons-matter (ABJM) theory. The fluctuation Lagrangian has constant coefficients, thus it allows to extend the computation of the free energy associated to such string solution up to two loops, from which we derive the null cusp anomalous dimension $f(\lambda)$ of the dual ABJM theory at strong coupling to the same loop order. The comparison between this perturbative result for $f(\lambda)$ and its integrability prediction results in the computation of the non-trivial ABJM interpolating function $h(\lambda)$, which plays the role of effective coupling in all integrability-based calculations in the $AdS_4/CFT_3$ duality. The perturbative result is in agreement with the strong-coupling expansion of an all-loop conjectured expression of $h(\lambda)$.

The last part of the thesis is devoted to a novel and genuinely field-theoretical way to investigate the $AdS_5 \times S^5$ superstring at finite coupling, relying on lattice field theory methods. Deeply inspired by a previous study of Roiban and McKeown, we discretize the $AdS_5 \times S^5$ superstring theory in the AdS light-cone gauge and perform lattice simulations employing a Rational Hybrid Monte Carlo algorithm. We measure the string action, from which we extract the null cusp anomalous dimension of planar $\mathcal{N} = 4$ SYM as derived from AdS/CFT, as well as the mass of the two AdS excitations transverse to the relevant null cusp classical solution. For both observables we find good agreement in the perturbative regime of the sigma-model at large ’t Hooft coupling. For small coupling, the expectation value of the action exhibits a deviation compatible with the presence of quadratic divergences. After their non-perturbative subtraction, the continuum limit can be taken and it suggests a qualitative agreement with the non-perturbative expectation from AdS/CFT. For small coupling we also detect a phase in the fermionic determinant that leads to a sign problem not treatable via standard reweighting. We explain its origin and also suggest an alternative fermionic linearization.
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Chapter 1

Introduction

Modern theoretical physics has been written in the language of two major scientific paradigms: theory of general relativity and quantum field theory.

The Einstein’s theory of gravitation provides an elegant geometric interpretation of gravitational attraction as a dynamical effect of the curvature of space and time, seen as interwoven in a single four-dimensional “fabric” called spacetime, determined by the distribution of energy and momentum carried by the matter and radiation filling the universe. Over the last century a number of physical phenomena has been derived from this principle and found consistent with experimental data at the current level of accuracy [1]. The first direct detections of gravitational waves, travelling as “ripples” of spacetime, has been confirmed recently by the LIGO and Virgo collaborations [2, 3]. Despite these successes, general relativity still defies all efforts to reconcile them with a microscopical description at a quantum level.

On a parallel route, non-gravitational forces have been incorporated into a theoretical framework where special relativity fits together with quantum mechanics and the concept of field quanta supersedes the classical idea of single particles. Quantum field theory (QFT) has evolved to start new trends in condensed matter physics, leading to the study of critical phenomena in connection with phase transitions using the renormalization group flow [4], with the benefit of providing a new viewpoint on renormalization in particle physics [5–7] ([8] for a review). The same symbiosis has developed in connection to special QFTs with spacetime conformal invariance, following earlier studies in two-dimensional critical systems [9].

The original focus of QFT arose within the first attempts to quantize gauge theories. The formulation of quantum theory of electrodynamics (QED) served as model for the development of quantum chromodynamics (QCD), which unravelling the puzzle behind the growing list of hadrons discovered in the late 1960s in terms of the strong interaction among constituents particles called quark and gluons. Subsequent efforts to describe weak interactions
as the exchange of heavy bosons culminated in the foundation of the best theoretical tool to investigate nature at short distances as we know it today, the Standard Model of elementary particle physics. Free of quantum anomalies and arguably theoretically self-consistent, it describes the dynamics of matter particles as the exchange of the force carriers of a non-abelian (Yang-Mills) theory with local (gauge) symmetry group $SU(3) \times SU(2) \times U(1)$ partially broken by the Higgs mechanism [10–12]. One of its greatest successes is the interpretation of the mysterious Feynman-Björken scaling as an effect of asymptotic freedom in non-abelian gauge theories [13], when quarks behave as non-interacting constituents in deep inelastic scattering. Since then, theoretical predictions have shown agreement with the experimental data with spectacular precision [14]. The process of experimental validation continues and recently led to the discovery of the last elusive particle, the Higgs boson, at the Large Hadron Collider [15, 16]. That being said, the Standard Model cannot be the last word on physical reality. The next future will likely shed light on many known inadequacies and unanswered questions, for instance the hierarchy problem of the fundamental forces, the phenomenon of neutrinos oscillations and cosmic observations hinting at the existence of dark matter and dark energy.

Most of the predictive power of the Standard Model is due to perturbative approximations around the free theory by means of Feynman diagrams. However, the hope of resumming loop expansions vanishes as soon as one realizes that they are typically asymptotic expansions with zero radius of convergence. Moreover, perturbation theory breaks down when applied to inherently strongly-coupled quantum phenomena, e.g. solitons and bound states. Of course, there are direct attempts to quantitatively understand the mechanism of quark confinement and arrive at reasonable approximations for the hadronic spectra, but they are the product of numerical simulations of effective theories, which may obscure a microscopic description in terms of the elementary constituents.

An alternative step consists in engineering a “toy model” that abandons the immediate ambition to describe the real world. The first step is to reduce the complexity of the problem and exploit enlarged number of symmetries to make non-trivial analytical statements. Secondly, the simplified model can be enriched with more features in order to transfer some of its properties back to the original system to some degree. This strategy has proven to be extremely useful in countless occasions throughout the history of science. For instance, it happened at the dawn of quantum mechanics when the development of the Hartree-Fock method (e.g. in [17]) to calculate wavefunctions for multi-electron atoms and small molecules was guided by earlier semi-empirical methods based on the exact Schrödinger solution for the hydrogen atom.

In order to gain a better theoretical understanding of QCD physics and to develop new computational tools in QFT in general, theoretical physicists have looked for the “most symmetric” interacting gauge theory in four dimensions. This role is arguably played by
$N = 4$ supersymmetric Yang-Mills (SYM) theory [18]: it describes a Minkowskian universe containing scalars and fermions interacting via non-abelian gluons. It possesses the maximal amount of $N = 4$ supercharges to be renormalizable in four dimensions, which fully constrains the precise form of the interactions, and it does not display any parameter other than the coupling constant and the gauge group. In addition to supersymmetry, the model exhibits exact conformal symmetry at the quantum level and it is conjectured to have an “electric-magnetic” Montonen-Olive $SL(2, \mathbb{Z})$ duality [20–22], one of the earliest instances of S-duality. Of course, we cannot expect to draw heavily on this analogy, as it is clear from the fact that $N = 4$ SYM has massless mass spectrum and no running coupling constant [23–26] – meaning neither a characteristic scale nor asymptotic freedom – leaving aside the fact that supersymmetry is not a feature of the Standard Model. However, there exist quantitative features of $N = 4$ SYM found to survive in QCD, for instance in the conformal dimension of local gauge-invariant operators and in the derivation of tree-level QCD scattering amplitudes from $N = 4$ SYM [30]. In the remainder of the chapter we will show that there are also other reasons that make $N = 4$ SYM a theoretical laboratory worth to be studied in its own right.

1.1 The $AdS_5/CFT_4$ and $AdS_4/CFT_3$ correspondences

One of the major breakthroughs of the recent years is the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [31–33]. The conjecture asserts the exact equivalence between a pair of models. On one side, there is a QFT with conformal spacetime symmetry in $d$ dimensions. On the other one, we have a superstring theory where strings move in the target space $AdS_{d+1} \times M_{9-d}$ including an anti de-Sitter space, a $(d+1)$-dimensional manifold with constant negative curvature, and a compact manifold $M_{9-d}$ in $9-d$ dimensions. The $d$-dimensional boundary of the background is a conformally-flat space on which the CFT is formulated. Note that the dimensions of the two factors (AdS and $M$) add up to yield a string theory with fermions in ten dimensions, which is the critical dimension to ensure the cancellation of conformal anomaly on the worldsheet. In this context the term “equivalence” is a synonym of one-to-one correspondence between aspects of the two models (e.g., global symmetries, operator observables, states, correlation functions). The claim

1In principle, one can also consider the instanton angle $\theta$ which combines with the YM coupling constant into a complex coupling $\tau = \frac{e}{g^2} + \frac{i}{\frac{4\pi}{g^2}}$. Through the AdS/CFT correspondence (section 1.1), the angle $\theta$ equals the expectation value of the axion field in the spectrum of the dual Type IIB superstring, e.g. [19].

2Twist-two (Wilson) operators play an important role in deep inelastic scattering in QCD as much as in $N = 4$. Their anomalous dimension for large spin is governed by the so-called scaling function of the theory in question, see section 7.1. The maximal transcendentality principle conjectured in [27] states that the $N = 4$ SYM scaling function has uniform degree of transcendentality $2l - 2$ at loop order $l$ and can be extracted from the QCD expression by removing the terms that are not of maximal transcendentality. A brief account of the subject and references are in [28, 29].

3Among the many reviews on the topic we suggest [19, 34–38] and the excellent textbook [39].
that the dynamics of the string degrees of freedom can be encoded in a lower-dimensional (non-gravity) theory at its boundary suggests to see it as a realisation of holographic duality. Since we will encounter CFTs that are gauge theories in this thesis (summarized in (1.1) and (1.5) below), we often refer to the correspondence also as gauge/gravity duality.

The first example [31] at the spotlight since 1997 – later named AdS5/CFT4 correspondence – relates

\[
\mathcal{N} = 4 \text{ super Yang-Mills in flat space } \mathbb{R}^{1,3} \\
\text{with Yang-Mills coupling constant } g_{YM} \text{ and gauge group } SU(N)
\]

and

\[
\text{type IIB superstring theory with string tension } T \text{ and coupling constant } g_s \text{ on } AdS_5 \times S^5 \text{ with curvature radii } R_{AdS_5} = R_{S^5} = R \\
\text{and } N \text{ units of Ramond-Ramond five-form flux through } S^5.
\]

Here, \(T\) is an overall factor in the string action and \(g_s\) is the genus-counting variable in the perturbative expansion over topologies of string theory. The AdS/CFT dictionary relates the gauge/string parameters through the dimensionless ’t Hooft coupling \(\lambda\)

\[
\lambda = g_{YM}^2 N, \quad \lambda = 4\pi^2 T^2 = 4\pi Ng_s = \frac{R^4}{\alpha'}.
\]

The constant \(\alpha'\) is the square of the string characteristic length and historically the slope parameter in the linear relationship between energy/angular momentum of rotating relativistic bosonic strings in flat space. The motivation behind the correspondence (1.1) arose from the investigation of a stack of \(N\) parallel Dirichlet branes (D3-branes), 3d objects sweeping out a \((1+3)\)-dimensional volume, separated by a distance \(d\) and embedded in type IIB string theory in \(\mathbb{R}^{1,9}\). D-branes can be viewed in two equivalent ways, fundamentally linked to the open/closed string duality, where \(\mathcal{N} = 4\) SYM theory and type IIB supergravity in \(AdS_5 \times S^5\) emerge as two (arguably equivalent) low-energy descriptions of the same physics in Maldacena limit for \(\alpha', d \to 0\) while holding \(\alpha'/d\) fixed. Relaxing the supergravity limit \(\alpha' \to 0\), the claim [31] is that the two models in (1.1) continue to be dual for any values of the parameters.

An immediate ‘check’ of the duality is the fact that the two models in (1.1) have the same global symmetry group \(PSU(2,2|4)\), namely the super-Poincaré and conformal invariance of \(\mathcal{N} = 4\) SYM and the superrisometry group of the string theory in \(AdS_5 \times S^5\). On operative level, one establishes the equivalence of the superstring partition function, subject to sources \(\phi\) for string vertex operators with boundary value \(\phi_0\), and the partition function in the CFT.
side with sources $\phi_0$ for local operators

$$Z_{\text{string}}[\phi|\partial(AdS_5) = \phi_0] = Z_{\text{CFT}}[\phi_0].$$  \hspace{1cm} (1.3)

The strongest version of the conjecture puts no restriction on the parameter space, but it is hard to check its validity if we do not work in certain simplifying limits to enable a perturbative approach. A unique parameter ($\lambda$) turns out to be a useful choice when considering the 't Hooft limit $[40]$

$$g_{\text{YM}} \to 0, \quad N \to \infty, \quad \lambda = \text{constant}.$$ \hspace{1cm} (1.4)

The Yang-Mills theory becomes a free non-abelian theory ($g_{\text{YM}} \to 0$) for infinitely-many “colors” ($N \to \infty$) where the class of planar graphs is dominant in the diagrammatical expansions. In the partner model, the joining and splitting of strings is suppressed ($g_s \to 0$) and only lowest-genus surfaces survive. For small $\lambda$ the string is subject to large quantum mechanical fluctuations ($T \to 0$) on a highly-curved $AdS_5 \times S^5$ ($R \ll \sqrt{\alpha'}$), conversely for large $\lambda$ the string behaves semiclassically ($T \to \infty$) in a flat-space limit ($R \gg \sqrt{\alpha'}$). For the latter interpretation we recall that $T$ is an overall factor of the string action and thus can be assimilated to a sort of inverse Planck constant. Conventional perturbative calculations on the gauge theory side are possible to a certain extent if we impose that $\lambda$ is small (weak coupling), while semiclassical methods can probe the string corrections to the classical supergravity theory ($\alpha' = 0$) when we adjust $\lambda$ to be large (strong coupling). This observation enables to make precise statements about a strongly-coupled regime of a gauge theory, typically lacking systematic quantitative tools previous to the AdS/CFT correspondence, as long as it admits a higher-dimensional string theory.

The seminal paper by Maldacena $[31]$ sparked a quest for other realizations of AdS/CFT duality. Following earlier works $[41, 42]$, Aharony, Bergman, Jafferis and Maldacena (ABJM) $[43]$ ($[44]$ for a review) established the equivalence between a theory of M2-branes in eleven dimensions and a certain three-dimensional gauge theory. The two parameters $k$ and $N$ (defined below) allow for a somewhat richer structure than the $AdS_5/CFT_4$ system. In this thesis we will limit ourselves to consider the duality between
\( \mathcal{N} = 6 \) super Chern-Simons theory with matter in flat space \( \mathbb{R}^{1,2} \) with integer Chern-Simons levels \( k \) and \(-k\) and gauge group \( SU(N)_k \times SU(N)_{-k} \) and type IIA superstring theory with string tension \( T \) and coupling constant \( g_s \) on \( AdS_4 \times \mathbb{CP}^3 \) with curvature radii \( 2R_{AdS_4} = R_{\mathbb{CP}^3} \equiv R \) and \( N \) units of Ramond-Ramond four-form flux through \( AdS_4 \) and \( k \) units of Ramond-Ramond two-form flux through \( \mathbb{CP}^1 \subset \mathbb{CP}^3 \), provided the identifications through the ’t Hooft coupling \( \lambda \)

\[
\lambda = \frac{N}{k}, \quad \lambda = \frac{R_{AdS_4}^6}{32\pi^2 k^2 R_p^2}.
\]

The duality (1.5) holds only in the analogue [43] of the ’t Hooft limit (1.4)

\[
N, k \to \infty, \quad \lambda = \text{constant}.
\]  

On the gauge-theory side, the ABJM theory is a supersymmetric extension of pure Chern-Simons theory, which is a broad subject with applications to 3d gravity theory [45] and knot theory [46]. The addition of \( \mathcal{N} = 6 \) supercharges \(^5\) renders ABJM a non-topological theory, but still retaining conformal invariance. The global symmetry group of the ABJM theory and the dual string theory is the orthosymplectic supergroup \( OSp(6|4) \).

The original “dictionary” proposal [43] for the string tension in terms of the ’t Hooft coupling \( \lambda \) reads

\[
T = \frac{R^2}{2\pi\alpha'} = 2\sqrt{2\lambda}, \quad g_s \propto \frac{N^{1/4}}{k^{5/4}}.
\]

As suggested in [48] and later quantified in [49], the relation between \( T \) and \( \lambda \) receives quantum corrections. The geometry of the background (and also the flux, in the ABJ theory [50], generalization of the ABJM theory with gauge group \( U(N) \times U(M) \)) induces higher-order corrections to the radius of curvature in the Type IIA description, which reads in the planar limit (1.7) of interest in this thesis

\[
T = \frac{R^2}{2\pi\alpha'} = 2\sqrt{2\left(\frac{\lambda - \frac{1}{24}}{24}\right)}.
\]

\(^4\)We will make clear the distinction between the ’t Hooft parameter \( \lambda_{YM} \) of \( \mathcal{N} = 4 \) SYM and the one \( \lambda_{ABJM} \) of ABJM when necessary, namely in chapter 6.

\(^5\)Supersymmetry is enhanced to \( \mathcal{N} = 8 \) at Chern-Simons level is \( k = 1, 2 \) [43, 47]. We can disregard this exception since we will be working in planar limit.
The anomalous radius shift by $-\frac{1}{24}$ in (1.9) is important at strong coupling, because it affects the corrections to the energy and anomalous dimensions of giant magnons and spinning strings starting from worldsheet two-loop order $O(\lambda^{-1/2})$. It will also turn out to be crucial in chapter 6 to translate the string tension $T$ into the gauge coupling $\lambda$.

Another instance of holography is the $AdS_3/CFT_2$ correspondence between superstring theories on backgrounds involving the $AdS_3$ space and two-dimensional superconformal field theories. The supersymmetric backgrounds of interest, especially because of their integrable properties, are the $AdS_3 \times S^3 \times S^3 \times T^1$ and $AdS_3 \times S^3 \times T^4$ supergravity backgrounds which preserve 16 real supercharges. However, in light of the work done in the next chapters, we will be mostly concerned with the other two dualities spelt out above, referring the reader to [51] (also [52]) and references therein for an account of the subject.

1.2 Integrable systems in AdS/CFT

Since its discovery, the AdS/CFT correspondence prompted a new interest in $\mathcal{N} = 4$ SYM and offered a (strong-coupling) perspective to study this gauge theory. Ideally, the aim of solving a QFT means to express arbitrary $n$-point correlation functions of any combination of fields in terms of elementary functions or integral/differential equations involving the parameters of the model. When this happens, it signals the presence of an infinite number of conserved charges and the theory in question is called classically integrable, and quantum integrable if the property persists at the quantum level. It is clear that this requirement is extraordinary difficult to satisfy, save for a few exceptions typically relegated to two-dimensional models. A less trivial occurrence, the first in four dimensions, emerges in high-energy QCD scattering.

Evidence of integrable structures in planar $\mathcal{N} = 4$ SYM later emerged in relation to single-trace operators (the only relevant ones at $N \to \infty$) and certain spin-chain models. Since the theory is conformal, the dynamical information is contained in the two- and three-point functions of local gauge-invariant operators. Conformal symmetry fixes their two-point correlators in terms of their eigenstates under the action of the dilatation operator $D \in \mathfrak{psu}(2, \mathbf{2} | 4)$, namely the spectra of scaling dimensions of all operators.

The breakthrough of [61] was realizing that single-trace operators in the flavour sector $SO(6)$ (i.e. traces of a product of any of the scalars of $\mathcal{N} = 4$ SYM) are mapped to states of a periodic spin-chain and the (one-loop) dilatation operator to the Hamiltonian of the spin-chain system. The spectrum of scaling dimensions at weak-coupling one-loop order was

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6We suggest [53, 54] for an extensive discussion of integrable systems and also [55, 56] for a focus on AdS/CFT.

7A transparent and concise introduction to the subject is in [60–62].

8Higher-point functions decompose into these elementary constituents [9].
set equivalent to the diagonalization problem of the auxiliary $SO(6)$ spin-chain Hamiltonian which, since it was known to be integrable, could be solved exactly using Bethe ansatz techniques [63–66]. Dropping the restriction to scalar operators, integrability was established for all operators at the one-loop order [67] in terms of an integrable $PSU(2,2|4)$ super spin-chain, later diagonalized in [68]. A further development concerned the generalization of the Hamiltonian method to two and three loops [69].

This program was pushed further to reveal classical integrability on the string theory side of AdS/CFT by explicitly rewriting the equations of motion of the non-linear sigma model on $AdS_5 \times S^5$ background [70] into a zero-curvature condition for a Lax pair operator [71]. Following the same approach the analogous set of non-local conserved charges was constructed in [72] in the pure-spinor formulation of the $AdS_5 \times S^5$ action [73–75] and the same Lax pair was found in [76].

With integrability becoming a solid fact at both weak and strong coupling, the focus shifted to speculate about this property holding true at all loops. In [83] a direct relationship between Bethe equations and classical string integrability was reinforced using the language of algebraic curves, interpreted as a sort of continuum version of Bethe equations. On the assumption of exact quantum integrability of the $AdS_5/CFT_4$ system, a set of Bethe equations valid at all loop-order was formulated [84] for all long local operators [85]. These results were complemented by the study of the so-called dressing factor [86–91] and collectively referred to as all-loop asymptotic Bethe ansatz (ABA), as their validity is limited to asymptotically long chains in the auxiliary picture. In principle this enabled to solve the spectral problem for the anomalous dimension of all long single-trace operators in planar $\mathcal{N} = 4$ SYM.

The understanding of the conjectured integrability has steadily advanced towards the inclusion of finite-size effects (wrapping effects) [93–96]. This ambitious program included the development of an infinite set of coupled integral equations called Thermodynamic Bethe ansatz (TBA) [94, 97–102] (also in [103, 104]) which are solvable in some cases for scattering amplitudes [105] and cusped Wilson lines [106–108]. This served as a basis for the so-called Y-system [99] (an infinite set of non-linear functional equations) and its successor FiNLIE (acronym for ‘finite system of non-linear integral equations”). The state-of-the-art in elegance and computational efficiency in solving the spectral problem seems to be achieved in the form of a set of Riemann-Hilbert equations that defines the quantum spectral curve (QSC) approach (or $P\mu$-system) [109, 110], where the so-called $Q$ functions are a sort of

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Footnotes:

9. The integrability of the string in the $AdS_5 \times S^5$ background has been mostly studied in the supercoset description [70] (e.g. in [77]) than in the pure spinor version. Some integrable properties in the former formalism will be discussed to some extent in sections 2.1.1, 2.1.2 and 2.1.3, while we refer the reader to a non-exhaustive selection of relevant references in the latter formalism in [78, 79] and in the reviews [80, 81]. Arguments that support the quantum integrability of the pure-spinor action were given in [82].

10. This is a function undetermined by the symmetries of the theory, but constrained by physical requirements such as crossing symmetry and unitarity, see [92] for a short review.
quantum generalization of pseudo-momenta in the algebraic curve construction. The potential of this machinery extends beyond the original scope of computing spectrum of anomalous dimensions, e.g. in the high-precision and non-perturbative numerical computation of the generalized cusp anomalous dimension of a cusped Wilson line [111, 112].

Almost all relevant statements that have been made about integrability for the planar $AdS_5/CFT_4$ system have been reworked almost in parallel for the lower-dimensional correspondence $AdS_4/CFT_3$ in the planar limit, see [113] for a comprehensive overview. The investigation started perturbatively at planar two-loop order for scalar operators by constructing the corresponding integrable spin-chain Hamiltonian [114, 115]. The extension to all operators (at two-loop order) was derived in [116, 117]. One of the most distinguishing differences between the integrable structure of ABJM and the one of $\mathcal{N} = 4$ is the fact that the transition from weak to strong coupling is more intricate due to the presence of the non-trivial (ABJM) interpolating function $h(\lambda)$, introduced and analysed in chapter 6. This function plays the crucial role of a “dressed” coupling constant that absorbs the dependence on the ‘t Hooft coupling $\lambda$ in all integrability-based computations, e.g. in the set of ABA equations for the complete spectrum of all long single-trace operators proposed in [118]. At strong coupling, the classical spectral curved was constructed in [119] and integrability was demonstrated for the supercoset action at classical level [120].

Echoing the developments in $\mathcal{N} = 4$ SYM, the Y-system was proposed in [99] along with the analogue one for $AdS_5/CFT_4$ system. The infinite set of nonlinear integral TBA equations encoding the anomalous dimensions spectrum was derived in [121, 122]. The QSC formalism was set up in [123] and used to put forward a conjecture for the exact form of $h(\lambda)$ in the ABJM model [124] and in its generalization, the ABJ model, in [125].

The concept of integrability has been rephrased in several contexts and its facets detected in a wide range of observables. Another realization is Yangian symmetry [126], a sort of enhancement of the Lie algebra symmetry $\text{psu}(2,2|4)$ of the theory, which benefited from the previous discovery of the duality [127] mapping scattering amplitudes of $n$ gluons to polygonal Wilson loops with $n$ light-like segments, see also [128–131] for some later developments. This duality was proposed at strong coupling and later noticed in perturbative computations at weak coupling [132] (also [133]) where it inspired the discovery of a hidden dual superconformal symmetry [134, 135]. Soon after, the latter and the conventional conformal symmetry were shown to combine into the Yangian symmetry [136]. This symmetry has been seen in color-ordered scattering amplitudes at tree level [136] and in loop quantum corrections [137–139], the dilaton operator [140] and supersymmetric extensions of Wilson loops [141, 142].

A further area rich of developments is the study of the dual polygonal light-like Wilson loops

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11 More references on the subject are below (2.35) in section 2.2.
12 We will test the strong-coupling expansion of the interpolating function in chapter 6.
loops at any coupling through a pentagon-block decomposition in the form of an OPE-like expansion \([143–147]\) which can be determined again on the basis of integrability arguments. An integrability-based framework to compute structure constants of higher-point correlation functions was recently established in \([148]\).

### 1.3 Quantization of strings in AdS/CFT

In the previous section we have seen that integrability offers a wide range of techniques to make quantitative predictions about the spectral problem and other observables in many AdS/CFT systems by means of analytical and high-accuracy numerical methods. All these statements are based on the conjectured all-loop integrability of the model. Without this assumption, a restricted class of supersymmetry-protected observables can be still computed at finite coupling and beyond the planar limit via supersymmetric localization techniques \([149]\), which are however only defined on the field theory side. Leaving aside the ambition of proving the assumptions of integrability from first principles, the natural question arising is whether one can check their predictions against perturbative results and non-perturbative ones obtained with different methods. Field theory computations maintain a crucial role in detecting the precise pattern of such functions of coupling and charges, as well as in checking the proposed all-loop formalisms. This viewpoint shifts the attention from exact methods to the development of computational tools, in principle flexible enough to work in different frameworks when neither integrability nor localization is available.

We shall pursue this goal in the AdS/CFT systems (1.1) and (1.5) considered in the respective \(^{t}\)Hooft limit (1.4) and (1.7) of their parameter spaces, exclusively working on the string theory side. Put in simple words, strings are objects spatially extended in one dimension, at variance with point-like particles of ordinary QFTs, and are embedded in a higher-dimensional ambient manifold (target space), for us the ten-dimensional \(AdS_5 \times S^5\) or \(AdS_4 \times \mathbb{CP}^3\). Note that quantum mechanical consistency guarantees the absence of conformal anomaly when the dimensionality of the spacetime is 10. Strings sweep out a \((1+1)\)-dimensional surface \(\Sigma\) (worldsheet) in their time evolution. Since all computations are in a limit where \(N\) is put to infinity, scattering of two or more strings does not occur and worldsheets are genus-0 surfaces (i.e. without “holes” and “handles”). All observables depend on the single tunable parameter \(T\) (or equivalently \(\lambda\)) of the model under investigation. The fields of a string theory consist of the bosonic embedding coordinates of \(\Sigma\) into the target space and their fermionic supersymmetric partners. From the worldsheet viewpoint, they are bosonic (collectively denoted by \(X\)) and fermionic fields (\(\Psi\)) propagating in the two-dimensional curved manifold \(\Sigma\).
From now on we shall focus on the prototypical duality (1.1), as the following statements hold for (1.5) after the necessary changes having been made. Since the string theory in $\text{AdS}_5 \times S^5$ includes a Ramond-Ramond five-form flux, the Neveu-Schwarz-Ramond (NSR) formalism [150, 151] is not applicable in a straightforward way. As we will see in chapter 2, the $\text{AdS}_5 \times S^5$ is 10d supersymmetric background of type IIB supergravity [152]. The Green-Schwarz (GS) approach [153, 154] seems to be adequate when the RR fields are not vanishing and would endow the string action with invariance under supersymmetry (manifestly realized as a target-space symmetry) and $\kappa$-symmetry (a local fermionic symmetry that ensures the correct number of physical fermionic degrees of freedom), but it is not very practical for finding the explicit form of the action in terms of the coordinate fields. For $\text{AdS}_5 \times S^5$ the superstring action is formulated [70] as a sigma-model on a supercoset target space. This is an highly-interacting two-dimensional field theory for which a first-principle quantization is a hard theoretical problem.

The quantization is more straightforward if one picks a suitable string vacuum (whose properties and/or quantum numbers depend on the particular string observable to study), fixes the gauge symmetries (2d diffeomorphisms and $\kappa$-symmetry) and expands the degrees of freedom of the superstring in terms of fluctuation fields around such vacuum.

As in ordinary quantum field theory, the fundamental object is the string partition function

\[ Z_{\text{string}} = \int \mathcal{D}g \mathcal{D}X \mathcal{D}\Psi e^{-S_{\text{IIB}}[g,X,\Psi]} . \] (1.10)

We work with $S_{\text{IIB}}$ being the sigma-model action of [70], where one has to integrate over the 2d metric $g_{ij}$ and fix the diffeomorphism-Weyl invariance of the action with the Faddeev-Popov procedure. The fluctuation string action is written in terms of the fluctuations $\delta X = X - X_{\text{cl}}$ and $\delta \Psi = \Psi$ around the non-trivial vacuum $(X_{\text{cl}}, \Psi = 0)$, where $X_{\text{cl}}$ is the chosen classical solution of the string equations of motion and fermions are set to zero on a classical configuration. The expansion of the action (1.10) delivers an infinite tower of complicated-looking interaction vertices organized in increasing inverse powers of $T$. Note that we have not made any assumption on the (small or finite) value of the coupling constant $T^{-1}$ up to this point.

\[ ^{13}\text{For a general curved target space, the string equations of motion are non-linear and the right and left oscillator modes of the string interact with themselves and with each other [155], see also [156] for further issues. As for the quantization of a generic field theory in curved spacetime, a good initial reference is the textbook [157].} \]

\[ ^{14}\text{The string sigma-model of [70] is non-linear because the curvature of the target space brings field-dependent coefficients of the kinetic terms. Expanding the path-integral around a classical solution generates standard quadratic kinetic terms (and interaction terms) for the fluctuation fields that make the sigma-model tractable.} \]

\[ ^{15}\text{One should also remember that a rigorous definition takes into account some factors associated with conformal Killing vectors and/or Teichmüller moduli. We defer the discussion to [158, 159] and the textbooks [156, 160].} \]
One can proceed with *perturbation theory* for large string tension $T \sim \sqrt{\lambda} \gg 1$, which indeed corresponds to the nearly-free regime of the sigma-model at small $T^{-1}$. To access the non-perturbative regime of the full quantum superstring, one can resort to techniques of lattice field theory and evaluate numerically the string observable of interest. One main objective of this thesis will be to present evidence that this route is indeed viable and that the data collected so far (chapter 7) is consistent with the expectations based on the integrability of the $AdS_5/CFT_4$ system.

Before addressing this important methodological distinction, we recall a few facts about the properties of classical backgrounds $X_{\text{cl}}$.

Solutions that are translationally invariant in the time and space coordinates $(\tau, \sigma)$ of the worldsheet, namely with constant derivatives of the background $X_{\text{cl}}$, are called *homogeneous*. In this case the effective action $\Gamma \equiv -\log Z_{\text{string}}$ is an extensive quantity – proportional to the area of the classical worldsheet – and the semiclassical analysis is highly simplified since the action turns out to have constant coefficients. Then the kinetic/mass-operator determinants entering the one-loop partition function are expressed in terms of characteristic frequencies which are relatively simple to calculate. We will see that computation of quantum corrections can be pushed to higher-loop order by standard diagrammatic methods. In this context, generalized unitarity techniques are a promising way to reproduce loop-level worldsheet amplitudes in terms of lower-loop ones [161–163]. Instances of such homogeneous cases are the rational rigid string solutions in [164–168]. Other cases can still fall under this category if it is possible to redefine coordinates and fields to make the coefficients in the fluctuation action constant, as in chapters 6 and 7, as well as in [169].

Next-to-simplest cases are *inhomogeneous solutions*, namely non-trivial solutions of the string sigma-model that are not translationally invariant in either the $\tau$- or $\sigma$-direction. Beyond the leading order, direct computations are generally difficult and one-loop corrections are already a daunting task that requires the diagonalization of many 2d matrix-valued differential operators using functional methods based on the notion of spectral zeta-function. The rigid spinning string elliptic solutions rotating with spin $S$ in $AdS_5$ and momentum $J$ in $S^5$ [170, 171] (and chapter 4) are well-known examples of inhomogeneous backgrounds. Another non-negligible difficulty to face is the appearance of non-trivial special elliptic functions in the fluctuation spectrum (and thus in the propagator) in [172–175] which depend on the worldsheet coordinates. However, there are non-homogeneous cases that become homogeneous in certain limits, as for the example above in the limit $S/\sqrt{\lambda} \gg 1$ with $J/(\sqrt{\lambda} \log S)$ fixed [176, 177].
Chapter 1. Introduction

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Figure 1.1: Sketchy depiction of two distinctive classes of string configurations that will be of interest in the next chapters: an open string protruding in the bulk of the AdS space and ending on the path of a supersymmetric Wilson loop operator at the AdS boundary (left panel, from [178]) and a closed string that is folded upon itself and rotating in a subregion of the $AdS_5 \times S^5$ space (right panel, from [170]).

The classical backgrounds studied the next chapters 4-7 assume a special relevance in the AdS/CFT correspondence since they will be dual to two types of gauge-theory observables.

One class of observables comprises gauge-invariant, non-local observables called Wilson loops. In ordinary (conformal or not) gauge theories they are obtained [179] from the holonomy of the gauge connection around a closed spacetime path and carry information on the potential between static quarks when defined on a rectangular contour. In $\mathcal{N} = 4$ SYM “quarks” are modelled by infinitely massive W-bosons arising from a Higgs mechanism and Wilson loops admit a supersymmetric extension (Maldacena-Wilson loops) locally invariant under half of the supercharges. The AdS/CFT formulation of the duality between Wilson loops and open strings [183, 184] states that the expectation value of a supersymmetric Wilson loop $\mathcal{W}[C]$ defined along a contour $C \subset \mathbb{R}^4$ equals the string partition function $Z_{\text{string}}[C]$ where the string embedding ends on $C$

$$\langle \mathcal{W}[C] \rangle = Z_{\text{string}}[C] \equiv \int Dg DX D\Psi e^{-S_{\text{IIB}}[g,X,\Psi]}.$$  (1.11)

The reader can consult the review papers [185, 186] for a (not latest though) collection of related works.

The second example comprises local gauge-invariant operators made of traces of fully-contracted products of fields of $\mathcal{N} = 4$ SYM. The AdS/CFT correspondence conjectures a relation between their conformal dimension and the energy of rotating string states in

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16 See section 5.1. Subsequent steps were made to construct further generalization called super Maldacena-Wilson loops [180]. The field theoretical description is explored in [142, 181] while the complementary view at strong coupling in [182].
$AdS_5 \times S^5$ with the same quantum numbers. Since the Cartan subalgebra of $su(2,2) \times su(4) \subset psu(2,2|4)$ has six commuting generators, an operator is labeled by a sextuplet of charges: the scaling dimension $\Delta$ under spacetime dilatations, the two spins $S_i$ ($i = 1, 2$) of the Lorentz group and the $J_i$ ($i = 1, 2, 3$) associated to the three commuting R-symmetry generators. In AdS/CFT the symmetry group $SU(2,2) \times SU(4)$ is identified with the isometry group of $AdS_5 \times S^5$ and the six charges above correspond to the energy $E$ conjugated to the $AdS$ global time $t$, the $AdS_5$ spins $S_i$ and the $S^5$ angular momenta $J_i$ of the string.

Not all parameters are independent due to the Virasoro conditions, so we can express the worldsheet energy $E = E(S_1, S_2, J_1, J_2, J_3)$ of closed spinning strings as a function of the given remaining charges. We will be interested in rigid strings, i.e. for which the shape does not change in time. Computing the expectation value of the energy, including its quantum corrections in the coupling constant (the string tension $T$), is conveniently done by taking a "thermodynamical" approach to connect the semiclassical computation of the energy to the perturbative evaluation of the worldsheet effective action $\Gamma = -\log Z_{\text{string}}$ expanded around the relevant classical configuration. This relation was elucidated in [187, 188] and synthetically reexplained in the review [189].

### 1.4 Perturbation theory for sigma-models

The perturbative approach to string quantization has proven to be an extremely useful tool for investigating the structure of the AdS/CFT correspondence [189, 190]. As a matter of fact, the first attempt in this direction was the determination of the strong-coupling correction [191] to the quark-antiquark potential of [183], although obstructed by an issue of UV divergences. The study of semiclassical partition functions was systematically set up in [192] and it has played an important role for spinning string states [164–168, 170–172, 193], worldsheet S-matrices [161–163, 194, 195], scattering amplitudes [130] and Wilson loops [173–175, 196–199].

In semiclassical quantization, observables are computed in worldsheet-loop series in $T^{-1}$, as we will do in chapters 3-6. In the example of (1.10), this means that we can truncate the fluctuation action at quadratic (aiming at a one-loop result in chapters 4-5), at quartic (in the two-loop example of chapter 6) or higher order in $\delta X$ and $\delta \Psi$, depending on the accuracy sought in the final result, and evaluate the path-integral in saddle-point approximation. The effective action takes into account semiclassical corrections around the background solution as

$$\Gamma = -\log Z_{\text{string}} = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} + \ldots .$$  

(1.12)
A covariant formalism for the one-loop semiclassical quantization of the action will be the topic of chapter 3. The supercoset action is not a necessary starting point because the complete (Nambu-Goto and Polyakov) bosonic action was well-known before and the covariant derivative in the Green-Schwarz action at quadratic order in fermions has already appeared in the Killing spinor equation of type IIB supergravity \[200\] \[17\]. The action at quadratic level is a free model for 8 physical bosons \((y^i)\) and 16 \(\kappa\)-symmetry fixed fermionic degrees of freedom (encapsulated in a new spinor that we call \(\Psi\) again)

\[ S_{\text{IIB}}[X, \Psi] = S_{\text{IIB}}[X = X_{\text{cl}}, \Psi = 0] + \int d\tau d\sigma \sqrt{h} \left( y_i (\mathcal{O}_B)^{ij} y_j + 2\bar{\Psi} \mathcal{O}_F \Psi \right) + \ldots \]  

(1.13)

and the one-loop approximation of the path-integral (1.10) can be put into the form (1.12) \[18\]

\[ \Gamma^{(0)} = S_{\text{IIB}}[X = X_{\text{cl}}, \Psi = 0], \quad \Gamma^{(1)} = -\frac{1}{2} \log \frac{\text{Det} \mathcal{O}_F}{\text{Det} \mathcal{O}_B}. \]  

(1.14)

When the background solution is homogeneous, the one-loop effective action can be formally expressed as a summation (integral) over the discrete (continuous) eigenvalues – usually called characteristic frequencies – the operators \(\mathcal{O}_B\) and \(\mathcal{O}_F\). The matching of the bosonic and fermionic degrees of freedom and a sum-rule for their masses (dictated by the geometry of the worldsheet and the target-space supersymmetry of the Green-Schwarz action) guarantee that the result is eventually finite. In the next section 1.4.1 we explain how one has to proceed to quantify the one-loop correction around inhomogeneous solutions. Then, in section 1.4.2 we will make some comments on higher-loop corrections \(\Gamma^{(\ell)}\) with \(\ell \geq 2\) in homogeneous backgrounds, which consist of vacuum Feynman diagrams computable via standard diagrammatical techniques.

### 1.4.1 Two-dimensional fluctuation operators

A fully two-dimensional definition of (the finite and divergent part of) a determinant can be achieved with the notion of heat kernel propagator \[203–208\] of a \(r \times r\) matrix operator \(\mathcal{O}\) defined on the classical worldsheet with Riemannian metric \(h_{ij}\). This object contains all spectral information on the operator (eigenvalues and eigenfunctions) \[209–211\] and it is defined as the unique solution \(K_{\mathcal{O}}(\tau, \sigma; \tau', \sigma'; t)\) of the heat equation, namely the evolutionary Schrödinger-type equation for the “Hamiltonian” \(\mathcal{O}\) evolving in the “Wick-rotated time”

\[ \text{Det} \mathcal{O}_F \]

\[ \text{Det} \mathcal{O}_B \]

---

\[17\] The Green-Schwarz action is known to quadratic order in fermions for any general type II supergravity background \[201\] and recently up to fourth order \[202\].

\[18\] The net contribution comes only from the kinetic operators \(\mathcal{O}_B\) and \(\mathcal{O}_F\) because we suppose that the determinant of the diffeomorphism ghosts cancels the one of the “unphysical” (longitudinal) bosons, see comments below (3.46) and (3.98), and we also ignore the caveats in footnote 15.
\[ t > 0 \]
\[
(\partial_t + \mathcal{O}(\tau, \sigma)) K_{\mathcal{O}} (\tau, \sigma; \tau', \sigma'; t) = 0, \tag{1.15}
\]
supplemented by an initial normalization à la Dirac delta (\(I_r\) is the \(r \times r\) identity matrix)
\[
\lim_{t \to 0^+} K_{\mathcal{O}} (\tau, \sigma; \tau', \sigma'; t) = \frac{\delta(\tau - \tau') \delta(\sigma - \sigma')}{\sqrt{h}} I_r \tag{1.16}
\]
and boundary conditions set by an operator \(B\) if the space boundary is not empty
\[
B_{\tau, \sigma} K_{\mathcal{O}} (\tau, \sigma; \tau', \sigma'; t) = 0. \tag{1.17}
\]
A vast literature has covered the Laplace \([212, 213]\) and the (square of the) Dirac operator \([214, 215]\) for Euclidean manifolds without boundary – flat spaces \(\mathbb{R}^d\), spheres \(S^d\) and hyperboloids \(H^d\) \(^{19}\) and products thereof – and with boundary, \(e.g.\) \([216]\).

The asymptotics of the traced heat kernel for small \(t\) is known for all Laplace and Dirac operators. In section 3.5.2 it will be used to estimate the infinite part of the functional determinants in (1.14). The cancellation of the logarithmic divergences proportional to the worldsheet Ricci curvature in the one-loop effective action was clarified in \([192]\) via a careful account of the Seeley-DeWitt coefficients of the operators, which exactly compensate the divergences arising in the measure factors of the string path-integral. The counting of anomalies in the GS string goes essentially as adding together the central charges of all the fields for the theory in flat space.

Since UV divergences are associated with conformal anomalies, the same mechanism can be seen as an explicit verification of the conformal invariance of the type IIB string theory on \(AdS_5 \times S^5\) background at one loop, which was actually argued to hold true to all order in the \(T^{-1}\) expansion \([70]\).

If we look instead for extracting the regularized part of the determinants in (1.14) via zeta-function regularization \([209, 211, 217]\), knowing only the small-time behaviour does not suffice. In this case the spectral information that we need is carried by the heat kernel as a function of \(t\), but finding a solution of the heat equation is practically impossible for most of the worldsheet geometries of interests. Some notable exceptions are represented by the spectral problems for the string worldsheets in \(AdS_5\) dual to the straight line and circular Wilson loop \([218, 219]\).

\(^{19}\)The three spaces listed here are \textit{maximally symmetric} because their metrics possess the maximal number \(d(d+1)/2\) of Killing vectors in \(d\) dimensions.
In the examples considered in chapters 4 and 5, the geometric properties of the classical surface deliver fluctuation operators that are translationally invariant in one variable, which we call \( \tau \) in what follows. The same coordinate does not appear in the 2d induced metric because the associated vector field \( \partial_{\tau} \) generates an isometry of the classical surface.

To proceed, let us call \( \phi \) the (scalar or spinor) field acted upon by the one-loop operator \( O \) in the free action \( \int d\tau d\sigma \phi^\dagger O \phi \). Without loss in generality, let us suppose that the isometry acts on the worldsheet points as a \( U(1) \) rotation with \( \tau \in [0,2\pi) \), so we can decompose the field into discrete Fourier modes

\[
\phi(\tau, \sigma) = \sum_\omega \phi(\omega, \sigma) \frac{e^{i\omega \tau}}{\sqrt{2\pi}}.
\] (1.18)

The frequencies \( \omega \) are integer or half-integer according to the periodicity or anti-periodicity of the field. In the case of non-compact symmetry, acting like as a translation along the \( \tau \)-direction, the frequency label turns into a continuous variable \( \tau \in \mathbb{R} \) and the Fourier integral takes the place of the Fourier series

\[
\phi(\tau, \sigma) = \int_{-\infty}^{+\infty} d\omega \phi(\omega, \sigma) \frac{e^{i\omega \tau}}{\sqrt{2\pi}}.
\] (1.19)

Plugging formula (1.18) or (1.19) into the Lagrangian, one effectively replaces \( \partial_\tau \rightarrow i\omega \) in the operator \( O(\tau, \sigma) \rightarrow O(\omega, \sigma) \). This trick eventually allows to trade the 2d spectral problem for \( O(\tau, \sigma) \) with infinitely-many 1d spectral problems for the Fourier-transformed differential operator \( O(\omega, \sigma) \). Our strategy can be summarized in two steps.

- At fixed Fourier mode \( \omega \), the evaluation of the determinant \( \text{Det}_{\omega}(O(\omega, \sigma)) \) is a one-variable eigenvalue problem on a certain line segment \( \sigma \in [a,b] \), which we solve using the Gel’fand-Yaglom method, a technique based on zeta-function regularization for one-dimensional operators. More precisely, we shall refer to an extension of the theorem elaborated by Forman in that encompasses also the case of odd-order differential operators. The theorem outputs the value of the ratio between two determinants in terms of solutions of an homogeneous ordinary differential equation, so it requires some preliminary work to pair bosonic and fermionic fluctuation determinants. It is worth mentioning that both the Gel’fand-Yaglom method and its corollaries do not naturally cope with infinite or semi-infinite

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20We are referring to (4.11) and (4.27) for the two examples of spinning strings studied in this thesis, (5.38) and (5.53) for the worldsheet dual to a 1/4-BPS latitude Wilson loop.

21This geometrically manifests as an axial symmetry of the minimal surface (5.25) in Figure 5.3.

22For the spinning strings of chapter 4, this is the invariance of the string surface under shifts of the AdS global time \( t \), see text above (4.1) and formula (4.22).

23This is of course just one possible route. It is a well-known result [221] that the logarithm of the determinant is equivalently computed by the on-shell vacuum energy, as obtained by summing over the frequency spectrum of the operator, e.g. [198], under certain assumptions. Alternatively, one can also employ the phaseshift method, see [225] for a recent application. We are grateful to Xinyi Chen-Lin and Daniel Medina Rincón for long discussions about this point.
intervals and with operators having singular coefficients for some \( \sigma \in [a, b] \). One usually needs some regulators on \( \sigma \) to deal with such situations in AdS/CFT applications, only then to prescribe an appropriate regularization scheme to eliminate or subtract them from the final result.

- The full 2d determinant is given by the sum over all discrete frequencies \( \omega \)

\[
\log \text{Det}(\mathcal{O}(\tau, \sigma)) = \left( \int_0^{2\pi} \frac{d\tau}{2\pi} \right) \sum_\omega \log \text{Det}_\omega(\mathcal{O}(\omega, \sigma)) = \sum_\omega \log \text{Det}_\omega(\mathcal{O}(\omega, \sigma)) \tag{1.20}
\]

or by an integral if \( \omega \) is continuous

\[
\log \text{Det}(\mathcal{O}(\tau, \sigma)) = \left( \int_{-\infty}^{+\infty} \frac{d\tau}{2\pi} \right) \int_{-\infty}^{+\infty} d\omega \log \text{Det}_\omega(\mathcal{O}(\omega, \sigma)) \tag{1.21}
\]

The infinite prefactor in the latter case is often neglected since it drops once bosonic and fermionic contributions are plugged into (1.14).

In the following we shall make a notational distinction between the algebraic determinant \( \text{det} \) and the functional determinant \( \text{Det} \), involving the determinant on the matrix indices as well as on the 2d space with coordinates \((\tau, \sigma)\). We also introduced the functional determinant \( \text{Det}_\omega \) over \( \sigma \) at a given Fourier mode \( \omega \). The issue of how to define an unambiguous infinite product (1.20) is addressed in the case of rotationally-invariant operators in \([228]\) and goes case by case in the main text.

We conclude the section with the observation that, while the algorithm described above is highly versatile and easy to implement either analytically or numerically, it does not treat the worldsheet directions on equal footing. On general grounds, an optimal algorithm for fluctuation determinants should preserve the classical symmetries (e.g. diffeomorphisms of the classical worldsheet and target-space supersymmetry of the Green-Schwarz action) at any steps of the regularization procedure. The Gel’fand-Yaglom theorem, paired up with cutoff regularization on \( \sigma \) and followed by the summation over the angular modes, breaks the invariance under 2d reparametrization \((\tau, \sigma) \to (\tau'(\tau, \sigma), \sigma'(\tau, \sigma))\) at least.

The heat kernel method does not suffer from this issue, but in the context of one-loop spectral problems its current level of development is limited to a very selected class of background solutions \([192, 218, 219]\). A fully two-dimensional method, based upon heat kernel techniques, is being developed in \([229]\) (see also chapter 8) when the worldsheet metric is a “small” deformation of the maximally symmetric hyperbolic space \( H^2 \).

\[\text{We thank Amit Dekel for informing us about this reference.}\]
1.4.2 String effective action beyond the next-to-leading order

The accuracy of the semiclassical approximation improves by including higher interaction vertices in the fluctuation Lagrangian in (1.10). Computations can be technically involved due to the large number of fields – 8 bosonic and 16 fermionic real off-shell degrees of freedom after gauge-fixing – and the lack of manifest two-dimensional Lorentz invariance, broken by the curved background $X_{cl}$, in the interactions terms.

Consistency at quantum level requires the theory to be finite. Perturbative calculations are crucial to directly address the non-trivial problem of divergences cancellation in a field theory that is expected to be finite, although it is not manifestly power-counting renormalizable beyond the one-loop level (see comments in [177, 230] and also [177, 188, 231, 232]). In fact, as soon as one expands around a particular vacuum and picks a proper gauge-fixing, the derivatives of the curved background generate a Green-Schwarz kinetic term $\partial X_{cl}\Psi\bar{\partial}\Psi$ and fermionic interactions that contain derivatives. While all fields are dimensionless and no dimensionful parameter appears in the action, derivatives of the background act as a dimensional scale that turns the fermion mass dimension from 0 to the canonical $1/2$ in two dimensions, therefore leading to non-renormalizable interactions and to the potential appearance of power-like and logarithmic divergences.

Loop computations with related tests of finiteness of the model may be a thorny issue. A particularly convenient choice to fix the diffeomorphism invariance and $\kappa$-symmetry of the coset action is the AdS light-cone gauge thoroughly discussed in [233, 234] and summarized in our sections 2.1.3 and 7.2.1. The resulting action has a relatively simple structure – where fermions are only quadratic and quartic – that was efficiently used, for instance, in [231] to compute the string effective action up to two-loops in the null cusp background. This is a homogeneous solution lying only in AdS that can be viewed as the light-like limit of the space-like cusp solution of [235].

The AdS light-cone gauge turns out to be perfectly applicable to the $AdS_4 \times \mathbb{C}P^3$ Lagrangian of [236, 237] in section 2.2. A side objective of chapter 6 is to begin the investigation of this quantum action, expanded about the $AdS_4 \times \mathbb{C}P^3$ counterpart of the null cusp solution above. Similarly to the $AdS_5 \times S^5$ case, here the AdS light-cone approach to the evaluation of the two-loop effective action turns out to be extremely efficient. Simplifications occur due to bosonic propagators being only diagonal, which reduces the number of Feynman graphs to be considered $^{25}$.

As a preparation for the two-loop study in chapter 6 following what was done in [231], let us emphasise that various non-covariant integrals with components of the loop momenta

$^{25}$ In the first two-loop calculation of [238] the conformal gauge was used, in which propagators are non-diagonal, implying the evaluation of a larger number of two-loop diagrams.
in the numerators originate from the combinations of vertices and propagators. This naturally bring us to the problem of handling potentially-divergent loop integrations. As for the $AdS_5 \times S^5$ coset action, the action is it not renormalizable by power-counting and naively it seems to lead to potential divergences. In principle, the expected finiteness of the theory implies that all their divergences cancel against a careful account of the contributions from the path-integral measure and $\kappa$-symmetry ghosts (see [177]). However, one typically does not attempt an exact evaluation of the terms crucial for the complete divergence cancellation. Alternatively, the use of dimensional regularization \(^{26}\) allows to automatically set all power-like divergences to zero. In doing so, we are effectively discarding these divergences in loop integrals, but we can non-trivially check the absence of logarithmically-divergent integrals in the final result.

A strong indication of consistency – in which the finite parts and the divergence cancellation are some of the ingredients – comes from the comparison with the integrability predictions of Bethe ansatz [118] in the ABJM theory. We also flash that the $AdS_4 \times CP_3$ action of [236, 237] in AdS light-cone gauge offers an efficient setting for computing the dispersion relation of worldsheet excitations on the same cusp background [239], which were found to essentially agree with the predictions from the Bethe ansatz of [240].

1.5 Lattice field theory for the $AdS_5 \times S^5$ string sigma-model

In the last sections we saw that the $AdS_5 \times S^5$ sigma-model is a non-trivial theory which, as virtually any interacting QFT, we do not know how to define rigorously without relying on perturbative expansions. The aim of this section is to set the ground for chapter 7, where we will see that lattice methods provide a concrete mean to non-perturbatively define the theory and evaluate observables on the two-dimensional (Wick-rotated) Euclidean worldsheet from the gravity side of AdS/CFT.

In general, the study of lattice models has a long and well-established tradition in QCD and condensed matter systems [241–243]. Such non-perturbative investigations are formulated in discrete rather than continuous spacetime. They provide a mathematically well-defined regularization of the theory of interest by introducing a momentum UV cutoff of order $a^{-1}$, where $a$ is the spacing between lattice sites in the spacetime grid. The central idea is to construct a discrete form of the action and the operators, which formally reduce to corresponding continuum counterparts when the regulator is removed. By working on a discrete spacetime, path-integrals defining observables become finite multi-dimensional integrals which can be evaluated through stochastic simulation techniques such as the Monte

\(^{26}\)In regularizing our integrals all manipulations of tensor structures in the Feynman integrands are however carried out strictly in two dimensions.
Carlo method. The aim is to eventually recover the values of the observables in the continuum model on finer and finer lattices, through an appropriate prescription on the continuum limit $a \to 0$ and the other technical parameters of the simulations.

The first finite-coupling calculation in the $\text{AdS}_5 \times \text{S}^5$ superstring sigma-model using purely (lattice) field theory methods was pioneered in [244] with the measurement of the universal scaling function $f(\lambda)$, one of the most important observables in the $\text{AdS}_5/\text{CFT}_4$ duality. It governs the (the renormalization of) a light-like cusped Wilson loop in $\mathcal{N} = 4$ SYM, while from the holographic viewpoint it is captured by the path-integral of an open string ending on two intersecting null lines at the AdS boundary. The convenient ground to set up such lattice investigation was the same of the two-loop perturbative analysis of [231] mentioned in section 1.4.1, i.e. the AdS light-cone gauge-fixed action of [233, 234] with no fermionic interactions of order higher than four and expanded around the null cusp background. The crucial difference is the fact that now all interactions must be kept in the fluctuation Lagrangian to be discretized. The numerical results of [244] for the scaling function were found in agreement with its exact integrability prediction from the Beisert-Eden-Staudacher (BES) equation [245] within reasonable numerical accuracy.

Our work in chapter 7 takes a deep inspiration from the route opened up by [244]. We will scrutinize the way one should extract the scaling function from the expectation value of the action, as well as inaugurate the study of the dispersion relations of worldsheet fields starting from the measurement of the physical mass of one bosonic field. We will also elucidate and enlarge the discussion related to many aspects related to the numerical simulations.

The investigation here and in [244] is not a non-perturbative definition of the worldsheet string model à la Wilson lattice-QCD. In this case, one should work with a Lagrangian which is invariant under the local symmetries of the model (bosonic diffeomorphisms and fermionic $\kappa$-symmetry), whereas we will make use of the action in the AdS light-cone gauge which fixes them all. However, there is a number of reasons that makes this model interesting for lattice investigations, potentially beyond the community interested in numerical holography [246].

- If the aim is a test of holography and integrability, it is computationally cheaper to simulate a two-dimensional model, rather than a four-dimensional one ($\mathcal{N} = 4$ SYM). Additionally, all fields are assigned to sites, since no gauge degrees of freedom are present and only (commuting and anti-commuting) scalar fields appear in the string action.

- Although we deal with superstrings, there is no subtlety involved with supersymmetry on the lattice because in the Green-Schwarz formulation of the action supersymmetry is manifest only in the target space, and also because $\kappa$-symmetry is gauge-fixed.

\footnote{In literature scaling function and cusp anomalous dimension are essentially synonyms, as we will explain in section 7.1.}
• At the same time, this is a computational playground interesting on its own, allowing in principle for explicit investigations/improvements of algorithms. In fact, we work with a highly non-trivial two-dimensional model for which relevant observables have not only, through AdS/CFT, an explicit analytic strong coupling expansion but also, through AdS/CFT and the assumption of integrability, an explicit numerical prediction at all couplings.

• In principle, the scope of numerical simulations is not limited to partition functions. An interesting program was outlined in section 2.2 of [244]. The goal is to provide a concrete mean to extract anomalous dimensions of local gauge-invariant operators in $\mathcal{N} = 4$ SYM from the numerical evaluation of worldsheet two-point functions of the dual integrated vertex operators [247, 248]. Significant work is necessary to investigate this direction in the next future.

On a different note, we conclude by mentioning – albeit not as much as it would deserve – that recent years have also witnessed rapid progress in constructing lattice discretizations of supersymmetric gauge theories [246, 249, 250], in particular of the $\mathcal{N} = 4$ SYM theory [251–255]. Large-scale calculations [256] have studied the static “interquark” potential [257, 258] and the anomalous scaling dimension of the Konishi operator [259] (see also [246]). Alternative numerical, non-lattice approaches include the study of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ as plane-wave (BMN) matrix model [260–266].

### 1.6 Plan of the thesis

Chapter 2 gives a short introduction to the construction of superstring theory in $AdS_5 \times S^5$ and discusses the advantages and the limits of the supercoset formalism in $AdS_4 \times \mathbb{C}P^3$.

In chapter 3, which is largely based on [267], we present a pedagogical discussion about the general structure of the quadratic fluctuation Lagrangian around arbitrary classical configurations in $AdS_5 \times S^5$, expressing the relevant differential operators in terms of geometric invariants of the background geometry.

Chapter 4 is based on [268] and exploits the analytical solution to the spectral problem of a type of fourth-order differential operators. This finds application in the exact evaluation of the semiclassical contributions to the one-loop energy of a class of two-spin spinning strings in $AdS_5 \times S^5$.

In chapter 5 we illustrate the perturbative computation in [269] aimed at the strong-coupling correction to the expectation value of a family of supersymmetric Wilson loops in $\mathcal{N} = 4$.
SYM. We comment on the unexpected discrepancy between our result and the all-loop prediction from supersymmetric localization.

Chapter 6 presents the derivation in [270] of the cusp anomalous dimension of the ABJM theory up to two loops at strong coupling, which provides support to a recent conjecture for the exact form of the interpolating function \( h(\lambda) \) in this theory.

Chapter 7 shows recent developments [271, 272] in the construction of consistent lattice discretizations of the two-dimensional \( AdS_5 \times S^5 \) string sigma-model in AdS light-cone gauge. We discuss the numerical results for the observables studied so far, test them against semiclassical and integrability predictions and comment upon the difficulties encountered in our approach.

In chapter 8 we summarize our results and conclude with an outlook on interesting directions for further research, in particular based on [229, 273–275].

Several appendices supplement the main text with important methodological and technical details. For this reason, we encourage the reader to consult them in parallel with the reading of the main text.

The structure of the individual chapters is outlined in their introductions. Sometimes the presentation has required to slightly deviate from the notation of the original references, but we will clearly point it out when confusion may arise.
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