

CURTIS HOMOMORPHISMS AND THE INTEGRAL BERNSTEIN CENTER FOR GL_n

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ABSTRACT. We describe two conjectures, one strictly stronger than the other, that give descriptions of the integral Bernstein center for $GL_n(F)$ (that is, the center of the category of smooth $W(k)[GL_n(F)]$ -modules, for F a p -adic field and k an algebraically closed field of characteristic ℓ different from p) in terms of Galois theory. Moreover, we show that the weak version of the conjecture (for $m \leq n$) implies the strong version of the conjecture for GL_n . In a companion paper [HM] we show that the strong conjecture for $n - 1$ implies the weak conjecture for n ; thus the two papers together give an inductive proof of both conjectures. The upshot is a description of the Bernstein center in purely Galois theoretic terms; previous work of the author shows that this description implies the conjectural “local Langlands correspondence in families” of [EH].

1. INTRODUCTION

In [EH], Emerton and the author describe a conjectural “local Langlands correspondence in families” for the group $GL_n(F)$, where F is a p -adic field. More precisely, we show that given a suitable coefficient ring A (in particular complete and local with residue characteristic ℓ different from p), and a family of Galois representations $\rho : G_F \rightarrow GL_n(A)$, there is, up to isomorphism, at most one admissible $A[GL_n(F)]$ -module $\pi(\rho)$ that “interpolates the local Langlands correspondence across the family ρ ” and satisfies certain technical hypotheses. (We refer the reader to [EH], Theorem 1.1.1 for the precise result.) We further conjecture that such a representation $\pi(\rho)$ exists for any ρ .

The paper [H2] gives an approach to the question of actually constructing $\pi(\rho)$ from ρ . The key new idea is the introduction of the integral Bernstein center, which is by definition the center of the category of smooth $W(k)[GL_n(F)]$ -modules. More prosaically, the integral Bernstein center is a ring Z that acts on every smooth $W(k)[GL_n(F)]$ -module, compatibly with every morphism between such modules, and is the universal such ring. The structure of Z encodes deep information about “congruences” between $W(k)[GL_n(F)]$ -modules (for instance, if two irreducible representations of $GL_n(F)$ in characteristic zero become isomorphic modulo ℓ , the action of Z on these two representations will be via scalars that are congruent modulo ℓ .)

Morally, the problem of showing that $\pi(\rho)$ exists for all ρ amounts to showing- for a sufficiently general notion of “congruence”- that whenever there is a congruence between two representations of G_F , there is a corresponding congruence on the other side of the local Langlands correspondence. It is therefore not surprising that one can rephrase the problem of constructing $\pi(\rho)$ in terms of the structure of

Z . Indeed, Theorem 7.4 of [H2] reduces the question of the existence of $\pi(\rho)$ to a conjectured relationship between the ring Z and the deformation theory of mod ℓ representations of G_F (Conjecture 7.2 of [H2].)

The primary goal of this paper, together with its companion paper [HM], is to prove a version of this conjecture, and thus establish the local Langlands correspondence in families. More precisely, we introduce a collection of finite type $W(k)$ -algebras R_ν that parameterize representations of the Weil group W_F with fixed restriction to prime-to- ℓ inertia, and whose completion at a given maximal ideal is a close variant of a universal framed deformation ring. We then conjecture that there is a map $Z \rightarrow R_\nu$ that is “compatible with local Langlands” in a certain technical sense (see Conjecture 10.2 below for a precise statement and discussion.) This conjecture, which we will henceforth call the “Weak Conjecture”, becomes Conjecture 7.2 of [H2] after one completes R_ν at a maximal ideal, and hence implies both that conjecture and the existence of $\pi(\rho)$.

If a map $Z \rightarrow R_\nu$ of the conjectured sort exists it is natural to ask what the image is. The “Strong Conjecture” (Conjecture 10.3 below) gives a description of this image (and in fact gives a description of the direct factors of Z in purely Galois-theoretic terms.) As the names suggest, the “Strong Conjecture” implies the “Weak Conjecture.”

The main result of this paper is that if the “Weak Conjecture” holds for all $\mathrm{GL}_m(F)$, with m less than or equal to a fixed n , then the “Strong Conjecture” holds as well for the groups $\mathrm{GL}_m(F)$. In the companion paper [HM], we will show that the “Strong Conjecture” for $\mathrm{GL}_{n-1}(F)$ implies the “Weak Conjecture” for $\mathrm{GL}_n(F)$. Since the case $n = 1$ is easy (it is essentially local class field theory), the two papers together will establish both conjectures for all n , and hence the local Langlands correspondence for GL_n in families.

Our approach in this paper is motivated by considerations from the representation theory of *finite* groups of Lie type, and in particular from the representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$. We detail these considerations in section 2. In particular, we use the Deligne-Lusztig induction and restriction functors of Bonnafé-Rouquier [BR] to give a description of the endomorphism ring of the Gelfand-Graev representation of $\mathrm{GL}_n(\mathbb{F}_q)$ in terms of a ring $\overline{A}_{q,n}$ constructed from the group rings of (the \mathbb{F}_q -points of) the maximal tori of GL_n over \mathbb{F}_q .

In the context of p -adic groups the natural analogue of the Gelfand-Graev representation is the space of Whittaker functions. In [H2] it is shown that the Bernstein center Z is the full ring of endomorphisms of this space of Whittaker functions. It is thus natural to expect a p -adic analogue of this finite group result that relates the ring Z to maximal tori of GL_n over F . In sections 3 and 4 we construct a p -adic analogue $A_{F,n,1}$ of the ring $\overline{A}_{q,n}$, and give criteria for a family of maps to rings $A_{F,n,1}$ (with F fixed and n varying) to be isomorphisms.

These criteria were mostly verified for the centers Z_n of the categories of smooth $W(k)[\mathrm{GL}_n(F)]$ -modules in [H1]; section 5 is devoted to recalling the necessary facts. The upshot is that *if* one has suitable maps $Z_n \rightarrow A_{F,n,1}$, then they must be isomorphisms. On the other hand it is not at all clear *a priori* how to construct such maps; in the setting of finite groups this is done via Deligne-Lusztig theory and there is no suitable analogue of this theory for p -adic groups. On the other hand it turns out that the rings R_ν admit subalgebras R_ν^{inv} that *do* admit naturally defined maps to $A_{F,n,1}$, and it is not hard to show that the conjectured map $Z_n \rightarrow R_\nu$

factors through R_ν^{inv} . Sections 6, 7, 8 and 9 are devoted to constructing these R_ν and establishing their basic properties, including the existence of the maps to $A_{F,n,1}$. In particular, if the Weak Conjecture holds and one has maps $Z_n \rightarrow R_\nu$, then they fit into a sequence:

$$Z_n \rightarrow R_\nu^{\text{inv}} \rightarrow A_{F,n,1},$$

and the results of section 4 show that the composition of such maps is an isomorphism (and hence that the Strong Conjecture follows.)

Throughout this paper we adopt the following conventions: F is a p -adic field with residue field \mathbb{F}_q , k is an algebraically closed field of characteristic ℓ , \mathcal{K} is the field of fractions of $W(k)$, and $\overline{\mathcal{K}}$ is an algebraic closure of \mathcal{K} . Algebraic groups over F will be denoted by uppercase mathcal letters \mathcal{T} , \mathcal{G} , etc.; for any such group the corresponding uppercase letters T , G , etc. will denote the groups of F -points of \mathcal{T} , \mathcal{G} , and so forth. In particular there is an implicit dependence of T on \mathcal{T} ; thus if S is a set of tori, then

$$\prod_{T \in S} T$$

denotes the product of the F -points of the tori in S .

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2. FINITE GROUPS

We motivate our approach to the description of the Bernstein center by studying an analogous problem in the representation theory of finite groups. Most of the ideas in this section originally appear in work of Bonnafé-Kessar [1], but we reproduce them here (with a slightly different point of view) in order to draw parallels with our approach to the analogous questions for p -adic groups.

Fix distinct primes p and ℓ , and a power q of p . Let $\overline{\mathcal{G}}$ be the group GL_n over \mathbb{F}_q , and let $\overline{G} = \overline{\mathcal{G}}(\mathbb{F}_q)$. We will consider the representation theory of \overline{G} over the Witt ring $W(k)$, where k is an algebraic closure of \mathbb{F}_ℓ . Let \mathcal{K} be the field of fractions of $W(k)$, and fix an algebraic closure $\overline{\mathcal{K}}$ of \mathcal{K} .

Our principal object of study in this section will be the Gelfand-Graev representation $\overline{\Gamma}$ of \overline{G} , with coefficients in $W(k)$. Fix a Borel $\overline{\mathcal{B}}$ in $\overline{\mathcal{G}}$, with unipotent radical $\overline{\mathcal{U}}$, and let \overline{B} , \overline{U} denote the \mathbb{F}_q -points of $\overline{\mathcal{B}}$ and $\overline{\mathcal{U}}$ respectively. Also fix a generic character $\Psi : \overline{U} \rightarrow W(k)^\times$. Then, by definition, we have $\overline{\Gamma} = \text{c-Ind}_{\overline{U}}^{\overline{G}} \Psi$, where Ψ is considered as a $W(k)[\overline{G}]$ -module that is free over $W(k)$ of rank one, with the appropriate action of \overline{U} . The module $\overline{\Gamma}$ is then independent of the choice of Ψ , up to isomorphism.

The objective of this first section is to study the endomorphism ring of $\overline{\Gamma}$. Our main tool for doing so will be the Deligne-Lusztig induction and restriction functors of Bonnafé-Rouquier [BR]. Let \overline{L} be the subgroup of \overline{G} consisting of the $\overline{\mathbb{F}_q}$ -points of a (not necessarily split) Levi subgroup $\overline{\mathcal{L}}$ of GL_n , and choose a parabolic subgroup $\overline{\mathcal{P}}$ of GL_n whose Levi subgroup is $\overline{\mathcal{L}}$. Let $\text{Rep}_{W(k)}(\overline{G})$ and $\text{Rep}_{W(k)}(\overline{L})$ denote the

categories of $W(k)[\overline{G}]$ -modules and $W(k)[\overline{L}]$ -modules, respectively. Then Deligne-Lusztig induction and restriction are functors:

$$\begin{aligned} i_{\overline{\mathcal{L}} \subseteq \overline{\mathcal{P}}}^{\overline{G}} : \mathcal{D}^b(\mathrm{Rep}_{W(k)}(\overline{L})) &\rightarrow \mathcal{D}^b(\mathrm{Rep}_{W(k)}(\overline{G})) \\ r_{\overline{G}}^{\overline{\mathcal{L}} \subseteq \overline{\mathcal{P}}} : \mathcal{D}^b(\mathrm{Rep}_{W(k)}(\overline{G})) &\rightarrow \mathcal{D}^b(\mathrm{Rep}_{W(k)}(\overline{L})). \end{aligned}$$

We will be concerned exclusively with the case where $\overline{\mathcal{L}}$ is a maximal torus in \overline{G} . In this case the effect of Deligne-Lusztig restriction on $\overline{\Gamma}$ has been described by Bonnafé-Rouquier when $\overline{\mathcal{L}}$ is a Coxeter torus and by Dudas [Du] in general.

Theorem 2.1 (Bonnafé-Rouquier, Dudas). *When $\overline{\mathcal{L}}$ is a torus, there is a natural isomorphism:*

$$r_{\overline{G}}^{\overline{\mathcal{L}} \subseteq \overline{\mathcal{P}}} \overline{\Gamma} \cong W(k)[\overline{L}](-\ell(w))$$

in $\mathcal{D}^b(\mathrm{Rep}_{W(k)}(\overline{L}))$, where w is the element of the Weyl group of G giving the $\mathrm{GL}_n(\mathbb{F}_q)$ -conjugacy class of the Borel $\overline{\mathcal{P}}$, and $\ell(w)$ is its length.

Proof. This is the main theorem of [Du]. □

An immediate consequence of this result is that, when \overline{T} is the $\overline{\mathbb{F}}_q$ -points of a torus in GL_n , then an endomorphism of $\overline{\Gamma}$ gives rise, by functoriality of Deligne-Lusztig restriction, to an endomorphism of $W(k)[\overline{T}]$ (or, equivalently, an element of $W(k)[\overline{T}]$.) We thus obtain homomorphisms:

$$\Phi_{\overline{T}} : \mathrm{End}_{W(k)[\overline{G}]}(\overline{\Gamma}) \rightarrow W(k)[\overline{T}]$$

for each torus \overline{T} in \overline{G} . These are integral versions of the classical ‘‘Curtis homomorphisms’’.

Over $\overline{\mathcal{K}}$, it is not difficult to describe the structure of $\overline{\Gamma} \otimes \overline{\mathcal{K}}$, its endomorphism ring, and the associated Curtis homomorphisms. Recall that an irreducible representation π of \overline{G} is said to be *generic* if π contains the character Ψ , or, equivalently, if there exists a nonzero map from $\overline{\Gamma}$ to π . The irreducible generic representations of \overline{G} over $\overline{\mathcal{K}}$ are indexed by semisimple conjugacy classes s in \overline{G}' , where \overline{G}' is the group of $\overline{\mathbb{F}}_q$ -points in the group \overline{G}' that is dual to \overline{G} . More precisely, given such an s , there exists a unique irreducible generic representation $\overline{\mathrm{St}}_s$ in the rational series attached to s .

The association of rational series to semisimple conjugacy classes in \overline{G}' depends on choices which we now recall: let $\mu^{(p)}$ denote the prime-to- p roots of unity in $\overline{\mathcal{K}}$, let $(\mathbb{Q}/\mathbb{Z})^{(p)}$ denote the elements of order prime to p in (\mathbb{Q}/\mathbb{Z}) , and fix isomorphisms:

$$\mu^{(p)} \cong (\mathbb{Q}/\mathbb{Z})^{(p)} \cong \overline{\mathbb{F}}_q^\times.$$

Now let t be a semisimple element in \overline{G}' , let $\overline{\mathcal{T}}'$ be a maximal torus containing s , and let $\overline{\mathcal{T}}$ be the dual torus in \overline{G} . Let X and X' denote the character groups of $\overline{\mathcal{T}}$ and $\overline{\mathcal{T}}'$ respectively. We have isomorphisms:

$$\begin{aligned} \overline{\mathcal{T}}(\mathbb{F}_q) &\cong \mathrm{Hom}(X/(\mathrm{Fr}_q - 1)X, \mathbb{G}_m) \\ \overline{\mathcal{T}}'(\mathbb{F}_q) &\cong \mathrm{Hom}(X'/(\mathrm{Fr}_q - 1)X', \mathbb{G}_m) \end{aligned}$$

where Fr_q is the endomorphism induced by q -power Frobenius. We also have a natural duality $X/(\mathrm{Fr}_q - 1)X \cong \mathrm{Hom}(X'/(\mathrm{Fr}_q - 1)X', (\mathbb{Q}/\mathbb{Z})^{(p)})$. The identifications we fixed above then give rise to isomorphisms:

$$\overline{\mathcal{T}}'(\mathbb{F}_q) \cong \mathrm{Hom}(X'/(\mathrm{Fr}_q - 1)X', \mathbb{G}_m) \cong X/(\mathrm{Fr}_q - 1)X \cong \mathrm{Hom}(\overline{\mathcal{T}}(\mathbb{F}_q), \mu^{(p)}).$$

In this way we associate, to any semisimple element t of $\overline{G}'(\mathbb{F}_q)$, and any $\overline{\mathcal{T}}'$ containing s , a character $\theta_{\overline{\mathcal{T}}',t} : \overline{\mathcal{T}}'(\mathbb{F}_q) \rightarrow \overline{\mathcal{K}}^\times$.

It is immediate (by applying the idempotent of $\overline{\mathcal{K}}[\overline{G}]$ corresponding to the rational series attached to s to Theorem 2.1) that we then have:

Proposition 2.2. *Let $\overline{\mathcal{T}}$ be a maximal torus of \overline{G} , and let $\overline{\mathcal{B}}$ be a Borel containing $\overline{\mathcal{T}}$. Then, up to a cohomological shift depending only on $\overline{\mathcal{B}}$, we have:*

$$r_{\overline{G}}^{\overline{\mathcal{T}} \subseteq \overline{\mathcal{B}}} \overline{St}_s \cong \bigoplus_{t \sim s; t \in \overline{\mathcal{T}}'} \theta_{\overline{\mathcal{T}}',t}.$$

Returning to $\overline{\Gamma}$, we have a direct sum decomposition:

$$\overline{\Gamma} \otimes \overline{\mathcal{K}} \cong \bigoplus_s \overline{St}_s$$

It follows immediately that the endomorphism ring of $\overline{\Gamma} \otimes \overline{\mathcal{K}}$ is isomorphic to a product of copies of $\overline{\mathcal{K}}$, indexed by the semisimple conjugacy classes s in \overline{G}' . As the endomorphism ring $\text{End}_{W(k)[\overline{G}]}(\overline{\Gamma})$ of $\overline{\Gamma}$ embeds in this product, we see immediately that $\text{End}_{W(k)[\overline{G}]}(\overline{\Gamma})$ is reduced and commutative.

Indeed, it is not difficult to describe the maps $\Phi_{\overline{\mathcal{T}}} \otimes \overline{\mathcal{K}}$. The isomorphism:

$$\overline{\Gamma} \otimes \overline{\mathcal{K}} \cong \bigoplus_s \overline{St}_s$$

where s runs over semisimple conjugacy classes in \overline{G}' , gives rise to an isomorphism:

$$\text{End}_{W(k)[\overline{G}]}(\overline{\Gamma}) \otimes \overline{\mathcal{K}} \cong \prod_s \overline{\mathcal{K}}.$$

On the other hand we have a direct sum decomposition:

$$W(k)[\overline{\mathcal{T}}] \cong \bigoplus_t \theta_{\overline{\mathcal{T}},t}$$

of $W(k)[\overline{\mathcal{T}}]$ -modules, and hence an algebra isomorphism:

$$W(k)[\overline{\mathcal{T}}] \cong \prod_t \overline{\mathcal{K}}.$$

It follows immediately from the previous paragraph that $\Phi_{\overline{\mathcal{T}}}$ maps the factor of $\overline{\mathcal{K}}$ of $\text{End}_{W(k)[\overline{G}]}(\overline{\Gamma})$ corresponding to s identically to each factor of $W(k)[\overline{\mathcal{T}}]$ that corresponds to a t in the \overline{G}' -conjugacy class s , and to zero in the other factors.

Now let $\overline{\mathcal{T}}$ range over all tori in \overline{G} , and consider the product map:

$$\Phi : \text{End}_{W(k)[\overline{G}]}(\overline{\Gamma}) \rightarrow \prod_{\overline{\mathcal{T}}} W(k)[\overline{\mathcal{T}}].$$

For each pair $(\overline{\mathcal{T}}', t)$, where t is an element of $\overline{\mathcal{T}}'$, we have a map:

$$\xi_{\overline{\mathcal{T}}',t} : \prod_{\overline{\mathcal{T}}} W(k)[\overline{\mathcal{T}}] \rightarrow \overline{\mathcal{K}}$$

given by composing the projection onto $W(k)[\overline{\mathcal{T}}]$ with the map: $\theta_{\overline{\mathcal{T}}',t} : W(k)[\overline{\mathcal{T}}] \rightarrow \overline{\mathcal{K}}$.

Define an equivalence relation on such pairs by setting $(\overline{\mathcal{T}}'_1, t_1) \sim (\overline{\mathcal{T}}'_2, t_2)$ if t_1 and t_2 are conjugate in \overline{G}' . Then our description of each $\Phi_{\overline{\mathcal{T}}}$ shows that, when $(\overline{\mathcal{T}}'_1, t_1) \sim (\overline{\mathcal{T}}'_2, t_2)$, one has $\xi_{\overline{\mathcal{T}}'_1, t_1} \circ \Phi = \xi_{\overline{\mathcal{T}}'_2, t_2} \circ \Phi$.

This suggests the following definition:

Definition 2.3. An element x of $\prod_{\overline{\mathcal{T}}} W(k)[\overline{\mathcal{T}}]$ is *coherent* if for all pairs $(\overline{\mathcal{T}}'_1, t_1) \sim (\overline{\mathcal{T}}'_2, t_2)$, one has $\xi_{\overline{\mathcal{T}}'_1, t_1}(x) = \xi_{\overline{\mathcal{T}}'_2, t_2}(x)$. Let $\overline{A}_{q,n}$ denote the subalgebra of coherent elements of $\prod_{\overline{\mathcal{T}}} W(k)[\overline{\mathcal{T}}]$.

In this language, the above discussion amounts to the assertion that the image of Φ is contained in $\overline{A}_{q,n}$. In fact, we have:

Theorem 2.4. *The map $\Phi : \text{End}_{W(k)[\overline{G}]}(\Gamma) \rightarrow \overline{A}_{q,n}$ is an isomorphism.*

Proof. Since every semisimple t in \overline{G}' is contained in some maximal torus $\overline{\mathcal{T}}'$, one easily sees from our description of $\Phi \otimes \overline{\mathcal{K}}$ that $\Phi \otimes \overline{\mathcal{K}}$ (and hence Φ itself) is injective. On the other hand, as a $\overline{\mathcal{K}}$ -vector space, the dimension of $\overline{A}_{q,n} \otimes \overline{\mathcal{K}}$ is equal to the number of equivalence classes of pairs $(\overline{\mathcal{T}}', t)$, where t is an element of $\overline{\mathcal{T}}'$. This is simply the number of semisimple conjugacy classes in \overline{G}' , and hence also the dimension of $\text{End}_{W(k)[\overline{G}]}(e\Gamma) \otimes \overline{\mathcal{K}}$. Thus $\Phi \otimes \overline{\mathcal{K}}$ is an isomorphism onto $\overline{A}_{q,n} \otimes \overline{\mathcal{K}}$. Since $\overline{\mathcal{K}}$ is faithfully flat over \mathcal{K} it follows that $\Phi \otimes \mathcal{K}$ is an isomorphism onto $\overline{A}_{q,n} \otimes \mathcal{K}$.

It remains at this point to show that Φ is surjective onto $\overline{A}_{q,n}$. Since Φ becomes so after inverting ℓ , it suffices to show that if x is in $\overline{A}_{q,n}$, and ℓx is in the image of Φ , then x is as well. Equivalently, it suffices to show that Φ is injective modulo ℓ ; that is, that $\Phi \otimes k$ is injective.

Let x be an element of the kernel of $\Phi \otimes k$. We may regard x as an endomorphism of $\Gamma \otimes k$. The condition that x is in the kernel of Φ implies that x annihilates the Deligne-Lusztig restriction of $\Gamma \otimes k$ to any torus in G . In particular, x annihilates $\text{RHom}_{k[\overline{\mathcal{T}}]}(k[\overline{\mathcal{T}}], r_{\overline{\mathcal{G}}}^{\overline{\mathcal{T}} \subseteq \overline{\mathcal{B}}}(\Gamma \otimes k))$ for any Borel $\overline{\mathcal{B}}$ with maximal torus $\overline{\mathcal{T}}$. By the adjunction of Deligne-Lusztig induction and restriction, this implies that x annihilates $\text{RHom}_{k[\overline{\mathcal{G}}]}(i_{\overline{\mathcal{T}} \subseteq \overline{\mathcal{B}}}^{\overline{\mathcal{G}}} k[\overline{\mathcal{T}}], \Gamma \otimes k)$.

On the other hand, a result of Bonnafé-Rouquier ([BR], Théorème A), shows that the set of all complexes of the form $i_{\overline{\mathcal{L}} \subseteq \overline{\mathcal{B}}}^{\overline{\mathcal{G}}} k[\overline{\mathcal{T}}]$, as $\overline{\mathcal{T}}$ varies over all maximal tori in $\overline{\mathcal{G}}$, generates the subcategory of perfect complexes in $\mathcal{D}^b(\text{Rep}_k(\overline{\mathcal{G}}))$. It thus follows from the previous paragraph that x annihilates $\Gamma \otimes k$. Hence $\Phi \otimes k$ is injective as claimed. \square

It will be necessary for us to understand the interaction of Φ with certain idempotents of $W(k)[\overline{G}]$. An ℓ -regular semisimple conjugacy class s in \overline{G}' gives rise, via the choices we have made above, to an idempotent e_s in $W(k)[\overline{G}]$, that acts by the identity on the rational series corresponding to those s' in \overline{G} with ℓ -regular part s , and zero elsewhere. Similarly, for any $\overline{\mathcal{T}}$, the primitive idempotents of $W(k)[\overline{\mathcal{T}}]$ are in bijection with ℓ -regular elements of $\overline{\mathcal{T}}$. It is not difficult to verify (for instance, via the description of $\Phi_{\overline{\mathcal{T}}} \otimes \overline{\mathcal{K}}$ above) that for the idempotent e_s of \overline{G} corresponding to s , one has:

$$\Phi_{\overline{\mathcal{T}}}(e_s) = \sum_{t \sim s} e_t$$

where t runs over the elements of \bar{T} that are \bar{G} -conjugate to s . Denote $\Phi_{\bar{T}}(e_s)$ by $e_{s,\bar{T}}$.

The idempotent $\Phi(e_s)$ of $\bar{A}_{q,n}$ projects to $e_{s,\bar{T}}$ for every \bar{T} . Let $\bar{A}_{q,n,s} = \Phi(e_s)\bar{A}_{q,n}$. We then have product decompositions:

$$\begin{aligned} \text{End}_{W(k)[\bar{G}]}(\bar{\Gamma}) &\cong \prod_s \text{End}_{W(k)[\bar{G}]}(e_s\bar{\Gamma}) \\ \bar{A}_{q,n} &\cong \prod_s \bar{A}_{q,n,s} \end{aligned}$$

compatible with the map Φ . In particular, we have:

Proposition 2.5. *The map Φ induces isomorphisms:*

$$\text{End}_{W(k)[\bar{G}]}(e_s\bar{\Gamma}) \cong \bar{A}_{q,n,s}$$

for every s .

We will primarily be interested in understanding $\bar{A}_{q,n,s}$ in the case where there is a cuspidal representation of \bar{G} over k that is not annihilated by e_s . This occurs precisely when s is the ℓ -regular part of an element s' of \bar{G}' whose characteristic polynomial is irreducible over \mathbb{F}_q . For any t in s the characteristic polynomial of t is a power of an irreducible polynomial over \mathbb{F}_q , of degree d dividing n , and the normalizer $\bar{\mathcal{G}}'_t$ of t in \bar{G}' is isomorphic to $\text{Res}_{\mathbb{F}_{q^d}/\mathbb{F}_q} \text{GL}_{\frac{n}{d}}/\mathbb{F}_{q^d}$.

Let $\bar{\mathcal{G}}_t$ be the dual of $\bar{\mathcal{G}}'_t$; it is a Levi in \bar{G} . Each maximal torus $\bar{\mathcal{T}}'$ in $\bar{\mathcal{G}}'_t$ is thus also a maximal torus of \bar{G}' containing the element t , and every maximal torus of \bar{G}' containing an element in s is conjugate to some $\bar{\mathcal{T}}'$ containing t .

We thus have an isomorphism

$$\Theta : \bar{A}_{q,n,s} \rightarrow \bar{A}_{q^d, \frac{n}{d}, s}$$

such that, for each x in $\bar{A}_{q,n,s}$, and each maximal torus $\bar{\mathcal{T}}_s$ in $\bar{\mathcal{G}}_s$, the projection of $\Theta(x)$ to $W(k)[\bar{\mathcal{T}}_s]$ is equal to the projection of x to $W(k)[\bar{\mathcal{T}}]$, where we now regard $\bar{\mathcal{T}}$ as a maximal torus in \bar{G} . “Twisting” this isomorphism by the inverse of the character $\theta_{\bar{\mathcal{T}}', t}$ of $\bar{\mathcal{T}}$ we obtain an isomorphism of $\bar{A}_{q,n,s}$ with $\bar{A}_{q^d, \frac{n}{d}, 1}$.

It is not hard to describe what this isomorphism does on $\bar{\mathcal{K}}$ -points. Indeed, maps $\bar{A}_{q,n,s} \rightarrow \bar{\mathcal{K}}$ are in bijection with semisimple conjugacy classes s' in \bar{G}' with ℓ -regular part s . Similarly, maps $\bar{A}_{q^d, \frac{n}{d}, 1}$ are in bijection with semisimple conjugacy classes ζ in \bar{G}'_s with trivial ℓ -regular part. The map $\zeta \mapsto t\zeta$ is a bijection between the latter and the former, and, under the bijections above, gives the map on $\bar{\mathcal{K}}$ -points induced by the isomorphism of $\bar{A}_{q,n,s}$ with $\bar{A}_{q^d, \frac{n}{d}, 1}$.

3. THE RING $A_{F,n,1}$

It is tempting, given the analogy between the Bernstein center and the endomorphisms of the Gelfand-Graev module in the finite group setting, to ask if one can establish an analogue of the results of the previous section in the context of smooth $W(k)[\text{GL}_n(F)]$ -modules, for a p -adic field F . The approach of the previous section is not available to us in this new context, as there are at present no satisfactory analogues of Deligne-Lusztig induction or restriction for smooth representations of p -adic groups.

We are thus forced to take a more circuitous approach. The first step will be to construct a suitable analogue of the ring $\overline{A}_{q,n}$ in the p -adic setting.

Let \mathcal{T} be a maximal torus of $\mathcal{G} = \mathrm{GL}_n/F$ that is split by an unramified extension of F . (In what follows, we will refer to such \mathcal{T} as *unramified* maximal tori.) Let T and G denote the F -points of \mathcal{T} and \mathcal{G} . Let $\theta : T \rightarrow \overline{\mathcal{K}}^\times$ be a smooth character. We may associate to θ a semisimple representation of W_F as follows: the group \mathcal{T} is isomorphic to a product of groups of the form $\mathrm{Res}_{F_i/F} \mathbb{G}_m$ where each F_i is a finite unramified extension of F . This gives a corresponding decomposition of T as a product of groups F_i^\times . With respect to the latter, the character θ decomposes as a product of characters $\theta_i : F_i^\times \rightarrow \overline{\mathcal{K}}^\times$, and we may regard each θ_i as a character of the corresponding Weil group W_{F_i} . Define a representation $\rho_{\mathcal{T},\theta}$ of W_F by:

$$\rho_{\mathcal{T},\theta} = \bigoplus_i \mathrm{Ind}_{W_{F_i}}^{W_F} \theta_i.$$

One checks easily that the representation $\rho_{\mathcal{T},\theta}$ is semisimple. Note that the representation $\rho_{\mathcal{T},\theta}$ depends, up to isomorphism, only on \mathcal{T} and θ .

We will be primarily interested in this construction for those θ whose ramification has ℓ -power order. More precisely, we call a $\theta : T \rightarrow \overline{\mathcal{K}}^\times$ ℓ -ramified if, for each i , $\theta_i(I_F)$ has ℓ -power order, where I_F is the inertia group of F , regarded as a subgroup of F_i^\times . It is easy to see:

Lemma 3.1. *Suppose that θ is ℓ -ramified. Then $\rho_{\mathcal{T},\theta}(I_F)$ has ℓ -power order.*

Proof. This is an easy consequence of the Mackey induction-restriction formula. \square

Define an equivalence relation on pairs (\mathcal{T}, θ) , where θ is a smooth character of T with values in $\overline{\mathcal{K}}^\times$, by setting $(\mathcal{T}_1, \theta_1) \sim (\mathcal{T}_2, \theta_2)$ if $\rho_{\mathcal{T}_1, \theta_1} = \rho_{\mathcal{T}_2, \theta_2}$.

For each unramified maximal torus \mathcal{T} , let $T^{(\ell)}$ denote the maximal compact subgroup of T of order prime to ℓ , and let $Z_{\mathcal{T},1}$ denote the group ring $W(k)[T/T^{(\ell)}]$. Then $\overline{\mathcal{K}}$ -points of $\mathrm{Spec} Z_{\mathcal{T},1}$ are in bijection with ℓ -ramified characters $\theta : T \rightarrow \overline{\mathcal{K}}^\times$.

We may now define the ring $A_{F,n,1}$, an analogue for p -adic groups of the ring $\overline{A}_{q,n,1}$ of the previous section.

Definition 3.2. An element x of the product $\prod_{\mathcal{T}} Z_{\mathcal{T},1}$ (where \mathcal{T} ranges over all unramified maximal tori of $\mathrm{GL}_n(F)$) is *coherent* if, for all pairs $(\mathcal{T}_1, \theta_1) \sim (\mathcal{T}_2, \theta_2)$, the projections $x_{\mathcal{T}_1}$ and $x_{\mathcal{T}_2}$ of x to $Z_{\mathcal{T}_1,1}$ and $Z_{\mathcal{T}_2,1}$ satisfy $\theta_1(x_{\mathcal{T}_1}) = \theta_2(x_{\mathcal{T}_2})$. We denote the subring of coherent elements of $\prod_{\mathcal{T}} Z_{\mathcal{T},1}$ by $A_{F,n,1}$.

We begin our study of $A_{F,n,1}$ by constructing certain natural elements. Observe:

Lemma 3.3. *Let w be an element of W_F . There exists an element $\mathrm{tr}_{\mathcal{T},w}$ of $Z_{\mathcal{T},1}$ with the property that for all ℓ -ramified $\theta : T \rightarrow \overline{\mathcal{K}}$, $\theta(\mathrm{tr}_{\mathcal{T},w})$ is equal to the trace of $\rho_{\mathcal{T},\theta}(w)$.*

Proof. Write T as a product of F_i^\times , with F_i/F unramified, and decompose θ as a product of characters θ_i of F_i^\times according to this identification.

For each i , choose a set of representatives $w_{i,1}, \dots, w_{i,d}$ for the cosets of w_{F_i} in W_F . For each coset fixed by right multiplication by w ; that is, for each j such that $w_{i,j}w$ lies in $W_{F_i} w_{i,j}$, let $t_{i,j}$ be the image of $w_{i,j}w w_{i,j}^{-1}$ under the map $W_{F_i} \rightarrow F_i^\times$. Then the trace of w on $\mathrm{Ind}_{W_{F_i}}^{W_F} \theta_i$ is equal to the sum of the $\theta_i(t_{i,j})$.

Let $\tilde{t}_{i,j}$ be the element of T that (under our identification of T with the product of F_i^\times) is equal to $t_{i,j}$ on the factor F_i and one on the other factors. Then the sum of all the $\tilde{t}_{i,j}$ projects to an element of $Z_{T,1}$ with the desired property. \square

The element tr_w of $\prod_{\mathcal{T}} Z_{T,1}$ whose projection to each $Z_{T,1}$ is $\text{tr}_{\mathcal{T},w}$ then lies in $A_{F,n,1}$.

Next we will describe the $\bar{\mathcal{K}}$ -points of $\text{Spec } A_{F,n,1}$.

An equivalence class (\mathcal{T}, θ) gives rise to a map $\xi_{\mathcal{T},\theta} : \text{Spec } A_{F,n,1} \rightarrow \bar{\mathcal{K}}$ defined by $\xi_{\mathcal{T},\theta}(x) = \theta(x_{\mathcal{T}})$; it follows immediately from the definition of coherence that this only depends on (\mathcal{T}, θ) up to equivalence. In fact we have:

Proposition 3.4. *The map $(\mathcal{T}, \theta) \mapsto \xi_{\mathcal{T},\theta}$ is a bijection between equivalence classes of (\mathcal{T}, θ) and $\bar{\mathcal{K}}$ -points of $\text{Spec } A_{F,n,1}$.*

Because the equivalence relation on pairs (\mathcal{T}, θ) is defined in terms of the map $(\mathcal{T}, \theta) \mapsto \rho_{(\mathcal{T},\theta)}$, Proposition 3.4 can be interpreted as giving a bijection between the $\bar{\mathcal{K}}$ -points of $\text{Spec } A_{F,n,1}$ and isomorphism classes of representations $\rho : W_F \rightarrow \text{GL}_n(\bar{\mathcal{K}})$ that are of the form $\rho_{(\mathcal{T},\theta)}$ for some (\mathcal{T}, θ) with θ ℓ -ramified. On the other hand, it is easy to see that any semisimple representation $\rho : W_F \rightarrow \text{GL}_n(\bar{\mathcal{K}})$ such that $\rho(I_F)$ has ℓ -power order is of the form $\rho_{(\mathcal{T},\theta)}$ for some unramified \mathcal{T} and ℓ -ramified θ . We thus have:

Corollary 3.5. *There is a unique bijection between $\bar{\mathcal{K}}$ -points of $\text{Spec } A_{F,n,1}$ and semisimple representations $\rho : W_F \rightarrow \text{GL}_n(\bar{\mathcal{K}})$ such that $\rho(I_F)$ has ℓ -power order that takes $\xi_{\mathcal{T},\theta}$ to $\rho_{\mathcal{T},\theta}$ for all pairs (\mathcal{T}, θ) .*

In order to prove Proposition 3.4, we first observe:

Lemma 3.6. *The map $A_{F,n,1} \rightarrow Z_{T,1}$ makes $Z_{T,1}$ into a finitely generated $A_{F,n,1}$ -module. In particular, the map $\text{Spec } Z_{T,1} \rightarrow \text{Spec } A_{F,n,1}$ is proper.*

Proof. Let Z' be the subalgebra of $Z_{T,1}$ generated over $W(k)$ by elements of the form $\text{tr}_{T,w}$ for $w \in W_F$. Then $Z_{T,1}$ is a finitely generated Z' -module. (Indeed, it is straightforward to see that $Z_{T,1}$ is even finitely generated over the subalgebra generated by $\text{tr}_{T,\text{Fr}}, \text{tr}_{T,\text{Fr}^2}, \dots, \text{tr}_{T,\text{Fr}^n}$, where Fr is a Frobenius element of W_F .) Since such elements are in the image of the map $A_{F,n,1} \rightarrow Z_{T,1}$, the claim is clear. \square

Proof of Proposition 3.4: Suppose that we have $(\mathcal{T}_1, \theta_1)$ and $(\mathcal{T}_2, \theta_2)$ such that $\xi_{\mathcal{T}_1, \theta_1} = \xi_{\mathcal{T}_2, \theta_2}$. Then in particular these maps agree on the elements tr_w of $A_{q,n,1}$. It follows that for any element w of W_F , the traces of w on $\rho_{\mathcal{T}_1, \theta_1}$ and $\rho_{\mathcal{T}_2, \theta_2}$ agree, and hence that these two semisimple representations of W_F are isomorphic. Thus $(\mathcal{T}_1, \theta_1)$ is equivalent to $(\mathcal{T}_2, \theta_2)$, and we have proven injectivity.

Conversely, the ring $A_{F,n,1}$ embeds in the product of the rings $Z_{T,1}$ as \mathcal{T} varies over a (finite) set of representatives for the conjugacy classes of maximal tori. It follows that the image of the map

$$\prod_{\mathcal{T}} \text{Spec } Z_{T,1} \rightarrow \text{Spec } A_{F,n,1}$$

is dense. But since this map is proper, it is surjective on $\bar{\mathcal{K}}$ -points. \square

Having classified the $\bar{\mathcal{K}}$ -points of $\text{Spec } A_{F,n,1}$, we turn to other fundamental structural questions about the ring $A_{F,n,1}$. In particular we will show that it is possible

to define the ring $A_{F,n,1}$ in terms of a considerably smaller number of maximal tori than we are currently considering.

To formulate the problem more precisely, suppose we have a collection S of tori in $\mathrm{GL}_n(F)$, and define $A_{F,n,S,1}$ to be the subring of $\prod_{\mathcal{T} \in S} Z_{\mathcal{T},1}$ consisting of those $x_{\mathcal{T}}$ such that, for all pairs $(\mathcal{T}_1, \theta_1) \sim (\mathcal{T}_2, \theta_2)$, with $\mathcal{T}_1, \mathcal{T}_2$ in S , we have $\theta_1(x_{\mathcal{T}_1}) = \theta_2(x_{\mathcal{T}_2})$.

There is an obvious map $A_{F,n,1} \rightarrow A_{F,n,1,S}$ that simply projects to the factors of $A_{F,n,1}$ corresponding to tori in S .

Proposition 3.7. *Suppose that for each unramified maximal torus T of $\mathrm{GL}_n(F)$ that is not conjugate to a torus in S , we have a map $\phi_T : A_{F,n,1,S} \rightarrow Z_{T,1}$ for which the composition*

$$A_{F,n,1} \rightarrow A_{F,n,1,S} \rightarrow Z_{T,1}$$

coincides with the usual projection of $A_{F,n,1}$ onto $Z_{T,1}$. Then the projection:

$$A_{F,n,1} \rightarrow A_{F,n,1,S}$$

is an isomorphism.

Proof. We define a map $A_{F,n,1,S} \rightarrow A_{F,n,1}$ as follows: given an element $x = \{x_{\mathcal{T}}\} : \mathcal{T} \in S$ of $A_{F,n,1,S}$, define an element y of $A_{F,n,1}$ by setting $y_{\mathcal{T}} = \phi_{\mathcal{T}}(x)$ if \mathcal{T} is not conjugate to an element of S , and $y_{\mathcal{T}} = gx_{\mathcal{T}'}g^{-1}$ if $\mathcal{T} = g\mathcal{T}'g^{-1}$ for some $\mathcal{T}' \in S$. This is independent of the choice of g since for $x \in A_{F,n,1,S}$, $x_{\mathcal{T}}$ is fixed by the normalizer of \mathcal{T} in G for all \mathcal{T} . The compositions

$$A_{F,n,1,S} \rightarrow A_{F,n,1} \rightarrow A_{F,n,1,S}$$

$$A_{F,n,1} \rightarrow A_{F,n,1,S} \rightarrow A_{F,n,1}$$

are clearly the identity by construction. \square

It will be useful to us to give a certain reinterpretation of the conclusion of Proposition 3.7 in terms of properties of the map $A_{F,n,1} \rightarrow \prod_{\mathcal{T} \in S} Z_{\mathcal{T},1}$.

Definition 3.8. Let $f : A \hookrightarrow B$ be an embedding of finitely generated, reduced, ℓ -torsion free $W(k)$ -algebras. We say that an element b of B is *f-consistent* if for any pair of maps $g_1, g_2 : B \rightarrow \overline{K}$ such that $g_1 \circ f = g_2 \circ f$, we have $g_1(b) = g_2(b)$. We say that f is \overline{K} -saturated (or A is \overline{K} -saturated in B) if every f -consistent element of B lies in A .

Note that $b \in B$ is f -consistent if, and only if, ℓb is f -consistent, so a \overline{K} -saturated map is saturated in the usual sense (i.e. b is in the image if, and only if, ℓb is.)

Proposition 3.9. *Under the hypotheses of Proposition 3.7, the map:*

$$A_{F,n,1} \rightarrow \prod_{\mathcal{T} \in S} Z_{\mathcal{T},1}$$

is \overline{K} -saturated.

Proof. By Proposition 3.7 this map is an embedding, and its image is $A_{F,n,1,S}$. The set of \overline{K} -points of $\prod_{\mathcal{T} \in S} Z_{\mathcal{T},1}$ is in bijection with pairs (\mathcal{T}, θ) , with $\mathcal{T} \in S$ and θ a character of T whose restriction to $T^{(\ell)}$ is trivial. The map $\mathrm{Spec} Z_{\mathcal{T},1} \rightarrow \mathrm{Spec} A_{F,n,1}$ takes a pair (\mathcal{T}, θ) to the point of $A_{F,n,1}$ corresponding to $\rho_{(\mathcal{T}, \theta)}$. Thus the consistent elements of $\prod_{\mathcal{T} \in S} Z_{\mathcal{T},1}$ are those x such that $x_{\mathcal{T}}(\theta) = x_{\mathcal{T}'}(\theta')$ whenever $\rho_{(\mathcal{T}, \theta)} = \rho_{(\mathcal{T}', \theta')}$; these are precisely the elements of $A_{F,n,1,S}$. \square

We now describe a reduction on the set of tori that need to be considered. Note that the conjugacy classes of unramified maximal tori in G are in bijection with partitions ν of n , where the partition $\nu = (\nu_1, \dots, \nu_r)$ corresponds to a torus whose F -points are isomorphic to $\prod_i F_i^\times$ with F_i/F unramified of degree ν_i .

We will call a partition ν *relevant* if each ν_i lies in the set $\{1, e_q, \ell e_q, \ell^2 e_q, \dots\}$, where e_q is the order of q modulo ℓ . Given a partition ν , define the *associated relevant partition* ν' by replacing each ν_i with ν_i/m_i copies of m_i , where m_i is the largest element of the set $\{1, e_q, \ell e_q, \ell^2 e_q\}$ dividing ν_i .

The importance of relevant partitions stems from the following result:

Lemma 3.10. *Let ν be a partition of n , and ν' the associated relevant partition. Let \mathcal{T} and \mathcal{T}' be unramified tori in the conjugacy classes associated to ν and ν' , respectively. Let $N_{\mathcal{T}'}$ be the subgroup of F -points in the normalizer of \mathcal{T}' . Then there is a map: $Z_{\mathcal{T}',1}^{N_{\mathcal{T}'}} \rightarrow Z_{\mathcal{T},1}$ such that the composition:*

$$A_{F,n,1} \rightarrow Z_{\mathcal{T}',1}^{N_{\mathcal{T}'}} \rightarrow Z_{\mathcal{T},1}$$

takes the $\bar{\mathcal{K}}$ -point of $\text{Spec } Z_{\mathcal{T},1}$ corresponding to a character θ to the point of $\text{Spec } A_{F,n,1}$ corresponding to $\rho_{\mathcal{T},\theta}$.

Proof. One reduces easily to the case where ν is the one-element partition $\{n\}$ of n . Let m be the largest element of $\{1, e_q, \ell e_q, \dots\}$ dividing n , and set $j = \frac{n}{m}$, so that ν' is the partition $\{m, m, \dots, m\}$ of n . Let a be the largest integer such that ℓ^a divides $q^m - 1$; then ℓ^a is also the largest power of ℓ dividing $q^n - 1$.

Let F_m and F_n be the unramified extensions of F of degree m and n , respectively, and fix a uniformizer ϖ_n of F_n such that ϖ_n^j is a uniformizer of F_m . Let ζ be a primitive ℓ^a -th root of unity in F_n ; then ζ lies in F_m and generates the subgroups of ℓ -power roots of unity in both fields. We thus have isomorphisms:

$$\begin{aligned} Z_{\mathcal{T},1} &\cong W(k)[\zeta, \varpi_n^{\pm 1}] / \langle \zeta^{\ell^a} - 1 \rangle \\ Z_{\mathcal{T}',1} &\cong \left[W(k)[\zeta, \varpi_m^{\pm 1}] / \langle \zeta^{\ell^a - 1} \rangle \right]^{\otimes j}. \end{aligned}$$

Let Q_i denote the element ϖ_m in the i th tensor factor of $Z_{\mathcal{T}',1}$, and let ζ_i denote the element ζ in the i th tensor factor, and consider these as elements of $Z_{\mathcal{T}',1}$ by tensoring them with 1 in the other tensor factors.

Let θ be an ℓ -ramified character: $W_{F_n} \rightarrow \bar{\mathcal{K}}^\times$. Then θ extends, in j different ways, to a character of W_{F_m} (corresponding to the j distinct j th roots of $\theta(\varpi_n)$), and the induction of θ to a representation of W_{F_m} is the direct sum of these j characters.

Define a map $Z_{\mathcal{T}',1} \rightarrow Z_{\mathcal{T},1}[Q] / \langle Q^j - \varpi_n \rangle$ by sending each ζ_i to ζ , and Q_i to $\eta^i Q$, where η is a primitive j th root of unity in $Z_{\mathcal{T},1}$. (Note that such a root always exists.)

The restriction of this map to the $N_{\mathcal{T}'}$ -invariant elements of $Z_{\mathcal{T}',1}$ has image in $Z_{\mathcal{T},1}$. Indeed, let Z' be the subalgebra of $Z_{\mathcal{T}',1}$ generated by the ζ_i and the elementary symmetric functions in the Q_i . Then the image of Z' is clearly contained in $Z_{\mathcal{T},1}$. On the other hand, any $N_{\mathcal{T}'}$ -invariant element in $Z_{\mathcal{T}',1}$ is the sum of an element of Z' and an element of the ideal of $Z_{\mathcal{T},1}$ generated by $\zeta_i - \zeta_j$, and the latter elements all map to zero in $Z_{\mathcal{T},1}$. We thus obtain a map

$$Z_{\mathcal{T}',1}^{N_{\mathcal{T}'}} \rightarrow Z_{\mathcal{T},1}$$

and one easily verifies that this map has the desired property. \square

Corollary 3.11. *Let S be any subset of the unramified maximal tori of G such that, for all relevant maximal tori \mathcal{T} of G , S contains a conjugate of \mathcal{T} . Then the map $A_{F,n,1} \rightarrow A_{F,n,1,S}$ is an isomorphism, and the embedding*

$$A_{F,n,1} \rightarrow \prod_{\mathcal{T} \in S} Z_{\mathcal{T},1}$$

is $\overline{\mathcal{K}}$ -saturated.

Proof. For any unramified maximal torus \mathcal{T} of G , the map $A_{F,n,1,S} \rightarrow Z_{\mathcal{T},1}$ factors through the $N_{\mathcal{T}}$ -invariants of $Z_{\mathcal{T},1}$. Thus the maps of Lemma 3.10 give rise to maps $A_{F,n,1,S} \rightarrow Z_{\mathcal{T},1}$ for any maximal torus \mathcal{T} in G . The claim then follows immediately from Proposition 3.7. \square

We now consider a certain natural class of maps relating the rings $A_{F,n,1}$ as n varies. Given a partition ν of n , let $A_{F,\nu,1}$ denote the tensor product

$$A_{F,\nu,1} := \bigotimes_i A_{F,\nu_i,1}.$$

When ν is a partition of n , there are natural maps $\text{Ind}_{\nu} : A_{F,n,1} \rightarrow A_{F,\nu,1}$ which we now describe. First note that there is an embedding:

$$A_{F,\nu,1} \hookrightarrow \prod_{\mathcal{T}_1, \dots, \mathcal{T}_r} \bigotimes_i Z_{\mathcal{T}_i,1}$$

where $\mathcal{T}_1, \dots, \mathcal{T}_r$ runs over all sequences of ℓ -ramified maximal tori \mathcal{T}_i of $\text{GL}_{\nu_i}(F)$. Given such a sequence, we let \mathcal{T} be the product of the \mathcal{T}_i , and we then have a map:

$$A_{F,n,1} \rightarrow Z_{\mathcal{T},1} \cong \bigotimes_i Z_{\mathcal{T}_i,1}.$$

Taking the product of these maps over all sequences $\mathcal{T}_1, \dots, \mathcal{T}_r$ gives a map:

$$A_{F,n,1} \rightarrow \prod_{\mathcal{T}_1, \dots, \mathcal{T}_r} \bigotimes_i Z_{\mathcal{T}_i,1}$$

and one verifies easily that this map factors through the image of $A_{F,\nu,1}$ in this product.

Note that a $\overline{\mathcal{K}}$ -point of $\text{Spec } A_{F,\nu,1}$ corresponds to a sequence ρ_i of representations $W_F/I_F^{(\ell)} \rightarrow \text{GL}_{\nu_i}(\overline{\mathcal{K}})$. The map Ind_{ν} takes such a point to the point of $\text{Spec } A_{F,n,1}$ corresponding to the direct sum of the ρ_i .

In certain cases we can say quite a lot about the image and kernel of the map Ind_{ν} . Given n , define the *maximal relevant partition* $\nu^{\max}(n)$ to be the coarsest possible partition of n , all of whose elements lie in $\{1, e_q, \ell e_q, \ell^2 e_q, \dots\}$; since each element of this set divides the next largest, this is well-defined, and every relevant partition of n refines $\nu^{\max}(n)$. When n is clear from the context, we will denote $\nu^{\max}(n)$ simply by ν^{\max} .

Proposition 3.12. *The map $\text{Ind}_{\nu^{\max}} : A_{F,n,1} \rightarrow A_{F,\nu^{\max},1}$ is injective, and $\overline{\mathcal{K}}$ -saturated.*

Proof. Let M be the standard Levi of G given by the partition ν^{\max} , and let S be the set of all unramified tori of G that are contained in M . Each such torus \mathcal{T}

factors as a product \mathcal{T}_i , where each \mathcal{T}_i is a maximal torus of the factor $GL_{\nu_i^{\max}}(F)$ of M . We have maps:

$$A_{F,n,1} \rightarrow A_{F,\nu^{\max},1} \hookrightarrow \bigotimes_i \prod_{\mathcal{T}_i} Z_{\mathcal{T}_i,1} \cong \prod_{\mathcal{T} \in S} Z_{\mathcal{T},1}$$

and the composition is the natural map $A_{F,n,1} \rightarrow \prod_{\mathcal{T} \in S} Z_{\mathcal{T},1}$. Since S contains all relevant tori, this natural map is injective and $\overline{\mathcal{K}}$ -saturated. Moreover, each map in the chain induces a surjection on $\overline{\mathcal{K}}$ -points. Thus every element of $A_{F,\nu^{\max},1}$ that is $\overline{\mathcal{K}}$ -consistent is also $\overline{\mathcal{K}}$ -consistent when considered as an element of $\prod_{\mathcal{T} \in S} Z_{\mathcal{T},1}$, and thus lies in $A_{F,n,1}$ as required. \square

Now let n be an element of $\{1, e_q, le_q, \dots\}$ (so that $\nu^{\max}(n) = \{n\}$). Suppose that $n > 1$, and let m be the largest element of $\{1, e_q, le_q, \dots\}$ that is strictly less than n . Set $j = \frac{n}{m}$, and let ν be the partition of n consisting of j copies of m . We then have the map

$$\text{Ind}_\nu : A_{F,n,1} \rightarrow A_{F,\nu,1} = A_{F,m,1}^{\otimes j}.$$

Proposition 3.13. *Let \mathcal{T} be a maximal torus of $GL_n(F)$, such that T is isomorphic to the multiplicative group of the unramified extension of F of degree n . Then the natural map:*

$$A_{F,n,1} \rightarrow Z_{T,1} \times A_{F,\nu,1}$$

is injective and $\overline{\mathcal{K}}$ -saturated.

Proof. Let M be the standard Levi of G given by the partition ν , and let S be the set of all unramified tori of G that are contained in M , plus the torus \mathcal{T} . The argument now proceeds in precisely the same way as the proof of Proposition 3.12, using the chain of maps:

$$A_{F,n,1} \rightarrow Z_{T,1} \times A_{F,\nu,1} \rightarrow Z_{T,1} \times \bigotimes_{i=1}^j \prod_{\mathcal{T}_i} Z_{\mathcal{T}_i,1} \cong \prod_{\mathcal{T} \in S} Z_{\mathcal{T},1}.$$

\square

As a consequence, note that projection onto $Z_{T,1}$ identifies the kernel of Ind_ν with a subset of $Z_{T,1}$. In order to obtain a description of this kernel in these terms, we will construct a subalgebra of $A_{F,n,1}$ that is isomorphic to $\overline{A}_{q,n,1}$. For a torus \mathcal{T} , let T^c denote the maximal compact subgroup of the F -points T of \mathcal{T} , and let $Z_{T,1}^c$ denote the subring $W(k)[T^c/T^{(\ell)}]$ of $Z_{T,1}$. Similarly, let $A_{F,n,1}^c$ denote the subalgebra of $A_{F,n,1}$ consisting of all elements x such that, for all unramified tori \mathcal{T} , the element $x_{\mathcal{T}}$ of $Z_{T,1}$ lies in $Z_{T,1}^c$. We will show that we can identify $A_{F,n,1}^c$ with $\overline{A}_{q,n,1}$ in a natural way.

For any positive integer m , let \mathcal{O}_m denote the ring of integers of the unramified extension of degree m of F . For a partition $\nu = \{\nu_1, \dots, \nu_r\}$ of m , define \mathcal{O}_ν to be the product $\prod_i \mathcal{O}_{\nu_i}$. For each such partition ν of n , fix an \mathcal{O}_F -basis of \mathcal{O}_ν ; this gives rise to an F -linear isomorphism $\mathcal{O}_\nu \otimes_{\mathcal{O}_F} F \cong F^n$ and an \mathbb{F}_q -linear isomorphism $\mathcal{O}_\nu \otimes_{\mathcal{O}_F} \mathbb{F}_q \cong \mathbb{F}_q^n$. We then obtain tori \mathcal{T}_ν and $\overline{\mathcal{T}}_\nu$, over F and \mathbb{F}_q respectively, as the automorphisms of $\mathcal{O}_\nu \otimes_{\mathcal{O}_F} F$ (resp. $\mathcal{O}_\nu \otimes_{\mathcal{O}_F} \mathbb{F}_q$) that commute with the action of \mathcal{O}_ν^\times .

Note that for each ν , the quotient $T_\nu^c/T_\nu^{(\ell)}$ is (via reduction modulo a uniformizer of F), canonically isomorphic to the group of ℓ -power torsion elements in \overline{T}_ν . This gives us a natural isomorphism:

$$e_1 Z_{\overline{T}_\nu} \cong Z_{T_\nu,1}^c.$$

Taking the product of all of these, we obtain an isomorphism:

$$\prod_\nu e_1 Z_{\overline{T}_\nu} \cong \prod_\nu Z_{T_\nu,1}^c$$

and we regard the target as a subalgebra of $\prod_\nu Z_{T_\nu,1}$.

Proposition 3.14. *Let S be the collection of tori $\{\mathcal{T}_\nu\}$ as ν varies over the partitions of n . Let \bar{x} be an element of $\prod_\nu e_1 Z_{\overline{T}_\nu}$, and let x be its image in $\prod_\nu Z_{T_\nu,1}$. Then x lies in $A_{F,n,1,S}$ if, and only, if, \bar{x} lies in the image of the map:*

$$\overline{A}_{q,n,1} \rightarrow \prod_\nu e_1 Z_{\overline{T}_\nu}.$$

In particular, there is a unique map:

$$\overline{A}_{q,n,1} \rightarrow A_{F,n,1}$$

making the diagrams:

$$\begin{array}{ccc} \overline{A}_{q,n,1} & \rightarrow & A_{F,n,1} \\ \downarrow & & \downarrow \\ e_1 Z_{\overline{T}_\nu} & \rightarrow & Z_{T_\nu,1} \end{array}$$

commute for all ν .

Proof. As in section 2, fix isomorphisms:

$$\mu^{(p)} \cong (\mathbb{Q}/\mathbb{Z})^{(p)} \cong \overline{\mathbb{F}}_q^\times.$$

This gives rise to a bijection, for each \overline{T}_ν , between elements s of \overline{T}'_ν and characters θ_s of \overline{T}_ν .

Now, given a partition ν and a character θ of $T_\nu/T_\nu^{(\ell)}$, the restriction of θ to T_ν^c is naturally a character of \overline{T}_ν , and thus corresponds to an element s of \overline{T}'_ν , which we regard as a semisimple element of \overline{G}' .

Let \mathcal{T}_1 and \mathcal{T}_2 be tori in S , and let θ_1 and θ_2 be characters of T_1 and T_2 respectively. We have semisimple elements s_1 and s_2 of \overline{G}' attached to $(\mathcal{T}_1, \theta_1)$ and $(\mathcal{T}_2, \theta_2)$. It is now straightforward to check that s_1 and s_2 are conjugate in \overline{G}' if, and only if, the restriction of $\rho_{\mathcal{T}_1, \theta_1}$ to I_F is isomorphic to the restriction of $\rho_{\mathcal{T}_2, \theta_2}$ to I_F . Note that this defines a bijection between n -dimensional representations of $I_F/I_F^{(\ell)}$ that extend to W_F and semisimple conjugacy classes in \overline{G}' .

Fix a $\mathcal{T} \in S$ and a character θ of $T/T^{(\ell)}$; then the pair (\mathcal{T}, θ) gives rise via this process to a pair $(\overline{\mathcal{T}}, s)$. By construction we have $x_{\mathcal{T}}(\theta) = \overline{x}_{\overline{\mathcal{T}}}(s)$; i.e. the image of $x_{\mathcal{T}}$ under the map $Z_{T,1} \rightarrow \overline{\mathcal{K}}$ corresponding to θ is equal to the image of $\overline{x}_{\overline{\mathcal{T}}}$ under the map corresponding to s .

In particular, if \bar{x} lies in $A_{q,n,1}$, then $\overline{x}_{\overline{\mathcal{T}}_1}(s_1) = \overline{x}_{\overline{\mathcal{T}}_2}(s_2)$ whenever s_1 and s_2 are conjugate, and this implies that $x_{\mathcal{T}_1}(\theta_1) = x_{\mathcal{T}_2}(\theta_2)$ whenever $\rho_{\mathcal{T}_1, \theta_1}$ is isomorphic to $\rho_{\mathcal{T}_2, \theta_2}$. Thus x lies in $A_{q,n,1,S}$. The converse is equally straightforward. \square

As S contains a representative of every conjugacy class of unramified torus, we have an isomorphism $A_{F,n,1,S} \cong A_{F,n,1}$. We thus obtain a natural map $\bar{A}_{q,n,1} \rightarrow A_{F,n,1}$ whose image is precisely $A_{F,n,1}^c$; since the tori $\{\bar{\mathcal{T}}_\nu\}$ represent all conjugacy classes of tori in GL_n/\mathbb{F}_q , this map is injective. As $A_{F,n,1}^c$ is, by definition, saturated in $A_{F,n,1}$, we have obtained a natural injection $\bar{A}_{q,n,1} \rightarrow A_{F,n,1}$, with saturated image. On $\bar{\mathcal{K}}$ -points, we can describe this map explicitly. In particular, if ρ is an n -dimensional representation of $W_F/I_F^{(\ell)}$, and s is the semisimple element of \bar{G}' that corresponds to the restriction of ρ to I_F via the bijection described in the proof of Proposition 3.14, then the injection $\bar{A}_{q,n,1} \rightarrow A_{F,n,1}$ takes the $\bar{\mathcal{K}}$ -point of $\text{Spec } A_{F,n,1}$ corresponding to ρ to the $\bar{\mathcal{K}}$ -point of $\text{Spec } \bar{A}_{q,n,1}$ corresponding to s .

Now fix a uniformizer ϖ of F . The scalar ϖI_n in G lies in every maximal torus of G , and hence gives rise to an element of $Z_{T,1}$ for all ℓ -ramified maximal tori \mathcal{T} . Let Q_n be the element of $\prod_{\mathcal{T}} Z_{T,1}$ such that $(Q_n)_{\mathcal{T}} = \varpi I_n$ for all ℓ -ramified tori \mathcal{T} , and zero otherwise. One verifies easily that Q_n lies in $A_{F,n,1}^\times$.

Note that although Q_n depends on the choice of ϖ , the subalgebra $\bar{A}_{q,n,1}[Q_n^{\pm 1}]$ of $A_{F,n,1}$ does not. (If $\varpi' = u\varpi$ for a unit u , and Q'_n is the unit corresponding to ϖ' , let \bar{u} denote the reduction mod ℓ of u . Then there is an element of $\bar{A}_{q,n,1}$ that projects to $\bar{u}I_n$ on every torus of \bar{G} , and this element is equal to $Q_n^{-1}Q'_n$.)

We now have:

Proposition 3.15. *Let m, n be consecutive elements of $\{1, e_q, \ell e_q, \dots\}$, and let $j = \frac{n}{m}$. Let ν be the partition of n consisting of j copies of m . Then the kernel of the map:*

$$\text{Ind}_\nu : A_{F,n,1} \rightarrow A_{F,\nu,1}$$

is contained in $\bar{A}_{q,n,1}[Q_n^{\pm 1}]$.

Proof. We have an embedding:

$$A_{F,n,1} \rightarrow Z_{T,1} \times A_{F,\nu,1},$$

where \mathcal{T} is a Coxeter torus and the map to the second factor is Ind_ν . Thus projection to $Z_{T,1}$ is injective on the kernel of Ind_ν .

Let ℓ^a be the largest power of ℓ dividing $q^n - 1$, and note that $Z_{T,1}$ is isomorphic to $W(k)[\zeta, Q]/\langle \zeta^{\ell^a} - 1 \rangle$, via the map that sends the class of a uniformizer in $T/T^{(\ell)}$ to Q and the class of a generator of the ℓ -power roots of unity of F_n^\times to ζ .

Let ℓ^b be the largest power of ℓ dividing $q^m - 1$ (so in particular $b < a$) and let I be the ideal of $Z_{T,1}$ generated by $\zeta^{\ell^b} - 1$. If x is in I , then x vanishes on all characters θ of T such that $\theta(T^c)$ has order dividing ℓ^b ; note that these are precisely the characters of $T/T^{(\ell)}$ such that $\rho_{\mathcal{T},\theta}$ is reducible. If in addition x is invariant under the automorphism $\zeta \mapsto \zeta^q, Q \mapsto Q$ of $Z_{T,1}$ then $x(\theta_2) = x(\theta_1)$ whenever $\rho_{\mathcal{T},\theta_1} = \rho_{\mathcal{T},\theta_2}$. It follows that the element $(x, 0)$ of $Z_{T,1} \times A_{F,\nu,1}$ lies in $A_{F,n,1}$, and is moreover in the kernel of Ind_ν .

Conversely, if x lies in the projection of the kernel of Ind_ν to $Z_{T,1}$, then x vanishes on all reducible n -dimensional representations ρ of $W_F/I_F^{(\ell)}$ (as the dimensions of irreducible representations of $W_F/I_F^{(\ell)}$ lie in the set $\{1, e_q, \ell e_q, \dots\}$). Thus x lies in I . Moreover, as x arises by projection from an element of $A_{q,n,1}$, x must be invariant under $\zeta \mapsto \zeta^q, Q \mapsto Q$. In particular, if we express x as a polynomial in Q , i.e. $x = \sum x_i Q^i$, then each x_i lies in I and is invariant under the automorphism $\zeta \mapsto \zeta^q$ of $W(k)[\zeta]/\langle \zeta^{\ell^a} - 1 \rangle$. Thus $(x_i, 0)$ lies in the kernel of Ind_ν . Moreover, as

x_i is supported on T^c for all i , each $(x_i, 0)$ lies in $A_{F,n,1}^c$ and hence in $\overline{A}_{q,n,1}$. Thus $(x, 0)$ lies in $\overline{A}_{q,n,1}[Q_n^{\pm 1}]$ as claimed. \square

We conclude by establishing the following useful technical result:

Lemma 3.16. *The ring $\overline{A}_{q,n,1}[Q_n^{\pm 1}]$ is saturated in $A_{F,n,1}$.*

Proof. It is clear that $\overline{A}_{q,n,1}$ is saturated in $A_{F,n,1}$, and hence that $Q_n^i \overline{A}_{q,n,1}$ is saturated in $A_{F,n,1}$ for all integers i .

We introduce a natural grading on $A_{F,n,1}$, induced by natural gradings on each $Z_{T,1}$, for unramified \mathcal{T} . Choose an isomorphism $T \cong F_i^\times$, with F_i/F unramified, and note that T/T^c is a free abelian group, with a basis given by the classes of the uniformizers ϖ_i of the F_i . We declare any element of $Z_{T,1}$ supported on the T^c -coset of T corresponding to the formal sum $\sum r_i[\varpi_i]$ to be homogeneous of degree $\sum r_i[F_i : F]$.

For $\alpha \in \overline{\mathcal{K}}^\times$, let χ_α be the character that sends an element $x_i \in \prod_i F_i^\times$ to $\alpha^{v(x_i)}$, where $v(x_i)$ is the sum of the valuations of $N_{F_i/F}(x_i)$. Then the elements of degree d in $Z_{T,1}$ are precisely those x that satisfy $x(\theta\chi_\alpha) = \alpha^d x(\theta)$ for any character $\theta : T/T^{(\ell)} \rightarrow \overline{\mathcal{K}}$.

From this homogeneity property it is easy to conclude that $A_{F,n,1}$ has a grading in which the homogeneous elements of degree d are those elements x such that $x(\rho \otimes \xi_\alpha) = \alpha^d x(\rho)$, where ξ_α is the unramified character of W_F that takes the values α on a uniformizer. Note that Q_n is homogeneous of degree n in this grading, and that $A_{F,n,1}$ is homogeneous of degree zero. Now each $Q_n^i \overline{A}_{q,n,1}$ is saturated in the homogeneous elements of degree ni , so $\overline{A}_{q,n,1}[Q_n^{\pm 1}]$ is saturated in $A_{F,n,1}$. \square

4. ISOMORPHISMS TO $A_{F,n,1}$

In this section, we consider a family of $W(k)$ -algebras $B_{F,n}$, together with maps $B_{F,n} \rightarrow A_{F,n,1}$ satisfying certain axioms, and show that when such axioms are satisfied the maps $B_{F,n} \rightarrow A_{F,n,1}$ are isomorphisms.

Our purpose in doing this is of course to verify this set of axioms for certain blocks of the Bernstein center, and thereby use our theory of the rings $A_{F,n,1}$ to obtain an explicit description of the Bernstein center. We will carry this program out in the next several sections.

We suppose we are given the following data:

- For each n , a finitely generated, reduced, ℓ -torsion free $W(k)$ -algebra $B_{F,n}$.
- For each n , a map $f_n : B_{F,n} \rightarrow A_{F,n,1}$ that induces a bijection on $\overline{\mathcal{K}}$ -points.
- For each n , a map:

$$\mathrm{Ind}_{\nu^{\max}} : B_{F,n} \rightarrow \bigotimes_i B_{F,\nu_i^{\max}},$$

where ν^{\max} is the maximal relevant partition of n .

- For each $n > 1 \in \{1, e_q, \ell e_q, \dots\}$, a map:

$$\mathrm{Ind}_{m,n} : B_{F,n} \rightarrow B_{F,m}^{\otimes j}$$

where m is the largest element of $\{1, e_q, \ell e_q, \dots\}$ less than n and $j = \frac{n}{m}$.

- For each $m \in \{1, e_q, \ell e_q, \dots\}$, a map $\phi_m : \overline{A}_{q,m,1}[T, T^{-1}] \rightarrow B_{F,m}$.

We require that this collection has the following properties:

- (1) For each n , the map $\text{Ind}_{\nu^{\max}}$ is injective, $\overline{\mathcal{K}}$ -saturated, and fits into a commutative diagram:

$$\begin{array}{ccc} B_{F,n} & \rightarrow & A_{F,n,1} \\ \downarrow & & \downarrow \\ \bigotimes_i B_{F,\nu_i^{\max}} & \rightarrow & A_{F,\nu^{\max},1} \end{array}$$

where the right hand vertical map is the natural map: $\text{Ind}_{\nu^{\max}} : A_{F,n,1} \rightarrow A_{F,\nu^{\max},1}$.

- (2) For each pair of consecutive elements m, n in $\{1, e_q, \ell e_q, \dots\}$, with $j = \frac{n}{m}$, the map $\text{Ind}_{m,n}$ fits into a commutative diagram:

$$\begin{array}{ccc} B_{F,n} & \rightarrow & A_{F,n,1} \\ \downarrow & & \downarrow \\ B_{F,m}^{\otimes j} & \rightarrow & A_{F,m,1}^{\otimes j} \end{array}$$

- (3) The map $f_1 : B_{F,1} \rightarrow A_{F,1,1}$ is an isomorphism.
 (4) For each $m \in \{1, e_q, \ell e_q, \dots\}$, the composition:

$$\overline{A}_{q,m,1}[T, T^{-1}] \rightarrow B_{F,m} \rightarrow A_{F,m,1}$$

takes T to the unit Q_n in $A_{F,m,1}$ and restricts to the inclusion of $\overline{A}_{q,m,1}$ in $A_{F,m,1}$ constructed in Proposition 3.14.

- (5) For any pair of consecutive elements m, n in $\{1, e_q, \ell e_q, \dots\}$, the image of the map $\text{Ind}_{m,n} : B_{F,n} \rightarrow B_{F,m}^{\otimes j}$ is “large” in the following sense: for any y in $B_{F,m}^{\otimes j}$ such that $\ell^b y$ lies in the image of $\text{Ind}_{m,n}$ for some $b \geq 0$, there exists \tilde{y} in $B_{F,n}$, $x \in \overline{A}_{q,n,1}[T^{\pm 1}]$, $a \geq 0$ such that $\text{Ind}_{m,n}(\phi_n(x)) = \ell^a(y - \text{Ind}_{m,n}(\tilde{y}))$.
 (6) For any pair of consecutive elements m, n in $\{1, e_q, \ell e_q, \dots\}$, the map $\text{Ind}_{m,n} : B_{F,n}[\frac{1}{\ell}] \rightarrow B_{F,m}^{\otimes j}[\frac{1}{\ell}]$ is $\overline{\mathcal{K}}$ -saturated.

Proposition 4.1. *If the above conditions are satisfied for all $m \leq n$, then the map $f_m : B_{F,m} \rightarrow A_{F,m,1}$ is an isomorphism.*

Proof. We proceed by induction on n ; the case $n = 1$ is trivial. Suppose that the claim holds for all $m < n$. If n does not lie in the set $\{1, e_q, \ell e_q, \dots\}$, we have a sequence of injections:

$$B_{F,n} \rightarrow \bigotimes_i B_{F,\nu_i^{\max}} \rightarrow \bigotimes_i A_{F,\nu_i^{\max},1}$$

where the first map is $\overline{\mathcal{K}}$ -saturated and the second is an isomorphism by the inductive hypothesis. Thus $B_{F,n}$ is $\overline{\mathcal{K}}$ -saturated in $A_{F,\nu_i^{\max},1}$, and is thus equal to all of $A_{F,n,1}$.

Suppose now that $n > 1$ lies in $\{1, e_q, \ell e_q, \dots\}$, and as usual let m be the largest element of this set strictly less than n , and $j = \frac{n}{m}$. We have a commutative diagram:

$$\begin{array}{ccc} B_{F,n} & \rightarrow & B_{F,m}^{\otimes j} \\ \downarrow & & \downarrow \\ A_{F,n,1} & \rightarrow & A_{F,m,1}^{\otimes j} \end{array}$$

and by the inductive hypothesis the right-hand vertical map is an isomorphism. This, together with (6) above, implies that the saturation of the image of $B_{F,n}$ in $A_{F,m,1}^{\otimes j}$ is $\overline{\mathcal{K}}$ -saturated in $A_{F,m,1}^{\otimes j}$. In particular (as the map $B_{F,n} \rightarrow A_{F,n,1}$ is a

bijection on $\overline{\mathcal{K}}$ -points), the saturation of the image of $B_{F,n}$ in $A_{F,m,1}^{\otimes j}$ contains the image of $A_{F,n,1}$ in $A_{F,m,1}^{\otimes j}$.

Now let z be an element of $A_{F,n,1}$, and let y be its image in $A_{F,m,1}^{\otimes j}$. Then, by (5), there exists an element \tilde{y} of $B_{F,n}$, an $x \in \overline{A}_{q,n}[T^{\pm 1}]$ and an $a > 0$ such that $\text{Ind}_{m,n}(\phi_n(x)) = \ell^a(y - \text{Ind}_{m,n}(\tilde{y}))$. Let z' be the image of \tilde{y} in $A_{F,n,1}$. Then the images of x and $\ell^a(z - z')$ in $A_{F,m,1}^{\otimes j}$ coincide. Since both $f_n(\phi_n(x))$ and the kernel of the map $A_{F,n,1} \rightarrow A_{F,m,1}^{\otimes j}$ are contained in $\overline{A}_{q,n,1}[Q_n^{\pm 1}]$, we find that $\ell^a(z - z')$ is contained in $\overline{A}_{q,n,1}[Q_n^{\pm 1}]$. On the other hand, the latter is saturated in $A_{F,n,1}$, so $z - z'$ must lie in $\overline{A}_{q,n,1}[Q_n^{\pm 1}]$, and hence in the image of $B_{F,n}$. As z' is in the image of $B_{F,n}$ by construction, we find that z must be as well. \square

5. THE INTEGRAL BERNSTEIN CENTER

We now turn to the main object of interest in this paper: the integral Bernstein center. As in the preceding sections let $G = \text{GL}_n(F)$, and denote by $\text{Rep}_{W(k)}(G)$ (resp. $\text{Rep}_{\overline{\mathcal{K}}}(G)$) the category of smooth $W(k)[G]$ -modules (resp. the category of smooth $\overline{\mathcal{K}}[G]$ -modules.)

By the phrase ‘‘integral Bernstein center’’ we mean the center of the category $\text{Rep}_{W(k)}(G)$. We recall what this means:

Definition 5.1. The *center* of an Abelian category \mathcal{A} is the ring of natural transformations $\text{Id}_{\mathcal{A}} \rightarrow \text{Id}_{\mathcal{A}}$.

By definition, if Z is the center of \mathcal{A} , then specifying an element of Z amounts to specifying an endomorphism of every object of \mathcal{A} , such that the resulting collection commutes with all arrows in \mathcal{A} . The center of \mathcal{A} is thus a commutative ring that acts naturally on every object in \mathcal{A} , and this action is compatible with all morphisms in \mathcal{A} .

Bernstein-Deligne [BD], give a complete and explicit description of the center \tilde{Z} of $\text{Rep}_{\overline{\mathcal{K}}}(G)$. We briefly summarize their results: first, define an equivalence relation on pairs $(M, \tilde{\pi})$, where M is a Levi of G and $\tilde{\pi}$ is an irreducible supercuspidal representation of M over $\overline{\mathcal{K}}$ by declaring $(M_1, \tilde{\pi}_1)$ to be *inertially equivalent* to $(M_2, \tilde{\pi}_2)$ if $\tilde{\pi}_1$ is G -conjugate to an unramified twist of $\tilde{\pi}_2$. One then has:

Theorem 5.2 ([BD], Proposition 2.10). *There is a bijection $(M, \tilde{\pi}) \mapsto e_{(M, \tilde{\pi})}$ between inertial equivalence classes of pairs $(M, \tilde{\pi})$ over $\overline{\mathcal{K}}$ and primitive idempotents of \tilde{Z} , such that for any irreducible smooth representation Π of G over $\overline{\mathcal{K}}$ $e_{(M, \tilde{\pi})}$ acts via the identity on Π if Π has supercuspidal support in the inertial equivalence class of $(M, \tilde{\pi})$, and by zero otherwise.*

The upshot is that \tilde{Z} decomposes as an infinite product of the rings $e_{(M, \tilde{\pi})}\tilde{Z}$ as $(M, \tilde{\pi})$ runs over all inertial equivalence classes of pairs. Denote $e_{(M, \tilde{\pi})}\tilde{Z}$ by $\tilde{Z}_{(M, \tilde{\pi})}$. Then Bernstein and Deligne give a complete description of the ring structure of $\tilde{Z}_{(M, \tilde{\pi})}$ that we now explain.

Let M_0 be the smallest subgroup of M containing every compact open subgroup of M . Then M/M_0 is a free abelian group of finite rank, and $\text{Spec } \overline{\mathcal{K}}[M/M_0]$ is a torus whose $\overline{\mathcal{K}}$ -points are in bijection with the characters $M/M_0 \rightarrow \overline{\mathcal{K}}^\times$. Let H be the subgroup of these characters consisting of those characters χ such that $\tilde{\pi} \otimes \chi$ is isomorphic to χ . Then H is a finite abelian group that acts on $\overline{\mathcal{K}}[M/M_0]$. The

torus $\text{Spec } \overline{\mathcal{K}}[(M/M_0)]^H$ is a quotient of $\text{Spec } \overline{\mathcal{K}}[M/M_0]$; its $\overline{\mathcal{K}}$ -points correspond to H -orbits of characters of M/M_0 .

Now let W_M be the subgroup of the Weyl group of G consisting of w such that $wMw^{-1} = M$. Let $W_M(\tilde{\pi})$ be the subgroup of W_M consisting of w such that the representation $\tilde{\pi}^w$ of M is an unramified twist of $\tilde{\pi}$. Then we have a natural action of $W_M(\tilde{\pi})$ on $\overline{\mathcal{K}}[(M/M_0)]^H$, characterized by $\tilde{\pi} \otimes \chi^w \cong (\tilde{\pi} \otimes \chi)^w$ for characters χ of M/M_0 . We then have:

Theorem 5.3 ([BD], Théorème 2.13). *There is a unique natural isomorphism:*

$$\tilde{Z}_{(M, \tilde{\pi})} \cong (\overline{\mathcal{K}}[(M/M_0)]^H)^{W_M(\tilde{\pi})}$$

such that, for any irreducible representation Π over $\overline{\mathcal{K}}$ whose supercuspidal support has the form $\tilde{\pi} \otimes \chi$, $\tilde{Z}_{(M, \tilde{\pi})}$ acts on Π via the map:

$$(\overline{\mathcal{K}}[(M/M_0)]^H)^{W_M(\tilde{\pi})} \rightarrow \overline{\mathcal{K}}[M/M_0] \rightarrow \overline{\mathcal{K}}$$

corresponding to the character $\chi : M/M_0 \rightarrow \overline{\mathcal{K}}^\times$. In particular $\tilde{Z}_{(M, \tilde{\pi})}$ is a reduced, finitely generated, and normal $\overline{\mathcal{K}}$ -algebra.

In particular, \tilde{Z} acts on irreducible two representations Π, Π' of G via the same map $\tilde{Z} \rightarrow \overline{\mathcal{K}}$ if, and only if, Π and Π' have the same supercuspidal support. This defines, for each $(M, \tilde{\pi})$, a bijection between the $\overline{\mathcal{K}}$ -points of $\text{Spec } \tilde{Z}_{(M, \tilde{\pi})}$ and supercuspidal supports in the inertial equivalence class of $(M, \tilde{\pi})$; that is, unramified twists of $\tilde{\pi}$ considered up to $W_M(\tilde{\pi})$ -conjugacy.

Now let L be a Levi in GL_n ; then L factors as a product of L_i isomorphic to $GL_{n_i}(F)$. For each i , let M_i be a Levi in L_i , and $\tilde{\pi}_i$ an irreducible supercuspidal $\overline{\mathcal{K}}$ -representation of M_i . We then have isomorphisms:

$$\tilde{Z}_{M_i, \tilde{\pi}_i} \cong (\overline{\mathcal{K}}[(M_i/(M_i)_0)]^{H_i})^{W_{M_i}(\tilde{\pi}_i)}.$$

Let M be the product of the M_i ; we may regard it as a Levi of L and hence as a Levi of $GL_n(F)$. Let $\tilde{\pi}$ be the tensor product of the $\tilde{\pi}_i$. The quotient M/M_0 factors naturally as a product of $M_i/(M_i)_0$, and this induces a map:

$$(\overline{\mathcal{K}}[(M/M_0)]^H)^{W_M(\tilde{\pi})} \rightarrow \bigotimes_i (\overline{\mathcal{K}}[(M_i/(M_i)_0)]^{H_i})^{W_{M_i}(\tilde{\pi}_i)}$$

and hence a map

$$\text{Ind}_{\{(M_i, \tilde{\pi}_i)\}} : \tilde{Z}_{(M, \tilde{\pi})} \rightarrow \bigotimes_i \tilde{Z}_{(M_i, \tilde{\pi}_i)}.$$

On $\overline{\mathcal{K}}$ -points this takes the \mathcal{K} -point of the tensor product that corresponds to the collection of supercuspidal supports $\{(M_i, \tilde{\pi}_i \otimes \chi_i)\}$ to the point of $\text{Spec } \tilde{Z}_{(M, \tilde{\pi})}$ corresponding to the supercuspidal support $(M, \otimes_i (\tilde{\pi}_i \otimes \chi_i))$.

We will have need of the following, nearly trivial, observation about this map:

Lemma 5.4. *The map $\tilde{Z}_{(M, \tilde{\pi})} \rightarrow \bigotimes_i \tilde{Z}_{(M_i, \tilde{\pi}_i)}$ is $\overline{\mathcal{K}}$ -saturated.*

Proof. Embed both $\tilde{Z}_{(M, \tilde{\pi})}$ and $\bigotimes_i \tilde{Z}_{(M_i, \tilde{\pi}_i)}$ in $\bigotimes_i \overline{\mathcal{K}}[M_i/(M_i)_0]$ as above. The target has an action of $W_M(\tilde{\pi})$, and hence also an action of the subgroup $\prod_i W_{M_i}(\tilde{\pi}_i)$, and $\tilde{Z}_{(M, \tilde{\pi})}$ and $\bigotimes_i \tilde{Z}_{(M_i, \tilde{\pi}_i)}$ are identified with those elements invariant under $W_M(\tilde{\pi})$ or this subgroup, respectively. But one can check whether an element in $\bigotimes_i \overline{\mathcal{K}}[M_i/(M_i)_0]$ is invariant simply by looking at its values on $\overline{\mathcal{K}}$ -points, so the result is immediate. \square

We now turn to the study of $\text{Rep}_{W(k)}(G)$; let Z denote the center of this category. In this setting there is an analogue of the Bernstein-Deligne characterization of the primitive idempotents of Z . By [H1], Theorem 11.8, such idempotents are parameterized by inertial equivalence classes of pairs (L, π) , where π is now an irreducible supercuspidal representation of L over k .

If we let $e_{[L, \pi]}$ denote the idempotent of Z corresponding to (L, π) , $\text{Rep}_{W(k)}(G)_{[L, \pi]}$ the corresponding block, and $Z_{[L, \pi]}$ the corresponding factor of the Bernstein center, then one has the following basic structure results:

Theorem 5.5 ([H1], Theorem 12.8). *The ring $Z_{[L, \pi]}$ is a finitely generated, reduced, flat $W(k)$ -algebra.*

It is important to note that, in contrast to the situation over $\bar{\mathcal{K}}$, the ring $Z_{[L, \pi]}$ is in general very far from being normal.

We also have a description of $Z_{[L, \pi]} \otimes \bar{\mathcal{K}}$ in terms of \tilde{Z} . This can be made precise as follows: if $(M, \tilde{\pi})$ is a pair over $\bar{\mathcal{K}}$, and Π is an irreducible integral representation of G over $\bar{\mathcal{K}}$ with supercuspidal support in the inertial equivalence class of $(M, \tilde{\pi})$, then there exists a (possibly proper) Levi subgroup L of M , and an irreducible supercuspidal representation π of L , such that every irreducible subquotient of the mod ℓ reduction of Π has supercuspidal support (L, π) . Moreover, the inertial equivalence class of (L, π) depends only on that of $(M, \tilde{\pi})$, and not on the particular choice of π . We say that $(M, \tilde{\pi})$ *reduces modulo ℓ* to (L, π) ; this defines a finite-to-one map from inertial equivalence classes over $\bar{\mathcal{K}}$ to inertial equivalence classes over k . One then has:

Theorem 5.6 ([H1], Proposition 12.1). *The natural map $Z \otimes \bar{\mathcal{K}} \rightarrow \tilde{Z}$ induces an isomorphism:*

$$Z_{[L, \pi]} \otimes \bar{\mathcal{K}} \cong \prod_{(M, \tilde{\pi})} \tilde{Z}_{(M, \tilde{\pi})},$$

where the product is over all pairs $(M, \tilde{\pi})$, up to inertial equivalence, that reduce modulo ℓ to the pair (L, π) .

From this and the description of the $\bar{\mathcal{K}}$ -points of $\text{Spec } \tilde{Z}_{(M, \tilde{\pi})}$ one immediately deduces:

Corollary 5.7. *The $\bar{\mathcal{K}}$ -points of $\text{Spec } Z_{[L, \pi]}$ are in bijection with the supercuspidal supports of irreducible smooth $\bar{\mathcal{K}}$ -representations in $\text{Rep}_{W(k)}(G)_{[L, \pi]}$.*

We now give a more precise description of $Z_{[L, \pi]}$. We first reduce to a more easily studied special case:

Definition 5.8. A pair (L, π) is *simple* if there exist r, m such that $n = rm$, L is isomorphic to $\text{GL}_m(F)^r$, and π , up to unramified twist, is of the form $(\pi')^{\otimes r}$ for an irreducible supercuspidal representation π' of $\text{GL}_m(F)$.

Note that any pair (L, π) factors uniquely as a product of simple pairs (L^i, π^i) , with $\pi^i \cong (\pi'_i)^{\otimes r_i}$, such that no π'_i is an unramified twist of any other. One then has:

Theorem 5.9 ([H1], Theorem 12.4). *Let $\{(L^i, \pi^i)\}$ be the natural decomposition of (L, π) as a product of simple pairs. Then there is a natural isomorphism:*

$$Z_{[L, \pi]} \cong \bigotimes_i Z_{[L^i, \pi^i]}$$

such that, for any sequence $\{(M^i, \tilde{\pi}^i)\}$ reducing modulo ℓ to $\{(L^i, \tilde{\pi}^i)\}$, the diagram:

$$\begin{array}{ccc} Z_{[L, \pi]} \otimes \overline{\mathcal{K}} & \rightarrow & [\otimes_i Z_{[L^i, \pi^i]}] \otimes \overline{\mathcal{K}} \\ \downarrow & & \downarrow \\ \tilde{Z}_{(M, \tilde{\pi})} & \rightarrow & \otimes_i \tilde{Z}_{(M^i, \tilde{\pi}^i)} \end{array}$$

commutes, where $(M, \tilde{\pi})$ is the product of the $(M_i, \tilde{\pi}_i)$, and the bottom horizontal map is the map $\mathrm{Ind}_{\{(M^i, \tilde{\pi}^i)\}}$ described above.

We thus focus our attention on the case where (L, π) is simple. Fix an integer n_1 and an irreducible supercuspidal representation π' of $\mathrm{GL}_{n_1}(F)$ over k . For each $m > 0$, let L_m be a Levi of $\mathrm{GL}_{n_1 m}(F)$ isomorphic to $\mathrm{GL}_{n_1}(F)^m$, and let π_m be the representation $(\pi')^{\otimes m}$ of L_m . We can then consider the family of rings $Z_m := Z_{[L_m, \pi_m]}$ as n varies.

Section 13 of [H1] contains detailed information about the structure of the family Z_m . In particular this structure theory is closely related to the endomorphism rings of certain projective objects $\mathcal{P}_{K_m, \tau_m}$ for particular m . More precisely, consider the group of unramified characters χ of $\mathrm{GL}_{n_1}(F)$ such that $\pi' \otimes \chi$ is isomorphic to π' . This is a finite group; denote its order by f' . Then attached to the system of pairs (L_m, π_m) we have a system of projective objects $\mathcal{P}_{K_m, \tau_m}$, where m lies in the set $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \ell^2 e_{q^{f'}}, \dots\}$. (We refer the reader to Sections 7 and 9 of [H1] for a construction and structure theory of these objects.) For brevity, denote the representation $\mathcal{P}_{K_m, \tau_m}$ by \mathcal{P}_m .

For such m , let E_m denote the endomorphism ring of \mathcal{P}_m . Then, by Corollary 9.2 of [H1], E_m is a reduced, finite type, ℓ -torsion free $W(k)$ -algebra. Moreover, we have a map $Z_m \rightarrow E_m$ that gives the action of Z_m on the object \mathcal{P}_m of $\mathrm{Rep}_{W(k)}(\mathrm{GL}_{n_1 m}(F))_{[L_m, \pi_m]}$.

If m is arbitrary, the relationship between the rings Z_m and E_m is more complicated. Let ν be the maximal relevant partition of m with respect to the set $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$ (relevant partitions were called admissible in [H1]). Let M_ν and P_ν be the standard Levi and (upper triangular) parabolic subgroups of GL_n attached to $n_1 \nu$, so that M_ν is a product of $\mathrm{GL}_{n_1 \nu_i}(F)$, and consider the representation $\otimes_i \mathcal{P}_{\nu_i}$ of M_ν . Then Z_m acts on the parabolic induction $i_{\overline{P}_\nu}^{\mathrm{GL}_{n_1 m}(F)} \otimes_i \mathcal{P}_{\nu_i}$, and we have:

Theorem 5.10 ([H1], Theorem 13.7). *The action of Z_m on $i_{\overline{P}_\nu}^{\mathrm{GL}_{n_1 m}(F)} \otimes_i \mathcal{P}_{\nu_i}$ factors through the action of $\otimes_i E_{\nu_i}$ on $\otimes_i \mathcal{P}_{\nu_i}$. Moreover, the resulting map:*

$$Z_m \rightarrow \bigotimes_i E_{\nu_i}$$

is injective with saturated image, and is an isomorphism if m lies in $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$. (Note that in this case ν is the one-element partition $\{m\}$ of m .)

For m in $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$ we thus have a natural identification of Z_m with E_m . For arbitrary m , we can regard the map $Z_m \rightarrow \otimes_i E_{\nu_i}$ as a map $Z_m \rightarrow \otimes_i Z_{\nu_i}$. Denote this map by Ind_ν . We then have:

Lemma 5.11. *The map Ind_ν is $\overline{\mathcal{K}}$ -saturated.*

Proof. This is immediate from the fact that the image of Ind_ν is saturated, together with Lemma 5.4. \square

For m in $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$, the results of Sections 7 and 9 of [H1] give very precise information about E_m , and hence Z_m . In particular there is an integer f dividing f' , and a cuspidal k -representation σ_m of $\mathrm{GL}_{\frac{m}{f}}(\mathbb{F}_{q^f})$ (attached to an ℓ -regular conjugacy class $(s'_1)^m$ with s'_1 irreducible of degree f' over \mathbb{F}_{q^f}), such that the projective \mathcal{P}_m is a compact induction $\mathrm{c}\text{-Ind}_{K_m}^{\mathrm{GL}_{n_1 m}(F)} \tilde{\kappa}_m \otimes \mathcal{P}_{\sigma_m}$, where $\tilde{\kappa}_m$ comes from type theory and \mathcal{P}_{σ_m} is the projective envelope of σ_m , inflated to a representation of K_m via a natural map $K_m \rightarrow \mathrm{GL}_{\frac{m}{f}}(\mathbb{F}_{q^f})$.

Section 5 of [H1] shows that \mathcal{P}_{σ_m} is the projection of the Gelfand-Graev representation of $\mathrm{GL}_{\frac{m}{f}}(\mathbb{F}_{q^f})$ to the block containing σ_m . In particular, the results of section 2 identify the endomorphisms of \mathcal{P}_{σ_m} with $\overline{A}_{q^{f'}, m, 1}$.

We thus obtain an embedding of $\overline{A}_{q^{f'}, m, 1}$ in E_m for such m . Furthermore, section 9 of [H1] constructs an invertible element $\Theta_{m,m}$ of E_m . We thus obtain a map

$$\overline{A}_{q^{f'}, m, 1}[T, T^{-1}] \rightarrow E_m$$

taking T to $\Theta_{m,m}$. It follows easily from the description of E_m as a Hecke algebra in section 9 of [H1] that the image of this map consists of the elements of E_m supported on double cosets of the form $K_m z_{m,m}^r K_m$ for various r . (In particular, this image is saturated in E_m .)

The image of $\overline{A}_{q^{f'}, m, 1}$ in Z_m is easy to describe. Indeed, we have:

Proposition 5.12. *Let m lie in $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$, and let x be an element of $\overline{A}_{q^{f'}, m, 1}$, where the latter is considered as a subalgebra of Z_m . Then for any irreducible $\overline{\mathcal{K}}$ -representations Π, Π' of $\mathrm{GL}_{n_1 m}(F)$ in the same block of $\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{GL}_{n_1 m}(F))$, the action of x on Π and Π' is via the same scalar.*

Proof. The ring Z_m annihilates both Π and Π' unless Π and Π' belong to a block of the form $\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{GL}_{n_1 m}(F))_{(M_s, \pi_s)}$ for a suitable s , in the notation of [H1], section 9. In this case the action of Z_m on Π and Π' factors through the action of Z_m on the summand $\mathrm{c}\text{-Ind}_{K_m}^{\mathrm{GL}_{n_1 m}(F)} \tilde{\kappa}_m \otimes \mathrm{St}_s$ of $\mathrm{c}\text{-Ind}_{K_m}^{\mathrm{GL}_{n_1 m}(F)} \tilde{\kappa}_m \otimes \mathcal{P}_{\sigma_m} \otimes \overline{\mathcal{K}}$. In particular the action of x on Π and Π' factors through the action of x on St_s , which is by a scalar. \square

We also make the following observation about the action of $\Theta_{m,m} \in Z_m$:

Proposition 5.13. *Let P be a parabolic subgroup of $\mathrm{GL}_{n_1 m}(F)$, with Levi subgroup M , and let π be an irreducible cuspidal $\overline{\mathcal{K}}$ -representation of M such that $i_P^G \pi$ lies in the block corresponding to L_m, π_m . Suppose that M decomposes as a product of groups $M_i = \mathrm{GL}_{n_1 m_i}(F)$, and let χ be an unramified character of M , of the form $\otimes_i (\chi_i \circ \det)$, where we regard $(\chi_i \circ \det)$ as a character of M_i .*

Let $x \in \overline{\mathcal{K}}^\times$ be the scalar by which $\Theta_{m,m}$ acts on $i_P^G \pi$. Then $\Theta_{m,m}$ acts on $i_P^G \pi \otimes \chi$ via $x \prod_i \chi_i^{f'}(\varpi_F)$.

Proof. For some s , the pair (M, π) is conjugate to an unramified twist of one of the pairs (M_s, π_s) described in section 9 of [H1]. Thus, by Theorem 9.4 of [H1], the action of $\Theta_{m,m}$ on π is via the element $\theta_{m,s}$ of Z_{M_s, π_s} defined in section 9 of [H1], and the claim is immediate from the definition of $\theta_{m,s}$ in that section. \square

Finally, let m' and m be two consecutive elements of $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$, and set $j = \frac{m}{m'}$. Theorem 13.5 of [H1] then provides a map:

$$\mathrm{Ind}_{m',m} : Z_m \rightarrow Z_{m'}^{\otimes j},$$

that is compatible with parabolic induction, in the sense that the action of x in Z_m on $i_P^{\mathrm{GL}_{n_1 m}(F)} \pi$ (where $P = MU$ is a parabolic such that M is isomorphic to $\mathrm{GL}_{n_1 m'}(F)^j$) is induced by the action of $\mathrm{Ind}_{m',m}(x)$ on π . We then have:

Proposition 5.14. *The image of $Z_m \otimes \mathcal{K}$ in $Z_{m'}^{\otimes j} \otimes \mathcal{K}$ is $\overline{\mathcal{K}}$ -saturated.*

Proof. This is an easy consequence of Lemma 5.4. □

It is necessary to invert ℓ to obtain saturation results such as the one above. Indeed, without inverting ℓ , the best that can be said is:

Theorem 5.15 ([H1], Theorem 13.6). *Let y be an element of $Z_{m'}^{\otimes j}$ such that, for some a , $\ell^a y$ lies in the image of $\mathrm{Ind}_{m',m}$. Then there exists an element \tilde{y} of Z_m , an element x of $\overline{A}_{q^{f'},m,1}[T^{\pm 1}]$, and an integer $b > 0$ such that $\mathrm{Ind}_{m',m}(\tilde{y}) = \ell^b(y - \mathrm{Ind}_{m',m}(x))$.*

6. THE RING $R_{q,n}$

The upshot of the previous section is that the rings Z_m admit many of the structures described in section 4. What is missing are maps $Z_m \rightarrow A_{F',m,1}$ for a suitable extension F' of F . We will show how such maps can be constructed, conditionally on a conjecture (Conjecture 10.2 below) relating the rings Z_m to Galois theory.

We begin by studying a family of rings with close connections to both Galois theory and the rings $A_{F,n,1}$. Let $X_{q,n}$ be the affine $W(k)$ -scheme parameterizing pairs of invertible n by n matrices (Fr, σ) such that $\mathrm{Fr} \sigma \mathrm{Fr}^{-1} = \sigma^q$, and let $X_{q,n}^0$ be the connected component of $X_{q,n}$ containing the k -point $\mathrm{Fr} = \sigma = \mathrm{Id}_n$. Let $R_{q,n}$ be the ring of functions on $X_{q,n}^0$, so that $X_{q,n}^0 = \mathrm{Spec} R_{q,n}$.

Lemma 6.1. *Let L be an algebraically closed field that is a $W(k)$ -algebra and x be an L -point of $X_{q,n}$ corresponding to a pair $(\mathrm{Fr}_x, \sigma_x)$ of elements of $\mathrm{GL}_n(L)$. Then x lies in $X_{q,n}^0$ if, and only if, the eigenvalues of σ_x are ℓ -power roots of unity.*

Proof. Consider the map $X_{q,n} \rightarrow \mathbb{A}_{W(k)}^n$ that takes a point x to the coefficients of the characteristic polynomial of σ_x . Let Y be the image of this map. Let $\tilde{Y} \subset \mathbb{A}_{W(k)}^n$ be the space of diagonal matrices that are conjugate to their q th powers; we then have a map $\tilde{Y} \rightarrow \mathbb{A}_{W(k)}^n$ that sends such a matrix to the coefficients of its characteristic polynomial. It is easy to see that (set-theoretically) the image of \tilde{Y} is equal to Y . On the other hand, $\tilde{Y}(\overline{\mathcal{K}})$ is a finite collection of points; indeed, the entries of any diagonal matrix that is conjugate to its q th power are roots of unity. Thus the ‘‘coordinates’’ of each $\overline{\mathcal{K}}$ -point of \tilde{Y} are integral over $W(k)$, and every point of $\tilde{Y}(k)$ is in the closure of some point of $\tilde{Y}(\overline{\mathcal{K}})$. It follows that the same is true for Y ; in particular Y is the closure of a finite set of $\overline{\mathcal{K}}$ -points, and the closure of any $\overline{\mathcal{K}}$ -point of Y meets the special fiber of Y . Therefore, the connected component Y^0 of Y containing the image of $X_{q,n}^0$ is the closure of the set of $\overline{\mathcal{K}}$ -points of Y that ‘‘specialize’’ mod ℓ to the characteristic polynomial $(X - 1)^n$ of the identity matrix. The only k -point of this component arises from the characteristic polynomial of the

identity matrix, and the \overline{K} -points of this component correspond to characteristic polynomials of elements of $\overline{Y}(\overline{K})$ whose roots reduce to 1 modulo ℓ . The roots of such a polynomial are ℓ -power roots of unity. Therefore, for x in $X_{q,n}^0(L)$ the roots of the characteristic polynomial of σ_x are ℓ -power roots of unity, as required.

Conversely, let x be an L -point of $X_{q,n}$, and suppose that the eigenvalues of σ_x are ℓ -power roots of unity. Note that $\mathrm{GL}_n(L)$ acts on $X_{q,n}(L)$, by conjugation on both F and σ , and this action preserves the connected components. We may thus assume σ_x is in Jordan normal form; in particular its entries lie in k or an integral extension \mathcal{O} of $W(k)$. Moreover, for a fixed σ_x , the set of Fr_x such that $\mathrm{Fr}_x \sigma_x = \sigma_x^q \mathrm{Fr}_x$ is a linear space; there is thus an invertible Fr'_x whose entries lie in k or $W(k)$, such that $\mathrm{Fr}'_x \sigma_x = \sigma_x^q \mathrm{Fr}'_x$ and $(\mathrm{Fr}'_x, \sigma_x)$ lies on the same connected component as x .

If L has characteristic ℓ , the above construction yields a k -point of $X_{q,n}$ in the same connected component as x . If L has characteristic zero, the closure of the point (Fr', σ) constructed above contains a k -point (Fr'', σ') of $X_{q,n}$ in the same connected component as x . Moreover, σ' is unipotent. We may again assume σ'' is in Jordan normal form; then in the closure of orbit of $(\mathrm{Fr}'', \sigma'')$ under conjugation by diagonal matrices there is a point where σ is the identity. It is clear that such a point lies in the connected component of the k -point x where $\mathrm{Fr}_x = \sigma_x = \mathrm{Id}_n$. \square

The ring $R_{q,n}$ is rather well-behaved from an algebraic standpoint. In particular, one has:

Proposition 6.2. *The ring $R_{q,n}$ is reduced and locally a complete intersection. Moreover, $R_{q,n}$ is flat as a $W(k)$ -algebra.*

Proof. This argument is a slight elaboration of an argument due to Choi [Ch]. We give a sketch here.

First note that $X_{q,n}$ is given by n^2 relations in a space of dimension $2n^2 + 1$. Consider the map $X_{q,n} \rightarrow \mathbb{A}_{W(k)}^{n^2}$ that sends a point x to the matrix σ_x . Let L be an algebraically closed field that is a $W(k)$ -algebra, and let x be an L -point. of $\mathrm{Spec} S_{q,n}$.

The group $\mathrm{GL}_n(L)$ acts on the set of L -points of $\mathrm{Spec} S_{q,n}$ by conjugation. Consider the locally closed subset U_{σ_x} of $\mathrm{Spec} \mathbb{A}_L^{n^2}$ consisting of those σ' conjugate to σ_x . For any L -point σ' of U_{σ_x} , the fiber of $X_{q,n} \times_{W(k)} L$ over σ' consists of pairs $(\mathrm{Fr}' h, \sigma')$, where Fr' is a fixed element of GL_n such that $\mathrm{Fr}' \sigma' (\mathrm{Fr}')^{-1} = (\sigma')^q$ and h commutes with σ' .

In particular, the dimension of the preimage of U_{σ} in $X_{q,n} \times_{W(k)} L$ is equal to the dimension of U_{σ} plus the dimension of the stabilizer of σ under conjugation; this is clearly n^2 . As σ varies over a finite list of conjugacy classes, the preimages of the U_{σ} cover $X_{q,n} \times_{W(k)} L$; thus $X_{q,n} \otimes_{W(k)} L$ is equidimensional of dimension n^2 . On the other hand the dimension of $X_{q,n}$ is at least $n^2 + 1$. It follows that the Zariski closures of the preimages of sets U_{σ} are irreducible components of $X_{q,n}$, and that no irreducible component of $X_{q,n}$ is contained in the special fiber (as it would then be a component of $X_{q,n} \otimes_{W(k)} k$ of dimension at most n^2 .) It also follows that every irreducible component of $X_{q,n}$ has dimension $n^2 + 1$, because if we had a component of larger dimension then its base change to \overline{K} would have dimension greater than n^2 . In particular $X_{q,n}$ is a complete intersection. It follows that $R_{q,n}$ is a local complete intersection.

An argument of Choi ([Ch], Theorem 3.0.13) shows that for any maximal ideal m of $R_{q,n}$, $(\mathrm{Spec} R_{q,n})_m[\frac{1}{\ell}]$ is generically smooth; in particular $X_{q,n}^0$ is generically reduced. By the unmixedness theorem the local complete intersection $X_{q,n}^0$ has no embedded points, so $R_{q,n}$ is reduced. As the generic points of $\mathrm{Spec} R_{q,n}$ all have characteristic zero, we may conclude that $R_{q,n}$ is flat over $W(k)$. \square

We have a universal pair of matrices (Fr, σ) in $\mathrm{GL}_n(R_{q,n})$. The above result immediately implies:

Corollary 6.3. *There exists a power ℓ^a of ℓ such that σ^{ℓ^a} is unipotent in $\mathrm{GL}_n(R_{q,n})$.*

Proof. Since $R_{q,n}$ is reduced and flat over $W(k)$, it suffices to check that σ^{ℓ^a} is unipotent for some a at each of the generic points of $\mathrm{Spec} R_{q,n}$, all of which lie in characteristic zero. This is an immediate consequence of Lemma 6.1. \square

Let L be a finite extension of \bar{K} . We call an L -point of $X_{q,n}^0$ *integral* if the corresponding map $R_{q,n} \rightarrow L$ factors through the ring of integers \mathcal{O}_L .

Lemma 6.4. *Let x be an L -point of $X_{q,n}^0$, and suppose that the eigenvalues of Fr_x are integral over $W(k)$. Then there is an integral point of $X_{q,n}^0$ in the GL_n -orbit of x .*

Proof. Extending L if necessary, we may assume that the eigenvalues of σ_x are in L , and hence \mathcal{O}_L . Then (for instance, by putting σ_x in Jordan normal form) we can find an \mathcal{O}_L -sublattice M of L^n preserved by σ_x . Using $\mathrm{Fr}_x \sigma_x \mathrm{Fr}_x^{-1} = \sigma_x^q$, we find that $\mathrm{Fr}_x M$, $\mathrm{Fr}_x^2 M$, etc. are also preserved by σ_x . Consider the lattice M' given by $M + \mathrm{Fr}_x M + \cdots + \mathrm{Fr}_x^{n-1} M$; it is clearly preserved by σ_x . On the other hand, since Fr_x is annihilated by a polynomial with integral coefficients, $\mathrm{Fr}_x^n M$ is contained in M' , and hence $\mathrm{Fr}_x M'$ is contained in M' . Thus M' is stable under both Fr_x and σ_x . Choosing a basis for M' , we find an integral point of $X_{q,n}^0$ in the same GL_n -orbit as x . \square

Proposition 6.5. *The images of the integral points of $X_{q,n}^0$ are dense in $X_{q,n}^0$.*

Proof. Since the integral points of GL_n are dense in GL_n , the closure of the integral points of $X_{q,n}^0$ is a union of GL_n -orbits, and hence, by the previous lemma, equal to the closure of the set of points x such that Fr_x has integral eigenvalues. We must show that such points are dense. But for any σ_x , the set of invertible Fr_x such that $\mathrm{Fr}_x \sigma = \sigma^q \mathrm{Fr}_x$ is an open subset of an affine space, so this is clear. \square

Corollary 6.6. *The ring $R_{q,n}$ is ℓ -adically separated; that is, the intersection of the ideals $\ell^i R_{q,n}$ is zero.*

Proof. Let f be an element of $R_{q,n}$ that is divisible by ℓ^i for all i . Then, for any integral point $x : R_{q,n} \rightarrow \mathcal{O}_L$, the image $x(f)$ is divisible by ℓ^i for all i and is therefore zero. In other words, f vanishes on a dense subset of $X_{q,n}^0$. Since $X_{q,n}^0$ is reduced, f is zero. \square

Now fix a Frobenius element $\tilde{\mathrm{Fr}}$ in W_F , and a topological generator $\tilde{\sigma}$ of the quotient $I_F/I_F^{(\ell)}$. Corollary 6.3 implies that there is a unique ℓ -adically continuous representation $\rho_{F,n} : W_F \rightarrow \mathrm{GL}_n(R_{q,n})$ that takes $\tilde{\mathrm{Fr}}$ to Fr and $\tilde{\sigma}$ to σ . (Recall that for an ℓ -adically separated ring A , a representation $\rho : W_F \rightarrow \mathrm{GL}_n(A)$ is ℓ -adically continuous if, for all i , the preimage of the subgroup $\mathrm{Id} + \ell^i M_n(A)$ of $\mathrm{GL}_n(A)$ is open in W_F .)

The pair $(R_{q,n}, \rho_{F,n})$ has the following universal property, which is easily seen to characterize the pair up to isomorphism:

Proposition 6.7. *For any finitely generated, ℓ -adically separated $W(k)$ -algebra A , and any framed, ℓ -adically continuous representation $\rho : W_F/I_F^{(\ell)} \rightarrow \mathrm{GL}_n(A)$, there is a unique map: $R_{q,n} \rightarrow A$ such that ρ is the base change of $\rho_{F,n}$.*

Proof. Given ρ , we have a pair of matrices $(\rho(\mathrm{Fr}), \rho(\tilde{\sigma}))$ in $\mathrm{GL}_n(A)$, satisfying

$$\rho(\mathrm{Fr})\rho(\tilde{\sigma})\rho(\mathrm{Fr})^{-1} = \rho(\tilde{\sigma})^q,$$

and hence a map $S_{q,n} \rightarrow A$. Moreover, since the restriction of ρ to I_F factors through $I_F/I_F^{(\ell)}$ and is ℓ -adically continuous, the eigenvalues of $\rho(\tilde{\sigma})$ are ℓ -power roots of unity. Thus the map $S_{q,n} \rightarrow A$ factors through $R_{q,n}$ and the result follows. \square

If we regard the $\overline{\mathcal{K}}$ -points of $X_{q,n}^0$ as framed representations of $W_F/I_F^{(\ell)}$, then one can show:

Proposition 6.8. *Let x be an $\overline{\mathcal{K}}$ -point of $X_{q,n}^0$. Then there is a point y in the closure of the GL_n -orbit of x such that the representation ρ_y is semisimple.*

Proof. Replacing x with a point in the same GL_n -orbit, we may assume that the framing on ρ_x is such that ρ_x is block upper triangular, with block sizes n_1, \dots, n_r , and that for $1 \leq i \leq r$, the restriction ρ_i of ρ_x to the i th diagonal block is irreducible. Let M be the block diagonal matrix whose i th block is given by t^i times the n_i by n_i identity matrix, for some parameter t . Then the limit, as t approaches zero, of $M\rho_x M^{-1}$ exists and is semisimple. \square

We will later need the following observation about the representation $\rho_{F,n}$.

Proposition 6.9. *As x varies over the $\overline{\mathcal{K}}$ -points of $X_{q,n}^0$, the restriction of ρ_x^{ss} to I_F is constant on connected components of $X_{q,n}^0 \times_{W(k)} \overline{\mathcal{K}}$.*

Proof. The restriction of ρ_x^{ss} to I_F is determined by the characteristic polynomial of σ_x ; since the eigenvalues of σ_x have bounded ℓ -power order there are only finitely possible characteristic polynomials of σ_x . \square

7. RELATING $R_{q,n}$ TO $A_{F,n,1}$

The ring $R_{q,n}$ and its associated representation $\rho_{F,n}$ turn out to be quite closely related to $A_{F,n,1}$. In particular $R_{q,n}$ has a natural subring $R_{q,n}^{\mathrm{inv}}$ consisting of GL_n -invariant functions on $X_{q,n}^0$; we will show that this subring admits a natural map to $A_{F,n,1}$.

The key is to consider certain natural subschemes of $X_{q,n}^0$. Fix a permutation w in S_n , considered as an n by n permutation matrix, and let $X_{q,n}^w$ denote the closed subscheme of $X_{q,n}^0$ consisting of pairs (Fr, σ) such that $w \mathrm{Fr}$ and σ are diagonal. Let $R_{q,n}^w$ be the ring of regular functions on $X_{q,n}^w$. Then $R_{q,n}^w$ is naturally a quotient of $R_{q,n}$.

Let D denote the standard maximal torus of GL_n . The conjugation action of D on $X_{q,n}$ preserves $X_{q,n}^w$. Let $R_{q,n}^{w,\mathrm{inv}}$ denote the invariant elements of $R_{q,n}^w$ under the action of D .

The ring $R_{q,n}^{w,\mathrm{inv}}$ is closely related to a certain torus in $\mathrm{GL}_n(F)$. In particular (regarding w as an element of the Weyl group of $\mathrm{GL}_n(F)$), let \mathcal{T}_w be a torus in

the conjugacy class of unramified maximal tori of $GL_n(F)$ corresponding to w . (Concretely, T_w is then a product of F_i^\times , where F_i is unramified over F and the degrees of the F_i correspond to lengths of cycles in w .)

We then have:

Proposition 7.1. *There is an isomorphism:*

$$R_{q,n}^{w,\text{inv}} \cong Z_{T_w,1}$$

such that the composition

$$R_{q,n}^{\text{inv}} \rightarrow R_{q,n}^{w,\text{inv}} \rightarrow Z_{T_w,1}$$

takes the \bar{K} -point of $\text{Spec } Z_{T_w,1}$ corresponding to the pair (T_w, θ) to the image in $\text{Spec } R_{q,n}^{\text{inv}}$ of the \bar{K} -point of $\text{Spec } R_{q,n}$ corresponding to $\rho_{T_w, \theta}$.

Proof. One easily reduces to the case when w consists of a single cycle; after conjugation by a suitable permutation matrix we may assume $w(e_i) = e_{i+1}$, $w(e_n) = e_1$, where the e_i are the standard basis of GL_n . Then the equations for $X_{q,n}^w$ can be made quite explicit: if f_1, \dots, f_n are the diagonal entries of $w \text{Fr}$, and ζ_1, \dots, ζ_n are the diagonal entries of σ , then the equation $\text{Fr} \sigma \text{Fr}^{-1} = \sigma^q$ becomes the equations $\zeta_i^q = \zeta_{i+1}$, $\zeta_n^q = \zeta_1$ with respect to these coordinates. The diagonal matrix with entries t_1, \dots, t_n acts by fixing each ζ_i , sending f_i to $t_i t_{i+1}^{-1} f_i$ and sending f_n to $t_n t_1^{-1} f_n$. Thus the ring $R_{q,n}^{w,\text{inv}}$ is generated by ζ_1 and $f_1 f_2 \dots f_n$, subject to the relation $\zeta_1^{\ell^a} - 1 = 0$, where ℓ^a is the largest power of ℓ dividing $q^n - 1$.

On the other hand, by definition $Z_{T_w,1}$ is isomorphic to $W(k)[(F')^\times / ((F')^\times)^{(\ell)}]$, where F' is unramified over F of degree n . Our chosen elements $\tilde{\text{Fr}}$ and $\tilde{\sigma}$ of W_F correspond, via local class field theory, to a uniformizer ϖ of F' and an element ζ of order ℓ^a in $\mathcal{O}_{F'}^\times$. Define a map from $R_{q,n}^{w,\text{inv}}$ to $Z_{T_w,1}$ sending ζ_1 to ζ and $f_1 f_2 \dots f_n$ to ϖ .

Now let x be an \bar{K} -point of $R_{q,n}^w$, and let θ be the character of $(F')^\times$ that takes ϖ to $x(f_1 f_2 \dots f_n)$ and ζ to $x(\zeta_1)$. It is then clear that the specialization ρ_x of $\rho_{F,n}$ to x is isomorphic to $\rho_{T_w, \theta}$, so the map $R_{q,n}^{w,\text{inv}} \rightarrow Z_{T_w,1}$ has the claimed property. \square

As an immediate consequence, we deduce:

Theorem 7.2. *There is a unique injection*

$$R_{q,n}^{\text{inv}} \rightarrow A_{F,n,1}$$

that takes the point of $\text{Spec } A_{F,n,1}$ corresponding to a semisimple representation ρ of W_F to the image, in $\text{Spec } R_{q,n}^{\text{inv}}$, of any point x of $X_{q,n}^0$ with ρ_x^{ss} isomorphic to ρ . Moreover,

- the image of $\det \text{Fr}$ under this map is the element Q_n of $A_{F,n,1}$.
- Let r be an element $R_{q,n}^{\text{inv}}$ such that $r(x)$ depends only on the restriction of ρ_x to I_F . Then the image of r in $A_{F,n,1}$ lies in the subalgebra $\bar{A}_{q,n,1}$.

Proof. It is clear from the previous proposition that the product, over all w , of the compositions:

$$R_{q,n}^{\text{inv}} \rightarrow R_{q,n}^{w,\text{inv}} \cong Z_{T_w,1}$$

is a map $R_{q,n}^{\text{inv}} \rightarrow \prod_w Z_{T_w,1}$ whose image is contained in the image of $A_{q,n,1}$ in $\prod_w Z_{T_w,1}$. It thus remains to prove the injectivity of the map $R_{q,n}^{\text{inv}} \rightarrow A_{q,n,1}$. An element r of the kernel of this map vanishes on all representations of W_F of the form $\rho_{T, \theta}$. Since any semisimple ℓ -ramified representation of W_F has this form, r

must vanish at every point of $X_{q,n}^0$ corresponding to a semisimple representation. On the other hand, there is such a point in the closure of every GL_n -orbit on $X_{q,n}^0$. Since r is GL_n -invariant, it must vanish at every $\overline{\mathcal{K}}$ -point of $X_{q,n}^0$; since the latter is reduced and such points are dense we have $r = 0$.

It is clear from the construction of the maps $R_{q,n}^{\mathrm{inv}} \rightarrow Z_{T_w,1}$ that the image of $\det \mathrm{Fr}$ in $Z_{T_w,1}$ is the element ϖI_n of the latter. Moreover, for any r such that $r(x)$ depends only on the restriction of ρ_x to I_F , the image of r in $Z_{T_w,1}$ is an element r_w of $Z_{T_w,1}$ such that the value of r_w at a character $\theta : T_w \rightarrow \overline{\mathcal{K}}^\times$ depends only on the restriction of θ to T_w^c . Hence the image of r in $A_{F,n,1}$ lies in $\overline{A}_{q,n,1}$. \square

8. DEFORMATION THEORY

In this section we examine the local deformation theory of a representation $\overline{\rho} : G_F \rightarrow \mathrm{GL}_n(k)$. As in previous sections, let $I_F^{(\ell)}$ denote the prime to ℓ part of the inertia group of F , and fix a topological generator $\tilde{\sigma}$ of $I_F/I_F^{(\ell)}$ and a Frobenius element Fr in $W_F/I_F^{(\ell)}$.

We first recall some results of Clozel-Harris-Taylor:

Proposition 8.1 ([CHT], Lemmas 2.4.11-2.4.13). *Let $\overline{\tau}$ be an irreducible representation of $I_F^{(\ell)}$ over k , and let $G_{\overline{\tau}}$ be the subgroup of G_F that preserves $\overline{\tau}$ under conjugation. Then:*

- (1) $\overline{\tau}$ lifts uniquely to a representation τ of $I_F^{(\ell)}$ over $W(k)$.
- (2) τ extends uniquely to a representation of $I_F \cap G_{\overline{\tau}}$ of determinant prime to ℓ .
- (3) τ extends (non-uniquely) to a representation of $G_{\overline{\tau}}$.

If we fix a representation τ of $G_{\overline{\tau}}$ as in part (3), we obtain an action of $G_{\overline{\tau}}/I_F^{(\ell)}$ on $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho)$ for any G_F -module ρ . Moreover, we have a direct sum decomposition of G_F -modules:

$$\rho \cong \bigoplus_{[\overline{\tau}]} \mathrm{Ind}_{G_{\overline{\tau}}}^{G_F} [\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho) \otimes \tau],$$

where $\overline{\tau}$ runs over G_F -conjugacy classes of irreducible representations of $I_F^{(\ell)}$ over k .

Fix, for each G_F -conjugacy class of $\overline{\tau}$, a τ as in the proposition. Suppose we are given a representation $\rho_A : G_F \rightarrow \mathrm{GL}_n(A)$. We then obtain a direct sum decomposition:

$$\rho_A = \bigoplus_{[\overline{\tau}]} \mathrm{Ind}_{G_{\overline{\tau}}}^{G_F} [\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho_A) \otimes \tau].$$

It is clear that $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho_A)$ is a free A -module for all τ , and that the collection of $G_{\overline{\tau}}$ -representations $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho_A)$ determines the representation ρ_A up to isomorphism.

Definition 8.2. A *pseudo-framing* of a continuous representation $\rho_A : G_F \rightarrow \mathrm{GL}_n(A)$ is a choice, for each $\overline{\tau}$, of a basis for each $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho_A)$. A *pseudo-framed deformation* of a continuous representation $\overline{\rho} : G_F \rightarrow \mathrm{GL}_n(k)$ (together with a chosen pseudo-framing) is a lift $\rho_A : G_F \rightarrow \mathrm{GL}_n(A)$ of $\overline{\rho}$, together with a pseudo-framing of ρ_A that lifts the chosen pseudo-framing of ρ .

Fix a $\bar{\rho}$ and a pseudo-framing of $\bar{\rho}$, and, for each $\bar{\tau}$, let $\bar{\rho}_{\bar{\tau}}$ be the $G_{\bar{\tau}}$ -representation $\text{Hom}_{I_F^{(\ell)}}(\tau, \bar{\rho})$. Let $R_{\bar{\rho}}^{\diamond}$ be the completed tensor product

$$\otimes_{[\bar{\tau}]} R_{\bar{\rho}_{\bar{\tau}}}^{\square}$$

of the universal framed deformation rings of the $\bar{\rho}_{\bar{\tau}}$. Over each such ring we have the universal framed deformation $\rho_{\bar{\tau}}^{\square}$ of $\bar{\rho}_{\bar{\tau}}$.

Using these, we construct a representation:

$$\rho^{\diamond} := \bigoplus_{[\bar{\tau}]} \text{Ind}_{G_{\bar{\tau}}}^{G_F} [\rho_{\bar{\tau}}^{\square} \otimes \tau]$$

that has a natural pseudo-framing induced by the universal framings of the representations $\rho_{\bar{\tau}}^{\square}$. One easily verifies that the pair $R_{\bar{\rho}}^{\diamond}, \rho^{\diamond}$ is a universal object for pseudo-framed deformations of ρ .

For each $\bar{\tau}$, the formal group $\mathcal{G}_{\bar{\rho}_{\bar{\tau}}}^{\square}$ acts on $\text{Spf } R_{\bar{\rho}_{\bar{\tau}}}^{\square}$ by “change of frame”. Let $\mathcal{G}_{\bar{\rho}}^{\diamond}$ be the product of the $\mathcal{G}_{\bar{\rho}_{\bar{\tau}}}^{\square}$. Then $\mathcal{G}_{\bar{\rho}}^{\diamond}$ acts on $\text{Spf } R_{\bar{\rho}}^{\diamond}$ by “change of pseudo-framing”.

For computational purposes it is often easier to work with $R_{\bar{\rho}}^{\diamond}$ rather than $R_{\bar{\rho}}^{\square}$, as $R_{\bar{\rho}}^{\diamond}$ can be made quite explicit. The two rings are related in a natural way: one has a ring $R_{\bar{\rho}}^{\square, \diamond}$ that is universal for triples consisting of a deformation ρ of $\bar{\rho}$, a framing of ρ lifting that of $\bar{\rho}$, and a pseudo-framing of ρ lifting that of $\bar{\rho}$. Then $\text{Spf } R_{\bar{\rho}}^{\square, \diamond}$ is a (split) $\mathcal{G}_{\bar{\rho}}^{\diamond}$ -torsor over $\text{Spf } R_{\bar{\rho}}^{\square}$ and a (split) $\mathcal{G}_{\bar{\rho}}^{\square}$ -torsor over $\text{Spf } R_{\bar{\rho}}^{\diamond}$.

We immediately deduce:

Corollary 8.3. *The ring $R_{\bar{\rho}}^{\square}$ is a reduced, ℓ -torsion free local complete intersection.*

Proof. The construction above shows that it suffices to prove the same claim with $R_{\bar{\rho}}^{\square}$ replaced by $R_{\bar{\rho}}^{\diamond}$. But the latter is a completed tensor product of rings of the form $R_{\bar{\rho}_{\bar{\tau}}}^{\square}$, and each of these is isomorphic to the completion of a ring of the form $R_{q,n}$ (with q and n depending on $\bar{\tau}$) at a maximal ideal. The result thus follows from the results of section 6. \square

Moreover, we may canonically identify both the $\mathcal{G}_{\bar{\rho}}^{\square}$ -invariant elements of $R_{\bar{\rho}}^{\square}$ and the $\mathcal{G}_{\bar{\rho}}^{\diamond}$ -invariant elements of $R_{\bar{\rho}}^{\diamond}$ with the $\mathcal{G}_{\bar{\rho}}^{\square} \times \mathcal{G}_{\bar{\rho}}^{\diamond}$ -invariant elements of $R_{\bar{\rho}}^{\square, \diamond}$. In particular these spaces of invariants are naturally isomorphic.

Given a choice of framing of ρ^{\diamond} , we get a map $R_{\bar{\rho}}^{\square} \rightarrow R_{\bar{\rho}}^{\diamond}$. When restricted to $\mathcal{G}_{\bar{\rho}}^{\square}$ -invariants this map is the isomorphism of $(R_{\bar{\rho}}^{\square})^{\mathcal{G}_{\bar{\rho}}^{\square}}$ with $(R_{\bar{\rho}}^{\diamond})^{\mathcal{G}_{\bar{\rho}}^{\diamond}}$ constructed above. Summarizing, we have:

Lemma 8.4. *For any choice of framing of ρ^{\diamond} , the induced map: $R_{\bar{\rho}}^{\square} \rightarrow R_{\bar{\rho}}^{\diamond}$ identifies the $\mathcal{G}_{\bar{\rho}}^{\square}$ -invariant elements of $R_{\bar{\rho}}^{\square}$ with the $\mathcal{G}_{\bar{\rho}}^{\square}$ -invariant elements of $R_{\bar{\rho}}^{\diamond}$. (In particular the image of this set of invariant elements is saturated in $R_{\bar{\rho}}^{\diamond}$.)*

9. THE RINGS R_{ν}

Let $\bar{\rho} : W_F/I_F^{(\ell)} \rightarrow \text{GL}_n(k)$ be a representation. Then we have a corresponding map $x : R_{q,n} \rightarrow k$, with kernel \mathfrak{m} . It follows easily from the universal property of the pair $(R_{q,n}, \rho_{F,n})$ that the completion $(R_{q,n})_{\mathfrak{m}}$ is isomorphic to $R_{\bar{\rho}}^{\diamond}$, and that this isomorphism is induced by the base change of $\rho_{F,n}$ to $(R_{q,n})_{\mathfrak{m}}$. In other words, $R_{q,n}$ is a global object that interpolates the formal deformation rings $R_{\bar{\rho}}^{\diamond}$ for $\bar{\rho}$ trivial on $I_F^{(\ell)}$.

We would like to construct similar objects for $\bar{\rho}$ whose restriction to $I_F^{(\ell)}$ is nontrivial. Let us define:

Definition 9.1. An ℓ -inertial type is a representation ν of $I_F^{(\ell)}$ over k that extends to a representation of W_F .

Note that (as $I_F^{(\ell)}$ is a profinite group of pro-order prime to ℓ), such a representation lifts uniquely to a representation of $I_F^{(\ell)}$ over $W(k)$, and this lift also extends to a representation of W_F . We will thus consider an ℓ -inertial type ν as a representation over $W(k)$ rather than over k whenever it is convenient to do so.

Now fix an ℓ -inertial type ν , and for each irreducible representation $\bar{\tau}$ of $I_F^{(\ell)}$ over k , let $n_{\bar{\tau}}$ be the multiplicity of $\bar{\tau}$ in ν (note that $n_{\bar{\tau}}$ depends only on the W_F -conjugacy class of $\bar{\tau}$.) Let $W_{\bar{\tau}}$ be the subgroup of W_F that fixes $\bar{\tau}$ under conjugation, let $F_{\bar{\tau}}$ be the fixed field of $W_{\bar{\tau}}$, and let $q_{\bar{\tau}}$ denote the cardinality of the residue field of $F_{\bar{\tau}}$.

We define R_ν to be the tensor product:

$$R_\nu := \bigotimes_{\bar{\tau}} R_{q_{\bar{\tau}}, n_{\bar{\tau}}}$$

where $\bar{\tau}$ runs over a set of representatives for the W_F -conjugacy classes of irreducible representations appearing in ν . For each $\bar{\tau}$ we have a representation $\rho_{F_{\bar{\tau}}, n_{\bar{\tau}}}$ over $R_{q_{\bar{\tau}}, n_{\bar{\tau}}}$, which we regard as a representation over R_ν in the obvious way.

Define the representation $\rho_\nu : W_F \rightarrow \mathrm{GL}_n(R_\nu)$ as follows:

$$\rho_\nu := \bigoplus_{\bar{\tau}} \mathrm{Ind}_{W_{\bar{\tau}}}^{W_F} \rho_{F_{\bar{\tau}}, n_{\bar{\tau}}} \otimes \tau,$$

where $\bar{\tau}$ runs over a set of representative for the W_F -conjugacy classes of irreducible representations appearing in ν , and for each such $\bar{\tau}$, we have chosen an extension τ of $\bar{\tau}$ to a representation $W_F \rightarrow \mathrm{GL}_n(W(k))$ as in Proposition 8.1. Note that ρ_ν inherits a pseudo-framing from the natural framings of the $\rho_{F_{\bar{\tau}}, n_{\bar{\tau}}}$, and that the restriction of ρ_ν to $I_F^{(\ell)}$ is given by ν .

For a map $x : R_\nu \rightarrow k$, the specialization $(\rho_\nu)_x$ is a pseudo-framed representation $W_F \rightarrow \mathrm{GL}_n(k)$, whose restriction to $I_F^{(\ell)}$ is given by ν . This defines a bijection between k -points of $\mathrm{Spec} R_\nu$ and such pseudo-framed representations. Moreover, it follows directly from the constructions of R_ν and $R_{(\rho_\nu)_x}^\circ$ that the completion of R_ν at the maximal ideal corresponding to x is naturally isomorphic to $R_{(\rho_\nu)_x}^\circ$, in a manner compatible with the universal family on the latter.

Moreover, the universal property for each $R_{q_{\bar{\tau}}, n_{\bar{\tau}}}$ immediately yields:

Proposition 9.2. *For any finitely generated, ℓ -adically separated $W(k)$ -algebra A , and any pseudo-framed, ℓ -adically continuous representation $\rho : W_F \rightarrow \mathrm{GL}_n(A)$ whose restriction to $I_F^{(\ell)}$ is isomorphic to ν , there is a unique map: $R_\nu \rightarrow A$ such that ρ is the base change of ρ_ν .*

For each $\bar{\tau}$, the group $\mathrm{GL}_{n_{\bar{\tau}}}$ acts on $R_{q_{\bar{\tau}}, n_{\bar{\tau}}}$. Let \mathcal{G}_ν be the product of the $\mathrm{GL}_{n_{\bar{\tau}}}$; then \mathcal{G}_ν acts on $\mathrm{Spec} R_\nu$ by “changing the pseudo-frame”.

10. MAPS FROM $Z_{[L, \pi]}$ TO R_ν

Now fix a pair (L, π) , where L is a Levi subgroup of $\mathrm{GL}_n(F)$ and π is an irreducible supercuspidal k -representation of L . The mod ℓ semisimple local Langlands

correspondence of Vigneras [Vi] attaches to π a semisimple k -representation ρ of W_F . Let $\bar{\nu}$ be the restriction of ρ to $I_F^{(\ell)}$. Then $\bar{\nu}$ lifts uniquely to a $W(k)$ -representation ν of $I_F^{(\ell)}$, and we have:

Proposition 10.1. *The irreducible $\bar{\mathcal{K}}$ -representations of $\mathrm{GL}_n(F)$ that are objects of $\mathrm{Rep}_{W(k)}(\mathrm{GL}_n(F))_{[L,\pi]}$ correspond, via local Langlands, to the $\bar{\mathcal{K}}$ -representations of W_F whose restriction to $I_F^{(\ell)}$ is isomorphic to ν .*

Proof. This is an easy consequence of the compatibility of Vigneras' mod ℓ correspondence with reduction mod ℓ . \square

This proposition shows that for any $\bar{\mathcal{K}}$ -point x of $\mathrm{Spec} R_\nu$, the representation ρ_x corresponds, via local Langlands (and Frobenius semisimplification if necessary) to an irreducible $\bar{\mathcal{K}}$ -representation Π_x in $\mathrm{Rep}_{W(k)}(\mathrm{GL}_n(F))_{[L,\pi]}$, and hence to a $\bar{\mathcal{K}}$ -point of $\mathrm{Spec} Z_{[L,\pi]}$. It is a natural question to ask whether this map is induced by a map $Z_{[L,\pi]} \rightarrow \mathrm{Spec} R_\nu$. Indeed, we conjecture:

Conjecture 10.2 (Weak local Langlands in families). *There is a map $Z_{[L,\pi]} \rightarrow R_\nu$ such that the induced map on $\bar{\mathcal{K}}$ -points takes a point x of $\mathrm{Spec} R_\nu$ to the $\bar{\mathcal{K}}$ -point of $Z_{[L,\pi]}$ that gives the action of $Z_{[L,\pi]}$ on the representation Π_x corresponding to ρ_x by local Langlands. (We will say such a map is compatible with local Langlands.)*

Since R_ν is reduced and ℓ -torsion free, such a map is unique if it exists. Note also that the image of any element of $Z_{[L,\pi]}$ under such a map is invariant under the action of \mathcal{G}_ν , and so any such map must factor through the subalgebra R_ν^{inv} of \mathcal{G}_ν -invariant elements of R_ν . We further conjecture:

Conjecture 10.3 (Strong local Langlands in families). *There is an isomorphism $Z_{[L,\pi]} \cong R_\nu^{\mathrm{inv}}$ such that the composition*

$$Z_{[L,\pi]} \rightarrow R_\nu^{\mathrm{inv}} \rightarrow R_\nu$$

is compatible with local Langlands.

If one completes at a maximal ideal of R_ν , corresponding to a representation $\bar{\rho}$ of W_F over k , and uses Lemma 8.4 to relate the invariant elements of $R_{\bar{\rho}}^\square$ and $R_{\bar{\rho}}^\circ$, one recovers Conjectures 7.5 and 7.6 of [H2]. In particular (c.f. Theorem 7.9 of [H2]), Conjecture 10.2 above implies the ‘‘local Langlands in families’’ conjecture of Emerton-Helm (conjecture 1.1.3 of [EH]).

These conjectures should be viewed as relating ‘‘congruences’’ between admissible representations (which are in some sense encoded in the structure of $Z_{[L,\pi]}$) with ‘‘congruences’’ between representations of W_F (encoded in R_ν). Since inverting ℓ destroys information about such congruences, one expects such conjectures to be relatively straightforward with ℓ inverted. We will show that, at least for Conjecture 10.2, this is indeed the case.

First, note that any map:

$$Z_{[L,\pi]} \otimes \bar{\mathcal{K}} \rightarrow R_\nu \otimes \bar{\mathcal{K}}$$

that is compatible with local Langlands is Galois equivariant, and hence descends to a map

$$Z_{[L,\pi]}[\frac{1}{\ell}] \rightarrow R_\nu[\frac{1}{\ell}]$$

compatible with local Langlands. It thus suffices to show:

Theorem 10.4. *There is a map $Z_{[L,\pi]} \otimes \bar{\mathcal{K}} \rightarrow R_\nu \otimes \bar{\mathcal{K}}$ compatible with local Langlands (and therefore a corresponding map over \mathcal{K} .)*

To prove this, we first work on the level of connected components. We have an isomorphism:

$$Z_{[L,\pi]} \otimes \bar{\mathcal{K}} \cong \prod_{M, \tilde{\pi}} \tilde{Z}_{(M, \tilde{\pi})},$$

by Theorem 5.6, where $(M, \tilde{\pi})$ varies over the inertial equivalence classes of pairs that reduce modulo ℓ to (L, π) . Thus the connected components of $\text{Spec } Z_{[L,\pi]} \otimes \bar{\mathcal{K}}$ are in bijection with the pairs $(M, \tilde{\pi})$. Via local Langlands, these correspond to representations of I_F . More precisely, let Π be an admissible representation of G , let $\rho : W_F \rightarrow \text{GL}_n(\bar{\mathcal{K}})$ correspond to Π via local Langlands, and let $\tilde{\rho} : W_F \rightarrow \text{GL}_n(\bar{\mathcal{K}})$ be the representation of W_F corresponding to $\tilde{\pi}$ via local Langlands. Then Π belongs to the block corresponding to $(M, \tilde{\pi})$ if and only if the restriction of ρ^{ss} to I_F coincides with the restriction of $\tilde{\rho}$ to I_F .

On the other hand, it is an easy consequence of Proposition 6.9 that as x varies over $\bar{\mathcal{K}}$ -points of $\text{Spec } R_\nu$, the restriction of $\rho_{\nu,x}^{\text{ss}}$ to I_F is constant on connected components of $\text{Spec } R_\nu \otimes \bar{\mathcal{K}}$. We can thus let $R_\nu^{\tilde{\rho}}$ be the direct factor of $R_\nu \otimes \bar{\mathcal{K}}$ corresponding to the union of the connected components of $\text{Spec } R_\nu \otimes \bar{\mathcal{K}}$ on which the restriction of $\rho_{\nu,x}^{\text{ss}}$ to I_F is isomorphic to the restriction of $\tilde{\rho}$ to I_F .

It then suffices to construct, for each $(M, \tilde{\pi})$, a map:

$$\tilde{Z}_{(M, \tilde{\pi})} \rightarrow R_\nu^{\tilde{\rho}}$$

compatible with local Langlands. Since $(M, \tilde{\pi})$ is only well-defined up to inertial equivalence, we may assume that $\tilde{\pi}$ has the form:

$$\tilde{\pi} \cong \bigotimes_i \tilde{\pi}_i^{\otimes r_i},$$

where the $\tilde{\pi}_i$ are pairwise inertially inequivalent representations of $\text{GL}_{n_i}(F)$. Unwinding the Bernstein-Deligne description of $\tilde{Z}_{(M, \tilde{\pi})}$, we obtain an isomorphism:

$$\tilde{Z}_{(M, \tilde{\pi})} \cong \bigotimes_i \bar{\mathcal{K}}[X_{i,1}^{\pm 1}, \dots, X_{i,r_i}^{\pm 1}]^{S_{r_i}},$$

where the symmetric group S_{r_i} acts by permuting the elements $X_{i,1}, \dots, X_{i,r_i}$.

For each i , and any $\alpha \in \bar{\mathcal{K}}$, let $\chi_{i,\alpha}$ denote the unramified character of $\text{GL}_{n_i}(F)$ that takes the value α on any element of $\text{GL}_{n_i}(F)$ with determinant ϖ_F . An irreducible Π in $\text{Rep}_{\bar{\mathcal{K}}}(M, \tilde{\pi})$ has supercuspidal support $(M, \tilde{\pi}')$ for some $\tilde{\pi}'$ of the form:

$$\tilde{\pi}' \cong \bigotimes_i \bigotimes_{r=1}^{j_i} \tilde{\pi}_i \otimes \chi_{i,\alpha_{i,j}}$$

for suitable $\alpha_{i,j}$. Then the d th elementary symmetric function in $X_{i,1}, \dots, X_{i,r_i}$, considered as an element of $\tilde{Z}_{(M, \tilde{\pi})}$, acts on Π via the d th elementary symmetric function in the $\alpha_{i,1}^{f'_i}, \dots, \alpha_{i,r_i}^{f'_i}$, where f'_i is the order of the group of unramified characters χ such that $\tilde{\pi}_i \otimes \chi$ is isomorphic to $\tilde{\pi}_i$.

For each i , the irreducible representation $\tilde{\rho}_i$ of W_F corresponding to $\tilde{\pi}_i$ via local Langlands decomposes, when restricted to I_F , as a direct sum of distinct irreducible representations of I_F , all of which are W_F -conjugate. Fix an irreducible representation $\tilde{\tau}_i$ of I_F contained in $\tilde{\rho}_i$, and let W_i be the normalizer of $\tilde{\tau}_i$ in W_F . Then there

is a unique way of extending $\tilde{\tau}_i$ to a representation of W_i such that the induction of the resulting extension to W_F is isomorphic to $\tilde{\rho}_i$. (Note that this implies that W_i has index f'_i in W_F .)

This choice of extension of $\tilde{\tau}_i$ to W_i gives rise to an action of W_i on the space $\text{Hom}_{I_F}(\tilde{\tau}_i, \rho_\nu)$. The quotient of this space that lives over $R_\nu^\tilde{\rho}$ is a free $R_\nu^\tilde{\rho}$ -module of rank r_i , with an unramified action of W_i .

Let Fr_i be a Frobenius element of W_i , and let $P_i(x) = \sum_{i=0}^{r_i} a_i X^i$ be the characteristic polynomial of Fr_i on $\text{Hom}_{I_F}(\tilde{\tau}_i, \rho_\nu)$ (over $R_\nu^\tilde{\rho}$). Consider the map $\tilde{Z}_{(M, \tilde{\pi})} \rightarrow R_\nu^\tilde{\rho}$ that sends the d th elementary symmetric function in $X_{i,1}, \dots, X_{i,r_i}$ to the element $(-1)^d a_{i-d}$ of $R_\nu^\tilde{\rho}$. One verifies easily that this map is compatible with local Langlands, establishing Theorem 10.4.

It is not hard to go slightly further, and show:

Theorem 10.5. *The image of $Z_{[L, \pi]}$ in $R_\nu[\frac{1}{\ell}]$ under the map of Theorem 10.4 lies in the normalization of R_ν .*

Proof. Fix an element x of $Z_{[L, \pi]}$, and let y be its image in $R_\nu[\frac{1}{\ell}]$. Let A be a discrete valuation ring that is a $W(k)$ -algebra, with field of fractions K of characteristic zero, and fix a map $R_\nu \rightarrow A$. This corresponds to a pseudo-framed representation ρ_A of W_F . Let Π_K denote the admissible K -representation corresponding to $\rho_A \otimes_A K$ via local Langlands. Since $\rho_A \otimes_A K$ admits an A -lattice, so does Π_K . In particular the action of x on Π_K is via an element of A , so y maps to an element of A under the map $R_\nu[\frac{1}{\ell}] \rightarrow K$. Since this is true for every A and every map $R_\nu \rightarrow A$, y lives in the normalization of R_ν as claimed. \square

11. MAIN RESULTS

The main objective of the paper is to show the following:

Theorem 11.1. *Suppose that Conjecture 10.2 holds for all $GL_m(F)$, $m \leq n$. Then Conjecture 10.3 holds for $GL_n(F)$.*

We first note that we have tensor factorizations:

$$Z_{[L, \pi]} \cong \bigotimes_i Z_{[L_i, \pi_i]}$$

$$R_\nu \cong \bigotimes_{\bar{\tau}} R_{q_{\bar{\tau}}, n_{\bar{\tau}}}$$

where the $[L_i, \pi_i]$ are simple blocks. The former factorization is compatible with parabolic induction and the latter arises from the direct sum decomposition:

$$\rho_\nu = \bigoplus_{\bar{\tau}} \text{Ind}_{W_{\bar{\tau}}}^{W_F} \rho_{F_{\bar{\tau}}, n_{\bar{\tau}}} \otimes \tau.$$

Since simple blocks correspond to types ν with only one $n_{\bar{\tau}}$ nonzero, these factorizations are compatible, in the sense that if we have maps $Z_{[L_i, \pi_i]} \rightarrow R_{\nu_i}$ for each i that are compatible with local Langlands, then their tensor product gives a map $Z_{[L, \pi]} \rightarrow R_\nu$ compatible with local Langlands. Thus both Conjecture 10.2 and Conjecture 10.3 reduce to the corresponding conjectures on simple blocks. We thus henceforth assume that $[L, \pi]$ is of the form $[L_n, \pi_n]$ with $\pi_n \cong \pi_1^{\otimes n}$ for a supercuspidal representation π_1 . Following section 5 we set $Z_n = Z_{[L_n, \pi_n]}$. The corresponding R_{ν_n} is then isomorphic to $R_{q_{\bar{\tau}}, n}$ for some fixed $\bar{\tau}$.

Suppose now that Conjecture 10.2 holds for $Z_{[L_m, \pi_m]}$, with $m \leq n$. We then have maps:

$$Z_m \rightarrow R_{\nu_m}^{\text{inv}} \cong R_{q_{\bar{\tau}}, m}^{\text{inv}} \rightarrow A_{F_{\bar{\tau}}, m, 1}$$

the composition takes the $\bar{\mathcal{K}}$ -point of $A_{F_{\bar{\tau}}, m, 1}$ corresponding to an ℓ -ramified representation ρ of $W_{F_{\bar{\tau}}}/I_F^{(\ell)}$ to the $\bar{\mathcal{K}}$ -point of Z_m giving the action of Z_m on the irreducible representation Π that corresponds, via local Langlands, to the representation $\text{Ind}_{W_{\bar{\tau}}}^{W_F} \rho \otimes \tau$ of W_F . (In particular, since the points of $\text{Spec } Z_m$ correspond to supercuspidal supports, and hence to semisimple representations of W_F , this map is a bijection on $\bar{\mathcal{K}}$ -points.) Moreover, we have commutative diagrams:

$$\begin{array}{ccc} Z_m & \rightarrow & A_{F_{\bar{\tau}}, m, 1} \\ \downarrow & & \downarrow \\ \bigotimes_{\nu} Z_{\nu_i} & \rightarrow & A_{F_{\bar{\tau}}, \nu, 1} \end{array}$$

for any partition ν of m . (In particular the maps $\text{Ind}_{\nu^{\text{max}}}$ and $\text{Ind}_{m, n}$ constructed in section 5 satisfy the compatibilities required in section 4.)

The maps $Z_m \rightarrow Z_{F_{\bar{\tau}}, m, 1}$, together with the maps $\text{Ind}_{\nu^{\text{max}}}$ and $\text{Ind}_{m, n}$ constructed in section 5 thus satisfy conditions (1), (2), (5), and (6) of section 4. Thus we will have proven Theorem 11.1 if we can also satisfy conditions (3) and (4).

Recall from section 5 that f' is the order of the group of unramified characters χ such that $\pi \otimes \chi$ is isomorphic to π . By compatibility of local Langlands with twists, this is equal to the degree of $F_{\bar{\tau}}$ over F . We may thus identify $\bar{A}_{q^{f'}, m, 1}$ with $\bar{A}_{q_{\bar{\tau}}, m, 1}$ for all m . Unfortunately, we cannot prove that the map

$$\bar{A}_{q^{f'}, m, 1}[T^{\pm 1}] \rightarrow Z_m$$

of section 5 satisfies condition (4) of section 4. (We strongly suspect, however, that this is the case.)

Fortunately, there is a workaround: we first note, by Proposition 5.12 and Theorem 7.2, the image of the composition:

$$\bar{A}_{q^{f'}, m, 1} \rightarrow Z_m \rightarrow R_{q_{\bar{\tau}}, m}^{\text{inv}} \rightarrow A_{F_{\bar{\tau}}, m, 1}$$

lies in $\bar{A}_{q_{\bar{\tau}}, m, 1}$ (which equals $\bar{A}_{q^{f'}, m, 1}$). We then invoke:

Lemma 11.2. *Let \bar{A} be a finite rank, reduced, ℓ -torsion free $W(k)$ -algebra, and let $f : \bar{A} \rightarrow \bar{A}$ be an injection. Then f is an isomorphism.*

Proof. Clearly f is an isomorphism after inverting ℓ . On the other hand, the hypotheses guarantee that $\bar{A}[\frac{1}{\ell}]$ is a product of finite extensions of \mathcal{K} , and f is an automorphism of this product. In particular there is some power of f that is the identity. \square

Thus all of $\bar{A}_{q^{f'}, m, 1}$ is in the image of the map $Z_m \rightarrow A_{F_{\bar{\tau}}, m, 1}$. Moreover, we may adjust the map $\bar{A}_{q^{f'}, m, 1} \rightarrow Z_m$ so that the composition with the map $Z_m \rightarrow \bar{A}_{q^{f'}, m, 1}$ is the identity. To extend this to a map $\bar{A}_{q^{f'}, m, 1}[T^{\pm 1}] \rightarrow Z_m$ satisfying condition (4) of section 4, we must show there is an element of Z_m that maps to the element Q_m of $A_{F_{\bar{\tau}}, m, 1}$, or, equivalently, that maps to the element $\det \text{Fr}$ of $R_{q_{\bar{\tau}}, m}$. Let Θ be the image of $\Theta_{n, n}$ in $R_{q_{\bar{\tau}}, m}$. Then, by Proposition 5.13 and Theorem 7.2, the element $\Theta^{-1} \det \text{Fr}$ of $R_{q_{\bar{\tau}}, m}^{\text{inv}}$ maps to an invertible element of $\bar{A}_{q^{f'}, m, 1}$ under the map $R_{q_{\bar{\tau}}, m} \rightarrow A_{F_{\bar{\tau}}, m, 1}$. This element lies in the image of Z_m , and therefore so does the image of $\det \text{Fr}$. There is thus an element Θ' of Z_m that

maps to $\det \text{Fr}$ in $R_{q_{\bar{\tau}}, m}$ and to Q_m in $A_{F_{\bar{\tau}}, m, 1}$; we extend the map $\bar{A}_{q_{f'}, m, 1} \rightarrow Z_m$ to a map on $\bar{A}_{q_{f'}, m, 1}[T^{\pm 1}]$ by sending T to Θ' . The newly constructed map clearly satisfies condition (4) of section 4.

It remains to verify condition (3) of section 4. On the other hand, note that $A_{F_{\bar{\tau}}, 1, 1}$ is generated over its subalgebra $\bar{A}_{q_{\bar{\tau}}, 1, 1}$ by $Q_1^{\pm 1}$, and that both $\bar{A}_{q_{\bar{\tau}}, 1, 1}$ and Q_1 are in the image of Z_1 . Thus the map $Z_1 \rightarrow A_{F_{\bar{\tau}}, 1, 1}$ is an isomorphism.

The results of section 4 now allow us to conclude that the composition:

$$Z_n \rightarrow R_{q_{\bar{\tau}}, n}^{\text{inv}} \rightarrow A_{F_{\bar{\tau}}, n, 1}$$

is an isomorphism. On the other hand, the map $R_{q_{\bar{\tau}}}^{\text{inv}} \rightarrow A_{F_{\bar{\tau}}, n, 1}$ is injective, so each of the individual maps must be isomorphisms.

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