We study contact angle hysteresis in a chemically heterogeneous microchannel by tracking static meniscus configurations in the microchannel upon varying the volume of liquid. We first construct a graphical force balance similar to a previous approach by Joanny and de Gennes for this system, though here with a straight contact line. It is shown that hysteresis is induced by wettability gradients above a finite threshold value. This is also visualized in a phase-plane plot enabling to easily predict stick-slip events of the contact line and the occurrence of hysteresis. Above the threshold and for non-overlapping Gaussian defects, we find good agreement with the expressions by Joanny and de Gennes for the hysteresis amplitude induced by a dilute system of defects. In particular, the hysteresis amplitude is found to be proportional to the square of the defect force and to the defect concentration. For a model sinusoidal heterogeneity, decreasing the ratio between the heterogeneity wavelength and the microchannel gap size, brings the system from a sub-threshold regime, to a stick-slip dominated regime, and finally to a regime with a quasi-constant advancing and receding angle. In the latter case, the hysteresis amplitude is found to be proportional to the defect force. We also consider an unusual heterogeneity for which the gradients of increasing and decreasing wettability are different. In such situation breaking the left/right symmetry, whether or not hysteresis is observed will depend on the side the liquid enters the microchannel.

I. INTRODUCTION

Key features of numerous technological innovations across a wide spectrum of applications and their associated industrial processes (coating of substrates is a well-known example) are the dynamics involved with liquid spreading on a solid surface, that is coverage of the surface by displacement, through wetting, of a gas (usually air) by a liquid – see for example the comprehensive reviews in Refs [1–3]. Contact angle hysteresis (CAH) is common place in wetting processes and it is characterized by a discontinuity between the receding and advancing contact angles [4, 5]. It is widely recognized that substrate heterogeneities are the principal source of CAH [5–7]. Such heterogeneities are either chemical or topographical or a combination of the two. Theoretical hydrodynamic analysis of contact line dynamics in the presence of chemical-topographical heterogeneities reveals complex dynamics, such as stick-slip motion and liquid motion due to strong heterogeneity gradients (e.g. uphill motion of a droplet) [8–15]), often in qualitative agreement with experiments. But despite the several developments and considerable attention that CAH has received for several decades, a large number of issues and problems have not been resolved. In particular, including rationally and systematically CAH in theoretical contact line models and/or numerical codes still eludes us [2, 16]. This is crucial for the control and optimization of technological processes exploiting CAH. Lack of fundamental understanding of CAH is due in large part to its inherently multiscale nature, a feature that plays a central role in the rapidly growing field of microfluidics [12, 17], as the largest typical lengthscales involved are below the capillary length, thus enabling capillary forces to dominate over body forces. For example, the motion of a gas-liquid interface in microsystems, and how it is affected by CAH and related issues such as pinning-depinning, still lacks fundamental understanding and rigorous characterization.

A number of theoretical efforts have shown that CAH is rooted in the statics of the system [6, 18]. In particular, it has been shown that CAH is related to the rugged free-energy landscape that is induced by the chemical imperfections of the solid, as the associated energy barriers
are able to trap the system in metastable configurations (defined as pairs of contact line position and contact angle) that depend on its history [18, 19]. This causes the moving contact line (CL) to display hysteresis, even in systems that are either translationally invariant in one direction and can be treated as two-dimensional (2D), in which case the CL is straight [20, 21], or in systems for which the CL has a constant curvature by axisymmetry [19].

An intrinsically three-dimensional (3D) approach was adopted in the theory of Joanny and de Gennes [6] which has formed the basis of many theoretical extensions as well as experimental investigations [2]. In their work, the profile of a CL, deformed by a single chemical defect on a substrate, is calculated such as to satisfy the balance of two forces acting on the CL: (i) the excess of capillary force exerted by the chemical defect, and (ii) the ‘elastic restoring force’ opposing the deformation of the CL. For a CL close to a ‘strong’ defect (i.e. exerting an excess capillary force above a certain threshold), this force balance is satisfied for two (meta)stable configurations, causing the system to display hysteresis.

In the present study we analyse the advancing and receding motion of the CL in a 2D microchannel with chemically heterogeneous walls. We compare our results with the main fundamental predictions of Joanny and de Gennes [6], including the existence of a threshold value of the heterogeneity strength, below which hysteresis is absent, and the case of a Gaussian defect, in which the threshold value shows a logarithmic dependence on the system size and heterogeneity wavelength, and vanishes in the limit of sharp edged (mesa) defects. It is important to remark, however, that here we are considering a completely different geometry to the 3D system studied in [6]. In particular, one of the goals of our work is to verify whether their theoretical predictions are still valid in our case. To this end, we only consider static hysteresis, neglecting the following effects: (i) thermal noise which can assist the system to overcome energy barriers [22–25] or external vibrations [19] which if strong enough may mitigate hysteresis [18], and (ii) dynamic effects which can also lead to the occurrence of hysteresis [26].

Another important, but still unresolved issue regarding CAH is the question of linking the properties of the substrate disorder to the hysteresis amplitude, which we denote as $H$. A decisive first stop on this question was taken in Ref. [6], where the case of a single strong defect was extended to a regime of sparsely spaced strong defects, and the following scaling law was proposed [2, 6, 27]

$$H = \gamma (\cos \theta_R - \cos \theta_A) \propto \frac{n F_{\text{max}}^2}{\gamma},$$  

(1)

where $F_{\text{max}}$ is the maximum excess of capillary force exerted by a single defect, and $n$ is the surface density of the defects. The quadratic dependence on $F_{\text{max}}$ and the proportionality with $n$ were verified experimentally and numerically [22, 28–30]. However, the extension of this expression to dense defects and to other types of heterogeneities is a challenging question that still remains open [29, 31].

Our aim here is precisely to address these issues and provide a quantitative description of CAH. For this purpose we adopt a 2D meniscus in a (2D) microchannel as a model system. To the best of our knowledge, this is the first attempt to address the existence of threshold and scaling laws of hysteresis in 2D systems.

In Section II A we introduce the model system, a liquid-gas meniscus in a microchannel with chemically disordered walls and we give the condition for mechanical equilibrium meniscus. Adopting the liquid volume as the control parameter, we offer an example in Section II B of how CAH can arise in the system. An alternative interpretation to this phenomenon is given in Section II C, which involves a graphical force balance, similar to [6]. This similarity is also present in some of the main results. In Section III A we show that weak heterogeneities do not produce hysteresis, and evaluate the hysteresis threshold for certain wettability distributions. In Section III B we explain the qualitatively different regimes of hysteresis upon varying the ratio between the heterogeneity wavelength and the microchannel gap size. We comment on the strong dependence of the threshold on the fluid configuration being considered, and argue that it does not exist for a drop fed by a sufficiently slow (quasi-static) liquid injection, for which all regimes of hysteresis are encountered while the drop grows. For heterogeneities with a strength sufficiently above the threshold, we explore in Section III C scaling laws between the parameters of the wettability distribution and $H$. Finally, conclusions and perspectives are offered in Section IV.

II. MODEL FORMULATION AND THEORETICAL FRAMEWORK

A. Definition of the system

We consider the 2D setup consisting of a liquid-gas meniscus moving inside a microchannel (Fig. 1). A gap of width $b^*$ separates two planar, rigid, topographically flat, but chemically heterogeneous walls. This chemical heterogeneity is characterized by a positional dependence of the interfacial tensions $\gamma_{sl}^* \text{ and } \gamma_{sg}^*$ between the walls and the liquid and the gas, respectively. The liquid-gas surface tension $\gamma^*$ is assumed constant. We thus impose an intrinsic contact angle $\theta_i$ distribution, identical on both walls, as:

$$\cos (\theta_i(x^*)) = \frac{\gamma_{sg}^*(x^*) - \gamma_{sl}^*(x^*)}{\gamma^*},$$  

(2)

where dimensional quantities are denoted with an asterisk.

We assume gravity to be negligible so that the static interface takes a constant curvature (circular arc). Any given pair of the contact line position and contact angle,
and the upper and bottom walls, respectively. As we limit meniscus length, and the subscripts $m$ is given, in dimensionless form, by:

$$V^* = x^* b^* + b^* f(\theta),$$

$$f(\theta) = \frac{2 \theta - \pi + \sin(2 \theta)}{8 \cos^2(\theta)}.$$  

The free energy $F^*$ of the system, per unit length of the contact line and up to a constant, consists of the three interfacial tensions, integrated along the length on which they act:

$$F^* = \gamma^* l^* + \int_0^{x^*} [\gamma_{sl,u}(x^*) - \gamma_{sg,u}(x^*)] dx^*$$

$$+ \int_0^{x^*} [\gamma_{sl,b}(x^*) - \gamma_{sg,b}(x^*)] dx^*,$$

where $l^* = (\pi - 2 \theta)/2 \cos \theta$ is the dimensional liquid-gas meniscus length, and the subscripts $u$ and $b$ represent the upper and bottom walls, respectively. As we limit our study to identical walls, the two integrals are equal, and

$$F^* = \gamma^* b^* \frac{\pi - 2 \theta}{2 \cos \theta} - 2 \gamma^* \int_0^{x^*} \cos(\theta_i(x^*)) dx^*.$$  

To find the extrema of $F^*$ at a fixed volume $V_0^*$, we minimize the function

$$G^* = F^* + \Lambda [x^* b^* + b^2 f(\theta) - V_0^*],$$

with respect to $x^*$ and $\theta$, where $\Lambda$ is a Lagrange multiplier. This calculation actually yields the natural condition

$$\cos(\theta) = \cos(\theta_i(x^*),$$

which states that at equilibrium, the meniscus angle must be equal to the local value of the Young’s angle, as expected. In the following, we refer to menisci satisfying Eq. (7) as being at equilibrium configurations. The volume of such equilibrium configurations is given, in dimensionless form, by:

$$V = \frac{V^*}{b^*} = x + f(\theta_i(x)).$$

where we have used $b^*$ as characteristic length scale. Similarly, it is convenient to scale the free energy as:

$$F = \frac{F^*}{b^* \gamma^*} = \frac{\pi - 2 \theta}{2 \cos \theta} - 2 \int_0^{x^*} \cos(\theta_i(x)) dx.$$  

**B. Occurrence of contact angle hysteresis**

For homogeneous walls ($\theta_i(x) = \theta_0$), $V$ increases linearly with $x$ [Eq. 8]. For non-constant $\theta_i(x)$, a nonlinear variation of $V$ with $x$ occurs, but $V$ might still be monotonously increasing with $x$ if variations of $\theta_i$ are weak. For sufficiently disordered walls (the condition will be derived in Section III A), $V(x)$ increases non-monotonously, enabling the occurrence of multiple equilibrium configurations for a given volume. In Fig. 2 we consider a model microchannel with periodically varying wetting properties, illustrating the latter case. Specifically, we take $\cos(\theta_i(x)) = 0.6 \sin(\frac{\pi}{\sqrt{2}} x)$.

For $V = 0.2$, the contact line will adopt one out of 7 equilibrium configurations, depending on the history of the system. This odd number of configurations corresponds alternatingly to minima (metastable configurations) and maxima (unstable configurations) of the free-energy curve, with the outermost configurations being metastable [see Figure 2 (c)]. These four metastable menisci are drawn in 2 (d). The difference between maxima and neighboring minima can be interpreted as energy barriers which trap the CL in a given metastable configuration [18, 32]. The magnitudes of these energy barriers are smallest for the metastable configurations with minimal and maximal $\theta$, in agreement with previous results on an axisymmetric drop [18, 32].

Therefore, by imposing that the volume is increasing or decreasing, we obtain the advancing and receding trajectories of the meniscus, respectively, shown in green and red lines in Fig. 2 (a). These trajectories are found with the assumption that the system has no ability to overcome energy barriers. For certain values of $V$, the curve corresponding to equilibrium configurations [black and gray curve in Fig. 2 (a)] has a limit (critical) point. At these values of the volume, say $V_c$, a saddle-node bifurcation takes place, where the local minimum of the free energy $F$ (black) collides with a neighboring maximum (gray) and annihilate each other. At these points, the CL suddenly jumps to the closest available metastable state giving rise to slip motion.

The corresponding trajectory followed by $\theta$ [Fig. 2 (b)] displays the typical experimentally observed sawtooth shapes [33, 34] associated with the stick-slip mechanism [11, 15, 20]. As expected, the advancing contact angle ($\theta_A$) is larger than the receding contact angle ($\theta_R$).

To quantify the global differences between $\theta_R$ and $\theta_A$,
we define the following quantities:

\[
\langle \cos \theta_R \rangle = \frac{1}{\Delta V} \int \cos \theta_R \, dV, \tag{10}
\]

\[
\langle \cos \theta_A \rangle = \frac{1}{\Delta V} \int \cos \theta_A \, dV, \tag{11}
\]

\[
H = \frac{H^*}{\gamma_s} = \langle \cos \theta_R \rangle - \langle \cos \theta_A \rangle, \tag{12}
\]

where \( \cos \theta_R \) and \( \cos \theta_A \) correspond to the solutions tracked upon respectively decreasing and increasing \( V \), and where the choice of \( \Delta V \) depends on the situation considered. For example, in the case of a periodic heterogeneity, it corresponds to an integer amount of wavelengths. In addition, we will use the Cassie-Baxter contact angle \( \theta_{CB} \), which is given by the average value of \( \cos \theta_i(x) \) over the whole microchannel with length \( L_x \) (or over a period of the heterogeneity \( \theta_i(x) \) in the periodic case).

\[
\cos \theta_{CB} = \frac{1}{L_x} \int \cos \theta_i \, dx, \tag{13}
\]

C. Graphical force balance

The phenomenon of CAH can also be interpreted by looking at the interplay between the two forces acting on the horizontal direction of the contact line, namely the spring-type elastic response of the contact line and the force due to the chemical heterogeneities [6]. However, it is important to note that the contact line in our 2D system is straight in the spanwise direction and so the spring-like behaviour is due to deformations of the meniscus in the cross-stream direction, as shown below, and not to corrugations of the contact line in the spanwise direction as in [6]. We will see here that the occurrence of hysteresis generically is connected to the multiplicity of solutions \( (x, \theta) \) for the balance between these forces at a fixed value of \( V \).

The presence of the solid implies a force \( f_s^* \) (per unit length of the contact line) in the positive x-direction:

\[
f_s^* = 2 (\gamma_{sg}^* - \gamma_{sl}^*) = 2 \gamma^* \cos \theta_i = \gamma^* f_s. \tag{14}
\]

At equilibrium, this force is balanced by the restoring force \( f_r^* \) due to the deformation of the meniscus

\[
f_r^* = -2\gamma^* \cos \theta = \gamma^* f_r. \tag{15}
\]

Clearly, equilibrium implies \( f_s^* + f_r^* = 0 \), i.e. \( \cos \theta = \cos \theta_i(x) \) as obtained in Eq. (7). Let us now define the deformation of the meniscus by

\[
\Delta x = x_V - x, \tag{16}
\]

where \( x_V \) is the mean position of the meniscus, defined as the position of a straight liquid/gas interface with the same liquid volume. Hence, \( \Delta x \) is geometrically related to \( \theta \) by:

\[
\Delta x = f(\theta), \tag{17}
\]

where \( f(\theta) \) is given by Eq. (3). The restoring force \( f_r = -2\cos \theta \) is an increasing function of the deformation \( \Delta x \), and can be written as:

\[
f_r = k(\theta) \Delta x. \tag{18}
\]

Therefore, we can interpret the meniscus as a spring with stiffness \( k \) that depends only on \( \theta \) (see Fig. 3), and is given by

\[
k(\theta) = \frac{-2\cos \theta}{f(\theta)} = \frac{16 \cos^3 \theta}{\pi - 2\theta - \sin(2\theta)}. \tag{19}
\]

Combining Eq. (14) and (18), the force balance then reads

\[
k(\theta) (x - x_V) = 2 \cos \theta_i(x). \tag{20}
\]

Figure 4 illustrates this force balance for the case of a single Gaussian-shaped defect:

\[
\cos \theta_i(x) = \cos \theta_0 + A_0 \exp \left(-\frac{(x - x_d)^2}{2\sigma^2}\right). \tag{21}
\]
The complete qualitative agreement of Fig. 4 with the graphical construction proposed by Joanny and de Gennes [6], despite the different nature of the spring constant, not only shows the robustness of this view, but also that the occurrence of hysteresis is generically associated with the existence of multiple static configurations upon varying an experimentally controllable parameter such as the liquid volume. We note, however, that in general, the curves representing $f_r$ are not just straight lines as the stiffness depends on the actual contact angle ($\theta = \theta_i(x)$ in Eq. 20). By assuming a weak heterogeneity (i.e. $\cos \theta_i(x) = \cos \theta_0 + \Delta \cos \theta_i(x)$ with $|\Delta \cos \theta_i(x)| \ll 1$), the force balance may be linearized by noting that $x-x_V = -f(\theta_i(x)) \approx -f(\theta_0) + f'(\theta_0)\Delta \cos \theta_i(x)/\sin(\theta_0)$, which is recast as

$$\frac{\sin(\theta_0)}{f'(\theta_0)}(x-x_V) = \Delta \cos \theta_i(x), \quad (22)$$

with

$$\ddot{x}_V = x_V - f(\theta_i). \quad (23)$$

Depending on the particular situation being considered, such a linearized force balance may be sufficient to describe the hysteresis behaviour.

### III. RESULTS

#### A. Hysteresis threshold and phase-plane plots

As is evident from Fig. 2, to have multiplicity of equilibrium configurations for a given volume, there must be at least one critical contact line position $x_c$, where a saddle-node bifurcation takes place. At one of these points, the meniscus jumps either forward in an advancing path or backwards in a receding path, as the volume varies. We note that alternatively, we can see these critical points as maxima and minima, respectively, of $V(x)$. Making use of Eq. (3) we can hence write:

$$V'(x_c) = 1 + \frac{\partial f(\theta_i)}{\partial \theta_i} \bigg|_{x=x_c} = 0, \quad (24)$$

where $V'(x_c) = \frac{dV}{dx} |_{x=x_c}$. Denoting $\hat{\theta}_i = \theta_i(x = x_c)$ and rearranging the above equation, we get

$$\hat{\theta}_i = -\frac{1}{\frac{\partial f(\theta_i)}{\partial \theta_i}} = \frac{2 \cos^2 \theta_i}{(\frac{\pi}{2} - \theta_i) \tan \theta_i - 1}. \quad (25)$$

Therefore, at critical points, the wettability gradient $\hat{\theta}_i$ exhibits a generic behaviour that does not depend on the particular form of $\theta_i(x)$ but only on the given value of $\theta_i$. The points where the curve of $\theta_i'(x)$ versus $\theta_i(x)$ crosses the generic curve $\theta_i(x_c)$ correspond to the critical values of $x$ at which jumps occur (either forward or backward). Thus, for a given distribution $\theta_i(x)$, visualizing both curves in a phase-plane plot will enable predicting whether and where the CL will jump. As $\theta_i$ is physically limited between 0 and $\pi$, we have that $\hat{\theta}_i(\theta_i)$ is limited between -2 (for $\theta = 0$ and $\pi$) and -6 (for $\theta = \frac{\pi}{2}$). Importantly, the negative sign of $\hat{\theta}_i$ implies that an advancing (receding) meniscus can only start jumping from a position at which the wettability is increasing with increasing $x$ (decreasing with decreasing $x$).

Another point to remark is that hysteresis occurs only if the strength of a (smooth) heterogeneity is above a finite threshold (a similar observation was reported in [6], albeit in a different physical system). If the condition $\theta_i'(\theta_i) = \hat{\theta}_i(\theta_i)$ is not satisfied for any $x$, then no stick-slip occurs and the same path of static solutions is followed during advancing and receding. These behaviours are illustrated in the following.
1. Smooth heterogeneities

We consider two examples for $\theta_i(x)$, namely a single Gaussian-shaped defect with amplitude $A_0$, width $\sigma$, and located at $x_d$:

$$\theta_i(x) = \theta_0 + A_0 \exp \left( -\frac{(x - x_d)^2}{2\sigma^2} \right),$$ (26)

and a periodic heterogeneity with amplitude $A_0$ and wavelength $\lambda$:

$$\theta_i(x) = \theta_0 + A_0 \sin \left( \frac{2\pi x}{\lambda} \right).$$ (27)

As just remarked, if condition (25) cannot be satisfied, a multiplicity of static solutions and hysteresis cannot occur. This is the case when $\theta_i'(x)$ is not negative enough, which is certainly true if $\min(\theta_i'(x)) > -2$. For the Gaussian defect (26) this condition leads to the relation $\frac{|A_0|}{\sigma} < 2\sqrt{e}$, and for the sinusoidal heterogeneity to $\frac{|A_0|}{\lambda} < 1/\pi$.

On the other hand, condition (25) is certainly satisfied if $\min(\theta_i'(x)) < -6$, which leads to the conditions $\frac{|A_0|}{\sigma} > 6\sqrt{e}$ for the Gaussian defect, and $\frac{|A_0|}{\lambda} > 3/\pi$ for the sinusoidal heterogeneity (27). For values of $\frac{|A_0|}{\sigma} \in (2\sqrt{e}, 6\sqrt{e})$ (or $\frac{|A_0|}{\lambda} \in (1/\pi, 3/\pi)$) hysteresis may or may not occur, depending on the background contact angle $\theta_0$. In that respect that condition 25 is more difficult to satisfy for a CA of $\pi/2$, because this is the angle around which $\theta_{ic}$ is most negative.

We hence conclude that the quantity $\frac{|A_0|}{\sigma}$ (or $\frac{|A_0|}{\lambda}$) determines whether hysteresis occurs or not, i.e. it is the gradient of wettability that matters. In particular, for a given defect amplitude $A_0$, hysteresis will always be observed at sufficiently small defect size (compared to the microchannel gap).

Fig. 5 depicts a comparison between two cases that are below and above the threshold for a single Gaussian defect [see panel (a)]. As is evident from the phase plot [Fig. 5 (b)], the system starts at the origin, and moves clockwise along a closed trajectory for an advancing CL (increasing $x$), or anticlockwise for a receding CL (decreasing $x$). If the defect is weak (dashed line), the phase plot does not intersect the generic line $\hat{\theta}_{ic}(\theta_i)$. On the other hand, for the stronger defect (full line), there are two intersections on the phase plot, which correspond to the minimum (left) and maximum (right) of $V(x)$ (cf. 5(c)). These are the points at which the receding and advancing jump start, respectively. The ending points of the jumps are found with the assumption of the constant volume, corresponding to a case where the externally imposed volume variation is of negligible rate compared to the velocity of the jumps. For values of the volume in between the values corresponding to the jumps, multiple equilibrium states exist and the system will choose one depending on whether the volume has been increasing or decreasing [Fig. 5 (c)]. More specifically, the advancing CL follows the branch corresponding to lower $x$, while the receding CL follows the branch corresponding to higher $x$. The middle branch is unstable as it corresponds to maxima of the free energy (as in Fig. 2). The contact angles corresponding to the two branches are shown in Figure 5 (d), revealing that when increasing $V$ (green), the CL first sticks on the hydrophobic defect, causing $\theta$ to increase progressively, and then slips, corresponding to a sudden decrease of $\theta$. When decreasing $V$ (red), the CL almost directly slips over the hydrophobic defect, corresponding to a sudden increase of $\theta$, and then sticks, causing $\theta$ to decrease progressively. Both for the hydrophobic case considered here, and for a hydrophilic defect, jumps correspond to a decrease of $\theta$ when increasing $V$, and to an increase of $\theta$ when decreasing $V$.

For the periodic heterogeneity given by Equation (27), we observe a similar behaviour in terms of the threshold-dependence of hysteresis. We will consider this case in more detail in Section III B.

2. Sharp heterogeneities

We now consider sharp-edged defects, or mesa defects adopting the term coined by Joanny and de Gennes [6]. In the limit case of symmetric mesa defects – i.e. defects involving both a step-like increase and decrease in wettability, as shown in Fig. 6 - $\theta_i'(x)$ will reach both plus and minus infinity, crossing the generic curve $\hat{\theta}_{ic}$ in between. Hence such defects, featuring infinitely large wettability gradients, always generate hysteresis, as was predicted by Joanny and de Gennes [6]. Asymmetric heterogeneities, however, of which the intrinsic CA decreases too slowly with increasing $x$ ($\min(\theta_i'(x)) > -2$), do not generate hysteresis, even if the increases of CA are step-like. We illustrate this with the example of a triangular heterogeneity with amplitude $A_0$ and wavelength $\lambda$ given by:

$$\theta_i(x) = \theta_0 + \frac{2A_0}{\pi} \arctan \left( \cot \left( \frac{\pi x}{\lambda} \right) \right).$$ (28)

and shown in Fig. 7(a). Any negative value of $A_0$ implies step-like decreases of the intrinsic CA with increasing $x$, which give rise to hysteresis. For a positive value of $A_0$ however, no hysteresis will occur if $\min(\theta_i'(x)) > -2 \equiv -\frac{2A_0}{\pi} > -2$, or $0 < A_0 < \lambda$. Figure 7 is an example of such a case. On the contrary, the same heterogeneity with an inverted sign of $A_0$ does generate hysteresis (Fig. 8). For this heterogeneity, inverting the sign of $A_0$ is equivalent to inverting the sign of the coordinate $x$. Therefore the situation in Fig. 8 can be obtained in exactly the same microchannel as in Fig. 7, by simply placing the liquid inlet at the right-hand side. Hence, for a microchannel with this particular type of heterogeneity, whether or not the system will suffer hysteresis and stick-slip, will depend on the side from which the liquid enters.
This analysis shows that the existence and nature of the threshold does not only depend on the wetting properties of the substrate $\theta_i(x)$, but also on the particular liquid configuration (in Appendix A we will contrast our predictions with the case of a droplet). However, recall that all lengths, and hence the typical length scale of the heterogeneity, are normalised with the gap width separating the two plates of the microchannel. As a result the weak Gaussian-shaped defect corresponding to the dashed lines in Fig. 5 would be a strong defect in Fig. 8 with $A_0 = 0.09$ and $\lambda = 0.1$.)

(a) Intrinsic contact angle distribution; (b) Phase-plane plot, the thick line represents $\theta_{ic}$; (c) CL position as a function of volume for (black: all static solutions, green: followed during advancing, red: followed during receding); (d) Contact angle corresponding to these paths.
a microchannel with a gap size that is five times larger. This points to the importance of the overall length scales, something that deserves more attention and was not considered in the work by Joanny and de Gennes [6], nor was it discussed in the more recent review by Bonn et al. [2].

B. From large to small heterogeneity length scale: the macroscopic limit.

Here we discuss qualitatively the effects on hysteresis of the ratio $\lambda$ between the characteristic lengthscale of the heterogeneity and the gap size of the microchannel (which is constraining the liquid-vapor interface in this configuration), by looking at a periodic distribution of the spreading coefficient of the walls as a model prototype

$$\cos(\theta(x)) = A_0 \sin \left( \frac{2\pi}{\lambda} x \right)$$

This heterogeneity is different to that given by Eq. 27, because we had imposed for simplicity a distribution for $\theta(x)$, as opposed to the spreading coefficient which is proportional to $\cos(\theta(x))$ and which gave simple analytical result for the value of the threshold, whereas here we want to obtain a scaling law for the hysteresis force. The results for this case are presented in Fig. 9. For a given value of $A_0$ and when $\lambda$ is sufficiently high (see first column of Fig. 9) the system is in a sub-threshold regime (i.e. the gradient $|A_0|/\lambda$ is too small). Although the contact angle varies with volume, hysteresis is absent. In particular, we observe that for each volume we find a unique static configuration $(x, \theta)$, the phase-plane plot of the heterogeneity does not intersect the generic curve $\hat{\theta}_c$, and the advancing and receding CL follow the same path, in clockwise and anticlockwise directions, respectively. The CA takes all possible values imposed in the distribution 29 in both the advancing and receding directions. Hence in this regime, advancing CAs lower than the Cassie-Baxter angle (here equal to 90°) are possible, and receding CAs higher than the Cassie-Baxter one, are possible.

If we decrease $\lambda$ by a factor three (second column of Fig. 9) by, e.g., increasing the gap width by the same factor, the system is then in a regime in which hysteresis occurs. We find that for most volumes there exists a unique static configuration $(x, \theta)$, but in certain ranges three static solutions are found. At these points, the branch followed by the system will depend on whether the CL advances or recedes. The contact angle of an advancing CL is on average higher. At the intersections between the phase plot of the heterogeneity and the generic curve $\hat{\theta}_c$, the contact line jumps (slips) to a point which has the same volume if we assume that such jumps occur instantaneously compared to the rate of volume variation. In the phase-plane plot, an advancing CL follows a different path than a receding CL.

A further decrease in $\lambda$ (third column of Fig. 9) increases the number of static solutions $(x, \theta)$ for a given volume. The path followed by an advancing CL is now totally different than the one followed by a receding CL. When $\lambda$ decreases, gradually, jumps get smaller and occur more frequently. The contact angles follow sawtooth shapes which get further separated. The advancing CAs are at all times higher than the Cassie-Baxter angle, while the receding CAs are lower. The bows followed in the phase diagram are separated further.

In the limit $\lambda \to 0$, which we refer to hereafter as the macroscopic limit, the number of static solutions $(x, \theta)$ at any given volume starts becoming arbitrarily high. This situation is represented for $\lambda = 0.001$ in the fourth column of Fig. 9. The CL advances with a nearly constant contact angle, corresponding in this case to the maximum of the imposed distribution, and recedes with a nearly constant contact angle (which is the minimum of the imposed distribution). The sawtooth amplitude of the contact angle is below 0.3° and hysteresis is maximum.

In the above analysis, we have assumed that the system has no ability to overcome energy barriers, which are also affected by varying $\lambda$. A lower value of this parameter increases the amount of extrema in the energy landscape for a fixed volume (Fig. 10, (a) and (b)). We define the maximum energy barrier $E_{BM}$ which needs to be overcome in order to attain the most stable contact angle $\theta_{MS}$, as indicated on Fig. 10(b). $E_{BM}$ is found to decrease with $\lambda^{-1}$ and reaches a value of the order $10^{-4}$ for $\lambda = 0.001$ (Fig. 10(c)). The most stable angle (Figure 10 (d)), approaches the Cassie-Baxter angle (90°). This result was previously reported in Refs [35, 36] for a configuration of a drop much larger than the heterogeneity wavelength.

C. Scaling laws of the hysteresis force

1. Near threshold behavior

Let $A_{0c}$ be the threshold value for the amplitude of the chemical defect. For $A_0 = A_{0c}$, at a certain contact line position, say $x_0$, we have that the volume satisfies:

$$V'(x_0) = V''(x_0) = 0.$$  (30)

Consider the deviations from this point for the quantities:

$$\left\{ \begin{array}{l} A_0 = A_{0c} + \mu, \\
 x = x_0 + \bar{x}. \end{array} \right.$$  (31)

By using (30), we expand the volume for $\mu \ll 1$ and $\bar{x} \ll 1$ as

$$V = x + f(\theta_0(x, A_0)) \approx V_0 + c_1\mu + c_2\mu\bar{x} + c_3\bar{x}^3.$$  (32)

The extrema of $V$ are then given by

$$\frac{dV}{dx} = 0 = c_2\mu + 3c_3\bar{x}^2,$$  (33)
FIG. 9. Qualitatively different regimes of hysteresis for the microchannel (assuming the system has no ability to overcome any energy barriers) caused by a model periodic heterogeneity given in Eq. (29) with $A_0 = 0.2$. The columns represent different values of $\lambda$, from left to right: 0.3, 0.1, 0.02 and 0.001. On the upper row: Black: equilibrium configurations, green: presumed advancing path, red: presumed receding path; on the middle row: the contact angle of the equilibrium configurations encountered in the advancing (green) and receding (red) path. Lower row: phase plots of the distribution in black, the thick line represents $\hat{\theta}_{ic}$, for an advancing contact line (green), the system moves in a clockwise fashion, while for a receding contact line (red), the motion is counterclockwise, as indicated with the large arrows. The small arrows (with the same color code) indicate jumps.

the solutions of which, assuming $c_2c_3 < 0$, are

$$\tilde{x}_\pm = \pm \left( -\frac{c_2\mu}{3c_3} \right)^{1/2} \propto \mu^{1/2}. \quad (34)$$

Inserting the above values into Eq. 32, we obtain:

$$V_{\text{max}} - V_{\text{min}} \propto \mu^{3/2}, \quad (35)$$

where $V_{\text{max}} = V(x_+, A_0)$ is the relative maximum of the volume at which the CL jumps in an advancing path, and $V_{\text{min}} = V(x_-, A_0)$ is the relative minimum of the volume, at which the CL jumps in a receding path. The magnitude of the jumps of the force scales as

$$\delta f \propto \cos(\theta_i(x_-, A_0)) - \cos(\theta_i(x_+, A_0)) \propto \theta_i(x_+, A_0) - \theta_i(x_-, A_0) \propto x_+ - x_- \propto \mu^{1/2}. \quad (36)$$

According to definitions (10)-(12), the hysteresis $H$ is directly related to the area between the curves of $\cos(\theta_R)$ and $\cos(\theta_A)$ versus $V$. Combining (35) and (36) we thus find that close enough to the threshold, the hysteresis amplitude scales as

$$H \propto \mu^2 \propto (\delta f)^4. \quad (37)$$

In the following, we explore numerically the emergence of scaling laws. We first focus on the case of hysteresis caused by an array of wetting defects, where we focus on the dilute limit at which we have a minimum number of equilibrium configurations for a given volume.

2. Dilute system of wetting defects

Consider a single Gaussian-shaped defect:

$$\cos \theta_i(x) = \cos \theta_0 + A_0 \exp \left( -\frac{(x - \frac{L}{2})^2}{2\sigma^2} \right), \quad (38)$$

where we fix $\theta_0 = \frac{\pi}{2}$. We assume $A_0 > 0$, i.e. a defect which is more wettable than the rest of the surface,
and vary the amplitude \( A_0 \) over 400 values, logarithmically spaced in the interval between 0.001 and 0.99, for four values of the defect width \( \sigma \). The computational domain \( L_x \) is fixed to a dimensionless length of unity and discretized in sections of \( \Delta x = 10^{-5} \). With this length the whole pinning-depinning process is captured for all calculated defects. For each value of \( \sigma \), a threshold amplitude \( A_{0c} \) is calculated numerically as the minimum \( A_0 \) for which the phase plot of the heterogeneity intersects the generic line (Section III A).

Fig. 11(a) shows that hysteresis depends only slightly on the width of the defect. The effect of defect amplitude \( A_0 \) on the hysteresis is twofold. A larger \( A_0 \) does not only imply a larger difference between the CA on the defect and the background angle, but also larger jumps (see the graphical force balance in Fig. 4), hence a larger range of \( V \) during which the CA differs from the background angle. Near the threshold (see Eq. 37) but also far from it, the hysteresis is found to scale as:

\[
H \propto (A_0 - A_{0c})^2,
\]

where \( A_{0c} \) is the threshold amplitude depending on \( \sigma \), which is 0.04945 and 0.00495 for the broadest and sharpest defect, respectively. As the maximum excess force \( F_{s,\text{max}} \) exerted on the contact line by the defect is proportional to \( \gamma^* A_0 \), we may write for the dimensional hysteresis force \( H^* \)

\[
H^* = \gamma^* H \propto \frac{F_{s,\text{max}}^*}{\gamma^*},
\]

where \( A_{0c} \) has been assumed negligibly small against \( A_0 \), e.g. in the macroscopic limit \( \sigma \to 0 \). This scaling law is in agreement with previous theoretical and numerical predictions [6, 30] and experiments [28], though in different configurations to the one studied here.

It is important to emphasise that in fact, the hysteresis caused by the wetting defect is mainly due to the increase of \( \langle \cos \theta_R \rangle \). When \( A_0 \) increases, while \( \langle \cos \theta_A \rangle \) stays nearly constant (see also the graphical force balance in Fig. 4). This is consistent with experimental findings [22, 34, 37] on surfaces with controlled chemical defects and can be understood by the fact that a wetting defect primarily affects [4] or more specifically pins [34] the CL in the receding direction. Fig. 11(b) shows that close enough to the threshold, the hysteresis amplitude scales with the fluctuation of the force caused by a jump to the power four, also in agreement with Section III C 1. Further from threshold, no clear scaling behaviour can be observed.

We note that as we quantify averaged hysteresis, the absolute value of \( H \) depends on the domain size. An increasing system size indeed enhances the dilution of the effects of the defect by the homogeneous background. This can be avoided by considering an array of such defects. We thus impose an array of \( N \) equally spaced and identical wetting defects.

\[
\cos \theta_i(x) = \cos \theta_0 + A_0 \sum_{i=1}^{N} \exp \left( -\frac{(x - x_{di})^2}{2\sigma^2} \right),
\]

where \( x_{di} = i \frac{L_x}{N+1} \). Fig. 12 shows that for low enough \( N \), the hysteresis force is directly proportional to \( N \), which is in agreement with experiments [29] and the proposed scaling law for the hysteresis caused by a dilute system.
The computational domain has a length of 20 and 11 start and end with a value at the CL jumps. Over which the contact angles are integrated (in Eq. 10) start and end with a value at the CL jumps.

The effects of the individual defects are simply additive as long as they are independent (non-overlapping). As the density of defects $N$ is increased however, this proportionality no longer holds as the stick-slip of the CL on a defect is affected by the neighboring defects. More specifically, the jumps of the CL do not end anymore on homogeneous parts between defects, but on other defects. This shadowing effect increases with $N$, decreasing the slope on the plot. For $N$ above 150, the defects start overlapping and hysteresis even starts decreasing as the substrate becomes less heterogeneous, though with a shifted background value.

3. Periodic heterogeneity

In our last example, we consider a periodic heterogeneity of the form:

$$\cos(\theta_i(x)) = A_0 \sin \left( \frac{2\pi x}{\lambda} \right).$$

(43)

The threshold amplitude $A_0c$ is 0.302 for the highest $\lambda$ and $9.56 \times 10^{-5}$ for the lowest, satisfying $A_0c/\lambda$ constant. We find that the scaling exponent depends on the wavelength $\lambda$ as follows. For the lowest values of $\lambda$ the hysteresis force is proportional to the defect force. For broader heterogeneities, the scaling exponent increases with $\lambda$ and reaches a value of around two when $\lambda$ is of the order of one (see Fig. 13). We then write:

$$H^* \propto F_{s,max}^*,$$

(44)

for $\lambda \to 0$. The linear behavior corresponds exactly to the macroscopic limit ($\lambda = \frac{\lambda^*}{V} \to 0$), in which the advancing and receding angle are nearly constant and equal to the minimum and maximum of the imposed distribution ($H \to 2A_0$), respectively. Also, for this infinitely sharp heterogeneity, the threshold amplitude vanishes ($A_0c \to 0$). This limit is plotted in black dashed lines and collapses with the data for sufficiently low $\lambda$, and at sufficiently high $A_0 - A_0c$ (although $A_0$ has an upper bound $A_0 = 1$, see Equation 43. This suggests that the value of $\lambda$ below which this limit is reached depends on $A_0$.

Away from this limit, i.e. for higher $\lambda$), the system is in a regime of observable stick-slip (see Section III B) where the difference between the advancing and receding angle depends on $V$. The exponent larger than one found for this regime is due to the same effect as in the case of a single defect (section III C 2); an increase of the amplitude of the heterogeneity $A_0$ increases both the dif-

FIG. 12. Logarithmic plot of the dimensionless hysteresis force $H = <\cos \theta_R> - <\cos \theta_A>$ for an array of equally spaced, identical wetting defects with width $\sigma = 0.01$ as a function of the amount of defects for six values of $A_0$: 0.075 ($\diamond$), 0.1 ($\triangle$), 0.125 ($\bigcirc$), 0.15 ($\bigtriangleup$), 0.175 ($\bigstar$) and 0.2 ($\ast$).

FIG. 13. Logarithmic plot of the dimensionless hysteresis force $H = <\cos \theta_R> - <\cos \theta_A>$ as a function of the heterogeneity amplitude for a periodic heterogeneity with $\theta_0 = \pi/2$ and 8 values of $\lambda$. Black symbols correspond to small wavelengths: $\lambda = 1 \cdot 10^{-4}$ ($\ast$), $8 \cdot 10^{-4}$ ($\bigstar$), $5.80 \cdot 10^{-3}$ ($\bigcirc$), 4.44, $10^{-2}$ and ($\bigcirc$). Gray symbols correspond to large wavelengths: $\lambda = 0.1$ ($\ast$), 0.178 ($\bigstar$), 0.316 ($\bigcirc$) and 0.562 ($\bigcirc$). The black dashed line (collapsing with data) corresponds to the macroscopic limit. The gray dashed line with slope two is drawn as visual aid.
ference between the advancing and receding angle at the value of $V$ at which the jumps occur, and the range of $V$ during which these angles differ.

IV. CONCLUSIONS AND PERSPECTIVES

We have analyzed in detail stick-slip motion and CAH of a liquid-gas meniscus in a chemically heterogeneous microchannel of fixed gap width by tracking stable and metastable states upon varying the liquid volume and for a wide range of heterogeneity distributions. To the best of our knowledge, this is the first time that such a study has been undertaken. Our goal has also been to examine to what extent previous results of wetting hysteresis hold in the present configuration.

We showed that the stick-slip mechanism can be studied using phase-plane plots of the heterogeneity. Weak heterogeneities do not produce hysteresis, a threshold for the heterogeneity strength is required. For values of the heterogeneity strength sufficiently above the threshold, we find good agreement with the previously reported expressions for the hysteresis amplitude for a diluted system of defects, even though the geometry and approach used here are notably different. For a model sinusoidal heterogeneity, we have found that the ratio between the heterogeneity wavelength and the liquid-gas interface length is a key parameter determining the qualitative aspects of the hysteresis. More specifically, decreasing gradually this ratio brings the system from a sub-threshold regime (i.e. no hysteresis), to a stick-slip dominated regime, and ultimately to a regime with a nearly constant advancing and receding angles. In the latter regime, which we referred to as the macroscopic limit, the hysteresis amplitude is shown to be proportional to the defect force.

Finally there are a number of interesting questions closely related to the analysis presented here. For example, extension of the static approach to investigate a microchannel configuration with non-identical upper and lower walls, but also account for the influence of thermal noise, especially in the macroscopic limit, where the energy barriers which keep the system in metastable states are relatively low, something that is also relevant in the context of nano-defects [25]. Moreover, noise can induce dynamic/state transitions in rugged energy landscapes and further hysteresis effects (noise-induced hysteresis) [38, 39]. Of particular interest will also be the corresponding dynamic study, addressing the question of how the hysteresis is amplified by the absolute velocity of the CL in each of the qualitatively differently different regimes of hysteresis. Extension of the present analysis to 3D configurations, though possible in principle, would require considering an infinite-dimensional generalization, which most likely will have to be treated numerically.

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Appendices

Appendix A: Other configurations

Here we briefly contrast the problem of a liquid-gas meniscus in a microchannel with that of a cylindrical droplet. Previous works on the statics of this configuration [7, 20, 40, 41] have predicted hysteresis and stick-slip motion upon varying the volume of the droplet, similarly to the present work. Recently, this behavior has also been observed when the droplet is fed through a pore [15].

In the cylindrical droplet configuration, the volume $V$ of the droplet is, as for our microchannel, an explicit function of the contact line position $x$: $V = x^2g(\theta)$, with $g(\theta)$ a geometric factor. A similar analysis as in Section IIIA would give as a criterion for a jump to occur the expression,

$$\left\{ \begin{array}{l}
\frac{\partial \theta}{\partial x} = \hat{\theta}_i \\
\hat{\theta}_i = \frac{-2g(\theta_i)}{g'(\theta_i)}
\end{array} \right.,$$

where $\frac{g(\theta_i)}{g'(\theta_i)}$ is positive for the physical range of $\theta_i$. Thus, for this configuration $\hat{\theta}_i$ is an explicit function of $x$, which was not the case for our microchannel. Whether a defect is strong or weak, will therefore depend on how far it is from the center of the droplet. Any defect which is far enough from the origin ($\hat{\theta}_i$ close enough to zero) will induce hysteresis and therefore no threshold exists for sufficiently large droplets, just like in the macroscopic limit $\lambda \to 0$ for the microchannel.

Furthermore, Fig. 14 shows that, for a given periodic intrinsic contact angle distribution, upon increasing the volume of the droplet, the system passes through the same qualitatively different regimes of hysteresis (introduced in Section III B) as for the microchannel. We neglect any external noise, which would enable the system to cross energy barriers, and we assume that the droplet center does not shift during the process (see Ref. [15] for details). Upon increasing $V$ (from left to right), the system goes from a sub-threshold regime, through a regime with pronounced stick-slip, and finally to the macroscopic limit, for droplets much larger than the length scale of the heterogeneity. In both this configuration and that of the microchannel, the ratio of the gas-liquid interface
length to the typical length scale of the heterogeneity seems to be an important physical parameter which determines the hysteresis behavior. In particular, a general conclusion seems to be that the advancing and receding contact angles become independent of the particular liquid configuration (drop, meniscus, bubble, ...) when this ratio becomes sufficiently large. In such macroscopic limit, the hysteresis range is fixed by the maximum and the minimum of the contact angle distribution, with the Cassie-Baxter angle falling in between, and stick-slip behavior is unobservable. This general conclusion supports the use of the common basic way of introducing CAH into static or time-dependent macroscopic models [3] by prescribing given constant values for the advancing and receding angles.

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FIG. 14. Qualitatively different regimes of hysteresis for a droplet (assuming the system has no ability to overcome any energy barriers) caused by a periodic heterogeneity described by Eq. 29 with $A_0 = 0.2$ and $\lambda = 0.1$. The columns represent zooms on different ranges of $V$ with qualitatively different behavior. The plotted quantities are the same as in Figure 9.


