Elastic imperfect cylindrical shells of varying length under combined axial compression and bending

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Abstract

This paper presents a comprehensive computational investigation into the elastic nonlinear buckling response of near-perfect and highly imperfect uniform thickness thin cylindrical shells of varying length under combined uniform compression and bending. In particular, the elastic ovalisation phenomenon in cylindrical shell of sufficient length under combined compression and bending was systematically investigated with finite elements for the first time. The study considered a representative range of practical lengths up to very long cylinders where ovalisation is fully developed under uniform bending and Euler column buckling controls under uniform axial compression. The imperfection sensitivity of the system was also studied by introducing a single idealised axisymmetric weld depression imperfection at the midspan of the cylinder.

The predictions permit an exploration of the nonlinear mechanics of the generally unfavourable interaction between bending and axial compression at the elastic nonlinear buckling limit state in thin long cylinders. The interaction is at its most unfavourable in cylinders where Euler column buckling is about to become critical, and is qualitatively very different from the favourable moment-force interaction at the reference plastic limit state of circular tubes. A simple closed-form algebraic characterisation of the interaction is proposed considering both imperfections and ovalisation.

Keywords

Axial compression; uniform bending; combined loading; cylindrical shells; imperfection sensitivity; length effects; N-M interaction; nonlinear mechanics; ovalisation.

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1. Introduction

A cylindrical shell under the combined actions of uniform axial compression and uniform bending constitutes a common generic system that finds widespread practical application, for example as wind turbine support towers, chimneys, tubular piles and circular hollow section (CHS) beam-column members. The ultimate resistance of these is known to be controlled by many phenomena, including material plasticity (Rotter and Sadowski, 2017), nonlinearity of the stress-strain relationship in metals other than mild steel (Zhao et al., 2016a; 2016b), elastic and elastic-plastic instability (Yamaki, 1984; Chen et al., 2008; Rotter et al., 2014), geometric imperfections (Rotter and Teng, 1989; Rotter and Al-Lawati, 2016; Wang and Sadowski, 2018; Fajuyitan and Sadowski, 2018) and, where long members are subject to bending, cross-sectional ovalisation (Brazier, 1927; Rotter et al., 2014; Xu et al., 2017; Wang and Sadowski, 2018; Wang et al., 2018). The ovalisation phenomenon may be especially pronounced in long thin-walled cylinders if the pre-buckling behaviour is dominated by the elastic fundamental response. However, no study appears to have fully explored the influence of ovalisation in thin cylindrical shells of varying length on the ‘N-M’ interaction between the compressive axial force and moment at the elastic nonlinear buckling limit state. This is partly due to the fact that it is only recently that length effects in elastic imperfect cylinders have become fully documented for the individual reference cases of uniform compression (Rotter and Al-Lawati, 2016) and uniform bending (Rotter et al., 2014; Fajuyitan and Sadowski, 2018).

This study thus aims to expand on the recent work of Rotter et al. (2014), Rotter and Al-Lawati (2016) and Fajuyitan and Sadowski (2018) by comprehensively investigating length effects and imperfection sensitivity in elastic cylindrical shells under combined axial compression and bending. To this end, an extensive series of geometrically nonlinear computational analyses of near-perfect and imperfect isotropic elastic cylindrical shells of varying length was performed using the ABAQUS (2017) commercial finite element software. The mechanics of length-dependent geometric nonlinearity, imperfection sensitivity and the N-M interaction of the system are explored in detail and the effects are characterised in a closed-form algebraic format.
2. Literature review

2.1. Stocky cylindrical shells under combined bending and compression

The existing literature on cylindrical shells, CHS and tubular members under combined axial compression and bending has focused predominantly on the inelastic buckling and plastic collapse behaviour of relatively thick tubes with cross-sections limited to radius to thickness ($r/t$) ratios of the order of 50 (Sherman et al., 1979; Prion and Birkemoe, 1992; Hu et al., 1993; Linzell et al., 2003; Nseir, 2015; Ma, 2016; Hayeck, 2016; Zhao et al., 2016a; 2016b; Pournara et al., 2017). The fully plastic state is an important reference against which the reductions due to different nonlinear phenomena can be assessed, and many algebraic proposals have been made to interpret the compressive load and moment resistance ($N$-$M$) interaction relationship of cylindrical tubes at different lengths, although the effect of elastic cross-sectional ovalisation does not appear to have ever been satisfactorily incorporated. Exact expressions and simplified but accurate approximations of the reference fully plastic $N$-$M$ envelope of tubes have recently been derived in Rotter and Sadowski (2017) based on a first principles algebraic treatment assuming an ideal elastic-rigid plastic constitutive law and an undeformed perfect geometry. However, the failure of relatively thick tubes is controlled by yielding that precedes the development of any significant ovalisation, and the effect of flattening on the favourable $N$-$M$ interaction at the plastic limit state is usually minimal (Wang et al., 2018).

For thin-walled ($r/t > 100$) cylindrical shell applications such as wind turbine support towers, chimneys, slender silos and long CHS members, elastic instability may strongly influence failure, and where such structures are subject to bending they may experience significant elastic ovalisation prior to buckling. For long cylindrical shells under combined axial compression and bending, ovalisation caused by bending may additionally interact with the ‘column $P$-$\Delta$ effect’ caused by Euler buckling. However, this phenomenon has not been studied in the past, and is therefore the focus of the current paper, as reported in detail in Sections 4 and 5.
2.2. Thin cylindrical shells of varying length under uniform compression

The elastic buckling behaviour of cylindrical shells under uniform compression has been the subject of a very large number of previous research studies, and it is well established that these shells present different buckling modes according to the length (Yamaki, 1984; Rotter, 2004; Rotter and Al-Lawati, 2016; Sadowski et al., 2018). The elastic nonlinear finite element calculations presented in Fig. 1, adapted from Rotter and Al-Lawati (2016), illustrate the variation of the predicted buckling stress $\sigma_k$ of a cylinder under uniform compression normalised by the classical elastic critical buckling stress $\sigma_{cl} \approx 0.605Et/r$ (assuming a Poisson ratio of $\nu = 0.3$, and where $E$, $r$ and $t$ are the elastic modulus, midsurface radius and thickness of the shell wall respectively) against the two dimensionless length groups $\omega = L/\sqrt{(rt)}$ and $\Omega = L/r\sqrt{(t/r)}$, such that $\Omega = \omega(t/r)$. At buckling, both edge boundary conditions were assumed to be radially restrained but free to rotate about the circumferential edge and to displace meridionally, classifying these as BC2f (EN 1993-1-6:A1, 2017) or S3 (Yamaki, 1984). This translates to an effective Euler column buckling length of $L$.

Fig. 1 – Variation of the elastic nonlinear buckling strength of imperfect cylindrical shells under uniform compression with dimensionless length parameters $\omega$ and $\Omega$ (figure adapted, with permission, from Rotter and Al-Lawati (2016) with added annotations).
Varying levels of imperfection amplitude were considered in the above study, from near-perfect ($\delta/t = 0.01$) to quite imperfect ($\delta/t = 2.5$), assuming the Rotter and Teng (1989) ‘Type A’ axisymmetric circumferential weld depression. This imperfection form is an idealised model of the inward curling as a result of material contraction during post-weld cooling, offering a realistic and widely-used representation of systematic manufacturing defect in welded cylindrical metal shells (Berry et al., 2000; Pircher et al., 2001; Teng et al., 2005; Sadowski et al., 2015). It has been used extensively as a modelling device in computational studies to explore imperfection sensitivity of cylindrical shells under load cases dominated by axial compression (e.g. Teng and Rotter, 1992; Song et al., 2004; Chen et al., 2008; Fajuyitan and Sadowski, 2018). Its functional form is defined later in this paper.

In Fig. 1, ‘medium-length’ near-perfect cylinders exhibit local short-wave buckling with a high circumferential wave number at a resistance $\sigma_k$ very close to the classical elastic critical buckling load $\sigma_{cl} \approx 0.605Et/r$, and the behaviour is characterised well by the dimensionless length $\omega = L/\sqrt{rt}$ (Yamaki, 1984; Rotter, 2004). Very ‘short’ near-perfect cylinders ($\omega < 1.7$) exhibit significantly higher buckling resistances than $\sigma_{cl}$ due to the formation of even a short-wave buckle being restrained by the short geometry and boundary conditions (Fajuyitan et al., 2018). By contrast, near-perfect cylinders longer than $\Omega \approx 0.5$ begin to exhibit a gradual reduction in their elastic nonlinear buckling resistance with increasing length together with a reduction in the circumferential wave number of the buckling mode, while those longer than $\Omega \approx 3.7$ (for these boundary conditions) are controlled by Euler column buckling. This behaviour is reasonably well captured by the current rule in EDR5 (ECCS, 2013) and EN 1993-1-6:A1 (2017) for this load condition. The study of Rotter and Al-Lawati (2016) was arguably the first to clearly illustrate that the sensitivity of this classical system to the weld depression imperfection is dependent on the length, with shorter ‘medium-length’ cylinders exhibiting the most severe relationship, and that the current design practice to characterise the imperfection sensitivity as independent of length through simple reduction factors deserves rethinking. It also found that Euler column buckling does not intervene over shell buckling until the imperfect buckling resistance becomes higher than the Euler buckling value of the perfect cylinder at a given length $\Omega$. 

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2.3. Thin cylindrical shells of varying length under uniform bending

Ovalisation in long cylindrical shells under bending can lead to up to a ~50% reduction in the elastic moment resistance compared to that of shorter cylinders where the cross-section remains undistorted. This phenomenon has been well-known for about a century (i.e. since von Karman, 1911; Brazier, 1927) and has been verified repeatedly by numerous analytical and numerical studies (e.g. Reissner, 1959; Reissner and Weinitschke, 1963; Fabian, 1977; Gellin, 1980; Boyle, 1981; Calladine, 1983; Libai and Bert, 1994; Karamanos, 2002). However, it was only recently that Rotter et al. (2014) offered a complete characterisation of the elastic nonlinear buckling resistance of perfect rotationally-clamped cylindrical shells (classified as BC2r or C3) for the full range of practical lengths. This work illustrated that the elastic nonlinear buckling behaviour should be categorised into four distinct domains (‘short’, ‘medium’, ‘transitional’ and ‘long’; one more than under uniform compression) given well-defined boundaries in terms of the dimensionless length parameters $\omega$ and $\Omega$, as illustrated in Fig. 2.

‘Short’ cylinders ($\omega < ~5$) achieve significantly higher moment capacities than the classical elastic critical buckling moment $M_{cl} \approx 1.901E_{rt}^2$, again due to the restraint imposed by the end boundaries on the formation of local short-wave buckles as is assumed by the expression for $M_{cl}$. ‘Medium-length’ cylinders are free to develop local short-wave buckling on the compressed side at a moment slightly below the classical critical buckling moment $M_{cl}$ due to slight (~10%) stress amplification near the loaded boundaries. In the ‘transitional’ range ($\Omega > ~0.5$), flattening of the compressed side due to increasingly severe ovalisation with increasing length reduces the local critical buckling stress and the elastic section modulus, and therefore also the critical buckling moment. However, this reduction eventually stabilises at a value of $\sim 0.51M_{cl}$ or 95% of the Brazier moment $M_{Braz} \approx 1.035E_{rt}^2 \approx 0.54M_{cl}$ in the ‘long’ length range ($\Omega > ~7$) and the buckling mode continuing to be in the form of local short-wave ripples on the now flattened compressed side. The imperfection sensitivity of the system (also to a single Rotter and Teng weld depression imperfection placed at midspan) was investigated by the follow-up studies of Fajuyitan et al. (2018) and Fajuyitan and Sadowski (2018), the results of which are also included in Fig. 2. It was similarly shown that the imperfection sensitivity of cylindrical shells under bending varies significantly with length, but that
the boundary between the ‘medium’ and ‘transitional’ domains at $\Omega \approx 0.5$ appears to present the most severe possible imperfection sensitivity relationship.

Fig. 2 – Variation of the elastic nonlinear buckling strength of imperfect cylindrical shells under uniform bending with dimensionless length parameters $\omega$ and $\Omega$, with data taken from Fajuyitan et al. (2018) and Fajuyitan and Sadowski (2018).

3. Numerical modelling

3.1. Scope of the study

The elastic nonlinear behaviour of thin cylindrical shells under combined compression and bending was investigated computationally using the finite element software ABAQUS (2017) through the following series of analyses:

- Geometrically nonlinear analyses of near-perfect elastic cylinders (GNAs) with a representative radius to thickness ($r/t$) ratio of 100 at 15 different lengths ranging from $\Omega = 0.3$ to 7, and 21 different load and moment combinations ranging from uniform axial compression to uniform bending.
• Geometrically nonlinear analyses of imperfect elastic cylinders (GNIAs) under
  the same geometric parameters as the GNAs with a single midspan instance of
  the ‘Type A’ weld depression imperfection of Rotter and Teng (1989) with 8
  amplitudes of \( \delta/t = 0.1, 0.25, 0.5, 0.75, 1, 1.5, 2 \) and 3.

The lower length boundary of \( \Omega = 0.3 \) represents the longer regions of the ‘medium’
length domains for both systems (Figs 1 and 2) where boundary effects have largely
decayed and the pre-buckling response both very linear and independent of the nature of
the local rotational restraint at the loaded edge. Similarly, the upper length boundary of
\( \Omega = 7 \) represents ‘long’ behaviour which is characterised by significant pre-buckling
nonlinearity that is qualitatively invariant with further a increase in length. The region
in-between these boundaries captures the transition from shell to Euler column buckling
for uniform compression and the onset of ovalisation for uniform bending, while also
encompassing the lengths of most practical cylinder applications. It is expected that the
\( N-M \) interaction will be almost linear for \( \Omega \approx 0.3 \), but may become very nonlinear with
increasing length. The large amount of analyses (several thousand simulations) was
handled by the model management and automation methodology of Sadowski \textit{et al.}
(2017).

3.2. Modelling details and assumptions

A schematic illustration of the finite element models implemented in ABAQUS (2017)
is shown in Fig. 3. Due to the symmetric nature of the combined loaded system, only a
quarter of the cylinder was modelled with circumferential and meridional symmetry
boundary conditions for computational efficiency. The 180° circumference of a quarter-
shell model permits buckling into any circumferential wave number under uniform
compression while not restricting the localised buckling modes under combined loading.
Such models have been widely used in the absence of torsion (Teng and Song, 2001; Cai
\textit{et al.}, 2003; Jayadevan \textit{et al.}, 2004; Song \textit{et al.}, 2004; Østby \textit{et al.}, 2005; Limam \textit{et al.},
2010; Rotter \textit{et al.}, 2011; 2014; Sadowski \textit{et al.}, 2018). A state of combined loading was
achieved by applying a point load \( N \) in the meridional direction through a reference
point located at a known eccentricity \( e \) away from the centroid of the cross-section and
linked via a rigid coupling to the circumferential edge of the cylinder, thus additionally
inducing an edge moment \( M = e \cdot N \) on the cylinder. The loaded edge was restrained
against rotation about the circumferential axis (though this assumption does not
influence the behaviour of tubes in the length domains studied here; Yamaki, 1984;
Fajuyitan and Sadowski, 2018), while the reference point was free to displace
meridionally but not transversely to the central axis of the cylinder. The end boundary
conditions are thus classified as BC2r or C3 and the system exhibits the same effective
Euler column buckling length of $L$ as Rotter and Al-Lawati (2016; Fig. 1). The chosen
load eccentricities were designed to give a gradual transition from uniform compression
($e = 0$) to uniform bending ($e \to \infty$), and were related to a ‘dimensionless eccentricity’ $\xi$
as follows (Fig. 4):

$$\xi = \tan^{-1}\left(\frac{M}{M_{cl}/N_{cl}}\right) = \tan^{-1}\left(\frac{e \cdot N}{M_{cl}/N}\right) = \tan^{-1}\left(\frac{e \cdot N_{cl}}{M_{cl}}\right) = \tan^{-1}\left(2 \cdot \frac{e}{r}\right)$$  \hspace{1cm} (1)

where $N_{cl} = A \cdot \sigma_{cl} \approx (2\pi r t) \cdot \left(0.605E\frac{t}{r}\right) \approx 3.803Et^2$  \hspace{1cm} (2)

and $M_{cl} = W \cdot \sigma_{cl} \approx (\pi r^2 t) \cdot \left(0.605E\frac{t}{r}\right) \approx 1.901Ert^2$  \hspace{1cm} (3)

are the classical elastic critical buckling load and moment (under linear conditions and a
pre-buckling purely membrane stress state that is free of length effects) for a ‘medium-
length’ thin-walled cylinder under uniform compression and bending respectively,
assuming $\nu = 0.3$. $A$ and $W$ are the cross-sectional area and elastic section modulus for a
thin-walled circular hollow section respectively. It should be added that $N_{cl} = N_{Euler}$ (Eq.
4, below) at a dimensionless length of $\Omega \approx 2.86$, consistent for a cylinder with pinned
‘column’ boundary conditions (Fig. 1).

$$N_{Euler} = A \cdot \sigma_{Euler} \approx (2\pi r t) \cdot \left(4.935E\left(\frac{r}{L}\right)^2\right) \approx 31.014Ert\left(\frac{r}{L}\right)^2$$  \hspace{1cm} (4)
Fig. 3 – Geometry, loading and boundary conditions of the finite element model.

Fig. 4 – Definition of the dimensionless eccentricity \( \xi \), load proportionality factor LPF and the characteristic buckling resistance \( R_k \).

The dimensionless eccentricity \( \xi \) may be visualised as the angle between the vertical \( N / N_{cl} \) axis and the line connecting the origin with the system resistance \( R_k \) computed by a finite element analysis as shown in Fig. 4, with \( \xi = 0 \) and \( \pi/2 \) representing the reference
loading states of uniform compression and uniform bending respectively. The 21
different load combinations investigated thus correspond to 20 intervals in \( \xi \) from 0 to
\( \pi/2 \). Without geometric nonlinearity or boundary effects, the \( N-M \) interaction is linear
and a linear bifurcation analysis (LBA) will always predict a critical buckling resistance
\( R_{cl} \) on a straight interaction line. Consequently, LBAs were not performed in this study.
The GN(I)As were performed as load-controlled path-tracing analyses with the Riks arc-
length algorithm. The magnitude of the applied point load \( N \) at any \( e \) was scaled by a
load proportionality factor (LPF) which also scaled the end moment \( e \cdot N \) by the same
amount, so that each analysis proceeds along a load path of constant gradient \( \pi/2 - \xi \) in
\( N / N_{cl} \) vs \( M / M_{cl} \) space (Fig. 4). The outcome of each GN(I)A is a critical LPF which
scales \( R_{cl} \) to identify the elastic nonlinear buckling resistance \( R_k \) under combined loading,
as well as the end force \( N_k = LPF \cdot N \) and end moment \( M_k = LPF \cdot e \cdot N \) at the elastic
nonlinear buckling limit state. As both geometric nonlinearity and imperfections always
have a detrimental effect under uniform compression and uniform bending, the LPF was
found to be always less than unity.

A single instance of a ‘Type A’ axisymmetric weld depression imperfection of Rotter
and Teng (1989) was placed at midspan \( (z = L/2) \) in every model (Fig. 3). The radial
position of the imperfect wall was defined as follows:

\[
\rho(z) = \rho \cdot e^{\frac{z - L}{2L}} \left( \cos \frac{\pi}{\lambda} \left| z - \frac{L}{2} \right| + \sin \frac{\pi}{\lambda} \left| z - \frac{L}{2} \right| \right)
\]  

(5)

where \( \rho \) is the imperfection amplitude and \( \lambda \approx 2.44 \sqrt{rt} \) is the linear axial bending half-
wavelength such that \( r \) and \( t \) are the shell midsurface radius and thickness respectively.
This choice of value for the axial half-wavelength of the imperfection has been justified
both experimentally as being close to the measured axial wavelengths of circumferential
weld depressions (Coleman \textit{et al.}, 1992; Pircher \textit{et al.}, 2001) and numerically as offering
the most critical imperfection sensitivity (Teng and Rotter, 1992). The midspan is the
location where the combined compression resulting from the applied force and the ‘\( P-\Delta \)’
bending moment is at a maximum and the cross-sectional ovalisation reaches the peak,
and is thus also the critical location for buckling. The weld depression was also used as a
mesh perturbation in ‘near-perfect’ GNA models where it was assigned a very small
amplitude of \( 10^{-2}t \) to facilitate in the correct detection of the critical buckling load
(corresponding to the appearance of the first negative eigenvalues in the global tangent stiffness matrix), a common modelling technique in the study of shells that exhibit bifurcation buckling (e.g. Stephens et al., 1975). All the models were meshed with the four-node reduced-integration bilinear S4R shell element, which has been shown in the authors’ previous studies to be capable of predicting the response of cylindrical tubes under global bending both accurately and efficiently (Sadowski and Rotter, 2013; Rotter et al., 2014; Fajuyitan et al., 2015; 2018; Wang and Sadowski, 2017; Fajuyitan and Sadowski, 2018). The mesh size was kept within 10% of the linear bending half-wavelength $\lambda \approx 2.44 \sqrt{rt}$ in the regions within $3\lambda$ of the loaded end boundary and the midspan to accurately represent local bending effects, while a coarser but smoothly-graded mesh was used in-between. An elastic modulus of $E = 200$ GPa was assumed.

4. Geometric nonlinearities in cylindrical shells under combined loading (GNAs)

A selection of computed nonlinear equilibrium paths of the normalised axial load $N / N_{cl}$ against normalised transverse displacement of the midspan centroid $\Delta / t$ for cylinders with dimensionless eccentricities of $\xi = 10^\circ$ (dominated by uniform compression), 30°, 50° and 75° (dominated by uniform bending) is illustrated in Fig. 5 for dimensionless lengths of $\Omega = 0.5$ (‘medium’ length), 1, 3 and 6 (‘long’ cylinders with significant ovalisation and/or Euler column effects). These equilibrium paths illustrate a very linear response at all levels of $N$-$M$ interaction within the ‘medium’ domain at $\Omega = 0.5$ and even those slightly longer at $\Omega = 1$, with local bifurcation buckling always occurring on the most compressed meridian at midspan (Fig. 6). By contrast, an increasingly nonlinear pre-buckling response is seen at all levels of the $N$-$M$ interaction for longer cylinders with $\Omega = 3$ (just above $\Omega \approx 2.86$ where Euler buckling becomes critical under linear conditions for the uniform compression reference system (Eqs 2 and 4) and the uniform bending reference system already exhibits significant pre-buckling ovalisation) and particularly $\Omega = 6$ (where Euler buckling and Brazier ovalisation now dominate the respective reference systems). The predicted GNA buckling points $N_k$ are denoted by grey triangles in Fig. 5, and correspond almost entirely to bifurcations except for very long cylinders where they are occasionally limit points.
Fig. 5 – Nonlinear equilibrium paths for cylinders under combined loading, computed by a GNA finite element analysis and calculated with simple formulae.

The finite element formulation in ABAQUS employs complete nonlinear kinematics to model the complete 3D shell (Budiansky and Sanders, 1963), offering an accurate representation of all geometric nonlinearities in the combined loaded system. Ignoring boundary effects, there are two sources of geometric nonlinearity in the combined system. The first is the well-known nonlinear amplification of the midspan bending moment by the product of the end load with the transverse centroidal displacement, known as the ‘P-Δ effect’ (although ‘P’ is ‘N’ in the current notation). This component of the response may be modelled very well by a classical ‘second-order’ theory of eccentrically-loaded columns, according to which the midspan transverse centroidal displacement is:

\[ \Delta = \varepsilon \left[ \sec \left( \frac{L}{2} \sqrt{\frac{N}{EI}} \right) - 1 \right] \]  

(6)
where \( I \approx \pi r^3 t \) is the second moment of area for a thin-walled cylinder, assumed to not change with increasing deformation. The equilibrium path predicted by Eq. 6 is represented by the dashed line in Fig. 5 and it correctly explains a significant portion of the initial nonlinearity of the pre-buckling response at all considered cylinder lengths. A first estimate of the failure load \( N_k \) at any eccentricity \( e \) may be obtained by invoking the ‘local buckling hypothesis’ (e.g. Seide and Weingarten, 1961; Calladine, 1983) that buckling occurs when the most compressed extreme fibre reaches the classical critical buckling stress \( \sigma_{cl} \) (Eq. 2; Fig. 6):

\[
\frac{N_k}{A} + \frac{(e+\Delta)N_k}{W} = \sigma_{cl}
\]

Taking a cross-sectional area of \( A = 2\pi rt \) and an elastic section modulus of \( W = I/r \approx \pi r^2 t \) corresponding to a thin-walled cylinder with an undeformed cross-section overpredicts the stiffness, because a simple ‘second-order’ analysis cannot easily account for cross-sectional distortion (the second source of geometric nonlinearity), and the intersection of Eqs 6 and 7 can thus only identify an unconservative upper-bound on \( N_k \) as denoted by the black circles in Fig. 5. However, if the GNA finite element model output is processed numerically to obtain the ovalised midspan geometry and updated \( A, W \) and \( \sigma_{cl} \) respectively which are then introduced into the above equations, the intersection of Eqs 6 and 7 will then correspond very closely to the computational GNA buckling loads.

The distribution of ovalisation along the cylinder length may be described well by the ‘out-of-roundness’ tolerance parameter \( U \) of EN 1993-1-6:A1 (2017), defined as:

\[
U = \frac{D_{\text{max}} - D_{\text{min}}}{D_{\text{nom}}}
\]

where \( D_{\text{max}} \) and \( D_{\text{min}} \) are the maximum and minimum deformed diameters at the same cross-section in a cylinder, and \( D_{\text{nom}} \) is the nominal undeformed diameter. The GNA model output was processed to obtain distributions of the \( U \) parameter at the point of buckling along the cylinder for \( \Omega = 3 \) and 6 under different dimensionless eccentricities \( \xi \), as plotted in in Fig. 7. Such curves were previously constructed for cylinders under uniform bending and global transverse shear (Xu et al., 2017; Wang and Sadowski, 2018; Liu et al., 2018), and Xu et al. (2017) had also used the original algebraic relationship of
Brazier (1927) to show that ‘infinitely long’ cylinders under uniform bending attain an upper bound of $U_{\text{max}} \approx 0.34$ at fully-developed ovalisation. These distributions confirm that cylinders with low load eccentricities (dominated by uniform compression) exhibit very little pre-buckling ovalisation, but those with higher eccentricities undergo increasingly higher levels of ovalisation approaching the $U_{\text{max}}$ upper bound as $\zeta \to \pi/2$ and $\Omega \to \infty$. However, the extent of elastic ovalisation is always beneath $U_{\text{max}}$ due to the non-uniform nature of the bending moment distribution in the combined system.

Fig. 6 – Illustration of the deformed configuration of a cylindrical shell under combined uniform compression and bending.

Fig. 7 – Variation of the ‘out-of-roundness’ parameter $U$ along half of the cylinder length for different combinations of uniform compression and bending.
5. N-M interaction in near-perfect cylindrical shells (GNAs)

A detailed set of individual N-M interactions for near-perfect cylinders at varying Ω and ξ are plotted in Fig. 8. The limiting cases of uniform compression (ξ → 0; vertical axis of Fig. 8) and uniform bending (ξ → π/2; horizontal axis of Fig. 8) are in broad agreement with the predictions of Rotter and Al-Lawati (2016; Fig. 1) and Fajuyitan and Sadowski (2018; Fig. 2) respectively. In a first representation (Fig. 8a), the axes show the computed GNA buckling loads $N_k$ and $M_k$ normalised by $N_{cl}$ and $M_{cl}$ respectively (Eqs 2 and 3). For cylinders longer than $Ω ≈ 0.5$, a significant departure from the normalising values is observed on both axes as a consequence of ‘long’ shell effects (Euler column buckling on the vertical axis, Brazier ovalisation on the horizontal axis, and interactions thereof in-between). An arguably more meaningful interpretation of the shape and severity of the N-M interaction may be obtained by normalising the computed $N_k$ and $M_k$ values for combined loading by their corresponding GNA predictions under uniform compression $N_{k,comp}$ and uniform bending $M_{k,bend}$ at the same Ω (Fig. 8b). The reference values of $N_{k,comp}$ and $M_{k,bend}$ are now well established and may be found in the literature (ECCS, 2013; Rotter et al., 2014; Rotter and Al-Lawati, 2016; Fajuyitan and Sadowski, 2018).

The reformulated N-M interaction shown in Fig. 8b permits the observation that, at the elastic nonlinear buckling limit state, the interaction is very linear when both reference conditions do not exhibit any ‘long’ length effects (i.e. $Ω < ~0.5$) and where the GNA resistances are close to the ‘medium’-length algebraic predictions. However, the interaction becomes increasingly unfavourable in the region $0.5 < Ω < ~3$ (i.e. the presence of a small compressive force leads to a disproportionate reduction in the moment resistance, and vice-versa) as the reference uniform bending system begins to exhibit increasing pre-buckling ovalisation. Interestingly, the interaction is at its most unfavourable at $Ω ≈ 3$ and becomes slightly less unfavourable for $Ω > ~3$ as the reference uniform compression system begins to exhibit critical Euler buckling, tending to what appears to be an ‘asymptotically’ unfavourable interaction beyond $Ω ≈ 5$ whose shape does not appear to change significantly with further increases in length.
The mechanics behind the ‘most unfavourable interaction’ at $\Omega \approx 3$ can be explored by reformulating the local buckling hypothesis (Eq. 7) into the form used in Fig. 8 (where $\sigma_k$ is the unknown critical buckling stress at the most compressed fibre):

$$\frac{N_k}{A} + \frac{(e+\Delta)\cdot N_k}{W} = \sigma_k$$

(9)

Next:

$$\frac{N_{k,\text{comp}}}{N_{k,\text{comp}}} \cdot \frac{N_k}{A\sigma_k} + \frac{e}{M_{k,\text{bend}}} \cdot \frac{(e+\Delta)\cdot N_k}{W\sigma_k} = 1$$

(10)

And:

$$\frac{N_{k,\text{comp}}}{A\sigma_k} \cdot \frac{N_k}{N_{k,\text{comp}}} + \left(1 + \frac{\Delta}{e}\right) \cdot \frac{M_{k,\text{bend}}}{W\sigma_k} = 1$$

(11)

Thus:

$$k_1 \cdot \frac{N_k}{N_{k,\text{comp}}} + k_2 \cdot \frac{M_k}{M_{k,\text{bend}}} = 1$$

(12)

where $k_1 = \frac{N_{k,\text{comp}}}{A\sigma_k}$ and $k_2 = \left(1 + \frac{\Delta}{e}\right) \cdot \frac{M_{k,\text{bend}}}{W\sigma_k}$

(13,14)

The changing shape of the $N$-$M$ interaction in Fig. 8 thus depends on the evolution of the coefficients $k_1$ and $k_2$ with the dimensionless length $\Omega$. Values of $k_1$ and $k_2$ equal to unity represent a linear interaction, while those less than and greater than unity represent favourable and unfavourable interactions respectively.
Fig. 9 – Variation of interaction factors $k_1$ and $k_2$ at buckling as a function of the cylinder dimensionless length $\Omega$ and loading eccentricity $\xi$.

Coefficients $k_1$ and $k_2$ calculated based on cross-sectional areas $A$, section moduli $W$, midspan transverse centroidal displacements $\Delta$ and buckling stress $\sigma_k$ estimated from the computed GNAs at the point of buckling are presented in Fig. 9 as a function of $\Omega$ and $\xi$. At $\Omega \approx 0.5$, both coefficients are near unity, explaining the linearity of the $N$-$M$ interaction in the ‘medium’ length domain. The onset of ovalisation for $\Omega > 0.5$ is manifest as a reduction in the section modulus and buckling stress, leading to an increase in $k_2$ (dashed lines in Fig. 9) since $W$ and $\sigma_k$ appear on the denominator of Eq. 14. The additional presence of an axial load ($\xi < 90^\circ$) exacerbates the cross-sectional deflection $\Delta$, which increases $k_2$ further as it appears on the numerator of Eq. 14. As the loading eccentricity $e$ and resistance of the reference uniform bending system $M_{k,bend}$ are constant for any individual $N$-$M$ curve, the coefficient $k_2$ at buckling thus always increases beyond unity with $\Omega$ and will thus always contribute unfavourably to the $N$-$M$ interaction in Eq. 12.
By contrast, the coefficient $k_1$ (solid lines in Fig. 9) exhibits a very minor increase above unity (minor unfavourable effect) as $\Omega > 0.5$ only because of the slight decrease in $\sigma_k$ due to the presence ovalisation in the combined system, since both $A$ and the resistance of the reference uniform compression system $N_{k,\text{comp}}$ are constant during any analysis. However, beyond $\Omega \approx 3$ the reference uniform compression system is governed by Euler column buckling (Fig. 1), so that for $\Omega > 3$ $N_{k,\text{comp}}$ becomes an increasingly small number that causes $k_1$ to tend to zero and thus have an increasingly favourable effect on the interaction. The combined effect of a decreasing (stabilising) $k_1$ and an increasing (destabilising) $k_2$ results in a progressively less severe and apparently asymptotically ‘stable’ $N$-$M$ interaction as $\Omega > 3$ (Fig. 8b). The most unfavourable $N$-$M$ interaction occurs at $\Omega \approx 3$ as this is where the $k_1$ coefficient is maximised, and this corresponds closely to the length $\sim 2.86$ (hereafter identified as $\Omega_E$) when the buckling mode of the reference uniform compression system changes from local shell to global Euler column buckling. It is likely that the same tendency holds for cylinders with different column boundary conditions and thus different effective Euler column lengths and values of $\Omega_E$.

6. N-M interaction in cylindrical shells with local weld depression imperfections (GNIAs)

The influence of weld depression imperfections on the $N$-$M$ interactions of cylindrical shells under combined compression and bending is investigated here through a series of geometrically nonlinear analyses with imperfections (GNIAs). A set of 3D surfaces denoting the dimensionless resistance $R_k / R_{cl}$ as a function of $\Omega$ and $\zeta$ is shown in Fig. 10 for $\delta/t = 0.01$ (GNAs; same data as in Fig. 8a), 0.5, 1.0 and 2.0. As may be expected, the GNIA buckling resistances suffer a marked reduction under the presence of an increasingly severe midspan imperfection. The imperfection sensitivities of the two reference systems correspond closely to the previous work of Rotter and Al-Lawati (2016) and Fajuyitan and Sadowski (2018), and the reader is invited to consult these references for further details. The surfaces in Fig. 10 show both the computed GNIA results (grey surface with white lines) as well as the predictions of a closed-form algebraic characterisation (white surface with black lines), for which more details are given in Section 7. Regions which appear white represent locations where the characterisation is slightly unconservative, but the overestimate is at most of the order of
2% relative to the computed GNIA value (with the exception of $\Omega < 0.5$ for $\delta/t > -1$, for reasons presented shortly).

An apparently anomalous behaviour may be seen for shorter cylinders ($\Omega \approx 0.5$) under combined loading with deep imperfections ($\delta/t > -1$), where a deeper imperfection leads to a rise in the buckling strength rather than a further reduction. There are a number of reasons for this. Firstly, Fajuyitan *et al.* (2018) demonstrated that where cylinders are very short, the unhindered formation of a local axial compression buckle of finite size requires a wider portion of the shell than is available and buckle formation is constrained by the edge boundary. This boundary restraint is quite potent and requires significantly more strain energy to overcome, and may even change the mode of buckling. Secondly, Rotter and Teng (1989), Fajuyitan and Sadowski (2018) and others illustrated that cylinders with progressively deeper imperfections exhibit progressively larger buckles, which eventually begin to encroach upon the end boundary if the shell is not very long. Thirdly, the thicker the shell, the larger the size of the buckle relative to either the length or the circumference (Fajuyitan *et al.*, 2018) such that thick shells are subject to boundary effects at higher values of $\Omega$ than thin shells. Presently, this has the effect of shifting the boundary between the ‘short’ and ‘medium’ length domains towards longer cylinders, such that increasingly imperfect but otherwise ‘medium-length’ cylinders (e.g. $\Omega \approx 0.5$, assuming a classification based on geometrically perfect behaviour) effectively behave as if they were ‘short’. For these reasons, the effect described above is visible in Fig. 2, which is based on a cylinder with $r/t = 100$, but not in Fig. 1 which assumed $r/t = 1000$. The choice of a shell with $r/t = 100$ instead of 1000 for the present calculations should be understood as a computational convenience since modelling larger buckles in thicker shells requires significantly fewer finite elements than modelling smaller buckles in thinner shells. The grouping according to the dimensionless group $\Omega$ ensures that the predictions apply to cylinders of any $r/t$. 


Fig. 10 – 3D representations of the computed and calculated dimensionless cylinder resistances $R_k/R_{cl}$ as a function of the dimensionless length $\Omega$, loading eccentricity $\xi$ and imperfection amplitude $\delta/t$.

The GNIA predictions in Fig. 10 are expressed in alternative form in Fig. 11 as $N$-$M$ interaction shapes employing the same transformation as in Fig. 8b but in terms of a slightly different definition of the load eccentricity $\xi_k$, to be defined in Section 7. Similar to the near-perfect case, the $N$-$M$ interactions for imperfect cylinders begin with a close to linear (or slightly convex) relationship at shorter lengths and become increasingly unfavourable (or concave) with increasing length. For each $\delta/t$ investigated here, there is a well-defined length $\Omega_E$ at which the interaction is at its most unfavourable, and this length corresponds very closely to the length at which the local shell buckling mode
transitions to the global column buckling mode in the reference uniform compression system (Fig. 1), for reasons explained previously.

Fig. 11 – 3D representations of the N-M interaction shapes as a function of the cylinder dimensionless length $\Omega$, loading eccentricity $\xi_k$ and imperfection amplitude $\delta/t$.

7. **Algebraic characterisation of the N-M interaction relationship**

In the European design rules for metal shells (ECCS, 2013; Rotter, 2016; EN 1993-1-6:A1, 2017), the elastic nonlinear buckling resistance $R_k$ of a shell system is given by:

$$R_k = \alpha R_{cr} \equiv \alpha_c \alpha R_{cr}$$  \hspace{1cm} (15)
where \( R_{cr} \) is the reference critical buckling resistance (presently equivalent to \( R_{cl} \), Eq. 4), while \( \alpha_G \) and \( \alpha_I \) are separate reduction factors accounting for the damaging effects of geometric nonlinearity and imperfection sensitivity respectively, such that overall \( \alpha = \alpha_G \cdot \alpha_I \). An algebraic characterisation of \( \alpha_G \) and \( \alpha_I \) for cylinders under combined compression and bending in terms of \( \Omega \), \( \delta/t \) and \( \xi \) would contain sufficient information for an analyst to estimate \( R_k \) reasonably accurately without recourse to an onerous finite element analysis. It is suggested here that detailed \( \alpha_G \) and \( \alpha_I \) characterisations need only be performed for the reference uniform compression and uniform bending systems, and that the shape of the \( N-M \) interaction in-between may be characterised by the \( k_1 \) and \( k_2 \) interaction factors introduced previously (Eqs 13 and 14). Indeed, detailed algebraic \( \alpha_G \) and \( \alpha_I \) characterisations have already been proposed in Rotter et al. (2014), Rotter and Al-Lawati (2016), EN 1993-1-6:A1 (2017) and Fajuyitan and Sadowski (2018) for the two reference systems. They are extensive, and it is not proposed to reproduce them here.

The current approach is to offer closed-form algebraic expressions for interaction factors \( k_1 \) and \( k_2 \) that satisfy the general interaction formula, applied at cylinder midspan:

\[
k_1 \frac{N_k}{N_{k,comp}} + k_2 \frac{e \cdot N_k}{M_{k,bend}} \leq 1
\]  

(16)

The \( k_1 \) interaction factor relates the nonlinear buckling load of the reference uniform compression system \( N_{k,comp} \) to the buckling load \( A\sigma_k \) of the system under combined loading (Eq. 13). Values of \( \sigma_k \) estimated from GNAs using Eq. 9 and accurate values of \( W \) and \( A \) are plotted in Fig. 12 for different \( \Omega \) in terms of a modified dimensionless eccentricity \( \xi_k \) (the same used in Figs 11 and 13) defined as:

\[
\xi_k = \tan^{-1}\left(\frac{M}{M_{k,bend}}/\frac{N}{N_{k,comp}}\right) = \tan^{-1}\left(\frac{e \cdot N_{k,comp}}{M_{k,bend}}\right)
\]  

(17)

It is proposed here to characterise the relationship between the unknown buckling stress \( \sigma_k \) that appears in \( k_1 \) and the known elastic nonlinear reference buckling resistances \( N_{k,comp} \) and \( M_{k,bend} \) using the following simple linear relationship with \( \xi_k \) (assumed to be in radians):

\[
23
\[ \sigma_k \approx \frac{N_{k,\text{comp}}}{A} + \left( \frac{M_{k,\text{bend}}}{W} - \frac{N_{k,\text{comp}}}{A} \right) \frac{2\xi_k}{\pi} \]  

(18)

where, for simplicity, \( W \) and \( A \) can be assumed to be for an undeformed cross-section. The same simple approach may be adopted for imperfect cylinders.

The \( k_2 \) interaction factor represents the nonlinear amplification of the midspan moment \( M_k = e\cdot N_k \) due to the ‘\( P-\Delta \) effect’ (Eq. 14). The theory of beam-columns subject to equal end-moments suggests that the moment amplification due to the ‘\( P-\Delta \) effect’ alone is approximated well by (Galambos and Ketter, 1959; Chen and Atsuta, 1976; Trahair, 1986):

\[ k_2 = \frac{1}{1 - \frac{N_k}{N_{\text{Euler}}}} \]  

(19)

This approximation is widely used in codified design rules (EN 1993-1-1, 2005; AISC 360-10, 2010) as the base formulation to estimate the maximum moment in beam-columns. Values of \( k_2 \) at buckling calculated with Eq. 14 using \( W, \sigma_k, M_{k,\text{bend}} \) and \( \Delta \) extracted from GNAs are plotted in Fig. 13, together with a slightly enhanced version of Eq. 19 as proposed in Eq. 20, shown in Fig. 13 to reproduce \( k_2 \) values for different lengths reasonably well. The functional form of the additional \( c \) exponent in Eq. 20 approximately captures the asymptotic stabilisation of the \( N-M \) interaction as \( \Omega \to \infty \), with coefficients established so as to achieve a conservative characterisation.

\[ k_2 \approx \frac{1}{1 - \left( \frac{N_k}{N_{\text{Euler}}} \right)^c} \quad \text{where} \quad c = 0.9 + 6e^{-0.9\Omega} \]  

(20)

While this simple characterisation of \( \sigma_k \) appears to bear only a limited resemblance to the extracted GNA results in Fig. 12 (the fit may always be improved by the choice of a higher-order functional form if desired), the proposed equations lead to surprisingly accurate and predominantly conservative characterisations of the \( N-M \) interaction for all \( \Omega \geq 0.5 \) and \( \delta l/\ell \) considered here as may be seen in Figs 10 and 12. The peak unconservative error is less than 2.6% except for very short imperfect cylinders (for...
reasons explained previously; Fig. 10d), while conservative errors are within 13%. The predictions shown in Fig. 10 were made based on the GN(I)A values of $N_{k,\text{comp}}$ and $M_{k,\text{bend}}$ computed here, but a similar (and completely algebraic) result may be obtained by substituting established algebraic expressions for the $\alpha_{\text{comp}}$ and $\alpha_{\text{bend}}$ parameters from the recent literature (Rotter et al. 2014; Rotter and Al-Lawati, 2016; EN 1993-1-6:A1, 2017; Fajuyitan and Sadowski, 2018) together with $N_{cl}$ and $M_{cl}$ (Eqs 2 and 3), such that:

$$N_{k,\text{comp}} = \alpha_{\text{comp}} N_{cl} \quad \text{and} \quad M_{k,\text{bend}} = \alpha_{\text{bend}} M_{cl}$$ (21,22)

Fig. 12 – Characterisation of the buckling stress $\sigma_k$ used in the $k_1$ coefficient with the dimensionless eccentricity $\xi_k$ for cylinders under combined bending and compression.
8. Conclusions

This paper has presented a comprehensive computational investigation into the elastic but geometrically nonlinear buckling response of perfect and imperfect cylindrical shells under combined axial compression and bending. The fully elastic response is an important reference result with which to interpret the complete structural response which may involve plasticity. It was shown that:

- The $N$-$M$ interaction at the elastic nonlinear buckling limit state varies strongly with length and is approximately linear whenever the reference uniform compression and uniform bending systems exhibit little pre-buckling nonlinearity.
- The $N$-$M$ interaction becomes increasingly unfavourable at lengths where the reference uniform bending system begins to exhibit pre-buckling ovalisation, and the interaction is at its most unfavourable at the length where Euler column buckling begins to be critical in the reference uniform compression system.
• Exceptionally long cylinders undergo a complex interaction between ovalisation, column buckling and the ‘P-Δ’ amplification of the midspan moment, and appear to exhibit a mostly invariant N-M interaction with further increases in length.

• It was found that cylinders with a midspan weld depression imperfection display qualitatively similar variations of the N-M interaction with length as near-perfect ones, while also exhibiting the most unfavourable interaction at the length where Euler column buckling begins to be critical in the reference uniform compression system. This length increases for cylinders with deeper local imperfections.

• An algebraic characterisation of the N-M interaction at the elastic buckling limit state of perfect and imperfect cylinders is presented, permitting a realistic and conservative calculation without recourse to an onerous finite element analysis.

Data Availability Statement

Some or all data, models or code generated or used during the study are available from the corresponding author by request.

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