

**M-theory on manifolds of  $G_2$  holonomy:  
the first twenty years<sup>1</sup>**

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**Abstract**

In 1981, covariantly constant spinors were introduced into Kaluza-Klein theory as a way of counting the number of supersymmetries surviving compactification. These are related to the *holonomy* group of the compactifying manifold. The first non-trivial example was provided in 1982 by  $D = 11$  supergravity on the squashed  $S^7$ , whose  $G_2$  holonomy yields  $N = 1$  in  $D = 4$ . In 1983 another example was provided by  $D = 11$  supergravity on  $K3$ , whose  $SU(2)$  holonomy yields half the maximum supersymmetry. In 2002,  $G_2$  and  $K3$  manifolds continue to feature prominently in the full  $D = 11$  M-theory and its dualities. In particular, singular  $G_2$  compactifications can yield chiral ( $N = 1, D = 4$ ) models with realistic gauge groups. The notion of generalized holonomy is also discussed.

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<sup>1</sup> Talk at the *Supergravity@25* conference, Stony Brook, December 2001.

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# 1 Introduction

For the *Supergravity@25* conference, the organizers requested that the speakers provide a blend of something historical and something topical. So I have chosen to speak about *holonomy*, especially  $G_2$  holonomy.

In 1981, Witten laid down the criterion for spacetime supersymmetry in Kaluza-Klein theory [1]. The number of spacetime supersymmetries is given by the number of covariantly constant spinors on the compactifying manifold. Covariantly constant spinors are, in their turn, related to the *holonomy*<sup>a</sup> group of the corresponding connection. It was well known that the number of massless gauge bosons was determined by the isometry group of the compactifying manifold, but it turned out to be the holonomy group that determined the number of massless gravitinos. This was further emphasized in [2].

The first non-trivial example was provided in 1982 [2] by compactifying  $D = 11$  supergravity on the squashed  $S^7$  [5], an Einstein space whose  $G_2$  holonomy yields  $N = 1$  in  $D = 4$ . (The round  $S^7$  has trivial holonomy and hence yields the maximum  $N = 8$  supersymmetry [6].) Although the phenomenologically desirable  $N = 1$  supersymmetry [1] and non-abelian gauge groups appear in four dimensions, the resulting theory was not realistic, being vectorlike with  $SO(5) \times SO(3)$  and living on  $AdS_4$ . It nevertheless provided valuable insight into the workings of modern Kaluza-Klein theories. Twenty years later,  $G_2$  manifolds continue to play an important role in  $D = 11$  M-theory for the same  $N = 1$  reason. But the full M-theory, as opposed to its low energy limit of  $D = 11$  supergravity, admits the possibility of *singular*  $G_2$  compactifications which can yield *chiral* ( $N = 1, D = 4$ ) models living in Minkowski space and with realistic gauge groups.

In 1983 another example was provided by  $D = 11$  supergravity on  $K3$  [7], whose  $SU(2)$  holonomy yields half the maximum supersymmetry. For the first time, the Kaluza-Klein particle spectrum was dictated by the topology (Betti numbers and index theorems) rather than the geometry of the compactifying manifold, which was four-dimensional, Ricci flat and without isometries. It was thus a forerunner of the six-dimensional Ricci flat Calabi-Yau compactifications [8], whose  $SU(3)$  holonomy yields ( $N = 1, D = 4$ ) starting from ( $N = 1, D = 10$ ).  $K3$  compactifications also continue to feature prominently in M-theory and its dualities.

## 2 D=11 supergravity

The low energy limit of M-theory (or  $D = 11$  supergravity as we used to call it) was introduced in 1978 by Cremmer, Julia and Scherk [9], not long after the discovery of supergravity itself [10, 11]. The unique  $D = 11, N = 1$  supermultiplet is comprised of a graviton  $g_{MN}$ , a gravitino  $\Psi_M$  and 3-form gauge field  $A_{MNP}$ , where  $M = 0, 1, \dots, 10$ , with 44, 128 and 84 physical degrees of freedom, respectively.

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<sup>a</sup>Holonomy had already found its way into the physics literature via gravitational instantons [3] and non-linear sigma models [4].

Already the rank of the form, dictated by supersymmetry to be three, presages deep connections with 2-branes and indeed  $G_2$  holonomy.

The supersymmetry transformation rule of the gravitino reduces in a purely bosonic background to

$$\delta\Psi_M = \tilde{D}_M \epsilon \quad (1)$$

where the parameter  $\epsilon$  is a 32-component anticommuting spinor, and where

$$\tilde{D}_M = D_M - \frac{i}{144} \left( \Gamma^{NPQR}{}_M + 8\Gamma^{PQR}\delta^N{}_M \right) F_{NPQR} \quad (2)$$

Here  $D_M$  is the usual Riemannian covariant derivative involving the usual Levi-Civita connection  $\omega_M$

$$D_M = \partial_M - \frac{1}{4} \omega_M^{AB} \Gamma_{AB}, \quad (3)$$

$\Gamma^A$  are the  $D = 11$  Dirac matrices and  $F = dA$ .

The bosonic field equations are

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{3} \left( F_{MPQR} F_N{}^{PQR} - \frac{1}{8} g_{MN} F^{PQRS} F_{PQRS} \right) \quad (4)$$

and

$$d * F + F \wedge F = 0 \quad (5)$$

Being at Stonybrook, I should not forget the paper that I wrote with Peter van Nieuwenhuizen [12] pointing out that the 4-form field strength can thus give rise to a cosmological constant  $\Lambda = -12m^2$  in the four dimensional subspace:

$$F_{\mu\nu\rho\sigma} = 3m\epsilon_{\mu\nu\rho\sigma} \quad (6)$$

where  $\mu = 0, 1, 2, 3$  and  $m$  is a constant with the dimensions of mass. A similar conclusion was reached independently by Aurilia et al [13]. This device was then used by Freund and Rubin [14] to effect a spontaneous compactification from  $D = 11$  to  $D = 4$ , yielding the product of a four-dimensional spacetime with negative curvature

$$R_{\mu\nu} = -12m^2 g_{\mu\nu} \quad (7)$$

and a seven-dimensional internal space of positive curvature

$$R_{mn} = 6m^2 g_{mn} \quad (8)$$

where  $m = 1, 2, \dots, 7$ . Accordingly, the supercovariant derivative also splits as

$$\tilde{D}_\mu = D_\mu + m\gamma_\mu\gamma_5 \quad (9)$$

and

$$\tilde{D}_m = D_m - \frac{1}{2} m\Gamma_m \quad (10)$$

If we choose the spacetime to be maximally symmetric but leave the internal space  $X^7$  arbitrary, we are led to the  $D = 11$  geometry  $AdS_4 \times X^7$ . The first example

was provided by the choice  $X^7 = \text{round } S^7$  [6, 15] which is maximally supersymmetric<sup>b</sup>. The next example was the round  $S^7$  with parallelizing torsion [17] which preserves no supersymmetry. However, it was also of interest to look for something in between, and this is where holonomy comes to the fore.

### 3 Killing spinors, holonomy and supersymmetry

The number of supersymmetries surviving compactification depends on the number of covariantly constant spinors [1]. To see this, we look for vacuum solutions of the field equations for which the the gravitino field  $\Psi$  vanishes. In order that the vacuum be supersymmetric, therefore, it is necessary that the gravitino remain zero when we perform a supersymmetry transformation and hence that the background supports spinors  $\epsilon$  satisfying

$$\tilde{D}_M \epsilon = 0 \tag{11}$$

In the case of a product manifold, this reduces to

$$\tilde{D}_\mu \epsilon(x) = 0 \tag{12}$$

and

$$\tilde{D}_m \eta(y) = 0 \tag{13}$$

where  $\epsilon(x)$  is a 4-component anticommuting spinor and  $\eta(y)$  is an 8-component commuting spinor. The first equation is satisfied automatically with our choice of  $AdS_4$  spacetime and hence the number of  $D = 4$  supersymmetries,  $0 \leq N \leq 8$ , devolves upon the number of *Killing spinors* on  $X^7$ . They satisfy the integrability condition

$$[\tilde{D}_m, \tilde{D}_n] \eta = -\frac{1}{4} C_{mn}{}^{ab} \Gamma_{ab} \eta = 0 \tag{14}$$

where  $C_{mn}{}^{ab}$  is the Weyl tensor.

The subgroup of  $Spin(7)$  (the double cover of the tangent space group  $SO(7)$ ) generated by this linear combination of  $Spin(7)$  generators  $\Gamma_{ab}$  corresponds to the *holonomy* group  $\mathcal{H}$  [2]. The number of supersymmetries,  $N$ , is then given by the number of singlets appearing in the decomposition of the 8 of  $Spin(7)$  under  $\mathcal{H}$  [18]. Some examples are given in Table 1.

Strictly speaking, the Weyl tensor characterizes the *restricted* holonomy group of  $\tilde{D}_m$ . If the space is not simply connected there may be further global obstructions to the existence of unbroken supersymmetries. For example, solutions of the form  $T^7/\Gamma$  and  $S^7/\Gamma$ , where  $\Gamma$  is a discrete group, have vanishing Weyl tensor but admit fewer than 8 Killing spinors [7].

The phenomenological desirability of having just one Killing spinor, and hence just one four-dimensional supersymmetry, is also discussed in [1].

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<sup>b</sup>The first Ricci flat ( $m = 0$ ) example of a compactification of  $D = 11$  supergravity was provided by the choice  $X^7 = T^7$  [16] which is also maximally supersymmetric.

$\mathcal{H}$	$8 \rightarrow$	$N$
$Spin(7)$	8	0
$G_2$	$1 + 7$	1
$SU(3)$	$1 + 1 + 3 + \bar{3}$	2
$SU(2)$	$1 + 1 + 1 + 1 + 2 + \bar{2}$	4
$\mathbf{1}$	$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$	8

Table 1: Examples of holonomy groups and the resulting supersymmetry

## 4 $G_2$

We see that the exceptional group  $G_2$  is of particular interest since it yields just  $N = 1$  supersymmetry. In fact, the first example of a Kaluza-Klein compactification with non-trivial holonomy was provided by the squashed  $S^7$  which does indeed have  $\mathcal{H} = G_2$  [2, 5].

This is probably a good time to say a word about terminology. When talking of the *holonomy* of a manifold, some authors take the word to refer exclusively to the Levi-Civita connection appearing in the Riemannian covariant derivative  $D_m$ . According to this definition, the holonomy of the squashed  $S^7$  would be  $Spin(7)$ . The group  $G_2$  would then correspond to what mathematicians call *weak holonomy* [19, 20]. A 7-dimensional Einstein manifold with  $R_{mn} = 6m^2g_{mn}$  has weak holonomy  $G_2$  if it admits a 3-form  $A$  obeying

$$dA = 4m * A \tag{15}$$

That such a 3-form exists on the squashed  $S^7$  can be proved by invoking the single (constant) Killing spinor  $\eta$ . The required 3-form is then given by [5]

$$A_{mnp} \sim \bar{\eta} \Gamma_{mnp} \eta \tag{16}$$

However, I prefer not to adopt this terminology for two reasons. First, from a strictly mathematical point of view, one should speak not of the holonomy of a manifold but rather of the connection on the manifold. Different connections on the same manifold can have different holonomies. Secondly, from a physical point of view, the whole reason for being interested in holonomy in the first place is because of supersymmetry, and here the relevant connection is not the Levi-Civita connection  $\omega$  appearing in  $D_m$  but rather the generalized connection appearing in

$$\tilde{D}_m = \partial_m - \frac{1}{4} \omega_m^{ab} \Gamma_{ab} - \frac{1}{2} m e_m^a \Gamma_a \tag{17}$$

So in the context of  $M$ -theory, when I speak loosely of the holonomy of a manifold, it is the supersymmetric connection that I have in mind <sup>c</sup>.

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<sup>c</sup>Some early papers used the term *Weyl holonomy* which should probably now be abandoned.

Owing to this generalized connection, vacua with  $m \neq 0$  present subtleties and novelties not present in the  $m = 0$  case [25], for example the phenomenon of *skew-whiffing* [5, 15]. For each Freund-Rubin compactification, one may obtain another by reversing the orientation of  $X^7$ . The two may be distinguished by the labels *left* and *right*. An equivalent way to obtain such vacua is to keep the orientation fixed but to make the replacement  $m \rightarrow -m$ . So the covariant derivative (17), and hence the condition for a Killing spinor, changes but the integrability condition (14) remains the same. With the exception of the round  $S^7$ , where both orientations give  $N = 8$ , at most one orientation can have  $N \geq 0$ . This is the *skew-whiffing theorem*. A corollary is that other *symmetric spaces*, which necessarily admit an orientation-reversing isometry, can have no supersymmetries. Examples are provided by products of round spheres. Of course, skew-whiffing is not the only way to obtain vacua with less than maximal supersymmetry. A summary of known  $X^7$ , their supersymmetries and stability properties is given in [15]. Note, however, that skew-whiffed vacua are automatically stable at the classical level since skew-whiffing affects only the spin  $3/2$ ,  $1/2$  and  $0^-$  towers in the Kaluza-Klein spectrum, whereas the criterion for classical stability involves only the  $0^+$  tower [62, 15].

Once again the squashed  $S^7$  provided the first non-trivial example: the left squashed  $S^7$  has  $N = 1$  but the right squashed  $S^7$  has  $N = 0$ . Interestingly enough, this means that setting the suitably normalized 3-form (16) equal to the  $D = 11$  supergravity 3-form provides a solution to the field equations, but only in the right squashed case. This solution is called the *right squashed  $S^7$  with torsion* [5] since  $A_{mnp}$  may be interpreted as a Ricci-flattening torsion. Other examples were provided by the left squashed  $N(1,1)$  spaces [24], one of which has  $N = 3$  and the other  $N = 1$ , while the right squashed counterparts both have  $N = 0$ .

All this presents a dilemma. If the Killing spinor condition changes but the integrability condition does not, how does one give a holonomic interpretation to the different supersymmetries? Indeed  $N = 3$  is not allowed by the usual rules. The answer to this question may be found in a paper [21] written before we knew about skew-whiffing. The authors note that in (17), the  $SO(7)$  generators  $\Gamma_{ab}$ , augmented by presence of  $\Gamma_a$ , together close on  $SO(8)$ . Hence one may introduce a generalized holonomy group  $\mathcal{H}_{gen} \subset SO(8)$  and ask how the 8 of  $SO(8)$  decomposes under  $\mathcal{H}_{gen}$ . In the case of the left squashed  $S^7$ ,  $\mathcal{H}_{gen} = SO(7)^-$ ,  $8 \rightarrow 1 + 7$  and  $N = 1$ , but for the right squashed  $S^7$ ,  $\mathcal{H}_{gen} = SO(7)^+$ ,  $8 \rightarrow 8$  and  $N = 0$ .

Kaluza-Klein compactification of supergravity was an active area of research in the early 1980s. Some early papers are [1, 2, 5, 6, 7, 17, 21, 22, 23, 26, 27] and I am glad to see many of the pioneers in the audience today: Leonardo Castellani, Bernard de Wit, Pietro Fre, Dan Freedman, Gary Gibbons, Bernard Julia, Ergin Sezgin, John Schwarz, Peter van Nieuwenhuizen, Nick Warner and Peter West. Reviews of Kaluza-Klein supergravity may be found in [15, 28].

## 5 Supermembranes with fewer supersymmetries

Interest in  $AdS_4 \times X^7$  solutions of  $D = 11$  supergravity waned for a while but was revived by the arrival of the  $D = 11$  supermembrane [29]. In 1991 this 2-brane was recovered as a solution of the  $D = 11$  supergravity theory preserving one half of the supersymmetry [30, 31]. Specifically, in the case that  $N$  branes with the same charge are stacked together, the metric is given by

$$ds^2 = (1 + Na^6/y^6)^{-2/3} dx^\mu dx_\mu + (1 + Na^6/y^6)^{1/3} (dy^2 + y^2 d\Omega_7^2) \quad (18)$$

and the four-form field strengths by

$$\tilde{F}_7 \equiv *F_4 = \pm 6Na^6 \epsilon_7 \quad (19)$$

Here  $d\Omega_7^2$  corresponds to the round  $S^7$ .

Of particular interest is the near horizon limit  $y \rightarrow 0$ , or equivalently the large  $N$  limit, because then the metric reduces to [32, 33, 34] the  $AdS_4 \times S^7$  vacuum with

$$m^{-6} = Na^6 \quad (20)$$

Thus

$$ds^2 = y^4 m^4 dx^\mu dx_\mu + m^{-2} y^{-2} dy^2 + m^{-2} d\Omega_7^2 \quad (21)$$

which is just  $AdS_4 \times S^7$  with the  $AdS$  metric written in horospherical coordinates.

Note that the round  $S^7$  makes its appearance. The question naturally arises as to whether the compactifications with fewer supersymmetries discussed above also arise as near-horizon geometries of  $p$ -brane solitons. The answer is yes and the soliton solutions are easy to construct [35, 36]. One simply makes the replacement

$$d\Omega_7^2 \rightarrow d\hat{\Omega}_7^2 \quad (22)$$

in (18), where  $d\hat{\Omega}_7^2$  is the metric on an arbitrary Einstein space  $X^7$  with the same scalar curvature as the round  $S^7$ . The space need only be Einstein; it need not be homogeneous [35]. (There also exist brane solutions on Ricci flat  $X^7$  [35]). Note, however, that these non-round-spherical solutions do not tend to Minkowski space as  $r \rightarrow \infty$ . Instead the metric on the 8-dimensional space transverse to the brane is asymptotic to a generalized cone

$$ds_8^2 = dy^2 + y^2 d\hat{\Omega}_7^2 \quad (23)$$

and 8-dimensional translational invariance is absent except when  $X^7$  is the round  $S^7$ . Note, however that the solutions have no conical singularity at  $y = 0$  since the metric tends to  $AdS_4 \times X^7$ . By introducing the Schwarzschild-like coordinate  $r$  given by

$$r^6 = y^6 + Na^6 \quad (24)$$

$\mathcal{H}$	Space	restrictions
$SO(n)$	$R^n$	$n \geq 2$
$U(m)$	$R^n \simeq C^m$	$m \geq 2, n = 2m$
$SU(m)$	$R^n \simeq C^m$	$m \geq 2, n = 2m$
$Sp(1) \times Sp(m)$	$R^n \simeq H^m$	$m \geq 1, n = 4m$
$Sp(m)$	$R^n \simeq H^m$	$m \geq 1, n = 4m$
$G_2$	$R^7$	
$Spin(7)$	$R^8$	
$Spin(9)$	$R^{16}$	

Table 2: Holonomies of Riemannian connections

we can see that the solutions exhibits an event-horizon at  $r = N^{1/6}a$ . Indeed the solution may be analytically continued down to  $r = 0$  where there is a curvature singularity, albeit hidden by the event horizon [33].

As a matter of fact, there is a one-to-one correspondence between Killing spinors on Einstein manifolds  $X^7$  satisfying  $\tilde{D}_m \eta = 0$ ,  $m = 1, 2 \dots 7$ , and Killing spinors on the Ricci-flat cone (23) satisfying  $D_M \eta = 0$ ,  $M = 1, 2 \dots 8$ , [37, 20]. So  $N = 1$  can then be understood as  $G_2$  holonomy on  $X^7$  or  $Spin(7)$  holonomy on the cone. Similarly, the weak holonomy 3-form (15) lifts to a covariantly constant self-dual 4-form on the cone.

## 6 A speculation on Spin(9)

In Berger's classification [38] of holonomy groups of Levi-Civita connections given in Table 2, there are three exceptional cases  $G_2$ ,  $Spin(7)$  and  $Spin(9)$ .  $G_2$  and  $Spin(7)$  have already made their appearance in our story, but  $Spin(9)$  seems an unlikely candidate since it corresponds to the holonomy of a *sixteen*-dimensional space (the Cayley plane), and that seems too high for  $D = 11$  supergravity. Nevertheless, as we shall now describe, there is a way in which subgroups of  $SO(16)$  can appear as holonomies in the theory. (The Cayley Plane, which is the 16-dimensional coset  $F_4/Spin(9)$ , has also featured in recent, but apparently unrelated, speculations on hidden mathematical structures in  $D = 11$  [39]. Recent discussions on the mathematics of  $Spin(9)$  may be found in [40, 41].)

The supermembranes discussed in the previous section preserve a fraction  $\nu = N/16$  of the spacetime supersymmetry where  $1 \leq N \leq 8$  is the number of Killing spinors on  $X^7$ . (In the near-horizon limit this doubles.) Following [30], we can attempt to quantify this in terms of a holonomy even more generalized than that discussed in section (4), namely

$$\mathcal{H}_{gen} \subset SO(16) \tag{25}$$

where  $N$  is then given by the number of singlets appearing in the decomposition of the 16 of  $SO(16)$  under  $\mathcal{H}_{gen}$ . In the case of the brane with a round  $S^7$ , we

have  $\mathcal{H}_{gen} = SO(8)$ , the 16 decomposes into an 8 plus 8 singlets and  $\nu = 1/2$ . Whereas in the case of the brane with a squashed  $S^7$ , we would have only one singlet and  $\nu = 1/16$ . The origin of this  $SO(16)$  was speculated in [42] and proved in [43]. After making a  $D = 11$  Lorentz-non-covariant three/eight split of the  $D = 11$  supergravity field equations, the tangent space group  $SO(1, 2) \times SO(8)$  gets enlarged to an  $SO(1, 2) \times SO(16)$  under which the supersymmetry parameter transforms as a  $(2, 16)$ . Note that the 16 is the *vector* representation, even though  $\epsilon$  is a spacetime spinor <sup>d</sup>. This can be understood by noting, as we have already done, that the supercovariant derivative (2) involves not merely the Levi-Civita connection but extra terms involving the 4-form field strength  $F_{MNPQ}$ , a fact that tends to be underemphasized in recent works on holonomy and  $D = 11$  supergravity (a notable exception being [44]).

Although we do not know of any examples, it is thus possible that some  $D = 11$  field configuration could have

$$Spin(9) = \mathcal{H}_{gen} \subset SO(16) \tag{26}$$

However, the embedding is such that  $16 \rightarrow 16$  and so such a configuration can preserve no supersymmetries, which may seem something of an anticlimax.

All this raises an interesting question: What is the number of possible supersymmetries allowed in M-theory? A priori, from the M-theory algebra [45],  $32\nu$  can be any integer from 0 to 32 [46]. Although there are supersymmetric backgrounds realizing the maximum 32, it was thought for a time that partially supersymmetric backgrounds were restricted to  $0 \leq \nu \leq 1/2$ , such as the above mentioned membranes. However, recent work on pp waves [47, 48, 49, 50, 51] and Gödel universes [52] has revealed certain values of  $32\nu$  lying between 16 and 32. Riemannian holonomy cannot account for these exotic fractions, so let us therefore take the generalized holonomy approach, discussed above. Of course, one is not obliged to make a three/eight split of M-theory, one could make any  $d/(11 - d)$  split. For example, after a four/seven split, the tangent space group  $SO(1, 3) \times SO(7)$  is enlarged to an  $SO(1, 3) \times SU(8)$  [42, 53]. Similar remarks apply to Type IIB. It is curious to note that if the 32 supercharges always belong to representations such as  $(2, 16)$ ,  $(4, 8)$  etc, then  $n = 32\nu$ , the number of singlets appearing in the decomposition, is restricted to 0, 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 32. This is consistent with the data. However, a better understanding of the two/nine and one/ten splits is necessary before ruling out other values.

## 7 Hopf dualities

In recent times, both perturbative and non-perturbative effects of ten-dimensional superstring theory have been subsumed by an eleven-dimensional  $M$  theory whose

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<sup>d</sup>This phenomenon had been noted long ago by Cremmer and Julia [16] when *dimensionally reducing*  $D = 11$  supergravity (or Type IIB supergravity) to  $D = 3$ , but here we are claiming  $SO(16)$  already in  $D = 11$  (or  $D = 10$ ).

low-energy limit is  $D = 11$  supergravity [54]. In particular, we have the duality

$$M \text{ on } X \equiv IIA \text{ on } Y \tag{27}$$

In the first examples  $X$  was just a direct product  $Y \times S^1$  and in this way one could give an  $D = 11$  M-theory origin to  $D = 10$  Type *IIA* objects by either wrapping around the  $S^1$  or by reducing. For example, the *IIA* string comes from the *M2*-brane by wrapping [55, 61, 57]; the *D6*-brane in  $D = 10$  may be interpreted as a Kaluza-Klein monopole in  $D = 11$  [60, 61] and the *D2* brane in  $D = 10$  may be interpreted as an *M2*-brane in  $D = 11$  by dualizing a vector into a scalar on the 3-dimensional worldvolume [58, 59].

However, the duality (27) may be generalized to the so-called *Hopf duality* where  $X$  is a twisted product or  $U(1)$  bundle over  $Y$  [63, 64, 65, 66, 67]. In [63], for example, such *M*-theory vacua with  $N > 0$  supersymmetry were presented which, from the perspective of perturbative Type *IIA* string theory, have  $N = 0$ . They can emerge whenever the  $X^7$  is a  $U(1)$  bundle over a 6-manifold  $Y^6$ . The missing superpartners are Dirichlet 0-branes. Someone unable to detect Ramond-Ramond charge would thus conclude that these worlds have no unbroken supersymmetry. In particular, the gravitinos (and also some of the gauge bosons) are 0-branes not seen in perturbation theory but which curiously remain massless however weak the string coupling. The simplest example of this phenomenon is provided by the maximally-symmetric  $S^7$ . Considered as a compactification of  $D = 11$  supergravity, the round  $S^7$  yields a four dimensional *AdS* spacetime with  $N = 8$  supersymmetry and  $SO(8)$  gauge symmetry, for either orientation of  $S^7$ . The Kaluza-Klein mass spectrum therefore falls into  $SO(8)$   $N = 8$  supermultiplets. In particular, the massless sector is described [15] by gauged  $N = 8$  supergravity [69]. Since  $S^7$  is a  $U(1)$  bundle over  $CP^3$  the same field configuration is also a solution of  $D = 10$  Type *IIA* supergravity [114]. However, the resulting vacuum has only  $SU(4) \times U(1)$  symmetry and either  $N = 6$  or  $N = 0$  supersymmetry depending on the orientation of the  $S^7$ . The reason for the discrepancy is that the modes charged under the  $U(1)$  are associated with the Kaluza-Klein reduction from  $D = 11$  to  $D = 10$  and are hence absent from the Type *IIA* spectrum originating from the massless Type *IIA* supergravity. In other words, they are Dirichlet 0-branes [68] and hence absent from the perturbative string spectrum. There is thus more non-perturbative gauge symmetry and supersymmetry than perturbative. (Here the words “perturbative” and “non-perturbative” are shorthand for “with and without the inclusion of Dirichlet 0-branes”, but note that the Type *IIA* compactification has non-perturbative features even without the 0-branes [63]). The right-handed orientation is especially interesting because the perturbative theory has no supersymmetry at all! A summary of perturbative versus non-perturbative symmetries is given in Table 2. In particular, the non-perturbative vacuum may have unbroken supersymmetry even when the perturbative vacuum has none.

Compactification	Perturbative Type <i>IIA</i>		Nonperturbative M-theory	
Left round $S^7$	$N = 6$	$SU(4) \times U(1)$	$N = 8$	$SO(8)$
Right round $S^7$	$N = 0$	$SU(4) \times U(1)$	$N = 8$	$SO(8)$
Left squashed $S^7$	$N = 1$	$SO(5) \times U(1)$	$N = 1$	$SO(5) \times SU(2)$
Right squashed $S^7$	$N = 0$	$SO(5) \times U(1)$	$N = 0$	$SO(5) \times SU(2)$
Left $M(3, 2)$	$N = 0$	$SU(3) \times SU(2) \times U(1)$	$N = 2$	$SU(3) \times SU(2) \times U(1)$
Right $M(3, 2)$	$N = 0$	$SU(3) \times SU(2) \times U(1)$	$N = 0$	$SU(3) \times SU(2) \times U(1)$

Table 2: Perturbative versus non-perturbative symmetries.

We have focussed in this paper on compactifications from  $D = 11$  but much of the discussion applies, *mutatis mutandis* to Type *IIB*. For example we have the Hopf T-duality

$$IIB \text{ on } S^5 \equiv IIA \text{ on } CP^2 \times S^1 \quad (28)$$

which untwists the  $S^5$  [64].

## 8 Recent developments

Following the M-theory revolution of 1995, it was noted [72, 73, 74, 76, 75, 81, 83] that non-chiral  $N = 1$  heterotic string compactifications can be dual to  $D = 11$  supergravity compactified on Ricci-flat seven-dimensional spaces of  $G_2$  holonomy [85, 86, 87, 84, 77]. See also [78, 79] for  $N = 1$  compactifications on 8-manifolds of spin(7) holonomy.

In 2002,  $G_2$  manifolds continue play an important role in  $D = 11$  M-theory for the same  $N = 1$  reason as in 1982 . Of course, for phenomenology we require not only  $N = 1$  but also chirality, and the lack of chirality on smooth seven-manifolds [80] was one of the main reasons that  $D = 11$  supergravity fell out of favor. But the full M-theory, as opposed to its low energy limit, admits the possibility of *singular*  $G_2$  compactifications which can yield *chiral* ( $N = 1, D = 4$ ) models living in Minkowski space, with realistic gauge groups [88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111] and doublet-triplet splitting [112].

## 9 K3

In 1983 a second example of the utility of holonomy was provided by the compactification of  $D = 11$  supergravity on the  $K3$  [7]. This manifold had recently made its appearance in the physics literature as a gravitational instanton [3]. It is a self-dual, and hence Ricci flat, solution with  $m = 0$  and  $\mathcal{H} = SU(2)$  instead of the  $SU(2) \times SU(2)$  of a generic 4-manifold. Hence  $N = N_{max}/2$ .  $K3$  provided a novel phenomenon in Kaluza-Klein theory: the appearance of massless particles as a consequence of the *topology*, as opposed to the *geometry* of the compactifying manifold, determined by Betti numbers and index theorems [7]. A discussion of

boson and fermion zero modes on  $K3$  and their relation to axial and conformal anomalies [70], may be found in [3, 71, 1].

$K3$  was thus the forerunner of the very influential *Calabi-Yau* compactifications<sup>e</sup> of ten-dimensional supergravity and string theory [8] and, indeed, the Ricci-flat  $G_2$  compactifications of  $M$ -theory mentioned above, some of which correspond to  $K3$  fibrations [108].

$K3$  compactification allows for the possibility of chirality in the lower dimensional spacetime and chiral compactification of the Type *IIB* supergravity was undertaken in [116] and the heterotic string theory in [117].

In 1986, it was pointed out [115] that  $D = 11$  supergravity on  $R^{10-n} \times K3 \times T^{n-3}$  [7] and the  $D = 10$  heterotic string on  $R^{10-n} \times T^n$  [82] not only have the same supersymmetry but also the same moduli spaces of vacua, namely

$$\mathcal{M} = \frac{SO(16+n, n)}{SO(16+n) \times SO(n)} \quad (29)$$

It took almost a decade for this “coincidence” to be explained but we now know that  $M$ -theory on  $R^{10-n} \times K3 \times T^{n-3}$  is dual to the heterotic string on  $R^{10-n} \times T^n$  [56, 57].

One way to understand this is to note that, when wrapped around  $K3$  with its 19 self-dual and 3 anti-self-dual 2-forms, the  $d = 6$  worldvolume fields of the  $M5$ -brane (or  $D5$ -brane)  $(B^-_{\mu\nu}, \lambda^I, \phi^{[IJ]})$  reduce to the  $d = 2$  worldsheet fields of the heterotic string in  $D = 7$  (or  $D = 6$ ) [120, 121, 122]. The 2-form yields (19, 3) left and right moving bosons, the spinors yield (0, 8) fermions and the scalars yield (5, 5) which add up to the correct worldsheet degrees of freedom of the heterotic string. A consistency check is provided by the derivation of the Yang-Mills and Lorentz Chern-Simons corrections to the Bianchi identity of the heterotic string starting from the  $M5$ -brane Bianchi identity [113].

Heterotic strings on  $K3$  can also be self-dual [118, 119]. Moreover, Heterotic strings on Calabi-Yau manifolds that are a  $T^3$  fibration over a base  $Q$  can be dual to  $M$ -theory on  $X^7$  that is  $K3$  fibered over  $Q$  and has  $G_2$  holonomy [108], which brings us back to where we started.

## 10 Conclusions

I first wrote about supergravity in a popular article for *New Scientist* in 1977 [123], where I said “Supergravity is theoretically very compelling, but it has yet to prove its worth by experiment”; a remark still unfortunately true at *Supergravity@25*. Let us hope that by the *Supergravity@50* conference, or before, we can say something different.

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<sup>e</sup>Indeed, the possibility of going from 10 dimensions to 4 on a Ricci-flat 6-manifold with  $SU(3)$  holonomy so as to get  $N = 1$  in  $D = 4$  occurred to Pope, Nilsson and myself while writing up the  $K3$  paper in 1983. Since we were at UT, Austin, at the time, we consulted one or two of the distinguished mathematicians in the Mathematics Department there, but were told they had never heard of such a thing! Consequently our paper states “We do not know any solutions with  $\mathcal{H} = SU(3)$ ”.

It is a privilege to have been a member of the Supergravity community these 25 years and I would like to say “Thanks for the memories” to its discoverers.

## 11 Acknowledgements

I would like to thank the conference organizers, Martin Rocek, George Sterman and Warren Siegel for their hospitality. I have enjoyed useful conversations on holonomy with Pietro Fre, Finn Larsen, Jim Liu, Hong Lu, Leo Pando-Zayas and Hisham Sati. Thanks to Gordy Kane and Jim Liu for suggesting improvements to the manuscript.

## 12 Notes added

- Generalized holonomy is developed further in [124, 125, 126, 127].
- In section (6) we conjectured, albeit on flimsy evidence, that the number of vacuum supersymmetries allowed by M-theory is restricted to

$$n = 0, 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 32 \quad (30)$$

Interestingly enough, after the completion of this work, a Gödel universe with  $n = 14$  was discovered [129] which completes this list. Furthermore:

- $n = 31$  has now been ruled out for both Type IIB [130] and Type IIA [131].
- $n = 30$  has now been ruled out for M-theory [132].
- The class of M-theory plane waves found in [132] has  $n = 16, 20, 26$  but not  $n = 28$ , although plane wave solutions with  $n = 28$  do appear in Type IIB [128].
- Backgrounds with  $n > 24$  are necessarily (locally) homogeneous. See [133] where it is also conjectured that 24 is the minimum number which guarantees this.
- In compactifying Type II strings from  $D = 10$  to  $D = 2$  we must allow for the possibility of asymmetric orbifolds where the left and right movers may experience different holonomies yielding  $D = 2$  supersymmetries  $(N_+, N_-)$  with  $N_+ \neq N_-$ . Berger’s [38] list of holonomies  $SO(8), Spin(7), G_2, SU(3), SU(2), I$  allows  $N_+ = 0, 1, 2, 4, 8, 16$  and  $N_- = 0, 1, 2, 4, 8, 16$  and hence  $n = N_+ + N_-$  can take on values

$$n = 0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, 20, 24, 32 \quad (31)$$

So if we broaden our definition of M-theory vacua to include asymmetric orbifold compactifications of Type II (as suggested to me in this context by Cumrun Vafa) then we must also include the cases  $n = 9, 17$  excluded in (30).

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