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A constant-*Q* model for general viscoelastic media

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SUMMARY

When seismic waves propagate through viscoelastic media, the viscoelastic response can be presented as a fractional-order derivative of the strain. This fractional order β controlling the degree of viscoelasticity of subsurface media is referred to as the viscoelastic parameter. However, the viscoelasticity is conventionally quantified by the quality factor Q, and there is a gap between the viscoelastic parameter β and the Q factor. Here this paper bridges the gap by establishing a relationship between these two parameters. An exact Q model is derived analytically based on the viscoelastic parameter β. Since the exact Q model is frequency dependent, a constant-Q model which is frequency independent is proposed under a small-dissipation assumption. This constant-Q model is applicable to seismic data with a narrow frequency band and is consistent with Kolsky's attenuation model. Furthermore, an inverse function of the constant-Q model is presented for evaluating the viscoelastic parameter β from any given Q factor. Thus, the viscoelastic parameter β has an intuitive physical meaning that is directly linked to the Q factor.

Key words: Non-linear differential equation; Seismic attenuation; Wave propagation.

INTRODUCTION

When seismic waves propagate through subsurface media, their energy gradually attenuates, because of the absorption effect attributable to the viscoelastic response of the Earth's interior. This wave attenuation or Earth absorption effect is conventionally quantified by the quality factor Q of the subsurface media. The quality factor is generally wave frequency dependent and is denoted as $Q(\omega)$, where ω signifies the angular frequency.

The viscoelasticity of a subsurface medium can be generalized in a compact form presented as a fractional-order derivative of the strain (Wang 2016). For an ideal viscous (Newtonian) medium, the stress is linearly related to the strain rate, the first-order derivative of the strain. However, in its general form, the stress is related to the fractional-order derivative rather than an integer-order derivative of the strain. Consequently, the generalized viscoelastic wave equation is formed by the fractional-order derivative of the particle displacement. The fractional parameter β controls the degree of viscoelasticity; equivalently, β controls the degree of wave attenuation. Thereafter, this fractional parameter is referred to as the viscoelastic parameter.

Although the generalized viscoelastic wave equation unifies the purely elastic wave equation and the viscoelastic wave equation into a single form, there is a gap between the viscoelastic parameter β and the conventional quality factor Q. Thus, the primary objective of this paper is to establish an intuitive relationship between these two parameters.

The quality factor $Q(\omega)$ is fundamentally frequency dependent. In practice, it can also be assumed to be frequency independent when seismic waves exhibit a relatively narrow frequency band. This frequency-independent Q factor is referred to as a constant-Q model (Kjartansson 1979). Therefore, a simple formula for directly evaluating the constant-Q value from the fractional parameter β or vice versa would be greatly useful for practical applications in reflection seismology. In this way, the physical meaning of the viscoelastic parameter β is intuitively related to the commonly used constant-Q factor.

Sun *et al.* (2019) presented approximate Q models for different ranges of the viscoelastic parameter. They used a constant-Q model, when the viscoelastic parameter is small, and a frequency-dependent Q model, when the viscoelastic parameter is large. They made the approximation based on Kolsky's linear Q model, in which the attenuation coefficient depends linearly on the frequency (Kolsky 1953; Wang & Guo 2004).

In this paper, the exact and analytic expression of $Q(\omega)$ is defined in terms of the viscoelastic parameter β. The derivation of the analytic expression is carried out first based on the complex modulus and subsequently through the complex wavenumber defined by the attenuation coefficient and the phase velocity. Then, a genuine rather than range-dependent constant-Q model is proposed based on a small-dissipation assumption. This constant-Q model depends upon only the viscoelastic parameter β and is thus independent of the wave frequency. Finally, for a given Q factor, a formula for constructing $\beta(Q)$ is presented. The viscoelastic parameter β can be applied directly to the viscoelastic wave equation.

THE ANALYTIC Q FUNCTION

This section defines the exact expression of $Q(\omega)$ in an analytic function. The derivation is carried out first through the complex modulus and subsequently based on the complex wavenumber.

Seismic waves propagate through subsurface media in a generalized form (Wang 2016). To reflect this generalized characteristic, the stress–strain relationship is described by a fractional-order instead of a first-order temporal derivative:

$$\sigma(t) = E\left(\varepsilon(t) + \beta\tau^{\beta}\frac{\mathrm{d}^{\beta}\varepsilon(t)}{\mathrm{d}t^{\beta}}\right), \tag{1}$$

where $\sigma(t)$ is the time-dependent tensile stress, $\varepsilon(t)$ denotes the corresponding strain, E is Young's modulus describing the elasticity of the media and τ is the retardation time describing the delay of the elastic response.

The compact form of a fractional derivative is used to describe the frequency and time dependencies of a viscoelastic system (Nutting 1921; Gemant 1936; Scott-Blair 1947; Smit & de Vries 1970; Bagley & Torvik 1983; Mainardi 2010; Wang 2016). The range of the viscoelastic parameter is $\beta \in [0, 1]$. When $\beta = 0$, eq. (1) is Hooke's law for purely elastic media, in which the stress–strain relationship is linear. When $\beta = 1$, eq. (1) is a viscoelastic case that combines the purely elastic case with the ideal viscous (Newtonian) case defined by a linear relationship between the stress and the strain rate.

In the frequency domain, eq. (1) may be written as

$$\sigma(\omega) = E\left(1 + \beta\tau^{\beta}(\mathrm{i}\omega)^{\beta}\right)\varepsilon(\omega), \tag{2}$$

where ω is the angular frequency and $\mathrm{i} = \sqrt{-1}$. Eq. (2) defines the complex modulus as

$$M(\omega) = E(1 + \mathrm{i}^{\beta}\beta\tau^{\beta}\omega^{\beta}). \tag{3}$$

Substituting $\mathrm{i}^{\beta} = \cos(\beta\pi/2) + \mathrm{i}\sin(\beta\pi/2)$ into eq. (3), the complex modulus may be rewritten as

$$M(\omega) = E\left[1 + \beta\left(\frac{\omega}{\omega_0}\right)^{\beta}\cos\frac{\beta\pi}{2} + \mathrm{i}\beta\left(\frac{\omega}{\omega_0}\right)^{\beta}\sin\frac{\beta\pi}{2}\right], \tag{4}$$

where $\omega_0 = \tau^{-1}$. Given this complex modulus, the dissipation factor that describes the phase lag of the strain behind the stress may be defined as $\xi(\omega) \equiv M_{\mathrm{Im}}(\omega)/M_{\mathrm{Re}}(\omega)$ (White 1965). This dissipation factor $\xi(\omega)$ is the inverse of the quality factor $Q(\beta, \omega)$:

$$Q^{-1}(\beta, \omega) = \frac{\beta\left(\frac{\omega}{\omega_0}\right)^{\beta}\sin\frac{\beta\pi}{2}}{1 + \beta\left(\frac{\omega}{\omega_0}\right)^{\beta}\cos\frac{\beta\pi}{2}}. \tag{5}$$

This constitutes the analytical definition of the Q factor and is defined in terms of the viscoelastic parameter β and the frequency ω.

Based on eq. (1), which represents the generalized form of the viscoelastic property, a generalized viscoelastic wave equation was developed (Wang 2016). The analytical solutions for the attenuation coefficient $\alpha(\beta, \omega)$ and the phase velocity $v(\beta, \omega)$ were also obtained by solving the generalized wave equation in the frequency domain.

The attenuation coefficient is expressed as

$$\alpha(\beta, \omega) = \frac{\omega}{\sqrt{2}c}\frac{\sqrt{A - B}}{A}, \tag{6}$$

and the phase velocity is

$$\frac{\omega}{v(\beta, \omega)} = \frac{\omega}{\sqrt{2}c}\frac{\sqrt{A + B}}{A}, \tag{7}$$

where

$$A = \left[1 + 2\beta\left(\frac{\omega}{\omega_0}\right)^{\beta}\cos\frac{\beta\pi}{2} + \beta^2\left(\frac{\omega}{\omega_0}\right)^{2\beta}\right]^{1/2}, \tag{8}$$

$$B = 1 + \beta\left(\frac{\omega}{\omega_0}\right)^{\beta}\cos\frac{\beta\pi}{2}, \tag{9}$$

and $c = \sqrt{E/\rho}$ is the elastic velocity, which is expressed in terms of Young's modulus E and the bulk density ρ. The attenuation coefficient and the phase velocity together define the complex wavenumber.

Eq. (7) clearly indicates that the elastic velocity c is the limit of the phase velocity when $\omega \to 0$, $c = \lim_{\omega\to 0} v(\omega)$. This velocity should be distinguished from the infinite limit of the phase velocity expressed as

$$v_{\infty} \equiv \lim_{\omega\to\infty} v(\omega) = \lim_{\omega\to\infty}\sqrt{\frac{M_{\mathrm{Re}}(\omega)}{\rho}}. \tag{10}$$

The latter is used in various Q models to define the phase velocity (Wang 2008).

Given the attenuation coefficient and the phase velocity, the quality factor Q is defined as (Wang 2008)

$$Q^{-1}(\omega) = 2\left(\frac{\omega}{\alpha(\omega)v(\omega)} - \frac{\alpha(\omega)v(\omega)}{\omega}\right)^{-1}. \tag{11}$$

According to eqs. (6) and (7), the Q factor may be expressed as follows:

$$Q^{-1}(\beta, \omega) = \frac{\sqrt{A^2 - B^2}}{B}. \tag{12}$$

Substituting A (eq. 8) and B (eq. 9) into eq. (12) leads to an expression for $Q^{-1}(\beta, \omega)$ which is identical to that presented in eq. (5). Note here that $Q^{-1}(\beta, \omega)$ is independent of the elastic velocity c and that the frequency-dependent Q factor depends only on the fractional parameter β.

Fig. 1 displays the analytic function of $Q^{-1}(\beta, \omega)$ derived from the generalized viscoelastic model. The frequency (the horizontal axis) is normalized by the reference frequency ω_0, and the viscoelastic parameter β varies from 0 to 1. When $\beta = 0$, there is no attenuation; when $\beta = 1$, there is a linear dependency on the frequency. Both these cases are represented by straight lines in Fig. 1.

The monotonically increasing feature of curves suggests that the analytic function of $Q^{-1}(\beta, \omega)$ is applicable in practice, for instance, for estimating an average value of $Q^{-1}(\beta)$ within a short frequency band, for any given β parameter. For any given frequency, $Q^{-1}(\beta)$ for various β values can be calculated directly using the analytic function (5).

THE CONSTANT-Q MODEL

For practical applications, it is necessary to establish the relationship between the constant-Q factor and the viscoelastic parameter β. The constant-Q or frequency-independent Q factor is valid within a narrow frequency band (Kjartansson 1979). For instance, the frequency band in reflection seismology is approximately on the order of hundreds of Hz. Once the relationship $Q(\beta)$ is established, the

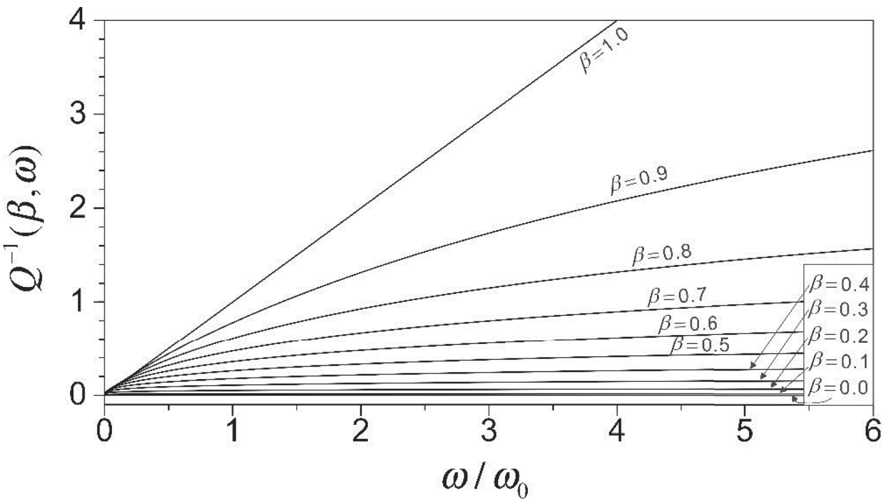

Figure 1. The analytic function of the $Q(\beta, \omega)$ factor varying with the viscoelastic parameter β and the frequency ω. The frequency (the horizontal axis) is normalized by the reference frequency ω_0.

viscoelastic parameter β, which is expressed in terms of the commonly used factor Q, will be represented intuitively. Meanwhile, the inverse relationship $\beta(Q)$ will be useful for wave simulations using the generalized wave equation.

Fig. 2 plots the analytic function $Q(\beta, \omega)$ where the horizontal axis is the viscoelastic parameter β and is different form that in Fig. 1. The topmost solid curve is the limit of Q when $\omega/\omega_0 \to \infty$:

$$Q^{-1}(\beta; \omega \to \infty) = \tan \frac{\beta\pi}{2}. \tag{13}$$

When $\omega/\omega_0 = 1$, eq. (5) becomes

$$Q^{-1}(\beta; \omega_0) = \frac{\beta \sin \frac{\beta\pi}{2}}{1 + \beta \cos \frac{\beta\pi}{2}}. \tag{14}$$

This function represents one of the solid red curves ($\omega/\omega_0 = 1$).

According to the definition $\omega_0 \equiv \tau^{-1} = (\eta/E)^{-1}$, where η represents the viscous coefficient, the reference frequency is proportional to the inverse of the viscosity. Therefore, normalizing the frequency for the various curves in Fig. 2 removes the influence of the viscosity η and differentiates the influence of the fractional parameter β on the Q model.

An inspection of eq. (14) indicates that $Q^{-1}(\beta; \omega_0) \leq 1$, which signifies the upper limit of the constant-Q model. Fig. 2 shows that $Q^{-1}(\beta; \omega_0)$ is the central curve between the two limits $Q^{-1}(\beta; \omega \to \infty)$ and $Q^{-1}(\beta; 0) = 0$ if a width between two limits is measured along the direction (the blue dashed straight line) perpendicular to this central curve $Q^{-1}(\beta; \omega_0)$.

In this paper, I propose to set $Q^{-1}(\beta; \omega_0)/\sqrt{2}$ as the constant-Q model for practical applications in reflection seismology:

$$Q^{-1}(\beta) = \frac{\frac{\beta}{\sqrt{2}} \sin \frac{\beta\pi}{2}}{1 + \beta \cos \frac{\beta\pi}{2}}. \tag{15}$$

I make this proposal based on the following two justifications.

1) $Q^{-1}(\beta; \omega_0)/\sqrt{2}$ represents the central curve between $Q^{-1}(\beta; \omega_0)$ and $Q^{-1}(\beta; 0)$, which are the upper and lower limits, respectively, of any acceptable constant-Q model.

2) An inspection of eq. (15) indicates that it satisfies the small-dissipation assumption $Q^{-1}(\beta) \ll 1$ since $Q^{-1}(\beta) = \frac{1}{\sqrt{2}} Q^{-1}(\beta; \omega_0) \leq \frac{1}{\sqrt{2}}$. This small-dissipation assumption is valid under most conditions of interest in geophysics, for instance, Kolsky (1953), Mason (1958) and Futterman (1962).

The abovementioned observation that all existing constant-Q models have an upper limit when $\omega = \omega_0$ is also consistent with the analysis in Wang & Guo (2004), who suggested modifying the Kolsky model by using the highest frequency of the available frequency band as the reference frequency ω_0 (denoted as ω_h). Therefore, the Kolsky model is comparable to other mathematical models presented in the literature.

Fig. 2 also indicates that the proposed constant-Q model is similar to the Q factor at one-half of the reference frequency. This observation endows an intuitive physical meaning to the constant-Q model.

Various constant-Q models $Q_r(\beta)$correspond to a variety of given β parameters. Applying these constant-Q models to Kolsky's model produces various attenuation coefficients. Let us now compare Kolsky's model with the generalized viscoelastic model.

Fig. 3(a) displays plots of the attenuation coefficient of the generalized viscoelastic model. The attenuation coefficient is normalized by ω_0/c:

$$\frac{\alpha(\omega)}{\omega_0/c} = \frac{\omega}{\sqrt{2}\omega_0} \frac{\sqrt{A - B}}{A}, \tag{16}$$

where A and B are given by eqs. (8) and (9), respectively. The frequency (the horizontal axis) is again normalized by the reference frequency ω_0.

The conventional Kolsky attenuation model is a linear function of the frequency for $\omega \geq 0$:

$$\alpha(\omega) = \frac{\omega}{2cQ_r}, \tag{17}$$

where Q_r is the constant-Q model given by eq. (15).

Fig. 3(b) demonstrates that Kolsky's attenuation model (dashed black curves) is comparable to the generalized attenuation model (solid red curves), which is defined in terms of the viscoelastic parameter β, if the proposed constant-Q model is employed. This comparison reveals the self-consistency of the proposed constant-Q model, despite there is a certain degree of difference when β is large, because the Kolsky's model is a linear function of frequency.

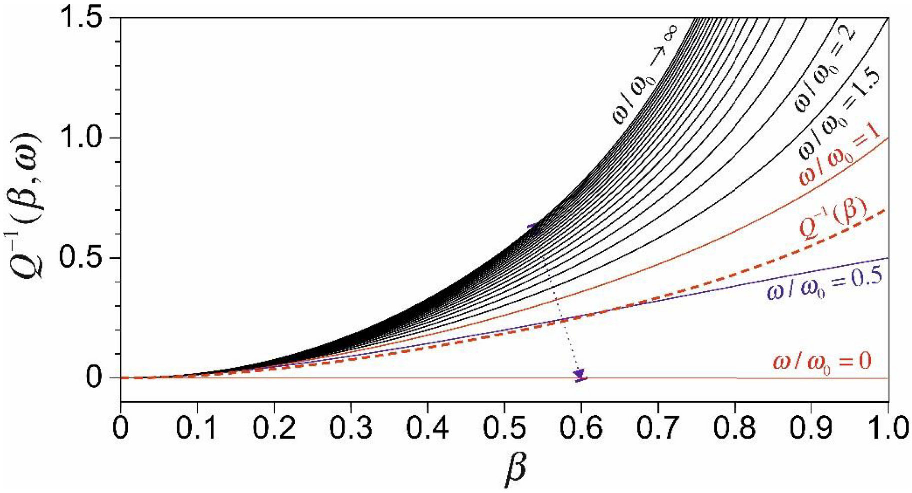

Figure 2. The analytic function of $Q(\beta, \omega)$ with the viscoelastic parameter as the horizontal axis. The topmost solid black curve is $Q^{-1}(\beta; \omega \to \infty)$. The two solid red curves are the upper limit $Q^{-1}(\beta; \omega_0)$ and the lower limit $Q^{-1}(\beta; 0)$, respectively, of any possible constant-Q model. The dashed red curve is the proposed constant-Q model $Q^{-1}(\beta)$ for practical applications in exploration seismology.

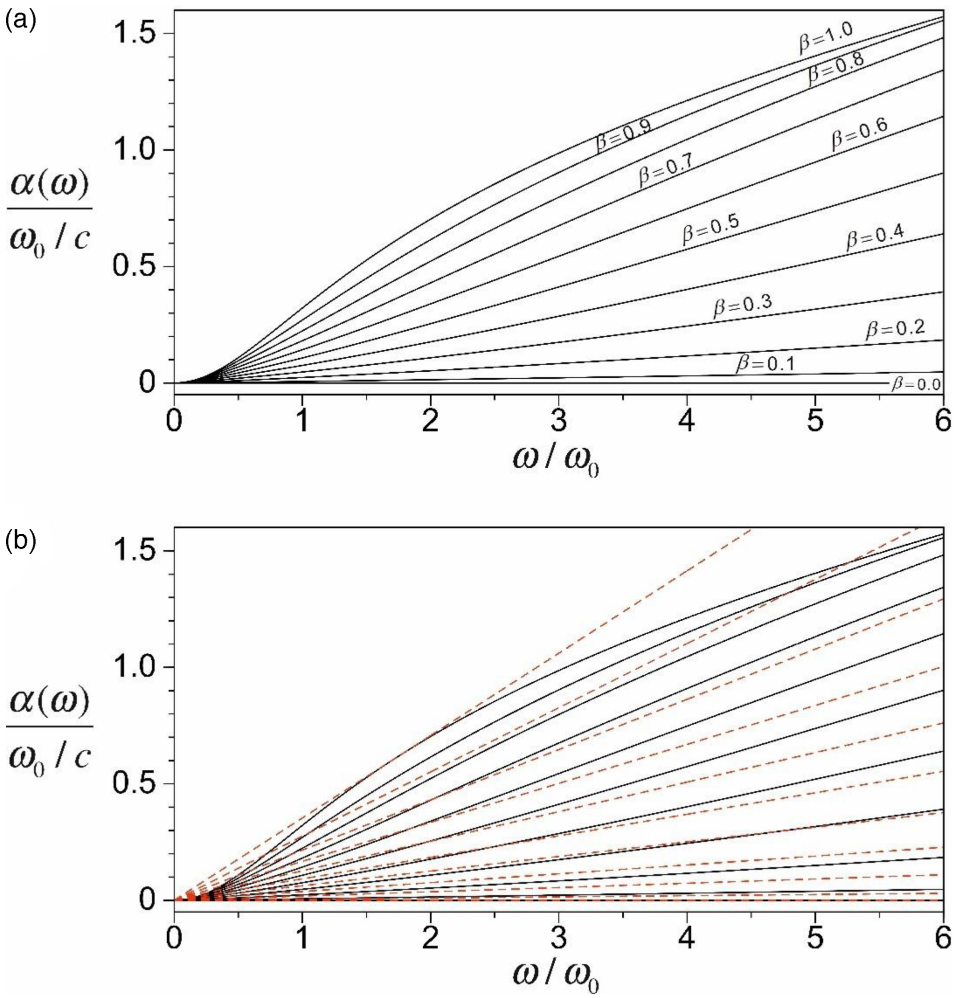

Figure 3. Different attenuation models. (a) Plots of the attenuation coefficient of the generalized viscoelastic model. (b) Comparison between Kolsky's attenuation model (dashed red curves) and the generalized viscoelastic model (solid black curves). The attenuation coefficient is normalized by ω_0/c, where ω_0 is the reference frequency and c is the elastic velocity. The frequency axis is also normalized by the reference frequency ω_0. The fractional parameter β varies within the range between 0 and 1.

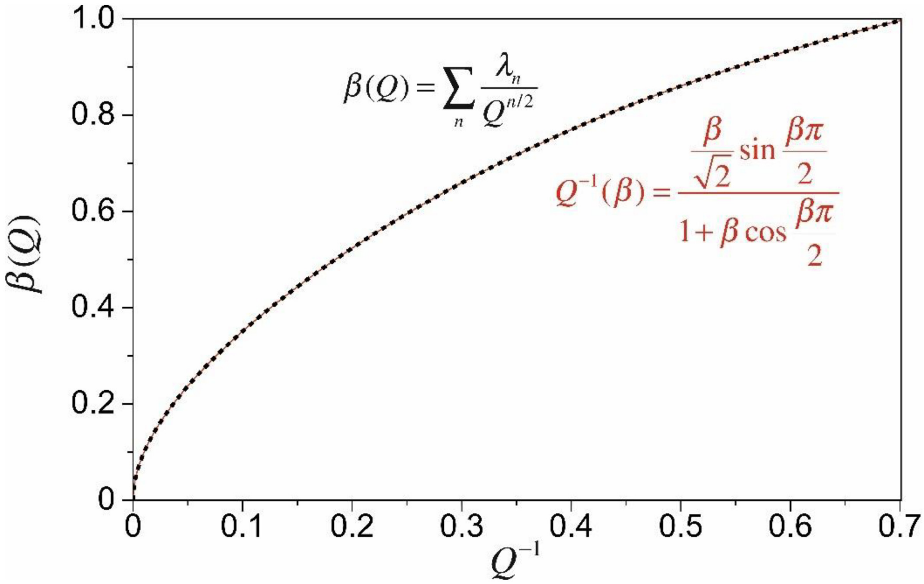

Figure 4. The viscoelastic parameter $\beta(Q)$ constructed from any given Q value. The solid red curve is the analytic function (the red equation), and the overlain dashed black curve is the numerical construction (the black equation).

THE VISCOELASTIC PARAMETER VERSUS Q

The constant-Q model $Q(\beta)$ in eq. (15) is defined in terms of the viscoelastic parameter β. However, an inverse function $\beta(Q)$ will be useful in practical applications. For example, for a given Q value, explicitly derived fractional parameter can be used in the generalized wave equation for seismic wave simulation. However, it is difficult to analytically construct the inverse function $\beta(Q)$ from $Q(\beta)$ in eq. (15). Therefore, I construct $\beta(Q)$ numerically and present $\beta(Q)$ as a polynomial function:

$$\beta(Q) = \sum_n \frac{\lambda_n}{Q^{n/2}}, \quad (18)$$

where $n = \{1, 2, 3, 4, 5, 6, 7\}$ and λ_n are coefficients to be determined. These seven coefficients are obtained numerically as

$$\{\lambda_n\} = \{0.947635, 0.486628, -0.071021, 1.34597, -3.36047, 1.95544, -0.168509\}. \quad (19)$$

Fig. 4 demonstrates a perfect fitting in a least-squares sense.

Note that eq. (18) is a polynomial function of $\sqrt{Q^{-1}}$. In eq. (15), the numerator of the fraction on the right-hand side is $\beta \sin \frac{\beta\pi}{2} \propto \beta^2$, which indicates that the fractional parameter β is proportional to the square root $\sqrt{Q^{-1}}$ rather than to Q^{-1} directly.

CONCLUSION

When seismic waves propagate through viscoelastic media, their energy gradually attenuates. Seismic wave propagation and attenuation may be described by a generalized viscoelastic wave equation in which the degree of viscoelasticity is controlled by the viscoelastic parameter β. This paper has established the relationship between the viscoelastic parameter β and the conventional Q factor.

1) This paper has derived analytically the exact $Q(\beta, \omega)$ model for generalized viscoelastic media. This analytical Q model is described by the viscoelastic parameter but is also frequency dependent.

2) For practical applications in reflection seismology using data with a narrow frequency band, this paper has proposed a constant-Q model presented in terms of the viscoelastic parameter but not the frequency.

3) The result of applying this constant-Q model to the conventional Kolsky model has shown self-consistency when comparing the Kolsky's attenuation model with the generalized attenuation model.

4) This paper has proposed also an explicit $\beta(Q)$ function for any given Q factor. Therefore, one can evaluate the β parameter and apply it directly to the generalized wave equation.

For Q model establishment, a practical procedure can be summarized into two steps: first, applying seismic waveform tomography to invert for the velocity model and the viscoelastic parameter model either sequentially or simultaneously, and then, converting the viscoelastic parameter model to a conventional Q model for seismic data processing and reservoir characterization.

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