The convex hull of finitely generable subsets and its predicate transformer

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Abstract—We consider the domain of non-empty convex and compact subsets of a finite dimensional Euclidean space to represent partial or imprecise points in Computational Geometry. The convex hull map on such imprecise points is given domain-theoretically by an inner and an outer convex hull. We provide a practical algorithm to compute the inner convex hull when there are a finite number of convex polytopes as partial points. A notion of pre-inner support function is introduced, whose convex hull gives the support function of the inner convex hull in a general setting. We then show that the convex hull map is Scott continuous and can be extended to finitely generable subsets, represented by the Plotkin power domain of the underlying domain. This in particular allows us to compute the convex hull of attractors of iterated function systems in fractal geometry. Finally, we derive a program logic for the convex hull map in the sense of the weakest pre-condition for a given post-condition.

1. Introduction

Convex hull computation is arguably the most important problem in computational geometry with a wide range of applications in many subjects including computer graphics, CAD and robotics. It has a huge literature and has been studied under different conditions and situations. The robustness of an algorithm in computational geometry is a fundamental property particularly when it is applied in critical situations. The impact of floating point system on numerical errors and robustness in algorithms and computations is a well known problem.

Here we study a version of the convex hull problem when the location of input points are not precise. When there is uncertainty in input data the robustness of algorithms becomes crucial. Non-robustness may affect practical algorithms and therefore their outputs. A main source of non-robustness stems from limitations of the current model of floating point computation in computers. Computers run algorithms on a model which uses a limited representation model for numbers. The floating point system with its limited precision creates problems which may lead to undesirable results in various situations (Figure 1). The numerical errors may lead to inconsistent structures in computational geometric algorithms. There is also an additional source of uncertainty which comes from limitations of measuring devices. There are methods and algorithms that try to give robust results for geometric problems. We will give a summary of these methods in the next section.

In this paper we use the model introduced in [1] which is a robust model and uses domains to represent geometric objects. We study properties of the so-called partial convex hull and prove theorems related to continuity, and hence computability, of the convex hull function in this model.

Program Logic, the study of the properties of programs, was pioneered by the works of Floyd [2], Hoar [3] and Dijkstra [4]. Dijkstra introduced the notion of a predicate transformer and the weakest-precondition of a program with a post-condition. In this paper, we also seek to develop the predicate transformer for the convex hull problem. A key basis of this work is grounded in domain theory which was introduced by Dana Scott as the mathematical foundation of computation [5]. In [6], Scott showed how one can develop finitary structures to represent domains of computations and maps between them. These finitary structures and maps between them is called information systems and approximable mappings, respectively. The domain of computation for Scott’s information systems are algebraic domains. A topological view of predicate transformers was provided by Mike Smyth using domain theory [7]. He represented open sets of a topological space as properties of a logic, points of the space as logical theories and frame homomorphism, i.e., the action of the inverse of a continuous function on the lattice of open subsets of its range space, as predicate transformers. The underlying logic has been called Geometric Logic. In [8], Abramsky provided a comprehensive account of domain theory in logical form for stably locally compact algebraic domains, which have a topological basis of compact-open sets allowing a geometric logic with finitary operations. The domain of computation for classical Hausdorff spaces are continuous domains which do not have such a basis of compact-open sets. Smyth [9], Abramsky and Jung [10] and Vickers [11] have extended the notion of Scott’s information systems to continuous domains. Jung and Sünderhauf have constructed a finitary geometric logic generated by so-called strong proximity lattices for representing stably locally compact spaces, which include basic
domain structures with Euclidean spaces. We will employ a basic information system approach in this paper to study the logical properties of the convex hull problem.

1.1. Related works

The algorithmic aspect of convex hull computation has been studied in depth by a number of authors [12], [13], [14], [15], [16]. Grunbaum studied properties of convex polytopes [17] and Klee studied topological properties of convex polytopes [18]. Ketner et. al demonstrated non-robustness of a set of computational geometry algorithms using experimental tests [19]. To overcome this problem several studies have used different methods to handle working with imprecise data [15], [20], [21], [22], [23], [24]. Some papers have used interval geometry methods to approximate with imprecise data [15], [20], [21], [22], [23], [24]. Some papers have used interval geometry methods to approximate with imprecise data [15], [20], [21], [22], [23], [24].

As in [34], from now on, by a domain we mean a continuous directed complete partial order. All domains in this paper will be bounded complete, also known as continuous Scott domains. Recall that a poset $D$ in which every directed subset has a supremum, denoted $\sqcup A$, is called a directed-complete partial order, or dcpo for short. In a dcpo the way-below relation $\ll$ is defined as $a \ll b$ if for all directed set $(c_i)_{i \in I}$ the relation $b \subseteq \sqcup_{i \in I} c_i$ implies that there exists $i \in I$ with $a \subseteq c_i$. A subset $B \subset D$ of a dcpo $D$ is called a basis if for all $a \in D$, the set \( \uparrow a := \{x \in B : x \ll a\} \) is directed with supremum $a$. If the dcpo $D$ has a (countable) base then it is called a (countably based) continuous dcpo, or domain for short. A domain is bounded complete if every pair $(a, b)$ of bounded elements have a supremum denoted by $a \sqcup b$. The Scott topology $\Omega D$ of a domain $D$ has basic open subsets of the form $\uparrow a = \{x \in B : a \ll x\}$ for any base $B \subset D$. Note that Scott open sets are upward closed and the separation property of Scott topology is only $T_0$. Given a basis $B \subset D$, the ordered set $(B, \ll)$ where $\ll$ is the restriction of the way-below relation to $B$, is called an information system for $D$. An ideal $A$ of $(B, \ll)$ is a downward closed subset such that for $x, y \in A$ there exists $z \in A$ with $x, y \ll z$. The collection of ideals of $(B, \ll)$ ordered by inclusion is a domain isomorphic to $D$.

In this paper, the underlying domain we use is the bounded complete countably based continuous dcpo $\mathbb{CIR}^N$ of all non-empty compact convex sets in $\mathbb{R}^N$, partially ordered by reverse inclusion (i.e., $a \sqsubseteq b$ if $b \subseteq a$) and augmented with $\mathbb{R}^N$ as its least element. In fact, $\mathbb{CIR}^N$ is a sub-domain of the upper space $\mathbb{UIR}^N$ of $\mathbb{R}^N$ [35] (consisting of non-empty compact subsets ordered by reverse inclusion) and inherits its properties. Each compact convex set represents an imprecise point of $\mathbb{R}^N$ and its refinement to a smaller one contained in it represents more information about the imprecise point. In $\mathbb{CIR}^N$, the set of maximal elements is the set of single points of $\mathbb{R}^N$, which with respect to the Euclidean topology and the Scott topology, is homeomorphic with $\mathbb{R}^N$. The supremum of a directed set is simply the intersection of all elements, i.e., compact convex sets, in the set and the way-below relation is given by $a \ll b$ iff $b \subseteq a$.

Next we need to define the domain $(\mathbb{CIR}^N)^{\omega}$ of tuples of elements of $\mathbb{CIR}^N$ or any finite length. For any ordered set $E$, for each positive integer $m \geq 2$, we obtain the ordered set $E^m$ of all $m$-tuples of elements of $E$ with component-wise ordering, i.e., $x = (x_1, \ldots, x_m) \subseteq y = (y_1, \ldots, y_m)$ is defined to hold if $x_i \subseteq y_i$ for $1 \leq i \leq m$. Then the supremums, if they exist, of directed subsets are computed component-wise and it follows that if $E$ is actually a domain then so is $E^m$ with $x \ll y$ iff $x_i \ll y_i$ for $1 \leq i \leq m$, and $\Omega(E^m) = (\Omega E)^m$.

We then can define the ordered set $E^{\omega}$ of all finite sequences of elements of the basic ordered set $E$ by putting

$$E^{\omega} = \bigcup_{m \geq 1} E^m$$

with the partial order $x \sqsubseteq y$ in $E^{\omega}$ defined if there exists

![Figure 1: Error in computed convex hull when points are almost collinear](image-url)
There is a well-known property of the convex hull map. Let $\{P_i : i \in I\}$ be a family of non-empty, convex and compact subsets $P_i \subset \mathbb{R}^n$ for some indexing set $I$ which may be infinite. We define $R(P) := \{p_i : i \in I\} : p_i \in P_i, i \in I\}$ be the collection of all possible subsets each containing exactly one point in the partial points of $P$.

**Definition 1.** The function $CH$ takes as input a family $P$ of partial objects in $\mathcal{CR}^n$ and returns an element $(CH^−(P), CH^+(P)) \in \mathcal{SCR}^n$ of the solid domain with interior and exterior given by

$$CH^−(P) = (\bigcap \{\Gamma((p : p \in R(P)))\})^o$$

$$CH^+(P) = (\bigcup \{\Gamma((p : p \in R(P)))\})^c$$

In words, $CH^−(P)$ is the interior of the set whose points are in the convex hull of any selection $p \in R(P)$, while
CH\(^+(P)\) consists of points that are in the complement of any such selection.

The partial convex hull function CH introduced in [31] is of type: \(\text{CH} : (\mathbb{IR}^N)^m \rightarrow \mathbb{SR}^N\) for \(m \geq 1\). An \(O(m \log m)\) algorithm for computing the interior and exterior of the partial convex hull of a set of \(m\) partial points in \(\mathbb{IR}^N\) for \(N = 2, 3\) is given in [31]. The algorithm also works in \(\mathbb{R}^N\) for \(N > 3\), but the complexity may no longer be \(O(m \log m)\). The algorithm computes the exterior by computing convex hull of the set of vertices of all partial points. The interior of the partial convex hull is the interior of the intersection of \(2^N\) convex hulls, each of which is the convex hull of the vertices of the same type of all the partial points (i.e., lower or upper end of the projection of a partial point in each dimension).

In [32] an algorithm is proposed to compute the partial convex hull when inputs are imprecise points in \(\mathbb{IR}^2\) represented by compact convex polytopes. In this setting each partial point is defined by the intersection of a set of half spaces whose normals are from a given finite set of unit vectors. In this section, we present an extension of this algorithm to \(\mathbb{R}^N\), which is therefore an extension of the work in [31] to polytopes as imprecise points.

Suppose \(P = (P_1, ..., P_m)\) is a list of polytopes in \(\mathbb{CR}^N\). We first note that the outer convex hull of \(P\) is simple to compute since \(\text{CH}^+(P) = (\Gamma(\bigcup P)^{\ominus})\), where \(\bigcup P = \bigcup_{1 \leq i \leq m} P_i\); thus we can simply compute the convex hull of all vertices of all partial points \(P_i\) for \(1 \leq i \leq m\). In the same way that an axes-aligned hyper-rectangle in \(\mathbb{R}^N\) is given by a lower and an upper end in each dimension, we use a set of opposite directions to define the polytopes in \(P\) as follows. Assume \(d = (d_1, ..., d_{2n})\), \(n \geq N\) be a list of \(2n\) unit vectors in \(\mathbb{R}^N\) with \(d_j = -d_{1+n}\) for \(1 \leq j \leq n\) with which all partial points in \(P\) are defined by the intersection of \(2n\) half spaces, i.e., such that the outer normal of each half space is \(d_j\) for some \(1 \leq j \leq 2n\). Therefore \(P_i\) has at most \(\binom{2n}{\lceil N/2 \rceil}\) vertices. Each vertex of polytope \(P_i\) is the intersection of at least \(N\) adjacent half spaces. We classify the vertices of \(P\) with outer normal vectors of their intersecting axes.

Suppose \(C\) is the convex hull of the unit vectors in \(d\). Then each face of \(C\) determines a cone in \(\mathbb{CR}^N\). We denote by \(\text{CN}\) the set of cones created by \(d\). For each cone \(c \in \text{CN}\) there exists \(-c \in \text{CN}\) such that for each unit vector \(x \in c\), we have \(-x \in -c\). We can now characterize the vertices of the partial points of \(P\) using \(\text{CN}\). For each \(P_i \in P\) we define \(P_{ic}\) as the vertex of \(P_i\) furthest away from the boundary of any half space with normal \(n \in c\) that contains \(P_i\). It is easy to see that \(P_{ic}\) exists for each \(P_i\) and is unique. However two cones in \(\text{CN}\) may have the same corresponding vertices.

Using this classification Algorithm 1 computes the intersection of a set of convex hulls of type-similar corners of the partial points. It provides an efficient way to compute the inner convex hull.

Algorithm 1 includes running of \(\text{CH}\) of the classic convex hull algorithm in \(\mathbb{IR}^N\) and then intersects \(\text{CN}\) convex polytopes. Thus the complexity of Algorithm 1 is \(O(|\text{CN}|T_{\text{CH}}(m, N) + \text{Int}(\text{CN}, m, N))\), in which \(T_{\text{CH}}(m, N)\) is the complexity of computing convex hull in \(\mathbb{R}^N\) for \(m\) points, and \(\text{Int}(\text{CN}, m, N)\) is the complexity to compute the intersection of \(|\text{CN}|\) polytopes with \(m\) vertices in \(N\) dimension. When \(N = 2, 3\) computing the convex hull can be done efficiently in \(O(m \log m)\), and there are linear algorithms to compute the intersection of a set of convex polygons [13]. Therefore when \(N = 2, 3\), algorithm 1 has complexity \(O(m \log m)\). For dimensions \(N > 3\) the number of facets of the convex polytopes in the algorithm may increase exponentially in the size of \(m\). It follows that different representations of the convex hull, e.g., by vertex or facet description, have different sizes. For \(N > 3\) computing the convex hull can be done efficiently in \(O(m^{\lceil N/2 \rceil})\) for \(m\) points in \(\mathbb{R}^N\) [14].

### 3. Convex and compact sets as partial points

In this paper we consider a general geometric framework in which an imprecise or partial point in \(\mathbb{IR}^N\) is given by a
Proof. We now define the infix conjunction operator \((\cdot ) \land (\cdot )\) : \((\mathbb{C} \mathbb{R}^N)^{\omega} \times (\mathbb{C} \mathbb{R}^N)^{\omega} \rightarrow (\mathbb{C} \mathbb{R}^N)^{\omega}\), which is defined for input values \((P, Q)\) when \(P\) and \(Q\) have the same length.

We now define the infix concatenation operator \((\cdot ) \land (\cdot )\) : \((\mathbb{C} \mathbb{R}^N)^{\omega} \times (\mathbb{C} \mathbb{R}^N)^{\omega} \rightarrow (\mathbb{C} \mathbb{R}^N)^{\omega}\). If \(P, Q \in (\mathbb{C} \mathbb{R}^N)^{\omega}\) then \(P = (P_1, \ldots, P_m)\) and \(Q = (Q_1, \ldots, Q_n)\) for some \(m, n \geq 1\), we define \(P \land + Q = (P_1, \ldots, P_m, Q_1, \ldots, Q_n) \in (\mathbb{C} \mathbb{R}^N)^{m+n+1} \subset (\mathbb{C} \mathbb{R}^N)^{\omega}\). It is straightforward to check the following properties.

**Proposition 3.** Suppose \(P, Q \in (\mathbb{C} \mathbb{R}^N)^{\omega}\). Then

\[
\begin{align*}
\text{CH}^{-}(P) \cup \text{CH}^{-}(Q) &\subseteq \text{CH}^{-}(P \land + Q) \\
\text{CH}^{+}(P) \cup \text{CH}^{+}(Q) &\supseteq \text{CH}^{+}(P \land + Q)
\end{align*}
\]

We see that with the ordering in \((\mathbb{C} \mathbb{R}^N)^{\omega}\), we do not get a monotone map if we add new partial points to our input. We will see in Section 5 that by moving to the Plotkin power domain of \((\mathbb{C} \mathbb{R}^N)^{\omega}\) we have the right ordering to increase the number of partial points and yet have a Scott continuous convex hull map. We however need a toolkit to prove the Scott continuity of the convex hull map and the next section provides us with the required tools.

### 4. Pre-inner support function

In this section, we introduce the notion of pre-inner support function which gives the support function of the inner convex hull as in Theorem 1. In obtaining this result, we use ideas from convex polarity [36, sections 14,15,16]. However, deriving the theorem from the classical material is not straightforward and, in contrast, our elementary and self-contained proof offers a clear view without requiring the reader to delve in convexity theory.

Consider a bounded collection \(P = \{P_i : i \in I\}\) of non-empty, convex and compact sets \(P_i \subseteq \mathbb{R}^N\), i.e., there exists \(K > 0\) such that the open ball of radius \(K\) centred at the origin contains \(P_i\) for all \(i \in I\). Define the pre-inner support function of \(P\) as the map \(S_P^r : \mathbb{R}^N \rightarrow \mathbb{R}\) with:

\[
S_P^r(v) = \sup_{i \in I} \inf_{x \in P_i} \langle v, x \rangle = \sup_{i \in I} \min_{x \in P_i} \langle v, x \rangle
\]

(1)

Recall that a function \(\phi : \mathbb{R}^N \rightarrow \mathbb{R}\) is positively homogeneous if

\[
\forall \lambda \geq 0, \phi(\lambda X) = \lambda \phi(X)
\]

From the definition (1) above, \(S_P^r(v)\) is positively homogeneous and Lipschitz with Lipschitz constant \(K\). Recall the definition of \(\text{CH}\) from Definition 1.

**Lemma 2.** One has the following equivalence:

\[
z \in \text{CH}^{-}(P) \iff \forall v \neq 0, \langle z, v \rangle < S_P^r(v) \iff \forall v, ||v|| = 1 \Rightarrow \langle z, v \rangle < S_P^r(v)
\]

**Proof.** For \(v \neq 0\) and \(i \in I\) there exists a point \(p_i(v) \in P_i\) such that \(\langle p_i(v), v \rangle = \min_{x \in P_i} \langle x, v \rangle\). Take \(z \in \text{CH}^{-}(P_i)\). From the definition of \(\text{CH}^{-}(P)\), one has \(z \in ((\text{CH}(\{p_i(v), i \in I\}))\) which implies:

\[
\langle z, v \rangle < \sup_{i \in I} \min_{x \in P_i} \langle p_i(v), v \rangle = \sup_{i \in I} \min_{x \in P_i} \langle x, v \rangle = S_P^r(v)
\]

Thus, we have: \(z \in \text{CH}^{-}(P) \Rightarrow \forall v \neq 0, \langle z, v \rangle < S_P^r(v)\).
Now, for each $i \in I$, pick a point $p_i \in P_i$. Since the convex hull of a set $X$ is the intersection of all half spaces containing $X$ one has:

$$(\text{CH}(\{ p_i : i \in I \}))^\circ = \bigcap_{v \neq 0} \left\{ z, \langle v, z \rangle < \max \{ \langle v, p_i \rangle \} \right\}$$

and since $\sup_{i \in I} \langle v, p_i \rangle \geq S_p^*(v)$ one has:

$$\text{CH}(\{ p_i : i \in I \})^\circ \supseteq \bigcap_{v \neq 0} \left\{ z, \langle v, z \rangle < S_p^*(v) \right\}$$

which gives the reverse inclusion. The second equivalence follows since $S_p^*(v)$ is positively homogeneous. \hfill \square

We now introduce the convex hull of a real-valued map.

**Definition 2.** (Convex Hull of a Function) [39] [40]. Given a function $\phi : \mathbb{R}^N \to \mathbb{R}$, the Convex Hull of $\phi$ with type $\mathcal{H}(\phi) : \mathbb{R}^N \to \mathbb{R} \cup \{ -\infty \}$ is the supremum of convex functions below $\phi$.

Note that if there exists a convex function below $\phi$ then $\mathcal{H}(\phi)$ is itself a convex function. Otherwise, $\mathcal{H}(\phi)$ is the constant extended real-valued function with value $-\infty$. In addition, a convex function below $\phi$ exists if and only if there exists an affine function below $\phi$. In particular, if $\phi(0) = 0$, then a convex function exists below $\phi$ if and only there is a linear function below $\phi$, in other words if there exists $h \in \mathbb{R}^N$ such that $\forall x \in \mathbb{R}^N, (x, h) \leq \phi(x)$. Denote by $\text{epi}(\phi)$ the epigraph (or supergraph or superlevel set) of $\phi$:

$$\text{epi}(\phi) = \{(X, t) \in \mathbb{R}^N \times \mathbb{R}, t \geq \phi(X)\}$$

Observe that a function $F : \mathbb{R}^N \to \mathbb{R}$ is convex if and only if the set $\text{epi}(F)$ is convex and $F \geq F'$ if and only if $\text{epi}(F) \subseteq \text{epi}(F')$. Therefore, since $\Gamma(\text{epi}(\phi))$ is the minimal convex set (with respect to inclusion) containing $\text{epi}(\phi)$, it is the epigraph of the largest convex function below $\phi$, i.e.,

$$\Gamma(\text{epi}(\phi)) = \text{epi}(\mathcal{H}(\phi)) \tag{2}$$

**Lemma 3.** Given a function $\phi : \mathbb{R}^N \to \mathbb{R}$, one has

$$\mathcal{H}(\phi)(X) = \inf \left\{ \sum_{i \in I} w_i \phi(X_i) : X_i \in \mathbb{R}^N, \right\}$$

$$\begin{align*}
& w_i \geq 0, \sum_{i \in I} w_i = 1, \sum_{i \in I} w_i X_i = X, I \text{ finite} \end{align*} \tag{3}$$

If moreover $\phi$ is positively homogeneous then:

$$\mathcal{H}(\phi)(X) = \inf \left\{ \sum_{i \in I} \phi(X_i) : X_i \in \mathbb{R}^N, \sum_{i \in I} X_i = X, I \text{ finite} \right\}.$$\

**Proof.** It is straightforward to check that the map on the right hand side of Equation (3) which has type $\mathbb{R}^N \to \mathbb{R}$ with

$$X \mapsto \inf \left\{ \sum_{i \in I} w_i \phi(X_i) : X_i \in \mathbb{R}^N, \right\}$$

is the largest convex function below $\phi$ and its epigraph is $\mathcal{H}(\text{epi}(\phi))$. Therefore Equation (3) follows from Equation (2). The second claim follows easily. \hfill \square

**Definition 3.** For a positively homogeneous function $\phi : \mathbb{R}^N \to \mathbb{R}$ the centre of $\phi$ is defined as:

$$\mathcal{I}(\phi) = \{ z \in \mathbb{R}^N : \forall v \in \mathbb{R}^N, \langle z, v \rangle \leq \phi(v) \}$$

Note that $\mathcal{I}(\phi)$ is convex but can be empty.

**Lemma 4.** For a positively homogeneous function $\phi : \mathbb{R}^N \to \mathbb{R}$:

$$\mathcal{I}(\phi) = \mathcal{I}(\mathcal{H}(\phi))$$

**Proof.** Since $\forall v \in \mathbb{R}^N, \mathcal{H}(\phi)(v) \leq \phi(v)$, the relation $\mathcal{I}(\phi) \supseteq \mathcal{I}(\mathcal{H}(\phi))$ is trivial. Furthermore, from Lemma 3, for any $v \in \mathbb{R}^N$ and $\epsilon > 0$ there are a finite number of vectors $v_i \in \mathbb{R}^N$ for $i \in I$, with $\sum_{i \in I} v_i = v$ such that:

$$\mathcal{H}(\phi)(v) + \epsilon > \sum_{i \in I} \phi(v_i).$$

It follows that if $z \in \mathcal{I}(\phi)$ then

$$\mathcal{H}(\phi)(v) + \epsilon > \sum_{i \in I} \langle z, v_i \rangle = \langle z, v \rangle$$

Since this is true for arbitrary small $\epsilon$ we have $\mathcal{H}(\phi)(v) \geq \langle z, v \rangle$ and therefore, since this is true for any $v \in \mathbb{R}^N$, we conclude that $z \in \mathcal{I}(\mathcal{H}(\phi))$. \hfill \square

### 4.1. Inner convex hull from pre-inner convex hull

We will now give now an explicit expression for the support function of $\mathcal{I}(\phi)$.

**Theorem 1.** For a positively homogeneous function $\phi : \mathbb{R}^N \to \mathbb{R}$, with a non-empty center, we have:

$$S_{\mathcal{I}(\phi)} = \mathcal{H}(\phi).$$

**Proof.** Since $\phi$ is positively homogeneous its epigraph $\text{epi}(\phi)$ is a cone with apex $0$. The convex hull of $\text{epi}(\phi)$ in $\mathbb{R}^N \times \mathbb{R}$ is the intersection of all half spaces containing it. We denote points in $\mathbb{R}^N \times \mathbb{R}$ by $(x, \lambda)$ with $x \in \mathbb{R}^N, \lambda \in \mathbb{R}$. Since $\phi$ is defined (i.e., it is real-valued) on $\mathbb{R}^N$, and since, by definition $\text{epi}(\phi)$ is not bounded above (i.e. it is unbounded in the positive direction of the last coordinate), all half spaces containing $\text{epi}(\phi)$ have an outer normal with a negative last coordinate. In addition, since $\text{epi}(\phi)$ is a cone with apex $(0, 0)$, if a half space $H$ contains $\text{epi}(\phi)$ it contains $(0, 0)$. We claim moreover that if $H$ is a half space that contains $\text{epi}(\phi)$, then the translate $H'$ of $H$ whose boundary hyper-plane goes through $(0, 0)$ still contains $\text{epi}(\phi)$. Indeed, assume that the line $0 \times \mathbb{R}$ in $\mathbb{R}^N \times \mathbb{R}$ cuts the boundary of $H$ at some point $(0, \mu)$ with $\mu < 0$ (note that if $\mu = 0$ then $H = H'$ and $\mu > 0$ would contradict $(0, 0) \in H$). Thus, we have $H = \{(x, \lambda), (h, x) + \lambda \geq 0 \}$ and $H' = \{(x, \lambda), (h, x) + \lambda \geq 0 \}$ for some vector $h \in \mathbb{R}^N$. Assume,
for a contradiction, that there exists \((x_0, \lambda_0) \in \text{epi}(\phi)\) such that \((x_0, \lambda_0) \in H' \setminus H'\). Then \(\mu \leq \langle h, x_0 \rangle + \lambda_0 < 0\). Now consider the point:

\[
(x_1, \lambda_1) = \left( \frac{-2\mu}{\langle h, x_0 \rangle + \lambda_0}, \frac{-2\mu}{\langle h, x_0 \rangle + \lambda_0} \right)
\]

By positive homogeneity one has \((x_1, \lambda_1) \in \text{epi}(\phi)\) and: \(\langle h, x_0 \rangle + \lambda_1 = 2\mu < \mu \) and \((x_1, \lambda_1) \notin H\) which contradicts \(\text{epi}(\phi) \subset H\). This proves the claim. Let us denote by \(H_h\) the half space:

\[
H_h = \{(x, \lambda) : \langle h, x \rangle + \lambda \geq 0\}
\]

From the claim, it follows that:

\[
\Gamma(\text{epi}(\phi)) = \bigcap_{(h \in \text{int}(\phi))} H_{-h}.
\] (4)

We have:

\[
\begin{align*}
\text{epi}(\phi) & \subset H_h \\
\forall x \in \mathbb{R}^N, \lambda \geq \phi(x) & \Rightarrow \langle h, x \rangle + \lambda \geq 0 \\
\forall x \in \mathbb{R}^N, \langle h, x \rangle + \phi(x) & \geq 0 \\
\forall x \in \mathbb{R}^N, \langle x, -h \rangle & \leq \phi(x)
\end{align*}
\]

which gives:

\[
\Gamma(\text{epi}(\phi)) = \bigcap_{h \in \text{int}(\phi)} H_{-h}.
\] (4)

Now consider \(\text{epi}(S_{I}(\phi))\) the epigraph of the support function of the center \(\text{int}(\phi)\) of \(\phi\). We obtain:

\[
(x, \lambda) \in \text{epi}(S_{I}(\phi)) \iff \lambda \geq S_{I}(\phi)(x) \iff \lambda \geq \sup \{ \langle x, h \rangle : h \in I(\phi) \} \iff \forall h \in I(\phi), \lambda \geq \langle x, h \rangle \iff \forall h \in I(\phi), (x, \lambda) \in H_{-h}.
\]

Therefore:

\[
\text{epi}(S_{I}(\phi)) = \bigcap_{h \in \text{int}(\phi)} H_{-h}.
\]

Comparing this with (4), we obtain:

\[
\Gamma(\text{epi}(\phi)) = \text{epi}(S_{I}(\phi))
\]

By Equation (2), we obtain \(\text{epi}(\mathcal{H}(\phi)) = \Gamma(\text{epi}(\phi)) = \text{epi}(S_{I}(\phi))\) and the result follows. \(\square\)

Taking \(\phi = S^*_p\), Lemma 2 gives

\[
\mathcal{I}(S^*_p) = \text{CH}^-(P),
\] (5)

where \(\mathcal{A}\) denotes the closure of the set \(A \subset \mathbb{R}^N\). We can then obtain a direct application of Theorem 1:

**Corollary 1.** If \(\text{CH}^-(P)\) is non-empty, the support function \(S_{\text{CH}^-(P)}\) of \(\text{CH}^-(P)\) is given by:

\[
S_{\text{CH}^-(P)} = S_{\text{CH}^-(P)} = \mathcal{H}(S^*_p)
\]

Now consider a directed and bounded family \((\phi_j)_{j \in J}\) of maps \(\phi_j : \mathbb{R}^N \rightarrow \mathbb{R}\), i.e., for each \(i, j \in J\) there exists \(k \in J\) such that \(\phi_i, \phi_j \leq \phi_k\) and \(\sup_{j \in J} \phi(x) < \infty\) for all \(x \in \mathbb{R}^N\). From Definition (3), we obtain

\[
\mathcal{I}(\sup_{j \in J} \phi_j) = \bigcup_{j \in J} \mathcal{I}(\phi_j)
\] (6)

**Proposition 4.** If \((\phi_j)_{j \in J}\) is a directed and bounded family of real-valued maps on \(\mathbb{R}^N\), then

\[
\sup_{j \in J} S_{I}(\phi_j) = S_{I}(\sup_{j \in J} \phi_j), \quad \sup_{j \in J} \mathcal{H}(\phi_j) = \mathcal{H}(\sup_{j \in J} \phi_j)
\]

**Proof.** By monotonicity, we have \(\sup_{j \in J} S_{I}(\phi_j) \leq \sup_{j \in J} \mathcal{H}(\phi_j) \leq \mathcal{H}(\sup_{j \in J} \phi_j)\). Using Equation (6), the first equation follows from Proposition 1(ii). The second equation then follows from Theorem 1. \(\square\)

Assume that for each \(j \in J\), with \(J\) an indexing set, we have a family \(P_j = \{P_{ji} : i \in I_j\}\) of non-empty convex and compact subsets \(P_{ji} \subset \mathbb{R}^N\) for \(i \in I_j\), where \(I_j\) is an indexing set for each \(j \in J\).

**Corollary 2.** If \((S^*_P)_{j \in J}\) is a directed family in the function space \((\mathbb{R}^N \rightarrow ([0, \infty]), \mathcal{S}_\mathbb{R}^N)\) with \(\sup_{j \in J} S^*_P(x) < \infty\) for all \(x \in \mathbb{R}^N\), then

\[
\sup_{j \in J} S_{\text{CH}^-(P_j)} = S_{\bigcup_{j \in J} \text{CH}^-(P_j)}
\]

**Proof.** Put \(\phi_j := S^*_P\) in Proposition 4. Then, we obtain:

\[
\sup_{j \in J} S_{\text{CH}^-(P_j)} = \sup_{j \in J} S_{I}(S^*_P) = S_{I}(\bigcup_{j \in J} S^*_P) = S_{I}(\bigcup_{j \in J} S^*_P) = S_{I}(\bigcup_{j \in J} S^*_P) = S_{\bigcup_{j \in J} \text{CH}^-(P_j)}
\]

Using Proposition 4, we can show that the domain-theoretic convex hull map is Scott continuous:

**Proposition 5.** The map \(\text{CH} : (\mathbb{C}^N)^m \rightarrow \mathbb{S}_\mathbb{R}^N\) for \(m \geq 1\) is Scott continuous.

**Proof.** The monotonicity of \(\text{CH}\) follows immediately from the Definition 1. Since \(\mathbb{C}^N\) is a countably based bounded complete domain it is sufficient to show that the supremum of an increasing sequence \((P_n)_{n \geq 0}\) with \(P_n \subset (\mathbb{C}^N)^m\) for all \(n \geq 0\) is preserved. Put \(P = \sup_{n \geq 0} P_n \in (\mathbb{C}^N)^m\).

First consider \(\text{CH}^+(P)\). Then \(P = \bigcap_{n \geq 0} \bigcup_{P_n}\) (Recall that for a set \(A\) of subsets of \(\mathbb{R}^N\), \(\bigcup_{A\text{ denotes the union of subsets in } A}\).) Therefore, \(d_H(\bigcup_{P_n}, P_n) \rightarrow 0\) as \(n \rightarrow \infty\), it follows by Lemma 1 that \(d_H(\bigcup_{P_n}, P_n) \rightarrow 0\) as \(n \rightarrow \infty\). Hence, \(d_H(\bigcup_{P_n}, P_n) = \bigcap_{\bigcup_{P_n}} = (\bigcap_{P_n})^c = (\bigcap_{n \geq 0} \bigcup_{P_n})^c = (\bigcap_{n \geq 0} \bigcup_{P_n})^c = U_{\geq 0} P_n = U_{\geq 0} \text{CH}^+(P_n)\).

Next consider \(\text{CH}^-(P)\). It follows from the definition of the pre-inner support function that \((S^*_P)_{n \geq 0}\) is an increasing sequence of bounded functions in \((\mathbb{R}^n \rightarrow \mathbb{R})\) with
sup_{n \geq 0} S_{P_n} = S_P$. In Proposition 4, let $\phi_n := S_{P_n}$. We have that $CH^{-}(P_n) = I(S_{P_n})$ and $CH^{-}(P) = I(S_P)$ by Equation 5. Also $I(\bigcup_{n \geq 0} S_{P_n}) = \bigcup_{n \geq 0} I(S_{P_n})$. Thus, $\sup_{n \geq 0} S_{CH^{-}(P_n)} = \sup_{n \geq 0} S_{CH^{-}(P)} = S_{CH^{-}(P)}$ and thus $CH^{-}(P) = \bigcup_{n \geq 0} CH^{-}(P_n)$.

\[ \square \]

5. Finitely generable subsets

In this section, we show that the domain-theoretic convex hull algorithm can be extended to finitely generable subsets as in non-deterministic semantics.

Recall that given a countably based domain $B \in D$, the Plotkin power domain $PD$ is constructed using finitely generable subsets of $D$. We can then define a pre-order $\leq_{EM}$ on finitely generable subsets $F(D)$ of $D$ by stipulating that $C_1 \subseteq EM C_2$ if for all finite subsets $A \in F(D)$ with $A \leq_{EM} C_1$ we have $A \leq_{EM} C_2$. The Plotkin power domain $PD$ of $D$ is then defined as the quotient $(F(D)/\leq_{EM})/\equiv_{EM}$ where $C_1 \equiv_{EM} C_2$ iff $C_1 \subseteq EM C_2$ and $C_2 \subseteq EM C_1$. The countable basis $B \in D$ provides a countable basis $F(B)$ for $PD$. Thus, $PD$ is a countably based domain. If $D$ is bounded complete and $C \in F(D)$ is a finitely generable set consisting of maximal elements of $D$, then the equivalence class of $C$ will have only one element namely $C$ which is itself a maximal element of $PD$. If $E$ is a dcpo then any monotone $g \in F(D) \to E$ can be extended, in the usual way using the way-below relation, to a Scott continuous map of type $PD \to E$.

Consider now the Plotkin power domain $PCRN$ of $CRN$. We introduce the notion of a finitely generable set of $RN$.

Definition 4. A subset $A \subset RN$ is finitely generable if it is an element of an equivalence class of $PCRN$, i.e., if it is a finitely generable subset of the domain $CRN$.

Let $T$ be a tree in the construction of an element of the power domain i.e., a finitely generable subset of $CRN$. Let $T_n$ denote the set of nodes of $T$ on level $n \geq 0$ and $T_\omega$ the set of leaves of the infinite branches of $T$. Then, $\bigcup T_n$ is the finite union of non-empty convex and compact sets and is compact, and thus $\bigcup T_\omega = \bigcap_{n \geq 0} \bigcup T_n$ is also compact. In addition, the following property shows the connection between finitely generable sets and compact sets.

Proposition 6. If $C \subset RN$ is any non-empty compact set, then the set containing all singleton sets contained in $C$ (i.e., $\{x \mid x \in C\} \subset C$) is a finitely generable set, a maximal element of $PCRN$.

Proof. Using compactness of $C$, construct the level $n$ of a finitely branching tree $T$ by induction as follows. Let $C_0$ be a finite open covering of $C$ with open balls of radius less than $1 = 1/2^n$, which is minimal, i.e., there is no proper subset of $C_0$ which covers $C$. Put $T_0 = \{O \mid O \in C_0\}$. Inductively, given the finite open covering $C_n$ of $C$, for $n \geq 1$, for each $x \in C$ we take an open ball centred at $x$ with radius less than $1/2^{n+1}$ that is contained in an element of $C_n$. By compactness there is a minimal finite covering $C_{n+1}$ which by construction refines $C_n$, i.e., for each $O \in C_{n+1}$, there exists $O’ \subset C_n$ such that $O \subset O’$. Put $T_{n+1} = \{O \mid O \in C_{n+1}\}$. Then $T_n \subseteq EM T_{n+1}$ for $n \geq 0$ and $T_\omega = \{x \mid x \in C\}$ which is clearly a maximal element of $PCRN$. \[ \square \]

5.1. Iterated Function Systems

A large class of finitely generable subsets is given by the attractors, i.e., fixed points, of Iterated Function Systems (IFS) consisting of affine maps [44]. We present a general domain-theoretic formulation here. Let $C_0 \subset RN$ be, say, a closed ball centred at the origin and let $C(C_0)$ be the bounded complete sub-domain of $CRN$ consisting of non-empty, convex and compact subsets of $C_0$ ordered with reverse inclusion. Consider a finite family of Scott functions $h_i : C_0 \to C_0$, for $i \in I$, that map $C_0$ into itself. Then the map $V : C(C_0) \to C(C_0)$ with $V(X) = \bigcup \{h_i(X) : i \in I\}$ is Scott continuous and thus has a least fixed point $\text{lfp}(V) = \sup_{n \geq 0} V^n(C_0) \in C(C_0)$. We also get a map $H : (F(C_0)), \leq_{EM}) \to PC(C_0)$ given by

$$H : \{A_j : j \in J\} \mapsto \{h_i(A_j) : i \in I, j \in J\}.$$
It is easy to check that $H$ is monotone and thus extends to map $\hat{H}: PC(C_0) \to PC(C_0)$ with
$\hat{H}(X) = \{V(Y) : Y \ll_{EM} X\}$.

Using the Scott continuity of $h_i$ for $i \in I$, it follows that $\hat{H}(X) = V(X)$ for $X \in \mathcal{P}(C(C_0))$ and we thus write $H$ for $\hat{H}$ for convenience. We obtain a least fixed point for $H: PC(C_0) \to PC(C_0)$ given by $\text{lfp}(H) = \sup_{n \geq 0} H^n(C_0)$.

The increasing sequence $(H^n(C_0))_{n \geq 0}$ provides a finitary branching tree, called the IFS tree, whose level $n$ is given by $H^n(C_0)$ for $n \geq 0$. This is depicted in Figure 4 for $I = \{1, \ldots, m\}$. The finitely generable set of the IFS tree consists of the intersection $\bigcap_{i \geq 1} h_i, h_{i_2}, \ldots, h_{i_m}(C_0)$ of the nodes in any infinite branch of the tree with $i_j \in I$ for $j \geq 1$. It is easy to check that $V^n(C_0) = \bigcup \{H^n(C_0)\}$ and thus $\text{lfp}(V) = \bigcup \text{lfp}(H)$.

5.2. The convex hull of finitely generable sets

Using the Plotkin power domain $PCR^N$, we can allow, in non-determinism of programs, for a partial point $P \in CR^N$ at a given stage of computation to be refined to several partial points $P_1, \ldots, P_m$ say with $P \subseteq P_i$ for $1 \leq i \leq m$. We will now show that the domain-theoretic convex hull map can be extended to $PCR^N$. Let $B$ be any basis of $CR^N$ including the case $B = CR^N$. The map $\text{CH} : (CR^N)^{\omega} \to SCR^N$ clearly does not depend on the order of the partial points $P_i$ for $1 \leq i \leq m$ in an input $(P_1, \ldots, P_m) \in (CR^N)^{\omega}$. In other words, this map is also well-defined on $\mathcal{P}(B)$ and in this section we consider it as a map $\text{CH} : \mathcal{P}(B) \to SCR^N$.

**Proposition 7.** If $X, Y \in \mathcal{P}(B)$, then $X \subseteq_{EM} Y$ implies $\text{CH}^-(X) \subseteq \text{CH}^-(Y)$ and $\text{CH}^+(X) \subseteq \text{CH}^+(Y)$.

**Proof.** Let $X = \{P_i : i \in I\}$ and $Y = \{Q_j : j \in J\}$. Since for each $j \in J$ there exists some $i \in I$ such that $P_i \supseteq Q_j$, it follows that $X \supseteq \cup Y$ and thus $\Gamma(X) \supseteq \Gamma(Y)$.

Hence, $\text{CH}^+(X) = (\Gamma(X))^c \subseteq (\Gamma(Y))^c = \text{CH}^+(Y)$. To prove $\text{CH}^-(X) \subseteq \text{CH}^-(Y)$, let $q_i \in Q_j$ for each $j \in J$ and define $p_i \in P_i$ for each $i \in I$. For each $i \in I$, choose $j \in J$ with $P_i \supseteq Q_j$ and put $p_i := q_j$. Then $\{p_i : i \in I\} \subseteq \{q_j : j \in J\}$ and thus $\Gamma(\{p_i : i \in I\}) \subseteq \Gamma(\{q_j : j \in J\})$. Hence,

$$\text{CH}^-(X) = \left(\bigcap_{p_i \in P_i} \Gamma(\{p_i : i \in I\})\right)^c \subseteq \left(\bigcap_{q_j \in Q_j} \Gamma(\{q_j : j \in J\})\right)^c = \text{CH}^-(Y).$$

Therefore, $\text{CH} : (\mathcal{P}(B), \subseteq_{EM}) \to SCR^N$ is monotone and we can thus extend it to a Scott continuous map $\widehat{\text{CH}} : PCR^N \to SCR^N$ by defining $\widehat{\text{CH}}(X) = \sup\{\text{CH}(Y) : Y \in \mathcal{P}(CR^N), Y \subseteq_{EM} X\}$.

**Proposition 8.** Let $P \in PCR^N$ be a finitely generable set constructed using a finitary branching tree $T$ with $P = \sup_{n \geq 0} T_n$. Then $S_{T_n}$ is an increasing sequence for $n \geq 0$ with $S_{T_n} = \sup_{n \geq 0} S_{T_n}$.

**Proof.** It follows from the definition of $S^*$ that $S^*_n \leq S^*_n$ if $X \subseteq_{EM} Y$. Thus, the sequence of functions $(S_{T_n})_{n \geq 0}$ is increasing for $n \geq 0$ with $S_{T_n} = \sup_{n \geq 0} S_{T_n}$. Let $P = \{P_i : i \in I\}$ and $v \in R^N$ be non-zero. Put $a := S^*_n(v) = \sup_{i \in I} \min_{x \in P_i} \{v, x\}$. Given $\mu > 0$, there exists $i \in I$ such that $\min_{x \in P_i} \{v, x\} > a - \mu/2$. Then $P_i = \bigcap_{n \geq 0} A_n$ where $A_n \in T_n$ and thus there exists $n \geq 0$ such that $\min_{x \in A_n} \{v, x\} > \min_{x \in A_n} \{v, x\} - \mu/2 > a - \mu$. It follows that $S_{T_n}(v) = \sup_{i \in I} \min_{x \in P_i} \{v, x\} > a - \mu$. Since $\mu > 0$ is arbitrary, we obtain $\sup_{n \geq 0} S_{T_n}(v) \geq a$ and the result follows.

We can now show our main result in this section.

**Theorem 2.** The map $\text{CH}$ computes the convex hull, i.e.,

$$\text{CH} = \text{CH}^+.$$

**Proof.** Let $X \in PCR^N$ be a finitely generable set constructed using a finitary branching tree $T$ with $T \subseteq_{EM} X$ and $X = \sup_{n \geq 0} T_n$. By definition, we have

$$\text{CH}(X) = \sup_{n \geq 0} \text{CH}^+(T_n) = \sup_{n \geq 0} \text{CH}^+(T_n).$$

First consider $\sup_{n \geq 0} (\text{CH}^+(T_n))^c$. Since $(\bigcup T_n)_{n \geq 0}$ is a decreasing sequence of compact sets with $\bigcup X = \bigcap_{n \geq 0} T_n$ and $\lim_{n \rightarrow \infty} d_h(\bigcup X, \bigcap T_n) = 0$, it follows from Lemma 1 that $\lim_{n \rightarrow \infty} d_h(\Gamma(X), \Gamma(T_n)) = 0$, i.e.,

$$\Gamma(X) = \bigcap_{n \geq 0} \Gamma(T_n).$$

Thus, $\text{CH}^+(X) = \Gamma(X) = \bigcap_{n \geq 0} \Gamma(T_n) = \bigcap_{n \geq 0} \text{CH}^+(T_n)$.

In other words, $(\text{CH}^+(X))^c = \bigcup_{n \geq 0} (\text{CH}^+(T_n))^c = \sup_{n \geq 0} (\text{CH}^+(T_n))^c$.

We conclude that $\text{CH}^+(X) = \text{CH}^+(X)$. Next consider $\sup_{n \geq 0} \text{CH}^-(T_n)$. By Proposition $8$,

$$S^*_n = \sup_{n \geq 0} T_n.$$

We therefore have:

$$\text{CH}^-(X) = \Gamma(X)$$

**Corollary 1.**

$$\text{CH}^+(X) = \Gamma(X)$$

**Equation (7).**

**Proposition 4.**

$$\text{CH}^-(X) = \Gamma(X)$$

**Corollary 1.**

Thus, from Proposition 1(ii), we finally have $\text{CH}^-(X) = \bigcup_{n \geq 0} \text{CH}^-(T_n)$. We conclude that $\text{CH}(X) = \text{CH}(X)$.

We finally note in this section that, as for any element of $\mathcal{P}(CR^N)$, the outer convex hull of any finitely generable set $X \in PCR^N$ is given by $CW^+(X) = (\Gamma(X))^c$. But in general $\Gamma(X)$ may be too complicated to compute directly and Theorem 2 tells that it can be computed by taking the countable intersection of the shrinking sets $\Gamma(T_n)$ for $n \geq 0$, i.e., the intersection of the convex hulls of its finitary branching tree levels. In addition, as for any element of $\mathcal{P}(CR^N)$, if a finitely generable set $X \in PCR^N$
only consists of singletons then it follows directly from the
definition that $CW^{-}(X) = (\Gamma(\bigcup X))^{\circ}$. Otherwise, by
Theorem 2, $CW^{-}(X)$ can be computed as the increasing
union of the inner convex hulls $CW^{-}(T_{n})$ of its finitary
branching tree levels.

5.3. Two subclasses of IFS

In this subsection, which is based on our notations in
Subsection 5.1 and 5.2, we will study two subclasses of IFS and see how the results of this section can be used to find the
inner convex hull of the finitely generable set induced by their IFS.

5.3.1. IFS with contracting affine maps. Suppose $h_{i} : \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ is affine (i.e., linear up to addition of a constant
vector) and contracting for $1 \leq i \leq m$, i.e., there exists $K$
with $1 > K \geq 0$ such that $\|h_{i}(x) - h_{i}(y)\| \leq K|x - y|$ for
all $x, y \in \mathbb{R}^{N}$ and $1 \leq i \leq m$. This is called a hyperbolic
IFS [44], [45]. In this case, we can always find a large
enough closed ball $C_{0}$ centred at the origin with radius $r$
that is mapped into itself by all $h_{i}$ for $1 \leq i \leq m$. In fact,
the constant $r$ can be computed as the increasing
family $\Omega(C) = (\Gamma(\bigcup C))^{\circ}, (\Gamma(\bigcup C))^{\circ}$. Nearly all examples of IFS treated
in [44] are of this form.

If $N = 1$ and $m = 2$, with $h_{1}(x) = x/3$ and
$h_{2}(x) = (x + 2)/3$ then we can take $C_{0} = [0, 1]$ and
the finitely generable set given by the IFS tree is precisely
the points of the classical Cantor set. If $N = 2, m = 3$
with $h_{1}, h_{2}, h_{3} : \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ given by $h_{1} : x \mapsto x/2,$
h2 : x \mapsto x/2 + (1/2, 0) and $h_{3} : x \mapsto x/2 + (1/4, \sqrt{3}/4),$
then the points of the finitely generable set are precisely
the points of the Sierpinski triangle. Figure 5 shows the
generation of the inner convex hull for the levels of the IFS
tree.

5.3.2. IFS with condensation. Suppose $h_{i} : \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ for
$1 \leq i \leq m$, are as in the previous section i.e., affine and contracting that map $C_{0}$ into itself. Assume further that we have
a finite number of constant maps $f_{j} : C(C_{0}) \rightarrow C(C_{0})$
with, say, constant values $C_{j}$ for $1 \leq j \leq k$. Then the
collection
\[
\{h_{i}, f_{j} : 1 \leq i \leq m, 1 \leq j \leq k\}
\]
provides an IFS on $C(C_{0})$, which induces a Scott continuous
map $H : PC(C_{0}) \rightarrow PC(C_{0})$ that is given on an input element $\{A_{i} : t \in I\} \in \mathscr{P}r(\mathbb{R}^{N})$ by
\[
\{h_{i}(A_{i}), f_{j}(A_{i}) : t \in I, 1 \leq i \leq m, 1 \leq j \leq k\}.
\]

In general in this case the finitely generable set contains
subsets that are not singletons, unless all the constant values
of $f_{j}$ for $1 \leq j \leq k$ are themselves singleton. In this case, it
can be easily seen that $\text{lfp}(H)$ is given by
\[
\{h_{i_{0}} \circ \ldots \circ h_{i_{n}}(C_{j}) : 1 \leq i_{1}, \ldots, i_{n} \leq m, n \geq 1, 1 \leq j \leq k\}.
\]

Putting $P_{n} = H^{n}(C_{0})$ for $n \geq 0$, we obtain a bounded
increasing family $(S_{P_{n}})_{n \geq 0}$ of pre-inner support functions,
and by Corollary 2, it follows that $CH^{-}(\text{lfp}(H)) =
\bigcup_{n \geq 0} CH(H^{n}(C_{0})).$

In the simplest case when $k = 1$, we have an IFS with condensation
where the constant value $C_{1}, \text{say, of } f_{1}$ and
its recursive image under all maps $h_{i}$ for $1 \leq i \leq m$ are
retained in the output of $H$. Consider a simple example with
$N = 3$ and $m = 1$ in which the value $C_{1}$ of the constant
map $f_{1}$ is the sphere of radius 1 with centre at $(0, 4, z_{0})$ with
$z_{0} \geq 0$, while $h_{1} : \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ is the linear map represented
by the matrix:
\[
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

with $\alpha = 0.95$ and $\theta = -2\pi/11$. The finitely generable
set $\text{lfp}(H)$ consists of an infinite sequence of spiralling and shrinking spheres around the origin, whose centres, for
$z_{0} > 0$, decrease in height $z$ and converge to the origin.
A 2D vertical view of this on the $(x, y)$ plane can be seen
in Figure 6, in fact, the figure presents the actual finitely
generable set when $z_{0} = 0$ and the spheres are represented
as circles. In this case, the inner convex hull of $\text{lfp}(H)$ will
be given by $CH^{-}(H^{10}(C_{1})) = CH^{-}(H^{n}(C_{1}))$ for $n \geq 10$
as seen in the figure.

When $z_{0} > 0$, however, $H^{n}(C_{1})$ is a strictly increasing
sequence in $\mathbb{R}^{3}$ for $n \geq 0$ and $CH^{-}(\text{lfp}(H)) =
\bigcup_{n \geq 0} CH_{n}(H)$ will have a complicated shape and will no
longer be a polytope.

6. Predicate transformer of the convex hull

Given an input $P \in \mathbb{P}CR^{N}$ and a Scott open neighbour-
hood $O \in \Omega(SCR^{N})$ of the output $CH(P)$, we ask the
A Scott open subset \( O \) of \( x \) is a family of open sets: \( O = \{ x \in B_D : \text{mapped into } E \} \). The proposition of geometric logics satisfy the following conditions [49]:

1. \( a \subseteq b \implies P_a \cup P_b \)
2. If \( S \) is a family of open sets: \( P_{\cup S} \models \forall a \in S P_a \)
3. If \( S \) is a finite family of open sets: \( \land a \in S P_a \vdash P_{\cap S} \)

Moreover, propositions of geometric logics satisfy the following conditions [49]:

1. \( a \subseteq b \implies P_a \cup P_b \)
2. If \( S \) is a family of open sets: \( P_{\cup S} \models \forall a \in S P_a \)
3. If \( S \) is a finite family of open sets: \( \land a \in S P_a \vdash P_{\cap S} \)

Next we will consider geometric logic in the context of domains as topological spaces. Let \( f : D \to E \) be a Scott continuous map of countably based domains \( D \) and \( E \) with countable bases \( B_D \) and \( B_E \) respectively. The frame homomorphism \( \Omega f : \Omega E \to \Omega D \) between the lattices \( \Omega E \) and \( \Omega D \) of open sets of \( E \) and \( D \) is given by \( \langle \Omega f \rangle(O) = f^{-1}(O) \), which preserves finite intersections (or meets) and arbitrary unions (or joins). Since any point of \( D \), respectively \( E \), can be obtained as the supremum of elements of \( B_D \), respectively \( B_E \), way below it, we can use the information systems \((B_D, \ll_D) \) and \((B_E, \ll_E) \), i.e., the set of basis elements with ordering induced by the way-below relation on them.

In terms of predicates, this means that we can restrict ourselves to the countable collection of open sets \( \uparrow x \) for \( x \in B_D \) and \( \uparrow y \) for \( y \in B_E \). To capture the frame homomorphism \( \Omega f \) in this setting, we make use of the fact that \( \Omega D \), a continuous lattice, is also a domain with a countable basis given by \( \{ (x, y) : \uparrow x \ll_D f^{-1}(\uparrow y) \} \). Therefore, we represent the predicate transformer \( \Omega f : \Omega E \to \Omega D \) by the relation, called an approximable mapping, \( R_f = \{ (x, y) : \uparrow x \ll_D f^{-1}(\uparrow y) \} \). We now use the following simple properties in domain theory.

**Lemma 5.** Suppose \( D \) is a domain and \( x, y \in D \) then:

\[
\uparrow x \ll y \implies \uparrow y \ll \Omega D \downarrow \uparrow x
\]

**Proof.** Recall that the Scott topology on the domain \( D \) has a basis consisting of \( \uparrow a \) for \( a \in D \). Suppose \( \{ \uparrow a_i \}_{i \in I} \) is a directed family of basic open sets with \( \uparrow x \subseteq \bigcup_{i \in I} \uparrow a_i \). By the interpolation property, there exists \( c \in D \) with \( x \ll c \ll y \). Since \( c \in \uparrow x \), it follows that there exists \( i \in I \) such that \( c \in \uparrow a_i \) and hence \( y \in \uparrow a_i \) as the latter set is upward closed. Using the upward closure of \( \uparrow a_i \), we obtain \( \uparrow y \subseteq \uparrow a_i \).

**Proposition 9.** Suppose \( f : D \to E \) is a Scott continuous function of domains \( D \) and \( E \) with \( x \in D \) and \( y \in E \). Then

\[
f(x) \gg y \iff \uparrow x \ll \Omega D \downarrow f^{-1}(\uparrow y)
\]

**Proof.** Suppose \( f(x) \gg y \). By the interpolation property, there exists \( c \in E \) such that \( f(x) \gg c \gg y \). Since \( f \) preserves the sup of directed sets and \( x = \sup \{ z : z \ll x \} \) where the latter set is directed, there exists \( z \in D \) such that \( z \ll x \) and \( y \ll c \ll f(z) \). Therefore \( z \in f^{-1}(\uparrow y) \), i.e., \( \uparrow z \subseteq f^{-1}(\uparrow y) \). By Lemma 5, we have \( \uparrow x \ll \Omega D \uparrow \uparrow z \subseteq f^{-1}(\uparrow y) \) as required. For the reverse direction, suppose \( \uparrow x \ll \Omega D \downarrow f^{-1}(\uparrow y) \). By the interpolation property, there exists \( c \in D \) with \( \uparrow x \ll \Omega D \uparrow \uparrow c \ll \Omega D \downarrow f^{-1}(\uparrow y) \). Thus, \( f(\uparrow c) \subseteq \uparrow y \) and hence \( f(x) \in \uparrow y \).

Now consider the Scott continuous convex hull map \( CH \) in the domain-theoretic setting. We use \( B_{SCR} \) as the set of convex compact polytopes with rational vertices to get the two information systems \( B_{SCR} \) and \( B_{PCR} := \langle (B_{SCR})^\omega, \ll \rangle \) and \( B_{PCR} := \langle \mathcal{P}(B_{SCR}), \ll_{EM} \rangle \). Let \( B_{SCR} \) denote the collection of convex open polytopes with rational vertices. Then we get an information system \((B_{SRCR}, \ll) \) of \( SCR \) where \( B_{SRCR} = (B_{SCR}^o \times B_{SCR}) \cap SCR \).
By Proposition 9, the predicate transformers of the maps \( \text{CH} : (\mathbb{CR}^N)^0 \to \mathbb{SCR}^N \) and \( \text{CH} : \mathbb{PCR}^N \to \mathbb{SCR}^N \) are equivalent respectively to the two relations

\[
\begin{align*}
R_{\text{CH}}^1 &= \{(x, y) \in B(\mathbb{CR}^N)^0 \times B(\mathbb{SCR}^N) : \text{CH}(x) \gg y\} \\
R_{\text{CH}}^2 &= \{(x, y) \in B(\mathbb{PCR}^N) \times B(\mathbb{SCR}^N) : \text{CH}(x) \gg y\}
\end{align*}
\]

Note that \( R_{\text{CH}}^1 \) and \( R_{\text{CH}}^2 \) are the same relations if we identify a finite tuple of elements of \( \mathbb{CR}^N \) with the finite set containing the components of the tuple, which, as we have mentioned previously, have the same image under CH. Since the way-below relation in \( \mathbb{SCR}^N \) is given by \((A_1, A_2) \ll (B_1, B_2) \) if \( A_1 \ll B_1 \) for \( i = 1, 2 \), it thus follows that for both convex hull maps the predicate transformer is equivalent to testing whether a convex polytope \( A \) is contained in the interior of another convex polytope \( B \), i.e., that \( A \subseteq B \) and \( A \) does not have any common points with the boundary of \( B \).

Given two convex polytopes \( C_1 \) and \( C_2 \) with \( m \) and \( n \) vertices in \( \mathbb{R}^2 \), there is an obvious \( O(n \log(N,n,m)) \) algorithm in which \( \text{LP}(N,m) \) is the time complexity of solving a linear programming problem with \( m \) variables and \( N \) constraints. This algorithm checks for containment of each vertex of \( C_2 \) in \( C_1 \), and therefore decides the way-below relation. For \( N = 2 \), we use a simple linear algorithm, i.e., with \( O(m+n) \) complexity, to check if a convex polygon \( C_2 \subset \mathbb{R}^2 \) is contained in the interior of another convex polygon \( C_1 \subset \mathbb{R}^2 \). Assume the vertices of \( C_1 \) and \( C_2 \) are, respectively, given in clockwise order as \((v_0, \ldots, v_m-1)\) and \((u_0, \ldots, u_n-1)\). Let \( v_m := v_0 \) and \( u_n := u_0 \).

**Definition 5.** We say a directed edge \( v_i v_{i+1} \), for some \( i \) with \( 1 \leq i \leq m-1 \), changes direction from left to right with respect to a directed edge \( u_j u_{j+1} \), for some \( j \) with \( 1 \leq j \leq n-1 \), if \( v_i \) is on the left side of \( u_j u_{j+1} \) and \( v_{i+1} \) is not (i.e., \( v_{i+1} \) is either on the line passing through \( u_j u_{j+1} \) or is on the right side of \( u_j u_{j+1} \)).

Observe that if \( u_j \) is on the right side of the directed edge \( v_{i+1} v_{i+1} \), we use this property to check whether the boundaries of the two polygons intersect. Algorithm 2 first checks if \( u_0 \) is \((C_1)^\circ\). If \( u_0 \) is not in \((C_1)^\circ\), then \( C_2 \) is not way-below \( C_1 \). Otherwise, it checks whether \( u_j u_{j+1} \) intersects the boundary of \( C_1 \) with the information that \( u_j \) is in \((C_1)^\circ\). For each edge that \( u_j u_{j+1} \) with \( j > 0 \), we only check edges from \( C_1 \) that have one endpoint on the right side of \( u_j u_{j+1} \), since at the end of the first loop if the algorithm is not terminated then \( u_j \) is still on the right side of \( u_j u_{j+1} \) and we check edges \( C_1 \) until an edge changes direction with respect to \( u_j u_{j+1} \). This analysis also shows that Algorithm 2 is of complexity \( O(m+n) \).

7. Concluding remarks

We have presented a general data type for representing imprecise or partial points in \( \mathbb{R}^N \) by non-empty convex and compact subsets. This data type unifies all the different notions of partial points in the literature. We have formulated in this context an algorithm to find the inner convex hull of a finite set of partial points represented by non-empty compact and convex polytopes in \( \mathbb{R}^N \). In order to derive the Scott continuity of the domain-theoretic convex hull map, we developed the notion of the pre-inner support function, whose convex hull gives the support function of the inner convex hull. We then extended the domain-theoretic convex hull map to the Plotkin power domain of the domain of non-empty convex and compact subsets of \( \mathbb{R}^N \) in order to allow for computing the convex hull of finitely generable sets i.e., for non-determinism in the computation of partial points. We showed that this can be used to find the inner and outer convex hull of the fixed points or attractors of iterated function systems in different settings. Finally, we characterised the predicate transformer for the convex hull map in the general setting of the Plotkin power domain and showed that it is reduced to deciding whether a polytope is contained in the interior of another polytope. We presented a linear algorithm to check the decidability in \( \mathbb{R}^2 \).

Future work in this area includes finding a logical characterisation for the inner convex hull of a finite number of partial points and for developing a more refined version of the predicate transformer for the convex hull by showing that the solid convex domain is a stably locally compact space and thus can be represented logically in a finitary way using semi-strong proximity lattices [50], [51].
References


