CHARACTER RATIOS FOR EXCEPTIONAL GROUPS OF LIE TYPE

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Abstract. We prove character ratio bounds for finite exceptional groups $G(q)$ of Lie type. These take the form $\frac{|\chi(g)|}{\chi(1)} \leq \frac{c}{q^k}$ for all nontrivial irreducible characters χ and non-identity elements g, where c is an absolute constant, and k is a positive integer. Applications are given to bounding mixing times for random walks on these groups, and also diameters of their McKay graphs.

1. Introduction

For a finite group G, a *character ratio* is a complex number of the form $\frac{\chi(g)}{\chi(1)}$, where $g \in G$ and χ is an irreducible character of G. Upper bounds for absolute values of character values and character ratios have long been of interest, for various reasons; these include applications to random generation, covering numbers, mixing times of random walks, the study of word maps, representation varieties and other areas. For a survey of such applications focussing particularly on simple groups, see [24].

The first significant bound on character ratios for groups of Lie type was obtained in 1993 by Gluck [13], who showed that $\frac{|\chi(g)|}{\chi(1)} \leq Cq^{-1/2}$ for any non-central element $g \in G(q)$, a group of Lie type over $\mathbb{F}_q$, and any non-linear irreducible character χ of $G(q)$, where C is an absolute constant. This has been improved in a number of subsequent papers, culminating in [2, 36] in which the following result is proved. Let $\mathcal{G}$ be a simple algebraic group of simply connected type over an algebraically closed field of good characteristic $p > 0$, and let $G = \mathcal{G}^F$ where F is a Frobenius endomorphism of $\mathcal{G}$. Let $\mathcal{L}$ be an F-stable (proper) Levi subgroup of $\mathcal{G}$. If $\mathcal{L}$ is not a torus, write $\mathcal{L}_{\rm unip}$ for the set of non-identity unipotent elements of $\mathcal{L}$, and define

$$\alpha(\mathcal{L}) = \max_{u \in \mathcal{L}_{\rm unip}} \frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}}.$$

If $\mathcal{L}$ is a torus, define $\alpha(\mathcal{L}) = 0$. Now [2, Thm. 1.1] together with [36, Cor. 1.11] show that if $x \in G$ is an element, semisimple if $\mathbf{Z}(\mathcal{G})$ is disconnected, such that $\mathbf{C}_{\mathcal{G}}(x) \leq \mathcal{L}$, then for any irreducible character χ of G,

$$|\chi(x)| \leq f(r) \cdot \chi(1)^{\alpha(\mathcal{L})}, \tag{1.1}$$

where $f(r)$ depends only on the rank r of $\mathcal{G}$.

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For example when $G = E_8(q)$, Theorem 1.7 of [2] shows that $\alpha(\mathcal{L}) \leq \frac{17}{29}$ for all proper Levi subgroups $\mathcal{L}$, while $\chi(1) \geq cq^{29}$ (where c is a positive constant – see Lemma 2.1); hence (1.1) gives

$$\frac{|\chi(x)|}{\chi(1)} < Cq^{-12}$$

for all nontrivial irreducible characters χ of G, and $x \in G$ such that $\mathbf{C}_{\mathcal{G}}(x) \leq \mathcal{L}$ for some F-stable proper Levi subgroup $\mathcal{L}$.

It is highly desirable to obtain such bounds on character ratios $\frac{|\chi(x)|}{\chi(1)}$ for *arbitrary* elements x (i.e. without the hypothesis on $\mathbf{C}_G(x)$). For classical groups $G(q)$, such bounds will be obtained in forthcoming work [30]. In this paper we obtain such bounds for exceptional groups of Lie type. Here is our main result.

Theorem 1. *Let $G = G(q)$ be a quasisimple group of exceptional Lie type E_8, E_7, E_6^ϵ or F_4 over $\mathbb{F}_q$, of simply connected type in good characteristic. Then for any nontrivial irreducible character χ of G, and any $g \in G \smallsetminus \mathbf{Z}(G)$, we have*

$$\frac{|\chi(g)|}{\chi(1)} \leq \begin{cases} \dfrac{c}{q^{a_1}}, & \text{if } g \text{ is a long root element,} \\ \dfrac{c}{q^{a_2}}, & \text{otherwise} \end{cases}$$

where a_1, a_2 are as in Table 1.1, and c is an absolute constant.

Remarks

(i) For the smaller exceptional groups of Lie type, the generic character tables are known and available in CHEVIE [12]. From this we see that the corresponding values of a_i such that the conclusion of Theorem 1 holds in all characteristics for these groups are as in Table 1.2, with the only exception marked by $^{(\sharp)}$ in the case $G = G_2(q)$, where $q \equiv \epsilon (\mathrm{mod}\ 3)$, χ a unique character of degree $q^3 + \epsilon$, g a unique (up to conjugacy) element of order 3 with $\mathbf{C}_G(g) = \mathrm{SL}_3^\epsilon(q)$, and $\chi(g)/\chi(1) = \epsilon q/(q^2 - \epsilon q + 1)$. Also note that we use the convention $q^2 = p^{2a+1}$ with $a \in \mathbb{Z}_{\geq 1}$ for types ${}^2B_2(q^2)$, ${}^2G_2(q^2)$, and ${}^2F_4(q^2)$ (with $p = 2$, 3, 2, respectively).

(ii) The hypothesis in Theorem 1 that the characteristic p is good is necessary for the proof, since we use [2] which, as mentioned above, requires this assumption.

Table 1.1

G	$E_8(q)$	$E_7(q)$	$E_6^\epsilon(q)$	$F_4(q)$
a_1	6	4	3	2
a_2	10	5	3	2

Table 1.2

G	$G_2(q)$	${}^2F_4(q^2)$	${}^2G_2(q^2)$	${}^2B_2(q^2)$	${}^3D_4(q)$
a_1	2	4	2	2	2
a_2	$2^{(\sharp)}$	6	3	2	3

We conclude the Introduction with two corollaries of our main result. The first concerns the theory of *mixing times* for random walks on finite quasisimple groups of Lie type corresponding to conjugacy classes. Let $G = G(q)$ be such a group, let $y \in G$ be a non-central element, and let $C = y^G$, the conjugacy class of y. Consider the random walk on the corresponding Cayley graph starting at the identity, and at each step moving from a vertex g to a neighbour gs, where $s \in y^G$ is chosen uniformly at random. Let $P^t(g)$ be the probability of reaching the vertex g after t steps. The mixing time of this random walk is defined to be the smallest integer $t = T(G, y)$ such that $||P^t - U|| < \frac{1}{e}$, where U is the uniform distribution and $||f|| = \sum_{g\in G} |f(g)|$ is the l_1-norm. There is a substantial literature on mixing times of such random walks (see for example [24] for a brief survey).

The character ratio bound of Gluck [13] can be used to show that for any $y \in G \setminus \mathbf{Z}(G)$, the mixing time $T(G, y)$ is bounded above by a quadratic function of the Lie rank of $G = G(q)$. Using Theorem 1, we can strengthen this bound for exceptional groups of Lie type, as follows.

Corollary 2. *Let $G = G(q)$ be a quasisimple group of exceptional Lie type over $\mathbb{F}_q$, in good characteristic, and let $\mathcal{G}$ be the corresponding simple algebraic group over $\bar{\mathbb{F}}_q$. Then for any $y \in G \setminus \mathbf{Z}(G)$, and for sufficiently large q, the mixing time $T(G, y)$ satisfies*

$$T(G, y) \le \left\lceil \frac{\dim \mathcal{G} + 1}{2a_i} \right\rceil,$$

where a_i is as in Tables 1.1 and 1.2, and $i = 1$ if y is a long root element, $i = 2$ otherwise.

The values of $M_i := \lceil \frac{\dim \mathcal{G}+1}{2a_i} \rceil$ are listed in Table 1.3.

The next corollary concerns the diameters of *McKay graphs* for exceptional groups of Lie type. For a finite group G, and a (complex) character α of G, the McKay graph $\mathcal{M}(G, \alpha)$ is defined to be the directed graph with vertex set $\mathrm{Irr}(G)$, there being an edge from χ_1 to χ_2 if and only if χ_2 is a constituent of $\alpha\chi_1$. By a classical result of Burnside (see [4]), $\mathcal{M}(G, \alpha)$ is connected if and only if α is faithful. A study of McKay graphs for finite simple groups was initiated in [29], and [29, Thm. 2] shows that the diameter of any McKay graph $\mathcal{M}(G, \alpha)$, where $G = G(q)$ is a simple group of Lie type and α a nontrivial irreducible character, is bounded above by a quadratic function of the Lie rank of G. The next result strengthens this bound for exceptional groups of Lie type.

Corollary 3. *Let $G = G(q)$ be a simple group of exceptional Lie type over $\mathbb{F}_q$, in good characteristic, and let $\mathcal{G}$ be the corresponding simple algebraic group over $\bar{\mathbb{F}}_q$. Let $d = \dim \mathcal{G}$, and $N = |\Phi^+(\mathcal{G})|$, the number of positive roots in the root system of $\mathcal{G}$. Then for any nontrivial irreducible character α of G, and for sufficiently large q,*

$$\mathrm{diam}\, \mathcal{M}(G, \alpha) \le 2 \left\lceil \frac{d - N + 1}{a_2} \right\rceil,$$

where a_2 is as in Tables 1.1 and 1.2.

The values of $D := 2\lceil \frac{d-N+1}{a_2} \rceil$ are listed in Table 1.3.

Table 1.3

G	$E_8(q)$	$E_7(q)$	$E_6^\epsilon(q)$	$F_4(q)$	$G_2(q)$	${}^2F_4(q^2)$	${}^2G_2(q^2)$	${}^2B_2(q^2)$	${}^3D_4(q)$
M_1	21	17	14	14	4	7	4	3	8
M_2	13	14	14	14	4	5	3	3	5
D	26	30	30	30	10	10	6	8	12

The layout of the paper is as folows. Section 2 contains preliminary results, and in Section 3 we study the action of a long root parabolic subgroup of $G(q)$ on its unipotent radical, which is an essential ingredient of the proof of Theorem 1. This proof is completed in Section 4. The final section 5 contains the proofs of Corollaries 2 and 3.

2. Preliminary results

We begin with some well-known information about the irreducible characters of minimal degree for exceptional groups. For a finite group G, denote the set of irreducible characters of G by $\mathrm{Irr}(G)$.

Lemma 2.1. *Let $G = G(q)$ be a quasisimple, simply connected group of exceptional Lie type E_8, E_7, E_6^ϵ or F_4. Define $l_1 = l_1(G), l_2 = l_2(G)$ as in the table below. Let $1 \neq \chi \in \mathrm{Irr}(G)$. Then either $\chi(1)$ is a polynomial in q of degree l_1, or $\chi(1) > cq^{l_2}$ for c a positive absolute constant.*

G	$E_8(q)$	$E_7(q)$	$E_6^\epsilon(q)$	$F_4(q)$
l_1	29	17	11	8
l_2	46	26	16	11

Proof. This follows by inspection of the lists of character degrees for these groups to be found in [31]. □

We shall also need to to identify the structure of some parabolic subgroups of groups $G = G(q)$ as in Lemma 2.1. Our notation for parabolics will be standard: P_i (resp. P_{ij}) is the standard parabolic that corresponds to deleting node i (resp. nodes i, j) from the Dynkin diagram of G, labelled as in [3]. Also for a parabolic subgroup P, we write $P = QL$ where Q is the unipotent radical and L a Levi factor.

Lemma 2.2. *Let $G = G(q)$ be as in Lemma 2.1, and let $P_0 = Q_0L_0$ be the maximal parabolic subgroup of G indicated in Table 2.1. Then $\mathbf{Z}(Q_0)$ has the structure of an irreducible $\mathbb{F}_qL_0$-module of the dimension indicated in the table; and $Q_0/\mathbf{Z}(Q_0)$ is an irreducible $\mathbb{F}_qL_0$-module for the entry in the first row, and is the sum of two irreducible 8-dimensional modules in the last row.*

Proof. This well-known information can be read off using [1], for example. □

Table 2.1

G	P_0	L_0'	$\dim \mathbf{Z}(Q_0)$	$\dim Q_0/\mathbf{Z}(Q_0)$
$E_8(q)$	P_1	$D_7(q)$	14	64
$E_7(q)$	P_7	$E_6(q)$	27	0
$E_6(q)$	P_1	$D_5(q)$	16	0
${}^2E_6(q)$	P_{15}	${}^2D_4(q)$	8	16

Finally, we need an elementary lemma. For a finite group X and a subgroup Y, denote by X/Y the set of right cosets of Y in X. And writing $\Omega = X/Y$, for $x \in X$ define the *fixed point ratio* of x acting on Ω by

$$\mathrm{fpr}(x, \Omega) = \frac{\mathrm{fix}(x, \Omega)}{|\Omega|}.$$

Lemma 2.3. *Let G be a finite group, and let $H < K < G$. Write $C = \mathrm{core}_K(H)$, the core of K in H. Let $y \in H$ and define*

$$M = \max\{\mathrm{fpr}(x, K/H) : x \in (y^G \cap K) \setminus C\}.$$

Then

$$\mathrm{fpr}(y, G/H) \le \frac{|y^G \cap C|}{|y^G|} + M\,\mathrm{fpr}(y, G/K).$$

Proof. Let $K = \bigcup_i Hk_i$ and $G = \bigcup_j Kg_j = \bigcup_{i,j} Hk_ig_j$, all disjoint unions. Then writing $y_j = g_jyg_j^{-1}$, we have

$$\begin{aligned}\mathrm{fpr}(y, G/H) &= \frac{1}{|G/H|}|\{(i,j) : k_ig_jyg_j^{-1}k_i^{-1} \in H\}| \\ &= \frac{|K/H|}{|G/H|}|\{j : y_j \in C\}| + \frac{1}{|G/H|}|\{(i,j) : y_j \notin C, k_iy_jk_i^{-1} \in H\}| \\ &= T_1 + T_2,\end{aligned} \tag{2.1}$$

where T_1, T_2 are the two terms on the second line. Observe that

$$T_1 = \frac{1}{|G/K|}|\{j : y_j \in C\}| = \frac{|y^G \cap C|}{|y^G|}. \tag{2.2}$$

As for T_2, the number of values of j such that $y_j \notin C$ and $k_iy_jk_i^{-1} \in H$ for some i is at most $|\{j : y_j \in K\}| = \mathrm{fix}(y, G/K)$. And given such a j-value,

$$|\{i : k_iy_jk_i^{-1} \in H\}| = \mathrm{fix}(y_j, K/H) \le M\,|K/H|,$$

where M is as defined in the lemma. It follows that

$$T_2 \le \frac{\mathrm{fix}(y, G/K) \cdot M\,|K/H|}{|G/H|} = M\,\mathrm{fpr}(y, G/K). \tag{2.3}$$

Now the conclusion follows using (2.1) together with (2.2) and (2.3). □

3. Long root parabolics

Let $\mathcal{G}$ be a simple algebraic group of type E_8, E_7, E_6 or F_4 over an algebraically closed field of odd characteristic p, and let $G(q) = \mathcal{G}^F$ be a corresponding group of Lie type over $\mathbb{F}_q$, where F is a Frobenius endomorphism of $\mathcal{G}$. Let Φ be the root system of $\mathcal{G}$ relative to a fixed maximal torus, and for $\alpha \in \Phi$ denote by U_α the corresponding root subgroup of $\mathcal{G}$. Let α_0 be the highest root in Φ. Then $\mathcal{P} = \mathbf{N}_{\mathcal{G}}(U_{\alpha_0})$ is a parabolic subgroup of $\mathcal{G}$, which we shall call a *long root parabolic*; likewise, taking $\mathcal{P}$ to be F-stable, $P = \mathcal{P}^F$ is a long root parabolic of $G(q)$.

The proof of Theorem 1 is based on the following results concerning long root parabolics of exceptional groups. These will be proved in the ensuing subsections.

There is some standard notation used in the statement: for a vector space W, and an element $g \in GL(W)$, we denote by $P_1(W)$ the set of 1-dimensional subspaces of W, and by $[W, g]$ the commutator space $\{w - wg : w \in W\}$.

Theorem 3.1. *Let $G = G(q)$ be a quasisimple, simply connected group of exceptional Lie type E_8, E_7, E_6^ϵ or F_4 in odd characteristic, and let $P = QL = \mathbf{N}_G(U_{\alpha_0})$ be a long root parabolic of G.*

(i) *We have $\mathbf{Z}(Q) = U_{\alpha_0}$, and $Q/\mathbf{Z}(Q)$ has the structure of an irreducible $\mathbb{F}_qL$-module of dimension as indicated in Table* 3.1.
(ii) *Let $W = \mathrm{Irr}(Q/\mathbf{Z}(Q))$. The orbits and stabilizers for the action of L' on $P_1(W)$ are as in Table* 3.1 *(one row for each orbit).*
(iii) *Let $g \in L \setminus \mathbf{Z}(L)$. Then for any L-orbit Δ on $P_1(W)$, we have*

$$\mathrm{fpr}(g, \Delta) \le \begin{cases} \dfrac{c_1}{q^{a_1}}, & \textit{if } g \textit{ is a long root element,} \\ \dfrac{c_2}{q^{a_2}}, & \textit{otherwise} \end{cases}$$

where a_1, a_2, c_1, c_2 *are as in Table* 3.2.

(iv) *Write* $\bar{W} = W \otimes \bar{\mathbb{F}}_q$. *For any* $g \in L \setminus \mathbf{Z}(L)$ *and any scalar* $\lambda \in \bar{\mathbb{F}}_q^*$, *we have*

$$\dim[\bar{W}, \lambda g] \geq \begin{cases} 2a_1, & \text{if } g \text{ is a long root element,} \\ 2a_2, & \text{otherwise.} \end{cases}$$

TABLE 3.1

G	L'	$\dim W$	stabilizers	containments
$E_8(q)$	$E_7(q)$	56	P_7	
			$E_6(q).2$	
			${}^2E_6(q).2$	
			$q^{1+32}.B_5(q).(q-1)$	$\leq P_1$
			$q^{26}.F_4(q).(q-1)$	$\leq P_7$
$E_7(q)$	$D_6(q)$	32	P_6	
			$A_5^\epsilon(q).2\ (\epsilon = \pm)$	
			$q^{1+16}.(A_1(q)B_3(q)).(q-1)$	$\leq P_2$
			$q^{14}.C_3(q).(q-1)$	$\leq P_6$
$E_6^\epsilon(q)$	$A_5^\epsilon(q)$	20	P_3	
			$(\mathrm{SL}_3^\epsilon(q) \wr \mathsf{S}_2)$	
			$(\mathrm{SL}_3(q^2).2)$	
			$q^{1+8}.\mathrm{Sp}_4(q).(q-1)(q-\epsilon)$	$\leq P_{15}$
			$q^8.\mathrm{SL}_3^\epsilon(q).(q-1)$	$\leq P_3$
$F_4(q)$ (q odd)	$C_3(q)$	14	P_3	
			$q^5.\mathrm{SO}_3(q).(q-1)$	$\leq P_3$
			$q^{1+4}.(\mathrm{SL}_2(q)^2.2).(q-1)$	$\leq P_1$
			$q^{1+4}.(\mathrm{SL}_2(q^2).2).(q-1)$	$\leq P_1$
			$\mathrm{SL}_3(q).2$	
			$\mathrm{SU}_3(q).2$	

TABLE 3.2

G	a_1	a_2	c_1	c_2
$E_8(q)$	6	10	1.04	1.5
$E_7(q)$	4	5	1.32	2
$E_6(q)$	3	3	2	2
${}^2E_6(q)$	3	3	1.4	1.4
$F_4(q)$	2	2	1.34	1.34

Proposition 3.2. *Let* G *and* $P = QL$ *be as in the statement of Theorem* 3.1.

(i) *Suppose* $G \neq {}^2E_6(q)$, *and let* $g \in G$ *be a non-identity unipotent element that is not a long root element. Then there is a* G*-conjugate* u *of* G *such that*
  (a) $u \in P$, *and*
  (b) $u \in Ql$, *where* $l \in L$ *is a non-identity unipotent element that is not a long root element.*

(ii) *Suppose* $G = {}^2E_6(q)$, *and let* $g \in G$ *be a non-identity unipotent element. Then there is a* G*-conjugate* u *of* G *such that* $u \in P \setminus Q$.

Proposition 3.3. *Let $P = QL$ be a long root parabolic of G as in Theorem* 3.1*, and let $\chi \in \mathrm{Irr}(G)$ be a nontrivial irreducible character, afforded by a $\mathbb{C}G$-module V. Let $g \in QL$ be an element with projection to L not lying in $\mathbf{Z}(L)$.*

(i) *Then*
$$V \downarrow QL = V^Q \oplus V_1 \oplus V_2,$$
where V^Q denotes the fixed point space of Q, $V_2 = [V, \mathbf{Z}(Q)]$ and $V^Q \oplus V_1 = V^{\mathbf{Z}(Q)}$. Let χ_{V_i} denote the character of V_i for $i = 1, 2$.

(ii) *We have $V^Q = {}^*R^G_L(\chi)$, the Harish-Chandra restriction of χ.*

(iii) *Let $W = \mathrm{Irr}(Q/\mathbf{Z}(Q))$, and let Δ_i $(1 \le i \le t)$ be the orbits of L on $P_1(W)$. Then*
$$\frac{|\chi_{V_1}(g)|}{\dim V_1} \le \max\{\mathrm{fpr}(g, \Delta_i) : 1 \le i \le t\}.$$

(iv) *We have*
$$\frac{|\chi_{V_2}(g)|}{\dim V_2} \le q^{-\frac{1}{2}\dim[W,g]}.$$

We shall give the proofs of these results in next subsections.

3.1. **Proof of Proposition 3.3.** Part (i) follows from the fact that $V = V^{\mathbf{Z}(Q)} \oplus [V, \mathbf{Z}(Q)]$, and (ii) is just the defnition of Harish-Chandra restriction.

Now consider part (iii). Write $V_1 = \bigoplus_\mu V_\mu$, a sum of weight spaces for nontrivial $\mu \in \mathrm{Irr}(Q/\mathbf{Z}(Q))$. These are permuted by g, and (iii) follows. For (iv), write $V_2 = \bigoplus_\lambda V_\lambda$, a sum of weight spaces for nontrivial $\lambda \in \mathrm{Irr}(\mathbf{Z}(Q))$. Then $|\chi_{V_\lambda}(g)| = |\mathbf{C}_W(g)|^{1/2}$, by [17, 2.4]. Part (iv) follows.

3.2. **Proof of Theorem** 3.1**.**

3.2.1. *Proof of Theorem* 3.1(i), (ii)*.* Part (i) is well-known and can be found for example in [8, Sec. 4].

Part (ii) follows from various references: [25, 4.3] for $L' = E_7(q)$; [19, Prop. 3] for $D_6(q)$ and [19, Prop. 7] for $C_3(q)$; [7, Thm. 2.1] for $A_5(q)$ (the twisted version follows from this using Lang's theorem).

3.2.2. *Proof of Theorem* 3.1(iii)*.* We consider each possibility for G separately. Recall that q is odd, by hypothesis. Let $W = \mathrm{Irr}(Q/\mathbf{Z}(Q))$ be as in the statement of the theorem, and note that the group induced by L on $P_1(W)$ is $L_1 := L/\mathbf{Z}(L)$, an adjoint group.

Case $G = E_8(q)$. Here $L' = E_7(q)$, and [21, Theorem 2] gives upper bounds for the fixed point ratios of elements of L_1 in all actions. These imply that for any faithful transitive action of L_1 on a set Δ, and any non-identity $g \in L_1$, we have
$$\mathrm{fpr}(g, \Delta) \le \begin{cases} \frac{1}{q^6 - q^3 + 1}, & \text{if } g \text{ is a long root element,} \\ \frac{1}{q^9(q-1)}, & \text{otherwise.} \end{cases}$$
Part (iii) follows immediately in this case.

Case $G = E_7(q)$. Here $L' = D_6(q)$. Let Δ be one of the orbits listed in Table 3.1, so that a point-stabilizer is contained in P_6, P_2 or$A_5^\epsilon(q).2$. In the last case we use [5, Theorem 1] (since in this case the point-stabilizer is not a subspace subgroup): this implies that for any $x \in L_1 \setminus \{1\}$, we have
$$\mathrm{fpr}(x, \Delta) < |x^{L'}|^{-\frac{1}{2} + \frac{1}{12} + \frac{1}{10}}.$$
The smallest class in $L_1 \setminus \{1\}$ consists of long root elements and has size less than $2q^{18}$. Hence it follows that $\mathrm{fpr}(x, \Delta) < \frac{1}{q^5}$ in this case, as required for Theorem 3.1(iii).

Now consider an orbit Δ for which the point-stabilizer lies in the parabolic P_6. Here we use information taken from [31], which gives the values of $1_{P_6}^{L_1}(x)$ for all $x \in L_1$. From these we read off that $\mathrm{fpr}(u_\alpha, \Delta) < \frac{1.32}{q^4}$, and that $\mathrm{fpr}(x, \Delta) < \frac{2}{q^5}$ if $x \in L_1 \setminus \{1\}$ is not a root element, as required.

Finally consider the orbit Δ in Table 3.1 for which the point-stabilizer is $H = q^{1+16}.(A_1(q)B_3(q)).(q-1) < P_2$. Again we can refer to [31] for the values of $1_{P_2}^{L_1}(x)$ for all $x \in L_1$. From this we see that there are several classes of elements x for which $\mathrm{fpr}(x, L_1/P_2)$ is of the order of q^{-4}. These classes are:

(1) root elements u_α;
(2) unipotent elements in the class labelled $(A_1^2)^{(1)}$ – on the natural D_6-module these have Jordan form (J_3, J_1^9);
(3) semisimple elements with centralizer in L_1 of type $D_5^\epsilon(q).(q-\epsilon)$ ($\epsilon = \pm$).

The remaining classes satisfy $\mathrm{fpr}(x, L_1/P_2) < \frac{5}{q^8}$, as required for Theorem 3.1(iii).

Let x be in one of the classes in (1), (2) or (3). By [31], we have $\mathrm{fpr}(x, L_1/P_2) < \frac{1.1}{q^4}$. Write $H = QA_1B_3T_1$, so that $H < P_2 = QA_1D_4T_1$. We shall apply Lemma 2.3. Since the core of P_2 in H is QA_1T_1, this gives

$$\mathrm{fpr}(x, L_1/H) \leq \frac{|x^{L_1} \cap QA_1T_1|}{|x^{L_1}|} + M\,\mathrm{fpr}(x, L_1/P_2), \tag{3.1}$$

where

$$M = \max\{\mathrm{fpr}(y, D_4(q)/B_3(q)) : y \in D_4(q) \setminus \mathbf{Z}(D_4(q))\}.$$

By [26, Thm. 1] we have $M \leq \frac{4}{3q}$. Hence the second term on the right hand side of (3.1) is less than $\frac{1.1}{q^4} \cdot \frac{4}{3q}$. Now consider the first term. Here the Levi factor $A_1D_4T_1$ acts on $Q/\mathbf{Z}(Q) = q^{16}$ as a tensor product $V_2 \otimes V_8$, and $x \in A_1T_1$ centralizes an 8-dimensional subspace. Hence any Q-class in $x^L \cap QA_1T_1$ has size at most q^9, and it follows that $|x^L \cap QA_1T_1| < q^{13}$. Therefore the first term on the right hand side of (3.1) is less than $\frac{2}{q^7}$. Part (iii) of Theorem 3.1 follows for the classes in (1),(2),(3) above. This completes the proof of Theorem 3.1(iii) for $G = E_7(q)$.

Case $G = E_6^\epsilon(q)$. Here $L' = A_5^\epsilon(q)$. Let Δ be one of the orbits listed in Table 3.1, so that a point-stabilizer is contained in P_3, P_{15}, $(\mathrm{SL}_3^\epsilon(q) \wr \mathsf{S}_2).(q-\epsilon)$ or $(\mathrm{SL}_3(q^2).2).(q-\epsilon)$. In the last two cases we use [5, Theorem 1] as above to get the result.

Consider an orbit Δ for which the point-stabilizer is contained in P_3. Here, for any element $x \in L_1 \setminus \{1\}$, [31] gives $\mathrm{fpr}(x, \Delta) < \frac{2}{q^3}$ if $\epsilon = +$, and $\mathrm{fpr}(x, \Delta) < \frac{2.35}{q^4}$ if $\epsilon = -$. Hence the conclusion holds for such orbits.

Finally consider the orbit Δ in Table 3.1 for which the point-stabilizer is $H = q^{1+8}.\mathrm{Sp}_4(q).(q-1)(q-\epsilon) < P_{15}$. Here [31] shows that $\mathrm{fpr}(x, L_1/P_{15})$ satisfies the bounds of Table 3.2 for all $x \in L_1 \setminus \{1\}$ except for the following classes:

(1) root elements u_α,
(2) semisimple elements with centralizer in L_1 of type $A_4^\epsilon(q).(q-\epsilon)$.

For both these classes, [31] gives $\mathrm{fpr}(x, L_1/P_{15}) < \frac{1.04}{q^2}$. We have $H = Q.\mathrm{Sp}_4(q).T_2 < P_{15} = Q.\mathrm{SL}_4^\epsilon(q).T_2$, where the unipotent radical $Q = q^{1+8}$. Now we can argue as above using Lemma 2.3 together with [26], that for the classes in both (1) and (2), we have

$$\mathrm{fpr}(x, L_1/H) \leq \frac{|x^L \cap QT_2|}{|x^L|} + \frac{1.04}{q^2}.\frac{4}{3q}.$$

The number of root elements in Q is less than $2q^5$; and an element of type (2) in QT_2 has Q-centralizer of order at least q^4. Hence the first term in the above sum is less than $\frac{1}{q^4}$. The conclusion follows.

Case $G = F_4(q)$**.** Here $L' = C_3(q)$ and the orbits listed in Table 3.1 have point-stabilizers contained in P_3, P_1 or $\mathbf{N}(\mathrm{SL}_3^\epsilon(q))$. In the last case we use [5, Theorem 1] as above to get the result.

For an orbit Δ with point-stabilizer contained in P_3, [31] gives $\mathrm{fpr}(x,\Delta) < \frac{1.3}{q^2}$ for all $x \in L_1 \setminus \{1\}$, giving the conclusion.

Finally, let Δ be an orbit with point-stabilizer $H < P_1$. We have $\mathrm{fpr}(x, L_1/P_1) < \frac{1.1}{q^2}$ for all $x \in L_1 \setminus \{1\}$ except long root elements u_α; and $\mathrm{fpr}(u_\alpha, L_1/P_1) < \frac{1}{q}$. We have $H = Q.DT_1.2 < P_1 = Q.C_2T_1$, where $Q = q^{1+4}$ and $D = \mathrm{SL}_2(q)^2$ or $\mathrm{SL}_2(q^2)$. Now argue as in the previous case that $\mathrm{fpr}(x, L_1/H) \le \frac{1.34}{q^2}$.

This completes the proof of Theorem 3.1(iii).

3.2.3. *Proof of Theorem* 3.1(iv). Consider first $G = E_8(q)$, where $L' = E_7(q)$ and V is the 56-dimensional L'-module $V_{L'}(\lambda_7)$. Write $\mathcal{G} = E_8$, $\mathcal{L} = E_7T_1$ for the corrsponding algebraic groups over $\bar{\mathbb{F}}_q$, and $\bar{V} = V \otimes \bar{\mathbb{F}}_q$. We aim to bound from below the dimension of $[V, \lambda g]$ for any $g \in \mathcal{L}' \setminus \mathbf{Z}(\mathcal{L}')$ and $\lambda \in \bar{\mathbb{F}}_q^*$. In doing this, we may assume that g is either semisimple or unipotent.

For semisimple elements g, we follow the method of [15, Section 8] (originally in [20]). Let Ψ be a subsystem of the root system Φ of $\mathcal{L}'$, and define an equivalence relation on the set of weights of $\bar{V} = V(\lambda_7)$ by saying that two weights are related if their difference is a sum of roots in Ψ. Call the equivalence classes Ψ-*nets*.

Now define $\Phi_g = \{\alpha \in \Phi \mid \alpha(g) = 1\}$, the root sytem of $\mathbf{C}_{\mathcal{L}'}(g)$. If $\Phi_g \cap \Psi = \emptyset$, then any two weights in a given Ψ-net that differ by a root in Ψ correspond to different eigenspaces for g.

The subsystem Φ_g is contained in a proper subsystem spanned by a subset of the nodes of the extended Dynkin diagram of $\mathcal{L}'$. Suppose $\Phi_g \ne A_7$. Then it is straightforward to check that there is a subsystem $\Psi = (A_1)^2$ such that $\Phi_g \cap \Psi = \emptyset$. For this Ψ the Ψ-nets are of size $4^2, 2^{16}, 1^{16}$, and so it follows from the observation in the previous paragraph that $\dim[\bar{V}, \lambda g] \ge 20$ for any $\lambda \in \bar{\mathbb{F}}_q^*$, as required for Theorem 3.1(iv). And if $\Phi_g = A_7$, then $g \in \mathbf{Z}(A_7)$ and $\bar{V} \downarrow A_7 = V_{A_7}(\lambda_2) \oplus V_{A_7}(\lambda_6)$ (see [28, 11.8]), and it follows that $\dim[\bar{V}, \lambda g] \ge 28$ for any λ.

Now suppose $g \in \mathcal{L}'$ is a unipotent element. If g is a root element, then $g \in A_1$, a fundamental SL_2 in $\mathcal{L}'$ and $V \downarrow A_1 = 1^{12} + 0^{32}$ (see [28, 11.8]). Hence $\dim[\bar{V}, g] = 12$. Now assume g is not a root element. Then the closure of the class $g^{\mathcal{L}'}$ contains the class A_1^2, by [34, p.452]. Hence for an element u in the latter class we have $\dim[\bar{V}, g] \ge \dim[\bar{V}, u]$ (see [16, 3.4]). Using [28, 11.8] it is routine to check that

$$V \downarrow A_1^2 = (1 \otimes 1)^2 + (1 \otimes 0)^8 + (0 \otimes 1)^8 + (0 \otimes 0)^{16}.$$

Hence $\dim[\bar{V}, u] = 20$. This completes the proof of Theorem 3.1(iv) for the case $G = E_8(q)$.

The other cases are as follows:

$\mathcal{G}$	$\mathcal{L}'$	$\bar{V}$
E_7	D_6	$V(\lambda_6)$
E_6	A_5	$V(\lambda_3)$
F_4	C_3	$V(\lambda_3)$

For these, the argument is similar (and more straightforward), and can be found in the proofs of Lemmas 2.4 and 4.6 of [22].

TABLE 3.3

class	A_6	$D_4(a_1)$	$D_4(a_1)A_1$	E_6	$E_6(a_1)$	$E_6(a_3)$
reps. in [34]	y_{36-37}	y_{92-94}	y_{87-90}	y_{21-24}	y_{25-27}	y_{54-57}

class	E_7	$E_7(a_1)$	$E_7(a_2)$	$E_7(a_3)$	$E_7(a_4)$	$E_7(a_5)$
reps. in [34]	y_{1-9}	y_{10-12}	y_{13-15}	y_{16-20}	y_{31-35}	y_{44-51}

The proof of Theorem 3.1 is now complete.

3.3. **Proof of Proposition 3.2.** (i) Let G and $P = QL$ be as in the statement of the proposition, and assume $G \neq {}^2E_6(q)$.

We give the proof for $G = E_7(q)$; the method is the same for all the other cases. Recall that q is odd, by hypothesis.

So let $G = E_7(q) = \mathcal{G}^F$, where $\mathcal{G} = E_7(\bar{\mathbb{F}}_p)$ and F is a Frobenius endomorphism. Here the long root parabolic is $P = P_1 = QL$, and the Levi subgroup $L' = D_6(q)$. Let $g \in G$ be a non-identity unipotent element that is not a long root element. The unipotent classes of $\mathcal{G}$ are listed in Table 22.1.2 of [28]. The label of each class is the semisimple part of a Levi subgroup in which a class representative is a regular element, and the corresponding classes in the finite group G are given in [28, Table 22.2.2] – of course there can be several G-classes arising from a single $\mathcal{G}$-class. If $\mathbf{C}_G(g)$ contains a subgroup $A = A_1(q)$ generated by root elements, then g lies in $\mathbf{C}_G(A) = D_6(q)$, a conjugate of L', and so the conclusion of Theorem 3.1(i) holds. So assume that $\mathbf{C}_G(g)$ contains no such subgroup. Then inspection of the tables shows that g is in one of the following $\mathcal{G}$-classes

$$\begin{aligned}&A_2A_1^3, A_2^2A_1, A_3A_2A_1, A_4A_2, A_5A_1, D_5A_1, D_5(a_1)A_1,\\ &A_6, D_4(a_1), D_4(a_1)A_1, E_6, E_6(a_1), E_6(a_3),\\ &E_7, E_7(a_1), E_7(a_2), E_7(a_3), E_7(a_4), E_7(a_5).\end{aligned}$$

Now $Q = \langle U_\alpha : \alpha = \sum c_i\alpha_i,\, c_1 \neq 0\rangle$, and L' is generated by the root groups $U_{\pm\alpha_i}$ for $i \neq 1$. In the first row of the above list, each $\mathcal{G}$-class corresponds to just one G-class, which can therefore be written as a product of root elements $U_\alpha(1)$ for roots α in a fundamental system of the listed Levi subgroup in G. Visibly, all such expressions belong to Ql, where $l \in L$ is a non-identity unipotent element that is not a root element.

The remaining entries in the above list require a different treatment, as each of these corresponds to more than one G-class of unipotent elements. The most convenient way to handle these is to refer to [34], where explicit representatives y_i for each unipotent class are given. In Table 3.3 we indicate which representatives y_i correspond to which class in $\mathcal{G}$. Inspection of these representatives in [34] shows that each is in Ql, where $l \in L$ is a non-identity unipotent element that is not a root element. This completes the proof of Proposition 3.2(i) for $G = E_7(q)$. The method is the same for the other exceptional groups (using [32, 35] for the unipotent conjugacy class representatives of $E_6(q)$ and $F_4(q)$).

(ii) Now assume that $G = {}^2E_6(q)$. (For this group there is no explicit list of unipotent class representatives (that we are aware of), which is why we are proving this weaker conclusion than in (i).) Let $g \in G$ be a non-identity unipotent element. There is certainly a conjugate u of G lying in the long root parabolic P. If $u \notin Q$ then we are done, so assume $u \in Q$. Then $\mathbf{C}_G(u)$ contains one of the stabilizers listed in Table 3.1. Moreover we can assume that $\mathbf{C}_G(u)$ does not contain a subgroup $A_1(q)$ generated by long root subgroups, since the centralizer of such a subgroup is $L' = {}^2A_5(q)$, and if $u \in L'$ the conclusion obviously holds.

Thus $\mathbf{C}_G(u)$ contains one of the stabilizers in Table 3.1, but does not contain a long root $A_1(q)$. Consulting the list of unipotent classes and centralizers in G given in [28, Table 22.2.3], we see that there is just one class u^G for which $\mathbf{C}_G(u)$ has these properties – namely the class labelled A_2 for which $|\mathbf{C}_G(u)| = 2q^{20}|A_2(q^2)|$. For u in this class, $\mathbf{C}_G(u)$ contains a subgroup $A_2(q^2)$ arising from a subsystem A_2^2, and $\mathbf{C}_G(A_2(q^2))$ is a subsystem subgroup $A = A_2(q)$ (see for example [27, Table 5.1]). Hence $u \in A$. If we take this A_2 subsystem to be spanned by the roots α_2, α_4, we see that u can be taken to be $u_{\alpha_2}(1)u_{\alpha_4}(1)$, which is an element of $P \setminus Q$, as required.

4. Proof of Theorem 1

We begin with a lemma.

Lemma 4.1. *Let $G = G(q)$ be as in Theorem 1, and let $g \in G \setminus \mathbf{Z}(G)$. Then one of the following holds:*

(i) *g is a long root element;*
(ii) *$g \in P = QL$, a long root parabolic, and $g \in Ql$, where $l \in L \setminus \mathbf{Z}(L)$; moreover, if $G \neq {}^2E_6(q)$, then the projection of l to $L/\mathbf{Z}(L)$ is not a long root element;*
(iii) *$|\mathbf{C}_G(g)| < q^m$, where $m = m(G)$ is defined as follows:*

G	$E_8(q)$	$E_7(q)$	$E_6^\epsilon(q)$	$F_4(q)$
m	38	24	16	12

(iv) *g is semisimple and $\mathbf{C}_G(g)$ is as in Table 4.1; moreover, g lies in the parabolic Q_0L_0 specified in the table.*

Proof. If g is unipotent, then (i) or (ii) holds, by Proposition 3.2(i). So we may assume that $g = g_sg_u$, where the semisimple part $g_s \in G \setminus \mathbf{Z}(G)$.

Suppose that $\mathbf{C}_G(g_s)$ contains a long root subgroup. Then if U is a Sylow p-subgroup of $\mathbf{C}_G(g_s)$, $\mathbf{Z}(U)$ contains a long root subgroup, by [14, Thm. 3.3.1]. Hence g_u centralizes a long root subgroup of $\mathbf{C}_G(g_s)$. It follows that g lies in a long root parabolic $P = QL$ with $g_s \in L$. If $g_s \notin \mathbf{Z}(L)$ then (ii) holds, so suppose $g_s \in \mathbf{Z}(L)$. Then $g_u \in \mathbf{C}_G(g_s)$, which is equal to L (or to $L'A_1$ if g_s is an involution). If the projection of g_u in L' is either 1 or a long root element, we can apply an element of the Weyl group of G to obtain a conjugate of g satifying (ii); and otherwise, g already satisfies (ii).

Now assume that $\mathbf{C}_G(g_s)$ does not contain a long root subgroup, and assume also that $|\mathbf{C}_G(g)| \geq q^m$, where m is defined as in (iii). Inspection of the lists of centralizers of semisimple elements in [9, 10, 32] then shows that $\mathbf{C}_G(g_s)$ is as in Table 4.1. Moreover, the lower bound on $|\mathbf{C}_G(g)|$ implies that $g_u = 1$, so $g = g_s$.

For exceptional groups, Lübeck [31] lists all the maximal tori lying in the centralizer of g and containing g. From this we can check that $\mathbf{C}_G(g)$ shares a maximal torus (containing g) with the Levi subgroup L_0 listed in Table 4.1. This completes the proof. □

Table 4.1

G	$\mathbf{C}_G(g)$	L_0
$E_8(q)$	${}^2A_4(q^2)$	$D_7(q).(q-1)$
$E_7(q)$	$A_3(q^2).T_1$	$E_6(q).(q-1)$
$E_6(q)$	$A_2(q^2).T_2$	$D_5(q).(q-1)$
${}^2E_6(q)$	$A_2(q^2).T_2$	${}^2D_4(q).(q^2-1)$

Proof of Theorem 1

Let $G = G(q)$ be as in Theorem 1, so that G is in one of the families $E_8(q)$, $E_7(q)$, $E_6^\epsilon(q)$, $F_4(q)$, with $q = p^a$ for a good prime p. Let $g \in G \setminus \mathbf{Z}(G)$, and let $1 \neq \chi \in \mathrm{Irr}(G)$. We consider the various possibilities for g given by Lemma 4.1.

Case 1 First suppose that $|\mathbf{C}_G(g)| < q^m$, as in Lemma 4.1(iii). Then $|\chi(g)| < q^{m/2}$, so

$$\frac{|\chi(g)|}{\chi(1)} < q^{\frac{m}{2}-l_1},$$

where $l_1 = l_1(G)$ is as in Lemma 2.1. Since $l_1 - \frac{m}{2} = a_2$ (as defined in Table 1.1), the conclusion of Theorem 1 follows in this case.

Case 2 Now suppose that g is as in (i) or (ii) of Lemma 4.1. Here $g \in Ql \subseteq QL = P$, where P is a long root parabolic and $l \in L \setminus \mathbf{Z}(L)$. Let V be a $\mathbb{C}G$-module affording the character χ, and write $V \downarrow QL = V^Q \oplus V_1 \oplus V_2$ as in Proposition 3.3. Let χ_0, χ_1, χ_2 be the characters of V^Q, V_1, V_2, respectively. Then $\chi_0 = {}^*R_L^G(\chi)$ by Proposition 3.3(i). An upper bound for the degree of $\chi_0 = {}^*R_L^G(\chi)$ is achieved in the proof of [2, Theorem 1.1], where it is shown that

$$\chi_0(1) \leq c\chi(1)^{\alpha(L)},$$

where $\alpha(L)$ is the maximum ratio $\frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}}$, taken over all non-identity unipotent elements of the Levi subgroup $\mathcal{L}$ of the same type as L in the algebraic group $\mathcal{G}$. The values of $\alpha(L)$ are computed in [2, Theorem 1.7], and are as follows:

G	$E_8(q)$	$E_7(q)$	$E_6^\epsilon(q)$	$F_4(q)$
$\alpha(L)$	$\frac{17}{29}$	$\frac{5}{9}$	$\frac{1}{2}$	$\frac{7}{15}$

Using Proposition 2.1, it follows that

$$|\chi_0(g)| \leq \chi_0(1) \leq c\frac{\chi(1)}{q^{a_2}}, \tag{4.1}$$

where a_2 is as defined in Table 1.1. Next, Proposition 3.3(iii), together with Theorem 3.1(iii) gives

$$|\chi_1(g)| \leq \frac{c_i}{q^{a_i}}\chi_1(1), \tag{4.2}$$

where $i = 1$ if g is a long root element, and $i = 2$ otherwise. Finally, Proposition 3.3(iv), together with Theorem 3.1(iv) gives

$$|\chi_2(g)| \leq \frac{1}{q^{a_i}}\chi_2(1), \tag{4.3}$$

where $i = 1$ if g is a long root element, and $i = 2$ otherwise.

Putting together (4.1), (4.2) and (4.3), we have

$$|\chi(g)| \leq |\chi_0(g)| + |\chi_1(g)| + |\chi_2(g)| \leq \frac{c}{q^{a_i}}\chi(1),$$

where $i = 1$ if g is a long root element, and $i = 2$ otherwise.

This completes the proof of Theorem 1, apart from elements g in the classes in Lemma 4.1(iv).

Case 3 It remains to prove Theorem 1 for g in one of the classes in Lemma 4.1(iv). Lemma 4.1(iv) shows that there is a Levi subgroup L_0 of a parabolic $P_0 = Q_0L_0$ as in Table 4.1 such that $g \in L_0$.

In each case we shall consider the restriction of χ to the parabolic P_0. The structure of Q_0 is given in Lemma 2.2. As in Proposition 3.3 we write

$$V \downarrow Q_0L_0 = V^{Q_0} \oplus V_1 \oplus V_2,$$

where $V_2 = [V, \mathbf{Z}(Q_0)]$ and $V^{Q_0} + V_1 = V^{\mathbf{Z}(Q_0)}$ (so that $V_1 = 0$ if Q_0 is abelian). Moreover,

$$V^{Q_0} \cong {}^*R^G_{L_0}(\chi),$$

and

$$\frac{|\chi_{V_2}(g)|}{\dim V_2} \le \max\{\mathrm{fpr}(g, \Psi_j) : 1 \le j \le t\},$$

where Ψ_j $(1 \le j \le t)$ are the orbits of L_0 on $P_1(\mathrm{Irr}\mathbf{Z}(Q_0))$, and if $V_1 \ne 0$, the bound in Proposition 3.3(iii) holds for $|\chi_{V_1}(g)|$.

First suppose $G = E_7(q)$. Here Q_0 is abelian, and has the structure of an irreducible 27-dimensional $\mathbb{F}_qL_0$-module, where $L_0' = E_6(q)$. Upper bounds for the fixed point ratios of elements of $E_6(q)$ in all primitive actions are given in [21, Theorem 2], and it follows that $\mathrm{fpr}(g) < \frac{1}{q^6-q^3+1}$ for all such actions. Now the conclusion of Theorem 1 follows for the element g as in Case 2.

Next consider $G = E_6(q)$. Again Q_0 is abelian, and is a 16-dimensional half-spin module for $L_0' = D_5(q)$. Here L_0' has two orbits on $P_1(\mathrm{Irr}(Q_0))$, with point-stabilizers an A_4-parabolic and a subgroup $q^8.B_3(q)T_1 < P_1$ (see [23, Lemma 2.9]). The element g has centralizer $A_2(q^2).T_2$ (see Table 4.1), hence is a regular semisimple element in $\mathbf{C}_G(A_2(q^2)) = {}^2A_2(q)$. From the embedding of ${}^2A_2(q)$ in $D_5(q)$, we see that in its action on the natural 10-dimensional $\bar{\mathbb{F}}_qD_5$-module, g has at least 4 distinct nontrivial eigenvalues. It follows that $\mathrm{fpr}(g, L_0'/P_1) < \frac{c}{q^4}$. Also $\mathrm{fpr}(g, L_0'/P_5) < \frac{c}{q^4}$ by inspection of the values of $1_{P_5}^{D_5(q)}$ given in [31]. The conclusion of Theorem 1 for the element g again follows.

Now suppose $G = {}^2E_6(q)$. This is similar to the previous case. The element g is a regular semisimple element in $\mathbf{C}_G(A_2(q^2)) = A_2(q)$. From the embedding of $A_2(q)$ in $L_0' = {}^2D_4(q)$, we see that in its action on the natural 8-dimensional module, g has at least 4 distinct nontrivial eigenvalues. Hence $\mathrm{fpr}(g, L_0'/P_1) < \frac{c}{q^4}$. Since each of the composition factors of $\mathbf{Z}(Q_0)$ and $Q_0/\mathbf{Z}(Q_0)$ is an 8-dimensional L_0'-module, the conclusion of Theorem 1 for the element g again follows.

The final case is $G = E_8(q)$, where the element g has centralizer ${}^2A_4(q^2)$ and lies in the Levi subgroup $L_0 = D_7(q)T_1$. Here g has order 5 dividing q^2+1, so in fact $g \in L_0'$. Now work in the algebraic group $\mathcal{G} = E_8$ over $\bar{\mathbb{F}}_q$, and the corresponding Levi subgroup $\mathcal{L}_0 = D_7$. We claim that on the natural 14-dimensional D_7-module, the element g has eigenvalues $\omega, \omega^2, \omega^3, \omega^4$ all with multiplicity 3, and eigenvalue 1 with multiplicity 2, where $\omega \in \bar{\mathbb{F}}_q^*$ is a fifth root of unity. Indeed, there is such an element $h \in L_0' = D_7(q)$ in the centre of a subgroup $\mathrm{GU}_3(q^2)$. To compute the centralizer of h, observe using [28, 11.3] that

$$L(E_8) \downarrow D_6 = L(D_6) + V(\lambda_1)^4 + V(\lambda_5)^2 + V(\lambda_6)^2 + V(0)^6.$$

A calculation with the weights of $V(\lambda_5) + V(\lambda_6)$ shows that the fixed point space of h on this sum has dimension 12. And the fixed point space of h on $L(D_6)$ has dimension 18. Hence $\dim \mathbf{C}_{L(E_8)}(h) = 48 = \dim A_4A_4$. Since $\mathbf{C}_G(h)$ contains $\mathrm{GU}_3(q^2)$, it follows by inspection of the list of semisimple element centralizers in [9] that $\mathbf{C}_G(h) = {}^2A_4(q^2)$. Hence h is conjugate to g, proving the claim.

Next, observe that $|\chi(g)| \le |\mathbf{C}_G(g)|^{1/2} < 2q^{24}$, so if the degree $\chi(1) > q^{35}$, then $\frac{|\chi(g)|}{\chi(1)} < \frac{1}{q^{10}}$, as required for Theorem 1. Hence we may assume that $\chi(1) \le q^{35}$, and therefore $\chi(1) < cq^{29}$ by Lemma 2.1. It follows from the proof of Proposition 3.3(iii) that

$$\frac{|\chi_{V_1}(g)|}{\dim V_1} \le \max\{\mathrm{fpr}(g, \Delta_i) : 1 \le i \le s\}.$$

where $\Delta_1, \ldots, \Delta_s$ are the orbits if L_0 on $P_1(Q_0/\mathbf{Z}(Q_0))$ of size at most cq^{29}. The orbits of the algebraic group D_7 on the half-spin module $V(\lambda_7)$ are determined in [33, Main Theorem, p.230], and only two of these have dimension less than 30. The point-stabilizers for these two orbits are connected groups lying in parabolic subgroups P_7 and P_3 of D_7. Hence these orbits correspond to unique orbits Δ_1, Δ_2 of $L_0' = D_7(q)$ on $P_1(Q_0/\mathbf{Z}(Q_0))$, with stabilizers contained in P_7 or P_3. Since our element $g \in L_0'$ fixes no totally singular 7-space or 3-space in the natural 14-dimensional module, it follows that $\mathrm{fpr}(g, \Delta_i) = 0$ for $i = 1, 2$. Finally, $\mathrm{fpr}(g, P_1(\mathbf{Z}(Q_0))) < \frac{1}{q^{10}}$. The conclusion of Theorem 1 follows for this element g.

This completes the proof of Theorem 1.

5. Proofs of Corollaries 2 and 3

Proof of Corollary 2. Let $G = G(q)$ be a quasisimple group of exceptional Lie type over $\mathbb{F}_q$ in good characteristic, and let $\mathcal{G}$ be the corresponding simple algebraic group over $\bar{\mathbb{F}}_q$. Let $y \in G \smallsetminus \mathbf{Z}(G)$, and let $P^t(g)$ be the probability of reaching g after t steps of the random walk described in the preamble to Corollary 2. By the Diaconis-Shashahani upper bound lemma (see for example [24, Prop. 1.7]),

$$||P^t - U||^2 \leq \sum_{1_G \neq \chi \in \mathrm{Irr}(G)} \left(\frac{|\chi(y)|}{\chi(1)}\right)^{2t} \chi(1)^2.$$

Hence by Theorem 1, we have

$$||P^t - U||^2 \leq (cq^{-a_i})^{2t} \sum_{1_G \neq \chi \in \mathrm{Irr}(G)} \chi(1)^2 \leq (cq^{-a_i})^{2t} \cdot q^{\dim \mathcal{G}},$$

where $i = 1$ if y is a long root element, and $i = 2$ otherwise (note that in view of Remark (i) after Theorem 1, for $G = G_2(q)$ there is an extra term of the order of $q^{-2t} \cdot q^6$ on the right hand side, but this is negligible for $t \geq 4$). Corollary 2 follows.

Proof of Corollary 3. We use the method of [29]. Let $G = G(q)$ be a simple group of exceptional Lie type over $\mathbb{F}_q$ in good characteristic p, and let α be a nontrivial irreducible character of G. If St denotes the Steinberg character of G, then by [18], St^2 contains every irreducible character of G as a constituent. As in [29, Lemma 2.3],

$$[\alpha^l, \mathsf{St}]_G = \frac{\alpha^l(1)}{|G|} \left(|G|_p + \sum_{1 \neq g \in G_{\mathrm{ss}}} \epsilon_g \left(\frac{\alpha(g)}{\alpha(1)}\right)^l |\mathbf{C}_G(g)|_p \right),$$

where G_{ss} is the set of semisimple elements of G and $\epsilon_g = \pm 1$. Hence $[\alpha^l, \mathsf{St}]_G \neq 0$ provided $\Sigma_l < |G|_p$, where

$$\Sigma_l := \sum_{1 \neq g \in G_{\mathrm{ss}}} \left|\frac{\alpha(g)}{\alpha(1)}\right|^l |\mathbf{C}_G(g)|_p.$$

By Theorem 1, we have

$$\Sigma_l \leq (cq^{-a_2})^l \sum_{1 \neq g \in G_{\mathrm{ss}}} |\mathbf{C}_G(g)|_p. \tag{5.1}$$

(Again by Remark (i) after Theorem 1, for $G = G_2(q)$ there is an extra term $q^{-l} \cdot q^3$ on the right hand side, but this is negligible assuming that $l \geq 4$.) For $1 \neq g \in G_{\mathrm{ss}}$, we have $\mathbf{C}_{\mathcal{G}}(g)^0 = T_k D$, where T_k is a rank k torus and D a semisimple group of rank $r - k$ (where $r = \mathrm{rank}(\mathcal{G})$). The number of such conjugacy classes is of the order of q^k, and their contribution to the sum in (5.1) is of the order of $|G : D(q)| \cdot |D(q)|_p$. It follows that

$$\Sigma_l \leq (cq^{-a_2})^l \cdot Cq^d,$$

where C is an absolute constant and $d = \dim \mathcal{G}$. Since $|G|_p = q^N$ where $N = |\Phi^+(\mathcal{G})|$, it follows that for q sufficiently large, $[\alpha^l, \mathsf{St}]_G \neq 0$ provided $a_2 l > d - N$. As remarked before, St^2 contains every irreducible character of G, and so Corollary 3 follows.

References

[1] H. Azad, M. Barry and G.M. Seitz, On the structure of parabolic subgroups, *Comm. Alg.* **18** (1990), 551-562.

[2] R. Bezrukavnikov, M.W. Liebeck, A. Shalev and P.H. Tiep, Character bounds for finite groups of Lie type, *Acta Math.* **221** (2018), 1–57.

[3] N. Bourbaki, *Lie groups and Lie algebras* (Chapters 4-6), Springer-Verlag, Berlin, 2002.

[4] R. Brauer, A note on theorems of Burnside and Blichfeldt, *Proc. Amer. Math. Soc.* **15** (1964), 31–34.

[5] T.C. Burness, Fixed point ratios in actions of finite classical groups, I. *J. Algebra* **309** (2007), 69–79.

[6] R.W. Carter, *Finite groups of Lie type: conjugacy classes and complex characters*, Wiley Interscience, 1985.

[7] A.M. Cohen and A.G. Helminck, Trilinear alternating forms on a vector space of dimension 7, *Comm. Alg.* **16** (1988), 1–25.

[8] C.W. Curtis, W.M. Kantor and G.M. Seitz, The 2-transitive permutation representations of the finite Chevalley groups, *Trans. Amer. Math. Soc.* **218** (1976), 1–59.

[9] D.I. Deriziotis, The centralizers of semisimple elements of the Chevalley groups E_7 and E_8, *Tokyo J. Math.* **6** (1983), 191–216.

[10] D.I. Deriziotis and M.W. Liebeck, Centralizers of semisimple elements in finite twisted groups of Lie type, *J. London Math. Soc.* **31** (1985), 48–54.

[11] F. Digne and J. Michel, *Representations of finite groups of Lie type*, London Math. Soc. Student Texts 21, Cambridge Univ. Press 1991.

[12] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer, CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras, *Appl. Algebra Engrg. Comm. Comput.* **7** (1996), 175–210.

[13] D. Gluck, Character value estimates for nonsemisimple elements, *J. Algebra* **155** (1993), 221–237.

[14] D. Gorenstein, R. Lyons and R. Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A. Almost simple K-groups.* Mathematical Surveys and Monographs, 40.3. American Mathematical Society, Providence, RI, 1998.

[15] R.M. Guralnick and R. Lawther, Generic stabilizers in actions of simple algebraic groups I, arXiv:1904.13375.

[16] R.M. Guralnick and G. Malle, Classification of $2F$-modules, II, *Finite groups 2003*, 117–183, Walter de Gruyter, Berlin, 2004.

[17] R.M. Guralnick and P.H. Tiep, Cross characteristic representations of even characteristic symplectic groups, *Trans. Amer. Math. Soc.* **356** (2004), 4969–5023.

[18] G. Heide, J. Saxl, P.H. Tiep and A.E. Zalesski, Conjugacy action, induced representations and the Steinberg square for simple groups of Lie type, *Proc. Lond. Math. Soc.* **106** (2013), 908–930.

[19] J. Igusa, A classification of spinors up to dimension twelve, *Amer. J. Math.* **92** (1970), 997–1028.

[20] D.J. Kenneally, On eigenvectors for semisimple elements in actions of algebraic groups, PhD Thesis, Cambridge 2010, https://doi.org/10.17863/CAM.16210.

[21] R. Lawther, M.W. Liebeck and G.M. Seitz, Fixed point ratios in actions of finite exceptional groups of Lie type, *Pacific J. Math.* **205** (2002), 393–464.

[22] M. Lee and M.W. Liebeck, Bases of quasisimple linear groups, *Algebra & Number Theory* **12** (2018), 1537–1557.

[23] M.W. Liebeck, The affine permutation groups of rank three, *Proc. London Math. Soc.* **54** (1987), 477–516.

[24] M.W. Liebeck, Applications of character theory of finite simple groups, in: *Local representation theory and simple groups* (eds. R. Kessar, G. Malle, D. Testerman), pp. 323-352, EMS Lecture Series, EMS Publishing House, Zürich, 2018.

[25] M.W. Liebeck and J. Saxl, On the orders of maximal subgroups of the finite exceptional groups of Lie type, *Proc. London Math. Soc.* **55** (1987), 299–330.

[26] M.W. Liebeck and J. Saxl, Minimal degrees of primitive permutation groups, with an application to monodromy groups of covers of Riemann surfaces, *Proc. London Math. Soc.* **63** (1991), 266–314.

[27] M.W. Liebeck, J. Saxl and G.M. Seitz, Subgroups of maximal rank in finite exceptional groups of Lie type, *Proc. London Math. Soc.* **65** (1992), 297–325.

[28] M.W. Liebeck and G.M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Math. Surveys and Monographs, Vol. **180**, Amer. Math. Soc., Providence, RI, 2012.

[29] M.W. Liebeck, A. Shalev and P.H. Tiep, On the diameters of McKay graphs for finite simple groups, submitted.
[30] M.W. Liebeck, A. Shalev and P.H. Tiep, Character ratios for classical groups, in preparation.
[31] F. Lübeck, Online data for groups of Lie type, `www.math.rwth-aachen.de:8001/~Frank.Luebeck/chev`.
[32] K. Mizuno, The conjugate classes of Chevalley groups of type E_6, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **24** (1977), 525–563.
[33] V.L. Popov, Classification of spinors of dimension 14, *Trans. Moscow Math. Soc.* (1980), no. 1, 181–232.
[34] K. Mizuno, The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8, *Tokyo J. Math.* **3** (1980), 391–461.
[35] T. Shoji, The conjugacy classes of Chevalley groups of type F_4 over finite fields of characteristic $p \neq 2$, *J. Fac. Sci. Univ. Tokyo* **21** (1974), 1–17.
[36] J. Taylor and P.H. Tiep, Lusztig induction, unipotent supports, and character bounds, arXiv:1809.00173.

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