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Approximation of the minimal robustly positively invariant set for discrete-time LTI systems with persistent state disturbances

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Abstract— **This paper provides a solution to the problem of computing a robustly positively invariant outer approximation of the minimal robustly positively invariant set for a discrete-time, linear, time-invariant system. It is assumed that the disturbance is additive and persistent, but bounded.**

Keywords: Set invariance, constrained control, robust control, linear systems.

I. Introduction and Notation

Set invariance plays a fundamental role in the control of constrained systems; see for instance [1], [2]. An important problem is how to compute the *minimal* robustly positively invariant (mRPI) set for a given discrete-time LTI system with additive state disturbances [3, Sect. IV]. The mRPI set is used as a target set in robust time-optimal control [4], in the design of robust predictive controllers [5] and in understanding the properties of the *maximal* robustly positively invariant set [3], [6]. The only results that allow one to compute the mRPI set exactly are given in [3, Rem. 4.2] and [4, Thm. 3], where it is assumed that the system dynamics are nilpotent. This paper presents new results that allow one to compute a robustly positively invariant, outer approximation of the mRPI set. A more detailed exposition and all proofs for the results stated in this paper can be found in [7].

The set of strictly positive integers is denoted by $\mathbb{N}_+ \triangleq \{1,2,\ldots\}$. $\|M\|_p$ and $\|v\|_p$ are the p-norms of the matrix M and vector v, respectively. The ∞-norm ball in $\mathbb{R}^n$ (hypercube) of size $r \geq 0$ is defined as $B_\infty(r) \triangleq \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq r\}$. The i'th standard basis vector $e_i \in \mathbb{R}^n$ in the Euclidean space has one as the i'th component and zero as all other components. If P and Q are subsets of $\mathbb{R}^n$, then the Minkowski (vector) sum is $P \oplus Q \triangleq \{p+q \mid p \in P,\ q \in Q\}$. The set $\bigoplus_{i=j}^{k} P_i$ is the Minkowski sum of the sets $\{P_j, \ldots, P_k\}$.

II. Problem Formulation

Consider the discrete-time, linear, time-invariant system:

$$x^+ = Ax + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, x^+ is the successor state, $w \in W$ is an unknown, additive and persistent disturbance. The standing assumptions are that the matrix $A \in \mathbb{R}^{n\times n}$ is strictly stable (the spectral radius $\rho(A) < 1$) and that the set W is a convex, compact subset in $\mathbb{R}^n$ containing the origin in its interior.

Definition 1: $\Omega \subset \mathbb{R}^n$ is a *robustly positively invariant* (RPI) set of (1) if $Ax + w \in \Omega$ for all $x \in \Omega$ and all $w \in W$.

Definition 2: The *minimal* robustly positively invariant (mRPI) set F_∞ of (1) is the set in $\mathbb{R}^n$ that is contained in every closed RPI set of (1).

It is possible to show [3, Sect. IV] that the mRPI set F_∞ exists, is compact, contains the origin in its interior and is given by $F_\infty = \bigoplus_{i=0}^{\infty} A^i W$. Since F_∞ is a Minkowski sum of infinitely many terms, it is generally impossible to obtain an explicit characterization of it. However, as noted in [3, Rem. 4.2], it is possible to show that if there exist an integer $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0,1)$ such that $A^s = \alpha I$, then $F_\infty = (1-\alpha)^{-1} \bigoplus_{i=0}^{s-1} A^i W$. It therefore follows trivially [4, Thm. 3] that if A is nilpotent with index s ($A^s = 0$), then $F_\infty = \bigoplus_{i=0}^{s-1} A^i W$.

In this paper, we relax the assumption that there exists an $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0,1)$ such that $A^s = \alpha I$. Since we can no longer compute F_∞ exactly, we address the problem of computing an RPI set $F(\alpha,s)$ that contains the mRPI set F_∞. We conclude with some remarks on computational issues if W is a polytope given by a finite set of affine inequalities.

III. Main Results

Proposition 1: [6] If the integer $s \in \mathbb{N}_+$ and scalar $\alpha \in [0,1)$ satisfy

$$A^s W \subseteq \alpha W, \tag{2}$$

then

$$F(\alpha,s) \triangleq (1-\alpha)^{-1} \bigoplus_{i=0}^{s-1} A^i W$$

is a convex, compact, RPI set of (1) containing F_∞.

Clearly, $F(\alpha_0,s) \subset F(\alpha_1,s) \Leftrightarrow \alpha_0 < \alpha_1$ for a given s. Note also that if A is not nilpotent, then $F(\alpha,s_0) \subset F(\alpha,s_1) \Leftrightarrow s_0 < s_1$ for a given α. These observations motivate the following discussion, which explains how one can obtain a better approximation of the mRPI set F_∞, given an initial pair (α,s).

Let

$$s^o(\alpha) \triangleq \inf_{s\in\mathbb{N}_+} \{s \mid A^s W \subseteq \alpha W\}, \tag{3a}$$

$$\alpha^o(s) \triangleq \inf_{\alpha\in[0,1)} \{\alpha \mid A^s W \subseteq \alpha W\} \tag{3b}$$

be the smallest values of s and α such that (2) holds for a given α and s, respectively. Clearly, $\alpha^0(s) \to 0$ as $s \to \infty$. Note that $s^0(\alpha) \to \infty$ as $\alpha \to 0$ if and only if A is not nilpotent. However, since A is strictly stable and W is a compact set containing the origin in its interior, the infimum in (3a) is guaranteed to exist and be contained in $\mathbb{N}_+$ for any choice of $\alpha \in (0,1)$. The infimum in (3b) is also guaranteed to exist and be contained in $[0,1)$ if s is sufficiently large.

By a process of iteration, one can use the above definitions and results to compute a pair (α,s) such that $F(\alpha,s)$ is a sufficiently good RPI, outer approximation of F_∞. For example, by starting with $s=1$, one can increment s until there exists an $\alpha \in [0,1)$ such that (2) holds. If necessary, one can increase s until $F(s,\alpha^o(s))$ is sufficiently small. Alternatively, one can take an initial value for α, compute $s^* \triangleq s^o(\alpha)$, proceed to compute $\alpha^* \triangleq \alpha^o(s^*)$ and test whether $F(\alpha^*,s^*)$ is small enough. It is clear that this iteration results in $F_\infty \subseteq F(\alpha^*,s^*) \subseteq F(\alpha,s^*) \subseteq F(\alpha,s)$. If $F(\alpha^*,s^*)$ is not small enough, then this procedure could be restarted by decreasing α. Of course, any other iteration can be implemented until a fixed point is reached or a sufficiently small $F(\alpha,s)$ has been obtained.

Because of the iterative nature of computing a suitable $F(\alpha,s)$ and the fact that $s^o(\alpha)$ may be large, it is desirable to have upper bounds on $s^o(\alpha)$ and the volume of $F(\alpha,s)$ that are easy to compute:

Proposition 2: Let $\beta_{\text{in}} \triangleq \max_{\beta\geq 0}\{\beta \mid B_\infty(\beta) \subseteq W\}$ and $\beta_{\text{out}} \triangleq \min_{\beta\geq 0}\{\beta \mid W \subseteq B_\infty(\beta)\}$. Let A be diagonizable with $A = V\Lambda V^{-1}$, where Λ is a diagonal matrix of the eigenvalues of A, and $\rho(A) \in (0,1)$. If $s \in \mathbb{N}_+$ and $\alpha \in (0,1)$ satisfy

$$s \geq \ln[\alpha\beta_{\text{in}}/(\beta_{\text{out}}\|V\|_\infty\|V^{-1}\|_\infty)]/\ln\rho(A), \qquad (4)$$

then $F(\alpha,s)$ is a convex, compact, RPI set of (1) containing F_∞. Furthermore, the set $F(\alpha,s)$ is contained in the ∞-norm ball (hypercube) $B_\infty(\eta)$, where

$$\eta \triangleq \beta_{\text{out}}\|V\|_\infty\|V^{-1}\|_\infty(1-\rho(A)^s)/[(1-\alpha)(1-\rho(A))].$$

Clearly, any s satisfying (4) is a (possibly conservative) upper bound for $s^o(\alpha)$ and η could be used to obtain a (possibly conservative) upper bound on the size of $F(\alpha,s)$.

IV. Computational Results if W is a Polytope

Before proceeding, recall that the *support function* [3] of a set $Z \subset \mathbb{R}^m$, evaluated at $a \in \mathbb{R}^m$, is $h_Z(a) \triangleq \sup_{z\in Z} a^T z$. Clearly, if Z is a polytope given by a finite set of affine inequalities, then $h_Z(a)$ is finite and can be computed by solving an LP. Recall also that if W is a polytope, then testing whether (2) holds can be implemented by evaluating the support function of W at a finite number of points [2], [3]. The set $F(\alpha,s)$ can then be computed using standard algorithms for computing the Minkowski sum of polytopes.

This section therefore considers the case when the set W is a polytope given by $W \triangleq \{w \in \mathbb{R}^n \mid f_i^T w \leq g_i,\ i \in \mathcal{I}\}$, where $f_i \in \mathbb{R}^n$, $g_i \in \mathbb{R}$ and $\mathcal{I}$ is a finite index set. It is easy to show that (2) holds if and only if $h_W((A^s)^T f_i) \leq \alpha g_i$ for all $i \in \mathcal{I}$. This observation implies that $s^o(\alpha)$ and $\alpha^o(s)$ can be computed efficiently by solving a finite number of suitably-defined LPs. For example, recall that W contains the origin in its interior if and only if $g_i > 0$ for all $i \in \mathcal{I}$. It then follows that $\alpha^o(s) = \max_{i\in\mathcal{I}} h_W((A^s)^T f_i)/g_i$.

In a similar fashion as above, it is also easy to check whether the set $F(\alpha,s)$ (and hence F_∞) is contained in a given polyhedron $X \triangleq \left\{x \in \mathbb{R}^n \;\middle|\; c_j^T x \leq d_j,\ j \in \mathcal{J}\right\}$, where $c_j \in \mathbb{R}^n$, $d_j \in \mathbb{R}$ and $\mathcal{J}$ is a finite index set, *without having to compute* $F(\alpha,s)$ *explicitly*. This is because the inclusion $F(\alpha,s) \subseteq X$ holds if and only if $h_{\mathcal{W}}((1-\alpha)^{-1}[A^0 \cdots A^{s-1}]^T c_j) \leq d_j$ for all $j \in \mathcal{J}$, where $\mathcal{W} \triangleq W^s \triangleq W \times \cdots \times W$. Proceeding in a similar fashion, it is possible to show that $\eta^o(\alpha,s) \triangleq \min_{\eta\geq 0}\{\eta \mid F(\alpha,s) \subseteq B_\infty(\eta)\} = \max_{i\in\{1,\dots,n\}} h_{\mathcal{W}}(\pm(1-\alpha)^{-1}[A^0 \cdots A^{s-1}]^T e_i)$ is the size of the smallest ∞-norm ball (hypercube) containing $F(\alpha,s)$, hence $\eta^o(\alpha,s)$ can be computed by solving $2n$ LPs.

We conclude this paper by referring back to Proposition 2. It is easy to show [8, Prop. 2] that $h_{B_\infty(\beta)}(f_i) = \beta\|f_i\|_1$, hence $\beta_{\text{in}} = \min_{i\in\mathcal{I}} g_i/\|f_i\|_1$. Note also that one can compute β_{out} by solving $2n$ LPs, since $\beta_{\text{out}} = \max_{i\in\{1,\dots,n\}} h_W(\pm e_i)$.

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VI. References

[1] F. Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999. Survey paper.

[2] E.C. Kerrigan. *Robust Constraint Satisfaction: Invariant Sets and Predictive Control.* PhD thesis, University of Cambridge, 2000.

[3] I. Kolmanovsky and E.G. Gilbert. Theory and computation of disturbance invariant sets for discrete time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications*, 4:317–367, 1998.

[4] D.Q. Mayne and W.R. Schroeder. Robust time-optimal control of constrained linear systems. *Automatica*, 33(12):2103–2118, 1997.

[5] W. Langson, I. Chryssochoos, S.V. Raković, and D.Q. Mayne. Robust model predictive control using tubes. *Automatica.* Accepted.

[6] K. Kouramas. *Control of linear systems with state and control constraints.* PhD thesis, Imperial College London, 2002.

[7] S.V. Raković, E.C. Kerrigan, K.I. Kouramas, and D.Q. Mayne. Approximation of the minimal robust positively invariant set for constrained discrete-time LTI systems with persistent disturbances. Technical Report EEE/C&P/SVR/8-b/2003, Imperial College London, August 2003.

[8] E.C. Kerrigan and J.M. Maciejowski. On robust optimization and the optimal control of constrained linear systems with bounded state disturbances. In *Proc. European Control Conference*, Cambridge, UK, September 2003.

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Coordinated Control and Information Architecture

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Abstract— **Coordinated control of vehicle formations is the motivating problem. The key idea is the use of Model Predictive Control (MPC) as a localized control law to connect the overall formation control performance and the inter-vehicle communication quality. The working problem is a simple 1-D vehicle formation with noisy channels. The measure of information quality is the covariance of the state estimates. A special form of MPC is used, which absorbs the estimate covariance into its probabilistically posed no-collision constraints. The resulting control law is deterministic and is adapted to the information quality.**

I. INTRODUCTION

In recent years, coordinated control of vehicle formation has become a significant topic in control. Many studies imply that this area needs fundamental developments in control theories and new tool sets. The purpose of this paper is to explore the interaction between the control performance and the information quality of different information flow architectures. The working problem is a simple 1-dimensional vehicle formation task obeying a no-collision condition with noisy inter-vehicle communication.

Information flow structure is a central issue of coordinated formation control. Since in large scale vehicle formation problem no individual vehicle has the access to the global information, centralized control strategies are not applicable. Hence using a distributed control law with certain inter-vehicle communication is the main track. Studies on how information structure affects the stability have been made. In their formulation of [2], the graph Laplacian of the information structure is connected to the stability of the formation and a Nyquist-like stability criterion is stated. All the interchanged data are assumed to be accurate. In this paper, the information architectures of interest are built on limited communication capacity. All types of inaccuracy of communication (e.g. quantization error, ambient disturbances etc.) are modelled as additive white noises. Each vehicle uses the covariance of the estimate of others' positions as the measure of its available information quality. This enables the study of how information quality affects the control performance, and furthermore, enables the study of what are the minimum requirements on the information structure to achieve a specific performance quality. The sophistication of the control task is dependent on the quality of the communication.

With limited and imperfect knowledge of the formation, each vehicle should apply a localized control law that accommodates the quality of its knowledge about others. Model Predictive Control (MPC) is an appropriate tool for this problem. Because it breaks a traditional multi-objective control task down to a single, online optimization task with multiple naturally posed constraints. Moreover, MPC controllers are reconfigurable, i.e. the formation task can be changed in flight via revising the optimization objective and the constraints. Due to the existence of random noises in our model, the constraints are posed in probabilistic form. Stochastic programming routines can be used to solve this probabilistically constrained MPC problem, as in [4]. There the problem is transformed into an equivalent nonlinear programming (NLP) problem which can be solved by a standard NLP solver. The method developed in [1] is adopted in this paper since only a deterministic Quadratic Programming solver is used. Moreover it uses the estimate covariance to change the probabilistic constraints. The resulting MPC control law accommodates the inaccuracy in estimates via the constraints. The control performance is thus adapted to the information quality. Application of MPC to coordinated multi-vehicle formations already exists, as in [3], where the application is focusing on how to stabilize the formation to a set of permissible equilibria.

In Section 2, two information structures and its corresponding MPC controllers are formulated. The simulation results are given in Section 3, which shows how the performance responds to the available information.

Notations: The number i in a superscript denotes that this variable is of the ith vehicle. And the subscripts stand for the time. $E_{\mathcal{I}_n^i}(\cdot)$ represents a conditional expectation $E(\cdot|\mathcal{I}_n^i)$ and $P_{\mathcal{I}_n^i}(\cdot)$ is a conditional probability $P(\cdot|\mathcal{I}_n^i)$, where $\mathcal{I}_n^i$ is the set of the information collected by vehicle v_i up to time n. In some subscripts $[n+k|n]$ may be seen, such a quantity represents a certain state estimate or estimate covariance conditioning on the corresponding $\mathcal{I}_n^i$ set.

II. WORKING PROBLEM DESCRIPTION

A. 1-D Coordinated Formation Scenario

Consider a simple 1-D coordinated formation control problem as the prototype coordinated flight problem. The formation consists of 3 mobile beads (vehicles) v_1, v_2 and v_3 on a wire with limited communication between them. Each bead's dynamic model is simply a controlled integrator with