DYNAMICS OF A RIGID SHAFT
SUPPORTED BY
ANGULAR CONTACT BALL BEARINGS

by

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Abstract

In this thesis, the dynamics of a rigid shaft supported by angular contact ball bearings is studied. Particular attention is paid to ball bearings which have wide industrial applications. Despite their frequent usage, little is known about the vibration characteristics of ball bearings. Indeed it is generally not widely appreciated that even a defect free ball bearing will cause vibrations. With the added effects of defects, the vibrations and noise produced can be difficult to analyse.

The method used to investigate the vibration characteristics of angular contact ball bearings was to consider the shaft bearing assembly as a spring-mass system, where the shaft acts as a mass and the raceway and balls act as massless nonlinear contact springs. The system therefore undergoes nonlinear vibrations under dynamic conditions. In this thesis the vibrations of such a system were assumed to occur in up to five degrees of freedom. A computer program was written to model the radial and axial vibrations. Results are obtained in both the time and frequency domains.

It is shown that the vibration characteristics of the shaft-bearing assembly change when the bearings operate in different regions of their nonlinear load deflection characteristics or when the bearing design or operating conditions are altered. The untoward effects of the natural frequency and the ball passage frequency, which is one of the main problems of shaft-bearing systems, are demonstrated for a perfect bearing where when there is coincidence of natural frequency and ball passage frequency, or their sub or superharmonics, resonances occur.

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In addition, the system behaviour was studied when defects such as: out of roundness, waviness, off-sized balls, misalignment and localised defects on the running surfaces, were present. The vibrations associated with these defects are demonstrated with resonances occurring at the natural frequency and certain of the shaft speeds, cage and ball speeds, ball passage frequency, their sub and super harmonics and their combinations.

Finally, the bearing outer races were supported on elastomer buttons, cartridges or O-rings. Since the dynamic properties of elastomers are shape dependent, the available data was not usable in this thesis. Therefore geometry dependent theoretical models were developed in order to estimate the stiffness and damping coefficients of elastomers from their material properties. One formula for cylindrical buttons, two formulae for ring cartridges and three formulae for O-rings were developed and shown to be consistent with available experimental data.

In the modelling of elastomers, a mechanical Voigt model was employed and the obtained spring dash-pot system was introduced to the shaft-ball bearing assembly. It was demonstrated that a marked reduction in vibration amplitude could be achieved by applying elastomeric dampers to shaft-ball bearing systems.
I dedicate this work to the people of Türkiye whose money has financed my studies abroad, in order that our country be benefited by the perfection of technological advancement to which I hope my research has contributed.

Many students for over a century have studied abroad but made limited contribution to the industry of our nation. I earnestly hope that the students of my generation will be better equipped in knowledge to enable them to enrich their nation.
ACKNOWLEDGEMENTS

First and foremost I am thankful to The Creator of the Heavens and the Earth, Who vibrates everything in order for them to survive, for giving me the health, courage and ability to complete this research.

I would like to express my deepest gratitude to my supervisor Dr. Ramsey Gohar for his continual support, guidance and help during the course of this research.

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<th>Units</th>
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<tr>
<td>$a$</td>
<td>the distance between the misalignment centre and bearing centre</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or ball waviness amplitude</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or cross-sectional area of button (Chapter 5)</td>
<td>m²</td>
</tr>
<tr>
<td></td>
<td>or width of the rectangular cross-sectional elastomer button (Chapter 5)</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or distance between the external forces and the LHS bearing (see Fig. (3.16))</td>
<td>m</td>
</tr>
<tr>
<td>$a_l$</td>
<td>position of the LHS bearing from the C.G. (see Fig. (3.16))</td>
<td>m</td>
</tr>
<tr>
<td>$A$</td>
<td>amplitude of the lobes</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or cross-sectional area of button (Chapter 5)</td>
<td>m²</td>
</tr>
<tr>
<td></td>
<td>or a constant (Chapter 5)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>or distance between raceway groove centres</td>
<td>m</td>
</tr>
<tr>
<td>$b$</td>
<td>breadth of the rectangular cross-sectional elastomer button (Chapter 5)</td>
<td>m</td>
</tr>
<tr>
<td>$b_l$</td>
<td>position of the RHS bearing from the C.G. (see Fig. (3.16))</td>
<td>m</td>
</tr>
<tr>
<td>$B$</td>
<td>total curvature, $A / d_b$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>or a constant (Chapter 5)</td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td>viscous damping factor</td>
<td>N s/m</td>
</tr>
<tr>
<td>$c_e$</td>
<td>equivalent viscous damping factor</td>
<td>N s/m</td>
</tr>
<tr>
<td>$d$</td>
<td>diameter</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or diameter of elastomer buttons (Chapter 5)</td>
<td>m</td>
</tr>
<tr>
<td>$d_b$</td>
<td>ball diameter</td>
<td>m</td>
</tr>
<tr>
<td>$d_i$</td>
<td>inner raceway diameter</td>
<td>m</td>
</tr>
<tr>
<td>$d_m$</td>
<td>pitch diameter</td>
<td>m</td>
</tr>
<tr>
<td>$d_o$</td>
<td>outer raceway diameter</td>
<td>m</td>
</tr>
<tr>
<td>$D$</td>
<td>diameter of the buttons</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or shaft diameter at groove base</td>
<td>m</td>
</tr>
<tr>
<td>$E$</td>
<td>Young's modulus</td>
<td>N/m²</td>
</tr>
<tr>
<td></td>
<td>or error function (Chapter 5)</td>
<td></td>
</tr>
<tr>
<td>$E_{eff}$</td>
<td>effective Young's modulus (Chapter 5)</td>
<td>N/m²</td>
</tr>
</tbody>
</table>
### Notation

- **$E_o$**  
  static Young's modulus (Chapter 5)  
  \( \text{N/m}^2 \)
- **$E'$**  
  complex Young's modulus of elasticity  
  \( \text{N/m}^2 \)
- **$E^*$**  
  real part of the complex Young's modulus of elasticity  
  \( \text{N/m}^2 \)
- **$E''$**  
  imaginary part of the complex Young's modulus of elasticity  
  \( \text{N/m}^2 \)
- **$f$**  
  frequency  
  \( \text{Hz} \)
- **$f_{bp}$**  
  ball passage frequency  
  \( \text{Hz} \)
- **$f_c$**  
  cage rotation frequency  
  \( \text{Hz} \)
- **$f_n$**  
  natural frequency  
  \( \text{Hz} \)
- **$f_w$**  
  wave passage frequency  
  \( \text{Hz} \)
- **$nf$**  
  \( n \)th super harmonic of one frequency  
  \( \text{Hz} \)
- **$F(p)$**  
  curvature difference  
  \( 1/\text{m} \)
- **$F$**  
  external forces  
  \( \text{N} \)
  or  
  external tensile force (Chapter 5)  
  \( \text{N} \)
- **$g$**  
  acceleration due to gravity  
  \( \text{m/s}^2 \)
- **$G$**  
  shear modulus of elasticity  
  \( \text{N/m}^2 \)
- **$G^*$**  
  complex shear modulus of elasticity  
  \( \text{N/m}^2 \)
- **$G'$**  
  real part of the complex shear modulus of elasticity  
  \( \text{N/m}^2 \)
- **$G''$**  
  imaginary part of the complex shear modulus of elasticity  
  \( \text{N/m}^2 \)
- **$h$**  
  the height of the buttons (Chapter 5)  
  \( \text{m} \)
  or  
  time increment  
  \( \text{s} \)
- **$H$**  
  compliance  
  \( \text{mN} \)
- **$H^*$**  
  complex compliance  
  \( \text{mN} \)
- **$H_1$**  
  real part of the complex compliance  
  \( \text{mN} \)
- **$H_2$**  
  imaginary part of the complex compliance  
  \( \text{mN} \)
- **$I$**  
  second moment of area  
  \( \text{m}^4 \)
- **$k$**  
  material constant, proportional to hardness
- **$K$**  
  contact stiffness factor  
  \( \text{N/m}^{3/2} \)
  or  
  Runge-Kutta interval constant
- **$K^*$**  
  complex stiffness  
  \( \text{N/m} \)
- **$K_1$**  
  real part of the complex stiffness  
  \( \text{N/m} \)
- **$K_2$**  
  imaginary part of the complex stiffness  
  \( \text{N/m} \)
- **$K_e$**  
  equivalent stiffness  
  \( \text{N/m} \)
- **$K_i$**  
  inner raceway/ball contact stiffness factor  
  \( \text{N/m}^{3/2} \)
- **$K_o$**  
  outer raceway/ball contact stiffness factor  
  \( \text{N/m}^{3/2} \)
- **$l$**  
  length of the specimen (Chapter 5)  
  \( \text{m} \)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>a characteristic length</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or mean circumferential length of a cartridge ring (Chapter 5)</td>
<td>m</td>
</tr>
<tr>
<td>$m$</td>
<td>number of balls</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or number of Voigt elements (Chapter 5)</td>
<td>m</td>
</tr>
<tr>
<td>$m_{er}$</td>
<td>mass of outer ring</td>
<td>kg</td>
</tr>
<tr>
<td>$M$</td>
<td>mass of the shaft</td>
<td>kg</td>
</tr>
<tr>
<td></td>
<td>or moment</td>
<td>Nm</td>
</tr>
<tr>
<td>$n$</td>
<td>mode order</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or shaft speed</td>
<td>r.p.m.</td>
</tr>
<tr>
<td>$n_i$</td>
<td>inner race (i.e., shaft) speed</td>
<td>r.p.m.</td>
</tr>
<tr>
<td>$n_o$</td>
<td>outer race speed</td>
<td>r.p.m.</td>
</tr>
<tr>
<td>$N$</td>
<td>number of data points</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or number of wave lobes (Chapter 5)</td>
<td>m</td>
</tr>
<tr>
<td></td>
<td>or number of buttons (Chapter 5)</td>
<td>m</td>
</tr>
<tr>
<td>$p$</td>
<td>empirical constant for a particular geometry</td>
<td>m</td>
</tr>
<tr>
<td>$q$</td>
<td>instantaneous centre of movement</td>
<td>m</td>
</tr>
<tr>
<td>$r$</td>
<td>radius</td>
<td>m</td>
</tr>
<tr>
<td>$r_{gi}$</td>
<td>inner groove radius</td>
<td>m</td>
</tr>
<tr>
<td>$r_{go}$</td>
<td>outer groove radius</td>
<td>m</td>
</tr>
<tr>
<td>$r_i$</td>
<td>inner raceway radius</td>
<td>m</td>
</tr>
<tr>
<td>$r_m$</td>
<td>pitch radius</td>
<td>m</td>
</tr>
<tr>
<td>$r_o$</td>
<td>outer raceway radius</td>
<td>m</td>
</tr>
<tr>
<td>$P_o$</td>
<td>the amplitude of forcing function</td>
<td>N</td>
</tr>
<tr>
<td>$P_{R}$</td>
<td>preload</td>
<td>N</td>
</tr>
<tr>
<td>$R$</td>
<td>ball centre focus radius under zero load</td>
<td>m</td>
</tr>
<tr>
<td>$R_i$</td>
<td>inner radius of the ring cartridge or O-ring (Chapter 5)</td>
<td>m</td>
</tr>
<tr>
<td>$R_o$</td>
<td>outer radius of the ring cartridge or O-ring (Chapter 5)</td>
<td>m</td>
</tr>
<tr>
<td>$\overline{R}$</td>
<td>mean radius of ring cartridge or O-ring (Chapter 5 –see Fig.(5.8))</td>
<td>m</td>
</tr>
<tr>
<td>$S$</td>
<td>entropy</td>
<td>m</td>
</tr>
<tr>
<td>$S_f$</td>
<td>shape factor equivalent to loaded area divided by force free area</td>
<td>m</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
<td>s</td>
</tr>
<tr>
<td></td>
<td>or thickness of the ring cartridge (Chapter 5)</td>
<td>m</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>time step</td>
<td>s</td>
</tr>
<tr>
<td>$T$</td>
<td>temperature</td>
<td>K</td>
</tr>
<tr>
<td>$V$</td>
<td>velocity</td>
<td>m/s</td>
</tr>
</tbody>
</table>
Notation

- or volume (Chapter 5) \( m^3 \)
- \( V_i \) inner raceway velocity \( m/s \)
- \( V_o \) outer raceway velocity \( m/s \)
- \( V_c \) cage velocity \( m/s \)
- \( w \) work done by the representative chain (Chapter 5) \( Nm \)
- \( w_1 \) weighting constant
- \( w_2 \) weighting constant
- \( W \) total contact force between a ball and its race \( N \)
  or the total work done (Chapter 5) \( Nm \)
- \( W_{off} \) total contact force due to off sized ball \( N \)
- \( x, y, z \) local coordinates attached to the shaft (see Fig.(3.17))
  or local coordinates attached to the reference ball (see Fig.(3.15))
- \( X, Y, Z \) fixed coordinates (see Fig.(3.17))
- \( X_i, Y_i, Z_i \) local fixed coordinates at the C.G. (see Fig.(3.17))
- \( X'_i, Y'_i, Z'_i \) intermediate coordinates following Euler's angles (see Fig.(3.17))
- \( \alpha \) contact angle \( \text{rad} \)
- \( \alpha_i \) instantaneous contact angle \( \text{rad} \)
- \( \alpha_o \) unloaded contact angle \( \text{rad} \)
- \( \alpha_p \) preloaded contact angle \( \text{rad} \)
- \( \beta \) shape factor
- \( \delta \) contact deflection \( m \)
  or phase angle as in Equ.(5.5) \( \text{rad} \)
- \( \delta_c \) deflection due to misalignment \( m \)
- \( \delta_i \) total instantaneous contact deflection for the \( i \) th ball \( m \)
- \( \Delta \delta \) diameter difference of the off sized ball \( m \)
- \( \delta_r \) amplitude of the vibration (Chapter 5) \( m \)
- \( \delta^* \) dimensionless contact deflection
- \( \kappa \) elliptical eccentricity parameter
- \( \lambda \) wave length \( m \)
  or extension ratio (Chapter 5) \( \text{rad} \)
- \( \nu \) Poisson's ratio
- \( \vartheta \) the angle between the fixed and moving reference axes (see Fig.(3.15)) \( \text{rad} \)
  or phase difference

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Notation

\( \varepsilon \)  convergence criterion  
\( \phi \)  rock angle of the shaft about \( y \) axis  
\( \gamma \)  angle between neighbouring two balls (see Fig.(3.15))  
\( \zeta \)  integration angle  
\( \sigma \)  damping ratio  
\( \theta \)  angular position of balls  
\( \theta' \)  arbitrary angular position of balls  
\( \rho \)  curvature  
\( \Sigma \rho \)  curvature sum  
\( \omega \)  angular velocity of the shaft  
\( \omega_a \)  ball angular velocity in the same plane as the cage speed  
\( \omega_b \)  ball angular velocity about its centre  
\( \omega_c \)  cage angular velocity  
\( \omega_{CL} \)  angular velocity about the instantaneous axis of zero velocity  
\( \omega_i \)  inner race angular velocity  
\( \omega_o \)  outer race angular velocity  
\( \omega_s \)  ball spinning velocity  
\( \Omega \)  The stiffness divided by the damping factor (\( K/c \)) (Chapter 5)  
\( \psi \)  rock angle of the shaft about \( x \) axis  
\( \chi_0 \)  the amplitude of vibration in \( x \) direction  
\( \chi_0 \)  the amplitude of vibration in \( y \) direction  
\( \Gamma_i \)  the amplitude of the waves at the contact angle  
\( \Gamma_o \)  initial wave amplitude  
\( \Gamma_p \)  maximum amplitude of the wave  
\( \kappa \)  complete elliptic integral of the first kind  
\( \mathcal{K} \)  complete elliptic integral of the second kind  
\( \mathcal{R}_i \)  radius of the loci groove centres of curvature for inner ring  
\( \mathcal{R}_o \)  radius of the loci groove centres of curvature for outer ring  
\( \ell \)  a characteristic length

\( m \)  mass
\( \text{rad.} \)  radians
\( \text{N/m}^2 \)  Pascal's (N/m²)
\( \text{rad}/s \)  radian per second (rad/s)
\( \text{m} \)  meter

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### Subscripts

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>initial conditions</td>
</tr>
<tr>
<td>1</td>
<td>relating to the real part of a complex variable</td>
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<tr>
<td>2</td>
<td>relating to the imaginary part of a complex variable</td>
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<tr>
<td>a</td>
<td>relating to axial</td>
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<tr>
<td>b</td>
<td>relating to ball</td>
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<tr>
<td>bp</td>
<td>ball passage frequency</td>
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<tr>
<td>c</td>
<td>relating to cage</td>
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<tr>
<td>e</td>
<td>equivalent</td>
</tr>
<tr>
<td>i</td>
<td>$i$ th ball</td>
</tr>
<tr>
<td></td>
<td>or $i$ th Voigt element (Chapter 5 and 7)</td>
</tr>
<tr>
<td></td>
<td>or iteration index</td>
</tr>
<tr>
<td></td>
<td>or relating to the initial conditions</td>
</tr>
<tr>
<td>j</td>
<td>relating to the Voigt element being calculated</td>
</tr>
<tr>
<td></td>
<td>or iteration index</td>
</tr>
<tr>
<td>k</td>
<td>relating to discrete time series</td>
</tr>
<tr>
<td>n</td>
<td>relating to the natural frequency</td>
</tr>
<tr>
<td>r</td>
<td>relating to discrete time series</td>
</tr>
<tr>
<td></td>
<td>or radial</td>
</tr>
<tr>
<td>wp</td>
<td>relating to the wave passage frequency</td>
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<tr>
<td>x, y, z</td>
<td>coordinate axes</td>
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<tr>
<td>$\omega$</td>
<td>frequency dependence</td>
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### Superscripts

<table>
<thead>
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<th>Symbol</th>
<th>Description</th>
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<tr>
<td>.</td>
<td>relating to the first derivative with respect to time</td>
</tr>
<tr>
<td>..</td>
<td>relating to the second derivative with respect to time</td>
</tr>
<tr>
<td>*</td>
<td>indicating a normalised variable</td>
</tr>
<tr>
<td></td>
<td>or relating to a complex number</td>
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## Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
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<tbody>
<tr>
<td>BPF</td>
<td>Ball Passage Frequency</td>
</tr>
<tr>
<td>C.G.</td>
<td>Centre of Gravity</td>
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<tr>
<td>deg.</td>
<td>degrees</td>
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<td>Equ.</td>
<td>equation</td>
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<td>equations</td>
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<td>Fig.</td>
<td>figure</td>
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<td>FFT</td>
<td>Fast Fourier Transform</td>
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<tr>
<td>LHS</td>
<td>Left Hand Side</td>
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<tr>
<td>rad.</td>
<td>radians</td>
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<tr>
<td>rms</td>
<td>root mean squares</td>
</tr>
<tr>
<td>rpm</td>
<td>revolutions per minute</td>
</tr>
<tr>
<td>RHS</td>
<td>Right Hand Side</td>
</tr>
<tr>
<td>wpc</td>
<td>waves per circumference</td>
</tr>
<tr>
<td>WPF</td>
<td>Wave Passage Frequency</td>
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CHAPTER 1

GENERAL INTRODUCTION

1.1 Introduction

A general introduction to the shaft-angular contact ball bearings assembly as a part of vibratory system and elastomers as external dampers to such a system will be given in the first part of this chapter. Later the scope and outline of the thesis will also be given and at the end of this chapter some of the important terminology used in this thesis will be defined.

1.2 Angular Contact Ball Bearings as a Source of Vibrations

In modern machine applications, ball bearings have become indispensable since, whenever a shaft rotates, it must be supported in such a way that it will be maintained in the correct position and be free to rotate in the machine's framework, thus ensuring the correct functioning of the machine.

In the transformation of rotating motion from one machine part to another, the main objectives are to minimise radial run-out and frictional losses since friction consumes energy as well as causing wear which brings about changes in dimensions until a machine becomes ineffective. The machine part employed to achieve minimum frictional losses and hence a free motion between moving and stationary parts of the machinery is called a bearing. The simplest bearing is made up of a shaft, called a journal, rotating in a mating hole which serves as the bearing. Even after making the running surfaces of the shaft and bearings as smooth as possible and using lubricant between them in order to achieve low friction torque, the losses due to friction are considerable. In engineering, rolling bearings, as an alternative to journal bearings, are therefore employed in order to minimise frictional losses, since their rolling resistance is much less than the journals' sliding resistance.
Amongst rolling element bearings, ball bearings are frequently used for applications, where load is not too high, on account of their exceptionally low friction and their ability to take a preload.

A ball bearing is composed of an inner race, an outer race, a set of balls and a cage or separator as shown in Fig. (1.1). Inner and outer races are basically two hardened steel rings with annular grooves which serve as a guide to the precessing balls. The inner race is generally force-fitted onto the shaft and the outer race is similarly fitted in a housing. Inner or outer races can sometimes be dispensed with, and the balls mounted in such a manner that they run on a hardened shaft itself or a supporting surface. This is needed in some applications because of the space limitations. The cage or separator prevents balls from contacting each other while they are precessing round the inner race and spaces them uniformly. The balls are spherical to a high accuracy and are the most important parts of a bearing since they transmit the forces from the moving parts of the machinery to the stationary parts.

Angular contact ball bearings are a special type for they are able to accommodate both axial and radial loads. The shafts under combined loading are generally supported by angular contact ball bearings with a built-in preload. Depending on the arrangement made or the type of angular contact bearing chosen, they can accommodate axial loads acting in one or both directions. Another advantage of angular contact ball bearings is their being producible with tight tolerances. In this case they are called precision angular contact ball bearings which are also used in the simulation model in this thesis.
Chapter 1

Ball bearings may appear simple and unsophisticated to the casual onlooker, yet their use in applications involving high speeds, high temperature, heavy, or unusual loading, requires a precise definition of their specification and performance. Kinematics, vibration analysis, theory of elasticity, hydrodynamics and elastrohydrodynamic lubrication, wear, friction and heat transfer are some of the topics which are employed in an analysis of bearing performance. Some of these topics are widely employed in the investigations of ball bearings while others remained relatively less used. In particular, kinematics and vibrations of ball bearings can be classified under the latter heading. Hence in this thesis vibrations associated with angular contact ball bearings will be investigated.

Angular contact ball bearings generally support a shaft rotating at a certain speed. Naturally they are involved in the dynamics of this shaft and the load it carries. For quite a long time researchers did not realise that ball bearings were one of the governing parameters for the dynamic characteristics of the system. Balls were assumed to be rigid or were taken as equivalent linear springs based on the complete bearing. One of the main reasons for not realising the importance of ball bearings as a vibratory system was that their measurements and simulations were not precise enough. As more precise applications were carried out, due attention was given to ball bearings in shaft dynamics.

It is generally not widely known that a rolling element bearing may be perfect (i.e. defect free) and yet still produce vibrations. With defects in addition, the vibrations and noise produced can be quite difficult to analyse. As a first step in identifying the noise problem caused by a ball bearing or in other words its vibration characteristics, a mechanical vibratory model of a shaft-bearing assembly should be considered as a spring-mass system, where the shaft acts as a mass and the raceways and balls act as nonlinear contact springs. Therefore, the system undergoes nonlinear vibrations under dynamic conditions.

Previous investigations also showed that when a shaft supported by ball bearings rotates, it undergoes vibrations. These vibrations can be quite severe if the wrong operational speeds or bearings are chosen. Researchers try to eliminate these vibrations for various reasons. Some may require a quiet environment, others very smooth running. Whatever the reason, they endeavour to overcome these vibrations by careful design. After each new design they would like to check whether their new design works. Without a simulation model, they have to build an experimental rig and keep testing with frequent modifications. This trial and error approach is very costly. The best thing is to have a simulation model, like the one in this thesis, in order to check whether the new idea will work at the design stage and then to verify the results with experiments afterwards. This saves money and time.
1.3 Damping of Vibrations

If the vibrations cannot be eliminated or lowered to an acceptable level by bearing design only, external dampers might have to be employed.

External dampers have become increasingly important in the control of shaft vibrations of rotating equipment. Previous researchers tried to eliminate or reduce the untoward effects of the vibrations by employing different dampers such as using elastrohydrodynamic lubrications or supporting the bearings on flexible mountings or by using oil squeeze bearings round the outer race.

Another type of external damping is by employing viscoelastic damping which is exhibited strongly in polymeric and glassy materials. In this thesis, elastomers are employed. Elastomer is a material that consists of long-chain organic molecules linked at various points along their length by strong chemical bonds giving it elastic properties akin to those of rubber.

Although elastomers have been used in different applications, mainly in vibration and shock isolation, for a long time, their dynamic properties were unknown for many years. Today, elastomers are used in designs involving increasingly complex applications. Thus accurate knowledge and a clear definition of elastomer dynamic properties (stiffness and damping) is becoming important.

However, there are three problems in employing elastomers. The first one is limited data since the dynamic properties are shape dependent. The second is the highly nonlinear damping exhibited in them and the fact that their dynamic characteristics are obtained only for a periodic excitation, whereas in the present study the initial transient vibrations contribute to the vibration behaviour of the system and are not necessarily periodic. The third is their relatively low rigidity that reduces one of the prime advantages of the ball bearing which is its relatively high stiffness compared with, say, gas bearings.

In order to investigate these difficulties, a pilot study was carried out during the course of this work and some interesting results were obtained. It is shown that the dynamic properties of elastomers can be derived from their material properties and, using this data, elastomer dampers can be successfully employed for attenuating untoward vibrations by careful design.

In summary it may be said that ball bearings are one of the essential parts of rotating machinery and their characteristics as well as imperfections in them, can cause severe vibration problems. Hence it is important to understand vibration response and to identify
undesired effects in order to eliminate them through design considerations or to avoid them by careful operational planning (i.e. shaft speed, preload, external forces etc.). Therefore, the object of this investigation is to study the vibration characteristics of angular contact ball bearings and to see whether elastomer dampers are able to reduce vibration levels.

1.4 Objectives of the Thesis

The main objectives of this thesis are to investigate:

1.4.1 The variation of the natural frequency due to different positions of the ball set in a bearing.

1.4.2 The effect of the ball passage frequency (it will be defined later in this chapter) and the effects of varying the number of balls and the preload on it.

1.4.3 The effect of system anomalies such as waviness, off-sized ball, misalignment and out of balance of shaft centre.

1.4.4 The presence of localised point defects on the running surfaces.

1.4.5 Obtaining the dynamic properties (stiffness and damping coefficients) of elastomers from their material properties (Young's modulus or shear modulus).

1.4.6 Whether elastomers can successfully be employed as external dampers in shaft-ball bearing assemblies.

1.5 Layout of the Thesis

This thesis is in eight chapters.

In Chapter 1, there is a general introduction, the outline of the project is defined and the terms used in the thesis are clarified.

Chapter 2 gives the survey of previous work done. Since the subject is broad, this review is done under different subheadings in order to show how the various topics interact with each other.

Chapter 3 deals with angular contact ball bearings that are used to support a rotating shaft. It gives a general definition of the dynamic problems associated with angular contact ball bearings. The theories used are discussed and the equations developed.
The resulting equations of motion are developed and their solution method is investigated.

Solutions of the equations of motion result in peak amplitudes at certain frequencies. These frequencies are investigated in Chapter 4. Some of the manufacturing anomalies and defects that can lead to severe vibrations are also considered and their effects are investigated. The effects of some external forces as well as the change in some of the system parameters on the dynamic behaviour of the system is also investigated in the fourth chapter.

In Chapter 5 elastomers as external dampers to angular contact ball bearing-shaft systems are studied. Because of limited data available, an investigation is carried out to obtain the dynamic behaviour of elastomers from their material properties and geometric shapes. Different types of elastomer dampers, namely buttons, cartridges and O-rings are introduced to the system as external dampers in order to eliminate or reduce untoward effects of natural frequencies of the system. In this a mechanical Voigt model is employed.

In Chapter 6 the simulation models described in the third and fourth chapters are used to obtain results which are then compared. First, a two degree of freedom system is investigated since its simplicity allows some very important system parameters to reveal themselves. The degrees of freedom are then increased, first to three and then to five in order to obtain close simulation of real systems. The results obtained are compared with other researchers' experimental and theoretical data and their differences are discussed.

In Chapter 7 the equations developed for elastomers are investigated and compared with experimental data presented by other researchers. Later elastomers are introduced to the simulation model and effect of employing elastomer as external dampers to the shaft-ball bearing system is studied.

Chapter 8 is the conclusion of the thesis. In this chapter the results obtained during the course of this research are summarised. The chapter concludes with suggestions for further work.

1.6 Definitions of Some Factors Influencing Ball Bearing Vibrations

Today, as new branches of science come into existence, each branch specialises in a particular topic, hence many new terms appear known only to specialists. Furthermore,
the same word is sometimes used with different meanings in other fields or by other researchers in the same field. In some cases different words are employed to describe the very same thing or occurrence. Therefore it has become necessary to clarify the meanings of the words used in reports and theses. Thus some of the important key words used in this thesis are defined here and have been kept unchanged throughout the thesis unless otherwise stated.

1.6.1 Natural Frequency of the Shaft and Bearings Assembly

If a shaft on ball bearings is displaced from its position of equilibrium and released, vibrations take place. These vibrations, which are maintained by the elastic force in the bearings alone, are called the free or natural vibrations of the shaft and its bearings. For this type of vibration there are certain frequencies where the amplitude of the shaft displacement becomes large (see Fig.(1.2)). If the shaft is assumed to be elastic (i.e., not rigid) there are a series of natural frequencies referred to as modal frequencies of the shaft and bearings. If any forcing functions coincide with one of the natural modes, the effect of resonance can be observed clearly.

![Graph showing first and second natural frequencies](image)

Fig.(1.2) Natural frequencies of a system for two degrees of freedom

If the shaft is assumed to be rigid, there is only one natural frequency for each degree of freedom it possesses.
1.6.2 Ball Passage Frequency

This is one of the important frequencies associated with ball bearings. Although it appears as ball pass frequency or variable compliance frequency in different references, it is referred to as the ball passage frequency (BPF for short) throughout this study. The vibrations associated with the BPF is called variable compliance vibrations or ball passage vibrations. While the shaft is rotating, applied loads are supported by a few balls restricted to a narrow load region defined by the radial position of the inner ring with respect to the outer ring and depending upon the elastic deflections at the ball to raceways contacts and the preload (see Fig.(1.3)). As the position of the balls change with respect to the applied load vector the load distribution on the balls changes, producing a relative radial movement between the inner and outer rings which repeats itself at a certain frequency that is called the ball passage frequency. In mathematical terms the BPF can be described as the cage speed times the number of balls. As a result of this, the system can go into resonance at a fraction of a natural frequency.

This untoward behaviour is inherent in the design of rolling element bearings and cannot be avoided even for a geometrically perfect bearing. More detailed information can be found in Chapter 4.

Fig.(1.3) Deformation of balls when they rotate round the inner ring

1.6.3 Roundness, Roughness and Waviness

A machine part is said to be round in a specific cross section if all points on the periphery are equidistant from the centre. If the cross section is not round, it is said to be out of
round. Out of roundness usually exists in the form of an irregular profile. The imperfections found on any surface generally take the form of peaks and valleys of varying height and width. In the simple case of out of roundness there are a certain number of lobes per circumference. It has been a common practise to describe different wavelength bands in different terms, referring to wavelengths of order of micrometers as roughness and the longer wavelengths as waviness (see Fig. (1.4)). In this thesis, if the wavelength of these lobes is of the order of the Hertzian contact width or less, in the description of the surface feature, the term roughness is employed, whereas for surfaces with longer wavelengths the term waviness is applicable. For longer wavelengths, rolling motion is continuous with the ball rolling surface over contours of the race. More information can be found in Chapter 4.

![Diagram of Roundness, Waviness and Roughness](image)

**Fig. (1.4) Roundness, Waviness and Roughness**

### 1.6.4 Contact Angle

For angular contact ball bearings, the most important parameter is the contact angle which is defined as the angle between a plane perpendicular to the bearing axis and an axis passing through the points of contact of the balls with the raceways (see Fig. (1.5)). The contact angle influences the axial and radial specifications for a bearing. Low contact angles are generally used for light axial loads and/or high speed applications, whereas higher contact angles are chosen for low speed applications and when high axial loads and/or axial rigidity are the main requirements.
1.6.5 Preload

*Preload* is the application of a permanent axial load on a bearing. It is achieved by applying internal loads to a pair of bearings, for instance by means of springs, or by preloading a pair of bearings against each other through a clamping nut. Preload should not be higher than is necessary for the application but should be sufficient to avoid the preload being completely relieved from any bearing by the action of external loads. The preload serves to guarantee a certain minimum load to seat all the balls and ensure firm rolling contact, because otherwise balls can skid and roll and produce a cage instability. High vibration levels can be produced by bearings which are too lightly loaded.

1.6.6 Bearing Internal Clearance

*Bearing internal clearance* is defined as the total distance through which one bearing ring can be moved relative to the other in the radial direction (radial internal clearance - Fig.(1.6.a)) or in the axial direction (axial internal clearance - Fig.(1.6.b)).

It is necessary to distinguish between the internal clearance of a bearing before mounting and the internal clearance in a mounted bearing which has reached its operating temperature (operational clearance). The initial clearance (before mounting) is greater than the operational clearance.
Chapter 1

Introduction

The radial clearance of a bearing is of considerable importance if satisfactory operation is to be obtained.

Depending on the application, it is necessary for there to be a positive or a negative operational clearance in a bearing arrangement. In the majority of applications, the operational clearance should be positive, i.e. when in operation, the bearing should have a residual clearance, however slight. There are some cases where normally a negative clearance, i.e. a preload is desirable in order to enhance the stiffness of the bearing arrangement or to increase running accuracy.

1.6.7 Misalignment

Ball bearing races are said to be misaligned when one bearing ring is not colinear or is angularly displaced relative to the other, about an axis at right angles to the bearing running axis (see Fig.(1.7)). This may be found described as tilted, canted, cocked, out of square or malaligned in other researcher's work but the term misalignment is employed in this thesis. Misalignment in ball bearings is a commonly occurring problem which has a predominant effect on basic running errors and can lead to lubrication problems, cage failure, premature raceway damage and/or vibration at the running speed.
In this chapter a general introduction to the subject was given, the thesis was outlined and some of the keywords used were defined. Taking this general introduction as a basis, in the next chapter, the previous research done on the subject will be investigated.
CHAPTER 2

PREVIOUS WORK

2.1 Introduction

The earliest form of ball bearing was found amongst the remains of ships dating from the time of Caligula (AD 42-54) [Dowson, 1979]. However, the idea of ball bearings for supporting a weight started to appear in written records only as early as 1490. The following text is an extract from two large unpublished manuscripts by Leonardo da Vinci (AD 1452-1520) discovered in the National Library in Madrid in 1967 [The Unknown Leonardo, 1974]:

"I affirm that if a weight with a flat surface moves on a similar surface, the movement will be facilitated by interposing between them balls or rollers ... and I do not see any difference between balls or rollers save the fact that balls have universal motion whereas rollers can move only in one direction. But if the balls or rollers touch each other in their motion, then movement will be more difficult than if there were no contact between them, because when they touch, the friction causes a contrariwise motion and for this reason the movements contradict each other. However, if the balls or rollers are pitched apart, they will touch only at one point between the load and its resistance and consequently it will be easy to produce this movement."
From the text it is obvious that Leonardo understood the importance of the cage or separator and advised its use for reducing friction (see Fig.(2.1)). The first patent to cover a ball bearing, taken out by Philip Vaughan with the British Patent No 2006, goes back to 1794. Although these very early records may suggest that roller bearings were common applications, they were not commonly adopted in machines until about 1900 [Roman and Gohar, 1975].

From Leonardo to the 18th Century, many types and forms of ball bearings are described in contemporary technical literature and presumably some of these designs have been manufactured with varying degrees of success. However, the application of ball bearings was largely empirical and specific designs of ball bearings did not emerge until the eighteenth century. By the 1880s because of the common usage and production of bicycles, ball bearings were subject to intense development. Scientific work on the development of rolling bearings also shows an increase in the latter part of the nineteenth century. Mass production of rolling bearings began in the early part of the twentieth century [Tindale, 1988]. A complete presentation on the early history of ball and rolling bearings can be found in a book by Dowson [1979].

After the Second World War the production of ball bearings as well as other type of bearings gained importance and scientific progress occurred at an exponential rate. With the advent in metallurgical processes and the increasing accuracy of machine tools, ball bearings have evolved to the form we know today.

After 1950s the vibration of shafts supported by ball bearings also came to the attention of researchers. An considerable increase in the theoretical research on the vibration of ball bearings was made possible by the advancement in computer technology and numerous research papers on the topic are available now. Although many books are
available on *Rolling Bearings*, only one book, entitled "Vibration of Bearings" [Ragulskis et al., 1989], investigates solely the vibration problems associated with ball bearings. But the importance of the topic is now realised and the vibration problems of rolling bearings are addressed in recent editions of standard texts [Harris, 1991].

As mentioned earlier much research has already been done on the vibration of ball bearings, such as; vibration monitoring, defects in bearings, natural frequencies of bearings, external damping of vibration due to ball bearings, etc. The same researchers have also worked on different aspects of ball bearing vibrations. Thus it will be more convenient to review the previous research works under subtitles which will now be reviewed individually.

### 2.2 Early Work on Ball Bearing Vibration

The first scientific papers on the subject were mainly concerned with producing perfect bearings and therefore most of them described experiments with more attention paid to stress levels, mechanical performance, wear etc. than to vibration problems, despite the fact that many research study were describing a rotating shaft supported by ball bearings (see, for example, Stribeck [1900], Hess [1908], SKF [1957]). A good example of the first extensive scientific papers on ball bearings is a report from the Central Laboratory for Scientific Technical Investigation written by Stribeck in 1900 under the title of "Ball Bearings for Various Loads". In this report there is reference to earlier work from 1897 and 1898. A summary of the development of roller bearing technology from 1907 to 1957 is given in a complimentary and extended article by SKF [1957] for the 50th year of establishment of SKF. However, in this latter article, there is no reference to the problems caused by the vibrations due to rolling bearings.

After solving major problems of production, researchers turned their attention to other factors. As a result of this, the noise problem caused by ball bearings became important. In any environment noise, even at a low level, from for example office apparatus, vacuum cleaners, fans, shafts rotating in cars or ship etc., is irritating. Hence the vibration characteristics of ball bearings attracted interest. Since the calculations involving vibrating ball bearings are very complicated, experiments were carried out, but no analytical approach was attempted for a long time. In the 1950s researchers tried to overcome the noise and wear problems with lubrication. This was successful to some extent [Hall, 1957]. Although some research workers approached the theoretical investigation of ball bearing dynamics pessimistically, even as recently as the 1970s [SKF, 1961; Wallin, 1966; Yhland and Johansson, 1970], some brilliant studies on the topic can be found as early as the 1950s [Perret, 1950a; 1950b; Meldau, 1951; 1952]. Perret and Meldau theoretically proved that even a perfect bearing may undergo vibrations due to ball
passage. But since they did not consider the shaft mass, their solutions were incomplete.

Meanwhile some researchers started comparing antifriction bearings, particularly ball bearings, with plain bearings for grinding spindles and concluded that although the plain bearings are superior to ball bearings for lower speeds, the latter would give better performance at higher speeds [Doi, 1958]. But this choice was not free of problems. A shaft would vibrate less when mounted on good quality properly designed plain bearings than on ball bearings [SKF, 1961]. Ball bearings certainly aggravated vibrations and make the design of spindles more difficult because the dynamic characteristics of the system were not well understood. Since there was not enough information on the dynamic behaviour of grinding spindles supported by ball bearings, the manufacturers tried to overcome the vibration problem by designing ball bearings with sufficient rigidity, assuming that in this case even a small damping factor would keep the amplitude of vibration well within permissible limits for most grinding conditions [Weimar, 1958]. They were not completely successful, since the problem was not as simple as they thought and vibration could not be avoided.

The importance of rolling bearings was recognised, but no attempt was made to include them in dynamic modelling of such problems as machine tool chatter, which were established by Tobias and Fishwick [1956]. The results were later experimentally verified by Tobias [1959] for drilling and face milling operations. These results were improved by Tobias et al. [1962] and published as book titled of "Machine Tool Vibrations" [Tobias, 1965]. However, Tobias et al. concentrated their research on the dynamic stability of the machines subjected to external forces due to friction between tool and work-piece, cutting force as a function of cutting velocity, rake angle variation, chip thickness variation without giving the full credit to the effect of bearing characteristics on the performance of machining spindles. Meanwhile, in the early 1960s, the vibration characteristics of ball bearings were analytically studied under various simplifying assumptions by different researchers [Igarashi, 1959; Tamura and Taniguchi, 1960]. Even with simplifications it was difficult to analyse theoretically the vibratory behaviour caused by ball bearings because of the complexity of the problem.

One of the most influential and extensive studies on vibration characteristics of rolling bearings was done in 1963 by Gustafsson et al. The purpose of this study was to investigate the vibration and noise producing characteristics of large roller bearings for propulsion machinery of ships, as well as smaller rolling bearings in order to reduce vibration and noise at all audio and subsonic frequencies. Gustafsson et al. investigated vibration of geometrically perfect bearings as well as those with some imperfections.
They showed that a shaft-ball bearing system may undergo vibrations even when they are geometrically perfect. They pointed out two causes for this; the first is the ball passage frequency (BPF) effect which was first realised and theoretically proven by Perret and Meldau in 1950, and the second is bending of the outer ring due to variable ball loads. Geometrical imperfections investigated in this report includes consideration of waviness and off-sized balls.

In the late 1960s some analytical studies of ball bearings appeared, but since there was no help from powerful computers which were then inaccessible, even relatively simple rolling bearing arrangements were difficult to analyse mathematically [Yhland and Johansson, 1970]. Some proposals made in these studies were either too simple to be generally applicable or too complicated to apply to the practical estimation of the spring and damping properties of ball bearings. As an excellent example of this era, the studies of Tamura and Shimizu can be given. In their first paper [1966] they investigated a bearing with two balls, in the second paper [Tamura and Shimizu, 1967] they increased the number of balls to three or four and in the third paper [Tamura and Shimizu, 1968] they extended the study to a large number of balls. Although the authors employed many equations and calculations, the vibration information was relatively limited.

From the late 1960s onwards, with the common use of powerful computers, much research was carried out on the vibration characteristics of ball bearings.

2.3 A Brief Look at Research post 1960

The research on ball bearing vibrations covers a large spectrum. Thus different researchers concentrated on various aspects of vibrations attributed to ball bearings. In particular, there are some groups that have produced major contributions. Some of the major research after 1960 is as follows:

2.3.1 Gustafsson and Tallian et al. [1963; 1965] carried out an investigation of the vibration and noise producing characteristics of ball bearings. They concentrated on some very important vibration characteristics of ball bearings such as: the BPF and vibration of the outer ring due to ball loads. The effects of waviness and off-sized balls on the vibration of ball bearings were also investigated.

2.3.2 Tamura and Shimizu [1967; 1968; 1969] studied the static stiffness of ball bearings. This research is important because it contributed towards investigations of the instabilities of shaft-bearing systems.

2.3.3 Gupta et al. [1977; 1979a; 1979b; 1979c; 1979d] modelled the rolling bearing in
cylindrical coordinates in six degrees of freedom. Since they did not consider the mass of the shaft, but the rolling elements' mass only they concentrated their investigation on the vibrations due to the balls and ball-cage interactions.

2.3.4 Yamamoto et al. [1974; 1975; 1977; 1981a; 1981b; 1984; 1985] worked on instabilities caused by ball bearings supporting an unsymmetric shaft. They also studied sub and super harmonic vibrations due to ball bearings.

2.3.5 Gad et al. [1983; 1984; 1985a; 1985b] employed a computer simulation model of ball bearing systems, including the shaft mass, in order to investigate the dynamic properties of ball bearings and pointed out instabilities around the natural frequencies.

2.3.6 Igarashi et al. [1982; 1983; 1985] investigated the vibration due to point defects in ball bearings.

2.3.7 Wardle et al. [1983; 1988a; 1988b] investigated the effect of distributed defects in a ball bearing. Particular attention was paid to waviness.


2.4 Experimental Work

Although research on ball bearing vibrations has a strong background, with many research papers available, there is no commonly accepted and justifiable standard experimental method for vibration tests. Each manufacturer or researcher has established his own rig and utilised his own method to extract information from ball bearing vibrations. Every researcher has his different reasons for the experiments and the rigs were designed to fit them. Therefore an acceptable method and test for one researcher may be useless for another. This increased the number of methods available. However, there is very little justification for the methods employed [Hemmings and Smith, 1976]. The standard test methods give little information that can be used in the manufacture of ball bearings or in their usage by the customer, but the results available are a useful general indicator that manufacturing quality is being maintained [Hemmings and Smith, 1976] or it can be used to check the re-usability of the dismounted ball bearings [Wallin, 1966].

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Ever since ball bearings were first manufactured, checks for vibrations in them have been carried out by spinning the bearings by hand. Since in normal bearing schemes the inner ring usually rotates, the checking procedure has been improved by mounting the bearing on a mandrel. The outer ring is then held and the inner ring rotated by means of the mandrel. Spinning bearings by hand is quite acceptable for a quick assessment of serviceability and this method is, to some extent, still in use today. However, the absolute vibration level of the bearing cannot be determined by this method [SKF, 1961]. Testing the bearing alone will only give information about manufacturing quality or re-usability of the bearing; when it is assembled the bearing will have all together different vibration characteristics. Therefore, tests should be performed on bearings that are functioning in the assembly [Watford and Stone, 1980]. When the bearings are in assembly, for experimental investigations, another difficulty arises; the interaction between predominant factors such as production accuracy, misaligned races, eccentric races, elliptic rings, unequal ball sizes, radial clearance and geometrical errors of the running surfaces, which are all of the same order of microns [Watford and Stone, 1980], cause complicated vibrations. Therefore, it is difficult experimentally to separate, measure and grasp the predominant factors clearly, and control the correspondence between the effect of each factor and the resulting vibration [Gad et al., 1984]. Bearing characteristics, therefore, have to be deduced indirectly from the behaviour of the system as a whole. Furthermore, the interfaces between the shaft and the inner race, and the housing and the outer race, can have a significant effect and must, therefore, be controlled carefully, and the bearing should be tested under its proper operating conditions of rotation and preload [Watford and Stone, 1980; Gad et al., 1984].

![Electrodynamic pickup](image)

*Fig. (2.2) Schematic presentation of test rig [Harris, 1991]*

When the bearing arrangement is running under normal operating conditions it vibrates and produces noise. Noise is a good indicator of vibrations. The major noise generation is due to vibration from the bearing ring travelling through the structure and resonating panels or sections of casing to produce noise. Hemmings and Smith [1976] argue that
noise produced by ball bearings is not useful in predicting bearing vibrations. This is because in the majority of installations the direct noise from a bearing is of low power and is generated inside a sealed item of machinery. Under these conditions negligible noise energy escapes through the casing or through seals and this source of noise may be ignored. Igarashi and Yabe [1983] argues that when sophisticated instruments are employed and experiments are carried out delicately, clear information can be obtained from the sound generated by ball bearings. However, the best way to get information from the shaft bearing assembly is by direct measurement of shaft vibration although this is not easily made either, when the bearings are assembled. As vibration forces are transmitted through bearings, measurements are best made on bearing housings as near to the shaft as possible [Dowson, 1970].

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**Fig.(2.3) Vibration Test Schematic [Harris, 1991]**

In spite of all these difficulties some rigs and experimental measurements are described in the references available. However, they are generally designed for a bearing in isolation.
and cannot represent the bearing in normal operating conditions. Some experimental results from the bearing in the assembly are given in the Section 2.7, health monitoring of bearings, of this thesis. The measurement method is the same in principle for all the tests available and as follows: the bearing to be measured is mounted on the mandrel against a shoulder. The shaft speed is thus imparted to the inner ring while the outer ring is held stationary. The radial and axial movements are registered by velocity pick-ups. This kind of device was used throughout the SKF organisation [SKF, 1961] and established a vibration standard in the early 1960s (see Fig. (2.2)). Very similar set-ups were also employed by Gustafsson et al. [1963], Cena and Hobbs [1972], Hemmings and Smith [1976], Braun and Danter [1979], Igarashi and Hamada [1982] and Karakurt [1989].

The schematic representation of a commonly used vibration tester is shown in Fig. (2.3). The inner ring raceway of the bearing mounts on a precision arbour fastened to the spindle, which rotates at 1800 rpm. A specified thrust load is applied to the side face of the non-rotating outer ring. The top of the velocity transducer is lightly spring-loaded on the outer diameter of the ring. The ring contacts the side face of the outer ring. The tool and load combinations are sufficiently compliant to allow radial motion of the outer ring to occur as balls roll over wavy surfaces or defects in races. The voltage signal from the transducer is input to the analysing instrument, which amplifies the signal, bandpass filters it, and displays the rms velocity values in each band. The three frequency bands are 50-300 Hz., 300-1800 Hz. and 1800-10000 Hz. Larger bearings are tested at slower rotational speed (700 rpm) with corresponding lower filter bands: 20-120 Hz., 120-700 Hz. and 700-4000 Hz. [Harris, 1991]. This basic testing technique has been successfully used by bearing manufacturers and customers for many years.

Watford and Stone [1980] developed a rig suitable for measuring bearing characteristics under controlled conditions of rotation, preload, and shaft and housing fit. In this rig they measured the relative deflection of the shaft centre with respect to housing, since the housing was no longer attached to a rigid abutment. They claim this gives better results for ball bearing vibration tests.

Fig. (2.4) The recurrence frequency of the pulses from noise and vibration [Igarashi and Yabe, 1983]
Igarashi and Yabe [1983] carried out their experiments in a soundproof chamber. They measured noise produced by ball bearings and employed the same sort of set used by Igarashi and Hamada [1982] in order to compare the results. Their experimental rig shows that the noise from a vibrating machine can be a very good source of information if handled correctly and vibrations caused by bearings can be isolated from the other vibrations. They showed that the recurrence frequency of the pulses was the same for both sound and vibration produced by ball bearings (see Fig.(2.4)).

Aini [1990] designed an experimental rig to investigate the vibration characteristics of a grinding spindle. A shaft was supported by angular contact ball bearings and a force could be exerted on the shaft-bearing assembly to simulate the generated forces in the grinding processes. Since this was a simulation of a spindle under working conditions, the vibration spectrum should be very similar to those obtained from a grinding machine.

![Image](image.png)

*Fig.(2.5) The influence of the sampling frequency on the discrete representation of a given analogue waveform [Xistris et al., 1980]*

Setting up the rig and obtaining the data is only one part of the problem in experimental research. The other is the acquisition of data and its analysis. Xistris et al. [1980] showed that if the data acquisition is not done correctly the results obtained could be misleading. Fig.(2.5) shows the influence of the sampling frequency on the discrete representation of a given analogue waveform. The digitized signal depicted in Fig.(2.5) were obtained by sampling a 10 V peak 1000 Hz sine wave at different frequencies. This shows what a wrongly selected sampling frequency may cause, for example. Filtering and windowing the data may also produce misleading results [Xistris et al., 1980; Mitchell, 1985].

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Even the data acquisition and transferring it into the frequency domain are performed correctly, the spectrum obtained may be very complicated to acquire any information from, since the effects of geometrical imperfections and natural frequencies are all present in the spectrum. Owing to the large number of factors, the experimental spectra (the FFT diagrams) have a large number of peaks, some of which correspond to different modulation effects. The successful isolation and attribution of certain effects to specific causes is difficult to achieve [Aini, 1990]. Fig.(2.6) shows an experimental vibration spectrum. Since waviness, misalignment, off-sized ball effect etc. produces peaks at side band frequencies \((\alpha_b \pm \alpha_c)\) and sub and super harmonics to the fundamental frequencies present in the spectrum [Wardle, 1988a; 1988b; Franco et al., 1992], almost any prediction can be fit to this experimental data with a little justification. Filtering the signal into specific frequency bands was suggested in order to avoid this problem [Braun and Danter, 1979; Li and Inasaki, 1988 and Karakurt, 1989]. This requires a good knowledge of the vibration of ball bearings since some important frequencies may be left out otherwise.
2.5 Elastic Contact Behaviour of Ball Bearings

The classical solution for the local stress and deformation of two elastic bodies apparently contacting at a single point was established by Hertz [1896] and today elastic contact deformations are called Hertzian in recognition of his achievement [Harris, 1991]. Hertz [1896] showed that the load-deflection relationship of the contacts between the balls and raceways can be represented in the form:

\[ F = K \delta^3 \]  

(2.1)

The characteristic spring action of the shaft and ball bearing oscillation system mainly comes from the Hertzian contact of balls and raceways, and it is characterised by a variable stiffness and hence nonlinear vibrations [Shimizu and Tamura, 1966].

Gupta [1975] modelled ball motion around the inner ring in six degrees of freedom. He assumed that the balls possess mass and the normal contact load comes from the Hertzian contact deflection. The interactions between the cage and the balls also modelled in terms of hydrodynamic and metallic contact interactions [Gupta, 1977; 1979a; 1979b; 1979c; 1979d].

Gad et al. [1983] followed a similar approach to Gupta in modelling the elastic reaction forces due to contact between balls and raceways; they, however, did not consider the mass of the ball and hydrodynamic interactions. The ball bearing modelled was ideal which means the inner and outer races were assumed to be perfectly circular, all balls were perfectly spherical and of equal diameter. Since the friction forces between balls and raceways was not considered the balls were investigated in five degrees of freedom.

However, most of the researchers did not follow this approach. Tamura and Shimizu [1966; 1967; 1968], Rahnejat and Gohar [1985], Aini and Gohar [1990] did not consider the ball in a six degrees of freedom system, rather they calculated the total deflection in the direction of ball to inner raceway and ball to outer raceway contacts. Having calculated the total force on one ball, this force was then resolved to its three components in x, y and z directions. Hence the total force and moment on a bearing were calculated by summing the forces and moments in three dimensions, resulting in five degrees of freedom system in general.

While the first approach may be useful in the calculation of “skidding” and “skewing” instabilities where skidding normally represents large sliding between the ball and the race, while skewing denotes rotation of the ball about its transverse axes, and elastrohydrodynamic contact lubrication effects, the latter will give a good estimation of...
lower band frequency of the vibration spectrum. The reason for this is in the first approach since the ball moves in five degrees of freedom, it may produce instabilities for certain loads and clearances [Gupta, 1979a; 1979c; Gad et al., 1983].

Although it was known that the load-deflection characteristic for ball to race contacts was nonlinear, it was assumed for a long time that a linear line can represent the actual curve in dynamical problems to avoid the difficulties caused by the non-linearity. Today, with the help of powerful computers, in dynamic as well as static problems, the relationship is modelled as a non-linear Hertzian contact. However, in contemporary studies of ball bearings, three different approaches for modelling the nonlinear ball to race contact relationship can be found. In the first, the nonlinear relationship is assumed to be linear over a narrow range of deflection, resulting in a piece-wise characteristic curve [Timoshenko et al., 1974; Matsubara, 1983]. Matsubara et al. [1985] employed this model to investigate the vibrations of an elastic shaft supported by a pair of radial ball bearings as seen in Fig.(2.7). The shaft/bearing model was formulated as an eigenvalue problem and the first (fundamental) and higher natural frequencies of the system were obtained under the application of various cyclic forces and shaft speeds. The model was convenient since the system was continuous, they needed to use a superposition technique which holds only for linear systems.

The second method uses the Taylor series in order to obtain an approximate curve fit to the Hertzian contact. This was first employed by Wardle [1988a; 1988b]. Wardle [1988a] assumed that the shaft and angular contact ball bearings oscillate about an equilibrium position. If the equilibrium position after preloading is given by the deflection $\delta_0$ at each contact point, the Hertzian contact force can be linearised as:

$$F = K \frac{3}{2} \left( \delta + \frac{3}{2} \frac{\delta}{\delta_0} \right)$$

The study [Wardle, 1988a] points out that for this linearisation, the ratio of $(\delta / \delta_0)$ has to be small in order to obtain accurate solutions. But if the same expansion is employed taking another term into account, the numerical error between the real solution and the approximation will be less than 2% in the case when the ratio of $(\delta / \delta_0)$ is less than 1.

Taking this study as a basis Franco [1991] showed that if the higher terms of the Taylor's expansion are taken into account, the Hertzian contact can be represented very closely. After the first six terms the effect of additional terms will not change the result very much whereas the process will increase the CPU time. Therefore, there is no need for use of more than six terms for practical purposes. In a paper by Franco et al [1992] this method was employed to investigate the the effect of multiple defects of ball bearings on system vibration characteristics.
Gustafsson and Tallian et al. [1963] employed a similar technique to the Taylor's series in order to linearise the Hertzian contact deflection curve as:

\[ F = K_L (\delta - \overline{\delta}) + \overline{F} \]  

(2.3)

where

\( \overline{\delta} \) is average deformation occurring under a load \( \overline{F} \) and,

\( K_L \) is linearised coefficient of contact deformation, defined as:

\[ K_L = \left( \frac{\partial F}{\partial \delta} \right)_{\delta=\overline{\delta}} = \frac{3}{2} K \overline{\delta}^{\frac{1}{2}} \]  

(2.4)

Equations (2.2-2.3) were assumed to be valid in the neighbourhood of the point \( \delta = \overline{\delta} \). Equ.(2.3) gives a satisfactory approximation of Eq.(2.1) as long as \( |\delta - \overline{\delta}| \) is small compared to the average deformation \( \overline{\delta} \) [Gustafsson and Tallian et al., 1963]. The second term on the RHS of Eq.(2.3) was introduced to compensate the minus values of \( (\delta - \overline{\delta}) \).

The third method employs the Hertzian contact relation. This is the most commonly used approach and will also be employed in this thesis (see Chapter 3).
2.6 Important Frequencies Associated with Ball Bearings

In the literature available there are certain identified frequencies of a shaft supported by ball bearings that should be avoided. Some of these frequencies have been thoroughly investigated while others remain largely unresearched.

Frequencies of shaft-ball bearing systems may be divided into two main categories: the frequencies of a geometrically perfect (i.e., defect free) system and the frequencies due to defects. In this section only the frequencies of the defect free system, or in other words the inherent frequencies of the system, will be investigated while the latter will be investigated in the next section.

2.6.1 Natural Frequency

The natural frequency is one of the most important frequencies of the system. However, it has been investigated by only few researchers for a shaft-ball bearing system. This is perhaps because it was taken for granted. Shimizu and Tamura [1966; 1967; 1968] made an attempt to calculate the static stiffness of ball bearings and showed that the difference in the overall stiffness comes from different positions of the ball set in the bearing with the effect being cyclic. Later, experimental research by Yamamoto and Ishida [1974] showed that the position of the ball set causes a change in the natural frequency of the system. Fig.(2.8) shows the experimental result of the change in the natural frequency when the position of the ball set is changed. Since the mass centre of the shaft they studied did not coincide with the geometrical centre of the shaft, a non-uniformity of the stiffness was present. Therefore the researchers obtained the natural frequency of the system in the direction of the maximum, \( p_{01} \) (the natural frequency in the \( x \) direction) and minimum stiffness, \( p_{02} \), (the natural frequency in the \( y \) direction) respectively by hitting the shaft in \( x \) or \( y \) direction and recording the vibrations. When the natural frequencies were plotted against different shaft positions as shown in Fig.(2.8), a cyclic change in the natural frequencies was observed. However, these results seem to be caused by wrongly selected experimental observation intervals (see Chapter 6 for a further discussion on this). From these results Yamamoto and Ishida [1974] concluded that when the shaft is rotated, the natural frequency will fluctuate periodically with shaft speed and its superharmonics and hence the system will be excited at the shaft rotating speed and its superharmonics.

Gupta [1974] has reported that ball bearings exhibit a hardening type spring and the change in natural frequency is therefore the result of change in the stiffness. Gad et al. [1984; 1985] observed jump behaviour around the natural frequencies and reported that this is due to the hardening type spring characteristic of ball bearings.
Yamamoto and Ishida [1974] and Yamamoto et al. [1977] reported that outer ring misalignment causes change in the spring characteristic of ball bearings and hence affect the natural frequency of the system. These effects resulted in jump behaviour over a large frequency range in the experimental researches by Yamamoto and Ishida [1974] and Yamamoto et al. [1977].

However these results of changes in the natural frequency were obtained for a non-rotating shaft. Later Watford and Stone [1980] and Yamamoto et al. [1981] did a series of experiments in order to obtain the stiffnesses and natural frequencies of ball bearings under oscillating conditions. Watford and Stone [1980] concludes that the stiffness and the damping of the system varies with the rotational speed as seen in Fig.(2.9). Stiffness
increases linearly as predicted, implying a hardening type stiffness. Yamamoto et al. [1981] confirms this and shows that the natural frequency increases parabolically as seen in Fig.(2.8), resulting in a system with a hardening type spring characteristic. Watford and Stone [1980] reported a big difference between the measured stiffnesses and the Hertzian contact spring. This was considered as a result of the interfaces between the races and housing, respectively. Although it is not very clear how they calculated the equivalent stiffness for the system, this big difference \((7 \times 10^8 \text{ N/m calculated}, 1 \times 2 \times 10^8 \text{ N/m from the practical tests})\) may come from modelling of the bearings as linear springs.

These experimental studies will be discussed in Chapter 6 in more detail and the comparison with the results of computer programme will be given.

Aktürk et al. [1992] showed that increasing preload would force the system to oscillate with a higher ball stiffness bearing and therefore the natural frequency of the system would be increased (see Fig.(2.10). This effect will also be discussed further in Chapter 6.

![Diagram showing some factors changing with preload](image)

\(\text{Fig.(2.10) Some Factors Changing with Preload [Aktürk et al., 1992]}\)

### 2.6.2 Ball Passage Frequency

The ball passage frequency (BPF) was first recognised by Perret-Meldau [1950] as a static running accuracy problem. Perret and Meldau suggested that an increase in the number of the balls in a bearing, reduces its untoward effect. However, its nature and importance was not properly understood until research by Gustafsson et al. [1963], who called this vibration "variable compliance". Due to the difference in the overall stiffness,
for the different positions of the ball set, a force imposed on the inner ring and it excites the system at the BPF as will be explained in more detail later in Chapter 4. This is an inherent frequency of the system, i.e., it is present even when the bearing is geometrically defect free. The experimental studies of Gustafsson et al. [1963] showed that clearance is an important parameter for the BPF. The greater the clearance, the more effective the ball passage vibrations become. This was later confirmed by the experimental research of Wardle and Poon [1983]. Wardle and Poon argue that the ball passage vibration levels depend on the number of balls and radial clearance. For radially loaded or misaligned bearings, running clearance determines the extent of ball passage vibration level [Wardle and Poon, 1983]. In general the ball passage vibration increases with the internal clearance as seen in Fig.(2.11).

Gustafsson et al. [1963] also observed that higher harmonics, which are multiples of the BPF, are also present in the vibration spectrum (see Fig.(2.12)). The results shown in Fig.(2.12) are approximate, higher harmonics, especially was used only for comparison of orders of magnitude. The radial loads were varied in the range from 100 N to 10000 N and the amplitudes of the ball passage vibrations in the figure represent the maximum amplitudes of each harmonic which occur under different radial loads. Although the physical explanation was not given, the researchers [Gustafsson et al., 1963] pointed out that the amplitudes at the harmonics of the BPF depend on the radial load, radial clearance, the rotational speed and the order of the harmonic. The same conclusion was later reached theoretically by Meyer et al. [1980] for a perfect ball bearing, with linear modelling of the spring characteristics of balls. Meyer et al. [1980] showed that for a perfect system, ball passage vibration and its superharmonics will appear in the spectrum (see Fig.(2.13) and Appendix 7).

Fig.(2.11) The vibration due to the BPF vs. internal clearance
[Wardle and Poon, 1983]
A resonance was reported by different researchers when the BPF coincides with the
natural frequency [Gustafsson et al., 1963; Meyer et al., 1980; Wardle and Poon, 1983;
Rahnejat, 1984; Gad et al., 1984; 1985; Rahnejat and Gohar, 1985; Wardle, 1988;
Aktürk, 1988; Aini, 1990; Aini et al., 1990; Aktürk et al., 1992]. This means the system
may resonate at a fraction of the natural frequency since the BPF is defined as the cage
speed which is about the half of the shaft speed, times the number of the balls in a
bearing. Rahnejat and Gohar [1985] showed that even in the presence of an
elastohydrodynamic lubricating film between the balls and the inner and outer races, a
peak at the BPF appears in the spectrum. Fig.(2.14) shows the steady-state frequency
spectrum of the oscillations in the vertical direction with a wavy inner race. The subharmonics of the BPF also appear in the frequency spectrum. However Rahnejat [1984] could not observe the BPF effect for a perfect ball bearing arrangement, probably because of the preload or heavy mass present or insensitivity of his computer program.

![Graph showing frequency spectrum for bearing without defects](Meyer et al., 1980)

![Graph showing steady-state frequency spectrum of oscillations with wavy inner race](Rahnejat and Gohar, 1985)

The study by Tamura and Gad et al. [1983] is a kind of exhibition of the Meldau-Perret problem. They showed that due to the different positions of the balls in a bearing there will be an oscillating force in the system at the frequency of the ball passage. This study was extended and republished by Tamura and Gad et al. [1985] with more attention paid to the BPF. Fig.(2.15) shows the presence of the BPF for different values of dimensionless characteristic parameter $\Gamma$, which is a combination of radial clearance, number of balls, the contact deflection proportionality and the radial load. Although the
shape of the inner ring motion changed, the oscillations show the effect of the BPF. This is further discussed in a paper by Tamura et al. [1986].

Gad et al. [1984b] investigated the BPF and its sub and super harmonics for perfect ball bearings. Fig.(2.16) shows a selection from this paper showing the effect of the BPF and its sub and super harmonics in the vertical and horizontal directions. In the figures, the
system exhibits the 3rd order superharmonics of the BPF. The locus of the shaft centre for these oscillations was also presented. Gad et al. [1984a] also point out beat like and chaotic vibrations of the shaft-ball bearing systems. As discussed earlier, it should be taken with precaution and further investigated since these may be due to the modelling of forces and moments for a single ball.

Fig. (2.16) Steady-state vibrations and paths of inner ring centre due to the BPF effect [Gad et al. 1984b]

Gad et al. [1984b] later showed that resonance occurs when the BPF coincides with the frequencies of system anomalies such as inner ring eccentricity and the misalignment of inner ring relative to the shaft axis. In this study, researchers pointed out that for certain speeds the BPF can exhibit its sub and super harmonic vibrations for a given shaft-ball bearing system.

Aini [1990] argues that the ball passage vibrations appear as a result of contact spring nonlinearity and become effective when there are system irregularities or when a forcing frequency excites the system. But the analytical linear model of Meyer et al. [1980] suggests the ball passage vibrations occur even for perfect bearings. The reason for this is when a radial clearance is present the balls of the bottom half of bearing will always be carrying the load and therefore the BPF effect will be present in the vibration spectrum. However, for the nonlinear model, even in the case of interference, the BPF will cause resonances due to the nonlinearity of the bearings. Aini [1991] argues that for perfect bearings the BPF does not necessarily appear in the spectrum since the levels of
vibration due to the BPF depend generally on the number of balls supporting the applied load, the greater the number of balls, the less the vibration amplitude. However, Aktürk et al. [1992] points out that the effect of the BPF is always present in the spectrum, however small it is. Aktürk et al. [1992] investigated the effect of number of balls and the preload on the vibrations due to ball passage. Fig.(2.18) shows the effect of changing the number of balls. When the number of balls is increased the BPF will coincide with the natural frequency at lower frequencies as seen in Fig.(2.18).

In spite of its importance, the effect of the BPF is generally not clear. Some researchers observed the BPF in their experimental researches but could not identify it as such. Three examples are given below:

Glöckner [BBJ 225] shows the effect of the BPF, even identifies its speed but did not recognise that it is one of the inherent frequencies of the system. Fig.(2.19) shows the the force acting on the shaft and the measured signal as a function of spindle load. The $f_n$ in Fig.(2.19) coincides to the BPF of the ball bearings but the author calls it "the over-rolling frequency" and $f_1$ coincides with the cage speed. As the figure suggest, although it is not mentioned in the paper, that the bearings had 10 rolling elements.
Fig. (2.19) Measuring signal as function of spindle load [Glöckner, BBJ 225]

Fig. (2.20) shows the vibrations resulting from a once per cage revolution averaged during a bearing test. The bearing had 8 balls. When the figure is investigated, 8 oscillations per cage rotation are observed, which means vibrations at $8 \omega_c$. This is the BPF. However, Hemmings et al. claim that this is due to surface waviness. This seems more unlikely (see the discussions on the BPF and waviness in this chapter). Perhaps the authors were unaware that even geometrically perfect bearings can undergo vibrations.

Karakurt [1989] identified the BPF which is marked with "ORDF - Outer Race Defect Frequency" in Fig. (2.21a) very clearly with a defect free bearing in his experimental investigation but attributed it to an outer race defect (see Fig. (2.21)). He admits that this is not a defect frequency as the bearings can be assumed defect free (defects present in the bearings are negligible when they are compared with the large local defects introduced by the researcher). It is known that the BPF and the outer race defect frequency have the same frequency. Hence the author wrongly identified the BPF as the outer race defect frequency.
2.6.3 Flexural Vibrations of the Outer Ring due to Ball Loads

This vibration was first studied by Gustafsson and Tallian et al. [1963]. They reported that flexural vibrations of the outer ring were induced by ball loads in an axial as well as a radially loaded ball bearing. It has the following characteristics [Gustafsson & Tallian et al., 1963]:

1. The fundamental frequency of vibrations is the frequency of balls passing a given point on the stationary outer ring of the bearing, i.e., the BPF, $f_{bp}$.

2. Higher harmonics, which are multiples of the fundamental frequency, are also present.

3. In a thin ring (such as the outer ring of most bearings) of mean radius $R$, and second moment of area of the ring cross-section $I$, loaded by an axial load $F_a$, the r.m.s. velocity amplitude of the $i$th harmonic of the flexural vibrations is given by [Gustafsson and Tallian et al., 1963]
As is suggested by Eq.(2.5) the velocity amplitude is directly proportional to the applied load and shaft speed and decreases inversely with the cube of the number of balls, \( m \) and with the cube of the order \( i \) of the harmonic. The fundamental frequency is therefore predominant. The velocity amplitude of the second harmonic is approximately 12% and the third approximately 3.5% of that of the fundamental [Gustafsson and Tallian et al., 1963]. Higher harmonics have successively lower amplitudes. Although the research suggests that this vibration is relatively small for normal applied loads, for loads exceeding these values, the effect of the flexural vibration due to ball loads may become an important contributor to the vibration level below 800 Hz.

### 2.6.4 Natural Frequencies of the Outer Ring

The natural frequency of rigid body motion of the outer ring was studied by Gustafsson and Tallian et al. [1963] for a free outer ring, supported only by the balls and also for the outer ring supported by the balls and mounted in a housing which is elastic both radially and angularly. In this study they assumed that the outer ring deforms elastically only where it contacts with the balls. Gustafsson and Tallian et al. [1963] calculated three natural frequencies of rigid body motion for radial, axial and angular mode of vibrations respectively. They also studied the natural frequencies of the elastic vibrations of the outer ring. In this case the outer ring deforms elastically, e.g., by bending. This natural frequency was computed under the assumption that the inner ring was rigid and joined to the outer ring by a set of equally spaced, identical, elastic balls. Gustafsson and Tallian et al. [1963] showed that a ring with multiple elastic supports (such as a bearing outer ring supported by balls) has a sequence of natural frequencies of flexural vibrations which are not integral multiples of a fundamental frequency and are dependent on ring mass and the elastic properties of the ring and supports.

Gustafsson and Tallian et al. [1963] suggested to use a bearing with a thicker outer ring to reduce the vibration level due to the rigid body motion of outer ring.

### 2.6.5 Natural Frequencies of Ball Motion

Gupta et al. [1977] modelled the motion of the ball in a cylindrical coordinate system. The simulation model showed two natural frequencies, one being a lower frequency and the other higher. Gupta et al. claimed that these were two unknown natural frequencies of the ball bearing systems and called them elastic contact frequency and bearing kinematic frequency (see Fig.(2.22)).
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Fig. (2.22) Variations of computed elastic contact frequency and bearing kinematic frequency as a function of ball-race contact loads [Gupta et al., 1977].

The elastic contact frequency is obtained from the linearised Hertzian contact deflection relation as follows [Gupta et al., 1977]:

$$\omega_e = \left( \frac{3Q}{2M_b \delta} \right)^{1/2}$$  \hspace{1cm} (2.6)

where $M_b$ is the ball mass and $Q$ is the load. The bearing kinematic frequency is obtained from the classical expression of the type $\omega = (g / l)^{1/2}$ as [Gupta et al., 1977]:

$$\omega_k = \left( \frac{Q}{M_b l} \right)^{1/2}$$  \hspace{1cm} (2.7)

where $l$ is the length of the oscillating pendulum and effective $l$ is very difficult to determine. Gupta et al. [1977] gave restricted experimental evidences for their theory. However, specially the bearing kinematic frequency seems to be arguable since any frequency can be matched playing with $l$. In the same paper's discussion [Gupta et al., 1977] Frarey argues that the predictions does not give reliable results since the experimental investigations showed that the predicted signals were not shifted for different load conditions although a shift was suggested by Eq.(2.7). Fig.(2.23a) is
submitted by Gupta et al. as the proof of the mentioned frequencies ($\omega_x=53.23$ KHz and $\omega_y=8.04$ KHz) and Fig.(2.23b) for the same bearings but for different load conditions shows no shift in the predicted vibrations hence they are fixed resonances and not the elastic contact frequency and the bearing kinematic frequency signals as suggested.

Fig.(2.23) Radial acceleration spectrum (Frarey's discussion[Gupta et al., 1977])

2.7 Vibration Monitoring and Frequencies due to the Imperfections of Ball Bearings

As bearing applications become increasingly complicated because of the requirements for continuous operation, automation, etc., failures in rotating machinery are expensive. The breakdown of this kind of machinery always causes additional expense, not only
because repairs are costly, but also because of the loss of operating time and production. Since bearings are a particularly critical component of machinery and their serviceability is usually essential for the operation of any machine into which they are fitted, it is important that bearing damage should be detected early enough to replace the bearing during a scheduled servicing period, thus reducing unscheduled servicing of the machine. This is particularly true where safety is a factor, as in aircraft engines, and certain process industries [Yhland and Johansson, 1970]. To illustrate the cost which can be involved it has been reported that a simple bearing failure in a fully integrated steel mill can lead to a total shut-down which at full output rate may cost £150-300 per minute [Braithwaite, 1969]. A similar bearing failure in a generator set could involve a loss of £1-20 per minute until the set was again operational [Scott, 1972]. In order to avoid failures and losses, "Machine Health Monitoring" can be the solution. When early failure of any part of bearings is diagnosed and located at an early stage the remedial action will be relatively inexpensive and shut down time for the replacement of the failed bearing can be planned in advance [Wallis, 1966; Karakurt, 1989]. The monitoring of rolling bearing installations can serve two important purposes: firstly they provide continuous 'health checks' on bearings without the need for dismantling the machinery; and secondly bearing damage can be detected at such an early stage that failure, and the possibly serious consequences of it, is avoided.

Health monitoring cannot be isolated from other areas of research as it has often concentrated on understanding vibration characteristics of ball bearings with the main objective of detecting problems of ball bearings after they are assembled. To obtain information while bearings are mounted is difficult. Perhaps the only way to get information from such bearings is to investigate the vibration and noise produced by them. The condition of ball bearings can be judged by the nature of their vibration pattern. If a change in the vibration pattern can be monitored and analysed, the health of the ball bearings can be determined.

In previous years some work was done on early detection of rolling bearing failure [Dowson, 1970; Hemmings and Smith, 1979; Braun and Danter, 1979]. In the early days of vibration monitoring, a simple but very effective method was employed. A wooden shaft acted as a good listening post because it transmits noise which relates to the condition of the bearing; it also cuts out much of the noise from other machine components which cause confusion and doubt when using a more sensitive stethoscope. This method is still used today in practical applications for simple cases. But for more complicated cases, the vibrations and sound produced by the machine should be recorded and investigated carefully [SKF, 1961; Wallin, 1966].
However, to obtain the vibration due to ball bearings alone is another difficult subject. As discussed earlier under Section 2.4 the noise produced by ball bearings may not be useful information to predict the bearing vibrations since in the majority of installations the direct noise from a bearing is of low power and is generated inside a sealed component. Under these conditions negligible noise energy escapes through the casing or through seals and this source of noise may be ignored. There are also other sources of conflicting signals and obstacles to their collection [Hampson, 1984]. Therefore the best way of getting information from the shaft-bearing assembly is direct measurement of shaft vibration but this is not easily made when bearings are assembled. As vibration forces are transmitted through bearings, measurements are generally made on bearing housings as near to the shaft as possible [Dowson, 1970]. However, obtaining the vibration signature is not enough in health monitoring since this signal is useful only when it is properly understood. To be able to investigate the vibration spectrum obtained by monitoring ball bearings, their vibration characteristics must be known. In practice the acceptable vibration levels for particular bearings and machines are largely a matter of experience [Dowson, 1970]. Therefore, when bearings are installed they should be monitored and the healthy bearing signals should be recorded. A trend of increasing vibration will indicate that the trouble is becoming worse. Different type of defects will cause different vibration patterns. Experience of various causes of bearing damage can be gained by highly qualified people over a long period. An additional difficulty is that the information obtained from the investigation of one design of bearing arrangement cannot necessarily be applied directly to another [Yhland and Johansson, 1970]. Once considerable experimental experience of a certain arrangement has been gained, it is usually known in which section of the spectrum the changes tend to lie and this reduces the work of analysis. On the other hand, it is not generally possible to predict a determined absolute level of the vibrations, because even modern mass produced assemblies with close tolerances on their components show great differences in their vibration spectra. Hence the importance of a clear understanding of vibrations associated with ball bearings is obvious.

Many researchers investigated the heath monitoring of ball bearings for different types of defects with varying degrees of success [Wallin, 1968; Hemmings and Smith, 1976; Taylor, 1980; Igarashi and Hamada, 1982; Igarashi and Yabe, 1983; Hampson, 1984; McFadden and Smith, 1984; Igarashi and Kato, 1985; Kanai et al., 1987; Karakurt, 1989; Khan, 1991]. Since there are many studies on the subject the papers will be reviewed under subtitles according to the defect type they consider.

2.7.1 Imperfections of Ball Bearings

It is generally accepted that it is impossible to produce a perfect surface or contour even with the best machine tools, and this applies also in ball bearing manufacture. The minute
surface irregularities of the bearing components and their effects on one another, result in vibrations of the bearing. Parts of a machine which resonate with a frequency produced by a bearing are a common source of high sound-level noise [Cena and Hobbs, 1972]. As a result of experiments a specific relationship has been found to exist between the internal dimensions of bearings and the vibration factors for the different components, the rolling elements being more important than the outer ring and inner ring [SKF, 1961; Cena and Hobbs, 1972].

Waviness, off-sized balls, misalignment can be cited for the manufacturing and assembling imperfections. These are discussed below.

2.7.1.1 Waviness

Anomalies may occur during the machining processes by the manufacturers which affect the form and the finish of the surfaces produced [Wardle et al., 1988; Aini, 1990]. Comparison of the surface features compared with the deformed contact (Hertzian contact) dimensions of the rolling element raceway, in terms of feature wavelength, leads to the terms “Roughness” and “Waviness” [Wardle and Poon, 1983] (see Chapter 1 for the definition of roughness and waviness).

The significance of the level of vibrations due to roughness is evident when asperities break through the lubricant film and contact the opposing surface. It consists of a random sequence of small impulses which can excite all natural modes of the bearing and the supporting structure. This phenomena was first recognised by Sayles and Poon [1981]. Vibration caused by surface roughness is not considered in this thesis.

The importance of running surface waviness from the vibration point of view was known for a long time. However, not until the 1960s was it studied. This was mainly because to determine the surface waviness was very difficult. In the 1960s, with the availability of vibration testing machines which could also measure the surface waviness (see the paper by SKF [1961]), the research on the contribution of waviness to the shaft-ball bearing vibrations increased. Gustafsson and Tallian et al. [1963] studied the effect of waviness and came to two important conclusions: Firstly, low order outer ring waviness affects the amplitudes of the vibrations at the BPF and secondly inner ring waviness of the order $k$ waves per circumference produces flexural vibrations of the outer ring (their pick-up was on the outer ring) with a predominant peak of $k$ times the rotational frequency. The amplitudes at other frequencies were small compared to the peak at times the rotational frequency and the amplitudes of vibrations were significant only for low orders of waviness. An experimentally obtained spectrum of a bearing with inner ring 2 and 3 waves per circumference is shown in Fig.(2.24).
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Gustafsson and Tallian et al. [1963] further concluded that the following formulae could be employed to find the approximate ranges of orders of waviness:

\[ k_o = \frac{f}{f_c} \]  \hspace{1cm} (2.8)

\[ k_i = \frac{f}{f_i} \] \hspace{1cm} (2.9)

\[ k_b = \frac{f}{f_b} \] \hspace{1cm} (2.10)

where \( k_o, k_i \) and \( k_b \) are the orders of inner ring, outer ring and ball waviness respectively, generating vibrations of frequency of \( f \) Hz. \( f_c \) is the rotational frequency of the cage with respect to outer ring, \( f_i \) the frequency of the cage with respect to the inner ring and \( f_b \) the polar rotation frequency of the ball, in Hz.

Yhland [1967] examined the correspondence between waviness and the resulting vibration spectrum. For a bearing with \( m \) rolling elements, if \( p \) and \( q \) are integers equal to or greater than 1 and 0, respectively, then for vibrations in the radial direction measured at...
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A point on the outer diameter of the outer ring, the vibration circular frequencies as functions of inner ring, outer ring, and roller waviness are given in Table (2.1) [Yhland, 1967; Harris, 1991].

<table>
<thead>
<tr>
<th>Component</th>
<th>Waviness of orders</th>
<th>Vibrations (caused by waviness)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner Ring</td>
<td>$k = qm \pm p$</td>
<td>$qm (\omega_i - \omega_c) \pm p \omega_i$</td>
</tr>
<tr>
<td>Outer Ring</td>
<td>$k = qm \pm p$</td>
<td>$qm \omega_c$</td>
</tr>
<tr>
<td>Roller</td>
<td>$k$ (even)</td>
<td>$k \omega_b \pm p \omega_c$</td>
</tr>
</tbody>
</table>

Table (2.1) Vibration frequency vs. waviness [after Harris, 1991]

The bearing outer ring moves as a rigid body when $p = 1$. For $p > 1$, vibrations are of the flexural type with $p$ equal to the number of lobes per circumference of the outer ring deflection curve. For a waviness spectrum obtained at an inner ring speed of 900 rpm, for a bearing with an accentuated inner ring waviness, Yhland [1967] obtained the vibration spectrum at 1800 rpm shown by Fig.(2.25). The Talyrond trace of the inner ring also shown in Fig.(2.25), the tested tapered roller bearing contained very smooth rollers and outer ring.

Fig.(2.25) Waviness and vibration spectra from inner ring with accentuated waviness [Yhland, 1967]
Cena and Hobbs [1972] concluded that ball waviness was of greater significance than raceway waviness on the vibration of ball bearing systems. Meyer et al. [1980] theoretically studied the outer ring waviness and concluded that the vibration spectrum of a ball bearing with a wavy surface would have peaks at $N \omega$, $\omega_c$, the BPF and its superharmonics and peaks around the wave passage frequency and its harmonics at $\pm N \omega$, where $N$ is the number of waves (see Fig.(2.26)).

Fig.(2.26) Frequency Spectrum for Waviness [Meyer et al., 1980]

Thomas [1982] pointed out that waviness contributes to low frequency noise, so at relatively low speeds it causes few problems with precision bearings. It only becomes a serious problem when components possess relatively large amplitude features or high speeds. However, Gupta [1988] argues that waviness does not have a significant effect on overall bearing dynamic performance and for practical applications it can be ignored.

However, in contrast, Wardle and Poon [1983] found that the waviness causes most of the severe vibrations and noise problem in bearings. They reported that waviness produces vibrations at frequencies up to approximately 300 times the rotational speed but is predominant at frequencies below about 60 times rotational speeds. Wardle and Poon [1983] also pointed out the relation between the number of balls and the waves for severe vibrations to occur. When the number of balls and waves are equal there would be severe vibrations since in this case there is symmetry of loading and all balls vibrate in phase. Later Wardle theoretically [1988a] and experimentally [1988b] showed that outer race waviness produces vibrations at the harmonics of outer race ball passage frequency, $m \omega_c$ as also calculated by Gustafsson and Tallian et al. [1963]. Rahnejat [1984] and Rahnejat and Gohar [1985] realised that inner ring waviness is somewhat more complicated than what Gustafsson and Tallian et al. [1963] and Meyer et al. [1980] have predicted. Wardle [1988a] showed that in the case of inner ring waviness the axial vibrations take place at frequencies harmonic with the ball to inner ring passage rate $m(\omega_c - \omega_i)$, whereas radial vibrations occur at frequencies $m(\omega_c - \omega_i) \pm \omega_i$. Wardle [1988a] argues that only specific orders of waviness generate vibrations. Axial vibration is
produced when the number of waves per circumference is an integer multiple of the number of balls, whereas radial vibration is produced by waviness of the order of $N = i \, m \pm 1$. Some of Wardle’s arguments were later proved by Aktürk [1988], Franco [1990] and Franco et al. [1992].

Wardle [1988a] showed that ball waviness produced vibrations in the axial direction at frequencies given by $2 \, i \, \omega_b$, while radial vibration occurs at frequencies given by $2 \, i \, \omega_b \pm \omega_c$. He points out that only even orders of ball waviness produce vibration; this was also pointed out by Yhland [1967]. This is understandable since otherwise, ball waviness would cancel itself out. In practice this is not the case of course. This problem arose from the assumption that the ball was assumed to remain rotating about an axis normal to the plane containing the centres of the inner and outer race contacts.

2.7.1.2 Off-Sized Balls

The existence of off-sized balls in a bearing introduces further untoward vibrations to the shaft-ball bearing system. It is impossible to produce a set of identical balls even with the best machine tools, there is always some difference between ball diameters (less than the machining tolerances). It is worth pointing out here that the balls for precision ball bearings are selected to be as near as possible the same within tight tolerances and surface finish. The cheaper the bearing, the more these tolerances are relaxed. These tolerances are not usually significant dimensionally as far as circular or elliptical contact footprints are concerned. However they may play an important role in the vibrations of ball bearings. The off-sized ball effect has been studied by different researchers. Tamura [1968], for example, experimentally showed that the ball bearing axial stiffness varies with the cage position, since the number of loaded balls at any instant is changed by the balls’ diameter differences. The appearance of rotor axial resonance was also reported by Tamura [1968] as a multiple of $i$ ($i = 1, 2, 3, \ldots$) times the angular velocity of the cage, when it was operating close to the natural frequency of the system. He further concluded that, in the case of a perfect bearing with no variation of diameter among balls, the resonance frequency did not appear.
Barash [1969] studied the effect of an off-sized ball on the ball speed variation. He reported that if one ball is bigger than the rest of the balls in a bearing, it will lag continuously since it will have a smaller contact angle and lower speed. This causes peculiar vibrations.

Meyer et al. [1980] analytically investigated the off-sized ball problem and come to the same conclusion as Tamura [1968]: it causes vibrations with the multiples of cage speed. It will interact with the BPF and cause an increase in the vibration amplitude as seen in Fig. (2.27). Yamamoto et al. [1981] also experimentally observed a peak at the cage rotational speed and another one at twice the cage speed.

Aktürk [1988] using angular contact ball bearings and Franco [1990] using radial ball bearings, studied the problem of an off-sized ball and both reported vibrations at cage speed and influence of an off-sized ball on the other frequencies. Although Aktürk [1988] observed vibrations at three times the cage speed for a ball bearing with 15 balls, 3 of which were 1 μm oversized and symmetrically distributed, he considered it simply as the superharmonic of the cage speed. Franco [1990] and Franco et al. [1992] observed vibrations due to off-sized balls at the cage speed due to the random distribution of off-sized balls within the bearing. They showed that when there is one oversized ball in a bearing the most dominant vibrations occur at the cage speed which is indicated by (a) in Fig. (2.28).
Gupta [1988] showed that when the off-sized ball size increases, some performance deterioration was observed. He reported that the shape of the cage whirl orbit changes from circular to somewhat polygonal with increasing variation in ball size. This may be the result of ball speed variation as also reported by Barash [1968].

### 2.7.1.3 Misalignment

Even bearings are themselves completely satisfactory, through inaccurate machining of shafts or bearing housing, or by adopting poor assembly techniques, the bearing can be deformed or the rings can be misaligned. The vibration caused by these factors can have exactly the same character as those arising from poor quality bearings [Johansson, BBJ 200]. Bearing distortions may take place when force fitting them into their housings. This takes place when the outer race of the bearing is skewed at an angle to the housing seating [Aini, 1990].

In earlier studies, misalignment was considered only because it imposed additional stresses on the ball bearings. The stress distributions in a misaligned bearing were obtained by different researchers [Harris, 1991]. When the research on bearing running problems and early failure started, misalignment began to be identified as a major source of bearing problems. For example a study by Barash [1968] into ball speed variation and its effect on cage design identified bearing misalignment as the most common cause of ball speed variation and cage failure.

An investigation into misalignment problems in ball bearings by Ellis [1970] showed that misalignment of ball bearings can produce adverse conditions and shorten bearing life. The cause of ball speed variations due to misalignment was found to be the contact angle variations [Ellis, 1970]. When there is contact angle variation, each ball tries to rotate about the bearing axis (cage rotation) at a different speed (see Fig. (2.29)). This is not a satisfactory situation, some balls are compelled to slow down and others to speed up by being forced to slide against the bearing rings. Each ball cycles through the total contact angle variation for each relative revolution between the misaligned bearing ring and cage. This effect was reported to be avoided by employing a plastic cage [Stolz, BBJ 228]. A flexible cage will, as a result of deformation, avoid the kinematically produced stress and permit the natural differences in ball velocity due to the variation in contact angles [Stolz, BBJ 228]. Smooth running of bearings will be impaired by misalignment causing additional vibrations and noise. The worst condition are likely to arise under combined loads [Barash, 1968; Ellis, 1970]. The severity of the condition increases with the degree of misalignment [Ellis, 1970]. When bearings failed due to misalignment are investigated,
a common running path can identify the cause of failure. Ellis [1970] gives an example of a common running path on radial and angular contact ball bearings under axial load (see Fig.(2.30)).

![Diagram](image)

*Fig.(2.29) Possible cage loading resulting from contact angle variations (frictional forces excluded) [Ellis, 1970]*

Further theoretical research into the forces and moments produced by misalignment was carried out by Andréason [1970]. Later Brändlein [1971] investigated the effect of misalignment on the life of ball bearings and calculated permissible misalignment for different ball bearings.

Research into vibration problems caused by misalignment did not start until the 1980s and today only a few papers are available on the subject. Meyer et al. [1980] theoretically investigated the vibrations due to misalignment. Fig.(2.31) shows the predicted frequency spectrum for a misaligned outer race [Meyer et al., 1980]. In the investigation, the researchers assumed that the bearing has a misaligned rotating outer race and a stationary inner race. Fig.(2.31) shows that the amplitudes at the BPF has not changed but the amplitudes of the sidebands will be lower than the ball passage harmonic components by an approximate factor of $P/2$ where $P$ is the degree of ball force variation due to the defect [Meyer et al., 1980].
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Fig. (2.30) Running path patterns on radial and angular contact ball bearings under axial load [Ellis, 1970]

Experimental research into the cause and cure of bearing noise showed that misalignment has a predominant effect on basic running errors (i.e., deviations in the path of the ball) and hence vibration at the running speed [Wardle and Poon, 1983; Ellis, 1970]. Fig. (2.32) shows an example of the effect of a combination of misalignment and thrust load on ball load distribution in a single row radial ball bearing. Wardle and Poon [1983] point out that misalignment produces regions of high ball loads as seen in Fig. (2.32) which leads to a secondary source of the BPF effect. Although it is always small compared to the basic running errors, it occurs at high frequencies, i.e., at the bearing's ball passage rate and its harmonics, where it may excite low order shaft resonance.

Fig. (2.31) Frequency spectrum for misalignment [Meyer et al., 1980]
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Fig. (2.32) Example of ball load distribution under thrust load and misalignment [Wardle et al., 1983]

Fig. (2.33) Resonance curve of axial vibration in the case of inner ring misalignment [Gad et al., 1984b]
Gad et al. [1984b] introduced inner ring misalignment into their model. Fig.(2.33) shows the effect of misalignment for different damping constants. The shaft-bearing system with misalignment shows periodic motions. It can be seen from Fig.(2.33) that the axial vibrations due to bearing inner ring misalignment is a forced vibration and exhibits resonance behaviour. Fig.(2.34) shows transient and steady-state vibrations of the shaft-ball bearing system with inner ring misalignment of 0.025 rad. at a speed of 850 r.p.m. and damping constant of 100 Ns/m.

![Fig.(2.34) Transient and steady-state motions in the case of inner ring misalignment](Gad et al., 1984b)

The harmonic axial and radial vibrations due to inner ring misalignment synchronise with shaft revolution. This is expected since the inner ring is rotating at the speed of shaft.

2.7.2 Defects

Defects in bearings can cause severe vibrations. By monitoring the vibration characteristics of a bearing, the surface roughness on the race or balls of ball bearings can be estimated [Kanai et al., 1987]. The sources of these defects can be manufacturing and assembling anomalies which was discussed in the previous section or external
effects. Some of the external sources of defects can be cited as wear of the contact surfaces, a defect on the surface of mating ball and cage surfaces, debris and contaminations in the contact etc. Defects can be local, i.e. point defect or discrete. Discrete defect refers to a wider range of faults, e.g. scores, indentations, corrosion pits, brinelling and contamination [Wardle and Poon, 1983]. Characteristically, discrete defects produce impulsive vibrations and noise. This can be considered as roughness as discussed earlier and will not be investigated in this thesis. In this section point defects will be discussed. Point defects may occur due to fatigue or chattering marks.

Fatigue begins as a minute crack at a certain depth below the surface and this leads to gradual flaking which sooner or later necessitates replacement of the bearing [Wallin, 1966]. If vibration monitoring is done correctly this crack can be identified long before it comes to the surface since structural changes associated with fatigue phenomena and cracks in the bearing components can affect the vibration spectrum even though they do not affect the bearing geometry [Yhland and Johansson, 1970]. This is because of the changes in the contact spring and damping characteristics of the bearing.

There are in fact at least three different but associated effects which occur when the rolling elements roll over a damaged raceway surface, or when damage on one of the rolling elements comes into contact with the raceway [Yhland and Johansson, 1970].

1. Characteristic changes in the vibration spectra at points on the boundaries of the bearing arrangements.

2. Impacts which through wave propagation phenomena in bearing components and the surrounding structure, take the form of transient oscillations at the boundaries of the bearing arrangements.

3. More or less pronounced vibration peaks.

Fig.(2.35) Vibration sequence in a damaged rolling bearing [Yhland and Johansson, 1970]
When a rolling element rolls over a damaged raceway surface, or when the damage on a rolling element contact the raceway, the edges of the microscopic geometric changes in the bearing caused by the damage give rise to sudden and large variations in the contact pressure. The resulting sequence of dynamic behaviour can most closely be described as an impact, or a series of impacts following each other closely (see Fig.(2.35)).

Yhland and Johansson [1970] investigated this dynamic behaviour and concluded that if the damaged bearing component is a ring, the damage is rolled over at approximately regular intervals. The frequency at which this occurs can easily be determined on the basis of the geometry of the bearing and its speed of rotation, and this is also true if the damage is on a roller of a roller bearing. In the case of a damaged ball in a ball bearing, there will be variable intervals during which the damaged point will periodically make contact with the raceways and variable intervals during which there will be no such contact. This due to the complex pattern of movements of the balls in a rotating ball bearing.

If there is damage at more than one point in a bearing, as may occur if secondary damage occurs fairly soon after the primary damage, several oscillation sequences will be superimposed on each other. This more or less linear superposition gives a correspondingly more complicated transducer signal, and this must be considered when the signal is being studied.

Braun and Danter [1979] investigated a bearing with developing localised defects. The vibration signature from each defect appeared with period $T_f$ for a constant rotational speed, as in Fig.(2.36). Taylor [1980] argues that a machine with a defective bearing will produce vibrations at least with five different period $T_f$, i.e., with frequency. They are shaft speed, cage speed, ball passage speeds of the outer and inner race, and ball spin frequency. Ball passage frequencies are generated as the balls pass over a defect on the raceways.

![Fig.(2.36) Typical bearing signature](Braun and Danter, 1979)
Igarashi et al. [1982; 1983; 1985] carried out exhaustive experimental studies of indentations caused by the debris in the lubricant on the raceway surfaces and the balls with a view to establish a procedure for diagnosing rolling bearing defects from their vibration and sound. Igarashi and Hamada [1982] used an anemometer, which is an instrument that is designed to make a measurement of radial vibration at one point on the outer ring of a rolling bearing while its inner ring is rotating, whereas Igarashi and Yabe [1983] used a sound measuring instrument to investigate the information obtained from the bearing with a single defect on the race surface of the inner or outer ring or on the ball surface. It was concluded that the main peaks in the bearing sound frequency spectrum coincide with those in the vibration acceleration frequency spectrum. McFadden and Smith [1984; 1985] studied single and multiple point defects in a rolling element bearing. They showed that in the case of inner case defect, for example, the vibration spectrum will accommodate peaks at the shaft speed, cage speed, their harmonics and their combinations (see Fig. (2.38)). Later Igarashi and Kato [1985] and Karakurt [1989] made a similar study on a bearing with multiple defects. Fig. (2.37) shows the envelope signal and related rms spectrum of a bearing with a defected inner race. The results of Karakurt [1989] are very similar to those of McFadden and Smith [1984; 1985]. All studies show that useful information about bearings can be obtained from their vibration spectrum.
2.8 Shaft-Ball Bearings with External Dampers

There are three commonly applied damping systems in the problem of the shaft-ball bearing arrangements. These are arbitrary damping for mathematical convenience, oil film damping and viscoelastic damping.

Generally when there is need for external damping in mathematical investigations, an arbitrary (convenient) damping are introduced to the system. This is particularly useful when the effects of the natural frequency is to be avoided in the problem. Gad et al. [1983; 1984a; 1984b], Gupta et al. [1975; 1979a; 1979b; 1979c; 1979d], Rahnejat and Gohar [1985], Aini et al. [1990], Franco et al. [1992], Aktürk et al. [1992] employed arbitrary damping in their theoretical investigations.

Rahnejat [1984] and Rahnejat and Gohar [1985] investigated the effect of the elastrohydrodynamic oil films between the ball to race ways contacts on the vibration characteristics of the shaft-bearing systems. They concluded that the effect of internal oil film damping is slight and can be omitted from a vibration point of view.

Fig.(2.39) shows the cyclic oil film behaviour in the contact of rolling members. It is composed of two unequal portions, of similar shape, with a common point A. The larger segment represents the film behaviour when the ball is in the least loaded region of the bearings, where there is a significant hydrodynamic contribution. The smaller segment, to the left of A, corresponds to the more highly loaded region which does not necessarily apply always to the top or the bottom region of the shaft contact, as its centre itself is
oscillating. Rahnejat and Gohar [1985] noted that as the loops on the RHS of the point A enclose larger areas than those on the LHS, there is more energy dissipation associated with them, implying that squeeze film damping action is more significant in the region of thicker films. Since damping in the oil film is caused by squeeze velocity, relatively larger damping is obtained with thicker oil films. However, thinner oil films means less speed and hence low damping. Therefore in Fig.(2.39) in the loaded zone there is less damping due to the thin oil film. This is in agreement with the findings of Dareing and Johnson [1975] who argue that lubrication film damping is greatest under conditions of separation and this indicates that film damping is generated primarily by the squeeze film mechanism. However, although Dareing and Johnson [1975] argue that in any case oil film damping will increase the overall damping at least 30% and it is more effective under conditions of severe vibration, Rahnejat and Gohar [1985] found it insignificant in vibration problems. This is perhaps because of the heavy load they employed in their model (500 kg of shaft mass) since fluid film damping may be more important and effective under light loading conditions where Hertz compression is small [Dareing and Johnson, 1975; Mehdigholi et al., 1990].

Fig.(2.39) Steady-state limit cycle [Rahnejat and Gohar, 1985]
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A dynamic analysis of the transient ball motion in an angular contact ball bearing, operating under elasto-hydrodynamic traction was formulated by Gupta [1975]. He assumed a cage-free bearing for his analysis, thus eliminating the cage influence on the stabilising/destabilising of the balls' motion. Gupta [1975] concluded that, his analysis presented a design tool, capable of simulation the dynamics of bearings subjected to skidding, due to the acceleration of races in lubricated contacts.

Later Gupta [1979c] and Aini [1990] reconfirmed that internal lubricant film damping in ball bearings was insignificant from the vibration point of view but reduces the noise caused by high frequency vibrations as suggested by Sayles and Poon [1981] and Wardle and Poon [1983].

Elastomers are employed as external dampers, particularly in the form of O-rings. Gustafsson and Tallian et al. [1963] suggested use of elastomer external dampers for shaft-ball bearing systems. However, elastomers have not been widely employed until more recently in practical applications since many engineers do not regard a moulded rubber component to be of sufficiently consistent dimensional accuracy for use in precision engineering and the information available on elastomers did not contain the required design data in terms of dynamic properties of elastomers and neither is this information easy to obtain experimentally, particularly in the case of O-rings [Powell and Tempest, 1968]. Elastomers have been employed in gas bearings running at high speeds with successful results [Powell and Tempest, 1968; Kazimierski and Jarzecki, 1979]. Hedman [1987] employed elastomer buttons as external dampers. His simulation also shows some reduction in the vibration amplitudes. Nevertheless, since he employed a relatively less sophisticated model, his results are not very reliable. There may be found other kinds of external dampers employed in the shaft-ball bearing systems. For example Gunter [1970] investigated the influence of damped externally and linear flexibly mounted rolling element bearings on dynamic shaft unbalance response. He showed that the untoward effects of the shaft unbalance can be dramatically reduced by proper design of bearing support characteristics.

Olsson [1986] analysed the effect of introducing an external lubricant squeeze film around the outer race as a damping element in an angular contact ball bearing assembly. Although the simulation did not suggest really successful application of this approach, the results showed some reduction in the vibration amplitudes at the natural frequencies.

In this thesis, different types of elastomer theoretical models are studied as alternative external damping elements for the shaft-ball bearing systems. Therefore there is need for further investigation into the literature on their dynamic and mechanical properties, in particular their stiffness and damping characteristics. This will be done in the next section.
2.9 Elastomers as External Dampers

Dampers are important in the control of shaft vibration of rotating equipment which must operate through one or more critical speeds. Although elastomers have been used in different applications, mainly in vibration and shock isolation for a long time, their dynamic properties had remained unresearched for many years. As today elastomers are used in the design of increasingly complex arrangements, accurate knowledge and clear definitions of elastomer dynamic properties (stiffness and damping) are becoming more important. For example, modern turbomachines are running at very high speeds (e.g. 120,000 rpm) and their size is reducing. These applications often lead to operation above one or more flexible rotor critical speeds [Taylor and Fehr, 1982]. For these kinds of applications it is very important to know the elastomer dynamic properties so that system critical speeds can be avoided or their corresponding amplitudes reduced. Therefore, there appears to be a need for a systematic study of the nonlinear dynamic behaviour of elastomers, and subsequent development and/or evaluation of an appropriate material model [Generiwal and Rotz, 1987].

Almost four decades ago the first papers on the dynamic applications of elastomers appeared. Some of the literature available deals only with experiments which show how to derive the dynamic stiffness and damping. A paper by Smalley et al [1975] investigates a number of methods and apparatuses which have been used to measure material dynamic properties for engineering application. More recently Kluesener [1986] tested two types of elastomers using five different methods and reported some significant differences between them. The literature review, made two decades ago contains many experimental papers on the subject but no attempt has been made to investigate the problem theoretically, since the subject is complicated and needs a computer. Snowdon [1963] tried to handle the problem analytically without any computation in 1963. He derived some equations that predict the response to vibration of rubberlike materials and structures but the results were not very accurate. In the last decade theoretical investigations on the subject have increased and some relevant literature is available.

Sommer and Meyer [1973] gave the brief summary of the changes in the spring stiffness and damping coefficients produced by variations in rubbers, fillers, oils, level of cross-linking and processing factors such as mixing, curing, and storage. Apart from these, the dynamic properties of elastomers are strongly dependent upon many factors; the most common of which are: frequency of the force exciting it, temperature, amplitude of vibrations and geometry of the dampers.
In the 1970s some important research was carried out on the calculation and prediction of the dynamic properties of elastomers by the National Aeronautics and Space Administration (NASA). Five contractor reports were produced during this research programme [Chiang et al., 1972; Gupta et al. 1974; Smalley and Tessarzik, 1975; Darlow and Smalley, 1977; Smalley et al., 1977]. In the first report [Chiang et al., 1972] the basic methods are studied and a rig for the experiments was built as seen in Fig.(2.40)

![Fig.(2.40) Schematic of base excitation resonant mass test rig [Chiang et al., 1972]](image)

Using this test rig Gupta et al [1974] showed that the variation of the complex dynamic stiffness with the frequency can be expressed in the form of a power function of the form shown below:

\[ Y = A \omega^B \]  

(2.11)

where \( A \) and \( B \) are constants. Gupta et al. [1974] also showed that this data can successfully be presented with Voigt or Maxwell mechanical models (see Appendix 6). Fig.(2.41) shows the experimental data and the mechanical model curve fit obtained for it. It is seen that most of the stiffness is determined by the static spring, and that the stiffness and damping behaviour of the model follows reasonably closely the mean level of the measured stiffness and damping values.
Smaley and Tessarzik [1975] investigated the effect of temperature, dissipation level and geometry using the test rig established by Chiang et al. [1972] (see Fig. (2.40)). Smaley and Tessarzik [1975] argue that if real and imaginary parts of complex stiffness, \( K_1(\omega) \) and \( K_2(\omega) \) respectively, are calculated from the following equations Equ.(2.12) and Equ.(2.13), an accurate curve fitting can be achieved. Note that Equ.(2.12) and Equ.(2.13) include dimension factors and hence are prediction formulae where \( G' \) is the real part and \( G'' \) is the imaginary part of the complex shear modulus \( G^* \). \( \beta' \) is the real part and \( \beta'' \) is the imaginary part of the shape factor.

\[
K_1(\omega) = 3G'(\omega) \frac{\pi D^2 N}{4h} [1 + \beta'(\omega)s^2]
\]

(2.12)

\[
K_2(\omega) = 3G''(\omega) \frac{\pi D^2 N}{4h} [1 + \beta''(\omega)s^2]
\]

(2.13)
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The experiment done by Smalley&Tessarzik[1975] and Darlow and Smalley [1977] showed that if $G', G'', \beta'$ and $\beta''$ in Equs.(2.12 and 2.13) are expressed at constant temperature of 32 degrees Celsius, as below, the error for dynamic stiffness would be less than 50% and with damping the error would be less than 20%.

\begin{align}
G'(\omega) &= 3.686 \times 10^6 \omega^{0.3037} \quad \text{N/m}^2 \quad (2.14) \\
G''(\omega) &= 8.333 \times 10^6 \omega^{-0.1277} \quad \text{N/m}^2 \quad (2.15) \\
\beta'(\omega) &= 12.33 \omega^{-0.290} \quad (2.16) \\
\beta''(\omega) &= 1.726 \omega^{0.0299} \quad (2.17)
\end{align}

Darlow and Smalley [1977] established another rig for rotational loading in order to compare their results with the previous findings and validate the previous results with the closer simulation of the real-life working conditions of elastomer dampers since most of the elastomer damper work in the designs where the loads are rotational (see Chapter 5 for these experimental arrangements). Their findings confirmed the previous results.

Smalley et al. [1975] tested elastomer O-rings of three different materials for dynamic properties with the rig established by Chiang et al. [1972] with small alteration. They showed that the dynamic properties of elastomer O-rings 4-15 times the corresponding static properties. A similar study done by Green and Etsion [1986] also confirmed the findings of this report.

Taking these reports as a basis, Tecza et al. [1979a, 1979b] designed elastomer dampers for a high-speed flexible shaft. They showed that when the dynamic properties are accordingly arranged, smooth running can be obtained with elastomer external dampers. These two papers have a very similar arrangement to the assumed grinding spindle model studied in this thesis. Therefore theoretical results obtained may be compared more readily with the experimental data obtained from these tests.

In these studies the dynamic properties of elastomers were obtained in the form of the real and imaginary parts of the complex stiffness. The real part represents the stiffness and the imaginary part represents the damping factor for the system. All reports and papers report that elastomers can be used as external dampers and they will be effective in damping out the untoward effects of natural frequencies and reducing forced vibration amplitudes if their dynamic properties are known and their application is made correctly.
Different ways of representation of obtained data were also studied by several researchers. Gupta et al. [1974], Dahl and Rice [1984], Fersht et al. [1986] employed mechanical models whereas Smalley and Tessarzik [1975], Darlow and Smalley [1977], Smalley et al. [1977] applied direct fits to the experimental data. Because the mechanical model is more convenient for vibration studies of the shaft-ball bearing system, a modified Voigt model is used for the presentation of the data in this thesis.

2.10 Background of the Project

Dr. R. Gohar started investigating the elastrohydrodynamic oil film behaviour under point contact of a vibrating bearing in the early 1980s. Rahnejat [1984] investigated the influence of vibrations on the oil film in the concentrated contacts under the supervision of Dr. Gohar. Rahnejat modelled gears and rolling element bearings as bodies in concentrated contact both with and without a lubricant film. During this work Rahnejat also mathematically modelled a horizontal milling machine supported by a pair of deep groove ball bearings. He studied the BPF and showed that it appears as a result of contact spring nonlinearity of the ball bearing. However, he could only observe it when the imperfections are present in the system. This may be because of his high preload and heavy mass, existence of which will considerably reduce the effect of the BPF [Aktürk et al., 1992]. Rahnejat [1984] also showed that damping of the oil film between the ball to raceway contacts comes mainly from the squeeze effect as discussed earlier (Section 2.9) and is negligible. Later, a dynamic simulation of a pair of deep groove ball bearings supporting a shaft, incorporating non-linear elastic behaviour of ball to race contacts, contact rolling/squeezing film lubrication and inclusion of surface defects of rolling members as well as eccentric rotation of inner raceway of radial ball bearings was presented by Rahnejat and Gohar [1985]. Major frequencies relating to applied load-stiffness characteristics of the bearings as well as the BPF, cage set rotational speed, wave passage frequency and some modulation effects between these were observed. The model was also applied to lubricated contact vibration problems in gears under hydrodynamic and elastrohydrodynamic conditions [Rahnejat, 1984].

Meanwhile Matsubara [1982] studied the vibrations of a heavy elastic shaft supported by a pair of radial contact ball bearings under the supervision of Dr. Gohar. Matsubara modelled ball bearings with linear rotating springs as linearised load deflection curves over a narrow region employing a piece-wise spring technique. Later, the results of this study were published in a paper by Matsubara et al. [1988].

The work presented by Rahnejat [1984] was extended by Eisenbeis [1985] under the supervision of Dr. Gohar. Eisenbeis substituted deep groove bearings support by a pair of angular contact ball bearings. He presented the effect of off-centre rotation, off-sized
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balls and waviness features on the vibration characteristics of the system as well as the effect of changing the number of balls and preload. However, he could not observe the effect of the BPF for a geometrically perfect bearing. This study was further extended by Aktürk [1988] under the supervision of Dr. Gohar. Aktürk showed some instabilities in the computer simulation of the previous researchers. This led to a new simulation programme which is used in this thesis. Some of the findings of this research were presented in a paper by Aktürk et al. [1992].

Further work in gear simulation involved the contact of wavy gears which was undertaken by Mehdigholi [1987] under the supervision of Dr. Gohar. He simulated the disc machine geometry reported by Dareing and Johnson [1975], where one disc was crowned and the other was corrugated to present a wavy surface. The result of the theoretical simulation and the experimental results confirmed closely to each other, except that the bandwidth of frequencies in the vicinity of the resonance is larger with the theoretical results, because of the failure in the oil film model under contact separation conditions. The result of this research was published in a paper by Mehdigholi et al. [1990].

Franco [1990] also studied radial contact ball bearings under the supervision of Dr. Gohar. He linearised the load deflection curve with Taylor's expansion and studied multiple defects of ball bearings. The anomalies he included in his simulation were off-sized balls, waviness, out-of-balance shaft centre. The results of this work was also presented in a paper by Franco et al. [1992].

Aini [1990] studied a perfect precision grinding spindle supported by a pair of angular contact ball bearings theoretically and experimentally. He also included an oil film into his calculations and concluded that the presence of the oil film along the line of contacts do not significantly alter the position of the major modes of the system. This is also in agreement with the results of Rahnejat [1984], Rahnejat and Gohar [1985], Mehdigholi [1987] and Mehdigholi et al. [1990].

Gohar et al. also employed different external damping systems to the shaft ball bearing systems. Olsson [1986] employed external oil squeeze film dampers and later Hedman [1987] employed elastomers as external dampers. Although his results showed some reductions in the vibration amplitudes, overall results were not satisfactory due to the difficulty in modelling of elastomers.

Later Larsson [1989] also employed elastomers as external dampers to aerostatic bearings and compared them with piezo-electric and oil film dampers. Since both Hedman [1987] and Larsson [1989] suffered from the lack of much solid information on the dynamic properties of elastomers, therefore, Grassano [1991] studied dynamic
characteristics of elastomer dampers for cylindrical buttons, ring cartridges and O-rings under the supervision of Dr. Gohar. He showed that close predictions for different geometries of elastomers were possible.

Closure

The dynamics of the shaft-ball bearing system was reviewed in this chapter. Due to the broadness of the subject and a plethora research papers available, only certain topics were reviewed in depth. Although there are many research papers quoted, there are some topics that have been only partially investigated. As was given in Section 2.10, the research done by Gohar et al. will be consolidated and the dynamic characteristics of shaft-ball bearing systems will be investigated in this thesis. This will also include the study of elastomers as external dampers to shaft-ball bearing systems. Therefore the following chapters 3, 4 and 5 will be outlining the theory involved in this fascinating subject.
CHAPTER 3

MECHANICS OF BALL BEARINGS

3.1 Introduction

In the second chapter the previous on ball bearings research was explained. In this and the next two chapters the theories for the forthcoming results and discussion will be established. In this chapter the main concern are the vibrations of ball bearings. However, vibration theory alone cannot handle the problem of the shaft-bearing system since the elasticity of bearings has to be defined by contact stress relations and geometrical considerations, hence the subject is very broad. Therefore there is a need to be involved in contact mechanics, kinematics, dynamics, numerical analysis and vibrations. Furthermore, the theory of vibrations should be placed in its appropriate position amongst others for correct handling of the problem.

The study of vibration is concerned with the oscillatory motion of mechanical systems and the dynamic conditions related thereto. This motion may be in a regular form or it may be irregular i.e., of a rather random nature. A vibratory system, in general, includes an element for storing potential energy, an element for storing kinetic energy, and an element by which energy is gradually lost.

Whilst for small amplitude linear vibrations, superposition techniques may be used with the mathematical solutions well developed, for nonlinear vibrations, the superposition principle is not valid, and suitable analytical techniques are less well known. Linear analysis is not always sufficient to describe the behaviour of a shaft-ball bearing assembly, since it is nonlinear and causing totally unexpected behaviour that is not predicted or even hinted at by linear vibration theory. Hence some knowledge of non-linear vibration theory is also desirable.
With nonlinear behaviour, the most difficult problem is usually to identify the source of nonlinearity, which is generally caused either by nonlinear springs or by nonlinear damping elements or both. In our case both are present. However, deflection dependent stiffness is often the more dominant cause of the nonlinearity.

The nonlinear damping element present in ball bearings can be caused by friction between the rolling/sliding elements, squeeze film lubrication or by an external damper. The first two are found to be insignificant [Ragulskis, 1975; Rahnejat, 1984; Rahnejat and Gohar, 1985] and the latter will be investigated in Chapter 5.

In order to predict the system vibrations correctly, it is necessary to analyse the dynamic behaviour of the shaft bearing systems using a mathematical model. The elastic properties of the shaft and the housing can be described reliably by established mathematical models. The bearings are however modelled less reliably. This is generally because the vibrations associated with ball bearings are very complex. Even with simplifications the theoretical model produced is difficult to analyse.

Two different approaches are employed in investigating the vibration characteristics of shaft-bearing assemblies. The first is called the superposition method in which the vibration of shaft and bearings are investigated separately while considering their interaction later. The second method is to investigate the system as a whole.

Ball bearings cannot be isolated from the shaft since its movement will cause deflection of balls resulting in forces that will contributed to its dynamic behaviour. Therefore the system should be modelled as a whole and as a result the contribution of bearing vibrations is included in overall vibrations interactively. In this method the shaft acts as a mass and the bearing balls act as nonlinear springs.

After the forces due to the deflection of a ball is found and the total force on a bearing is calculated, the equations of motion for shaft centre vibrations are developed using these forces, as well as other forces which will be defined. Finally the equations of motion will be defined and their numerical solution will be discussed.

3.2. Elastic Contact Modelling

3.2.1 Contact Mechanics

The solution to the elastic problem for the concentrated contact of two elements was established by Hertz [1896]. Some general knowledge of the stress-strain relationship of contact mechanics is needed to investigate the problem.
If two elastic bodies in concentrated contact are considered as shown in Figure (4.1), the following definitions may be employed:

The curvature is generally defined as inversely proportional of the radius of curvature and given as

$$\rho = \frac{1}{r}$$

(3.1)

where curvature is positive for surfaces whose centres of curvature are on opposite sides of the tangent plane and is negative for surfaces whose centres of curvature are on the same side with the tangent plane.

\[\sum \rho = \rho_{A1} + \rho_{A2} + \rho_{B1} + \rho_{B2}\]

(3.2)
where $A$ and $B$ refers to the body and index 1 and 2 refers to the planes. The curvature difference is defined as (see Fig.(3.1)):

$$F(\rho) = \frac{(\rho_{A1} - \rho_{A2}) + (\rho_{B1} - \rho_{B2})}{\sum \rho}$$

(3.3)

The curvature difference for a Hertzian contact is found to be also a function of the elliptical parameters as follows (for more information see [Harris, 1991]):

$$F(\rho) = \frac{(\kappa^2 + 1)\mathfrak{I} - 2\kappa}{(\kappa^2 - 1)\mathfrak{I}}$$

(3.4)

where $\kappa^2$ is the elliptical eccentricity parameter and $\mathfrak{I}$ and $\mathfrak{S}$ are complete elliptic integrals of the first and second kind respectively.

By assuming values of the elliptical eccentricity parameter, a table of $\kappa^2$ versus $F(\rho)$ can be obtained and then it can be shown that the local force and deflection relationship of two bodies may be written as follows [Harris, 1991]:

$$\delta = \delta^* \left( \frac{3}{2} \sum w \left[ \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right] \right)^{1/3}$$

(3.5)

where $\delta^*$ is dimensionless deflection and is expressed as:

$$\delta^* = \frac{2\kappa}{\pi} \left( \frac{\pi}{2\kappa^2\mathfrak{I}} \right)^{1/3}$$

(3.6)

$\delta^*$ is given as a function of $F(\rho)$ in a table and is plotted on three figures in [Harris, 1991].

In order to write the computer programme the following 4th order polynomial was fitted along the curve represented by Eq.(3.6) [Aktürk,1988]:

$$\delta^* = -327.6145 + 1883.338(F(\rho)) - 3798.1121(F(\rho))^2$$

$$+ 3269.6154(F(\rho))^3 - 1026.96(F(\rho))^4$$

(3.7)
For angular contact ball bearings, the curvature difference is generally between 0.9 and 0.99. Therefore in order to achieve maximum accuracy the curve was fitted over this range of data as seen in Fig. (3.2) and each time the curvature difference is checked if it is within the limits. If it is out of range then the other two curves fitted over the data between 0 and 0.9 and 0.99 and 1 are employed as it is appropriate.

The curve fitting over 0 to 0.9 is:

\[
\delta^* = 0.99878 + 0.097536(F(\rho)) - 0.94627(F(\rho))^2 + 1.6462(F(\rho))^3 - 1.2784(F(\rho))^4
\]  

(3.8)

The curve fitting over 0.99 to 1.0 is:

\[
\delta^* = 397890 - 1205000(F(\rho)) + 1216500(F(\rho))^2 - 409370(F(\rho))^3
\]  

(3.9)
3.2.2 Load-deflection Relationship and Calculation of the Stiffness Coefficient

From Equ.(3.5) it can be seen that for a given ball-raceway point contact loading there is a relation between the deflection and the load. This can be expressed as:

\[ \delta = \frac{2}{W^3} \]  
(3.10)

Inverting Equ.(3.9) and expressing it in equation format yields to [Harris, 1991]:

\[ W = K \delta^2 \]  
(3.11)

If Equ.(3.5) is written in the form of Equ.(3.11) for the same material properties of the two contacting bodies:

\[ W = \left[ \frac{2\sqrt{2}}{3} \left( \frac{E}{1 - \nu^2} \right) \left( \frac{1}{\delta^2} \right) \left( \frac{1}{\delta^2} \right) \right] \delta^2 \]  
(3.12)

Hence from Equ.(3.12) and the similarity of Equ.(3.11) and Equ.(3.12), the stiffness coefficient can be given as:

\[ K = \frac{2\sqrt{2}}{3} \left( \frac{E}{1 - \nu^2} \right) \left( \frac{1}{\delta^2} \right) \]  
(3.13)

where \( \nu \) is Poisson's ratio

For the \( i \) th ball, there are two contact surfaces: outer raceway/ball and ball/inner raceway. Hence, the total deflection at the \( i \) th ball is summation of these two local relative movement:

\[ \delta_a = \delta_i + \delta_o \]  
(3.14)

Considering \( \delta = \left( \frac{W}{K} \right)^2 \) and \( W_a = W_i + W_o \) and using Equ.(3.9):
This is the total proportionality of deflection and is used in this thesis. In the previous researches [Eisenbeis, 1985; Akturk, 1988; Aini, 1990] the stiffness factor is assumed to be constant as the change in it due to contact angle variation is relatively small. However, Fig. (3.3) shows that for certain contact angles the change becomes quite significant. In this research, therefore, a subroutine was written in order to determine the stiffness factor for each new contact angle.

$$K_n = \left[ \frac{1}{\left( \frac{1}{K_i} \right)^{1/n} + \left( \frac{1}{K_o} \right)^{1/n}} \right]$$

(3.15)

Fig.(3.3) The effect of changing contact angle on the stiffness factor

3.3 Bearing Geometry

3.3.1 Calculation of the Preload Contact Angle

When the bearing is axially preloaded, the preload will cause the same amount of deflection at each ball because it is uniformly supported by all the balls. In this case,
considering the Hertzian contact relationship, from Fig.(3.4), the relation between the preload and the preloaded contact angle is given as:

\[ P_{as} = m K_i (\delta_0)^{3/2} \sin(\alpha_p) \]  \hspace{1cm} (3.16)

where \( \delta_0 \) is the preloaded contact deflection and given as

\[ \delta_0 = B d_b \left( \frac{\cos(\alpha_o)}{\cos(\alpha_p)} - 1 \right) \] \hspace{1cm} (3.17)

Substituting Eqn.(3.17) into Eqn.(3.16):

\[ P_{as} = m K_i \left[ B d_b \left( \frac{\cos(\alpha_o)}{\cos(\alpha_p)} - 1 \right) \right]^{3/2} \sin(\alpha_p) \] \hspace{1cm} (3.18)

Equation (3.18) includes sine and cosine simultaneously. Therefore, the preloaded contact angle \( \alpha_p \) is solved by trial and error.
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Mechanics of Ball Bearings

The initial axial deflection $z_0$ due to the preload can then be calculated from (see Fig. (3.4)):

$$z_0 = \frac{\delta_p}{\sin(\alpha_p)} + B d_p \left( \frac{\sin(\alpha_0)}{\sin(\alpha_p)} - 1 \right) = B d_p \left( \frac{\sin(\alpha_p - \alpha_0)}{\cos(\alpha_p)} \right)$$

(3.19)

### 3.3.2 Calculation of Deflection

The objective is to calculate the net force on a bearings due to the displacement of shaft centre since this force can then be used in the equations of motion in order to find the movement of the shaft centre.

In order to calculate the total force, the deflection at the $i$th ball in Fig.(3.5) will be calculated first and this will be used in the calculation of the total force.

As seen in Fig.(3.5) the ball is rotating between inner and outer rings. During this rotation the ball is continuously in contact with different points in circular grooves in each race. In the initial position, without any preload, the loci of raceway groove centres of curvature will produce circles as shown in Fig.(3.6).

If a ball is compressed by a force, for example the weight of the shaft, since the centres of the curvature of the raceway grooves are fixed with respect to the corresponding raceway, the distance between the centres is increased by the amount of the normal approach between the raceways. Fig.(3.6) shows a ball bearing prior to the application of load and after preloading. It can be determined from Fig.(3.6) that the locus of the centres of the inner ring raceway groove curvature radii is expressed by:

$$\mathcal{R}_i = \frac{d_a}{2} + \left( r_2 - \frac{d_b}{2} \right) \cos \alpha_0$$

(3.20)

$$\mathcal{R}_o = \mathcal{R}_i - B d_p \cos \alpha_0$$

(3.21)

or

$$\mathcal{R}_o = \frac{d_a}{2} + \left( -r_1 + \frac{d_b}{2} \right) \cos \alpha_0$$

(3.22)
If the outer ring of the bearing is considered fixed in space as load is applied to the bearing, then the inner ring will be displaced. It can be shown that the distance between the centres of curvature of the inner and outer ring raceway grooves at $i$th ball before and after the preload is equal to the contact deflection and can be found from Fig.(3.7) as (also see Fig.(3.8)):

$$\delta_o = B d_o \left( \frac{\cos \alpha_2}{\cos \alpha_p} - 1 \right)$$  \hspace{1cm} (3.23)

The preloaded contact angle is:

$$\tan(\alpha_p) = \frac{B d_o \sin(\alpha_0) + z_0}{B d_o \cos(\alpha_0)}$$  \hspace{1cm} (3.24)

The actual deflections caused by vibration in the $x$, $y$ and $z$ directions will be dealt with from this point onwards. Fig.(3.9) describes the situation after this three dimensional movement of the inner ring centre has taken place. The deflections along the $x$ and $y$ axes can always be combined as a single radial deflection.
The radial deflection for the $i$ th ball (see Fig. (3.9)):

$$\delta_r = x \cos(\theta_i) + y \sin(\theta_i)$$

(3.25)
Fig. (3.8) View normal to a plane at the angle $\theta_i$ from the $x$ axis (see Fig. (3.7))

Fig. (3.9) Loci groove centres of curvature after three dimensional displacement of the inner ring centre ($x$, $y$ and $z$)

From Fig. (3.10) the deflection:

$$\delta'_i = \left[ (Bd_b \sin (\alpha_0) + z_0 + z)^2 + (Bd_b \cos (\alpha_0) + \delta_i)^2 \right]^{\frac{1}{2}} - Bd_b$$  \hspace{1cm} (3.26)
and the contact angle:

\[
\tan(\alpha') = \frac{Bd_0 \sin(\alpha_0) + z_0 + z}{Bd_0 \cos(\alpha_0) + \delta_r}
\] (3.27)

Deflection of the inner ring centre due to rocking motion of the shaft may also be considered in terms of deflections in three axes. For the LHS bearing the radial and axial deflections will be (see Fig.(3.10)):

\[
\delta' = a_i \sin(\phi_i) \cos(\theta_i) + a_i \cos(\phi_i) \sin(\psi_i) \sin(\theta_i)
\] (3.28)

\[
z' = a_i (1 - \cos(\phi_i)) \cos(\theta_i) + a_i (1 - \cos(\psi_i)) \cos(\phi_i) \sin(\theta_i)
\] (3.29)

Since the \(i\) th ball centre is at a distance, \(R\), from the bearing rotation axis this will cause additional axial and radial displacements. Assuming small \(\phi\) and \(\psi\):

\[
\delta'' = -R(1 - \cos(\phi_i)) \cos(\theta_i) - R(1 - \cos(\psi_i)) \sin(\theta_i)
\] (3.30)

\[
z'' = \phi_i R \cos(\theta_i) - \psi_i R \sin(\theta_i)
\] (3.31)
Fig. (3.11) Displacement of the bearing centre due to rocking motion

Fig. (3.12) Effect of rocking angles
Fig. (3.13) The final displacement as seen from the plane at the angle $\theta_i$

The resulting deflections for the $i$th ball of the left hand side bearing will be:

$$\begin{align*}
(\delta_i)_{w} & = \left( B d_s \sin(\alpha_o) + z_0 - z + a_i \left( 1 - \cos(\phi_i) \right) \cos(\theta_i) \\
& + a_i \left( 1 - \cos(\psi_i) \right) \cos(\phi_i) \sin(\theta_i) + (\phi_i) R \cos(\theta_i) \\
& - (\psi_i) R \sin(\theta_i) \right)^{1/2} - B d_s
\end{align*}$$

(3.32)
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The contact angle for the $i$th ball of the LHS bearing is:

$$
\tan[(\alpha_{i})_{LHS}] = \frac{\left( B_d b \sin(\alpha_0) + z_0 - z + a_i(1 - \cos(\phi_i))\cos(\theta_i) \right) + a_i(1 - \cos(\psi_i))\cos(\phi_i)\sin(\theta_i) + (\phi_i)R\cos(\theta_i)}{B_d b \cos(\alpha_0) + x\cos(\theta_i) + y\sin(\theta_i) - (\psi_i)R\sin(\theta_i)}
$$

$$
\tan[(\alpha_{i})_{LHS}] = \frac{\left( B_d b \sin(\alpha_0) + z_0 + z + a_i(1 - \cos(\phi_i))\cos(\theta_i) \right) + a_i(1 - \cos(\psi_i))\cos(\phi_i)\sin(\theta_i) + (\phi_i)R\cos(\theta_i)}{B_d b \cos(\alpha_0) + x\cos(\theta_i) + y\sin(\theta_i) - (\psi_i)R\sin(\theta_i)}
$$

(3.33)

Similar expressions can be derived for the RHS bearing. The deflections and contact angles for the RHS bearing are

$$
(\delta_{i})_{RHS} = \frac{\left( B_d b \cos(\alpha_0) + x\cos(\theta_i) + y\sin(\theta_i) \right)^2}{-B_d b}\left[ \begin{array}{c}
B_d b \sin(\alpha_0) + z_0 + z + a_i(1 - \cos(\phi_i))\cos(\theta_i) \\
- b_i(\phi_i)\cos(\theta_i) + b_i(\psi_i)\cos(\phi_i)\sin(\theta_i) \\
+ R(1 - \cos(\phi_i))\cos(\theta_i) + R(1 - \cos(\psi_i))\sin(\theta_i) \\
\end{array} \right]
$$

$$
\tan[(\alpha_{i})_{RHS}] = \frac{\left( B_d b \sin(\alpha_0) + z_0 + z + a_i(1 - \cos(\phi_i))\cos(\theta_i) \right) + a_i(1 - \cos(\psi_i))\cos(\phi_i)\sin(\theta_i) + (\phi_i)R\cos(\theta_i)}{B_d b \cos(\alpha_0) + x\cos(\theta_i) + y\sin(\theta_i) - (\psi_i)R\sin(\theta_i)}
$$

(3.34)

$$
\tan[(\alpha_{i})_{RHS}] = \frac{\left( B_d b \sin(\alpha_0) + z_0 + z + a_i(1 - \cos(\phi_i))\cos(\theta_i) \right) + a_i(1 - \cos(\psi_i))\cos(\phi_i)\sin(\theta_i) + (\phi_i)R\cos(\theta_i)}{B_d b \cos(\alpha_0) + x\cos(\theta_i) + y\sin(\theta_i) - (\psi_i)R\sin(\theta_i)}
$$

(3.35)
3.3.3. Calculation of Total Contact Force on a Bearing

The deflection for the $i$ th ball was calculated in the previous section. The force due to this deflection for a single ball can be found and then the total force in the bearing can be calculated as this ball rotates round the inner ring. In the model, the contacts of balls to the inner and outer races are represented by nonlinear contact springs, i.e. balls act as massless springs. The elastic model of the bearing is represented by Fig.(3.14). Any deflections in any direction will produce a compressive force in that direction. Therefore, the first step in calculating the total force is to find the deflection of each ball due to the movement of the shaft centre.

![Fig.(3.14). Nonlinear Elastic Model of the Bearing](image)

If the force on the $i$ th ball in Fig.(3.5) is calculated, the reference axes should be set and the total deflection and hence forces with respect to these axes should be calculated. The deflection for each ball can be calculated as described in the previous section (see Section 3.3.2). Having calculated the deflection for the $i$ th ball in its contact direction, the force in the same direction ($F_i$) can easily be found. This force can be split into two components in the radial ($F_{ri}$) and axial ($F_{ai}$) directions for angular contact ball bearings.

Hence the force in the $X$ direction is:

$$F_x = F_r \cos \theta_i$$

(3.36)

where $\theta_i$ is the angle between the $X$ axis and the axis of the $i$ th ball and this angle is combination of different angles as shown in Fig.(3.15).
Fig. (3.15) The Reference axes

It is difficult to find the angle $\theta_i$ since the balls are also rotating round the inner ring as time passes. Therefore, three sets of reference axes are described as shown in Fig. (3.15). The first one ($X, Y$ and $Z$) is fixed in space and the $X$ axis is vertically downwards in the gravitational force direction. The second set of the axes ($u, v$ and $w$) can be defined arbitrarily in the space. It is considered stationary with the cage turning relative to it. In our case the $u$ axis has an off-set angle of $\nu$ anti-clock-wise difference to the $X$ axis. But in this thesis $\nu$ is assumed to be equivalent to zero unless otherwise stated. The last set ($x, y$ and $z$) are set on the shaft and rotating at the speed of one of the balls without any disturbance from the rotations of the ball around its own axes. Under normal conditions all balls are rotating in a cage at the cage speed around the inner ring and this set of axes is also rotating at the cage speed. Hence this set of axes is apart from the second reference axes at the angle of $\omega_c t$. The ball calculated ($i$ th ball) is, now, found easily since it is fixed with respect to the last set of axis. If the angle between two balls is defined as $\gamma$, it will be:

$$\gamma = \frac{2\pi}{m}$$

(3.37)

where $m$ is the number of balls in the bearing.
As a result the angle $\theta_i$ can be expressed as

$$\theta_i = \theta + \omega_z t + i \gamma$$ \hspace{1cm} (3.38)

Substituting Eqn.(3.38) into Eqn.(3.36), the forces along the main fixed axes are:

$$W_{x_i} = W_i \cos(\theta + \omega_z t + i \gamma)$$ \hspace{1cm} (3.39)

$$W_{y_i} = W_i \sin(\theta + \omega_z t + i \gamma)$$ \hspace{1cm} (3.40)

$$W_{z_i} = W_i$$ \hspace{1cm} (3.41)

The total force in any direction will be the summation of all forces in that direction:

$$W_{x_i} = \sum W_i \cos(\theta + \omega_z t + i \gamma)$$ \hspace{1cm} (3.42)

$$W_{y_i} = \sum W_i \sin(\theta + \omega_z t + i \gamma)$$ \hspace{1cm} (3.43)

$$W_{z_i} = \sum W_i$$ \hspace{1cm} (3.44)

### 3.4 Dynamic Modelling of a Vibrating Shaft Supported by Ball Bearings

Vibrations can be classified in several ways, for example free and forced vibration, undamped and damped vibration, linear and nonlinear vibration etc. If all basic components of a vibratory system behave linearly, the resulting vibration is known as "linear vibration". In linear systems cause and effect are related linearly whereas in a nonlinear system this relationship is no longer proportional. Hence there is a number of oscillatory behaviours that cannot be explained by linear theory [Thomson, 1988; Stoker, 1950].

The differential equation describing a nonlinear oscillatory system may have the general form:

$$\frac{\partial^2 x}{\partial t^2} + f \left( \frac{\partial x}{\partial t}, x, t \right) = 0$$ \hspace{1cm} (3.45)
For this sort of equation the principal of superposition does not hold for their solution [Thomson, 1988; Stoker, 1950].

The vibrations of a shaft-bearing system has a nonlinear nature due to the Hertzian contact relation of ball to race elastic contacts and the nonlinearity of the elastomers employed as external dampers.

3.4.1 Assumptions

The real shaft-ball bearing system described above is generally very complicated and difficult to model. Thus, in order to study this real system a number of simplifying assumptions have to be made. The assumptions made in this thesis are as follows:

3.4.1.1 A maximum of five degrees of freedom system is assumed for the shaft with radial vibrations in the $x$ and $y$ directions, axial vibrations in the $z$ direction and two rocking motions about the $x$ axis and $y$ axis ($\psi$ and $\phi$ respectively). The sixth degree of freedom is omitted by assuming that the friction between the rolling elements and races is negligible (i.e., torsional vibrations —about the $z$ axis— are omitted).

3.4.1.2 The rolling elements (i.e. balls) are assumed to be massless in order to eliminate $6m$ degrees of freedom (where $m$ is the number of balls in a bearing).

3.4.1.3 The rings are flexurally rigid and undergo only local deformation due to the contact stresses.

3.4.1.4 Deformations occur according to the Hertzian theory of elasticity (i.e. elasto-static footprints and pressure distributions are perfectly elliptical).

3.4.1.5 Rolling elements are positioned equi-pitched around the inner ring and there is no interaction between them.

3.4.1.6 The sources of damping contributions either from the elastrohydrodynamic film at the contacts or any friction present, between the cage and the rolling elements and in various races to shaft and housing joints are assumed to be negligible since their contribution to vibration damping is small within the frequency range of interest in this thesis.

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3.4.1.8 The outer ring is assumed to be firmly fitted to a rigid housing, thus eliminating its flexural vibrations which generally cause high frequency vibration, while the inner ring is firmly fitted to the shaft and can be assumed to be a part of it.

3.4.1.9 The shaft is considered to be rigid in order to eliminate its flexural natural modes and hence only the first natural mode (rigid body mode) of the shaft appears in the vibration spectrum. Since the other modes are not in the range of running speed, with this assumption, while the problem is relatively easier to solve, only the small effects of the higher modes on the peaks in the frequency range of interest are neglected.

3.4.1.10 The pitching and yawing angles \( \phi \) and \( \psi \) are assumed to be small, as are \( x \), \( y \) and \( z \).

3.4.1.11 For a three degrees of freedom system analysis, both bearings are positioned symmetrically such that they move in the same direction simultaneously. In other words the balls are assumed to be in phase (i.e. there is no rocking motions about the \( x \) or \( y \) axis).

### 3.4.2 Equations of Motion

The equations of motion are based on the second law of dynamics which leads to the net force or moment on a mass balanced by the acceleration of mass. In a shaft bearing system there are three kinds of applied forces. The first is external force. The second force is due to deflected balls acting on the inner and outer rings and was discussed earlier in this chapter. The third is due to damping in the system. In this research elasto-hydrodynamic forces are omitted and the external damping forces due to the external elastomer dampers will be investigated in Chapter 5.

The equations of motion in five degrees of freedom system can be written as follows (see Figs. (3.16), (3.17) and Fig. (3.18)):

\[
M \ddot{x} + \sum_{i=1}^{3} \left( K_i (\delta_i)^3 \cos(\alpha_i) \cos(\theta_i) \right)_L + \sum_{i=1}^{3} (K_i (\delta_i)^3 \cos(\alpha_i) \cos(\theta_i) )_R + Q_x - Mg = 0
\]

(3.46)
Fig. (3.16) Forces on the shaft-ball bearing system
Fig (3.17) Vectorial representation of the speeds due to the movement of the shaft centre
\[ M \dddot{y} + \sum_{i=1}^{n} \left( K_i (\delta_i)^2 \cos(\alpha_i) L \sin(\theta_i) \right) + Q_y = 0 \]  
(3.47)

\[ M \dddot{z} + \sum_{i=1}^{n} \left( K_i (\delta_i)^2 \sin(\alpha_i) L \right) + \sum_{i=1}^{n} \left( K_i (\delta_i)^2 \sin(\alpha_i) R \right) + Q_z = 0 \]  
(3.48)

\[ I_{yy} \dddot{\phi} + \sum_{i=1}^{n} \left( K_i (\delta_i)^2 \cos(\alpha_i) L \cos(\theta_i) \right) a_i 
- \sum_{i=1}^{n} \left( K_i (\delta_i)^2 \cos(\alpha_i) R \cos(\theta_i) \right) b_i + \sum_{i=1}^{n} \left( K_i (\delta_i)^2 \sin(\alpha_i) L \cos(\theta_i) \right) R 
- \sum_{i=1}^{n} \left( K_i (\delta_i)^2 \sin(\alpha_i) R \cos(\theta_i) \right) R - Q_y (a_i + a) - I_{xx} \dot{\psi} \omega = 0 \]  
(3.49)

**Fig. (3.18) Definition of axes and rocking angles**
\[ I_m \ddot{\psi} + \sum_{i=1}^{n} \left( K_i (\delta_i)^3 \cos(\alpha_i) \sin(\theta_i) L \right) a_i \]
\[ -\sum_{i=1}^{n} \left( K_i (\delta_i)^3 \cos(\alpha_i) \sin(\theta_i) R \right) b_i + \sum_{i=1}^{n} \left( K_i (\delta_i)^3 \sin(\alpha_i) \sin(\theta_i) L \right) R \]
\[ -\sum_{i=1}^{n} \left( K_i (\delta_i)^3 \sin(\alpha_i) \sin(\theta_i) R \right) R - Q_0 (a_i + a) + I_\omega \dot{\phi} \omega = 0 \quad (3.50) \]

where \( \theta_i \) is as defined in Equ.(3.43) and therefore equations of motion are time dependent.

\[ \theta_i = \dot{\theta} + \omega_i t + i \gamma \quad (3.43) \]

### 4.4 Dimensionless Groups

For the convenience, the equations of motion are normalised. This is necessary because otherwise a change in one value of parameters implies a change from one system to another within the given class of problem and hence the comparison of the results becomes inadequate. This is particularly true for relatively complicated system such as for five degrees of freedom in our case.

Redefining the variables in the equations of motion as:

\[ x^* = \frac{x}{\delta_o}, \quad y^* = \frac{y}{\delta_o}, \quad z^* = \frac{z}{\delta_o}, \quad \delta_i^* = \frac{\delta_i}{\delta_o}, \quad \phi^* = \phi, \quad \psi^* = \psi, \quad t^* = \omega t, \quad K_i^* = \frac{K_i}{K_o} \]

and the constants as:

\[ a_i^* = \frac{a_i}{D}, \quad b_i^* = \frac{b_i}{D}, \quad R^* = \frac{R}{D} \]

Substituting these definitions in Equ.(3.46) through Equ.(3.50) will reveal the equations of motion in nondimensional form as follows:

\[ M^* \ddot{x}^* + \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^3 \cos(\alpha_i) \cos(\theta_i) L \right) \]
\[ + \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^3 \cos(\alpha_i) \cos(\theta_i) R \right) + Q_0^* = 0 \quad (3.51) \]
\[ M^* \dddot{y}^* + \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \cos(\alpha_i) \sin(\theta_i)_L \right) + \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \cos(\alpha_i)_R \sin(\theta_i)_R \right) + Q_i^* = 0 \]  
(3.52)

\[ M^* \dddot{z}^* + \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \sin(\alpha_i)_L \right) + \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \sin(\alpha_i)_R \right) + Q_i^* = 0 \]  
(3.53)

\[ I^* \dddot{\phi}^* + a \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \cos(\alpha_i) \cos(\theta_i)_L \right) - b \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \cos(\alpha_i)_R \cos(\theta_i)_R \right) + R \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \sin(\alpha_i)_L \cos(\theta_i)_L \right) - R \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \sin(\alpha_i)_R \cos(\theta_i)_R \right) - T^*_i - \frac{1}{\kappa^*} \ddot{\psi}^* = 0 \]  
(3.54)

\[ I^* \dddot{\psi}^* + \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \cos(\alpha_i)_L \sin(\theta_i)_L \right) - \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \cos(\alpha_i)_R \sin(\theta_i)_R \right) + \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \sin(\alpha_i)_L \sin(\theta_i)_L \right) - \sum_{i=1}^{n} \left( K_i^* (\delta_i^*)^2 \sin(\alpha_i)_R \sin(\theta_i)_R \right) - T^*_i + \frac{1}{\kappa^*} \ddot{\phi}^* = 0 \]  
(3.55)

where dimensionless groups can be defined as follows:

\[ M^* = \frac{M \omega^2}{K_0 \delta_0^2} \]  
(3.56)

(dimENSIONLESS MASS)

\[ Q_i^* = \frac{Q_i}{K_0 \delta_0^2} \]  
(3.57)

(dimENSIONLESS EXTERNAL FORCE IN THE X DIRECTION)
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\[ Q_y^* = \frac{Q_y}{K_0 \delta_0^2} \]  \hspace{1cm} (3.58)

(dimensionless external force in the \( Y \) direction)

\[ Q_z^* = \frac{Q_z}{K_0 \delta_0^2} \]  \hspace{1cm} (3.59)

(dimensionless external force in the \( Z \) direction)

\[ g^* = \frac{Mg}{K_0 \delta_0^2} \]  \hspace{1cm} (3.60)

(dimensionless acceleration due to gravity)

\[ I^* = \frac{I_x \omega^2}{3} = \frac{I_y \omega^2}{3} \quad \frac{DK_0 \delta_0^2}{DK_0 \delta_0^2} \]  \hspace{1cm} (3.61)

(dimensionless moment of inertia about the \( X \) or \( Y \) axis)

\[ T^*_x = \frac{Q_z(a+y)}{K_0 \delta_0^2 D} \]  \hspace{1cm} (3.62)

(dimensionless moment stiffness factor about the \( X \) axis)

\[ T^*_y = \frac{Q_z(a+y)}{K_0 \delta_0^2 D} \]  \hspace{1cm} (3.63)

(dimensionless moment stiffness factor about the \( Y \) axis)

\[ I^*_u = \frac{I_u \omega^2}{3} \quad \frac{DK_0 \delta_0^2}{DK_0 \delta_0^2} \]  \hspace{1cm} (3.64)

(dimensionless moment of inertia about the \( Z \) axis)

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4.6 Computational Solution of the Equations

The solution of equations of motion (Equ.(3.51) through Equ.(3.55)) is obtained using the Runge-Kutta iterative method, which was described earlier, since Equ.(3.51) through Equ.(3.55) are nonlinear and the direct substitution technique does not hold for them. In the Runge-Kutta method the initial conditions have crucial importance and have to be considered carefully.

4.6.1 Initial Conditions

For numerical solutions, the iteration starts from an initial point at which the conditions are known as initial conditions. Initial conditions and step size are very important for successive and economic computational solutions. Particularly for nonlinear systems, different initial conditions mean a totally different system and hence different solutions. Wrongly selected initial conditions can also cause larger computation time (CPU) or sometimes wrong results.

The larger the time step, the faster the CPU time. On the other hand the time step should be small enough to achieve an adequate accuracy. Moreover, very small time steps can increase the truncation errors and result in wrong results. Therefore an optimisation should be made between them.

At time \( t = 0 \) the following assumptions are made:

1. The shaft is held at the centre of the bearing such that there is no radial load on the balls and all balls are assumed to have equal axial preload and a preloaded contact angle \( \alpha_p \). The preload, as described earlier in this chapter, is:

\[
P_{\text{rs}} = m K_i \left[ B d_b \left( \frac{\cos(\alpha_p)}{\cos(\alpha_p)} - 1 \right) \right]^{\frac{3}{2}} \sin(\alpha_p)
\]  

(3.18)

and the initial deflection, as described earlier in this chapter, is:

\[
\delta_0 = B d_b \left( \frac{\cos \alpha_p}{\cos \alpha_p} - 1 \right)
\]  

(3.23)

The preloaded contact angle \( \alpha_p \) can be calculated numerically as described earlier in this chapter using Equ.(3.18).
2. For fast convergence the initial displacements are set to the following values:

\[ x_0^* = \frac{10^{-6}}{\delta_0}, \quad y_0^* = \frac{10^{-7}}{\delta_0}, \quad z_0^* = \frac{10^{-4}}{\delta_0}, \quad \phi_0^* = 0 \quad \text{and} \quad \psi_0^* = 0 \]

The velocities are assumed to be zero:

\[ \dot{x}_0^* = 0, \quad \dot{y}_0^* = 0, \quad \dot{z}_0^* = 0, \quad \dot{\phi}_0^* = 0 \quad \text{and} \quad \dot{\psi}_0^* = 0 \]

As the shaft is released, the displacements, velocities and accelerations are obtained. The initial accelerations are obtained using Equ.(3.51) through Equ.(3.55) and the above initial displacements and velocities.

3. When \( t > \Delta t \) the initial conditions have already passed and the normal procedure commences.

### 4.6.2 Computer Flow Chart and the Solution Procedure

For the successful solution of the equations of motion, the following procedure is followed:

1. Read all the necessary data of the shaft-bearing system.
2. All the variables are calculated using the above stated initial conditions.
3. The contact stiffness factor is calculated for these conditions.
4. Calculate the stiffness and damping coefficients of external dampers (i.e. elastomers).
5. Calculate the approximate natural frequencies of the shaft bearing system using a linearised model.
6. The Runge-Kutta method is commenced to calculate values of \( x^*, y^*, z^*, \phi^*, \psi^* \) and their derivatives with respect to time.
7. The deflections in the contact direction of each ball and the corresponding contact angles are calculated from Equ.(3.32) through Equ.(3.35).
8. Considering new contact angles the new dimensionless stiffness coefficients are calculated for each ball.
9. The dimensionless contact load in the direction of each contacting member is next calculated using Eqn. (3.11).

\[ W_i^* = K_i^* (\delta_i^*)^3 \quad (3.11) \]

10. The dimensionless total restoring forces in the \(X\), \(Y\), \(Z\) directions are obtained from:

\[ W_{x_i}^* = \sum_{i=1}^{m} W_i^* \cos(\alpha_i) \cos(\vartheta + \omega_c t + i \gamma) \quad (3.42) \]

\[ W_{y_i}^* = \sum_{i=1}^{m} W_i^* \cos(\alpha_i) \sin(\vartheta + \omega_c t + i \gamma) \quad (3.43) \]

\[ W_{z_i}^* = \sum_{i=1}^{m} W_i^* \sin(\alpha_i) \quad (3.44) \]

11. Equations of motion Eqns. (3.51) through (3.55) are solved to find new values of \(x^*\), \(y^*\), \(z^*\), \(\phi^*\), and \(\psi^*\).

12. Steps 7 to 11 are repeated for the required time steps.

13. The results are recorded and their FFT is computed to obtain the vibrations in the frequency domain.

**Closure**

In this chapter the deflection and the force due to this deflection on the \(i\)th ball was calculated from contact mechanics and geometrical considerations as a precursor to vibrations of the shaft-ball bearing system. The method to find the total force on a bearing was also defined. The equations of motion were derived in terms of dimensionless groups. The solution procedure was also shown.

Solutions can be represented in time and frequency domains. When the solution is investigated a series of peaks at different frequencies can be observed. Some of them will be investigated in more detail in the next chapter.
CHAPTER 4

VIBRATION FREQUENCIES
OF A SHAFT-BEARING SYSTEM

4.1 Introduction

The differential equations representing the shaft-bearing system and their solution were defined in the previous chapter. The solutions to these equations result in shaft oscillations that possess a number of peaks at different frequencies. These frequencies will be investigated in this chapter.

The frequencies resulted from the differential equations may be divided into two main categories [Gustafsson and Tallian et al, 1963]:

4.1.1 Frequencies generated by a geometrically perfect (i.e. defect free) bearing. These frequencies are inherent of the system or in other words they are the characteristics of the system. This can be further divided into two subcategories. The first is the natural frequency of the system due to the shaft mass and nonlinear contact stiffness and the second is the vibrations induced by the ball passage which is one of the important system characteristics and occurs at all shaft speeds.

4.1.2 Frequencies generated by geometrical imperfections (i.e. manufacturing malfunctions). These frequencies may be considered as a result of vibrations that are forced by the geometrical imperfections. Vibrations due to wavy rolling surfaces or an out of balance shaft centre can be cited as good examples of these vibrations.
4.2 The Inherent System Frequencies

4.2.1 System Natural Frequency

The system natural frequency is a function of mass and the stiffness. For linear systems this relation is defined as:

\[ f_n = \frac{1}{2\pi} \left( \frac{K}{M} \right)^{\frac{1}{2}} \sqrt{1 - \zeta^2} \]  \hspace{1cm} (4.1)

where \( \zeta \) is the viscous damping factor and defined as:

\[ \zeta = \frac{1}{2\sqrt{K M}} \]  \hspace{1cm} (4.2)

Damping in moderate amounts has little influence on the natural frequency and may be neglected in its calculations. The effect of damping is mainly evident in the diminishing of the vibration amplitude with time. For most systems the vibratory mass of the system is constant for a given system and the stiffness coefficient, \( K \), will therefore be the predominant cause of the natural frequency. While for a given linear system, the natural
frequency is constant, the initial deflection of the vibratory mass will determine the natural frequency for nonlinear systems where the amount of nonlinearity of the springs will have crucial importance. An approximate natural frequency for a shaft-ball bearings system around a constant deflection was given in Appendix 10.

Nonlinear springs are classified into two groups. The first group is hardening springs and the second is softening springs as shown in Fig.(4.1). For a linear system the restoring force curve linearly changes with the displacement (see Fig.(4.1)). If the slope increases as the load increases, the spring is said to be hardening (see Fig.(4.1)) whereas if the slope decreases as the load increases, the spring is called a softening spring. For variable stiffness, there are certain regions where the system is unstable. For a softening spring, with increasing excitation frequency, the amplitude gradually increases until a point where a sudden jump to a larger value takes place. In the case of hardening spring a sudden jump to a smaller value is observed [Harris and Crede, 1961]. The shaded region in Fig.(4.2) is "unstable" where a sudden increase or decrease in the amplitude is observed as the significant feature. The extent of instability depends on a number of factors such as the amount of damping present, the rate of the change of the excitation frequency, the extent of spring nonlinearity etc. [Thomson, 1981].

In the case of ball bearings, jump behaviour can only be observed when there is a very large mass present which will force the system to oscillate at large amplitudes and hence it will exhibit the effect of nonlinearity due to Hertzian contact. For small masses the system will tend to act more as a linear system since the local deflection is small (see Fig. (4.3)).

![Fig.(4.2) Jump Behaviour of Nonlinear Systems](image-url)
Another important behaviour of nonlinear vibrations is that permanent oscillations whose frequency is a fraction \(1/2, 1/3, \ldots, 1/n\) of the exciting force or displacement can also occur. It is termed as *sub-harmonic vibration* of frequency. In the same manner, *super harmonic vibration* of frequency equal to some multiple \(2, 3, \ldots, n\) of the exciting frequency, can also occur [Stoker, 1950; Den Hartog, 1985; Harris and Crede, 1976].

It should be noted that sub-harmonics may occur only under very special conditions while super harmonics always occur and damping may completely inhibit the existence of subharmonic vibrations for a linear system [Stoker, 1950]; but for a nonlinear system it can occur even in the presence of viscous damping [Harris and Crede, 1976].

![Graph](image1)

*Fig.(4.3) Large and small oscillations*

System stability is always of prime importance for linear as well as nonlinear systems. In the case of nonlinear systems, due to time dependent variables, slight disturbance from the equilibrium state causes peculiar vibrations which are referred as "parametric vibrations". System can easily become unstable, if this disturbance takes place close to the jump behaviour region [Harris and Crede, 1976; Rao, 1984].

### 4.2.2 Cage Speed

Considering pure rolling, the absolute speed of the point \(i\) (see Fig.(4.4)), located on the circumference of a rolling element, is equal to the circumferential speed \(V_i\) of the inner ring raceway at the same point \(i\) which is on the circumference of the inner ring race way as well. In the same manner the point \(o\), which is the contact point of a rolling element and the outer ring circumferences and share them both has the absolute speed of \(V_o\). The absolute speed of each point on the ball is a combination of the speed of the ball around its spin axis and its orbital speed round the bearing axis.
Therefore, the absolute velocities of points between \( i \) and \( o \) are, as shown in Fig. (4.4), parallel to each other and vary linearly about the instantaneous axis of zero velocity. The velocity of inner and outer races respectively can be expressed as follows:

\[
V_i = \omega_i r_i \tag{4.3}
\]

\[
V_o = \omega_o r_o \tag{4.4}
\]

Two other equations can be written using the angular speed about the instantaneous axis \( CL \), of zero velocity, as follows:

\[
V_i = \omega_{CL} \left( r_m - q - \frac{d_b}{2} \cos \alpha \right) \tag{4.5}
\]

\[
V_o = \omega_{CL} \left( r_m + q + \frac{d_b}{2} \cos \alpha \right) \tag{4.6}
\]

where \( \alpha \) is the bearing contact angle (see Fig. (4.5)). Solving Equ. (4.5) and Equ. (4.6), the unknowns \( q \) and \( \omega_{CL} \) can be found to be:
\[ q = r_m - \frac{V_m + V_i}{V_m - V_i} \frac{d_k}{2} \cos \alpha \]  
(4.7)

\[ \omega_c = \frac{V_m - V_i}{d_k \cos \alpha} \]  
(4.8)

The velocity of the ball centre can be calculated from

\[ V_m = \omega_c (r_m - q) \]  
(4.9)

Substituting Equs.(3 and 5) into Equ.(6):

\[ V_m = \frac{V_i + V_x}{2} \]  
(4.10)

The centre of the ball rotates around the bearing axis at a speed which is known as cage speed because the cage also rotates at the same speed. The cage speed can be expressed as

\[ \omega_c = \frac{V_m}{r_m} \]  
(4.11)

From Fig.(4.5)

\[ r_i = r_m - \frac{d_k}{2} \cos \alpha \]  
(4.12)

\[ r_o = r_m + \frac{d_k}{2} \cos \alpha \]  
(4.13)

Substituting Equs.(4.3, 4.4, 4.12 and 4.13) into Equ.(4.11) the cage speed can be obtained as

\[ \omega_c = \frac{1}{2} \omega_i \left[ 1 - \frac{d_k}{d_m} \cos \alpha \right] + \frac{1}{2} \omega_o \left[ 1 + \frac{d_k}{d_m} \cos \alpha \right] \]  
(4.14)

where

\[ \omega_i = \frac{2 \pi n_i}{60} \quad \text{and} \quad \omega_o = \frac{2 \pi n_o}{60} \]
4.2.3 Ball Rotational Speed

Considering the inner and outer rings have different rotational speeds and the speed vectors are perpendicular to the rotation plane, it can be shown that the outer and inner rings and the ball centres, i.e. cage, have their angular speed vectors as shown in Fig.(4.5).

The balls rotate around their own axis as well as instantaneous axes of zero velocity. Therefore the ball speed can be split into its two components as shown in Fig.(4.5). The vector $\omega_b$ is the actual ball rotating speed around its own axis and $\omega_a$ is the speed at which the balls spin about their own axis while they are precessing round the inner race. Combination of these two speeds is $\omega_*$ which is the ball rotational speed in the cage speed plane. Hence from Fig.(4.5):

$$\omega_* = \omega_a \cos(\alpha)$$

(4.15)
This is caused by the rotation of the ball centre about the bearing centre. Therefore, the effect of this speed and the cage speed should produce the same velocity with respect to the instantaneous axis of zero velocity. Hence

\[ \omega_s = \frac{q}{r_m - q} \omega_c \]  

(4.16)

Substituting Eqn.(4.16) into Eqn.(4.15) and considering Eqn.(4.7) and Eqn.(4.14):

\[ \omega_h = \frac{1}{2} \frac{d_m}{d_b} (\omega_c - \omega_i) \left[ 1 - \frac{d_b^2}{d_m^2} \cos^2 \alpha \right] \]  

(4.17)

4.2.4 Relative Speeds

Assuming that both inner ring and outer rings are rotating at speeds of \( \omega_i \) and \( \omega_o \) respectively, cage to inner ring relative speed is

\[ \omega_{ci} = \frac{1}{2} \omega_i \left[ 1 + \frac{d_b}{d_m} \cos \alpha \right] + \frac{1}{2} \omega_o \left[ 1 + \frac{d_b}{d_m} \cos \alpha \right] \]  

(4.18)

and cage to outer ring relative speed is

\[ \omega_{co} = \frac{1}{2} \omega_i \left[ 1 - \frac{d_b}{d_m} \cos \alpha \right] + \frac{1}{2} \omega_o \left[ 1 - \frac{d_b}{d_m} \cos \alpha \right] \]  

(4.19)

4.2.5 Ball Passage Frequency

The effect of the BPF is present even if the rolling surfaces are perfect. While the shaft is rotating, applied loads are supported by a few balls restricted to a narrow load region, Fig.(4.6), and the radial position of the inner ring with respect to the outer ring depends on the elastic deflections at the ball to race way contacts. Balls are deformed as they enter the loaded zone where the mutual convergence of bearing rings takes place and the balls rebound as they move to the unloaded region as shown in Fig.(4.6). As the positions of the balls change with respect to the applied load vector when they move from the loaded to the unloaded zone, the load distribution on the shaft changes thus producing a relative movement between inner and outer rings, i.e. a periodic relative motion between the rings must occur even though the bearing is geometrically perfect.
The situation is best illustrated by considering the change in angular position of a ball set with respect to a point on the load line as seen in Fig. (4.6).

The bearing on the left in Fig. (4.6) shows the situation of the balls for a set of 6 balls at time \( t = 0 \). Ball 1 is immediately under a gravity load. The bearing in the middle in Fig. (4.6) shows the situation at the time \( t = \frac{1}{2} \times \frac{\text{time for a complete cage rotation}}{m} \) where \( m \) is the number of balls. In this case only two balls are supporting the gravity load as the ball on the right and left become perpendicular to the load line. The radial deflection is different on account of their being now 2 loaded bearing balls instead of 3 as previously. The shaft therefore moves downwards in response to the decreased contact spring stiffness force (which is also nonlinear) as shown in Fig. (4.6). This change occurs cyclically as shown with the shadowed line in Fig. (4.6). To do this the shaft must accelerate so a vibration occurs as it is a mass supported by springs.

The shaft and inner raceway have approached closer to the outer race and housing in the time taken for one half of the ball spacing to pass a point on the outer raceway to reach the position of the ball set shown in the middle bearings in Fig. (4.6).

When Ball 6 comes to the initial location of Ball 1, the shaft regains its initial position as shown in Fig. (4.6) for the bearing on the right. The time passed for the shaft's regaining its initial position is:

\[
t = \frac{\text{time for a complete rotation of the cage}}{m} \quad (4.20)
\]
Hence the shaft will be excited at the frequency of \( m \omega_c \) where \( m \) is the number of balls and \( \omega_c \) is the cage speed. This frequency is called \textit{ball passage frequency (BPF for short)}. In mathematical terms, the BPF can be described as the cage speed times the number of balls.

\[
\omega_{np} = m \omega_c
\]  

(4.21)

It is relatively more difficult to visualise the case for the \textit{inner ring ball passage frequency}. It is described as the passage of ball set from a point on the inner ring surface and can be given as:

\[
\omega_{np} = m (\omega_i - \omega_c)
\]  

(4.22)

The vibrations associated with the BPF is called the elastic compliance vibrations and two dimensional since the rotating ball springs and asymmetry. The vibration is also non-sinusoidal because of spring nonlinearity.

The effect of the BPF can be worst when it coincides with a natural frequency of the shaft-bearing system. This is also demonstrated in Appendix 7.

\textbf{4.3 Manufacturing Malfunctions}

In spite of very high standards in the manufacturing of ball bearings, there are still many errors that are unavoidable. Therefore certain production anomalies in the geometric shape or dimensions exist even in high precision ball bearings. Some of the malfunctions are as follows:

\textit{4.3.1. Waviness,}
\textit{4.3.2. Out of round rolling members,}
\textit{4.3.3. The presence of off-sized rolling elements,}
\textit{4.3.4. Misalignment of the bearing rings,}
\textit{4.3.5. Out of balance of the shaft centre.}

Some of these anomalies were investigated by different research workers [Wardle, 1988; Ellis, 1970; Meyer et al., 1980 etc.]. However they need further investigations in order to understand them better. Some other topics were not thoroughly investigated such as the flexural vibrations of outer race and the influence of damping and friction on vibration levels. Particularly for the high speed applications, these effects influence the vibration of the system but since these anomalies were found to be less related to the aims of the research, they were left out.
In this thesis the available theories for some anomalies such as waviness, off-sized ball, defects on the running surfaces, misalignment will be extended and some new attempts will be made in order to understand the effect of these anomalies on the vibration characteristics of shaft bearings systems.

### 4.3.1 Waviness

Waviness is a manufacturing imperfection. It may be caused by different manufacturing malfunctions such as uneven wear of the wheel in grinding operations, variable interactions between the tool and work piece, vibrations of machine elements or movements of the work in the fixture.

![Waviness of the bearing](image)

*Fig.(4.7) Waviness of the bearing*

Waviness is normally in the form of peaks and valleys of varying height and width. This causes problems in the mathematical modelling of waviness effect. A statistical approach is necessary in order to have a complete solution. If the rings are assumed to bend due to the ball loads then the flexural vibrations of the rings as well as the rigid body motion have
to be considered. To avoid these problems the inner and outer rings are assumed not to bend under the ball loads and a perfect sinusoidal wavy surface is assumed with a regular peaks and valleys of constant height and width. Furthermore, the wave length is assumed to be much greater than the ball to race foot-print width and the wave geometry itself is assumed to be unaffected by contact distortion.

\[ \text{Fig. (4.8) One wave of the inner ring} \]

Waves are described in terms of two important parameters; the wavelength, $\lambda$, which is the distance taken up by a single cycle of a wave, and its amplitude, $I$.

For geometrically perfect surfaces the wavelength is infinite ($\lambda = \infty$). For an imperfect surface with $N$ waves, the wavelength is inversely proportional to the number of waves, $N$.

When a ball is moving round the inner ring, it follows the rolling surface contours continuously. The point of interest is the amplitude of the wavy surface with respect to central point at a certain angle from the reference axis. Waviness exists on all surfaces. To simplify the problem, first the inner ring waviness and then the outer ring and ball waviness will be considered.

4.3.1.1 Inner ring Waviness

If it is assumed that the inner race surface has a circumferential sinusoidal wavy feature, the radial clearance consists of a constant part and a variable part. The amplitude at the angle $\theta$ (see Fig. (4.7)) is directly related to the wavelength. This can be clearly seen if
one wave is flatten out (see Fig.(4.8)). Since it is assumed the wave is perfectly sinusoidal the amplitude will vary between zero and the maximum amplitude and can be express as follows:

$$\Gamma_i = \Gamma_p \sin \left( \frac{2\pi L}{\lambda} \right)$$  \hspace{1cm} (4.23)

where $\Gamma_p$ is the amplitude of the wave. If there is a constant interference or clearance applied, Eq.(4.23) will take the form of:

$$\Gamma_i = \Gamma_0 + \Gamma_p \sin \left( \frac{2\pi L}{\lambda} \right)$$  \hspace{1cm} (4.24)

In the case of angular contact ball bearings, $\Gamma_0$ is replaced by the deflection due to the preload. Eq.(4.24) consists of a constant part and a variable part. From Fig.(4.7):

$$L = r \theta'$$  \hspace{1cm} (4.25)

Fig.(4.9) Relation of the angles
Chapter 4

Vibration Frequencies

Since the wavelength is the length of the inner race circumference divided by the number of waves on the circumference.

\[ \lambda = \frac{2\pi r}{N} \]  
(4.26)

Substituting Equ.(4.25) and Equ.(4.26) into Equ.(4.24):

\[ \Gamma_i = \Gamma_0 + \Gamma_p \sin\left(N \theta'\right) \]  
(4.27)

Since the inner race is moving at the speed of the shaft and the ball centre at the speed of cage, for the inner race waviness, \( \theta' \) should be replaced with an angle \( \theta_i \) for the \( i \)th ball (see Fig.(4.9)).

\[ \theta_i = \theta + (\omega_c - \omega_z)t + \gamma i \]  
(4.28)

If a point \( a \) on the circumference of the outer race and a point \( b \) at the ball centre are assumed at the initial time and initial position at an angle \( \theta \) apart from a reference axis, as seen in Fig.(4.9), after the time \( t \) taken, the cage, i.e., the point \( b \), will lag the shaft, i.e., the point \( a \), and as a result of this, the \( i \)th ball will be at the angle of \( -(\omega t - \omega_c t) \). Hence the instantaneous height at the angle of interest:

\[ (\Gamma_i) = \Gamma_0 + \Gamma_p \sin \left[N (\theta + (\omega_c - \omega)t + \gamma i)\right] \]  
(4.29)

4.3.1.2 Outer race waviness

In the case of outer race since the outer race is assumed to be stationary and the balls are rotating at the speed of cage, Equ.(4.29) will take the form of:

\[ (\Gamma_i)_0 = \Gamma_0 + \Gamma_p \sin \left[N (\theta + \omega_z t + \gamma i)\right] \]  
(4.30)
4.3.1.3 Ball Waviness

Balls are free to spin about any axis and this axis may even change during the rotation. In order to calculate the waviness of balls, a simple case is considered where a ball with a perfectly sinusoidal wavy surface rotates about an axis normal to the plane containing the centres of the inner and outer race contacts. In this case the instant amplitude of the ball waviness is (see Fig. (4.10)):

\[ a_i = a_p \sin (N \theta') \]  
\[ \text{(4.31)} \]

This will cause a change in the ball diameter through inner and outer contacts:

\[ \Delta d_b = 2 a_p \sin (N \theta') \]  
\[ \text{(4.32)} \]

where \( N = 2, 4, 6, \ldots \) (for \( N = 1, 3, 5, \ldots \), \( \Delta d_b = 0 \)).

The effect of ball waviness on the general clearance can be given from Fig. (4.10) as:

\[ (\Gamma_i)_b = 2 a_p \sin [N \omega_b t] \]  
\[ \text{(4.33)} \]

Note that in Eqn. (4.33) the location of ball has no effect on the ball waviness contribution to the clearance.
4.3.2 Off-sized Balls

An off-sized ball causes a different force excitation on the inner ring from the rest of the balls. This different force can be larger or smaller than the rest depending on the off-sized ball diameter. Fig.(4.11) shows that one ball has a greater diameter from the rest of the balls in the set. This ball will be forced to squeeze more in relation to the other balls and hence will produce a greater force from the rest. Hence for this ball the force produced will be:

$$W_{\text{eff}} = K(\delta + \Delta\delta)^{3/2}$$

(4.34)

where $\Delta\delta$ is the diameter difference of the off-sized ball.

As explained in Section 4.2.5 under the title of Ball Passage Frequency, each ball inserts its force on the shaft in turn. The off-sized ball will insert a different magnitude of force and hence will disturb the movement of the shaft. As the balls are moving at the cage speed, this will repeat itself for each cage rotation and produce a peak at the cage speed. The same thing is true for more than one off-sized ball but this may cause a change in the
amplitude of the peak. However, if the position of the balls in a set is arranged symmetrically, vibrations will be observed at the superharmonics of the cage speed. When the cage speed or its superharmonics coincide with the natural frequency, the system will resonate. More information on the vibrations associated with the off-sized balls can be found in Chapter 6.

### 4.3.3 Misalignment

Ball bearings are normally assumed to be correctly aligned (see Fig.(1.5.a)). Under operating conditions this is not always true. It is possible for the bearing to operate as shown in Fig.(1.5.b) where one ring is angularly displaced relative to the other, about an axis at right angles to the bearing running axis. This is a common fault in bearing operation and can severely affect the performance of the system. It can cause vibrations or excite vibrations at other frequencies.

![Misalignment of angular contact ball bearings](image)

*Fig.(4.12) Misalignment of angular contact ball bearings*
Chapter 4

There are many causes of misalignment. These can be placed into five categories [Ellis, 1970]: bearing errors, deflection of shaft and bearing housing, poor bearing arrangement or selection, machining errors and fitting errors.

Since misalignment was not commonly known to cause the severe vibrations and noise encountered, it was investigated from the view of cage and bearing failure [Barash, 1969; Ellis, 1970; Andréason, 1970; Brändlein, 1985]. More recently Wardle [Wardle and Poon, 1983; Wardle, 1988] identified the misalignment as a predominant effect on bearing noise and vibration.

The main reason for taking up misalignment into the scope of this work is that the detection of bearing misalignment before equipment is put into service is desirable. As the maximum amount of misalignment cannot be seen by the naked eye even if the bearing concerned is accessible, there is no universal, simple, quick method to examine misalignment. Even the experienced engineer can hardly detect misalignment without direct measurement, which is time consuming and expensive, and only used when large expensive equipment is involved. Moreover, this is only a static measurement and operational loads may cause misalignment that is unpredictable under dynamic conditions.

Therefore if the vibration signature due to a certain amount of misalignment is known, vibration monitoring can tell us of the presence of the misalignment and its magnitude.

4.3.3.1 Deflection due to Misalignment

Misalignment of the bearing is generally defined about an axis at right angles to the bearing running axis. This can be mathematically transformed to the bearing rotation centre as a deflection and an angle. Hence misalignment can be expressed in terms of an angular displacement from the centre of rotation axis.

In the general case, the maximum displacement due to misalignment will be at an angle \( \theta_c \) which depends on the way it is misaligned and can be found using the cosine rule (see Fig.(4.12)).

\[
\delta_c = \left[ \left( (R_1 + a) \cos(\rho) - (R_s + a) \right)^2 + \left( (R_1 + a) \sin(\rho) + B d_s \sin(\alpha_0) \right)^2 \right]^{1/2} - B d_s
\]

(4.35)
where $\rho$ is the misalignment angle. The resulting deflection on the $i$th ball is (see Fig. (4.12)):

$$\delta_i = \delta \cos \theta \cos \theta_i + \delta \sin \theta \sin \theta_i$$  \hspace{1cm} (4.36)

\textbf{4.3.3.2 Contact Angle Variation due to Misalignment}

In order to determine the effect of misalignment, the contact angle for each ball must be calculated. Considering a ball bearing whose inner ring is misaligned, as described in Fig. (4.12), the maximum contact angle variation is experienced by a ball at the angle $\theta_c$ (see Fig. (4.13)) or right opposite to it. The contact angle at the angle $\theta_c$ may be written from the definition of $\rho_i$ as:

$$\alpha_{\text{max}} = \arctan \left( \frac{\alpha_i\sin(\rho) + Bd_\alpha\sin(\alpha)}{\alpha_i\cos(\rho) - (\alpha_i + a)} \right)$$  \hspace{1cm} (4.37)

where $-R_i < a < R_i$ for the inner ring misalignment and $-R_o < a < R_o$ for the outer ring misalignment. From this, the contact angle due to misalignment for the $i$th ball can be found using effective misalignment angle for the $i$th ball, $\rho_{ei}$

$$\alpha_i = \arctan \left( \frac{(\alpha_i + a \cos(\pi + \theta_i - \theta_c))\sin(\alpha) + Bd_\alpha\sin(\alpha)}{(\alpha_i + a \cos(\pi + \theta_i - \theta_c))\cos(\alpha) - (\alpha_i + a)} \right)$$  \hspace{1cm} (4.38)

where

$\theta_c = \text{constant}$ \hspace{1cm} (for outer race misalignment) and

and

$\theta_c = \omega t$ \hspace{1cm} (for inner race misalignment)

From Fig. (4.13) for small $\rho$ and $\rho_{ei}$:

$$\tan(\alpha_i) = \frac{\delta_i}{R} = \frac{\delta \cos(\theta)}{R} = \tan(\rho \cos(\theta_i - \theta_c))$$  \hspace{1cm} (4.39)
Or considering the location (see Fig.(4.13)):

\[ \rho_{ei} = \rho \cos(\pi + \theta_e - \theta_c) \]  

(4.40)

Substituting Eqn.(4.40) into Eqn.(4.38):

\[ \alpha_i = \arctan \left( \frac{(R_i + a \cos(\pi + \theta_i - \theta_c)) \sin(\rho \cos(\pi + \theta_i - \theta_c)) + Bd \sin(\alpha_c)}{(R_i + a \cos(\pi + \theta_i - \theta_c)) \cos(\rho \cos(\pi + \theta_i - \theta_c)) - (\overline{R}_e + a)} \right) \]

(4.41)

Note that this is only for observing the variation of contact angle due to misalignment, because otherwise the change in contact angle is automatically included in the equations via the deflection due to misalignment.
4.3.4 Out of Balance of Mass Centre

In rotating machines a common source of vibration excitation is the off-centre or eccentricity of the mass centre, where the geometric centre of inner ring, o, is off-set by an eccentricity, e, from the geometrical centre of the shaft, as is shown in Fig.(4.14).

A theoretical investigation of the response of a rotating rigid shaft supported by a pair of radial deep groove lubricated ball bearings having rotating unbalance shaft centre was carried out by Rahnejat [1984]. The research showed that rotating unbalance at the shaft centre, excites the system at the shaft rotating speed and is more severe at high frequencies. When the shaft speed coincides with the natural frequency, resonance occurs. Some other researchers also reached the same conclusion [Wardle and Poon, 1983; Eisenbeis, 1985; Franco, 1990].

In modelling this behaviour, it was assumed that the force due to unbalance at the shaft centre was $M \omega^2 e$ in the direction o-o' and the gravitational force was in the x direction due to the mass of the shaft. The angle between the centrifugal force vector and the x axis is a function of the shaft speed and time. Therefore the out of balance forces acting
in the \( x \) and \( y \) directions are:

\[
(F_x)_T = F_x + Mg = M\left(g + \omega^2 e \cos(\omega t)\right) \tag{4.42}
\]

\[
(F_y)_T = F_y = M\omega^2 e \sin(\omega t) \tag{4.43}
\]

where subscript \( T \) stands for \textit{total}.

There is also a moment produced by this movement. This is (see Fig.(4.14)):

\[
T = Mg e \sin(\omega t) \tag{4.44}
\]

### 4.3.5 Defects on the Rolling Surfaces

The vibration signature produced by bearings can be very informative if its meaning is well understood. Since failure in rotating machines can be dangerous (in aircraft for example) and can lead to loss of time and money, machine health monitoring is employed by many companies to get the necessary information out of the vibration signature. Regular monitoring of machines points out the change in the state of the machine immediately since the condition of ball bearings can be judged by the nature of their vibration pattern [Karakurt, 1989].

The experimental studies showed that defects on bearing raceways, the cage, or balls cause unique vibration signals [Taylor, 1980]. However, the theoretical model for defect detection is not well established although some analytical solutions were presented [Igarashi and Hamada, 1982; Igarashi and Yabe, 1983; Igarashi and Kato, 1985; McFadden and Smith, 1984; 1985]. This is mainly because the problem is complex and more than one defects makes the problem very difficult to analyse. For this reason an attempt was made to model defects in the present study, in particular to see the effect of surface defects on system vibrations, since the available computational approach makes it easy to handle.

First a model is established for a single defect on the outer race surface, then a single defect on the inner race surface and finally a defect on the ball surface is modelled. The combination of these defects then is applied in the simulation.
4.3.5.1 A Defect on the Outer Race Surface

Let there be a defect on the surface of outer race at the angle $\varphi$ from the horizontal axis $X$. If $\omega_c i + \gamma (i = 1, m)$ coincides with the defect angle $\varphi$, a ball at its contact will have additional deflection $\delta_d$. Hence (see Fig.(4.15)):

$$\delta_d = \delta_i + \delta_d$$ (4.45)

For the computational convenience, following definitions are made:

1. Since the defect is not at a single point, to define the defect at a certain angle may cause problems in computation. Thus a tolerance band put on the angles. For example, for the outer race defect, the defect angle is defined as

$$\varphi_d = \varphi \pm \frac{w_d}{2r_o}$$ (4.46)

where $w_d$ is the width of the defect.
2. Since $\omega t + i \gamma$ is increasing continuously, its sine and cosine are simultaneously checked with those of the defect angle.

4.3.5.2 A Defect on the Inner Race Surface

Let there be a defect on the surface of inner race. This defect will rotate at the shaft speed, $\omega$, as the inner ring is forced fitted to the shaft. If the defect angle, $\omega t \pm w_d / (2r_i)$, coincides with one of the balls, $\omega_c t + i \gamma (i = 1, m)$, the deflection on that ball will be (see Fig. (4.16)):

$$\delta_{id} = \delta_i + \delta_d \quad (4.47)$$

![Fig.(4.16) Defect on the surface of inner race](image)

4.3.5.2 A Defect on the Ball Surface

From Fig. (4.17) it can be seen that when the cosine of the angle $(\omega_b + \omega_c) t$ equals zero, for that particular ball the deflection will be:

$$\delta_{id} = \delta_i + \delta_d \quad (4.48)$$
System frequencies can be classified under two main categories. The first being the vibrations of a geometrically perfect system and the second is vibrations due to the imperfections in the system. In this chapter these frequencies were mathematically defined.

In order to reduce the untoward effects of these frequencies to an acceptable level, elastomer external dampers will be employed in this thesis. Their dynamic properties will be investigated in the next chapter.
CHAPTER 5

ELASTOMERS AS EXTERNAL DAMPERS

5.1 Introduction

In the last two chapters, a vibratory model of a rigid shaft supported by a pair of angular contact ball bearings having no external damping was described. In the previous chapter, it was shown that the solution of the equations of motion would result in a series of vibration peaks at certain frequencies. One method of reducing untoward effects at these frequencies, is to introduce elastomers as external dampers. Hence, in this chapter, the dynamic properties of elastomers will be investigated and the system characteristic equations including external dampers, will be solved for a three degrees of freedom system.

Elastomers are an artificial rubber made up of polymer chains pinned together at points along their length by blobs of polystyrene formed by the coalescence of polystyrene sections along each chain. Unlike natural rubbers, they can be melted and formed into any desired shapes. Compared to thermo-setting polymers elastomers do not have enough cross-links to make them completely rigid, but do have enough to ensure they do not break under the heavy strains typical of natural rubbers [McCrum et al., 1989]. Like all polymers, elastomers are viscoelastic in nature, which means that they are very much rate and temperature dependent, a high strain rate giving a high stiffness in the same way that cooling the material down to below its transition temperature would do. Therefore, their internal damping mechanism enables them to be used as external dampers in many industrial applications.
Elastomers have advantages over oil-squeeze dampers because no oil pump circuitry is needed. They are suitable for shafts supported by ball bearings which have a lubricant sealed within them. Elastomers can also be used in engineering applications in order to keep down the amplitude of the vibration of a shaft running in the vicinity of its critical speeds, and/or reduce the vibrations of a shaft, and/or reduce or eliminate the noise problems caused by vibrations of a shaft. For these reasons, in this thesis different designs of elastomers, mainly cylindrical buttons, rectangular cartridges and O-rings are studied as external dampers to shaft-ball bearing systems.

In order to study elastomers one should have a general knowledge of their dynamic properties. Section 5.2 of this chapter describes the dynamic properties of elastomers. Since the dynamic properties of elastomers are shape and deflection dependent along with other dependencies, formulations for obtaining dynamic properties of elastomers from their material properties (i.e., shear and Young's moduli) is given in Section 5.3 for various geometries. The material properties of elastomers used in this thesis are obtained from a series of contractor reports for NASA. The experimental procedure for obtaining data is summarised in Section 5.4 and the introduction of this data to the shaft-ball bearing system as a mechanical model (Voigt model) is given in Section 5.5. A brief description of the subroutine written for this purpose is included in Section 5.6. Finally the obtained equivalent spring and damping factors are employed in the solution of equations of the motion for an externally damped shaft-ball bearing system in three degrees of freedom.

### 5.2 Properties of Elastomers

Some properties of elastomers have to be well understood before proceeding with the calculations. Some of the unique properties of elastomers are given below:

#### 5.2.1 Large Deformations

It is important to note that deviations from linearity of the load-compression curves for elastomers are usually ignored for strains up to 10%. At higher strains, the value of Young's modulus is to be modified as follows [Allen et al., 1967]:

\[
E_{\text{eff}} = p E_s \left(1 + k_S f_s^2\right)
\]

(5.1)

where

\(p\) is an empirical constant for a particular geometry,  
\(E_{\text{eff}}\) is the effective Young's modulus,
$E_o$ is Young's modulus,

$k$ is a material constant proportional to hardness and

$S_f$ is a shape factor equivalent to the loaded area divided by the force free area.

### 5.2.2 Dynamic Properties of Elastomers

In many applications, elastomers are subject to oscillatory or impact stresses. When examined thoroughly, the factors controlling the dynamic properties of a given elastomeric product fall into four categories: the composition of the material in the product; the manner in which the material is processed; the size, shape and configuration of the product (design factor); the manner in which it is used or tested (strain rate, strain magnitude and temperature).

It is important to note that the amplitude effect becomes significant only at relatively high amplitudes of vibration since it relates to the strain in the elastomer.

Important compounding parameters are the degree and type of cross-linking and the amount and type of carbon black used to stiffen the compound.

### 5.2.3 Geometry Effect

The specific value of the dynamic spring rate and damping coefficient in the overall dynamic response of a elastomeric product, are controlled in a major way by the design of that product.

It has been proved that the static spring rate of a rubber assembly may be changed by a factor of over 100 by changing the ratio of free to loaded area (principle reviewed by Allen et al. [1967]). Similarly the dynamic stiffness changes with shape factor, but data collected by Puydak and Auda [1967] indicate that the relationship is complex.

According to Sommers and Meyer [1973] damping is even more difficult to treat in design with the best approach most probably being the use of $\tan(\delta)$ to calculate the approximate damping coefficient, since it has been shown to be approximately constant for a given elastomer.

Care should be taken also when considering the level of preload (governs constant stiffness $K_0$ (see Appendix 6)), noting that its variations could be important when operating on the nonlinear portion of the load deflection curve.
5.2.4 Frequency Effect

The viscoelasticity of the material is also seen in its time dependent response to stress or strain. Creep and relaxation are always present to a large degree in elastomers, which means that their response to continual or sinusoidal loading is further complicated since the stress and strain are not in phase.

The best way of representing the response to a sinusoidal input is in terms of two rotating vectors \( \sigma_o \) and \( \varepsilon_o \) as shown in Fig.(5.1).

![Fig.(5.1) Vector representation of stress and strain during oscillatory loading](image)

As can be seen, the stress has two components, one in phase with the strain and one 90 degrees out of phase. Therefore it would be best to describe the response in terms of a modulus and a phase angle.

\[
E^* = E' + iE'' = E'(1 + i \tan(\delta))
\]  

(5.2)

where

\[
E' = \frac{\sigma}{\varepsilon} \cos(\delta) \quad \text{(in phase modulus)}
\]  

(5.3)

\[
E'' = \frac{\sigma}{\varepsilon} \sin(\delta) \quad \text{(out of phase modulus)}
\]  

(5.4)

and

\[
\tan(\delta) = \frac{E''}{E'}
\]  

(5.5)
In Eq. (5.2) $E'$ is called the *storage modulus*, and $\tan(\delta)$ the *loss tangent*. Fig. (5.2) is obtained by plotting these functions: $E$ and $\tan(\delta)$, as functions of frequency. Where, on these curves, the particular frequency range of interest lies, depends on the material in question, but overall it shows that the stiffness and loss factor both increase with frequency.

From Fig. (5.2) the section *a* (rubbery region) would represent a low damping material, and the section *b* (transition region) a high damping material which would be much more frequency dependent in its properties than the low loss material. The rubbery and transition regions are those normally encountered in elastomers and offer the best opportunity for use in vibration control [Smalley and Tessarzik, 1975].

An aspect of rubber and elastomers that is not shared by other polymers is their high dependence on their geometry. Because rubbers have a Poisson's ratio of nearly 0.5, they are virtually incompressible so the amount of constraint they are subjected to determines how stiff they will be, up to the point when they are fully constrained they will be completely rigid and offer no damping at all.

This level of constraint is normally described by the use of a *shape factor*. This is defined as:
Chapter 5

Elastomers

\[ \text{Shape Factor, } S_f = \frac{\text{Area of one of the loaded faces}}{\text{Total area of all unloaded faces}} \quad (5.6) \]

With this definition, the actual modulus can be estimated from an equation such as Equ.(5.1).

5.3. Formulation of Elastomers

The work presented here is derived from a property of rubbers that is not so commonly known; high extensibility and recovery due to deformation changes in entropy.

The differential of the Helmholtz free energy with respect to the length \( L \) of the specimen is equal to an external tensile force \( F \) [McCrum et al., 1989]:

\[ F = \left( \frac{\partial E}{\partial L} \right)_{T,V} - T \left( \frac{\partial S}{\partial L} \right)_{T,V} \quad (5.7) \]

For an ideal rubber while there are changes in the entropy \( S \) with the change in \( L \), there is no change whatsoever in \( E \). Since the second term on the RHS of Equ.(5.7) is negative an increase in \( F \) is generated by an increase in \( L \).

If the elastomers are made up of chains, for an ideal chain detached from the network with one end placed at the origin of the coordinate system as seen in Fig.(5.3), the internal energy is independent of the chain end-to-end vector.

For the specimen undergoing deformation as seen in Fig.(5.3), assuming that the randomness of change in molecular shape with the passage of time is reproduced by the Gaussian theory, this will lead to (see Appendix 3):

\[ W = \frac{V G}{2} \left( \lambda_x^2 + \lambda_y^2 + \lambda_z^2 - 3 \right) \quad (5.8) \]

where

- \( W \) is the total work done during the deformation of rubber,
- \( V \) is the volume of rubber undergoing deformation,
- \( G \) is the shear modulus of the rubber,
- \( \lambda_x, \lambda_y, \) and \( \lambda_z \), are the ratio of deformed dimensions to undeformed dimensions in \( x, y, \) and \( z \) axes in Cartesian coordinates respectively (see Appendix 3).
By definition, the force applied is linked to the strain energy function by a first order partial differential with respect to the change in the length. In the same way, the stiffness is related to the force applied by a first order partial derivative with respect to the change in length. Hence:

\[ K = \frac{\partial}{\partial \lambda} \left( \frac{\partial W}{\partial \lambda} \right) \frac{\partial \lambda}{\partial l} \]  

(5.9)

where \( l \) is the length of the specimen.

![Diagram](image)

*Fig.(5.3) The representative chain OA detached from the network (after McCrum et al. [1989]*)

Since the change in length, \( l \), is also involved in the extension ratios, Equ.(5.9) is used in the following form:

\[ K = \frac{\partial}{\partial \lambda} \left( \frac{\partial W}{\partial \lambda} \frac{\partial \lambda}{\partial l} \right) \frac{\partial \lambda}{\partial l} \]  

(5.10)

If Equ.(5.8) is substituted into Equ.(5.10), the form of resulting function will be as follows:

\[ K = G f(\text{Geometry, deflection}) \]  

(5.11)

Since the vibrations take place with a small amplitude, it can be assumed that the dynamic stiffness will also be in the form of [Goebel, 1954; Frederick and Payne, 1978; Smalley and Tessarzik, 1975; Grassano, 1991]:

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\[ K^* = G^* f(Geometry, \text{ deflection}) \]  \hspace{1cm} (5.12)

where \( K^* \) is the complex stiffness coefficient and \( G^* \) is the complex shear modulus. Hence it will be possible to derive the stiffness of a given elastomer damper from its shear or Young's modulus.

\textbf{5.3.1 Rectangular Cross-section Buttons}

The total work to deform a rectangular cross-section button, as seen in Fig.(5.4), is (see Appendix 3):

\[ W = \frac{VG}{2} [\lambda_x^2 + \lambda_y^2 + \lambda_z^2 - 3] \hspace{1cm} (5.13) \]

Since the volume is constant \( \lambda_x \lambda_y \lambda_z = 1 \). The deformation in the \( x \) and \( y \) directions from the deflection in the \( z \) direction may be assumed as equal for small deformations because in this case neither the material, being isotropic, nor the mode of stressing favours either direction [McCrum et al., 1989]. Hence:

\[ \lambda^2 \lambda_z = 1 \hspace{1cm} (5.14) \]

where \( \lambda = \lambda_x = \lambda_y \). Substituting Eq.(5.14) into Eq.(5.13)
\[ W = \frac{VG}{2} \left[ \frac{2}{\lambda_z^2} + \lambda_z^2 - 3 \right] \]  
(5.15)

\[ K = \frac{\partial}{\partial \lambda_z} \left( \frac{\partial W}{\partial \lambda_z} \right) \frac{\partial \lambda_z}{\partial z} = \frac{VG}{z_i} \left[ \frac{2}{\lambda_z^3} + 1 \right] \]  
(5.16)

Considering \( z_i = h \) and \( \lambda_z = 1 + \frac{\delta h}{h} \) Equ.(5.16) becomes:

\[ K = \frac{ab}{2h} \frac{G}{\left( \frac{2}{1 + \frac{\delta h}{h}} \right)^2 + 1} \]  
(5.17)

Since

\[ (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots \]  
(5.18)

Ignoring the higher order terms and considering compression and shear cases, Equ.(5.17) will take the form of:

\[ K = \frac{3}{2} \frac{ab}{h} (E + G) \left( 1 - 2 \frac{\delta h}{h} \right) \]  
(5.19)

5.3.2 Cylindrical Buttons

The work done in the axial direction will be (see Appendix 3):

\[ W_i = \frac{vkT}{2} \frac{\langle z^2 \rangle}{\langle z^2 \rangle_0} \left[ \lambda_z^2 - 1 \right] \]  
(5.20)

The work in the radial direction, considering Equ.(5.21) will be as in Equ.(5.22) [Aklonis and MacKnight, 1983].

\[ \langle r^2 \rangle_i = \frac{2}{3} \langle t^2 \rangle_i \]  
(5.21)
\[ W_r = \frac{v k T (r^2)}{2} \left( \frac{r^2}{r_o^2} \right) \left[ \lambda_r^2 - 1 \right] \]  
(5.22)

Hence the total work (see Appendix 3):

\[ W = \frac{V G}{2} \left[ 2 \lambda_r^2 + \lambda_s^2 - 3 \right] \]  
(5.23)

Since the volume is accepted to be constant Eqn.(5.23) becomes

\[ W = \frac{V G}{2} \left[ \frac{2}{\lambda_s} + \lambda_s^2 - 3 \right] \]  
(5.24)

![Fig.(5.5) Cylindrical Button](image)

Calculation for a cylindrical button using the Cartesian coordinate system, with the assumption that \( \lambda_r = \lambda_x = \lambda_y \), will also result in Eqn.(5.24) [Grassano, 1991]. This is the same equation as Eqn.(5.15) and results in:

\[ K = \frac{3 \pi d^2}{4} \frac{(E + G)}{h} \left( 1 - 2 \frac{\delta h}{h} \right) \]  
(5.25)
5.3.3 Ring Cartridge

For ring cartridges, two equations for stiffness were derived. The first is the modified form of the equation for a rectangular cross-sectional button. It is assumed that during the deformation, the cross-sectional shape follows the same pattern as shown in Fig.(5.4). When the bearing outer ring moves in any direction the situation in Fig.(5.6) will arise where the cartridge outer ring is fixed and the cartridge inner ring (i.e. the bearings outer race) is displaced by $\delta r$ in one direction. In this case, only the half of the cartridge goes under deformation as seen in Fig.(5.6). Hence from Equ.(5.19):

$$K = \frac{3 \pi (R_i + R_o) t}{4 (E + G)} \left(1 - \frac{\delta r}{R_o - R_i}\right)$$

(5.26)

The second equation is derived as follows: For the cartridge, Equ.(5.13) will take the form of (see Fig.(5.6) and Appendix 3)

$$W = \frac{VG}{2} \left[\lambda_2^2 + \lambda_3^2 + \lambda_r^2 - 3\right]$$

(5.27)

where (see Fig.(5.6))
\[
\lambda_L = 1 - \frac{\delta r}{L} \quad (5.28)
\]
\[
\lambda_s = 1 - \frac{\delta r}{2a} \quad (5.29)
\]
\[
\lambda_i = \frac{1}{\lambda_L \lambda_s} = \frac{2aL}{\delta r^2 - (L+2a)\delta r + 2aL} \quad (5.30)
\]

In these equations (see Fig.(5.6)):

\[
L = \frac{\pi}{2}(R_i + R_s) \quad (5.31)
\]
\[
a = R_s - R_i \quad (5.32)
\]

Hence

\[
K = \frac{\partial}{\partial \delta r} \left( \frac{\partial W}{\partial \delta r} \right) = \frac{V(E+G)}{2} \left[ \frac{1}{2(R_s - R_i)^2} + \frac{8}{\pi^2(R_s + R_i)^3} \right. \\
\left. + \frac{96\pi^2(R_s - R_i)^2(R_s + R_i)^2}{(2(R_s - R_i) - \delta r)^3(\pi(R_s + R_i) - 2\delta r)^4} \right. \\
\left. + \frac{64\pi^2(R_s - R_i)^2(R_s + R_i)^2}{(2(R_s - R_i) - \delta r)^3(\pi(R_s + R_i) - 2\delta r)^2} \right. \\
\left. + \frac{24\pi^2(R_s - R_i)^2(R_s + R_i)^2}{(2(R_s - R_i) - \delta r)^4(\pi(R_s + R_i) - 2\delta r)^2} \right] \quad (5.33)
\]

5.3.4 O-rings

For O-rings three equations were derived. The first one is the modified form of the equation that was derived for cylindrical buttons. It was assumed that during the deformation the cross-sectional shape remains unchanged but the diameter of the O-ring (i.e. \(R_o - R_i\)) changes (see Fig.(5.7)). In this case Equ.(5.25) becomes

\[
K = \frac{3}{32} \frac{\pi^2d^2(R_s + R_i)}{(R_o - R_i)^2}(E+G) \left( 1 - \frac{\delta r}{R_o - R_i} \right) \quad (5.34)
\]

The second procedure followed for O-rings is very similar to the one for the cartridges. In
order to reduce the mathematics involved, the deformation shape is assumed to be as shown in Fig.(5.7). In Fig.(5.7) $\delta r$ is assumed to be vibration amplitude. Following the same procedure as for the ring cartridges results in:

$$K = \frac{V(E+G)}{2} \left[ \frac{1}{2d^2} + \frac{8}{\pi^4 (R_i + R_o)^4} + \frac{96\pi^2 d^2 (R_i + R_o)^2}{(2d - \delta r)^4 (\pi (R_i + R_o) - 2\delta r)^4} \right]$$

$$+ \frac{64\pi^2 d^2 (R_i + R_o)^2}{(2d - \delta r)^3 (\pi (R_i + R_o) - 2\delta r)^3}$$

$$+ \frac{24\pi^2 d^2 (R_i + R_o)^2}{(2d - \delta r)^4 (\pi (R_i + R_o) - 2\delta r)^4}$$

(5.35)

Fig. (5.7) Assumed deformation of the O-ring cross-section

The third method is similar to the beam-column method including radius effects. Basically it follows the same procedure as in Appendix 4.

$$\delta r \cos(\zeta) = \int_{r=r_i}^{r=R} \frac{dF_r}{E2u d\zeta} \frac{dr}{r}$$

(5.34)

$$\delta r \sin(\zeta) = \int_{r=r_i}^{r=R} \frac{dF_r}{G2u d\zeta} \frac{dr}{r}$$

(5.35)

where
\[
\begin{align*}
\mu &= \left[ 1 - \frac{(r - \overline{R})^2}{(d + \delta r \cos(\zeta))^2} \right]^{\frac{1}{2}} \left( \frac{d - \delta r \cos(\zeta)}{2} \right) \\
(5.36)
\end{align*}
\]

where \( \overline{R} = (R_i + R_o)/2 \) (see Fig.(5.8)). Hence Equs.(5.34 and 5.35) becomes respectively

\[
\begin{align*}
\delta r \cos(\zeta) &= \frac{dF_c}{E(d - \delta r \cos(\zeta))} \int_{r_{ri}}^{r_{ri + \delta r}} \frac{1}{r} \left[ 1 - \frac{(r - \overline{R})^2}{(d + \delta r \cos(\zeta))^2} \right]^{\frac{1}{2}} \, dr \\
(5.37)
\end{align*}
\]

\[
\begin{align*}
\delta r \sin(\zeta) &= \frac{dF_c}{G(d - \delta r \cos(\zeta))} \int_{r_{ri}}^{r_{ri + \delta r}} \frac{1}{r} \left[ 1 - \frac{(r - \overline{R})^2}{(d + \delta r \cos(\zeta))^2} \right]^{\frac{1}{2}} \, dr \\
(5.38)
\end{align*}
\]

\[\text{Fig.(5.8) O-ring}\]
The total force on the element is

\[ dF = dF_e \cos(\zeta) + dF_e \sin(\zeta) \]  

(5.39)

Hence

\[ K = \frac{3}{4} \left( \frac{dF_e \cos(\zeta) + dF_e \sin(\zeta)}{dF_e \cos(\zeta) + dF_e \sin(\zeta)} \right) \]

(5.40)

Equations (5.37 and 5.38) were solved using Mathematica®. However, Mathematica® failed to solve Equation (5.40) and it is too complicated to solve with the classical methods. Therefore, Equations (5.37 and 5.38) are simplified by assuming that, since \( \delta r \) is small, sine and cosine of the angle are ignored. For this Equation (5.40) will take the form of

\[ K = \frac{d}{\delta r} \left[ \frac{3}{4} \int_{\zeta=0}^{\pi} \frac{(d-\delta r)E\cos^2(\zeta) + G\sin^2(\zeta)}{r^2} \frac{dr}{r} \right] \]

(5.41)

Equation (5.41) was solved using Mathematica® and it gave the successful results presented in Chapter 7.

5.4 Experimental Results Used in This Thesis

The experimental data used in this thesis is obtained from five contractor reports [Chiang et al., 1972; Gupta et al., 1974; Smalley and Tessarzik, 1975; Darlow and Smalley, 1977; Smalley et al., 1977] prepared for NASA (National Aeronautics and Space Administration) between 1972 and 1977.

The first report [Chiang et al., 1972] investigates the basic methods. An experimental arrangement was set up during the preparation of this report and it was used for all reports.
except the fourth one. It was a forced vibration, resonant mass type of apparatus. It was designed as a base excitation, electromagnetic shaker driven mass-spring system which could be brought to resonance for a range of differently sized vibrating masses on top of the elastomer "springs". One of the major features of this rig was that the tests were performed at near-resonance conditions. Since phase angle between base excitation and resonant mass response is an accurate indicator of the amount of damping in the region of resonance, measurements were preferably made in the phase angle range between approximately 15 to 165 degrees [Gupta et al., 1974]. This required that the test frequency be approximately 0.9 to 1.5 times the critical frequency of the elastomer mass resonant system. In order to cover the frequency range of 100 to 1000 Hz. with sufficient test points, the size of the mass was therefore changed.

Gupta et al. [1974] used this test rig to measure the dynamic characteristics of shear and compression elements. The research showed that direct power fitting to the...
experimental data gave satisfactory results. It was also postulated that mechanical models can represent the elastomer.

Smalley and Tessarzik [1975] investigated the effect of temperature, dissipation level and geometry. This report also used the same test rig. In particular, the prediction models presented in this report and the comparison with the experimental results, were useful tool for this thesis.

However, in this thesis most of the comparisons were made with the experimental data given in Report 4 [Darlow and Smalley, 1977] and Report 5 [Smalley et al., 1977], since the experimental rig employed in Report 4 is very relevant to the grinding spindle model presented in this thesis. While Darlow and Smalley [1977] reinvestigated the previous test results namely cylindrical buttons, ring cartridges in their new rig, Smalley et al. [1977] investigated the dynamic properties of O-rings employing the rig used in the first three reports with some modifications and additional O-ring mounter as seen in Fig.(5.11) (see also Fig.(2.40)).

The method employed in Report 4 [Darlow and Smalley, 1977] was essentially the same as the Base Excitation Resonant Mass Technique employed in the other reports (see Fig.(2.40) and Fig.(5.9)), except that here the elastomer specimen was subjected to a rotating load rather than a reciprocating load. The rotating shaft was supported by a pair of preloaded angular contact ball bearings so that the only significant deflection came from the elastomer support. This test rig employed a forced vibration, resonant mass principle in which rotating unbalance provided the exciting force. The test rig was designed to allow the frequency of resonance to be modified by changing the amount of resonant mass being supported by the elastomer cartridge specimen. Acquisition of data was, however, not limited to the resonance condition of the system. In fact, data obtained at the resonance frequency of each mass-spring combinations was just one of several data points acquired at several vibration frequencies around resonance, where significant amplitude ratios between shaft and resonant mass vibration existed.

The test method used in Report 5 [Smalley et al., 1977] employs a pair of O-rings to support a mass on an electromagnetic shake table so that the mass and O-rings form a one degree of freedom, damped system for vertical motion. When the table is shaken near the resonance frequency of the O-ring mass system, the relative motion across the ring is an amplification of the table motion and there is a phase shift across the elastomer of the order of 90 degrees. The important sensors for the test method are accelerometer signals, at a particular frequency, and from the known value of supported mass, the stiffness and damping are inferred.
5.4.1 Material Selection

The initial selection of polybutydiene, which is classed as a Broad Temperature Range (BTR) material, was primarily based upon the desire to test a material that was not very sensitive to small variation in ambient temperature variations induced in the sample through vibration testing [Gupta et al., 1974; Smalley and Tessarzik, 1975; Darlow and Smalley, 1977; Smalley et al., 1977].

This material, which carried the manufacturer's (Nichols Engineering Inc., Shelton, Connecticut) designation NEX1564, has a nominal hardness of 70 durometers (or Shore A hardness) and has the highest hardness and damping of those available from the manufacturer. The values for $G'$, $G''$ and $G^*$ were determined from the results of shear specimen tests conducted over a frequency range from 100 to 1000 Hz., for a constant energy dissipation rate.

5.4.2 Cylindrical Button Cartridges Test Samples

Each of the test specimens consisted of two rows of three cylindrical elastomer compression elements (or buttons) evenly distributed around the origin of the rotating force vector (see Fig.(5.9)) [Gupta et al., 1974; Smalley and Tessarzik, 1975]. The buttons were held between the flat surfaces of two shell structures.

![Cylindrical Button Cartridge](image)

In all cases buttons were cemented to the inner housing and in all but one case, the buttons were also cemented to the outer housing. The buttons were located such that a line drawn from the origin of the rotating force vector through the centre of any button
would form a right angle with either face of that button. Table (5.1) shows the selected dimensions for the test configuration.

<table>
<thead>
<tr>
<th>Sample #</th>
<th>Inner diameter</th>
<th>Outer diameter</th>
<th>Button diameter</th>
<th>Button height</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0635 m</td>
<td>0.0699 m</td>
<td>0.0127 m</td>
<td>0.00318 m</td>
</tr>
<tr>
<td>2</td>
<td>0.0635 m</td>
<td>0.0683 m</td>
<td>0.0127 m</td>
<td>0.00238 m</td>
</tr>
<tr>
<td>3</td>
<td>0.0635 m</td>
<td>0.0730 m</td>
<td>0.0127 m</td>
<td>0.00476 m</td>
</tr>
</tbody>
</table>

Table (5.1) Dimensions of the buttons

Sample 3 is the one that was not cemented to the outer housing hence in theory it should only undergo compression unlike Samples 1 and 2 which undergo both compression and shear.

5.4.3 Ring Cartridge Test Samples

Each of the test specimens consisted of two continuous axisymmetric rings of rectangular cross-section (see Fig.(5.10)) [Smalley and Tessarzik, 1975; Darlow and Smalley, 1977]. Each ring was held between two shell structures and cemented to the inner housing. The outer housings were split to facilitate the assembly of these test elements and to allow a small positive radial preload during testing.

![Fig.(5.10) Ring Cartridge](image)

Two sets of dimensions were selected for the test configuration as in Table (5.2).
5.4.4 O-ring Test Samples

A pair of O-rings with nominal outer diameter of 6.35 cm were tested [Smalley et al., 1977]. Three different materials (Viton-70, Viton-90, Buna N-70) were used. A parameter perturbation test programme was executed for each material. Imposed squeeze and stretch, cross-sectional diameter, O-ring groove diameter and temperature were varied in turn, about a nominal value.

![Diagram of O-ring and components](image)

Table (5.2) Dimensions of the ring cartridges

<table>
<thead>
<tr>
<th>Sample #</th>
<th>Inner diameter</th>
<th>Outer diameter</th>
<th>Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.064 m</td>
<td>0.073 m</td>
<td>0.0048 m</td>
</tr>
<tr>
<td>2</td>
<td>0.064 m</td>
<td>0.073 m</td>
<td>0.0024 m</td>
</tr>
</tbody>
</table>

Fig.(5.11) O-ring (after Smalley et al. [1977])

Nominal values are as follows:

- Temperature : 25 degrees of Celsius
- Vibration amplitude : $7.62 \times 10^{-6}$ m
- Squeeze : 15%

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Stretch : 5%
Cross-sectional diameter : 0.00353 m
Groove width : 135% of original cross-sectional diameter
O-ring outer diameter : 0.0635 m

The stretch and squeeze are defined as follows:

\[ \text{Stretch} = 100 \left( \frac{D_i - D_i}{D_i} \right) \% \]  \hspace{1cm} (5.42)

\[ \text{Squeeze} = 100 \left( \frac{d - L}{d} \right) \% \]  \hspace{1cm} (5.43)

5.5 Representation of the Experimental Data

Tan (δ), the tangent of the phase angle between the real and imaginary part of complex stiffness, \( K_2/K_1 \) in Fig.(5.1), or is the ratio of \( G'/G'' \), \( E'/E'' \) and is often used to describe the damping characteristics. Various combinations of the above terms may be used to describe a given rubbery material [Sommer and Meyer, 1973].

\[ G^* = G' + i \, G'' \]  \hspace{1cm} [Shear Modulus, N/m²]

\[ E^* = E' + i \, E'' \]  \hspace{1cm} [Young Modulus, N/m²]

\[ K^* = K_1 + i \, K_2 \]  \hspace{1cm} [Stiffness, N/m]

\[ c = K_2/\omega \]  \hspace{1cm} [Damping Coefficient, Ns/m]

where \( \omega \) is frequency in radians per second.

Sommer and Meyer [1973] gave the brief summary of the changes in the spring stiffness and damping coefficients produced by variations in rubbers, fillers, oils, level of cross-linking and processing factors such as mixing, curing, and storage.

Basically it can be said that the real part of the complex stiffness \( K_1 \) and the imaginary part of the complex stiffness \( K_2 \) (i.e. damping of the elastomer) can be obtained from the relationships between the force and displacement in the frequency domain.

\[ K^* = K_1(\omega) + i \, K_2(\omega) \]

where \( K^* \) is a complex number. This gives rise to discrete data points. For mathematical
purposes it is convenient to seek, for an element, a continuous relationship between force and displacement which closely matches experimental behaviour. There are several possible approaches available to this problem.

### 5.5.1 Generalised Viscoelastic Models

An approach to representing the variation of stiffness and damping properties of an elastomer element with frequency, is to replace the elastomer with a mechanical analogue, consisting of a series of springs and viscous dash-pots. The simplest of these models are the Maxwell and Voigt elements (see Fig. (5.12))

However, in practise those two models are generally combined. For more than one element model, the Voigt element model is more convenient since it readily defines compliance. In this thesis, for computational purposes, a generalised Voigt model system is used as shown in Fig. (5.13) (see Appendix 6).

\[
K^*(\omega) = K_1(\omega) + iK_2(\omega)
\]  

(5.44)

![Fig.(5.12) Basic Elements of Viscoelastic Model](image)

![Fig.(5.13) General Voigt Model](image)
The equivalent complex compliance for the system:

\[ H^* = H_1^* + H_2^* + H_3^* + \ldots + H_n^* \quad (5.45) \]

where

\[ H^* = \frac{1}{K} \quad (5.46) \]

Substituting Equ.(5.44) into Equ.(5.46) will give the complex compliance for a single Voigt element:

\[ H^*(\omega) = H_1(\omega) + i H_2(\omega) \quad (5.47) \]

where

\[ H_1(\omega) = \frac{K_1}{K_1^2 + K_2^2} \quad (5.48) \]

\[ H_2(\omega) = \frac{K_2}{K_1^2 + K_2^2} \quad (5.49) \]

### 5.5.2 Generalised Force-Displacement Relationships

An alternative approach for representing the variation with frequency is to assume a general partial differential relationship between force and displacement:

\[
\left[ a_0 + a_1 \left( \frac{\partial}{\partial t} \right) + a_2 \left( \frac{\partial^2}{\partial t^2} \right) + \ldots + a_n \left( \frac{\partial^n}{\partial t^n} \right) \right] F
\]

\[ = \left[ b_0 + b_1 \left( \frac{\partial}{\partial t} \right) + b_2 \left( \frac{\partial^2}{\partial t^2} \right) + \ldots + b_m \left( \frac{\partial^m}{\partial t^m} \right) \right] \delta \quad (5.50) \]

For sinusoidal loading and displacement a close connection between this generalised force displacement relationship and the generalised viscoelastic models exists (see Appendix 6).

### 5.5.3 Direct Fit of Data to Mathematical Expressions

A direct approach to the problem is to impose upon the data some correlation function and to seek those coefficients which give the best match between the function and the
data. Polynomial, trigonometrical, or power law relationships are some of the options available for this approach.

Gupta et al [1974] showed that the variation of complex dynamic stiffness with frequency can be expressed in the form of a power function of the form shown below:

\[ Y = A \omega^B \]  \hspace{1cm} (5.51)

where \( A \) and \( B \) are constants.

Smalley and Tessarzik [1975] curve fitted the data with consideration of geometrical shape factors. Although this is also a prediction method, if \( K_1(\omega) \) and \( K_2(\omega) \) in Eq. (5.1) are calculated from the functions below, quite accurate curve fitting can be satisfied [Smalley and Tessarzik, 1975]:

\[ K_1(\omega) = 3G'(\omega) \frac{\pi D^2 N}{4h} \left[ 1 + \beta'(\omega) s^2 \right] \]  \hspace{1cm} (5.52)

\[ K_2(\omega) = 3G''(\omega) \frac{\pi D^2 N}{4h} \left[ 1 + \beta''(\omega) s^2 \right] \]  \hspace{1cm} (5.53)

where \( G' \) is the real part and \( G'' \) is the imaginary part of the complex shear modulus \( G \). \( \beta' \) is the real part and \( \beta'' \) is the imaginary part of the geometrical factor and \( s \) is a shape factor defined as the ratio of the loaded area to the unloaded (see Appendix 5). The experiment done by Smalley&Tessarzik[1975] and Darlow and Smalley [1977] showed that if \( G', G'', \beta', \) and \( \beta'' \) in Eqs. (5.3&5.4) are expressed at constant temperature of 32 degrees Celsius as below, the error for dynamic stiffness would be less than 50% and with damping the error would be less than 20%.

\[ G'(\omega) = 3.686 \times 10^6 \omega^{0.2057} \text{ N/m}^2 \]  \hspace{1cm} (5.54)

\[ G''(\omega) = 8.333 \times 10^6 \omega^{-0.1777} \text{ N/m}^2 \]  \hspace{1cm} (5.55)

\[ \beta'(\omega) = 12.33 \omega^{-0.290} \]  \hspace{1cm} (5.56)

\[ \beta''(\omega) = 1.726 \omega^{0.0299} \]  \hspace{1cm} (5.57)
5.6 Brief Description of the Subroutine

A subroutine called **DAMPER** was written in Fortran77 to find the coefficients of damping and stiffness for the equivalent mechanical system. The number of elements in the Voigt model, the frequency range and the frequency step size are required as inputs.

The subroutine basically minimises the error between the theoretical and experimental results.

Let a generalised the Voigt model be defined by \( n \) Voigt elements connected in series with a constant stiffness coefficient \( K_0 \) and the \( i \) th the Voigt element have a stiffness coefficient \( K_i \) and a damping coefficient \( c_i \). If this model is applied to solid materials such as elastomers and there is no preload \( (K_0 = 0) \), the stiffness coefficients in all elements must have non-zero values. For this model the equivalent complex compliance is the summation of the complex compliance of each Voigt element and can be given as:

\[
H^*(\omega) = \left\{ \frac{1}{K_0 + \sum_{i=1}^{n} b_i \frac{1}{1 + \left( \frac{\omega}{\Omega} \right)^2}} \right\} - i \left\{ \sum_{i=1}^{n} b_i \left( \frac{\omega}{\Omega} \right)^2 \right\} 
\]

where \( n \) is the number of Voigt elements and

\[
b_i = \frac{1}{K_i} \tag{5.59}
\]

\[
\Omega_i = \frac{K_i}{c_i} \tag{5.60}
\]

The error function can be given as the sum of the squares of the difference between the theoretical and experimental complex compliance [Gupta et al., 1974]:

\[
E = w_1 \sum_{i=1}^{k} [H_{1\omega}(\omega) - H_{1\omega}(\omega)]^2 + w_2 \sum_{i=1}^{k} [H_{2\omega}(\omega) - H_{2\omega}(\omega)]^2 \tag{5.61}
\]

where \( k \) is the number of discrete point used, \( w_1 \) and \( w_2 \) are weighting functions and \( H_{1\omega}(\omega) \) and \( H_{2\omega}(\omega) \) for the theoretical model are the same as in Equations 5.48 and 5.49 for a single Voigt element for example. \( H_{1\omega}(\omega) \) and \( H_{2\omega}(\omega) \) for the experimental results are in
the form of stiffness and damping, and can easily be converted into the complex compliance form as in Equ.(5.47).

To minimise the error function the least square method is used. Detailed information about the technique can be found in Ref. [Gupta et al., 1974].

The first step in solving the equations, is to obtain the constant stiffness $K_0$, of the Voigt element created by preloading. For a perfect solution, the experimental and theoretical results should be the same. Hence from Equ.(5.47) and Equ.(5.49):

$$\left\{ \frac{1}{K_0} + \sum_{i=1}^{n} b_i \frac{1}{1 + \left( \frac{\omega}{\Omega} \right)^2} \right\} - i \left\{ \sum_{i=1}^{n} b_i \frac{\left( \frac{\omega}{\Omega} \right)}{1 + \left( \frac{\omega}{\Omega} \right)^2} \right\} = \frac{K_1}{K_1^2 + K_2^2} - i \frac{K_2}{K_1^2 + K_2^2}.$$  

(5.62)

Equating real and imaginary parts of Equ.(5.62):

$$K_0 = \left( K_1^2 + K_2^2 \right) \left( 1 + \sum_{i=1}^{n} \frac{1}{K_2 \left( \frac{\omega}{\Omega} \right) - 1} \right)^2 + \left[ \sum_{i=1}^{n} \frac{\left( \frac{\omega}{\Omega} \right)}{K_2 \left( \frac{\omega}{\Omega} \right) - 1} \right]^2 \right\}^{\frac{1}{2}}.$$  

(5.63)

Note that this equation always results in a positive solution.

For $b_1$, the same procedure is followed with a small alteration to be able to introduce the weighting functions $w_1$ and $w_2$. The idea is that if a Voigt element being calculated is left out, the equivalent of the rest of the elements plus the one exempted should be equal to experimental result. Hence

$$b_j = \frac{A}{B}$$  

(5.64)

where

$$A = w_1 \left[ H_1 - H_0 - \sum_{i=1}^{\infty} b_i f_i(\Omega) \right] f_j(\Omega) + w_2 \left[ H_2 - \sum_{i=1}^{\infty} b_i g_i(\Omega) \right] g_j(\Omega)$$  

(5.65)

$$B = (f_j(\Omega))^2 + (g_j(\Omega))^2$$  

(5.66)
in these equations

\[ f(\Omega) = \frac{\left( \frac{\Omega}{\omega} \right)^2}{1 + \left( \frac{\Omega}{\omega} \right)^2} \]  

(5.67)

and

\[ g(\Omega) = \frac{\left( \frac{\Omega}{\omega} \right)^2}{1 + \left( \frac{\Omega}{\omega} \right)^2} \]  

(5.68)

Equation (5.61) is called the sum of the squared deviations of the error and it has to be minimised. For this:

\[ \frac{dE}{d\Omega} = 0 \]  

(5.69)

\[ E \] should be differentiated with respect to all variables in turn and all of them should simultaneously be equal to zero. Hence;

\[ \frac{\partial E}{\partial \Omega_i} = p_i = 0 \quad (i = 1,n) \]  

(5.70)

Since the damping, or in other words \( \Omega \) in the equations, is nonlinear, the solution must be obtained by a numerical method. In this study, the Newton-Raphson Method was employed to solve the equations. For successful solutions below, a Jacobian matrix must also be solved.

\[
\left[ \frac{\partial p_i}{\partial \Omega_{j_{\nu}}} \right] \{ \Omega_{j_{\nu}} - \Omega_i \} = -[p_i]
\]  

(5.71)

For the Newton-Raphson Method, the initial values are very important as the solution can converge to different results. Therefore, the initial value for \( \Omega \) is taken as below.
\[
\Omega_0 = \frac{K_0}{C_0} = \frac{K_0(\omega)}{K_2(\omega)} \quad \text{(5.72)}
\]

where \(\omega = \frac{\omega_{\text{min}} + \omega_{\text{max}}}{2}\) \(\text{(5.73)}\).

The iterative solution procedure can be summarised as follows: The linear parameters, \(K_0, K_i\) (where \(1 \leq i \leq n\)) are solved for any set of iterates \(\Omega_{i,j+1}\) (where \(1 \leq i \leq n, 1 \leq j \leq n\) and \(i \neq j\)) are obtained by solving the linear algebraic equations (Equ.(5.71)).

It should be noted that such a least squares analysis may sometimes result in negative coefficients, which are physically meaningless. Under these conditions, it might seem reasonable to calculate the linear parameters by carrying out a simple linear regression.

### 5.7 Modelling of The System

The elastomer damper investigated in this thesis may be in the form of cylindrical buttons, ring cartridges or O-rings. Such a system with external dampers can be modelled as in Fig.(5.14) where \(m_{or}\) simulates the mass of outer rings and \(K_e\) and \(c_e\) are the stiffness and damping of elastomer dampers respectively.

![Fig.(5.14 Simulation of the system with external dampers](image)

The equations of motion in a three degrees of freedom system can be written as follows:

\[
M \ddot{\mathbf{x}} + \sum_{i=1}^{n} W_{i} \cos(\alpha_{i}) \cos(\theta_{i}) - Mg = Q_{a}(t) \quad \text{(5.74)}
\]
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Fig.(5.15) Deflection of the i th ball with external dampers

\[ M \ddot{y} + \sum_{i=1}^{n} W_i \cos(\alpha_i) \sin(\theta_i) = Q_x(t) \]  
(5.75)

\[ M \ddot{z} + \sum_{i=1}^{n} W_i \sin(\alpha_i) = Q_z(t) \]  
(5.76)

\[ m_x \ddot{x}_2 + C_x \dot{x}_2 + K_x x_2 - \sum_{i=1}^{m} W_i \cos(\alpha_i) \cos(\theta_i) - m_x g = Q_{x_2}(t) \]  
(5.77)

\[ M \ddot{y}_2 + C_y \dot{y}_2 + K_y y_2 - \sum_{i=1}^{m} W_i \cos(\alpha_i) \sin(\theta_i) = Q_y(t) \]  
(5.78)

\[ M \ddot{z}_2 + C_z \dot{z}_2 + K_z z_2 - \sum_{i=1}^{m} W_i \sin(\alpha_i) = Q_z(t) \]  
(5.79)

where \( W_i \) denotes the restoring force of the \( i \) th ball as given in Equ.(4.1). From Fig.(5.15) which is the modified form of Fig.(3.10), the deflection, \( \delta_i \) and contact angle, \( \alpha_i \) of the \( i \) th ball will respectively be

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\[ \delta_i = \left( B d_i \sin(\alpha_0) + z_0 + (z - z_2) \right)^2 \left[ +((x - x_2) \cos(\theta_i) + (y - y_2) \sin(\theta_i) + B d_i \cos(\alpha_0)) \right]^{1/2} - B d_i \]  
(5.80)

\[ \alpha_i = \frac{B d_i \sin(\alpha_0) + z_0 + (z - z_2)}{(x - x_2) \cos(\theta_i) + (y - y_2) \sin(\theta_i) + B d_i \cos(\alpha_0)} \]  
(5.81)

The solution procedure is similar to that employed in Chapter 3.

5.8 Excitation Frequency

As was explained earlier the dynamic properties of elastomers vary with the frequency they are subjected to. This means the current frequency they are subjected to will determine the system steady-state vibration characteristics. During transient vibrations period, the excitation frequency the elastomers dampers see will continuously change, hence the steady-state vibrations may be totally different from the one calculated. Therefore the instantaneous speed, the elastomer dampers are subjected to, should be introduced to the system.

![Fig.(5.16) Transient and the steady-state oscillations](image)

However all available data is obtained for the cyclic excitation forces. Since the oscillation during the transient vibrations are not cyclic, the data available are not useful. Therefore during the computation the following procedure is followed:

For any motion of the shaft-ball bearing system such as in this case, the amplitude of the vibrations will have instantaneous maximum and minimum points as shown in Fig.(5.16).
In Fig. (5.16) the maximum amplitudes are indicated with 1 and minimum amplitudes are with 0. It is known that for the points 1s and 0s, \( \dot{x}_2(t) = 0 \). From 1 to 0, \( \dot{x}_2(t) < 0 \) and from 0 to 1, \( \dot{x}_2(t) > 0 \).

Therefore the outer race centre movement is controlled and for the first \( \dot{x}_2(t) = 0 \), the time, \( t_0 \) is recorded. When the next \( \dot{x}_2(t) = 0 \) is observed, the time, \( t_1 \), is also recorded.

With the assumption that this movement will repeat itself, the first excitation frequency is found as follows:

\[
\omega_1 = \frac{\pi}{\Delta t_1}
\]

(5.82)

where \( \Delta t_1 = t_1 - t_0 \). Until the next \( \dot{x}_2(t) = 0 \) point, this will be the excitation frequency and it will follow as:

\[
\omega_i = \frac{\pi}{\Delta t_i}
\]

(5.83)

where \( \Delta t_i = t_i - t_{i-1} \). When the steady-state oscillations are reached, if at all:

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where \( \omega_{or} \) is the speed of the outer race centre. Note that the same result may be reached using \( y_2(t) \) oscillations instead of \( x_2(t) \) (see Fig.(5.17)).

**Closure**

Since, design-orientated data on the dynamic behaviour of elastomers is limited and there is no adequate theory describing the dynamic behaviour of elastomers, a method of obtaining the stiffness and damping characteristics of elastomers from their material properties (i.e., shear or Young's modulus) for different designs namely, cylindrical buttons, cartridges and O-rings was developed in this chapter. The results obtained will be compared with other researchers' predictions and used in the vibration model available to see the overall effect of elastomer dampers on the vibration characteristics of a shaft supported by angular contact ball bearings in the forthcoming *Results and Discussion* chapter.
CHAPTER 6

RESULTS AND DISCUSSION:
BALL BEARINGS

6.1 Introduction

In the previous three chapters, the necessary theories for studying the dynamics of ball bearings were established. Employing these theories, a computer simulation of a shaft-bearing assembly was obtained. In Chapter 4 some anomalies of ball bearings, namely waviness, off-sized ball, misalignment and local point defects were also simulated. In this chapter these simulation results will be compared with the other sources of experimental and theoretical predictions. Results and discussions will be presented in the same order as in Chapter 4 i.e., natural frequency of the system, ball passage frequency, waviness, off-sized ball, misalignment and defects on the rolling surfaces.

6.2 Shaft-Ball Bearing Specifications

In order to test the simulation programme a grinding spindle is chosen, as seen in Fig.(6.1). The shaft (2) is driven by a pulley (5) in order to operate the grinder (1). The shaft is supported by a pair of precision angular contact ball bearings (3 and 4). The specifications of the bearings and the shaft were chosen as the Aini's experimental arrangement [Aini, 1990] since the particulars of both the shaft and bearings were readily available.

Particulars of the bearings (see Fig.(6.2)):

- Type : Angular contact ball bearings
- Inner race bore : 0.04 m
- Inner race diameter : 0.046 m
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Fig. (6.1) Assumed grinding spindle assembly
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- Outer race diameter: 0.062 m
- Outer race outside diameter: 0.068 m
- Inner race radius curvature: 0.00408 m
- Outer race radius curvature: 0.00461 m
- Bearing width: 0.015 m
- Ball diameter ($d_b$): 0.00794 m
- Maximum number of balls: 16
- Unloaded contact angle: 15 deg.
- Pitch diameter of the ball set: 0.054 m
- Both bearings are identical.

Fig.(6.2) Ball Bearing

Particulars of the shaft:

- Diameter of the shaft between the bearings: 0.04 m
- Length of the shaft: 0.421 m
- Mass of the shaft: 5.5 kg
- Distance between the support bearings: 0.215 m
- Position of the LHS bearing from the C.G.: 0.0875 m
- Position of the RHS bearing from the C.G.: 0.174 m
- Moment of inertia of the shaft about $x$ or $y$-axis: 0.05177 kg m$^2$
- Moment of inertia of the shaft about $z$-axis: 0.0044 kg m$^2$
6.3 System Natural Frequency

First a bearing with five balls was considered in order to calculate the variation in the natural frequency of the system due to different positions of ball set. Since for five balls, the positions of balls repeat themselves after each 72 degrees, a span of 72 degrees of position change was observed as shown in Fig.(6.3). In order to calculate the natural frequency of the system the procedure below was followed: the shaft speed was set to zero and the shaft centre was released from an arbitrary position. The natural frequency values were calculated from the transient vibrations as the system must vibrate at its natural frequency because there is not any external force or damping. The natural frequency values were obtained as described in Appendix 2.

![Graph showing natural frequency vs. offset angle](image)

*Fig.(6.3) The effect of the position of balls on the natural frequency (m=5, Pr=5 N)*

As is seen in Fig.(6.3) the change in the natural frequency is cyclic and varies with the offset angle (see Fig.6.12 for the definition of offset angle) as was also observed by Shimizu and Tamura [1966; 1967; 1968]. Some points in Fig.(6.3) seem to be slightly misplaced. This is due to an accuracy problem in measuring the natural frequency and associated with the time step size. When the time increment was decreased, better results (i.e. a smoother curve) were obtained with the cost of longer computing time. Then the position of the ball set was varied for 360 degrees in order to obtain a full picture for a cycle. Fig.(6.4) shows that the natural frequency changes cyclically \( m \) times (where \( m \) is the number of balls). This was further observed with a bearing with 8 balls (see Fig.(6.5)). It was also observed that as the number of balls increases the change in the amplitude of the natural frequency decreases because the bearing becomes stiffer. This is particularly correct for a bearing with relatively higher number of balls (e.g. 16 balls).
where the variation in the natural frequency is almost insignificant. This small variation of natural frequency will be discussed further later in this section.

\[ \text{Fig.(6.4) The effect of the position of balls on the natural frequency (m=5, Pr=5 N)} \]

\[ \text{Fig.(6.5) The effect of the position of balls on the natural frequency (m=8, Pr=5 N)} \]
At this point a further discussion on the experimental results from Yamamoto and Ishida [1974] will be useful. They experimentally showed that the natural frequency changes cyclically with the position of the balls once per cage rotation (see Fig.(2.8)). This seems likely to be due to wrongly selected experimental position intervals [Xistris et al., 1980] e.g., as shown in Fig.(2.5). The possible result of wrongly selected intervals is also shown in Fig.(6.6) for ball bearings. If experimental points are selected as shown with the diamond shaped points in Fig.(6.6), the results obtained can be misleading since the actual variation is totally different and shown by dotted line. The larger the number of balls the greater the possibility of making this mistake, since relatively small changes in the natural frequency and larger cyclic variations take place.

Fig.(6.6) The effect of wrongly selected experimental intervals on the natural frequency (m=8, Pr= 5 N)

Due to the nonlinear Hertzian contact conditions, the preload changes the natural frequency. The effect of axial preload on the natural frequency is shown in Fig.(6.7). For lower preloads the change is fast since the balls are relatively softer. As the preload is increased the balls get stiffer and allow smaller deflections for the same amount of preload. This is clearly observed in Fig.(6.7).

Fig.(6.8) shows the effect of changing the number of balls. As the number of balls are increased the system get stiffer since larger number of balls support the shaft. For relatively larger number of balls, the change in the natural frequency is relatively less, since in this case larger number of balls in the heavily loaded region allow relatively small deflections.
The approximate natural frequency for the system (see Appendix 10) can be written in two forms:

\[ \omega_n = A \left( \frac{1}{B^2} + \frac{1}{4} \frac{\Delta \delta}{B^2} \right) \]  

or

\[ \omega_n = A \left( \frac{1}{C^6} + \frac{1}{4} \frac{C^2 \Delta \delta}{C^6} \right) \]  

where

\[ A = \left( \frac{3}{M} \sum_{i=1}^{n} K \left( \cos(\theta_i - \beta) \cos^2 \alpha_i + \sin^2 \alpha_i \right) \cos \alpha_i \cos \theta_i \right)^{\frac{1}{2}} \]  

\[ B = \frac{P_{re}}{mK \sin \alpha} \]  

\[ C = \frac{1}{B} \]

Fig. (6.7) The effect of the axial preload on the natural frequency (m=8)
In Equ.(6.1) the first term on the RHS will be effective for relatively larger values of B, whereas for small values of B, the second term in RHS will also contribute to the change. Fig.(6.7) shows the behaviour suggested by Equ.(6.1). The curve shows a big similarity with that of a curve obtained only by the first term on the RHS of Equ.(6.1). This is particularly true for preload values greater than 30 N. In our case $\Delta \delta$'s being very small also contributes to making the first term dominant in Equ.(6.1).

In order to study the effect of the number of balls on the natural frequency, Equ.(6.1) can be written in the form of Equ.(6.2) which is more convenient for this purpose since the effect of the number of balls has a proportional relation in Equ.(6.2). It is obvious that the first term on the RHS is less dominant for changing the number of balls as the number of balls is always greater than 1. It will be more so when the number of balls is increased. Hence the second term on the RHS of Equ.(6.2) will be more dominant implying the behaviour in Fig.(6.8). This curve is similar to that obtained only by the second term on the RHS of Equ.(6.2).

Figs.(6.7&6.8) were obtained from the simulation model by a similar method to that which was explained earlier at the beginning of this section. For preloads the number of balls was assumed to be 8 and their position were kept same during the calculations for both cases and was such that the first ball was always set to the bottom of the bearings. The effect of this can be seen in Fig.(6.8) as the natural frequency values flicker slightly for odd and even numbers of balls.

*Fig.(6.8) The effect of the number of balls on the natural frequency (Pr=5 N)*
6.4 Ball Passage Frequency

Ball passage frequency is one of the most important frequencies of rolling element-shaft systems (see Chapter 1 for its definition and Chapter 4 for its calculation). In order to display its effect, a defect free system was considered with three degrees of freedom. The particulars of the arrangement was given earlier in this chapter (see Section 6.2). The axial preload value was set to 5 N, the mass centre was displaced \( x_0 = 1 \mu m \), \( y_0 = 0.1 \mu m \) and \( z_0 = 0.01 \mu m \) and oscillations along the \( x \), \( y \) and \( z \) axes were recorded when the shaft was rotating at a speed of 5000 rpm. For this speed the BPF is 280 Hz (from Equ.(4.21)) and the natural frequency for the system is about 500 Hz (from Figs.(6.8,6.9)).

![Graph showing vibrations along the x-axis in the time and its FFT](image)

**Fig.(6.9) Vibrations along the x-axis in the time and its FFT**

\( m=8, P=5 \text{ N}, n=5000 \text{ rpm}, \omega_{bp}=280 \text{ Hz, } \omega_n=500 \text{ Hz} \)
Chapter 6

Results and Discussion: Ball Bearings

The results obtained in both the time and frequency domains are given below. Fig.(6.9) shows the oscillations along the x-axis. The effect of the natural frequency along the z-axis appears at about 200 Hz (a) (see also Fig.(6.11)).

Fig.(6.10) Vibrations along the y-axis in the time and its FFT

(m=8, Pr=5 N, n=5000 rpm, $\omega_{bp}=280$ Hz, $\omega_n=500$ Hz)

The second most dominant peak (c) after the peak at the natural frequency in the vibration spectrum is at the BPF. Its first superharmonic ($2 \omega_{bp}$) (e) also appears in the spectrum. Although the ball passage effect is relatively small in amplitude, it should be noted that when it coincides with the natural frequency the system resonates, as will be shown later in this chapter.
The natural frequency is about 510 Hz. (see Fig.(6.7)) for a shaft supported by bearings with 8 balls when an axial preload of 5 N is applied on the system. The peak at the natural frequency (d) is at about 490 Hz. This slight difference comes from the operating conditions, since in the calculations of natural frequency (see previous section) the system assumed to be static. As is also seen in Fig.(6.9) the first superharmonic of the natural frequency \(2\omega_n\) appears in the frequency spectrum.

\[0.0115\]

\[0.0015\]

\[0.0000\]

\[0.0100\]

\[0.0010\]

\[0.0005\]

\[0.0000\]

\[0.20\]

\[0.125\]

\[0.30\]

\[0.35\]

\[200\]

\[300\]

\[400\]

\[500\]

\[181\]

Fig.(6.11) Vibrations along the z-axis in the time and its FFT

\(m=8, Pr=5\ N, n=5000\ \text{rpm}, \omega_{bp}=280\ \text{Hz}, \omega_n=200\ \text{Hz}\)
It is also important to note that the second (c) and third (f) superharmonics of the effect of
the natural frequency along the z-axis also appear in the spectrum. Although the
frequency spectrum for the x-oscillations presented in Fig.(6.9) is the simplest case,
since there are no defects present in the system, the effects of vibrations along the y and
z axes make the spectrum relatively complicated. This is the indicator of the complex
nature of vibrations associated with ball bearings.

Fig.(6.10) is the vibration spectrum for y-oscillations. It is similar to Fig.(6.9) with the peaks
at the natural frequency along the y-axis (c), the effect of the natural frequency in the z
direction (a) and its superharmonic (e), the BPF (b) and its first superharmonic (d). These
frequencies are almost the same as along the x-axis. This is understandable since the
system along the x and y axes are symmetrical with the difference that the force due to
mass of the shaft acts only along the x-axis and hence causes a difference in the vibration
amplitudes.

The z-oscillations produce the frequency spectrum in Fig.(6.11). Due to the small initial
displacement in the z-direction and because opposing bearings were identical, the
resulting vibrations had relatively small amplitudes. The only apparent peak in the
frequency spectrum is at the system natural frequency along the z-axis. The z-oscillations
will be further investigated in this chapter.

In order to study the effect of the BPF in a more detailed form, the angular contact ball
bearing employed in this thesis was reduced to a radial ball bearing for the time being and
modelled as in Fig.(6.12). The angles and reference axes were set as in Fig.(3.15) in
Chapter 3. The off-set angle in Fig.(6.12) is an arbitrary reference point on the cage.

Balls were radially preloaded in order to ensure the continues contact of all balls and the
raceways, since otherwise, a chaotic behaviour may be observed [Gad et al., 1984]. The
preloaded deflection for each ball, $\delta_p$, was assumed to be 5 \( \mu \text{m} \). The centre of the inner
race was shifted 2 \( \mu \text{m} \) in the x direction and 2 \( \mu \text{m} \) in the y direction with respect to the
outer race centre. For different off-set angles (see Fig.(6.12)), the total static net force
and the phase angle were recorded.

Fig.(6.13) shows the change in the total net force and phase angle for a bearing with 5
balls. The off-set angle was changed over a span of 360 degrees. It is shown that the
total force vector and its direction (i.e., phase angle) changes \( m \) times (\( m \) is the number of
balls) implying vibrations at the BPF. As the number of balls increases the amplitude of
change in the force vector and phase angle becomes small, implying small amplitude
variations with relatively larger number of balls. The experimental force monitoring of ball bearings gave similar results (see Fig.(2.19)) in Chapter 2 and the paper by Glöckner [BBJ 225]). This model will be heavily used for the waviness and off-sized ball anomalies later in this chapter in order to show their effects as well.

The change in the phase angle follows the same pattern as seen in Fig.(6.13). This implies a similar excitation force (i.e. excitation due to the BPF) in the $y$ direction and hence vibrations at the BPF. The effect of the BPF is apparent in this simplified model. When the forcing frequency coincides with the natural frequency the system undergoes severe vibrations. In order to study this effect the results presented in Figs.(6.9 through 6.11) are reinvestigated after the natural frequency effect is damped out. For this, different arbitrary damping coefficients were introduced to the system and 300 Ns/m was finally chosen since the smaller coefficients took very long to get to steady-state vibrations hence expensive computation, whereas larger damping coefficients did not allow the important peaks to reveal themselves.
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Fig. (6.13) The change in the total net force and phase angle for different off-set angles

(m=5)

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A time step is an important parameter in the solution of the equations of motion. There should be sufficient number of points to describe accurately an oscillation of the shaft centre. On the other hand unnecessarily larger points will cause a delay in reaching to steady-state vibrations. The longer the time to reach steady state vibrations, the longer CPU time needed. This means more expensive computation. Therefore before each run, a check for the number of points was made to ensure that about 40 points will be present in a complete oscillation. For example for the shaft speed of 10000 rpm the time step is taken 0.00005 s. Here the BPF for the system with 8 balls is taken as a basis since the resulting oscillations will be at this speed. Therefore there are 36 points in an oscillation for the given example.

The shaft was again rotated at a speed of 5000 rpm and the \( x \) and \( y \) oscillations were recorded. The FFT of the oscillations are shown in Figs.(6.14 and 6.15) for the \( x \) and \( y \) oscillations respectively. The BPF for this system for the given speed is 280 Hz which is apparent in Fig.(6.14) (a). Its first superharmonic (b) is also present at 560 Hz. For the \( y \) oscillations presented in Fig.(6.15) the same pattern is observed with the BPF (a) at 280 Hz, the first, second and third superharmonics of the BPF (c, d and e respectively) are observed at 560 Hz, 840 Hz and 1120 Hz respectively.

Fig.(6.14) FFT of the vibrations in the \( x \)-axis after the natural frequency is damped out
\( (m=8, \; Pr=5 \; N, \; n=5000 \; rpm, \; \omega_{bP}=280 \; Hz, \; \omega_n=500 \; Hz, \; c=300 \; Ns/m) \)
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6.4.1 Effect of Varying the Number of Balls

Increasing the number of balls means increasing the system stiffness and reducing the vibration amplitude. This is exhibited in Fig.(6.16) which shows the phase-plane diagram of the $x$ oscillations for different number of balls, when the shaft is released from an arbitrary position for free oscillations without damping. When the number of balls are increased, the centre of oscillations approaches zero implying a stiffer system. From this it can be predicted that increasing the number of balls will reduce the effect of the BPF. However, the BPF is the cage speed times the number of balls (see Chapter 4), therefore it coincides with the natural frequency at a lower shaft speed.

The preload has a significant effect on the system vibrations. When an axial preload is applied to a small number of balls its effect will be relatively greater. Therefore the axial preload was chosen to be 10 N (very low), so that its contribution would be negligible.

By introducing damping to the system, the transient vibrations were eliminated and peak to peak steady-state amplitudes were recorded as shown in Fig.(6.17).
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Fig.(6.17) shows the effect of varying the number of balls. For a relatively small number of balls, the peak to peak amplitudes of vibrations at the BPF is more significant. Since a small number of balls supporting the shaft, the natural frequency of the system is relatively low. For example, for the system having bearings with 5 balls the natural frequency coincides with the BPF at a shaft speed of 12700 rpm. This means the system natural frequency is 445 Hz. This value can also be obtained from Fig.(6.8) by considering an axial preload of 10 N. The effect of the first subharmonic of the BPF ($1/2 \omega_{bp}$) also appears in a spectrum at the speed of 6350 rpm.

![Graph showing the effect of varying the number of balls](image)

For a system having bearings with 8 balls, the peak to peak vibration amplitude decreases and the natural frequency is pushed to a higher value of 510 Hz and it coincides with the BPF at a speed of 9200 rpm. Although the natural frequency increases with an increase in the number of balls, the BPF coincides with the natural frequency at a lower shaft speed as explained earlier. The first superharmonic of the BPF ($2 \omega_{bp}$) also appear at a speed of 18400 rpm while the first and second subharmonics ($1/2 \omega_{bp}$ and $1/3 \omega_{bp}$ respectively) clearly exhibit themselves at the speeds of 4600 rpm and 3070 rpm respectively.
Fig. (6.17) The effect of varying the number of balls on the BPF (Pr=10 N, c=300 Ns/m)
When the number of balls are further increased, a peak at the BPF always appears in the vibration spectrum but at relatively lower peak to peak amplitudes as seen in Fig.(6.18). While the effects of superharmonics disappear, the subharmonic effects get more influential. In the case of 15 balls, the first, second, third and fourth subharmonic effects are all apparent in the spectrum with relatively smaller amplitudes as seen in Fig.(6.18).

**Effect of Varying the Preload**

The preload is assumed to be axially applied through springs. When a preload is applied, the initial contact angle and deflections will change. Increasing preload therefore means increasing the difference between the constant initial contact angle ($\alpha_0$) and the preloaded contact angle ($\alpha_p$). This will cause an increase in the initial axial displacement $z_0$. The larger the initial deflection, the more dominant it will be in the deflection equation (Equ.(3.26)). Therefore for larger preloads, the vibration amplitudes associated with the BPF will be reduced.

$$\delta_i' = \left( (Bd_b \sin(\alpha_0) + z_0 + z)^2 + (Bd_b \cos(\alpha_0) + \delta_r)^2 \right)^{\frac{1}{2}} - Bd_b$$  \hspace{1cm} (3.26)
Fig.(6.19) Frequency response showing the effect of varying the preload
(m=8, c=300 Ns/m)
Fig. (6.19) shows the set of results for a constant number of balls \((m=8)\) and mass \((M=5.5 \text{ kg})\). When there is an increase in the preload from 10 N to 30 N, there is a sharp decrease in the peak to peak amplitude at the resonant frequency (i.e. the BPF) as also proven experimentally [Wardle and Poon, 1983]. This can be predicted because the bearings get stiffer and they allow lower vibration amplitudes in the radial and axial directions. The effect of the natural frequency in the \(z\) direction is increased as observed at the natural frequency of 3000 rpm.

In Fig. (6.19) for a preload value of 50 N, which is given as the low preload value by the manufacturer, the amplitudes of the vibrations at the subharmonics of 6550 rpm and 3275 rpm show no decrease and are slightly higher than the amplitude at the BPF. The effect of the natural frequency in the \(z\) direction is dominant in the spectrum at a shaft speed of 3050 rpm. For a heavy preload value (100 N), as described by the manufacturer, the effect of the BPF is again dominant with its first and second subharmonics as shown in Fig. (6.19). The peak to peak amplitude at the resonant frequency shows an increase. This is because the heavy preload resulted in a high stiffness and the effect of damping force has lessened because of lower velocities.
6.5 Waviness

At relatively low speeds waviness causes few problems with precision bearings [Thomas, 1982]. Waviness only becomes a serious problem when components possess relatively high-amplitude features such as chatter marks, or where a precision bearing is fitted in a housing with poor geometry. At high speeds, however, the situation can radically change, a doubling of speed producing a doubling of vibration amplitude [Thomas, 1982]. A further effect with increased speed is causing the BPF and its harmonics to appear in the spectrum as the excitation frequency (due to “wave-pass”) near the BPF and can resonate with it. The amplitude of vibration depends on the relation between the rolling elements and the number of waves of the circumference of the ring.

6.5.1 Outer ring Waviness

The same procedure of observing the total net force and the phase angle described earlier in this chapter (see Fig.(6.12)) was applied to a bearing with a wavy outer race surface in order to study the resulting force variation due to waviness. Since the inner ring rotation is difficult to model in this manner, the inner ring waviness was not studied with this model. The waviness amplitude was set to 2 µm and the number of waves round the outer race circumference was varied for a bearing with 8 balls. The total net force and phase angle changes were recorded, as shown in Fig.(6.20).

For a relatively low number of waves (e.g. 1, 2, 3, 4 or 5 waves) the effect of waviness is small as shown in Fig.(6.20). However, when investigated closely, the total net force and phase angle are changing at a period equalling of the number of balls (i.e. 8 times in this case). When the number of waves increased to 6, the change in both the total net force and phase angle follow almost a sinusoidal pattern with a relatively higher amplitude as shown in Fig.(6.20).

For seven waves the variation in both the total net force and phase angle are large, the total net force varies between 100 N and 500 N and phase angle between 0 and 90 degrees. Phase angle almost linearly increases until 90 degrees and suddenly drops to 0 degrees in order to start linearly increasing again. This sharp change take place when the total net force is about 100 N. Therefore the change of the total net force follows a relatively smooth pattern as it goes through large forces. When the phase angle is 0 degrees the total force is equal to the force along the x-axis and when it is 90 degrees, it is equal to the force along the y-axis. However, the total net force and phase angle still change m times (m is the number of balls).

When the number of balls and the number of waves are equal, the change in the phase angle is negligible. This was also experimentally observed by Wardle and Poon [1983] and Wardle [1988]. Since the balls and the waves are in phase, the vibrations produced will be important. If the change in the phase angle is investigated in detail, minute changes may be observed as shown in Fig.(6.21). The jump from 0 to 90 degrees
disappears and as the system suggest (see Fig.(6.12)) the phase angle is almost steady at 45 degrees.
Fig. (6.20) ...continued on the next page
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For 9 waves in Fig. (6.20) the system behaves almost like the system with 7 waves but this time shift occurs from 0 to 90 degrees and then it decreases almost linearly to 0 degrees again. Continuing increasing the number of waves first to 10 and then to 11 causes
almost the same sort of effect in an opposite sense to the increasing of the number of waves from 4 to 6. When the number of waves are 11, 12 or 13 the change in the total net force and phase angle are small as seen in Fig.(6.20). However, when the case of 11 waves is, for example, investigated in detail, commencement of a radical change is observed as seen in Fig.(6.22). The number of oscillations of the total net force and phase angle are shifted to 16 which is twice the number of balls in this case.

Fig.(6.21) The change in the force for different off-set angles (m=8, N=8)

Fig.(6.22) The change in the force for different off-set angles (m=8, N=11)

Fig.(6.23) The change in the force for different off-set angles (m=8, N=16)
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The behaviour of the system for 14, 15 and 16 waves is similar to the system with 6, 7, and 8 waves. The only difference is that the oscillation of the total net force and phase angle is twice the number of balls. For 16 waves per circumference the phase angle change has reduced considerably, as it had for 8 waves. When it is observed closely, a similar pattern to that of 8 waves is seen as shown in Fig.(6.23).

![Graph showing time domain vibrations and FFT](image)

These results are in agreement with the predictions shown in Table (2.1) in Chapter 2 for the outer race waviness. The pattern of vibrations for a bearing with a wavy outer race is given as (see Table (2.1)):

\[
Waviness \text{ of orders: } k = q m \pm p \\
\text{Vibrations caused by outer ring waviness: } q m \omega_c
\]

(6.6)
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Fig. (6.25) ...continued on the next page
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Fig. (6.25) The FFT of vibrations due to outer race waviness of different orders
\(m=8, Pr=10 N, n=5000 \text{ rpm}, c=300 \text{ Ns/m}, \omega_{bp}=280 \text{ Hz}\)

For example if \(q=1\) and \(p=2\) in Equ.(6.6) the result matches with the Fig.(6.20) quite favourably. However, for \(q=1\) and \(p=4\) the results barely matches since for \(k=12\) the variations are transient and changing from 8 cycles to 16 cycle (see Fig.(6.21)). The same result can also be obtained from \(q=2\) and \(p=4\) where \(k\) is again 12 but vibrations are predicted to occur at twice the BPF. However, this is in an acceptable level since the amplitudes of the variations are small.

In order to see whether the predicted vibrations would occur in the simulation model, outer race waviness was introduced to the system. The outer races of both, right and left hand side, bearings were assumed to be wavy by the same amount, so that the shaft would move in a cylindrical mode.

The time and frequency domain vibrations of a shaft which is supported by bearings with 8 balls and rotating at a speed of 5000 rpm, are given in Fig.(6.24). In this case the outer race wave number is 8 with an amplitude of 3 \(\mu\text{m}\). As predicted from Fig.(6.20) the most dominant peak is at the BPF which is 280 Hz for the given shaft speed. Peaks at the super harmonics of the BPF are also present at 560 Hz, 840 Hz and 1120 Hz.

In order to observe whether the other predicted frequencies would also be observed, a set of results were obtained for a selected number of waves. The amplitude of the waves was 2 \(\mu\text{m}\) and the number of balls was 8.

When the number of waves is 7, a peak at the BPF appears as seen in Fig.(6.25). The amplitude of the peak is about 3 \(\mu\text{m}\). For 8 waves the main dominant peak is again at the BPF (see also Fig.(6.24)) with its super harmonics.

Experimental researchers have pointed out that severe vibrations occur for the outer race waviness when the number of waves equals to the number of balls [Wardle, 1988b].
Fig.(6.20) suggests that the variation of the total net force and phase angle are maximum when the order of waviness \( k = i \pm 1 \) where \( i \) is a positive integer. The amplitude of the peak at the dominant frequency for \( N=8 \) is less than the one for \( N=7 \) for the simulation model as shown in Fig.(6.25). Wardle [1988a] concluded from his linear model that the radial vibrations occur only when \( k = i m \pm 1 \) whereas the axial vibrations take place when \( k = m \). This is partially correct since the radial vibrations have larger amplitudes when \( k = i m \pm 1 \).

\[ \text{Fig.(6.26) The changing of the vibration phase from the BPF to twice the BPF} \]

\( (m=8, P=10, N=8, n=5000 \text{ rpm}, c=300 \text{ Ns/m}, \omega_{bp}=280 \text{ Hz}) \)

For \( N=12 \) in Fig.(6.25) the vibrations are very small as also predicted by the total net force change in Fig.(6.20). The pattern of vibration spectrums observed for \( N=16 \) and \( N=17 \) in Fig.(6.25) are similar to those that are predicted by the total net force and phase angle variations in Fig.(6.20).
A change of dominant vibration frequency from the BPF to twice the BPF \((2 \omega_b p)\) is of particular interest. The results presented in Fig.(6.26) shows this in the frequency domain. As seen in Fig.(6.26) when \(N=11\) the dominant vibrations are at the BPF. However, for \(N=12\) the dominant vibrations are observed at twice the BPF. This transformation was also predicted from the the total net force and phase angle variations in Fig.(6.20).

### 6.5.2 Inner ring Waviness

Inner ring waviness causes more complicated vibrations than outer ring waviness. In order to study the inner ring waviness, the bearings of the simulation model were assumed to have waviness in their inner rings of the same order and magnitude such that both bearings are identical. The amplitude of the waviness was 3 \(\mu\)m. As the inner rings of the bearings are fitted to the shaft, they rotate at the shaft speed.

Fig.(6.27) shows the vibration spectrums obtained for different orders of waviness. For relatively low wave orders (in this case it is also called out of roundness), the vibration spectrum due to waviness was first experimentally studied in detail by Gustafsson and Tallian et al. [1963]. They reported that vibration spectrum was dominated by the number of waves times the inner ring rotational speed (see Fig.(2.24)). This is also predicted from Table 2.1 in Chapter 2.

The predicted vibrations are clearly observed in the case of two waves in Fig.(6.27). A dominant peak at \(2 \omega\) (166 Hz) is observed besides peaks at the super harmonics of the shaft speed (249 Hz, 332 Hz, 415 Hz, 498 Hz), at inner race to cage relative speed \(((\omega-h) = 48 Hz)\) and the wave passage frequency (384 Hz). Peaks at more complicated speeds \((\omega_{wp}+2\omega =550\) Hz and \(2 \omega_{wp}+2\omega =934\) Hz) also appear. However, a more complicated frequency spectrum is observed for 3 waves. A peak at \(3 \omega\) (249 Hz) is observed at a relatively smaller amplitude implying a shape change in the vibration spectrum. While two relatively larger peaks are observed at \(\omega_{wp}-2\omega\) (218 Hz) and \(\omega_{wp}+\omega\) (467 Hz).

For 4 waves there is only one peak at the WPF (384 Hz). This is one of the turning points for the system behaviour. From this waviness order onwards the vibration frequency becomes more obedient to the formula given in Table 2.1:

\[
\begin{align*}
\text{Waviness of orders} & \quad k=q m \pm p \\
\text{Vibrations caused by inner ring waviness} & \quad q m (\omega-\omega_c) \pm p \omega
\end{align*}
\]

(6.7)
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The vibrations due to the waviness of order 5 are relatively small in amplitude (0.5 μm) and do not match with Eqn.(6.7). The peaks are at \( \omega_{wp}+2\omega \) (550 Hz) and \( 2\omega_{wp}-\omega \) (685 Hz). For 6 waves the frequency spectrum has a closer match with the equation. For \( q=1 \) and \( p=2 \) a peak at \( \omega_{wp}-2\omega \) (218 Hz). However, there are also peaks at \( \omega_{wp}-4\omega_c \) (244 Hz), \( \omega_{wp}+\omega \) (467 Hz), \( 2\omega_{wp}-\omega \) (685 Hz) and \( 2\omega_{wp}-\omega_c \) (715 Hz) which are not predicted by Eqn.(6.7).

Fig.(6.27) ...continued on the next page
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Fig. (6.27) The FFT of vibrations due to the inner race waviness of different orders
(m=8, Pr=10 N, n=5000 rpm, c=300 Ns/m, \( \omega_{bp}=280 \text{ Hz} \), \( \omega_{wp}=384 \text{ Hz} \))

Wardle [1988a] predicted from his linearised equations of motion that the vibrations would take place only at waviness of order \( N = i m \pm 1 \). However, the vibrations for nonlinear systems are relatively more complicated. Vibrations at his prediction speeds are the most severe ones and the formula given above matches with the simulation model perfectly for these values of waviness order. For example for 7 waves there is a peak at \( \omega_{wp}-\omega \) (301 Hz) where \( q=1 \) and \( p=1 \) and for 9 waves the peak is at \( \omega_{wp}+\omega \) (467 Hz) where \( q=1 \) and \( p=1 \). From above relationship for 8 balls vibrations are predicted at the
WPF (384 Hz). Its superharmonics at $2\omega_{wp}$ (768 Hz) and $3\omega_{wp}$ (1152 Hz) are also apparent in Figs.(6.27 & 6.28). For 10 waves the relation still holds and for $q=1$ and $p=2$ a peak at $\omega_{wp}+2\omega$ (550 Hz) is observed.

From 11 waves to 13 waves a transformation is observed from $q=1$ to $q=2$. Vibration amplitudes become negligible ($<0.08$ µm). For 11 waves, peaks at $11(\omega-\omega_c)+\omega$ (611 Hz), $22(\omega-\omega_c)-\omega$ (973 Hz) and $33(\omega-\omega_c)$ (1585 Hz) are observed. For the waviness of order 12, the predicted peak is either at $\omega_{wp}+4\omega$ or at $2\omega_{wp}-4\omega$ depending on the parameters chosen for $p$ and $q$ in Equ.(6.7). The peaks for 12 waves are at $2\omega_{wp}-4\omega$ (436 Hz), $12(\omega-\omega_c)$ (576 Hz), $\omega_{wp}+4\omega$ (716 Hz) and $24(\omega-\omega_c)$ (1152 Hz). A clear transformation from $q=1$ to $q=2$ can be observed in the vibration spectrum obtained for the waviness of order 12.

For 13 waves the dominant peak is at $2\omega_{wp}-3\omega$ (519 Hz) as predicted from the Equ.(6.7). However, there are still peaks at $3\omega_{wp}+2\omega$ (1318 Hz) and $5\omega_{wp}-\omega$ (1837 Hz).

There can be found four stages in the results presented in Fig.(6.27). From $N=3$ to $N=5$ is a transformation from relatively less vibration amplitude to larger vibration amplitudes. From $N=6$ to $N=10$, there are predictable vibrations for $q=1$ in Equ.(6.7). From $N=11$ to $N=13$, there is a transformation from $q=1$ to $q=2$ in Equ.(6.8) and from $N=14$ to $N=17$ is
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again predictable vibrations for \( q=2 \). The same trend for larger orders of waviness can be expected.

### 6.5.3 Ball Waviness

In order to study the effect of ball waviness, one ball from each bearing was assumed to have a wavy surface. The case was simplified as the ball was assumed to rotate around an axis that passes through its centre and parallel to the bearings axis. The amplitude of the waviness was 3 \( \mu \text{m} \). The wavy balls were in phase in each bearing.

*Fig.(6.29) Vibrations due to ball waviness (\( N=4, \ m=8, \ n=5000 \ \text{rpm}, \ Pr=10 \ N, \ c=300 \ \text{Ns/m}, \ \omega_c=35 \ \text{Hz}, \ \omega_b=278 \ \text{Hz}, \ 4\omega_b=1112 \ \text{Hz})*
In the case of ball waviness there are two important frequencies. The ball set rotates at the cage speed round the inner ring and the ball with the wavy surface acts like an oversized ball, hence since the ball set comes to the same position after one cage rotation, the system undergoes vibrations at the cage speed (for more information see off-sized-balls in Section 6.6). Other important frequencies are the ball rotation frequency. When the ball rotates, the position of the balls will repeat itself after each $2\pi/N$, where $N$ is the number of waves per circumference of the ball. Therefore, the
vibrations due to ball waviness will take place at the speed of $N\omega_b$.

Fig.(6.29) shows the vibrations for the bearing with a ball having waviness of order 4. The shaft rotates at a speed of 5000 rpm. For this speed, the cage speed is 35 Hz and the ball rotating speed is 278 Hz. When the time domain vibrations are investigated, there are two main frequencies: one at 35 Hz (the cage speed) and the other is at about 1110 Hz ($4\omega_b$). The frequency spectrum shows similar results. When the peak at about 1110 Hz is investigated in detail, a set of peaks at $N\omega_b \pm i\omega_c$ is observed as seen in Fig.(6.29). Relatively higher peaks appear at $N\omega_b \pm \omega_c$.

![Figure 6.31](image.png)

**Fig.(6.31) The FFT of the vibrations due to ball waviness**

(m=8, n=5000 rpm, N=2, Pr=10 N, c=300 Ns/m, $\omega_c=35$ Hz, $\omega_b=278$ Hz, $2\omega_b=556$ Hz)

Increasing the number of waves means making the ball smoother with a larger diameter. This can be seen from Fig.(6.30) as well. As the number of waves are increased the vibration frequency is dominated by the cage speed as clearly seen in the time and frequency domain vibrations shown in Fig.(6.30). For a more detailed discussion on vibrations of this sort see Section(6.6).

When the $N\omega_b \pm i\omega_c$ coincides with the natural frequency of the system severe vibrations take place. This is shown in Fig.(6.31) where a bearing with a wavy ball of order 2 causes vibrations at a speed close to the natural frequency of the system and hence resulting in severe vibrations.
6.6 Off-sized Ball

First, for further simplification of the problem the contact stiffness coefficient between the balls and the raceways were considered to be linear and constant. For a simple radial ball bearing as seen in Fig.(6.12) the angles and reference axis were set as was in Chapter 3 (see Fig.(3.15)).

Balls were radially preloaded in order to ensure the continuous contact of all balls and the raceways, since otherwise a chaotic behaviour might be observed. The preloaded deflection for each ball, $\delta_0$, was assumed to be $5 \mu m$. The centre of the inner race is shifted $2 \mu m$ in the $x$ direction and $2 \mu m$ in the $y$ direction with respect to the outer race centre. For different off-set angles the total net force and the angle between the net force and the $x$ axis (phase angle) as seen in Fig.(6.12) were recorded.

![Fig.(6.32) The change in the force for different off-set angles for a linear system (n=5, 3rd ball is 1 \mu m oversized)](image)

Fig.(6.32) shows the change in the force and the phase angle for such a bearing with 5 balls. In this case the third ball in the set (see Fig.(6.34) for numbering) is $1 \mu m$ oversized. The results show that the changes are periodic with a frequency of cage speed, implying vibrations at the cage speed. The results also imply that the most severe vibration for this system will occur when the cage speed coincides with the natural frequency. The shapes of the change are perfectly sinusoidal with a small shift. This is understandable since the other forces will be cancelled out by each other and the only force acting on the bearing will be that of the off-sized ball's. This force will increase as the ball enters to the loaded zone and decreases as it enters to the relief zone. This is also observed in Fig.(6.32).

Later, similar results were obtained for a bearing with nonlinear ball to race contact stiffness proportionality as seen in Fig.(6.33). The shape of the curves show slight
change due to the nonlinearity. The change in the force remained cyclic with the speed of cage. Although Tamura [1968] and Meyer et al. [1980] suggested that the off-sized ball would cause vibrations at the cage speed and its multiples, the study on the ball passage frequency indicated that the vibrations at the multiples of cage speed would take place only for special cases. Therefore a series of different cases was tested resulting in following outputs.

![Fig. (6.33) The change in the force for different off-set angles for nonlinear system (m=5, 3rd ball is 1 μm oversized)](image)

Figs. (6.35–6.39) show the change in the total net force and phase angle for different locations of off-sized balls. In Fig. (6.35) a bearing with 5 balls is tested. The second and the fifth balls (see Fig. (6.34) for ball numbers) are 1 μm oversized from the rest. As Fig. (6.35) shows the change in the combination of off-sized balls or numbers of off-sized balls will only cause a change in the shape of the curve. The total net force will change cyclically at the cage speed. This is also true for a bearing with 8 balls as seen in Fig. (6.36), in the case of the second, fourth and sixth balls are 1 μm oversized.
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Results and Discussion: Ball Bearings

For a bearing with 6 balls, it is shown that vibrations will take place at twice the cage speed in the case of the second and fifth balls are 1 μm oversized as seen in Fig.(6.37) and three times the cage speed in the case of the second, fourth and sixth balls are oversized as seen in Fig. (6.38). These results may explain the experimental studies of Tamura [1968] and Yamamoto et al. [1981]. The former observed vibrations at twice the cage speed in the axial direction when there are two neighbouring balls oversized but wrongly concluded that off-sized balls would cause vibrations at the multiples of the cage speed and the latter observed vibrations at the cage speed and twice the cage speed.

The results indicate that off-sized balls will cause vibrations at the multiples of cage speed only when the off-sized balls are symmetrically distributed. In this case, balls will repeat their pattern as explained in Section (6.4). This was further confirmed with Fig.(6.39)
where a bearing with 8 balls produced cyclic forces at four times the cage speed for the second, fourth, sixth and eighth balls are oversized.

![Graph](image1)

*Fig.(6.37) The change in the force for different off-set angles (m=6, 2nd and 5th balls are 1 μm oversized)*

![Graph](image2)

*Fig.(6.38) The change in the force for different off-set angles (m=6, 2nd, 4th and 6th balls are 1 μm oversized)*

Table (6.1) shows the possible multiples of cage vibrations for a bearing with different number of balls. The difference in the size of balls within the off-sized ball set produced the same effect of having only one off-sized ball. However, the vibration amplitude was affected by this change.

Having these initial results in hand, further research was carried out in order to obtain the vibrations due to off-sized balls from the simulation model. The off-sized balls were located symmetrically in both bearings such that they will move in the same direction simultaneously (i.e., the balls are assumed to be in phase).
Results and Discussion: Ball Bearings

Fig.(6.39) The change in the force for different off-set angles (m=8, 2nd, 4th, 6th and 8th balls are 1 μm oversized)

<table>
<thead>
<tr>
<th>No of Balls</th>
<th>Possible Speeds at Peaks</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$\omega_c$</td>
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<td>3</td>
<td>$\checkmark$</td>
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<td>4</td>
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<td>17</td>
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</tbody>
</table>

Table (6.1) Frequencies of the possible vibrations for different numbers of balls in a bearing
First, two balls were assumed to be 1 μm oversized for a bearing with 8 balls. These were the second and the fourth balls in the set for both right and left bearings. An external damping of 300 N s/m was introduced to the system in order to eliminate the effect of the natural frequency and an axial preload of 100 N was applied to ensure the continuous contact of the raceways and balls. Peak to peak amplitudes were recorded when steady-state vibrations were reached. Fig.(6.40) shows the peak to peak amplitudes against the different shaft speeds.

From Fig.(6.41) for a shaft supported by angular contact ball bearings with 8 balls and preloaded with 100 N, the natural frequency can be found to be 840 Hz (≈ 50400 rpm).
As predicted earlier for a bearing with 2 oversized balls (1st and 3rd) the system is forced at the cage speed. When the cage speed coincides with the natural frequency, the system resonates as shown in Fig.(6.40). For the shaft speed of about 120000 rpm the cage frequency is about 50400 rpm and this is clearly observed in Fig.(6.40) as predicted from Fig.(6.41). The curve indicates jumping behaviour for hardening type stiffness. This is because the oversized ball goes through a large deflection, hence this changeover a large deflection range causes a clear exhibition of hardening type behaviour and a jump in the amplitude.

In order to verify the predicted behaviour made by the total net force observations, the off-sized balls are placed as such, the first and fifth balls are 1 μm oversized for a bearing with 6 balls. From Fig.(6.37) the system is expected to resonate when the $2\omega_c$ coincides with the natural frequency.

Fig.(6.41) shows that at a shaft speed of 60000 rpm, twice the cage speed which is 50400 rpm (840 Hz) coincides with the natural frequency. However, the resonant amplitude is relatively small at about 2 μm. At the speed twice the off-sized ball passage frequency ($4\omega_c$), the cage speed also coincides with the natural frequency and results in a vibration amplitude of about 10 μm.

\[
\begin{align*}
\text{rotational shaft speed, r.p.m.} & \\
\text{peak-peak amplitude, m} & \\
0 & 0.000e+0 \\
1.0e-6 & 0.000e+0 \\
2.0e-6 & 0.000e+0 \\
3.0e-6 & 0.000e+0 \\
\end{align*}
\]

Fig.(6.42) Peak to peak amplitudes of the system vibrations for a bearing with 2nd, 5th and 8th balls are 1 μm oversized ($m=9, Pr=10 N, c=300 Ns/m, \omega_n=580 Hz$)

In order to show the off sized ball effect further, and to check the predictions of the total net force investigation, another set of results for a bearing with 9 balls were obtained. The first, fourth and seventh balls were made 1 μm oversized and the preload value was set to 10 N. It is predicted by Fig.(6.38) that the resonance would occur at the speed of $3\omega_c$ which is 27500 rpm ($\omega_n=580 Hz$) for the given system.
As seen in Fig.(6.42) the system goes into resonance at this speed. The first super harmonic of the off sized ball passage frequency is at 55000 rpm and its first and second subharmonics at frequencies 13750 rpm and 9165 rpm respectively are also observed in Fig.(6.42). Since the super harmonic is close to the natural frequency, the dominant peak is at this speed.

![Graphs showing peak-to-peak amplitudes for various bearing conditions.](image)

*Fig.(6.043) Peak to peak amplitudes of the system vibrations for a bearing with 1st, 3rd, 5th and 7th balls are 1 µm oversized (m=8 Pr=10 N, c=300 Ns/m)*
In order to study the effect of the amplitude of the oversized ball diameter, a case was studied. The preload was reduced to 10 N and the first, third, fifth and seventh balls were made 0.5 μm, 1 μm and 2 μm oversized in turn. Fig. (6.39) suggests that the resonance will occur at the speed of $4 \omega_c$.

In Fig. (6.43) for the bearing with 0.5 μm oversized balls, the resonance occurs at 21000 rpm which means a natural frequency of 600 Hz. This is relatively higher with respect to 510 Hz of a perfect system in the same conditions. The difference is due to the off-sized balls. The diameter difference of the balls due to waviness acts as preload to the system. This point is further proven when the results from 1 μm and 2 μm oversized ball are investigated. For the former the natural frequency is 630 Hz and for the latter it is 700 Hz as seen in Fig. (6.43).

Fig. (6.43) shows that increasing the diameter difference for oversized balls decreases the resonance amplitude resulting in the vibration spectrum becoming more complicated. For example for 2 μm diameter difference the sub and superharmonics of the off-sized ball passage frequency also appears in the spectrum. Another important point is that in this example the balls are symmetrically distributed. If only one ball is oversized, when the diameter difference increases, the peak to peak amplitude of the vibrations will also increase.
6.7 Misalignment

6.7.1 Both Bearings are Misaligned

6.7.1.1 Outer ring misalignment

In order to see the effect of misalignment a simple case of outer ring misalignment is studied. In Equ.(4.35) and Equ.(4.36) in Chapter 4, \( a \) and \( \theta_c \) were both set to 0. The misalignment angle \( \rho \) was set to be 0.02 rad. and both bearings were misaligned in the opposite sense in order to eliminate the \( z \) oscillations and shaft rotated at a speed of 3000 rpm.

Fig.(6.44) show the FFT of the \( x \) oscillations. Vibrations occur at two frequencies: one is the cage speed (21 Hz) and the other is the system natural frequency (820 Hz). The outer ring misalignment produces a case that the same position of balls can only be obtained after one cage revolution and the system is therefore excited at this frequency. This results in vibrations at the cage speed. Since the superharmonics of the cage speed excites the natural frequency, vibrations at the natural frequency are also observed. In the case of misalignment, the system natural frequency is shifted to a relatively higher frequency since the misalignment acts to preload the system. As studied earlier in this chapter varying the positions of balls alters the natural frequency. This becomes more significant when the misalignment is applied. While the balls in the upper half of the bearing are preloaded more, the balls in the lower half are relieved. This results in a relatively more nonlinear system. This is not very significant in the \( z \) direction since the \( z \) movement is cancelled out.

![Fig.(6.44) The FFT of the vibrations due to outer ring misalignment](image-url)

\( m=8, \rho=0.02, P_r=10 \text{ N}, n=3000 \text{ rpm}, c=300 \text{ Ns/m}, \omega_c=21 \text{ Hz}, \omega_n=820 \text{ Hz} \)
6.7.1.2 Inner ring misalignment

Fig. (6.45) shows the inner ring misalignment effect on the system vibrations. Three significant shaft speeds were tested. In each case the resulting vibrations were at the shaft speed. Since the effect of position of the most loaded ball is more dominant than the position of the ball set itself, the vibrations at the shaft speed overcome the vibrations at the inner ring ball passage frequency. A small peak was observed at the inner ring ball passage frequency when the FFT of the oscillations was obtained but the amplitude of the peak was very small with comparison to the peak at the shaft rotating speed. As pointed out by the experimental studies of Wardle and Poon [1983] this may excite the low order shaft resonances.

![Image of vibrations due to inner ring misalignment](image)

Fig.(6.45) Vibrations due to the inner ring misalignment
(m=8, ρ=0.02, Pr=10 N, c=300 Ns/m)

6.7.2 LHS Bearing is Misaligned

When only the inner ring of the LHS bearing is misaligned, the vibrations produced is violent. For a shaft speed of 5000 rpm the superharmonics of the shaft speed coincide with the natural frequency of the system in the x direction and excite it. Due to the misalignment, the natural frequencies in the x direction and y direction are no longer the same or are very close. For the x oscillations, small peaks at ω_n ±ω appear in the spectrum. The vibrations in the y direction are at the shaft speed and its superharmonics. However, the vibrations in the z direction are at twice the shaft speed and its harmonics as seen in Fig.(6.46).
Fig. (6.46) Vibrations due to the inner ring misalignment on the LHS bearings only
\( m=8, \rho=0.02, Pr=10 \ N \ n=5000 \ rpm, \ c=300 \ Ns/m, \ \omega_x=83 \ Hz, \ \omega_n=35 \ Hz, \ \omega_n=750 \ Hz \)

### 6.8 Defects

A crack or debris was assumed to be located on the running surface. It was assumed to have 3 \( \mu \)m depth and 1 degree of width.
6.8.1 Outer race Defect

When the defect is on the running surface of the outer ring, vibrations take place at the BPF and the system natural frequency. When it or its harmonics coincide with the natural frequency of the system severe vibrations are observed. At lower shaft speeds, it is almost certain that one of the superharmonics of the BPF will coincide with the natural frequency.

![Graph showing frequency spectrum of vibrations due to defected outer ring for different shaft speeds](image)

*Fig. (6.47) The frequency spectrum of vibrations due to defected outer ring for different shaft speeds (*Pr*=10 N, *m*=8, *c*=300 Ns/m)*

For a shaft speed of 2000 rpm the system vibrates at the natural frequency of 560 Hz as seen in Fig. (6.47). For this speed the BPF is 112 Hz. \(5\omega_{bp}\) coincides with the natural frequency and the system resonates. For a shaft speed of 3000 rpm the BPF is 168 Hz. Therefore peaks at \(i\omega_{bp}\) where \(i = 1, 2, \ldots\) appear in Fig. (6.47) for this speed. The dominant peak is at 504 Hz \(3\omega_{bp}\) since it is the closest harmonic to the natural frequency. A small peak at the natural frequency appears as well. As for a speed of 5000 rpm the BPF is 280 Hz and \(2\omega_{bp}\) coincides with the natural frequency, the vibrations take
place at the natural frequency for this speed as seen in Fig.(6.47).

In order to confirm the point made above, another shaft speed (7000 rpm) was chosen since for it the BPF is 390 Hz and \(2\omega_{bp}\) is 780 Hz, both being further away from the natural frequency. As seen in Fig.(6.47) the dominant peak is at the BPF for this speed. Peaks at its superharmonics also appear. A peak at the natural frequency is also present in the spectrum. However, when the amplitudes are compared, the vibrations for a shaft speed of 7000 rpm are not significant as the maximum peak amplitude is only about 0.3 \(\mu m\), compared to 4 \(\mu m\) for \(n=2000\) rpm or \(n=5000\) rpm.

The results obtained from the simulation model are similar to experimental results reported by Yhland and Johansson [1970], Braun and Danter [1979], Igarashi et al. [1982; 1983; 1985], McFadden and Smith [1985] and Karakurt [1989]. They all observed that when a defect exists on the outer ring running surface, the resulting vibrations are at the BPF and the system resonates when it or its harmonics coincide with the natural frequency.

### 6.8.2 Inner ring defect

More complicated vibrations take place when the defect is on the inner ring running surface since the defect itself rotates at the shaft speed. Some researchers predicted that it would cause vibrations at the inner ring ball passage frequency [Braun and Danter, 1979; Karakurt, 1989]. Some others [McFadden and Smith, 1985] pointed out that in the case of inner ring defect, the spectrum would accommodate peaks at the shaft speed, cage speed, their harmonics and their combinations. The frequency spectrums shown in Fig.(6.48) exhibit peaks at shaft speed, cage speed, ball passage speeds of the outer and inner race and their harmonics as also pointed out by Taylor [1980].

For a shaft speed of 1000 rpm peaks are at the shaft speed (17 Hz), the BPF (56 Hz), the inner ring ball passage frequency \((m (\omega -\omega_2) = 77\) Hz) and their harmonics. As some of these coincide with the natural frequency, a resonant peak at the natural frequency (540 Hz) appears as shown in Fig.(6.48).

For a shaft speed of 3000 rpm, peaks at the shaft speed (50 Hz), the BPF (168 Hz), the inner ring ball passage frequency (232 Hz) and their sub and superharmonics appear in the frequency spectrum. The most dominant peak is at 504 Hz \((3\omega_{bp})\) as it is close to the natural frequency (540 Hz) and excites it. Peaks at the inner ring ball passage frequency and its harmonics (116 Hz, 232 Hz, 348 Hz, 696 Hz) are less dominant as seen in Fig.(6.48). When the vibration peaks in Fig.(6.48) for the shaft speed of 3000 rpm and 5000 rpm are observed overall, the peaks form almost a semi-parabola having the centre at the natural frequency.
For a shaft speed of 4000 rpm the most dominant peak in the frequency spectrum in Fig.(6.48) is at the natural frequency as it coincides with $19 \, \omega_c$ (530 Hz). A relatively small peak at the shaft speed (67 Hz) and a peak at the BPF (224 Hz) also appear in the spectrum.

For a shaft speed of 5000 rpm the most dominant peak is again at the natural frequency since it is close to the $2 \omega_{bp}$ (560 Hz). Other peaks at the shaft speed (83 Hz), the BPF (280 Hz), the inner ring ball passage frequency (387 Hz). The second dominant peak in
the spectrum is at $\omega_n - \omega$ (457 Hz) and the third dominant peak is at $\omega_n + 4\omega$ (680 Hz).

A similar frequency spectrum was obtained for a shaft speed of 7000 rpm as shown in Fig.(6.48). However, in this case the natural frequency coincides with the inner ring ball passage frequency $(8(\omega - \omega_1) = 544$ Hz). The second and third dominant peaks are at $\omega_n - \omega$ (423 Hz) and $\omega_n + \omega$ (657 Hz). Another dominant peak is at the shaft speed (117 Hz).

The vibration spectrum obtained for a shaft speed of 9000 rpm in Fig.(6.48) is similar to the one for 1000 rpm or 4000 rpm in Fig.(6.48). The BPF (504 Hz) resonates with the natural frequency so the vibrations are at the natural frequency.

A close similarity was observed between the frequency spectrums in Fig.(6.48) and the experimental spectrum due to the inner ring defect presented in Fig.(2.38) in Chapter 2.

6.8.3 Defect on the Ball Surface

One of the balls on each ball bearing was assumed to have a scratch of 3 $\mu$m on its running surface. The scratch was assumed to be rotating around an axis passing through its centre and parallel to bearing axis, hence the scratch was passing the same points on the inner and outer raceways at regular intervals.

The vibrations caused by the defect will have a frequency of twice the ball rotating speed since it inserts the same amount of force once each half rotation.

Fig.(6.49) shows the frequency spectrum obtained for different shaft speeds. For a shaft speed of 2000 rpm the ball rotating speed is 111 Hz. Since $5 \omega_b$ (555 Hz) is in the neighbourhood of the system natural frequency, it resonates with the natural frequency and results in vibrations at 555 Hz. Peaks at the ball rotating speed and its superharmonics (111 Hz, 222 Hz, 333 Hz, 444 Hz, ...) appear in the spectrum. The peaks at $2\omega_b$ and its superharmonics (222 Hz, 444 Hz, 666 Hz, ...) have relatively higher amplitudes, implying the actual forcing frequency and its superharmonics. For a shaft speed of 3000 rpm in Fig.(6.49), the frequency spectrum exhibits peaks at only $2\omega_b$ and its harmonics. In this case the ball rotating speed is 166.61 Hz. Peaks at 333 Hz ($2\omega_b$), 666 Hz ($4\omega_b$), 999 Hz ($6\omega_b$) and 1333 Hz ($8\omega_b$) are present in the spectrum. However, the dominant peak is at 666 Hz since it is the closest one to the natural frequency.

For a shaft speed of 4000 rpm in Fig.(6.49) the dominant peak is at $2\omega_b$ (444 Hz). Peaks at its superharmonics (888 Hz and 1333 Hz) also appear in the spectrum. However, since $3\omega_b$ (666 Hz) is closer enough to the natural frequency to excite it, a peak at the natural frequency (540 Hz) and peaks at 222 Hz ($\omega_b$), 666 Hz ($3\omega_b$) and 1110 Hz ($5\omega_b$) also appear in the spectrum in Fig.(6.49).
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Fig.(6.49) The FFT of the vibrations due to defected ball (F=10 N, m=8, c=300 Ns/m)

For a shaft speed of 5000 rpm the ball rotating speed is 277 Hz. 2\(\omega_b\) (554 Hz) is the main excitation force and is close to the natural frequency. Therefore the resulting vibrations are at the natural frequency of the system.

6.9 Vibrations in 5 Degrees of Freedom

As was seen in the previous section in this chapter, the vibrations associated with ball bearings are complicated even with 3 degrees of freedom. Therefore although the vibrations of a shaft supported by ball bearings were simulated in 5 degrees of freedom, only the simulation in three degrees of freedom was generally employed in the investigation.

Fig.(6.50) shows the vibrations for a geometrically perfect bearing in 5 degrees of freedom. Since shaft also vibrates in the z direction, the shaft centre no longer moves in synchronisation so the effect of the BPF becomes insignificant. This can be clearly observed in Fig.(6.51) for the FFT of the vibrations along the x and y axes. Rocking and yawing angle variations are also very small.
Fig.(6.50) The vibrations of a shaft supported by ball bearings in 5 degrees of freedom
(Pr=10 N, m=8, M*=31124, n=5000 rpm)
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Results and Discussion: Ball Bearings

Fig.(6.51) The FFT of the vibrations of a shaft supported by ball bearings in 5 degrees of freedom along the x and y axes (Pr=10 N, m=8, M=31124, n=5000 rpm)

Hence in this thesis no further runs were made in 5 degrees of freedom. However, when actual experimental results of other workers are compared with the simulation model, it is probable that simulation in 5 degrees of freedom would give closer predictions.

For the perfect system the frequency spectrums of vibrations presented in Fig.(6.9) (or Fig.(9.10)) show a similarity to the frequency spectrum in Fig.(6.51) implying that since the system is defect free, the vibrations of the shaft is not disturbed. However, since the oscillations in the z direction becomes larger, they become dominant in the deflection equations (i.e. Equ.(3.32) and Equ.(3.34)), hence the effect of the BPF reduces.
However, in the \( y \) oscillations a peak at the BPF (280 Hz) appears in Fig. (6.51) but it is not significant.

**Closure**

In this chapter the natural frequency, the BPF, the effects of varying number of balls and preload on the BPF are studied with the simulation model. It was shown that with a careful design many problems caused by the effects of the natural frequency and the BPF can be avoided. The imperfections of the shaft ball bearings namely out of roundness, waviness, misalignment, defects on the running surfaces, were also investigated. The conclusions of these studies will be given in Chapter 8. In the next chapter, the dynamic properties of elastomers will be investigated and elastomers as external dampers will be introduced to the system in order to eliminate or reduce the untoward effects of the frequencies discussed in this chapter to an acceptable level.
CHAPTER 7

RESULTS AND DISCUSSION:
ELASTOMERS AS EXTERNAL DAMPERS

7.1 Introduction

In the previous chapter, it was shown that even a shaft-ball bearings system equipped with geometrically perfect ball bearings, undergoes vibrations due to variations in the overall contact stiffness of the ball bearings. With defects in addition, vibrations can be quite severe. Therefore in this chapter elastomers as external dampers will be introduced to the simulation model in order to eliminate the untoward effects of vibrations associated with ball bearings.

There is limited data available on the dynamic properties of elastomers since the stiffness and damping coefficients of elastomers depend on their geometry. The experimental data available in the references were obtained for certain geometries and could not be directly used in this thesis. There is to date no acceptable, sufficiently correct theory that gives the dynamic properties of elastomers for different geometries.

Therefore in order to obtain the dynamic properties of elastomer for cylindrical buttons, ring cartridges and O-rings used in this thesis, a study was undertaken as discussed in Chapter 5. In that study dynamic properties of elastomers were obtained from their material properties (i.e., Young's or shear modulus). In this chapter the predictions of the formulae obtained in Chapter 5 will be compared with experimental findings and other researchers' predictions.
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Results and Discussion: Elastomers as External Dampers

The properties obtained will later be applied to the shaft-ball bearing system as external dampers. For this purpose, a three degrees of freedom system will be considered and the effect of elastomer dampers on reducing the untoward effects of natural frequency, the BPF and out of balance of the shaft centre will be investigated.

7.1 Dynamic Properties of Elastomers

The formulae developed in Chapter 5 will be employed in predicting the stiffness and loss coefficients of the elastomer buttons, ring cartridges and O-rings respectively.

For the buttons and cartridges the material used was polybutadiene which is classed as a Broad Temperature Range (BTR) elastomer. This material is chosen because of the availability of the experimental data in the NASA contractor reports [Chiang et al, 1972; Gupta et al., 1974; Smalley and Tessarzik, 1975, Darlow and Smalley, 1976]. For O-rings four different materials are used namely polybutadiene, Viton 70, Viton 90 and Buna–N. Since the all necessary experimental data for Viton 70, Viton 90 and Buna–N are not available, the O-rings used in this thesis are thought to be in the same size as in the NASA contractor report [Darlow and Smalley, 1976]. This is not a major problem since the dimensions of O-rings tested in the report were very close to the ones required in this thesis.

7.2.1 Cylindrical Buttons

Three different button dimensions are studied as given in Table 5.1. All had the same diameter of 0.0127 m but each had different height. Each of the button cartridges had six elastomer buttons arranged in two rows, with three buttons equally spaced circumferentially in each row as seen in Fig.(5.9).

For this arrangement the compressive and shear stiffness of a ring of three buttons are both equal to three halves of the corresponding stiffness of a single button for a single row [Darlow and Smalley, 1977]. Since there are two rows in this case, Equ.(5.25) will be multiplied by 3. The Young’s modulus (E) in Equ.(5.25) is assumed to be about three times the shear modulus (G) [Smalley and Tessarzik, 1976; Darlow and Smalley, 1977].

Fig.(7.1) shows the prediction values for the button cartridges of buttons with a height of 0.00318 m. Predictions of formulae given by Smalley and Tessarzik [1975] (Equ.(5.52) and Equ.(5.53)) and the predictions of the formula developed in this thesis are very close. The predictions for the stiffness and experimental data matches quite favourably.

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For relatively low speeds the prediction, including shear effect, is closer to the experimental data while for relatively higher frequencies the predictions, neglecting shear effect, is closer to the experimental data. In this experiment both ends of the buttons were glued in place, it was therefore expected that they would undergo shear as well as compression loading. Experimental results suggest that for higher frequencies the shear effects are small [Darlow and Smalley, 1976].

![Graph showing stiffness and loss coefficients as a function of frequency]

For the loss coefficients it is clear that the predictions from Equ.(5.25) are much closer to the experimental data than the predictions of NASA (i.e., Equ.(5.53)) as shown in Fig.(7.1).

Fig.(7.1) Stiffness and loss coefficients of button cartridge no 1 (0.00318 m) as a function of frequency

For the loss coefficients it is clear that the predictions from Equ.(5.25) are much closer to the experimental data than the predictions of NASA (i.e., Equ.(5.53)) as shown in Fig.(7.1).
Fig. (7.2) shows the results for a button height of 0.00238 m. Since in this case both sides of the buttons were glued in place it was expected the predictions including shear effect would match better with the experimental results. As seen in Fig. (7.2) the predictions for stiffness including shear effect matches with the experimental data almost precisely.

In this case predictions from Eq. (5.52) including shear effect is slightly higher whereas the predictions neglecting shear effect matches with the experimental data quite favourably. Prediction for the loss coefficient are slightly higher than was experimentally...
observed. However the predictions obtained from the formula developed in this thesis show the same trend of reduction with an increase in the frequency.

Fig.(7.3) shows the results for a button height of 0.00476 m. In this case buttons were only glued to the inner ring and hence it was expected that shear effect would be negligible. However, predictions for stiffness coefficients were slightly higher than the experimental data although the trend of the data and the prediction from the formula derived in this thesis were the same. The predictions for the loss coefficients were favourably close to the experimental data.

![Graph showing stiffness and loss coefficients](image)

*Fig.(7.3) Stiffness and loss coefficients of button cartridge no 3 (0.00476 m) as a function of frequency*
7.2.2 Ring Cartridges

Two different dimensions were tested for ring cartridges. Both cartridges had the same inner and outer diameter but had different thicknesses as shown in Table (5.2). For the cartridges the material used was polybutadiene.

The first cartridge had a thickness of 0.0048 m. The results for this ring cartridge is given in Fig. (7.4). As is clearly seen, the closest predictions are obtained from two formulae developed in this thesis.

![Stiffness and loss coefficients of ring cartridge no 1 (0.0048 m) as a function of frequency](image-url)
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Again, for the cartridge with a thickness of 0.0024 m, the closest predictions are obtained from the formulae developed in this thesis as shown in Fig.(7.5).

For both cartridges the predictions for the loss coefficients were the same from all prediction formulae.

![Graph showing stiffness and loss coefficients of ring cartridge no 2 (0.0024 m) as a function of frequency.]

Fig.(7.5) Stiffness and loss coefficients of ring cartridge no 2 (0.0024 m) as a function of frequency

7.2.3 O-rings

There were no direct experimental results available for O-rings. Therefore, an O-ring was assumed with the dimensions given in Chapter 5 for the O-ring. The material is assumed to be the same as that of the ring cartridges (i.e. polybutadiene).
Three different formulae were derived in Chapter 5 for O-rings. As shown in Fig.(7.6) the predictions for stiffness coefficients formed two groups. The first group are the predictions from relatively simpler approaches and in the second group there are two predictions. One of them is the empirical formula of Lindley (Equ.(A5.)) and the second one is the direct solution of the integral given in Equ.(5.41).

As seen in Fig.(7.6) the predictions from the Lindley’s empirical formula and that from the solution of Equ.(5.41) are almost the same. The predictions for the loss coefficient from all the formulae were the same as seen in Fig.(7.6).

*Fig.(7.6) Stiffness and loss coefficients of O-ring as a function of frequency*
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7.3 Simulation of Elastomer External Dampers with a Mechanical Voigt Model

In the previous section the dynamic properties of elastomer dampers were obtained from their material properties. In this section a mechanical Voigt model will be introduced in order to represent the dynamic behaviour of elastomer dampers in the form of cylindrical buttons and O-rings. The polybutadiene was used for the button cartridges and Viton 70, Viton 90 and Buna-N were used for O-rings. For cylindrical buttons the experimental data was obtained at 32 °C and for the all elastomer O-rings the testing temperature was 25 °C. The experimental results were obtained from NASA contractor reports [Darlow and Smalley, 1976; Smalley et al., 1977]

![Graph of Voigt Model](image)

**Fig.(7.7) Correlation of theoretical results of Equ.(5.25) with a single Voigt Model for the compliance and loss angle of the elastomer buttons**

\[ K_0 = 4.086 \times 10^7 \text{ N/m} \]
\[ K_1 = 6.974 \times 10^7 \text{ N/m} \]
\[ c_1 = 58052 \text{ Ns/m} \]


7.3.1 Elastomeric Cylindrical Buttons

The prediction method developed in this thesis for buttons was employed in order to obtain the Voigt model elements. A one element Voigt model is employed and the results shown in Fig.(7.7) were obtained. The figure shows the Voigt element curve-fit with the theoretical compliance and the loss coefficient.

![Voigt Model and Experimental Data](image)

\[ K_0 = 4.086 \times 10^7 \, \text{N/m} \]

\[ K_1 = 6.974 \times 10^7 \, \text{N/m} \]

\[ c_l = 58052 \, \text{Ns/m} \]

*Fig.(7.8) Correlation of experimental data with a single Voigt Model for the stiffness and loss coefficients of the elastomer buttons*

The curve-fit obtained from the Voigt model is almost the same as the theoretical results for the compliance and very close for the loss angle (i.e. loss coefficient), as seen in Fig.(7.7). At low frequencies, a slight deviation from the experimental prediction is observed.
This curve-fit is obtained for a single element Voigt model (see Fig.(7.7) for coefficients). Increasing the number of Voigt elements will give a better curve-fit to the prediction values.

Fig.(7.8) shows the comparison of the results obtained from the same Voigt element coefficients for the real part of the complex stiffness and loss coefficients with the actual experimental data. As seen in Fig.(7.8) the curve-fit is slightly higher than the experimental data for the loss coefficient while the stiffness coefficients agree with the experimental data favourably. This is because the curve-fit is obtained from prediction formulae that result in slightly higher predictions for loss coefficients, as seen in Fig.(7.2).

The loss angle in Fig.(7.7) and the loss coefficient in Fig.(7.8) are the same. The loss angle is obtained from

\[ \tan \delta = \frac{H_2(\omega)}{H_1(\omega)} \]  

(7.1)

and the loss coefficient is obtained from

\[ \eta = \frac{K_2(\omega)}{K_1(\omega)} \]  

(7.2)

Considering (see Chapter 5)

\[ H_1(\omega) + H_2(\omega) = \frac{K_1(\omega)}{K_1^2(\omega) + K_2^2(\omega)} + i \frac{K_2(\omega)}{K_1^2(\omega) + K_2^2(\omega)} \]  

(7.3)

It can be shown that

\[ \frac{H_2(\omega)}{H_1(\omega)} = \frac{K_2(\omega)}{K_1(\omega)} \quad \text{or} \quad \tan \delta = \eta \]  

(7.4)

It is clearly demonstrated that a close prediction of the stiffness and damping coefficients of elastomers and hence the coefficients of the mechanical Voigt model is possible from their material properties for a given geometry. The procedure followed in order to obtain the coefficients of the mechanical Voigt model elements can also be employed for rectangular cartridges and O-rings. Therefore there is no need to repeat the same calculations any further. Since the objective of the research is to see whether elastomer dampers are effective in eliminating or reducing the untoward effects of the system resonance frequencies, as many elastomeric material and geometry as possible will be investigated. For this reason a mechanical Voigt model is fitted to the experimental data provided by the NASA contractor report [Smaile et al., 1977] for O-rings.
Fig.(7.9) Correlation of experimental data with a single Voigt Model for the stiffness and loss coefficients of the O-rings (Viton70)

7.3.2 O-rings

In the case of the O-rings, curve-fits are obtained for the empirical formulae given in the NASA contractor report [Smalley et al., 1977] as

\[ K_I(\omega) = A_1 \omega^{B_1} \]  
\[ \eta_I(\omega) = A_2 \omega^{B_2} \]  

For each materials and the test conditions a set of \( A_1, A_2, B_1 \) and \( B_2 \) coefficients are given by Smalley et al. [1977].

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Fig. (7.10) Correlation of experimental data with a single Voigt Model for the stiffness and loss coefficients of the O-rings (Viton 90)

Fig. (7.9) shows the comparison of the curve-fit values with the actual experimental data for Viton 70. The curve-fit for the real part of the stiffness is in good agreement with the experimental data. However, a deviation from the experimental data can be observed for the lower and higher frequencies in the case of the loss coefficient values.

Fig. (7.10) and Fig. (7.11) show similar curve-fits for O-rings made of Viton 90 and Buna–N respectively. In both figures, the curve-fit for the real part of complex stiffness agree quite favourably with the experimental data whereas the loss coefficients show deviations. For Viton 90, deviations are observed for the low frequencies of the loss coefficient. These deviations can be reduced by employing more than one Voigt elements [Gupta et al., 1974].
Fig. (7.11) Correlation of experimental data with a single Voigt Model for the stiffness and loss coefficients of the O-rings (Buna-N)

7.4 Elastomer Dampers in Assembly

Elastomer dampers were assumed to be placed round the outer ring of the both left hand and right hand side ball bearings. For example elastomer buttons in assembly are shown in Fig. (7.12). The buttons can be preloaded through screw mechanisms shown in the figure. Similar arrangements can be designed for ring cartridges and O-rings.

In the previous section, elastomer dampers were modelled in terms of a mechanical Voigt model. However, it is generally more convenient to model elastomers as an equivalent
stiffness and dash-pot system for mathematical purposes. Any number of Voigt element models can easily be transformed into an equivalent stiffness and dash-pot system by adding as many two element Voigt models (see Appendix 1) as is deemed necessary. Equivalent damping and stiffness coefficients for a single element Voigt model, such as the one employed in this thesis, are given as (see Appendix 1):

\[
K_{eq} = \frac{(K_0 + K_1)K_0K_1 + K_0c_1^2 \omega^2}{(K_0 + K_1)^2 + c_1^2 \omega^2}
\]

(7.7)

\[
c_{eq} = \frac{K_0^2c_1}{(K_0 + K_1)^2 + c_1^2 \omega^2}
\]

(7.8)

where \( \omega \) is the damper excitation frequency and an important parameter; \( \omega \) is obtained as described in Chapter 5.

\[\text{Fig. (7.12) The elastomer buttons in the assembly}\]

The outer ring of the bearing was assumed to be mounted on an elastomer button cartridge as seen in Fig. (7.12). The cartridge has two rows of elastomer buttons, each of which has three elastomer buttons that are equally distributed around the outer ring.
In order to see the effect of the elastomer damper more clearly the shaft was assumed to be 0.5 μm off-centred. This was introduced into the simulation model as described in Chapter 5.

![Graph of x oscillations vs time](image)

**Fig.(7.13) Transient and steady-state oscillations of the centre of the shaft supported by bearings mounted on elastomer buttons (n=5000 rpm, e=0.5 μm, m=5.5 kg, Pr=10 N)**

The transient and steady-state oscillations for a shaft speed of 5000 rpm are given in Fig.(7.13) for a shaft-ball bearing assembly supported on elastomer buttons as seen in Fig.(7.12). The number of balls in each bearing was 8 and a 10 N of axial preload was applied. The response shown in Fig.(7.13) suggests steady-state vibrations. However, in order to see the effect of external dampers the vibrations of the system without external dampers and the system with different external elastomer dampers, having different geometries and materials should be obtained.

The vibration response curve for the system without external dampers was obtained by introducing an arbitrary damping of 300 Ns/m in order to eliminate the effect of the natural frequency. A resonance peak appears when the shaft rotating speed coincides with the natural frequency of the system, as seen in Fig.(7.14).

The natural frequency of the system is between 38000 and 40000 rpm. It is higher than the natural frequency of the geometrically perfect system which is about 34000 rpm. The difference comes from the force exerted by the out of balance of the shaft centre. The out of balance of the mass centre forces the balls into regions where the stiffness is greater and therefore might affect the resonance frequency. Therefore the natural frequency of the system is pushed to relatively higher frequencies. Jump behaviour is clearly observed in the vicinity of the natural frequency as the system oscillates at large amplitudes which forces balls to have large deflections over their nonlinear load-deflection curve.
Fig. (7.14) Shaft centre vibration in the x-y plane for the system without external damping
The transient and steady-state oscillations of the shaft centre were obtained for three shaft speeds along the frequency response curve for the system without external dampers. The speeds were chosen such that the first (n=20000 rpm) was well below the resonance frequency, the second (n=36000 rpm) was very close to resonance frequency and the third (n=50000 rpm) was well above the resonance frequency as seen in Fig.(7.14). As seen in Fig.(7.14), when the steady-state oscillations are reached, the shaft centre locus for a shaft speed of 20000 rpm is elliptical whereas for a shaft speed of 36000 rpm it is almost circular.

![Graph showing peak to peak oscillation amplitudes for different shaft rotating speeds](image)

*Fig.(7.15) Peak to peak oscillation amplitudes for different shaft rotating speeds when the bearings are mounted on the elastomer buttons (e=0.5 μm, Pr=10 N)*

The procedure described in Fig.(7.13) was followed for different shaft speeds and the peak to peak amplitudes were recorded as presented in Fig.(7.15) for elastomer buttons for example.

The resulting vibration spectrum shows slight indication of jump behaviour. Since the outer ring of the ball bearing is assumed to be 100 grams, only one peak is observed at about 31000 rpm. The first superharmonic of the natural frequency also appears in the spectrum. It is at about 59000 rpm. It is slightly less than twice the natural frequency. This may be because the actual natural frequency was affected by large oscillations and pushed to a slightly higher frequency.
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The peak to peak amplitudes of the outer ring centre vibrations are in-phase with the shaft centre peak to peak amplitudes but at relatively lower values. The peak to peak amplitudes of the outer ring and the shaft centre oscillations are also given in Fig.(7.15). This behaviour is expected as for the steady-state oscillations the shaft and the outer ring movements are in-phase.

![Graph of peak to peak oscillation amplitudes for different shaft rotating speeds for O-rings and buttons external dampers](image)

*Fig.(7.16) Peak to peak oscillation amplitudes for different shaft rotating speeds for O-rings and buttons external dampers*

In addition, the peak to peak amplitude for O-rings of Viton70, Viton90 and Buna-N were obtained in a similar way. For each one, when the shaft rotational speed coincides with the natural frequency, the system resonates. Since the data available for elastomer dampers are between 100 and 1000 Hz, the results are obtained for speeds between 10000 and 60000 rpm. Since the damping becomes very low for the speeds less than 10000 rpm vibration amplitudes become relatively larger. The outer ring mass is small, therefore the natural frequency associated with this mass does not appear in the vibration spectrum. Hence only one peak appears as the natural frequency. The system also resonates at the first superharmonic of the natural frequency. This quite clear for Buna-N. Vibration spectrums of the system with and without external dampers are given to the same scale in Fig.(7.16).
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Fig. (7.17) The effect of external dampers on the shaft centre vibrations in the x-y plane
Fig. (7.16) shows that the correct choice of elastomer external dampers is important. For example, button dampers have worsened the vibration amplitudes rather than eliminating them. The shaft rotating speed is another important parameter to consider in determining the material for the external dampers. As seen from Fig. (7.16) for a shaft speed of 30000 rpm, or greater speeds, the best material for the O-ring dampers is Viton 70. Buna-N should be considered very carefully as it has violent vibrations at the first superharmonic of its natural frequency. But for lower shaft speeds (less than 25000 rpm) all external dampers have a worsening effect.

The natural frequency of the system with external dampers can roughly be determined by a simple technique. If the equivalent stiffness of the system without external dampers is \( K_{\text{wod}} \), considering the Voigt model stiffness and neglecting outer ring mass and dash-pot system the new equivalent stiffness for the system is given as

\[
\frac{1}{K_{\text{eq}}} = \frac{1}{K_{\text{wod}}} + \frac{1}{K_0} + \frac{1}{K_1}
\]  

(7.9)

The calculations of the natural frequency from Eq.(7.9) roughly matches with the natural frequencies obtained from the simulation model. For example, the equivalent linear stiffness coefficient for the system without external dampers is \( 8.7 \times 10^7 \) N/m (its calculation was based on a natural frequency of 38000 rpm). For O-rings of Buna-N, the approximate natural frequency can be calculated from Eq.(7.9) as 15000 rpm assuming that two O-ring cartridges are employed; one for the LHS bearing and the other for the RHS bearing. Similar calculations result in a natural frequency of 26000 rpm for elastomer buttons and 25500 rpm for Viton 90 (see Figs.(7.8-7.11) for the mechanical Voigt model coefficients). Although these values are lower than the actual values found from the vibration response curves, they are of correct magnitude and order as seen in Fig.(7.16). The difference may be due to the calculation of effective number of O-rings and buttons or calculation of equivalent stiffness value of elastomers.

In order to see the comparison the shaft centre loci are given for the system without external dampers and for the system with Viton 70 elastomeric O-rings as external dampers at a speed of 36000 rpm which is very close to the natural frequency of the system without dampers. As seen in Fig.(7.17) a smaller peak to peak amplitude and more stable and smooth shaft oscillations are obtained by the use of elastomeric O-rings as external dampers.
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Finally, the locus of the shaft centre for Buna-N O-rings is given at a speed of 36000 rpm as an alternative to Viton 70 O-rings. However, as also seen from Fig.(7.16) Buna-N resonates at about this speed. Fig.(7.18) clearly shows that the system is on the verge of instability. For example in this case employing Buna-N instead of Viton 70 will have disastrous effect on the system.

Closure

In the previous chapter, the effect of elastomers as external dampers on the shaft-ball bearing vibratory system was investigated. An out-of-balance of shaft centre was considered as a forcing function in the system. Jump behaviour due to the ball to raceways contact nonlinearity and the nonlinearities of elastomer dynamic properties were also demonstrated. It was concluded that in order to employ elastomers as external dampers, their dynamic properties have to be known in advance and their material and geometry have to be chosen at the design stage. Elastomers have to be seriously investigated before selection as the wrong choice may worsen the vibrations.

The conclusions of the results obtained in this chapter as well as the previous chapter will be summarised in the next chapter.
8.1 Introduction

In the previous two chapters the results and discussion of the theories presented in Chapters 3, 4 and 5 were given. Some of the important conclusions reached will be summarised in this chapter. The conclusions will be given in the same order that was followed in the results and discussion chapters. At the end of this chapter suggestions for further work can also be found.

8.2 Conclusions for Ball Bearings

8.2.1 System Natural Frequency

8.2.1.1 The natural frequency varies for different positions of the ball set and this variation is cyclic.

8.2.1.2 The number of variations per cage rotation is equal to the number of balls in the bearing.

8.2.1.3 The amplitude of the change considerably reduces for larger number of balls in a bearing.

8.2.1.4 Preload changes the natural frequency quite considerably. For relatively low preloads the change is dramatic whereas for larger preload values the change is almost linear (see Fig.(6.7)).
8.2.2 Ball Passage Frequency

8.2.2.1 The BPF is one of the important system characteristics and running speed should be design to avoid from the resonance.

8.2.2.2 The number of balls in the bearings can be significant in the shaft-bearing dynamics and should be considered from the vibration perspective as well as mechanics at the design stage.

8.2.2.3 Increasing the number of balls in an angular contact ball bearing will significantly reduce the peak to peak amplitude at the resonant frequencies when there is a fixed external damping coefficient.

8.2.2.4 Increasing the number of balls will decrease the resonant frequency and this has to be carefully considered at the design stage since the natural frequency coincides with the excitation frequency at relatively lower speeds.

8.2.2.5 Preload is one of the important parameters in bearing dynamics and can be useful in controlling the vibrations of the system.

8.2.2.6 Increasing the preload will reduce the vibration amplitude at the resonant frequencies when there is a fixed external damping coefficient.

8.2.2.7 Preload values should be carefully selected for the best performance from a vibration point of view.

8.2.3 Outer ring Waviness

8.2.3.1 The experimental studies of other researchers are confirmed by the simulation model presented in this thesis.

8.2.3.2 In the case of outer ring waviness the most severe vibrations occur when the BPF and its harmonics coincide with the natural frequency.

8.2.3.3 Severe vibrations are observed for a order of waviness $k = i \cdot m \pm 1$ and $k = m$. For some orders of waviness, the vibrations produced due to waviness are negligible (see Fig.(6.20)).
8.2.3.4 The frequency of the vibrations due to outer ring waviness depends on the waviness order and can be given as

\[ k = q m \pm p \quad q m \omega_c \]

However, this equation is only applicable to major peaks. Since the load deflection relation for balls is nonlinear, for some cases this equation does not hold as explained in Chapter 6.

8.2.4 Inner Ring Waviness

8.2.4.1 Available prediction methods are useful to detect peaks at the major frequencies due to inner ring waviness while peaks at other frequencies are also present but they cannot be detected by available prediction methods.

8.2.4.2 For transformations the peaks can be at (see Chapter 6):

\[ i \omega_{wp} \pm j \omega \text{ or } i (\omega - \omega_b) \pm j \omega_c. \]

8.2.4.3 The formula

\[ k = q m \pm p \quad q m (\omega - \omega_b) \pm p \omega \]

holds for the major peaks (see Fig.(6.27)).

8.2.4.4 The most severe vibrations occur for waviness of order \( i m \pm 1 \).

8.2.4.5 For the waviness of order \( i m \), superharmonics of the peak at the WPF are observed.

8.2.5 Ball Waviness

8.2.5.1 A ball with a wavy surface in the set, will cause vibrations at two frequencies: the cage speed (\( \omega_c \)) and wave passage frequency of the ball (\( N / \omega_b \)).
8.2.5.2 Peaks at $N \omega_b \pm i \omega_c$ will also appear.

8.2.5.3 When $N \omega_b \pm i \omega_c$, or the cage speed coincides with the natural frequency of the system, a resonance will occur.

8.2.5.4 Increasing the order of waviness will diminish the vibrations at $N \omega_b \pm i \omega_c$ and only vibrations at the cage speed will remain in the spectrum, implying a similar case to bearings possessing oversized balls.

8.2.5.5 Relatively higher peaks will appear at $N \omega_b \pm \omega_c$.

8.2.6 Off-sized Balls

8.2.6.1 A single off-sized ball within a bearing produces vibrations at the cage speed. This is true for linear and nonlinear ball to race deflection coefficients.

8.2.6.2 More than one off-sized ball will produce vibrations according to their arrangement within the bearing. Symmetric combinations will produce vibrations at the multiples of the cage speed as shown in Table (6.1). All other combinations will produce vibrations at the cage speed.

8.2.6.3 Difference in the off-sized ball's size for symmetric combinations will reduce the vibration speed to that the cage speed unless they are symmetric within themselves.

8.2.6.4 For bearings with 3, 5, 7, 11, 13, ... balls, for any combination or arrangements of off-sized balls, vibrations due to the off-sized balls will take place only at the cage speed (see Table (6.1)).

8.2.6.5 For any ball bearing, the highest radial vibrations due to the off-sized balls is at a speed of half of the number of balls times the cage speed ($\frac{1}{2} m \omega_c$).

8.2.7 Misalignment

8.2.7.1 When the outer races of both bearings are misaligned, variation occur at the cage speed as the same position of the ball set can only be
obtained after one cage revolution. The system resonates when the cage speed coincides with the natural frequency.

8.2.7.2 Outer ring misalignment increases the system natural frequency as it acts to preload the bearings.

8.2.7.3 When the inner rings of both bearings are misaligned, vibrations occur at the shaft speed. The system resonates when the shaft speed coincides with the natural frequency.

8.2.7.4 When only the inner ring of the LHS bearing (or RHS bearing) is misaligned, the natural frequencies of the system along the $x$ and $y$ axes are no longer the same. For the $x$ oscillations, small peaks at $\omega_n \pm \omega$ appear in the spectrum. The vibrations in the $y$ direction are at the shaft speed and its superharmonics while the vibrations in the $z$ direction take place at twice the shaft speed and its superharmonics.

8.2.8 Defective Running Surfaces

8.2.8.1 When the outer race running surface has a defect, the vibrations take place at the BPF.

8.2.8.2 When a defect is present on the inner ring running surface of a bearing, vibrations will take place at four frequencies and their harmonics and combinations. These frequencies are: the shaft rotating speed, the cage speed, the ball passage speeds of inner and outer races.

8.2.8.3 When any of the above frequencies coincides with the natural frequency of the system, a resonance occurs. For relatively low speeds the vibrations will occur at the natural frequency of the system. That is, the defect will excite the natural frequencies.

8.2.8.4 A ball defect produces vibrations at twice the ball rotating speed. When $\omega_b$ or $2\omega_b$ or their harmonics, coincide with or are close enough to the natural frequency, resonance occurs and the resulting vibrations are at the natural frequency.
8.3 Conclusions for Elastomers as External Dampers

8.3.1 In this research the dynamic stiffness and damping coefficients of elastomers from their material properties were derived for different geometries. These are as follows:

For rectangular cross-section buttons:

\[ K = \frac{3}{2} \frac{ab}{h} (E + G) \left(1 - 2 \frac{Sh}{h}\right) \]

For cylindrical buttons:

\[ K = \frac{3 \pi d^2}{4} \frac{1}{h} (E + G) \left(1 - 2 \frac{Sh}{h}\right) \]

For ring cartridges, two equations were derived. The first is the modified form of the equation for a rectangular cross-sectional button:

\[ K = \frac{3 \pi (R_1 + R_o)t}{4} \frac{1}{R_o - R_i} (E + G) \left(1 - \frac{\delta r}{R_o - R_i}\right) \]

The second equation is as follows:

\[
K = \frac{V(E + G)}{2} \left[ \begin{array}{c}
1 \\
\frac{8}{2(R_o - R_i)^2 + \pi^2(R_i + R_o)^2} \\
\frac{96 \pi^2(R_o - R_i)^2(R_i + R_o)^2}{(2(R_o - R_i) - \delta r)^4(\pi(R_i + R_o) - 2 \delta r)^4} \\
\frac{64 \pi^2(R_o - R_i)^2(R_i + R_o)^2}{(2(R_o - R_i) - \delta r)^3(\pi(R_i + R_o) - 2 \delta r)^3} \\
\frac{24 \pi^2(R_o - R_i)^2(R_i + R_o)^2}{(2(R_o - R_i) - \delta r)^2(\pi(R_i + R_o) - 2 \delta r)^2}
\end{array} \right]
\]

For O-rings three equations were derived. The first one is the modified form of the equation that was derived for cylindrical buttons:

\[ K = \frac{3}{32} \frac{\pi^2 d^2(R_1 + R_o)}{(R_o - R_i)^2} \frac{1}{(E + G)} \left(1 - \frac{\delta r}{R_o - R_i}\right) \]
The second equation is:

\[
K = \frac{V(E+G)}{2} \left[ \frac{1}{2d^2} + \frac{8}{\pi^2(R_1+R_2)^2} + \frac{96\pi^2 d^2(R_1+R_2)^2}{(2d-\delta r)^2(\pi(R_1+R_2)-2\delta r)^4} \right. \\
+ \left. \frac{64\pi^2 d^2(R_1+R_2)^2}{(2d-\delta r)^3(\pi(R_1+R_2)-2\delta r)^3} \right. \\
+ \left. \frac{24\pi^2 d^2(R_1+R_2)^2}{(2d-\delta r)^4(\pi(R_1+R_2)-2\delta r)^2} \right]
\]

The third method is similar to the \textit{beam-column method} including radius effects:

\[
K = \frac{\partial}{\partial \delta r} \left[ 4 \int_{\zeta=0}^{\pi/2} \frac{1}{r^2} \left( \frac{1}{r^2} \right)^{1/2} \left( d - \delta r \right) \delta r \left( E \cos^2(\zeta) + G \sin^2(\zeta) \right) d\zeta \right]
\]

This equation was solved using \textit{Mathematica}\textregistered{} and it gave the successful results presented in Chapter 7.

8.3.2 All predictions match with the experimental data available quite favourably.

8.3.3 It was shown that in order to employ elastomers as external dampers, their dynamic properties have to be known in advance and their material, geometry and working environment have to be chosen at the design stage. They have to be seriously investigated before selection, as the wrong choice may worsen the vibrations.
8.4 Recommendation for Further Research

8.4.1 In this thesis the cases investigated are generally for steady-state vibrations. However, for cutting machines, especially for surface processing machines, transient vibrations are important and of interest. Therefore a careful investigation of this behaviour is needed. Particularly for the dynamic properties elastomers the transient vibrations have to be taken into account.

8.4.2 In this thesis the anomalies of the system and the defects are investigated separately. In real life more than one defect is simultaneously present in the system. Moreover, different variation of the same defect such as more than one waviness order, more than one defect on the inner, outer and ball surfaces, the inner ring and the outer ring misalignment together can be present in the system. Their interaction has to be investigated.

8.4.3 In this thesis the masses of the balls are not considered. Particularly for high speed applications they are important and should be modelled. This can easily be added to the model presented in this thesis.

8.4.4 The shaft was assumed to be rigid. The behaviour of an elastic shaft supported by ball bearings has to be studied. This can be done by linearising the load-deflection relation of balls over a narrow band of deflection as presented in Chapter 2 and Chapter 3.

8.4.5 In this thesis the lubricant film between the rolling elements and the raceways was not considered. But in real life all ball bearings naturally include (hopefully) a lubricant film. Investigation of the effect of lubrication on the misalignment and varying preload may be a subject to study.

8.4.6 Different types of excitation forces are also of interest. For example random excitation and step force excitation are very important for CNC cutting machines.

8.4.7 Above all, of course, it is necessary to prove the results presented in this thesis by more experimental findings. In particular, natural frequency variation, off-sized balls and misalignment are investigated less than other characteristics. This is also true for the dynamic properties of elastomers.

8.4.8 Elastomers have to investigated more in detail. Particularly the stiffness and damping properties of elastomeric O-rings are not very well known.
8.4.9 The response of elastomeric materials to the non-periodic excitation has to be investigated since little is known about it.

8.4.10 Interaction between material temperature and vibration amplitude has to be considered as the temperature dependence of elastomers is one of the main drawbacks of the usage of elastomers.

8.4.11 Available data was for a frequency range of 100 to 1000 Hz. A check has to be done whether the prediction formulae presented in this thesis are valid for other frequency ranges.


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APPENDIX 1

THE EQUIVALENT STIFFNESS AND DAMPING COEFFICIENTS

Fig.(A1.1) Two element Voigt model and equivalent dash-pot system

A two element Voigt model is assumed to represent the elastomer. For this, the force-equilibrium equation can be given as

\[(x_1 - x_2)K_1 + (\dot{x}_1 - \dot{x}_2)c_1 = F_1 \]  \hspace{1 cm} (A1.1)
\[(x_2 - x_3)K_2 + (\dot{x}_2 - \dot{x}_3)c_2 = F_2 \]  \hspace{1 cm} (A1.2)

And the force equilibrium equation for the equivalent dash-pot system can be given as

\[(x_1 - x_3)K_{eq} + (\dot{x}_1 - \dot{x}_3)c_{eq} = F_3 \]  \hspace{1 cm} (A1.3)

knowing \( F_1 = F_2 = F_3 = 0 \); \( x_3 = \dot{x}_3 = 0 \). and letting \( x_1 = A e^{i\omega t} \) and \( x_2 = B e^{i\omega t} \)

The equivalent stiffness and damping can be obtained as

\[K_{eq} = \frac{-(K_1 + K_2)(c_1c_2\omega^2 - K_1K_2) + (K_1c_2 + K_2c_1)(c_1 + c_2)\omega^2}{(K_1 + K_2)^2 + (c_1 + c_2)^2\omega^2} \]  \hspace{1 cm} (A1.4)

\[c_{eq} = \frac{(c_1 + c_2)(c_1c_2\omega^2 - K_1K_2) + (K_1 + K_2)(K_1c_2 + K_2c_1)}{(K_1 + K_2)^2 + (c_1 + c_2)^2\omega^2} \]  \hspace{1 cm} (A1.5)
APPENDIX 2

PHASE PLANE REPRESENTATION

Phase plane representation is a very useful and widely used tool in the investigation of nonlinear systems. In the phase plane representation, the displacement of the shaft centre, \( x(t) \), is plotted against its derivative with respect to time, \( \frac{dx}{dt} \), at small time intervals of \( \Delta t \).

The response of the bearing system is initiated with \( x_0 \) and \( (\frac{dx}{dt})_0 \), and the oscillations eventually reach a stable amplitude known as the limit cycle. There is only one closed trajectory for the system which is independent of the initial conditions. Initial conditions may be inside or outside the closed trajectory. For conservative nonlinear systems, the limit cycles will respectively be more elliptical than those of respectively less nonlinear systems to the same scale. The limit cycle occurs over one steady-state cycle, indicating that the net energy input from the excitation is equal to the energy dissipated within the system [Rao, 1988].

For a simple equation of motion

\[
\ddot{x} + f(x(t), x(t)) = 0 \quad (A2.1)
\]

when \( \frac{dx}{dt} = 0 \) and \( f(x(t)) \neq 0 \), the slope of \( \frac{dx}{dt} \) becomes

\[
\frac{\partial \dot{x}(t)}{\partial x(t)} = f\left(\frac{x(t)}{\dot{x}(t)}\right) \quad (A2.2)
\]

For example,

\[
M \frac{\partial^2 x(t)}{\partial t^2} + c \frac{\partial x(t)}{\partial t} + K x(t) = 0 \quad (A2.3)
\]
Appendix 2

Phase Plane Representation

Since $\frac{\partial x}{\partial t} = 0$, Equ.(A2.3) becomes

$$M \frac{\partial^2 x(t)}{\partial t^2} + K x(t) = 0 \tag{A2.4}$$

or

$$\frac{\partial \dot{x}(t)}{\partial x(t)} = -\frac{K \left( x(t) \right)}{M \left( \dot{x}(t) \right)} \tag{A2.5}$$

As Equ.(A2.2) suggests the slope becomes infinite when $\frac{\partial x}{\partial t} = 0$ with the trajectories of the phase plane crossing the $x$ -axis at right angles. These phase plane trajectories are symmetrical about the $x$ -axis and the amount of deviation from a circular path trajectory indicates the extent of nonlinear behaviour in the system.

![Fig.(A2.1) Phase-Plane trajectory](image)

Fig.(A2.1) shows the phase plane trajectory for a nonlinear system. For this phase plane trajectory

$$\frac{\left[ \dot{x}(t) \right]^2}{b^2} + \frac{(x-c)^2}{a^2} = 1 \tag{A2.6}$$

where $2a$ is the length of the major axis, $2b$ is that corresponding to the minor axis, and $c$ is the centre of the trajectory (see Fig.(A2.1)).

Let $V(x)$ be the potential energy of the system. The energy equation for the conservative system at any instant can be written as
\[
\frac{1}{2}M[\dot{x}(t)]^2 + V(x(t)) = E = \text{Constant}
\]  
(A2.7)

For the minimum potential energy level

\[
\frac{\partial V(t)}{\partial x(t)} = 0
\]  
(A2.8)

Hence for \(\frac{\partial^2 x}{\partial t^2} = 0\), \(x(t) = c\).

For the phase-plane trajectory, the full period of oscillation can be obtained from the closed loop as [Rahnejat, 1984]:

\[
T = \oint_{x=a}^{x=b} \frac{dx(t)}{\dot{x}(t)}
\]  
(A2.9)

Because of the symmetric nature of the trajectory, the period of oscillations, \(T\), can be evaluated using

\[
T = 4 \int_{x=c}^{x=b} \frac{dx(t)}{\dot{x}(t)}
\]  
(A2.10)

If the trajectory is obtained for the undamped system for a step load at zero speed, then

\[
\omega_n = \frac{1}{T}
\]  
(A2.11)

Solving Equ.(A2.6) for \(\frac{\partial x}{\partial t}\) will give

\[
\dot{x}(t) = \frac{b}{a} \left[ a^2 - (x - c)^2 \right]^{\frac{1}{2}}
\]  
(A2.12)

Substituting Equ.(A2.12) into Equ.(A2.10) will give

\[
T = 4 \frac{a}{b} \int_{x=c}^{x=b} \frac{dx(t)}{\left[ a^2 - (x - c)^2 \right]^{\frac{1}{2}}}
\]  
(A2.13)

Let
Appendix 2

Phase Plane Representation

\( x - c = a \sin \xi \)  \hspace{1cm} (A2.14)

\( dx(t) = a \cos \xi \, d\xi \) \hspace{1cm} (A2.15)

Substituting Equ.(A2.14) and Equ.(A2.15) into Equ.(A2.13)

\[ T = \frac{4}{b} \int_{\xi_1}^{\xi_2} d\xi \] \hspace{1cm} (A2.16)

\( \xi_1 \) and \( \xi_2 \) in Equ.(A2.16) can be obtained from Equ.(A2.14), considering \( x_{\text{max}} = c + a \)

\( \xi_1 = 0 \) and \( \xi_2 = \frac{\pi}{2} \)

Hence Equ.(A2.16), the period of oscillations, will take the form of

\[ T = 2\pi \frac{a}{b} \] \hspace{1cm} (A2.17)
APPENDIX 3

LARGE DEFORMATIONS OF ELASTOMERS

A specimen of initial dimensions $x_i, y_i$ and $z_i$ is deformed so that the dimensions become $x, y$ and $z$. End A of the representative chain at $x_i, y_i$ and $z_i$ before deformation; after deformation is at $x, y$ and $z$. The end-to-end vector changes in exactly the same way as the specimen dimensions (see Fig.(A3.1)). For this single chain, the work done by external forces to deform it from $x_i, y_i, z_i$ dimensions to $x, y, z$, can be found from, for example in the $x$ direction:

$$w_x = \int_{x=x_i}^{x} K x \, dx$$  \hspace{1cm} (A3.1)

where

$$K = \frac{2kT}{\rho^2}$$  \hspace{1cm} (A3.2)

Fig.(A3.1) Representation of elastomer deformation
Hence for a single chain, the work done is

$$w_x = \frac{2kT}{\rho^2} x_i^2 (\lambda_i^2 - 1)$$  \hspace{1cm} (A3.3)

The total work in the \(x\) direction is

$$W_x = \sum_i^v w_x = \frac{kT}{\rho^2} (\lambda_x^2 - 1) \sum_i^v x_i^2$$  \hspace{1cm} (A3.4)

By definition [Mc Crum et al., 1989]:

$$\sum_i^v x_i^2 = v \langle x^2 \rangle_i$$  \hspace{1cm} (A3.5)

where \(\langle x^2 \rangle_i\) is the mean square value of \(x_j\) in the undeformed state. The undeformed state is isotropic, so:

$$\langle x^2 \rangle_i = \langle y^2 \rangle_i = \langle z^2 \rangle_i = \frac{\langle \ell^2 \rangle_i}{3}$$  \hspace{1cm} (A3.6)

For a Gaussian chain \(\rho^2\) controls the mean square end-to-end distance \(\langle x^2 \rangle_0\) [Mc Crum et al., 1989]:

$$\langle \ell^2 \rangle_o = \frac{3}{2} \rho^2$$  \hspace{1cm} (A3.7)

Hence

$$W_x = v \frac{kT}{2} \frac{\langle \ell^2 \rangle_i}{\langle \ell^2 \rangle_o} (\lambda_x^2 - 1)$$  \hspace{1cm} (A3.8)

The total work will the summation of all work done in three axes:

$$W = W_x + W_y + W_z$$  \hspace{1cm} (A3.9)

or

$$W = \frac{VG}{2} [\lambda_x^2 + \lambda_y^2 + \lambda_z^2 - 3]$$  \hspace{1cm} (A3.10)

where

$$G = NkT \frac{\langle \ell^2 \rangle_i}{\langle \ell^2 \rangle_o} \text{ and } N = \frac{v}{V}$$

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APPENDIX 4

OTHER METHODS OF OBTAINING THE STIFFNESS COEFFICIENTS OF RING CARTRIDGES

In Chapter 5 a relatively simple predictive method was given for a practical elastomer configuration, such as radially loaded ring cartridges. In this appendix a number of alternative methods of prediction of radial stiffness are developed. Numerical values for the particular cartridge configurations were obtained and compared with experimental results and the method developed in this thesis.

Fig.(A4.1) Cartridge showing Beam-Column element
(after Smalley and Tessarzik [1975])
The methods developed in this appendix are "static" prediction methods; that is, they seek a ratio of element stiffness to storage modulus which is a function of the geometry only and that ratio applies to the determination of dynamic stiffness and damping. The approach is not rigorously justifiable, but may be given credibility by empirical verification [Smalley and Terssarzik, 1975].

**Beam-Column Method**

This method was developed and used in a report by Smalley and Terssarzik [1975]. The cartridge is assumed to be made up of a series of non-interacting, elemental column/beams, whose variation of thickness due to radius effects is small. Fig.(4.1) illustrates the cartridge and the Beam-Column element. Consider the rigid inner member to be displaced radially by a small radial distance $\delta r$, relative to the outer member. For this the force caused will be:

$$F = \sigma A$$  \hspace{1cm} \text{(A4.1)}

The force on the element is:

$$df = \sigma da$$  \hspace{1cm} \text{(A4.2)}

Considering the element to act independently of adjacent elements, the elemental compression and shear forces are

$$df_c = \left(\frac{\delta r}{(R_o - R_i)}\right) \left[\left(\frac{1}{2}(R_o + R_i) d\zeta\right)(t)\right] \cos(\zeta)$$  \hspace{1cm} \text{(A4.3)}

$$df_s = \left(\frac{\delta r}{(R_o - R_i)}\right) \left[\left(\frac{1}{2}(R_o + R_i) d\zeta\right)(t)\right] \sin(\zeta)$$  \hspace{1cm} \text{(A4.4)}

The total force exerted at the mean radius is:

$$F = \int_{\zeta=0}^{2\pi} \frac{(R_o + R_i)}{2(R_o - R_i)} \delta r \left[E \cos^2(\zeta) + G \sin^2(\zeta)\right] d\zeta$$  \hspace{1cm} \text{(A4.5)}

Hence

$$F = \frac{\pi (R_o + R_i)}{2(R_o - R_i)} \delta r (E + G)$$  \hspace{1cm} \text{(A4.6)}
Appendix 4

Other Methods for Ring Cartridges

If it is assumed that $E = 3G$ (Poisson's ratio $\nu = 0.5$ implied)

$$F = 2\pi \frac{(R_o + R_i)}{(R_o - R_i)} G \delta t$$  \hspace{1cm} (A4.7)

Then

$$K = \frac{\partial F}{\partial \delta r} = 2\pi \frac{(R_o + R_i)}{(R_o - R_i)} G t$$  \hspace{1cm} (A4.8)

**Beam-Column Method (the radius effect accounted for)**

When the radius effect is added, a slightly different result is obtained (see also Method of Gōbel below):

$$K = 4\pi \frac{G t}{\ln \left(\frac{R_o}{R_i}\right)}$$  \hspace{1cm} (A4.9)

However, the reduction in stiffness due to radius effects was found to be insignificant by Smalley and Tessarzik [1975], and neglected.

**Method of Gōbel**


Total elongation for a body with a varying cross-section is:

Compression or Tension : $\delta r_c = \int \epsilon \, dr$  \hspace{1cm} (A4.10)

Shear : $\delta r_s = \int \gamma \, dr$  \hspace{1cm} (A4.11)

Considering $\sigma = \frac{dF_c}{da}$ and $\tau = \frac{dF_s}{da}$ and using mean length $t$.

$$\delta r \cos(\zeta) = \int_{r_i}^{r_o} \frac{dF_c}{Et \, d\zeta} \, dr$$  \hspace{1cm} (A4.12)
\[
\delta r \sin(\zeta) = \int_{r_i}^{r_o} \frac{dF_c}{Gt} \frac{dr}{r}
\]

Hence the total force on the element calculated:

\[
dF = dF_c \cos(\zeta) + dF_s \sin(\zeta)
\]

\[\text{(A4.13)}\]
\[\text{(A4.14)}\]

Fig. (A4.2) Cartridge showing Göbel's element (after Göbel [1974])

Hence

\[
K = \frac{\partial F}{\partial \delta r} = \pi \frac{t}{\ln \left(\frac{R_o}{R_i}\right)} (G + E)
\]

\[\text{(A4.15)}\]

However, Göbel then claims that, \( t / (R_o - R_i) = 1 \) the effective Young's modulus is given by:

\[E_{\text{eff}} = 6.5G\]

\[\text{(A4.16)}\]

and in addition that, for other length to radius ratios, a correction factor, \( f_1 \), should be applied, so that
Appendix 4

Other Methods for Ring Cartridges

\[ K = 7.5 \pi \frac{G f_1}{\ln \left( \frac{R_s}{R_i} \right)} \]  
(A4.17)

Göbel [1955] presented \( f_1 \) as a graphical function of \( t/(R_s - R_i) \). Payne and Scott [1960] present a cubic fit to Göbel's function:

\[ f_1 = 1 + \frac{0.0097 t^3}{(R_s - R_i)^3} \]  
(A4.18)

Equ. (A4.18) agrees closely with Göbel's graphical presentation. The basis for \( f_1 \) or for the statement \( E_{eg} = 6.5G \) is not made clear in Göbel's book [Göbel, 1974]. Smalley and Tessarzik [1975] reported that although Göbel's method shows good agreement at higher values of \( t/(R_s - R_i) \) with the test data, at lower \( t/(R_s - R_i) \), the Beam-Column method appeared to come closer, due to a much sharper drop in stiffness of the test element. This comparison suggested that \( E_{eg} = 6.5G \) is not a valid asymptotic condition for short cartridges.
APPENDIX 5

PREVIOUSLY DERIVED PREDICTION METHODS FOR ELASTOMERS

For the shear element (Fig.(A5.1), the stressed area, $A$, is its width times its length ($a \times b$) and the strained dimension is its thickness, $h$. For this element the stiffness and damping can be related to effective moduli as follows [Smalley and Tessarzik, 1975]:

\[
K_1(\omega) = G'_{\text{eff}} \frac{A}{h} \quad (A5.1)
\]

\[
K_2(\omega) = G''_{\text{eff}} \frac{A}{h} \quad (A5.2)
\]

\[
K^* = G'_{\text{eff}} \frac{A}{h} \quad (A5.3)
\]

Fig.(A5.1) Shear element (after Smalley and Tessarzik [1975])

Hence
where $G'_\text{eff}$, $G''\text{eff}$ are shear moduli, commonly referred to as the effective storage and loss moduli, respectively. $G'_\text{eff}$ and $G''\text{eff}$ are generally functions of the material, the frequency, the amplitude of loading, the temperature, the initial strain and geometry. They are, therefore, component rather than material properties. The relationships between effective and true moduli can be given as follows, as used many researchers [Gent and Lindley, 1960; Hattori and Takei, 1950; Smalley and Tessarzik, 1975; Freakley and Payne, 1978]:

$$\left(G'_\text{eff}\right)_o = G'_o \left(1 + \beta s^2\right) \quad (A5.4)$$

where $s$ is a shape factor defined as the ratio of the loaded area to the unloaded area ($D / (4 h)$ for cylinder) and $b$ is a coefficient. Subscript, $o$, implies static loading.

For the compression element, stiffness and damping can be related to the effective moduli as follows:

**Stiffness**

$$K_1(\omega) = 3 G'_\text{eff} \frac{A}{h} \quad (A5.5)$$

**Damping**

$$K_2(\omega) = 3 G''\text{eff} \frac{A}{h} \quad (A5.6)$$

where the factor 3 arises from the classical elasticity relationship $E = 2(1 + \nu)\ G$ and
the fact that elastomer Poisson's ratios are typically in the range 0.4945 to 0.4999 [Holownia, 1970].

**Cylindrical Buttons**

For cylindrical buttons the real and imaginary parts of the complex dynamic stiffness can be given as follows [Smalley and Tessarzik, 1975]:

\[
K_1(\omega) = 3G'\frac{\pi d^2}{4h}[1 + \beta' s^2] + G''\frac{\pi d^2}{4h} \tag{A5.7}
\]

\[
K_2(\omega) = 3G''\frac{\pi d^2}{4h}[1 + \beta'' s^2] + G''\frac{\pi d^2}{4h} \tag{A5.8}
\]

where the second terms on the right hand side of the Equs. (A5.7 and A5.8) are added in order to account for the shear forces. \(G'(\omega)\) is the real part and \(G''(\omega)\) is the imaginary part of the complex shear modulus \(G^*\).

Values for \(G', G'', \beta'\) and \(\beta''\) are required to make predictions of dynamic stiffness and damping of the elastomer buttons. These values are determined from experimental results, as a function of frequency (and temperature). For example, Smalley and Tessarzik [1975] empirically determined \(G', G'', \beta'\) and \(\beta''\) for polybutadiene as a function of frequency, by testing at 32 degrees Celsius as follows:

\[
G'(\omega) = 3.686 \times 10^6 \omega^{0.2037} \text{ N/m}^2 \tag{A5.9}
\]

\[
G''(\omega) = 8.333 \times 10^6 \omega^{-0.1277} \text{ N/m}^2 \tag{A5.10}
\]

\[
\beta'(\omega) = 12.33 \omega^{-0.290} \tag{A5.11}
\]

\[
\beta''(\omega) = 1.726 \omega^{0.0299} \tag{A5.12}
\]

where \(\omega\) is frequency in radian per second.

**Ring Cartridges**

For ring cartridges there are five prediction formulae derivated.

1. Göbel's Formula (see Appendix 4)

\[
K'(\omega) = 7.5\pi \frac{G'(\omega)l}{\ln\left(\frac{R_2}{R_1}\right)} f_1 \tag{A5.13}
\]
where

\[ f_1 = 1 + \frac{0.0097t^3}{(R_o - R_i)^3} \]  
(A5.14)

2. **Beam-Column Formula** (see Appendix 4)

\[ K^*(\omega) = 2\pi \left( \frac{R_o + R_i}{R_o - R_i} \right) \frac{G^*(\omega)t}{G_v} \]  
(A5.15)

3. **Beam-Column Formula** (the radius effect accounted for—see Appendix 4)

\[ K^*(\omega) = 4\pi \frac{G^*(\omega)t}{\ln \left( \frac{R_o}{R_i} \right)} \]  
(A5.16)

4. **Lindley's Formula**

Lindley [Lindley, 1966] started his derivation from a simple case that under static conditions

\[ K = \frac{\partial F}{\partial \delta} = \frac{\sigma \partial A}{\partial \delta} \]  
(A5.17)

which leads to

\[ K_1(\omega) = 3\pi \left( \frac{R_o^2 - R_i^2}{t} \right) G^*(\omega) \left[ 1 + k \left( \frac{R_o - R_i}{2t} \right)^2 \right] \]  
(A5.18)

5. **Grassano's Formula** [Grassano, 1991]

\[ K_{sg}^*(\omega) = \frac{4}{9} \frac{V}{(R_o - R_i)^2} \left( \frac{G^*(\omega)}{ \left( \frac{\lambda r + A}{\lambda r^2 + 2\lambda r A} \right)^3 + \frac{6(\lambda r + A)^3}{\left( \lambda r^2 + 2\lambda r A \right)^4} } \right) \]  
(A5.19)

where \( \lambda_r = 1 - \frac{\delta r}{R_o - R_i} \) and \( A = \frac{\delta r R_i}{R_o^2 - R_i^2} \)

**O-Rings**

There are two prediction formulae for O-rings.
1. Lindley's Formula

A similar treatment to the one which was applied to ring cartridges is applied to the O-rings to result in Equ.(A5.25) (see Fig.(A5.3) for notation):

\[ K^*(\omega) = 4G^*(\omega)\frac{(2\delta r/d)^{\frac{1}{2}}}{d - \delta r} \]  \hspace{1cm} (A5.20)

Lindley has also given an empirical formula for O-rings [Freckley and Payne, 1978]:

\[ \frac{F}{\pi d DE_0} = 1.25\left(\frac{\delta r}{d}\right)^3 + 50\left(\frac{\delta r}{d}\right)^6 \]  \hspace{1cm} (A5.21)

Hence

\[ K^*(\omega) = 4\pi(R_0 + R_1)G^*(\omega) \left[ 1.875\left(\frac{\delta r}{d}\right)^{\frac{1}{2}} + 300\left(\frac{\delta r}{d}\right)^5 \right] \]  \hspace{1cm} (A5.22)


\[ K_{G}^*(\omega) = \frac{4}{9} \frac{V}{(R_0 - R_i)^2} G^*(\omega) \left[ 1 - \frac{1}{(\lambda_i^2 + 2\lambda_i, A)} + \frac{6(\lambda_i + A)^2}{(\lambda_i^2 + 2\lambda_i, A)^4} \right] \]  \hspace{1cm} (A5.23)

where \( \lambda_i = 1 - \frac{\delta r}{R_0 - R_i} \) and \( A = \frac{\delta r}{R_0 - R_i} \)
VOIGT MODEL

The Voigt model is a way of representing the material in terms of springs and dash-pots. It can be traced back to the generalized Hook's law which can be mathematically represented as [Moore, 1972]

\[
\left[ a'_0 + a_1 \frac{\partial}{\partial t} + a_2 \left( \frac{\partial^2}{\partial t^2} \right) + \ldots + a_n \left( \frac{\partial^n}{\partial t^n} \right) \right] \sigma = \left[ b'_0 + b_1 \left( \frac{\partial}{\partial t} \right) + b_2 \left( \frac{\partial^2}{\partial t^2} \right) + \ldots + b_n \left( \frac{\partial^n}{\partial t^n} \right) \right] \varepsilon \tag{A6.1}
\]

or

\[
\left[ a'_0 + a_1 \frac{\partial}{\partial t} + a_2 \left( \frac{\partial^2}{\partial t^2} \right) + \ldots + a_n \left( \frac{\partial^n}{\partial t^n} \right) \right] F = \left[ b'_0 + b_1 \left( \frac{\partial}{\partial t} \right) + b_2 \left( \frac{\partial^2}{\partial t^2} \right) + \ldots + b_n \left( \frac{\partial^n}{\partial t^n} \right) \right] \delta \tag{A6.2}
\]

where \( a_i, a'_i, b_i \) and \( b'_i \) are constants (\( i = 0, 1, 2, \ldots, n \)). If the force and the deflection vary sinusoidally,

\[
\frac{F}{\delta} = \frac{b'(\omega) + jb''(\omega)}{a'(\omega) + ja''(\omega)} \tag{A6.3}
\]

where \( a'(\omega), a''(\omega), b'(\omega) \) and \( b''(\omega) \) are function of frequency and can be obtained from Equ.(A6.2) as

\[
a'(\omega) = a_0 - a_2 \omega^2 - a_4 \omega^4 - \ldots \tag{A6.4}
\]

\[
a''(\omega) = a_1 \omega + a_3 \omega^3 + a_5 \omega^5 - \ldots \tag{A6.5}
\]

\[
b'(\omega) = b_0 - b_2 \omega^2 - b_4 \omega^4 - \ldots \tag{A6.6}
\]
Appendix 6

Voigt Model

\( b''(\omega) = b_1 \omega + b_2 \omega^3 + b_3 \omega^5 \ldots \)  \hfill (A6.7)

Equation (A6.3) can be reduced to

\[
\frac{F}{\delta} = K_1(\omega) + j K_i(\omega) = K^*(\omega)
\]  \hfill (A6.8)

where

\[
K_1(\omega) = \frac{a'(\omega) b'(\omega) + a''(\omega) b''(\omega)}{[a'(\omega)]^2 + [a''(\omega)]^2}
\]  \hfill (A6.9)

\[
K_2(\omega) = \frac{a'(\omega) b''(\omega) - a''(\omega) b'(\omega)}{[a'(\omega)]^2 + [a''(\omega)]^2}
\]  \hfill (A6.10)

For a modified Voigt model as seen in Fig.(A6.1)

\[
F = K_0 (\delta - \delta_1) = K_1 \delta + c_1 \left( \frac{\partial \delta_1}{\partial t} \right)
\]  \hfill (A6.11)

Hence

\[
\left[ 1 + \left( \frac{c_1}{K_0 + K_1} \right) \left( \frac{\partial}{\partial t} \right) \right] F = \left[ \left( \frac{K_0}{K_0 + K_1} \right) + \left( \frac{K_0 c_1}{K_0 + K_1} \right) \left( \frac{\partial}{\partial t} \right) \right] \delta
\]  \hfill (A6.12)

From the similarity of Equation (A6.2) and Equation (A6.12)

\[
K_1(\omega) = \frac{(K_0 + K_1) K_0 K_1 + K_0 c_1^2 \omega^2}{(K_0 + K_1)^2 + c_1^2 \omega^2}
\]  \hfill (A6.13)

\[
K_2(\omega) = \frac{[(K_0 + K_1) K_0 - K_0 K_1] c_1 \omega}{(K_0 + K_1)^2 + c_1^2 \omega^2}
\]  \hfill (A6.14)
APPENDIX 7

BALL PASSAGE FREQUENCY

The ball passage frequency can be exhibited from the basic theory of vibrations as done by Meyer et al. [1980]. General differential equations for a vibrating system in x and y directions can be written:

\[ M \frac{\partial^2 x}{\partial t^2} + c \frac{\partial x}{\partial t} + K x = Q_x(t) \]  

(A7.1)

\[ M \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} + K y = Q_y(t) \]  

(A7.2)

Assuming that:
1. balls to race way contacts have linear spring characteristics,
2. the shaft is not flexible,
3. system does not accommodate any damping,
4. the forcing functions are in the form of:

\[ Q_x(t) = P_0 \cos(n \theta_i) \]  

(A7.3)

\[ Q_y(t) = P_0 \sin(n \theta_i) \]  

(A7.4)

where

\[ \theta_i = \omega_c t + i \gamma \]  

The differential Equ.(A7.1) and Equ.(A7.2) will give successive solutions to any

\[ x_c = x_0 \cos(n \omega_c t + n i \gamma) \]  

(A7.5)
\( y_i = Y_0 \sin(n \omega_c t + ni \gamma) \) \hfill (A7.6)

Substituting Equ.(A7.5) and Equ.(A7.6) into Equ.(A7.1) and Equ.(A7.2):

\[ x_0 = Y_0 = \frac{P_0}{n^2 \Omega^2 \left[ \left( \frac{\omega_n}{n \omega_c} \right)^2 - 1 \right]} \] \hfill (A7.7)

Substituting Equ.(A7.7) into Equ.(A7.5) and Equ.(A7.7):

\[ x_n = \frac{P_0}{n^2 \Omega^2 \left[ \left( \frac{\omega_n}{n \omega_c} \right)^2 - 1 \right]} \cos(n \omega_c t + ni \gamma) \] \hfill (A7.8)

\[ y_n = \frac{P_0}{n^2 \Omega^2 \left[ \left( \frac{\omega_n}{n \omega_c} \right)^2 - 1 \right]} \sin(n \omega_c t + ni \gamma) \] \hfill (A7.9)

For a single mode, the deflection in one of the ball to race way contacts can be given as:

\[ \delta_i = x_n \cos(n \omega_c t + ni \gamma) + y_n \sin(n \omega_c t + ni \gamma) \] \hfill (A7.10)

For the total deflection, the effects of other modes should also be considered, hence the total deflection in the direction of \( i \) th ball is given as the summation of all deflections due to the different modes:

\[ \delta_i = \sum_{n=1}^{\infty} \left[ x_n \cos(n \omega_c t + ni \gamma) + y_n \sin(n \omega_c t + ni \gamma) \right] \] \hfill (A7.11)

A deflection \( \delta_i \) of the inner ring relative to the outer ring in the radial direction with respect to the reference axes at an arbitrary angle \( \Theta \) can be obtained by summation of the ball to race way contact deflections for all balls:

\[ \delta_r = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \left[ \frac{P_0 \cos(n \omega_c t + ni \gamma)}{n^2 \Omega^2 \left( \frac{\omega_n}{n \omega_c} \right)^2 - 1} \cos(\Theta) + \frac{P_0 \sin(n \omega_c t + ni \gamma)}{n^2 \Omega^2 \left( \frac{\omega_n}{n \omega_c} \right)^2 - 1} \sin(\Theta) \right] \] \hfill (A7.12)
Hence,

\[ \delta_r = \sum_{i=1}^{n} \sum_{n=1}^{\infty} \left[ \frac{P_0}{M \left( \omega_n^2 - (n \omega_c)^2 \right)} \cos(n \omega_c t + ni \gamma - \vartheta) \right] \]  
(A7.13)

Since the balls are assumed to be linear, unless \( n \) is equivalent to \( m \) or its multiples, all summations will be zero. Hence,

\[ \delta_r = \sum_{i=1}^{n} \sum_{n=1}^{\infty} \left[ \frac{P_0}{M \left( \omega_n^2 - (nm \omega_c)^2 \right)} \cos(nm \omega_c t + n2 \pi i - \vartheta) \right] \]  
(A7.14)

As a result;

\[ \delta_r = \sum_{n=1}^{\infty} \left[ \frac{mP_0}{M \left( \omega_n^2 - (nm \omega_c)^2 \right)} \cos(nm \omega_c t - \vartheta) \right] \]  
(A7.15)

The term \( \cos(nm \omega_c t - \vartheta) \) in Equ.(4.35) implies that the frequency band of deflection will have a series of peaks at, and multiples of, ball passage frequency with a trend of reduction in the amplitudes. If the shaft is assumed to be linear as in our case, only the first mode will be effective. Therefore, there will be only one peak in the frequency spectrum at the ball passage frequency as long as the system is not disturbed by any other external excitations. Hence,

\[ \delta_r = \frac{mP_0}{M \left( \omega_n^2 - (m \omega_c)^2 \right)} \cos(m \omega_c t - \vartheta) \]  
(A7.16)

\( \delta_r \) is infinity when \( m \omega_c = \omega_n \)
A signal of a vibrating system can be described either in the time domain or in the frequency domain. The amplitude of the signal, \( h \), will be a function of time or frequency depending on the domain. The Fourier Transform can easily relate these two domains.

Since the equations in this thesis are nonlinear, a time domain solution has to be performed. Therefore it is quicker and more accurate to estimate the spectra directly from the original time series. However, it is sometimes easier and more convenient to investigate the results in the frequency domain.

It should be remembered that the Fourier series is an approximate one. Nevertheless it can reproduce the original discrete-data-point time series exactly [Mitchell, 1985].

Fig. (A8.1) shows the relationship between time and frequency domains. The signal in the time domain is the sum of many ideal sinusoidal waves at different frequencies and amplitudes. In other words the amplitudes in the frequency domain coincide with the peak to peak amplitudes of sinusoidal waves at that frequency. Hence the signal in the time domain is the sum of the signals in the frequency domain:

\[
h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{-i\omega t} d\omega
\]

(A8.1)

or in other words

\[
H(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t) e^{i\omega t} dt
\]

(A8.2)

\( H \) is generally a complex number. The Fast Fourier Transform (FFT for short) is a remarkably efficient way of calculating the Fourier transform of a time series [Newland, 1975]. The FFT algorithm saves a considerable time and increases the accuracy by reducing the errors caused due to truncation.
Appendix 8

Fast Fourier Transform

In this thesis the displacement histories of the shaft centre are obtained by the solution of the differential equations in the time domain. The sampling interval, $\Delta t$ (constant) and the discrete values of $x(t)$, at time $t = k \Delta t$, $X_r$ are used and the sequence $\{X_r\}, r = \ldots, -1, 0, 1, 2, \ldots, N-1$, is obtained and called a "discrete time series". From this sequence a new finite sequence $\{X_k\}$ is defined as

$$X_k = \frac{1}{N} \sum_{r=0}^{N-1} X_r e^{-\frac{2\pi kr}{N}}$$  \hspace{1cm} (A8.3)

which is obtained by using

$$X_k = \frac{1}{2} \left[\left( \frac{1}{N/2} \sum_{r=0}^{N/2-1} X_{2r} e^{-\frac{2\pi kr}{N/2}} \right) + W^k \left( \frac{1}{N/2} \sum_{r=0}^{N/2-1} X_{2r+1} e^{-\frac{2\pi kr}{N/2}} \right) \right]$$  \hspace{1cm} (A8.4)

$$X_{k+N/2} = \frac{1}{2} \left[\left( \frac{1}{N/2} \sum_{r=0}^{N/2-1} X_{2r} e^{-\frac{2\pi kr}{N/2}} \right) - W^k \left( \frac{1}{N/2} \sum_{r=0}^{N/2-1} X_{2r+1} e^{-\frac{2\pi kr}{N/2}} \right) \right]$$  \hspace{1cm} (A8.5)

in half of the computing time of that of a normal discrete Fourier transform.

In these equations

$k = 0, 1, 2, \ldots, (N/2-1)$ and $W = e^{-(2\pi i/N)}$

More detailed information about the FFT and derivation of these formulas may be found in Ref. [Newland, 1975].

Fig. (A8.1) The relation between the frequency and time domains
APPENDIX 9

RUNGE-KUTTA METHOD

The Runge-Kutta Method is a numerical solution technique that is employed in the solution of equations of motion. It is widely used in computational solutions to differential equations since the truncation error of the method is of the order five of the interval $h$ (therefore it is called 4th order) it is a relatively accurate method.

In this method, the differential equation has its solution extended forward from known conditions by an increment of the independent variable without using information outside of this increment. The solution is obtained from the weighted average of a number of estimates of the change in the variable. Hence the next iteration will be:

$$x_{i+1} = x_i + (K_1 + K_2 + K_3 + K_4)$$  \hspace{1cm} (A9.1)

where

$$K_1 = hf(x_i, t_i)$$  \hspace{1cm} (A9.2)

$$K_2 = hf\left(x_i + \frac{1}{2}K_1, t_i + \frac{1}{2}h\right)$$  \hspace{1cm} (A9.3)

$$K_3 = hf\left(x_i + \frac{1}{2}K_2, t_i + \frac{1}{2}h\right)$$  \hspace{1cm} (A9.4)

$$K_4 = hf(x_i + K_3, t_{i+1})$$  \hspace{1cm} (A9.5)

where $h$ is the time increment.
APPENDIX 10

THE EFFECT OF PRELOAD AND THE NUMBER OF BALLS ON THE NATURAL FREQUENCY

Fig. (A10.1) The force on the $i$th ball due to the shaft centre deflection

Let us assume that the bearing in Fig. (A10.1) is an angular contact ball bearing. The maximum deflection will caused at the angle $\beta$ due to the displacement of the inner ring centre in the $x$, $y$ and $z$ directions. This can be given as:

$$\delta = \sqrt{x^2 + y^2 + z^2}$$  \hspace{1cm} (A10.1)

and the angle $\beta$ can be calculated from (see Fig. (A10.1)):
Appendix 10

Natural Frequency Variations

\[ \tan \beta = \frac{y}{x} \quad (A10.2) \]

and the radial deflection on the \( i \)th ball can be calculated from (see Fig.(A10.1)):

\[ \delta_i = \delta \left( \cos(\theta_i - \beta)\cos^2 \alpha_i + \sin^2 \alpha_i \right) \quad (A10.3) \]

If the total forces in the \( x, y \) and \( z \) directions are calculated:

\[ W_x = \delta^2 \sum_{i=1}^{m} K A_i \cos \alpha_i \cos \theta_i \quad (A10.4) \]

\[ W_y = \delta^2 \sum_{i=1}^{m} K A_i \cos \alpha_i \sin \theta_i \quad (A10.5) \]

\[ W_z = \delta^2 \sum_{i=1}^{m} K A_i \sin \alpha_i \quad (A10.6) \]

where \( m \) is the number of balls and \( A_i = \cos(\theta_i - \beta)\cos^2 \alpha_i + \sin^2 \alpha_i \)

From these forces the stiffness coefficients in the \( x, y \) and \( z \) directions can be derived. For brevity here only the \( x \) direction will be derived. The force in the \( x \) direction can be written as:

\[ W_x = S \delta^2 \quad (A10.7) \]

where

\[ S = \sum_{i=1}^{m} K (\cos(\theta_i - \beta)\cos^2 \alpha_i + \sin^2 \alpha_i) \cos \alpha_i \cos \theta_i \quad (A10.8) \]

As Equ.(A10.7) suggest for a given time \( t = t^* \), \( S \) is constant. Therefore the derivation of Equ.(A10.7) with respect to the deflection will give an instantaneous linear stiffness around a value of \( \delta_o \) as

\[ (K_s)_x = \frac{\partial W_x}{\partial \delta} = \frac{3}{2} S (\delta)^\frac{1}{2} \delta_o \quad (A10.9) \]

The approximate natural frequency of the system in the \( x \) direction, around the deflection \( \delta_o \) will be (see Equ.(4.2)):
\[(\omega_n)_{\delta=\delta_0} = \left(\frac{3S}{M}\right)^{\frac{1}{2}} (\delta_0)^{\frac{1}{4}}\]  
(A10.10)

If the deflection is considered in terms of a constant deflection \(\delta_0\) and a small variation \(\Delta\delta\) around \(\delta_0\), Equ.(A10.10) will take the form of:

\[(\omega_n)_{\delta=\delta_0} = \left(\frac{3S}{M}\right)^{\frac{1}{2}} (\delta_0)^{\frac{1}{4}} \left(1 + \frac{1}{4} \frac{\Delta\delta}{\delta_0}\right)\]  
(A10.11)

\[P_{rs} = mK(\delta_0)^{\frac{3}{2}} \sin \alpha_p\]  
(3.16)

Considering Equ.(3.16), Equ.(A10.3) will become:

\[\left(\frac{\omega_n}{\delta_0}\right)_{\delta=\delta_0} = \left(\frac{3S}{M}\right)^{\frac{1}{2}} \left(\frac{P_{rs}}{mK \sin \alpha_p}\right)^{\frac{1}{6}} + \frac{1}{4} \left(\frac{\Delta\delta}{\left(\frac{P_{rs}}{mK \sin \alpha_p}\right)^{\frac{1}{2}}}\right)\]  
(A10.12)