P-ADIC ASAI L-FUNCTIONS OF BIANCHI MODULAR FORMS

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Abstract. The Asai (or twisted tensor) L-function of a Bianchi modular form Ψ is the L-function attached to the tensor induction to $\mathbb{Q}$ of its associated Galois representation. In this paper, when Ψ is ordinary at p we construct a p-adic analogue of this L-function: that is, a p-adic measure on $\mathbb{Z}_p^\times$ that interpolates the critical values of the Asai L-function twisted by Dirichlet characters of p-power conductor. The construction uses techniques analogous to those used by Lei, Zerbes and the first author in order to construct an Euler system attached to the Asai representation of a quadratic Hilbert modular form.

1. Introduction

1.1. Background Several of the most important conjectures in modern number theory, such as the Bloch–Kato and Beilinson conjectures, relate the special values of L-functions to arithmetic data. In much of the work on these conjectures to date, an important role has been played by *p-adic L-functions*: measures or distributions on $\mathbb{Z}_p^\times$, for a prime p, interpolating the special values of a given complex L-function and its twists by Dirichlet characters of p-power conductor. Such functions are expected to exist in wide generality, but in practice they can be difficult to construct, and there are large classes of L-functions which at present are not known to have a p-adic analogue. In this paper, we provide such a construction for a new class of L-functions: the *Asai*, or *twisted tensor*, L-functions attached to Bianchi modular forms (automorphic forms for GL_2/F, where F is imaginary quadratic).

In order to construct our p-adic L-function, we use the Betti cohomology of a locally symmetric space associated to GL_2/F. Work of Ghate [Gha99] shows that the critical values of the Bianchi Asai L-function and its twists are computed by certain special elements in Betti cohomology, which are pushforwards of cohomology classes for $\mathrm{GL}_2/\mathbb{Q}$ associated to Eisenstein series. However, interpolating such classes p-adically is not straightforward. The key novelty in our construction is to *simultaneously* vary two parameters: the choice of Eisenstein series, and the choice of embedding of $\mathrm{GL}_2/\mathbb{Q}$ in GL_2/F. This allows us to reduce the interpolation problem to a (much simpler) compatibility property of the $\mathrm{GL}_2/\mathbb{Q}$ Eisenstein series.

Our construction uses techniques that are closely related to those those found in [LLZ14] and [LLZ16], in which Lei, Zerbes and the first author constructed Euler systems (certain compatible families of étale cohomology classes) for Rankin–Selberg convolutions of modular forms, and for the Asai representation of a Hilbert modular form over a real quadratic field. In the Bianchi setting, there is no étale cohomology to consider, since Bianchi manifolds (the symmetric spaces associated to GL_2/F) are not algebraic varieties. However, we show in this article that applying the same techniques in this setting instead gives compatible families of classes in the Betti cohomology of these spaces. Hence the same techniques used to construct an Euler system for GL_2/F when F is real quadratic also give rise to a p-adic L-function when F is imaginary quadratic.

We hope that these techniques can be extended to build other new p-adic L-functions as "Betti counterparts" of known Euler system constructions; in particular, we are presently

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exploring applications of this method to the standard L-function of (possibly non-self-dual) cohomological automorphic representations of $\mathrm{GL}_3/\mathbb{Q}$.

Note While working on this project, we learned that Balasubramanyam, Ghate and Vangala have also been working on a construction of p-adic Asai L-functions for Bianchi cusp forms [BGV]. Their work is independent of ours, although both constructions rely on the same prior work [Gha99] of Ghate.

1.2. Outline of the construction We give a brief outline of the construction in the simplest case, when Ψ is a Bianchi modular eigenform of weight 0 (i.e. contributing to cohomology with trivial coefficients) for some imaginary quadratic field F, and the Hecke eigenvalues $\lambda_{\mathfrak{m}}$ of Ψ for ideals $\mathfrak{m}$ are in $\mathbb{Z}$ (as in the case of the eigenforms conjecturally associated to elliptic curves). We assume that the level $\mathfrak{n}$ of Ψ is divisible by all primes $\mathfrak{p} \mid p$ of F.

The Asai L-function of Ψ is defined by[1]

$$L^{\mathrm{As}}(\Psi, s) := L(\varepsilon_{\Psi,\mathbb{Q}}, 2s-2) \sum_{n \geq 1} \lambda_{n\mathcal{O}_F} n^{-s},$$

where $\varepsilon_{\Psi,\mathbb{Q}}$ is the restriction to $\widehat{\mathbb{Z}}^\times$ of the nebentypus character of Ψ, and $\lambda_{\mathfrak{m}}$ is the Hecke eigenvalue of Ψ at the ideal $\mathfrak{m}$. It is the L-function of the *Asai representation* of Ψ, that is, the tensor induction to $\mathbb{Q}$ of the Galois representation attached to Ψ.

We assume that Ψ is ordinary at p (i.e. $\lambda_{p\mathcal{O}_F}$ is a p-adic unit). From Ψ we construct a class $\phi_\Psi^* \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}^*(\mathfrak{n}), \mathbb{Z}_p)$, where $Y_{F,1}^*(\mathfrak{n})$ is a Bianchi manifold with appropriate level structure. This cohomology group is Poincaré dual to $\mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), \mathbb{Z}_p)/(\text{torsion})$. In [Gha99], Ghate showed that critical values of the Asai L-function can be obtained by pairing ϕ_Ψ^* with certain classes in this H^2 coming from classical weight 2 Eisenstein series. The main new ideas in the present paper arise in controlling integrality of these Eisenstein classes as the level varies, thus putting them into a compatible family from which we build a p-adic measure.

The first input in our construction is a collection of maps, one for each $m \geq 1$ and $a \in \mathcal{O}_F$, defined by

$$Y_{\mathbb{Q},1}(m^2N) \xrightarrow{\ \iota\ } Y_{F,1}^*(m^2\mathfrak{n}) \xrightarrow{\ \kappa_{a/m}\ } Y_{F,1}^*(\mathfrak{n}),$$

where ι is the natural embedding, and $\kappa_{a/m}$ is obtained by twisting the natural quotient map by $\left(\begin{smallmatrix} 1 & a/m \\ 0 & 1\end{smallmatrix}\right)$. Here $Y_{\mathbb{Q},1}(m^2N)$ is the usual (open) modular curve for $\mathrm{GL}_2/\mathbb{Q}$ of level m^2N, where $N = \mathfrak{n} \cap \mathbb{Z}$.

The second input is a collection of special cohomology classes ("Betti Eisenstein classes") ${}_cC_{m^2N} \in \mathrm{H}^1(Y_{\mathbb{Q},1}(m^2N), \mathbb{Z})$. These are constructed using Siegel units. The theory of Siegel units shows that these classes satisfy norm-compatibility properties as m varies, and that their images in de Rham cohomology are related to the Eisenstein series used in [Gha99]. (The factor c refers to an auxiliary choice of integer which serves to kill off denominators from these classes).

With these definitions, we set

$$\begin{aligned} {}_c\Xi_{m,\mathfrak{n},a} &:= (\kappa_{a/m} \circ \iota)_* \left({}_cC_{m^2N}\right) \in \mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), \mathbb{Z}), \\ {}_c\Phi^r_{\mathfrak{n},a} &:= \sum_{t \in (\mathbb{Z}/p^r\mathbb{Z})} {}_c\Xi_{p^r,\mathfrak{n},at} \otimes [t] \in \mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), \mathbb{Z}) \otimes \mathbb{Z}_p[(\mathbb{Z}/p^r)^\times]. \end{aligned}$$

The key theorem in our construction (Theorem 3.13) is that the classes ${}_c\Phi^r_{\mathfrak{n},a}$ satisfy a norm-compatibility relation in r. Both the statement of this norm-compatibility relation, and its

[1] This should not be confused with the standard L-function $L^{\mathrm{std}}(\Psi, s) := \sum_{\mathfrak{m} \trianglelefteq \mathcal{O}_F} \lambda_{\mathfrak{m}} \mathrm{Nm}(\mathfrak{m})^{-s}$. The construction of a p-adic counterpart of the standard L-function of a Bianchi eigenform is the main result of the paper [Wil17] of the second author.

proof, are very closely analogous to the norm-compatibility relations for Euler system classes in [LLZ14, LLZ16].

From this, it follows that (after renormalising using the Hecke operator U_p) the classes ${}_c\Phi^r_{\mathfrak{n},a}$ form an inverse system. In particular, they fit together to define an element

$$ {}_c\Phi^\infty_{\mathfrak{n},a} \in e_{\mathrm{ord}}\mathrm{H}^2(Y^*_{F,1}(\mathfrak{n}),\mathbb{Z}) \otimes \mathbb{Z}_p[[\mathbb{Z}_p^\times]], $$

where e_{ord} is Hida's ordinary projector, which we view as bounded measure on $\mathbb{Z}_p^\times$ with values in the ordinary part of $\mathrm{H}^2(Y^*_{F,1}(\mathfrak{n}),\mathbb{Z})$. We then define the p-adic Asai L-function to be the measure

$$ {}_cL_p^{\mathrm{As}}(\Psi) := \langle \phi^*_\Psi, {}_c\Phi^\infty_{\mathfrak{n},a}\rangle \in \mathbb{Z}_p[[\mathbb{Z}_p^\times]]. $$

That this measure interpolates the critical values of the (complex) Asai L-function then follows from [Gha99] together with certain twisting maps (to obtain twisted L-values).

The case of higher-weight Bianchi forms (contributing to cohomology with non-constant coefficients) is similar, although unavoidably a little more technical. Suppose Ψ is such a form of weight (k,k). Using [Gha99] and the same twisting methods as in the weight $(0,0)$ case, one can prove algebraicity for the critical value $L^{\mathrm{As}}(\Psi,\chi,j+1)$, where $0 \leq j \leq k$ and $\chi(-1)(-1)^j = 1$, by pairing with classes in H^2 arising from Eisenstein series of weight $2k-2j+2$. For each such j, we define a compatible system of cohomology classes with coefficients in a suitable algebraic representation of GL_2/F by applying a p-adic moment map to our Siegel-unit classes, obtaining classes ${}_c\Phi^{\infty,j}_{\mathfrak{n},a}$ analogous to ${}_c\Phi^\infty_{\mathfrak{n},a}$ in the weight $(0,0)$ construction. Again, this is a "Betti analogue" of a construction for étale cohomology which is familiar in the theory of Euler systems [Kin15, KLZ17].

Pairing ϕ^*_Ψ with ${}_c\Phi^{\infty,j}_{\mathfrak{n},a}$ gives a p-adic measure on $\mathbb{Z}_p^\times$, as above. Using Kings' theory of p-adic interpolation of polylogarithms, it turns out that after a twist by the norm this measure is actually independent of j, and we define the p-adic Asai L-function ${}_cL_p^{\mathrm{As}}(\Psi)$ to be the measure for $j=0$. Moreover, the class ${}_c\Phi^{\infty,j}_{\mathfrak{n},a}$ can be explicitly related to weight $2k-2j+2$ Eisenstein series, so that integrating the function $\chi(x)x^j$ against ${}_cL_p^{\mathrm{As}}(\Psi)$ computes the value $L^{\mathrm{As}}(\Psi,\chi,j+1)$ (under the parity condition above). The above can be summarised in the following theorem, which is the main result of this paper.

Theorem 1.1. *Let Ψ be an ordinary cuspidal Bianchi eigenform of weight (k,k) and level $\mathfrak{n}$, where $\mathfrak{n}$ is divisible by the primes of F above p. Let R be the ring of integers in the finite extension $L/\mathbb{Q}_p$ generated by adjoining the Hecke eigenvalues of Ψ. Then there exists a p-adic measure*

$$ {}_cL_p^{\mathrm{As}}(\Psi) \in R[[\mathbb{Z}_p^\times]] $$

on $\mathbb{Z}_p^\times$ satisfying the following interpolation property: if χ is a Dirichlet character of conductor p^r, and $0 \leq j \leq k$, then we have

$$ \int_{\mathbb{Z}_p^\times} \chi(x)x^j \mathrm{d}_c L_p^{\mathrm{As}}(\Psi)(x) = \begin{cases} (*)L^{\mathrm{As}}(\Psi,\overline{\chi},j+1) & : \chi(-1)(-1)^j = 1, \\ 0 & : \chi(-1)(-1)^j = -1, \end{cases} $$

where $(*)$ is an explicit factor (which is always non-zero if $r \geq 1$).

The assumption that all primes $\mathfrak{p} \mid p$ of F divide $\mathfrak{n}$ leads to no loss of generality, since we do not require that Ψ be a newform, and hence we may apply our results to $\mathfrak{p}$-stabilisations of newforms of level prime to $\mathfrak{p}$. The precise interpolation theorem is Theorem 7.4 of the main text.

It is possible to remove the dependence on c, at the cost of possibly passing to a slightly larger space of "pseudo-measures", which may be interpreted as meromorphic (rather than analytic) functions on p-adic weight space. The details of this are contained in §6.2. We show that the resulting element of $\operatorname{Frac} R[[\mathbb{Z}_p^\times]]$ has at worst two simple poles, and in many important cases it has none at all (i.e. it is a measure).

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2. Preliminaries and notation

2.1. Basic notation We fix notation for a general number field K, which will either be $\mathbb{Q}$ or an imaginary quadratic field. (We'll generally denote this imaginary quadratic field by F to distinguish it from the rationals in the notation). Denote the ring of integers by $\mathcal{O}_K$, the adele ring by $\mathbb{A}_K$ and the finite adeles by $\mathbb{A}_K^f$. We let $\widehat{\mathcal{O}}_K := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_K$ be the finite integral adeles, and $K^{\times +}$ the totally-positive elements of $K^\times$ (so that $K^{\times +} = K^\times$ for $K = F$).

Let $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the usual upper half-plane, with $\mathrm{GL}_2(\mathbb{R})_+$ (the group of 2×2 matrices of positive determinant) acting by Möbius transformations in the usual way; we extend this to all of $\mathrm{GL}_2(\mathbb{R})$ by letting $\left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right)$ act via $x + iy \mapsto -x + iy$.

Define the *upper half-space* to be

$$\mathcal{H}_3 := \{(z,t) \in \mathbb{C} \times \mathbb{R}_{>0}\},$$

with $\mathrm{GL}_2(\mathbb{C})$ acting via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (z,t) = \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}t^2}{|cz+d|^2 + |c|^2t^2}, \frac{|ad-bc|t}{|cz+d|^2+|c|^2t^2} \right).$$

We embed $\mathcal{H}$ in $\mathcal{H}_3$ via $x + iy \mapsto (x,y)$, which is compatible with the actions of $\mathrm{GL}_2(\mathbb{R})$ on both sides.

Throughout, p will denote a rational prime. Let F be an imaginary quadratic field of discriminant $-D$, with different $\mathcal{D} = (\sqrt{-D})$, and fix a choice of $\sqrt{-D}$ in $\mathbb{C}$. Let $\mathfrak{n} \subset \mathcal{O}_F$ be an ideal of F, divisible by all the primes of F above p; this will be the level of our Bianchi modular form. We assume throughout that $\mathfrak{n}$ is small enough to ensure that the relevant locally symmetric space attached to $\mathfrak{n}$ is smooth (see Proposition 2.6). Let N be the natural number with $(N) = \mathbb{Z} \cap \mathfrak{n}$ as ideals in $\mathbb{Z}$ (noting that $p \mid N$).

For an integer $n \geq 0$ and a ring R, define $V_n(R)$ to be the space of homogeneous polynomials of degree n in two variables X, Y with coefficients in R, with $\mathrm{GL}_2(R)$ acting on the right via $(f \mid \gamma)(X,Y) = f(\gamma \cdot (X,Y))$.

2.2. Locally symmetric spaces

Definition 2.1. Let U be an open compact subset of $\mathrm{GL}_2(\widehat{\mathcal{O}}_K)$. We define locally symmetric spaces of level U as follows:

- If $K = \mathbb{Q}$ we set
$$Y_{\mathbb{Q}}(U) := \mathrm{GL}_2(\mathbb{Q})_+ \backslash \left[\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^f) \times \mathcal{H}\right] /U,$$
where $\mathrm{GL}_2(\mathbb{Q})_+$ acts from the left on both factors in the usual way, and U acts on the right of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^f)$.
- If $K = F$ is imaginary quadratic, we set
$$Y_F(U) := \mathrm{GL}_2(F) \backslash \left[\mathrm{GL}_2(\mathbb{A}_F^f) \times \mathcal{H}_3\right] /U.$$
- Again for $K = F$, we write $\mathrm{GL}_2^*(\mathbb{A}_F^f) = \{g \in \mathrm{GL}_2(\mathbb{A}_F^f) : \det(g) \in (\mathbb{A}_{\mathbb{Q}}^f)^\times\}$ and similarly $\mathrm{GL}_2^*(F)_+ = \{g \in \mathrm{GL}_2(F) : \det(g) \in \mathbb{Q}^\times, \det(g) > 0\}$, and define
$$Y_F^*(U) := \mathrm{GL}_2^*(F)_+ \backslash \left[\mathrm{GL}_2^*(\mathbb{A}_F^f) \times \mathcal{H}_3\right] /U^*,$$
where $U^* := U \cap \mathrm{GL}_2^*(\mathbb{A}_F^f)$.

Remark 2.2: Respectively, these spaces correspond to the algebraic groups GL_2, $G := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, and $G^* := G \times_{\mathbb{D}} \mathbb{G}_m$, where $\mathbb{D} := \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ and the map $G \to \mathbb{D}$ is determinant. We will always work explicitly with (adelic or global) points of these groups.

Each of these spaces has finitely many connected components, each of which is the quotient of $\mathcal{H}$ or $\mathcal{H}_3$ by a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ (resp. $\mathrm{PGL}_2(\mathbb{C})$). If U is sufficiently small, these discrete subgroups will act freely, so in particular the quotient is a manifold.

Definition 2.3. Let $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{a}$ be ideals in $\mathcal{O}_K$, and define:

(ii) $U_K(\mathfrak{m},\mathfrak{n}) := \{\gamma \in \mathrm{GL}_2(\widehat{\mathcal{O}}_K) : \gamma \equiv I \pmod{\left(\begin{smallmatrix}\mathfrak{m} & \mathfrak{m}\\ \mathfrak{n} & \mathfrak{n}\end{smallmatrix}\right)}\}$,
(iii) $U_K(\mathfrak{m}(\mathfrak{a}),\mathfrak{n}) := \{\gamma \in \mathrm{GL}_2(\widehat{\mathcal{O}}_K) : \gamma \equiv I \pmod{\left(\begin{smallmatrix}\mathfrak{m} & \mathfrak{m}\mathfrak{a}\\ \mathfrak{n} & \mathfrak{n}\end{smallmatrix}\right)}\}$,

We write $Y_K(\mathfrak{m},\mathfrak{n}) := Y_K(U(\mathfrak{m},\mathfrak{n}))$ and similarly $Y_K(\mathfrak{m}(\mathfrak{a}),\mathfrak{n})$. We will be particularly interested $Y_K(\mathfrak{m},\mathfrak{n})$ for $\mathfrak{m} = (1)$, which we abbreviate as $Y_{K,1}(\mathfrak{n})$.

Example 2.4: The following three locally symmetric spaces are of particular importance in the sequel, so here we describe them explicitly (and record some of their other basic properties) for reference later in the paper. In particular, if F is an imaginary quadratic field:

(i) $Y_{\mathbb{Q},1}(N)$ is the usual (open) modular curve of level $\Gamma_1(N)$. It has one connected component, isomorphic to $\Gamma_1(N)\backslash\mathcal{H}$.
(ii) The space $Y_{F,1}^*(\mathfrak{n})$ also has a single connected component, isomorphic to $\Gamma_{F,1}^*(\mathfrak{n})\backslash\mathcal{H}_3$, where
$$\begin{aligned}\Gamma_{F,1}^*(\mathfrak{n}) &:= \mathrm{GL}_2^*(F)_+ \cap U_{F,1}(\mathfrak{n})\\ &= \left\{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) : c = 0, a = d = 1 \bmod \mathfrak{n}\right\}.\end{aligned}$$
(iii) Since $\det(U_{F,1}(\mathfrak{n})) = \widehat{\mathcal{O}}_K^\times$, the space $Y_{F,1}(\mathfrak{n})$ has h_F connected components, where h_F is the class number of F. The identity component is isomorphic to $\Gamma_{F,1}(\mathfrak{n})\backslash\mathcal{H}_3$, where
$$\Gamma_{F,1}(\mathfrak{n}) := \mathrm{GL}_2(F) \cap U_{F,1}(\mathfrak{n}).$$

Suppose $N = \mathfrak{n} \cap \mathbb{Z}$. Then there are natural maps
$$Y_{\mathbb{Q},1}(N) \xrightarrow{\ \iota\ } Y_{F,1}^*(\mathfrak{n}) \xrightarrow{\ \jmath\ } Y_{F,1}(\mathfrak{n})$$
induced by the natural maps $\mathcal{H} \hookrightarrow \mathcal{H}_3$ and $\mathrm{GL}_2^*(\mathbb{A}_F^f) \to \mathrm{GL}_2(\mathbb{A}_F^f)$ respectively. The composition $\jmath \circ \iota$ is *never* injective, since $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) \in \Gamma_1(\mathfrak{n})$ acts on $\mathcal{H}$ by $x + iy \mapsto -x + iy$, and the (distinct) images of $\pm x + iy$ in $Y_{\mathbb{Q},1}(N)$ are identified when mapped to $Y_{F,1}(\mathfrak{n})$. Indeed, we see directly that:

Proposition 2.5. *The map $\jmath : Y_{F,1}^*(\mathfrak{n}) \to Y_{F,1}(\mathfrak{n})$ has image equal to the identity component $\Gamma_{F,1}\backslash\mathcal{H}_3$. Its fibres are the orbits of the finite group $\left\{\left(\begin{smallmatrix}\epsilon & 0\\ 0 & 1\end{smallmatrix}\right) : \epsilon \in \mathcal{O}_F^\times\right\}$ acting on $\Gamma_{F,1}^*\backslash\mathcal{H}_3$.*

By contrast, we *do* have injectivity of ι, providing a key reason for introducing the space $Y_{F,1}^*(\mathfrak{n})$.

Proposition 2.6. *If $\mathfrak{n}$ is divisible by some integer $q \geq 4$, then $Y_{F,1}^*(\mathfrak{n})$ is a smooth manifold, and*
$$\iota : Y_{\mathbb{Q},1}(N) \hookrightarrow Y_{F,1}^*(\mathfrak{n})$$
is a closed immersion.

Proof. First, the smoothness assertion. It suffices to prove that $\Gamma_{F,1}^*(\mathfrak{n})$ has no non-trivial torsion elements. Since $\Gamma_1^*(\mathfrak{n})$ is a subgroup of $\mathrm{SL}_2(\mathcal{O}_F)$, any torsion element γ must have eigenvalues ζ, ζ^{-1} where ζ is a (non-trivial) root of unity, defined over an extension of F of degree at most 2. Since $\zeta + \zeta^{-1} = a + d = 2 \bmod \mathfrak{n}$, we conclude that $\mathfrak{n}$ divides $\zeta + \zeta^{-1} - 2$.

A case-by-case check shows that this implies ζ has order $2, 3, 4$ or 6, and $\mathfrak{n}$ must contain one of the integers 1,2,3.

Let us now prove the injectivity assertion. Let $z, z' \in \mathcal{H}_3$ be such that $\gamma z = z'$, for some $\gamma \in \Gamma_1^*(\mathfrak{n})$. Then $\gamma^{-1}\bar{\gamma}z = z$, so either $\gamma^{-1}\bar{\gamma} = \mathrm{id}$, or $\gamma^{-1}\bar{\gamma}$ is a non-trivial torsion element in $\mathrm{SL}_2(\mathcal{O}_F)$. Since γ is upper-triangular modulo some integer $q \geq 4$, the same is true of $\bar{\gamma}$ and thus also of $\gamma^{-1}\bar{\gamma}$; but we have just seen that $\Gamma_{F,1}^*(q)$ has no torsion elements for $q \geq 4$.

We can therefore conclude that $\gamma^{-1}\bar{\gamma} = \mathrm{id}$, in other words that $\gamma \in \Gamma_{F,1}^*(\mathfrak{n}) \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma_{\mathbb{Q},1}(N)$. Hence $z = z'$ as elements of $Y_{\mathbb{Q},1}(N)$. $\square$

Remark: Henceforth, we will always assume that $\mathfrak{n}$ is divisible by such a q, or, more generally, is small enough to avoid the possibility that these spaces are (non-smooth) orbifolds.

2.3. Hecke correspondences We can define Hecke correspondences on the symmetric spaces $Y_{K,1}(\mathfrak{n})$, for $\mathfrak{n}$ an ideal of $\mathcal{O}_K$, as follows. Firstly, we have diamond operators $\langle w \rangle$ for every $w \in (\mathcal{O}_K/\mathfrak{n})^\times$, which define an action of $(\mathcal{O}_K/\mathfrak{n})^\times$ on $Y_{K,1}(\mathfrak{n})$; this even extends to an action of the narrow ray class group modulo $\mathfrak{n}$, although we shall not use this.

Secondly, let $\mathfrak{a}$ be a square-free ideal of $\mathcal{O}_K$. Consider the diagram

$$\begin{array}{ccc} & Y_K(1(\mathfrak{a}), \mathfrak{n}) & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ Y_{K,1}(\mathfrak{n}) & & Y_{K,1}(\mathfrak{n}), \end{array}$$

where π_1 is the natural projection map, and π_2 is the 'twisted' map given by the right-translation action of $\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}$ on $\mathrm{GL}_2(\mathbb{A}_K^f)$, where $\varpi \in \widehat{\mathcal{O}}_K$ is any integral adèle which generates the ideal $\mathfrak{a}\widehat{\mathcal{O}}_K$. (If $K = \mathbb{Q}$ and $\varpi = a$ is the positive integer generating $\mathfrak{a}$, then this map π_2 corresponds to $z \mapsto z/a$ on $\mathcal{H}$.) We then define

$$(T_\mathfrak{a})_* := (\pi_2)_* \circ (\pi_1)^*$$

$$(T_\mathfrak{a})^* := (\pi_1)_* \circ (\pi_2)^*$$

as correspondences on $Y_{K,1}(\mathfrak{n})$. When $\mathfrak{a}$ divides the level $\mathfrak{n}$, we denote these operators instead by $(U_\mathfrak{a})_*$ and $(U_\mathfrak{a})^*$. The definition may be extended to non-squarefree $\mathfrak{a}$ in the usual way.

The same construction is valid for the more general symmetric spaces $Y_K(\mathfrak{m}, \mathfrak{n})$, but it is no longer independent of the choice of generator ϖ of $\mathfrak{a}$ (it depends on the class of ϖ modulo $1 + \mathfrak{m}\widehat{\mathcal{O}}_K$). We will only use this in the case where $\mathfrak{a}$ is generated by a positive integer a, in which case we of course take $\varpi = a$. With this convention, the Hecke operators $(T_a)_*$ and $(T_a)^*$ for positive integers a also make sense on the "hybrid" symmetric spaces $Y_{F,1}^*(\mathfrak{m}, \mathfrak{n})$.

Remark: The maps $(T_\mathfrak{a})^*$ and $(U_\mathfrak{a})^*$ are perhaps more familiar, as their action on automorphic forms is given by simple formulae in terms of Fourier expansions, as we shall recall below. The lower-star versions $(T_\mathfrak{a})_*$ and $(U_\mathfrak{a})_*$ are the transpose of the upper-star versions with respect to Poincaré duality; this duality explains the key role played by $(U_p)_*$ in our norm relation computations.

2.4. Bianchi modular forms and Asai L-functions We briefly recall the definition of Bianchi modular forms; for further details see [Wil17, §1]. As above, let F be an imaginary quadratic field, and U an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_F^f)$. Then, for any $k \geq 0$, there is a finite-dimensional $\mathbb{C}$-vector space $S_{k,k}(U)$ of *Bianchi cusp forms* of weight (k, k) and level U, which are functions

$$\Psi : \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/U \longrightarrow V_{2k+2}(\mathbb{C})$$

transforming appropriately under the subgroup $\mathbb{C}^\times \cdot \mathrm{SU}_2(\mathbb{C})$, and satisfying suitable harmonicity and growth conditions.

These forms can be described by an appropriate analogue of q-expansions (cf. [Wil17, §1.2]). Let $e_F : \mathbb{A}_F/F \to \mathbb{C}^\times$ denote the unique function whose restriction to $F \otimes \mathbb{R} \cong \mathbb{C}$ is

$$x_\infty \longmapsto e^{2\pi i \mathrm{Tr}_{F/\mathbb{Q}}(x_\infty)},$$

and let $W_\infty : \mathbb{R} \to V_{2k+2}(\mathbb{C})$ be the real-analytic function defined in 1.2.1(v) of *op.cit.* (involving the Bessel functions K_n).

Theorem 2.7. *Let Ψ be a Bianchi modular form of weight (k,k) and level U. Then there is a* Fourier–Whittaker expansion

$$\Psi\left(\begin{pmatrix}\mathbf{y} & \mathbf{x}\\ 0 & 1\end{pmatrix}\right) = |\mathbf{y}|_{\mathbb{A}_F} \sum_{\zeta \in F^\times} W_f(\zeta \mathbf{y}_f, \Psi) W_\infty(\zeta \mathbf{y}_\infty) e_F(\zeta \mathbf{x}),$$

where $W_f(-,\Psi)$ is a vector in the Kirillov model, that is, a locally constant function on $(\mathbb{A}_F^f)^\times$, with support contained in a compact subset of $\mathbb{A}_F^f$.

If $U = U_{F,1}(\mathfrak{n})$ for some $\mathfrak{n}$, then $W_f(-,\Psi)$ is supported in $\mathcal{D}^{-1}\widehat{\mathcal{O}}_F$. For $\mathfrak{m}$ an ideal of $\mathcal{O}_F$, we define a coefficient $c(\mathfrak{m},\Psi)$ as the value $W_f(\mathbf{y}_f,\Psi)$ for any $\mathbf{y}_f$ generating the fractional ideal $\mathcal{D}^{-1}\mathfrak{m}\widehat{\mathcal{O}}_F$; this is independent of the choice of $\mathbf{y}_f$.

Exactly as for elliptic modular forms, the space $S_{k,k}(U_{F,1}(\mathfrak{n}))$ has an action of (commuting) Hecke operators $(T_\mathfrak{m})^*$ for all ideals $\mathfrak{m}$; and if Ψ is an eigenvector for all these operators, normalized such that $c(1,\Psi) = 1$, then the eigenvalue of the $\mathfrak{m}$-th Hecke operator on Ψ is $c(\mathfrak{m},\Psi)$.

We end this subsection by defining the Asai L-function. The space $S_{k,k}(U_{F,1}(\mathfrak{n}))$ has an action of diamond operators $\langle d\rangle$, for all $d \in (\mathcal{O}_F/\mathfrak{n})^\times$; and on any Hecke eigenform Ψ these act via a character $\varepsilon_\Psi : (\mathcal{O}_F/\mathfrak{n})^\times \to \mathbb{C}^\times$. Let $\varepsilon_{\Psi,\mathbb{Q}}$ denote the restriction of this character to $(\mathbb{Z}/N\mathbb{Z})^\times$.

Definition 2.8. Let Ψ be a normalized eigenform in $S_{k,k}(U_{F,1}(\mathfrak{n}))$, and χ a Dirichlet character of conductor m. Define the *Asai L-function* of Ψ by

$$L^{\mathrm{As}}(\Psi,\chi,s) := L^{(mN)}(\chi^2\varepsilon_{\Psi,\mathbb{Q}}, 2s-2k-2) \cdot \sum_{\substack{n\geq 1\\ (m,n)=1}} c(n\mathcal{O}_F,\Psi)\chi(n)n^{-s},$$

where $L^{(mN)}(-,s)$ is the Dirichlet L-function with its Euler factors at primes dividing mN removed.

Remark: This Dirichlet series converges for $\Re(s) > k+2$, and has meromorphic continuation to all $s \in \mathbb{C}$. As we have defined it, $L^{\mathrm{As}}(\Psi,\chi,s)$ may not satisfy a functional equation (because it may have the wrong local factors at primes dividing mND). However, if Π is the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ generated by Ψ, then we can write

$$L^{\mathrm{As}}(\Psi,\chi,s) = L^{\mathrm{As}}(\Pi,\chi,s) \prod_{\ell \mid mND} C_\ell(\Psi,\chi,s)$$

where $C_\ell(\Psi,\chi,s)$ is a polynomial in $\{\ell^s,\ell^{-s}\}$ and the "primitive" L-function $L^{\mathrm{As}}(\Pi,\chi,s)$ satisfies a functional equation relating $L^{\mathrm{As}}(\Pi,\chi,s)$ and $L^{\mathrm{As}}(\Pi,\varepsilon_\Pi^{-1}\chi^{-1},2k+3-s)$. If Π is not of dihedral type and not a twist of a base-change from $\mathrm{GL}_2/\mathbb{Q}$, then $L^{\mathrm{As}}(\Pi,\chi,s)$ is entire.

2.5. The modular symbol attached to Ψ The Bianchi modular forms we consider in this paper are *cohomological*, in the following sense.

Definition 2.9. Let A be an F-algebra, and let $V_{kk}(A) := \mathrm{Sym}^k(A^2) \otimes_F \mathrm{Sym}^k(A^2)^\sigma$ be the $F[\mathrm{GL}_2(F)]$-module on which $\gamma \in \mathrm{GL}_2(F)$ acts in the usual way on the first component and via its complex conjugate γ^σ on the second component. Via this action, the space $V_{kk}(A)$ gives rise to a local system on $Y_{F,1}(\mathfrak{n})$.

Theorem 2.10 (Eichler–Shimura–Harder)**.** *There is a Hecke-equivariant injection*

$$S_{k,k}(U_{F,1}(\mathfrak{n})) \hookrightarrow \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{n}), V_{kk}(\mathbb{C}))$$

whose cokernel is Eisenstein. In particular, this map is an isomorphism after restriction to the Ψ*-eigenspaces for the Hecke operators.*

Proof. See [Har87]. □

If $\Psi \in S_{k,k}(U_{F,1}(\mathfrak{n}))$, write ω_Ψ for the associated cohomology class. This can be described concretely as the class of a harmonic V_{kk}-valued differential form constructed from Ψ; for a summary of the construction using our conventions, see [Wil17, §2.4].

We want to work with integral rather than rational coefficients, but in the above, we defined the local systems $V_{kk}(A)$ only for F-algebras A. We extend this as follows.

Definition 2.11. Let $I_1, ..., I_h$ be a complete set of class group representatives for F, with each I_i coprime to $p\mathfrak{n}$. There is a decomposition $Y_{F,1}(\mathfrak{n}) = \sqcup_{i \in \mathrm{Cl}_F} \Gamma_i \backslash \mathcal{H}_3$ into connected components, where $\Gamma_i := \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2(F) : a, d \in 1 + \mathfrak{n}, b \in I_i, c \in \mathfrak{n}I_i^{-1}\}$. Now let E be the number field generated by the Hecke eigenvalues of Ψ. Fix a finite extension $L/\mathbb{Q}_p$ large enough that E embeds into L, and fix such an embedding; this distinguishes a prime $\mathscr{P}$ of L above p. Let R' be the valuation ring in E corresponding to $\mathscr{P}$; this is integrally closed and, under our fixed embedding, has image in the ring of integers R of L. Each of the arithmetic groups Γ_i acts on $V_{kk}(R')$ in the same manner as before, giving rise to a local system on each $\Gamma_i \backslash \mathcal{H}_3$ and hence on $Y_{F,1}(\mathfrak{n})$.

Proposition 2.12 (Hida)**.** *Let* Ψ *be a Bianchi modular form of weight* (k, k) *that is an eigenform for all the Hecke operators. Let* $R' \subset E$ *be as above. There exists a complex period* $\Omega_\Psi \in \mathbb{C}^\times$ *such that*

$$\phi_\Psi := \omega_\Psi/\Omega_\Psi \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{n}), V_{kk}(R')).$$

Proof. The $\mathrm{H}^1_{\mathrm{c}}$ of a manifold with coefficients in a torsion-free locally constant sheaf is torsion-free, and hence the extension of scalars map

$$\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{n}), V_{kk}(R')) \longrightarrow \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{n}), V_{kk}(\mathbb{C}))$$

is injective. In particular, we view the former as a lattice in the latter. In [Hid94], just after equation (8.5), Hida shows that the Ψ-eigenspace $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{n}), V_{kk}(R'))[\Psi]$ is free of rank one over R'. By multiplicity one, the analogous eigenspace in $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{n}), V_{kk}(\mathbb{C}))$ is one-dimensional over $\mathbb{C}$, and is generated by ω_Ψ. We let ϕ_Ψ denote a basis element of the integral cohomology; since this also generates the complex cohomology, the existence of Ω_Ψ follows immediately. □

Remarks: (i) Hida actually defines the space $\mathrm{H}^1_{\mathrm{cusp}}(Y_{F,1}(\mathfrak{n}), V_{kk}(R'))$ and shows that the Ψ-eigenspace for *this* is 1-dimensional. From the definition (§5 *op. cit.*), however, and the fact that the extension of scalars map is injective on compactly supported cohomology, one sees that this eigenspace is simply equal to $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}(\mathfrak{n}), V_{kk}(R'))[\Psi]$.

(ii) All of the above can be seen more explicitly in this setting using "modular symbols". In particular, for very general local systems M, there is a Hecke-equivariant isomorphism between $\mathrm{H}^1_{\mathrm{c}}(\Gamma_i \backslash \mathcal{H}_3, M)$ and the space of Γ_i-equivariant maps $\mathrm{Div}^0(\mathbb{P}^1(F)) \to M$ (see [BSW17, Lemma 8.4], generalising [AS86, Proposition 4.2]). The torsion-free statement, and injectivity of extension of scalars maps, follow easily.

After extending scalars, we view this class as having R-coefficients. The pullback $\phi_\Psi^* := \jmath^*(\phi_\Psi)$ lies in $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}^*(\mathfrak{n}), V_{kk}(R))$. Note that this is independent of any choice of class group representatives made in defining the local system.

3. Siegel units and weight 2 Asai–Eisenstein elements

3.1. Modular units Let $U \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be an open compact subgroup, with associated symmetric space $Y_{\mathbb{Q}}(U)$. In this section, we work exclusively over $\mathbb{Q}$, so we shall drop the subscript $\mathbb{Q}$ from the notation. As is well known, the manifolds $Y(U)$ are naturally the complex points of algebraic varieties defined over $\mathbb{Q}$.

Definition 3.1. A *modular unit* on $Y(U)$ is an element of $\mathcal{O}(Y(U))^\times$, that is, a regular function on $Y(U)$ with no zeros or poles. (This corresponds to a rational function on the compactification $X(U)$ whose divisor is supported on the cusps).

Modular units are *motivic* in the sense that there are *realisations* of modular units in various cohomology theories. In particular, to a modular unit $\phi \in \mathcal{O}(Y_1(N))^\times$ one can attach:

- its *de Rham* realisation $C_{\mathrm{dR}}(\phi) \in \mathrm{H}^1_{\mathrm{dR}}(Y_1(N), \mathbb{Q})$, which is the class of the differential form $d\log\phi = \frac{\mathrm{d}\phi}{\phi}$;
- its *Betti* realisation $C(\phi) \in \mathrm{H}^1(Y_1(N), \mathbb{Z})$, which is the pullback along $\phi : Y_1(N)(\mathbb{C}) \to \mathbb{C}^\times$ of the generator of $\mathrm{H}^1(\mathbb{C}^\times, \mathbb{Z}) \cong \mathrm{H}^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$.

These are closely related:

Proposition 3.2. *There is a comparison isomorphism*

$$\mathrm{H}^1(Y_1(N), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathrm{H}^1_{\mathrm{dR}}(Y_1(N), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

that sends

$$2\pi i \cdot C(\phi) \longmapsto d\log\phi.$$

3.2. Eisenstein series The de Rham realisations of modular units give rise to weight 2 Eisenstein series in the de Rham cohomology. In the next section, we'll exhibit a canonical system of modular units – the *Siegel units* – whose de Rham realisations can be written down very explicitly in terms of the following Eisenstein series.

Definition 3.3 (cf. [Kat04, §3])**.** Let $\tau \in \mathcal{H}$, k an integer ≥ 2, and $\alpha \in \mathbb{Q}/\mathbb{Z}$, with $\alpha \neq 0$ if $k = 2$. Define

$$F_\alpha^{(k)}(\tau) := \frac{(k-1)!}{(-2\pi i)^k} \sideset{}{'}\sum_{(m,n)\in\mathbb{Z}^2} \frac{e^{2\pi i \alpha m}}{(m\tau+n)^k},$$

where the prime denotes that the term $(m, n) = (0, 0)$ is omitted. This is a modular form of weight k and level $\Gamma_1(N)$, for any N such that $N\alpha = 0$.

3.3. Siegel units

Definition 3.4. For $N \geq 1$, and $c > 1$ an integer coprime to $6N$, let

$${}_c g_N \in \mathcal{O}(Y_1(N))^\times$$

be Kato's Siegel unit (the unit denoted by ${}_c g_{0,1/N}$ in the notation of [Kat04, §1]).

Slightly abusively, we shall use the same symbol ${}_c g_N$ for the pullback of this unit to $Y(M, N)$, for any $M \geq 1$ (we shall only need this when $M \mid N$).

As in *op.cit.*, we note that if c, d are two integers that are both > 1 and coprime to $6N$, then we have the identity

$$(d^2 - \langle d\rangle)\,{}_c g_N = (c^2 - \langle c\rangle)\,{}_d g_N. \tag{1}$$

It follows that the dependence on c may be removed after extending scalars to $\mathbb{Q}$: there is an element $g_N \in \mathcal{O}(Y_1(N))^\times \otimes \mathbb{Q}$ such that ${}_c g_N = (c^2 - \langle c\rangle) \cdot g_N$ for any choice of c.

Proposition 3.5.

(i) *The Siegel units are* norm-compatible, *in the sense that if $N'|N$ and prime(N) = prime(N'), where prime(N) is the set of primes dividing N, then under the natural map*

$$\mathrm{pr} : Y(M,N) \longrightarrow Y(M,N')$$

we have

$$(\mathrm{pr})_*({}_c g_N) = {}_c g_{N'}.$$

(ii) *The de Rham realisation of g_N is the Eisenstein series*

$$d\log(g_N)(\tau) = -2\pi i\, F^{(2)}_{1/N}(\tau)\,\mathrm{d}\tau.$$

Proof. The first part is proved in [Kat04], Section 2.11. The second part is Proposition 3.11(2) *op.cit.*. □

One important use of Siegel units comes in the construction of *Euler systems*; for example, see [Kat04], [LLZ14], and [KLZ17]. The basic method in each of these cases is similar; one takes cohomology classes attached to Siegel units under the realisation maps and pushes them forward to a different symmetric space, then exploits the norm compatibility to prove norm relations for these cohomology classes. We will do something similar in the Betti cohomology. In particular, we make the following definition:

Definition 3.6. Let ${}_cC_N := C({}_c g_N) \in \mathrm{H}^1(Y_1(N),\mathbb{Z})$ be the Betti realisation of ${}_c g_N$.

From Proposition 3.5(i), we see that if $p \mid N$, the classes ${}_cC_{Np^r}$ for $r \geq 0$ are compatible under push-forward, and define a class

$${}_cC_{Np^\infty} \in \varprojlim_r \mathrm{H}^1(Y_1(Np^r),\mathbb{Z}_p).$$

Lemma 3.7. *The class ${}_cC_N$ is invariant under the involution of $Y_1(N)$ given by $\tau \mapsto -\bar{\tau}$ on $\mathcal{H}$.*

Proof. This follows from the fact that there is a canonical model of $Y_1(N)$ as the $\mathbb{C}$-points of an algebraic variety over $\mathbb{Q}$ such that the above involution is complex conjugation; and, with respect to this model, the units ${}_c g_N$ are defined over $\mathbb{Q}$. □

3.4. Asai–Eisenstein elements in weight 2 Now let F be an imaginary quadratic field, and $\mathfrak{n}$ an ideal of $\mathcal{O}_F$ divisible by some integer ≥ 4. Recall that we have

$$Y^*_{F,1}(\mathfrak{n}) = \Gamma^*_{F,1}(\mathfrak{n})\backslash\mathcal{H}_3,$$

and that we showed in Proposition 2.6 that the natural map

$$\iota : Y_{\mathbb{Q},1}(N) \hookrightarrow Y^*_{F,1}(\mathfrak{n})$$

is a closed immersion. We also need the following map:

Definition 3.8. Let $m \geq 1$ and $a \in \mathcal{O}_F$. Consider the map

$$\kappa_{a/m} : Y^*_{F,1}(m^2\mathfrak{n}) \to Y^*_{F,1}(\mathfrak{n})$$

given by the left action of $\left(\begin{smallmatrix}1 & a/m\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{SL}_2(F)$ on $\mathcal{H}_3$. This is well-defined[2], since it is easy to see that

$$\begin{pmatrix}1 & a/m\\ 0 & 1\end{pmatrix}\Gamma^*_{F,1}(m^2\mathfrak{n})\begin{pmatrix}1 & -a/m\\ 0 & 1\end{pmatrix} \subset \Gamma^*_{F,1}(\mathfrak{n}),$$

and it depends only on the class of a modulo $m\mathcal{O}_F$.

[2] Note that $\kappa_{a/m}$ is *not* in general well-defined on $Y_{F,1}(m^2\mathfrak{n})$, since

$$\left(\begin{smallmatrix}1 & a/m\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & -a/m\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}-1 & 2a/m\\ 0 & 1\end{smallmatrix}\right),$$

which is not in $\Gamma_{F,1}(\mathfrak{n})$ if $2a \notin m\mathcal{O}_F$.

The elements we care about are the following. Assume that $\mathfrak{n}$ is divisible by some integer $q \geq 4$, as above.

Definition 3.9. For $a \in \mathcal{O}_F/m\mathcal{O}_F$, $m \geq 1$, and $c > 1$ coprime to $6mN$, define

$$ {}_c\Xi_{m,\mathfrak{n},a} \in \mathrm{H}^2\left(Y_{F,1}^*(\mathfrak{n}), \mathbb{Z}\right) $$

to be the pushforward of ${}_cC_{m^2N}$ under the maps

$$ Y_{\mathbb{Q},1}(m^2N) \hookrightarrow Y_{F,1}^*(m^2\mathfrak{n}) \xrightarrow{\kappa_{a/m}} Y_{F,1}^*(\mathfrak{n}). $$

Note that the pushforward of ${}_cC_N$ to $Y_1(\mathfrak{n})$ lands in the second degree Betti cohomology, since the locally symmetric space $Y_{F,1}^*(\mathfrak{n})$ is a 3-dimensional real manifold.

Proposition 3.10. *We have* $\left(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right)^* \cdot {}_c\Xi_{m,\mathfrak{n},a} = {}_c\Xi_{m,\mathfrak{n},-a}$.

Proof. It is clear that $\left(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right) \circ \kappa_{a/m} = \kappa_{-a/m} \circ \left(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right)$ as maps $Y_{F,1}^*(m^2\mathfrak{n}) \to Y_{F,1}^*(\mathfrak{n})$. Moreover, the action of $\left(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix}\right)$ on $Y_{F,1}^*(m^2\mathfrak{n})$ preserves the image of $Y_{\mathbb{Q},1}(m^2N)$, and the involution of $Y_{\mathbb{Q},1}(m^2N)$ it induces preserves the class ${}_cC_{m^2N}$, by Lemma 3.7. ☐

Definition 3.11. Define

$$ {}_c\Phi^r_{\mathfrak{n},a} \in \mathrm{H}^2\Big(Y_{F,1}^*(\mathfrak{n}), \mathbb{Z}\Big) \otimes_{\mathbb{Z}} \mathbb{Z}_p[(\mathbb{Z}/p^r)^\times] $$

by

$$ {}_c\Phi^r_{\mathfrak{n},a} := \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi_{p^r,\mathfrak{n},at} \otimes [t]. $$

Lemma 3.12. *If* $\mathfrak{n} \mid \mathfrak{n}'$ *are two ideals of* $\mathcal{O}_F$ *with the same prime factors, then pushforward along the map* $Y_{F,1}(\mathfrak{n}') \to Y_{F,1}(\mathfrak{n})$ *sends* ${}_c\Phi^r_{\mathfrak{n}',a}$ *to* ${}_c\Phi^r_{\mathfrak{n},a}$ *(for any valid choices of c, a, r).*

Proof. This is immediate from the norm-compatibility of the Siegel units; compare [LLZ14, Theorem 3.1.2]. ☐

We now come to one of the key theorems of this paper, which shows that if $m = p^r$ with r varying, then the above elements fit together p-adically into a compatible family. We now impose the assumption that $\mathfrak{n}$ is divisible by all primes $v \mid p$ of $\mathcal{O}_F$.

Theorem 3.13. *Let* $r \geq 1$*, let* a *be a generator of* $\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z})$*, and let*

$$ \pi_{r+1} : \mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p[(\mathbb{Z}/p^{r+1})^\times] \longrightarrow \mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p[(\mathbb{Z}/p^r)^\times] $$

denote the map that is the identity on the first component and the natural quotient map on the second component. Then we have

$$ \pi_{r+1}({}_c\Phi^{r+1}_{\mathfrak{n},a}) = (U_p)_* \cdot {}_c\Phi^r_{\mathfrak{n},a}, $$

where the Hecke operator $(U_p)_*$ *acts via its action on* $\mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), \mathbb{Z})$*. Similarly, when* $r = 0$ *we have*

$$ \pi_1({}_c\Phi^1_{\mathfrak{n},a}) = ((U_p)_* - 1) \cdot {}_c\Phi^0_{\mathfrak{n},a}. $$

Remark: Before embarking on the proof, which will occupy the next section of the paper, we pause to give a brief description of how this is important for the construction of p-adic Asai L-functions, in the simplest case of a Bianchi eigenform of weight $(0,0)$. Define $e_{\mathrm{ord}} := \lim_{n\to\infty}(U_p^{n!})_*$ to be the ordinary projector on cohomology with $\mathbb{Z}_p$ coefficients, so that $(U_p)_*$ is invertible on the (p-ordinary) space $e_{\mathrm{ord}}\mathrm{H}^2(Y_1(\mathfrak{n}), \mathbb{Z}_p)$. Given the theorem, we see that the collection

$$ [(U_p)_*^{-r} e_{\mathrm{ord}} \cdot {}_c\Phi^r_{\mathfrak{n},a}]_{r \geq 1} $$

forms an element ${}_c\Phi^\infty_{\mathfrak{n},a}$ in the inverse limit

$$ e'_{\mathrm{ord}}\mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), \mathbb{Z}_p) \otimes \mathbb{Z}_p[[\mathbb{Z}_p^\times]]. $$

If we pair this with a modular symbol in $\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{n}), \mathcal{O}_E)$ arising from a p-ordinary weight (0,0) Bianchi eigenform Ψ, then we obtain a measure on $\mathbb{Z}_p^\times$ with values in $\mathcal{O}_E \otimes \mathbb{Z}_p$. This will be our p-adic L-function. By construction, its values at finite-order characters are given by integrating Ψ against linear combinations of Eisenstein series on $Y_{\mathbb{Q},1}(m^2N)$; and these will turn out to compute the special values of the Asai L-series.

4. Proving the norm relations (Theorem 3.13)

Theeorem 3.13 is directly analogous to the norm-compatibility relations for Euler systems constructed from Siegel units; specifically, it is the analogue in our context of [LLZ14, Theorem 3.3.2]. Exactly as in *op.cit.*, it is simplest not to prove the theorem directly but rather to deduce it from a related result concerning cohomology classes on the symmetric spaces $Y_F^*(m, m\mathfrak{n})$, analogous to Theorem 3.3.1 of *op.cit.*. Note that these symmetric spaces are not connected for $m > 2$, but have $\phi(m)$ connected components; this will allow us to give a tidy conceptual interpretation of the sum over $j \in (\mathbb{Z}/p^r\mathbb{Z})^*$ appearing in the definition of ${}_c\Phi^r_{\mathfrak{n},a}$.

4.1. Rephrasing using the spaces $Y_F^*(m, m\mathfrak{n})$

Proposition 4.1. *For any $a \in \mathcal{O}_F$, the element $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ normalises $U^*(m, m\mathfrak{n}) \subset \mathrm{GL}_2^*(\mathbb{A}_F^f)$.*

Proof. Easy check. □

We can therefore regard right-translation by $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ as an automorphism of $Y_F^*(m, m\mathfrak{n})$, and we can consider the composite map

$$\iota_{m,\mathfrak{n},a} : Y_{\mathbb{Q}}^*(m, mN) \hookrightarrow Y_F^*(m, m\mathfrak{n}) \xrightarrow{\left(\begin{smallmatrix} 1 & -a \\ 0 & 1 \end{smallmatrix}\right)} Y_F^*(m, m\mathfrak{n}),$$

where the first arrow is injective (as soon as $m\mathfrak{n}$ is divisible by some integer ≥ 4) by the same argument as in Proposition 2.6. Note also that the components of $Y_F^*(m, m\mathfrak{n})$ are indexed by $(\mathbb{Z}/m\mathbb{Z})^*$, with the fibre over j corresponding to the component containing the image of $\left(\begin{smallmatrix} j & \\ & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2^*(\mathbb{A}_F^f)$; and the action of $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ preserves each component.

Remark: The change of sign appears because we are comparing left and right actions.

Definition 4.2. We define ${}_c\mathcal{Z}_{m,\mathfrak{n},a}$ to be the image of ${}_cC_{mN} \in H^1(Y_{\mathbb{Q}}(m, mN), \mathbb{Z})$ under pushforward via $\iota_{m,\mathfrak{n},a}$, and ${}_c\mathcal{Z}_{m,\mathfrak{n},a}(j)$ the projection of ${}_c\mathcal{Z}_{m,\mathfrak{n},a}$ to the direct summand of $H^1(Y_F^*(m, m\mathfrak{n}), \mathbb{Z})$ given by the j-th component, so that

$${}_c\mathcal{Z}_{m,\mathfrak{n},a} = \sum_j {}_c\mathcal{Z}_{m,\mathfrak{n},a}(j).$$

Exactly as in the situation of Beilinson–Flach elements, these $\mathcal{Z}$ elements turn out to be closely related to the Φ's defined above (compare [LLZ14, Proposition 2.7.4]). We consider the map

$$s_m : Y_F^*(m, m\mathfrak{n}) \to Y_{F,1}^*(\mathfrak{n})$$

given by the action of $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ (corresponding to $(z, t) \mapsto (z/m, t/m)$ on $\mathcal{H}_3$).

Proposition 4.3. *We have $(s_m)_* \left({}_c\mathcal{Z}_{m,\mathfrak{n},a}(j)\right) = {}_c\Xi_{m,\mathfrak{n},ja}$, and hence*

$${}_c\Phi^r_{\mathfrak{n},a} = \sum_j (s_{p^r})_* \left({}_c\mathcal{Z}_{p^r,\mathfrak{n},a}(j)\right) \otimes [j].$$

Before proceeding to the proof, we note the following lemma:

Lemma 4.4. *The pushforward of ${}_cC_{m^2N}$ along the map*

$$Y_{\mathbb{Q},1}(m^2N) \to Y_{\mathbb{Q}}(1(m), mN),$$

given by $z \mapsto mz$ on $\mathcal{H}$, is ${}_cC_{mN}$.

Proof. This follows from the well-known norm-compatibility relations of the Siegel units, cf. [Kat04, Lemma 2.12]. □

Proof of Proposition 4.3. For each $j \in (\mathbb{Z}/m\mathbb{Z})^\times$, we have a diagram

$$\begin{array}{ccc} Y_F^*(m, mN)^{(1)} & \xrightarrow{\left(\begin{smallmatrix}1 & -ja\\ 0 & 1\end{smallmatrix}\right)} & Y_F^*(m, m\mathfrak{n})^{(1)} \\ \downarrow{\left(\begin{smallmatrix}j & 0\\ 0 & 1\end{smallmatrix}\right)} & & \downarrow{\left(\begin{smallmatrix}j & 0\\ 0 & 1\end{smallmatrix}\right)} \\ Y_F^*(m, mN)^{(j)} & \xrightarrow{\left(\begin{smallmatrix}1 & -a\\ 0 & 1\end{smallmatrix}\right)} & Y_F^*(m, m\mathfrak{n})^{(j)} \end{array}.$$

In other words, if we identify $Y_F^*(m, mN)^{(j)}$ with $\Gamma_F^*(m, mN)\backslash\mathcal{H}_3$ via $\left(\begin{smallmatrix}j & 0\\ 0 & 1\end{smallmatrix}\right)$, the restriction to this component of the right action of $\left(\begin{smallmatrix}1 & -a\\ 0 & 1\end{smallmatrix}\right)$ on the adelic symmetric space corresponds to the left action of $\left(\begin{smallmatrix}1 & ja\\ 0 & 1\end{smallmatrix}\right)$ on $\mathcal{H}_3$.

With these identifications, we see that the map

$$\kappa_{ja/m} : Y_{F,1}^*(m^2\mathfrak{n}) \to Y_{F,1}^*(\mathfrak{n})$$

used in the definition of $\Xi_{m,\mathfrak{n},ja}$ factors as

$$\begin{aligned} Y_{F,1}^*(m^2\mathfrak{n}) &\xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & m\end{smallmatrix}\right)} Y_F^*(1(m), m\mathfrak{n}) \\ &\xrightarrow{\cong} Y_F^*(m, m\mathfrak{n})^{(j)} \\ &\xrightarrow{\left(\begin{smallmatrix}1 & -a\\ 0 & 1\end{smallmatrix}\right)} Y_F^*(m, m\mathfrak{n})^{(j)} \\ &\xrightarrow{s_m} Y_{F,1}^*(\mathfrak{n}). \end{aligned}$$

Pushforward along the first map is compatible with pushforward along the corresponding map on $\mathcal{H}$, which sends ${}_cC_{m^2N}$ to ${}_cC_{mN}$ by the previous lemma. □

Corollary 4.5. *The classes ${}_c\Xi_{m,\mathfrak{n},a}$ and ${}_c\mathcal{Z}_{m,\mathfrak{n},a}$ depend only on the image of a in the quotient $\mathcal{O}_F/(m\mathcal{O}_F + \mathbb{Z})$.*

Proof. If $b \in \mathbb{Z}$, the action of $\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$ on $Y_{\mathbb{Q}}(m, mN)$ fixes the cohomology class ${}_cC_{mN}$, as this class is the pullback of a class on $Y_{\mathbb{Q},1}(mN)$. Since the actions of $\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$ on $Y_{\mathbb{Q}}(m, mN)$ and $Y_F^*(m, m\mathfrak{n})$ are compatible, we see that ${}_c\mathcal{Z}_{m,\mathfrak{n},a} = {}_c\mathcal{Z}_{m,\mathfrak{n},a+b}$ for any $a \in \mathcal{O}_F$ and $b \in \mathbb{Z}$, as required. The corresponding result for ${}_c\Xi_{m,\mathfrak{n},a}$ now follows from the previous proposition. □

4.2. A norm relation for zeta elements In this section, we formulate and prove a norm relation for the zeta elements ${}_c\mathcal{Z}_{m,\mathfrak{n},a}$ which is analogous to Theorem 3.13, but simpler to prove.

Definition 4.6. For p prime, define a map

$$\tau_p : Y_F^*(pm, pm\mathfrak{n}) \longrightarrow Y_F^*(m, m\mathfrak{n})$$

by composing the right-translation action of $\left(\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{GL}_2^*(\mathbb{A}_F^f)$ with the natural projection.

Theorem 4.7. *Suppose p is a prime with $p \mid m$, and suppose that $a \in \mathcal{O}_F$ maps to a generator of the quotient $\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. Then we have the norm relation*

$$(\tau_p)_*({}_c\mathcal{Z}_{pm,\mathfrak{n},a}) = (U_p)_*(\mathcal{Z}_{m,\mathfrak{n},a}).$$

For simplicity, we give the proof under the slightly stronger hypothesis that $p \mid \mathfrak{n}$ (rather than just that every prime above p divides $\mathfrak{n}$, which is our running hypothesis). This only makes a difference if p is ramified in F, and the proof can be extended to handle this extra case at the cost of slightly more complicated notation; we leave the necessary modifications to the interested reader.

Firstly, note that there is a commutative diagram

$$\begin{array}{ccccc} Y_F^*(pm, pm\mathfrak{n}) & \xrightarrow{\mathrm{pr}_1} & Y_F^*(pm, m\mathfrak{n}) & \xrightarrow{\mathrm{pr}_2} & Y_F^*(m(p), m\mathfrak{n}) \\ & \searrow^{\tau_p} & & \swarrow^{\pi_2} & \downarrow{\scriptstyle \pi_1} \\ & & Y_F^*(m, m\mathfrak{n}) & & Y_F^*(m, m\mathfrak{n}) \end{array} \tag{2}$$

where the top maps are the natural projection maps, τ_p is the twisted degeneracy map of the previous section, and π_1, π_2 are the degeneracy maps of Section 2.3.

Lemma 4.8. *Let $\mathfrak{n}' = (p)^{-1}\mathfrak{n}$. Under pushforward by the natural projection map*

$$\mathrm{pr}_1 : Y_F^*(pm, pm\mathfrak{n}) \longrightarrow Y_F^*(pm, m\mathfrak{n}) = Y_F^*(pm, pm\mathfrak{n}'),$$

we have

$$(\mathrm{pr}_1)_*({}_c\mathcal{Z}_{p^{r+1},\mathfrak{n},a}) = {}_c\mathcal{Z}_{p^{r+1},\mathfrak{n}',a}.$$

Proof. This is immediate from the corresponding norm-compatibility property of the Siegel units, which is Proposition 3.5. Compare [LLZ14, Theorem 3.1.1]. □

So we need to compare the classes $\mathcal{Z}_{pm,\mathfrak{n}',a}$ and $\mathcal{Z}_{m,\mathfrak{n},a}$. Note that these both involve the same Siegel unit ${}_cg_{0,1/mN}$. Let us write u_a for the element $\left(\begin{smallmatrix} 1 & -a \\ 0 & 1 \end{smallmatrix}\right)$.

Definition 4.9. (i) Let $\alpha_{pm,\mathfrak{n},a}$ denote the composition of the maps

$$Y_{\mathbb{Q}}(pm, mN) \hookrightarrow Y_F^*(pm, m\mathfrak{n}) \xrightarrow{u_a} Y_F^*(pm, m\mathfrak{n}) \xrightarrow{\mathrm{pr}_2} Y_F^*(m(p), m\mathfrak{n}).$$

(ii) Let $\iota_{m,\mathfrak{n},a}$ denote, as above, the composition of the maps

$$Y_{\mathbb{Q}}(m, mN) \hookrightarrow Y_F^*(m, m\mathfrak{n}) \xrightarrow{u_a} Y_F^*(m, m\mathfrak{n}).$$

The following lemma is the key component in the proof of Theorem 4.7.

Lemma 4.10. *Suppose that $a \in \mathcal{O}_F$ is a generator of $\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z})$. Then:*

(i) *The map $\alpha_{pm,\mathfrak{n},a}$ is injective.*
(ii) *The diagram*

$$\begin{array}{ccc} Y_{\mathbb{Q}}(pm, mN) & \xhookrightarrow{\alpha_{pm,\mathfrak{n},a}} & Y_F^*(m(p), m\mathfrak{n}) \\ \downarrow & & \downarrow{\scriptstyle \pi_1} \\ Y_{\mathbb{Q}}(m, mN) & \xhookrightarrow{\iota_{m,\mathfrak{n},a}} & Y_F^*(m, m\mathfrak{n}) \end{array}$$

is Cartesian, where the left vertical arrow is the natural projection.

The proof of this lemma is taken essentially verbatim from [LLZ16, Lemma 7.3.1], where the analogous result is proved for real quadratic fields.

Proof. To prove part (i), note that the image of $\alpha_{pm,\mathfrak{n},a}$ is the modular curve of level

$$\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^f) \cap u_a^{-1} U_F(m(p), m\mathfrak{n}) u_a.$$

This intersection is the set of $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ such that

$$\begin{pmatrix} r-at & s+a(r-u)-a^2t \\ t & at+u \end{pmatrix} \equiv I \mod \begin{pmatrix} m & pm \\ m\mathfrak{n} & m\mathfrak{n} \end{pmatrix}.$$

We want to show that this is equal to $U_{\mathbb{Q}}(pm, mN)$. Clearly, any $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$ in the intersection satisfies

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod \begin{pmatrix} m & * \\ mN & mN \end{pmatrix},$$

whilst

$$s + a(r-1) \equiv 0 \pmod{mp}.$$

Suppose $m = p^q m'$ with m' coprime to p. We know that both summands are zero modulo m, so it suffices to check that they are both zero modulo p^{q+1}. Since a generates $\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z})$, $\{1, a\}$ is a basis of $(\mathcal{O}_F/p^{q+1}\mathcal{O}_F) \otimes \mathbb{Z}_p$ as a module over $\mathbb{Z}/p^{q+1}\mathbb{Z}$; so both summands must be individually zero modulo p^{q+1}. But this means precisely that $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right) \in U_{\mathbb{Q}}(mp, mN)$, as required.

Part (ii) follows from the observations that the horizontal maps are both injections and that both vertical maps are finite of degree p^2. □

Remark: Since this lemma is crucial to the proof, we expand slightly on what part (i) really says. The map $\alpha_{mp,\mathfrak{n},a}$ is p-to-1 on connected components, in the sense that the preimage of a single component of $Y_F^*(p^r(p), p^r\mathfrak{n})$ contains p connected components of $Y_F^*(p^{r+1}, p^rN)$. The condition on a ensures that the map u_a 'twists' these p components away from each other inside that single component of the target space, so that their images are disjoint. In particular, the result would certainly fail without this condition; for instance, if $a = 0$ then the map factors through $Y_{\mathbb{Q}}(m(p), mN)$.

Proof of Theorem 4.7. The Cartesian diagram of Lemma 4.10 shows that

$$(\alpha_{mp,\mathfrak{n},a})_*({}_cC_{mN}) = (\pi_1)^*({}_c\mathcal{Z}_{m,\mathfrak{n},a}).$$

But by definition,

$$\begin{aligned} (\alpha_{mp,\mathfrak{n},a})_*({}_cC_{mN}) &= (\mathrm{pr}_2)_*({}_c\mathcal{Z}_{mp,\mathfrak{n}',a}) \\ &= (\mathrm{pr}_2)_*(\mathrm{pr}_1)_*({}_c\mathcal{Z}_{mp,\mathfrak{n},a}), \end{aligned} \tag{3}$$

where the second equality follows from Lemma 4.8. From the commutative diagram (2), we know that $(\tau_p)_* = (\pi_2)_*(\mathrm{pr}_2)_*(\mathrm{pr}_1)_*$, whilst by definition $(U_p)_* = (\pi_2)_*(\pi_1)^*$. Hence applying $(\pi_2)_*$ to equation (3) gives the result. □

We can now deduce the proof of the main theorem:

Proof of Theorem 3.13. We need to show that, for each $j \in (\mathbb{Z}/p^r\mathbb{Z})^\times$, we have

$$\sum_{\substack{k \in \mathbb{Z}/p^{r+1}\mathbb{Z} \\ k = j \bmod p^r}} {}_c\Xi_{p^{r+1},\mathfrak{n},ka} = (U_p)_* \cdot {}_c\Xi_{p^r,\mathfrak{n},ja}. \tag{†}$$

We have a commutative diagram

$$\begin{array}{ccc} \bigsqcup_k Y_F^*(p^{r+1}, p^{r+1}\mathfrak{n})^{(k)} & \xrightarrow{\ \tau_p\ } & Y_F^*(p^r, p^r\mathfrak{n})^{(j)} \\ \downarrow{\scriptstyle s_{p^{r+1}}} & & \downarrow{\scriptstyle s_{p^r}} \\ Y_{F,1}^*(\mathfrak{n}) & \longrightarrow & Y_{F,1}^*(\mathfrak{n}). \end{array}$$

The left-hand side of (†) is exactly the pushforward of $\sum_k {}_c\mathcal{Z}_{p^{r+1},\mathfrak{n},a}(k)$ along the left vertical arrow (where again we are using the notation $x(k)$ for the projection of x to the k-th component). Theorem 4.7 shows that the pushforward of the same element along τ_p is

$(U_p)_{*c}\mathcal{Z}_{p^r,\mathfrak{n},a}(j)$. So it suffices to check that the operators $(U_p)_*$ on $Y_F^*(p^r, p^r\mathfrak{n})^{(j)}$ and on $Y_{F,1}^*(\mathfrak{n})$ are compatible under s_{p^r}, which is clear by inspecting a set of single-coset representatives (using our running assumption that all primes above p divide $\mathfrak{n}$).

The case where $r = 0$ is a special case, since we must exclude the term ${}_c\Xi_{p^1,\mathfrak{n},0}$ from the above sum, introducing the -1 term of the theorem. □

5. Asai–Eisenstein elements in higher weights

In the previous sections, we have defined compatible systems of classes in the Betti cohomology of the spaces $Y_{F,1}^*(\mathfrak{n})$ with trivial coefficients. We now extend this to coefficients arising from non-trivial algebraic representations.

We fix, for the duration of this section, a prime p and a finite extension L of $\mathbb{Q}_p$ large enough that F embeds into L (and we fix such an embedding). We let R be the ring of integers of L. We also choose an ideal $\mathfrak{n}$ of $\mathcal{O}_F$ divisible by all primes above p. For convenience, we also assume that $\mathfrak{n}$ is divisible by some integer $q \geq 4$; note that this is automatic if p is unramified in F and $p \geq 5$.

5.1. Coefficients and moment maps

As above, we let $V_k(R) = \mathrm{Sym}^k R^2$ be the left $R[\mathrm{GL}_2(\mathbb{Z})]$-module of symmetric polynomials in 2 variables with coefficients in R. We will be interested in the dual $T_k(R) = V_k(R)^*$ (the module of symmetric tensors of degree k over R^2). We view this as a local system of R-modules on $Y_{\mathbb{Q},1}(N)$, for any $N \geq 4$, in the usual way.

Similarly, we have $R[\mathrm{GL}_2(\mathcal{O}_F)]$-modules $V_{kk}(R) = \mathrm{Sym}^k R^2 \otimes (\mathrm{Sym}^k R^2)^\sigma$, where $\mathrm{GL}_2(\mathcal{O}_F)$ acts on the first factor via the given embedding $\mathcal{O}_F \hookrightarrow R$ and on the second via its Galois conjugate. We let $T_{kk}(R)$ be the R-dual of $V_{kk}(R)$. These give local systems on $Y_F^*(U)$ and $Y_F(U)$ for sufficiently small levels U.

There is a canonical section

$$e_{F,k,r} \in \mathrm{H}^0\left(Y_{F,1}^*(\mathfrak{n}p^r), V_{kk}(R/p^r)\right),$$

and since R is p-adically complete, cup-product with this section defines a "moment" map

$$\mathrm{mom}^{kk} : \varprojlim_t \mathrm{H}^\bullet(Y_{F,1}^*(\mathfrak{n}p^t), \mathbb{Z}) \otimes R \to \mathrm{H}^\bullet(Y_{F,1}^*(\mathfrak{n}), V_{kk}(R)).$$

This is the Betti cohomology analogue of the moment maps in étale cohomology of modular curves considered in [KLZ17, §4].

By Lemma 3.12, the family of classes $\left(\Phi^{k,r}_{\mathfrak{n}p^t,a}\right)_{t\geq 0}$ is compatible under pushforward, so it is a valid input to the maps mom^{kk} (after tensoring over R with the group ring to $R[(\mathbb{Z}/p^r)^\times]$).

Definition 5.1. We let ${}_c\Phi^{k,r}_{\mathfrak{n},a} \in \mathrm{H}^2\Big(Y_{F,1}^*(\mathfrak{n}), V_{kk}(R)\Big) \otimes_R R[(\mathbb{Z}/p^r)^\times]$ be the image of the compatible system $\left(\Phi^{k,r}_{\mathfrak{n}p^t,a}\right)_{t\geq 0}$ under mom^{kk}.

The action of the Hecke operator $(U_p)_*$ is well-defined both on $\mathrm{H}^2\Big(Y_{F,1}^*(\mathfrak{n}), V_{kk}(R)\Big)$ and on the inverse limit $\varprojlim_t \mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}p^t), \mathbb{Z}_p)$, and the maps mom^{kk} commute with this operator (cf. [KLZ17, Remark 4.5.3]). So we deduce immediately from Theorem 3.13 that the classes ${}_c\Phi^{k,r}_{\mathfrak{n},a}$, for any fixed $k \geq 1$ and varying r, satisfy the same norm-compatibility relation as the $k = 0$ classes.

5.2. Relation to the weight $2k$ Eisenstein class We will later relate the ${}_c\Phi^{k,r}_{\mathfrak{n},a}$ to values of L-functions. For this purpose the definition above, via a p-adic limiting process, is inconvenient; so we now give an alternative description of the same classes via higher-weight Eisenstein series for $\mathrm{GL}_2/\mathbb{Q}$, directly generalising the classes obtained in weight 2 from realisations of Siegel units.

Let $k \geq 0$. The local system $T_k(\mathbb{Q})$ is exactly the flat sections of a vector bundle $T_{k,\mathrm{dR}}$ with respect to a connection ∇ (the Gauss–Manin connection). There is a comparison isomorphism

$$\mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_k(\mathbb{Q})) \otimes \mathbb{C} \cong \mathrm{H}^1_{\mathrm{dR}}(Y_{\mathbb{Q},1}(N), T_{k,\mathrm{dR}}, \nabla) \otimes \mathbb{C}.$$

Proposition 5.2. *There exists a class $\mathrm{Eis}^k_N \in \mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_k(\mathbb{Q}))$ such that under the comparison isomorphism, we have*

$$\mathrm{Eis}^k_N \longmapsto -(2\pi i)^{k+1} N^k F^{(k+2)}_{1/N}(\tau)\, \mathrm{d}z^{\otimes k}\, \mathrm{d}\tau,$$

where $\mathrm{d}z^{\otimes k}$ is a basis vector of $T_{k,\mathrm{dR}}$.

Proof. See [KLZ15, §4] for further details. □

These classes are in general not p-adically integral, but for any $c > 1$ as above, there exists a class ${}_c\mathrm{Eis}^k_N \in \mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_k(\mathbb{Z}))$ such that the equality

$${}_c\mathrm{Eis}^k_N = \left(c^2 - c^{-k}\langle c\rangle\right) \mathrm{Eis}^k_N$$

holds in $\mathrm{H}^1(Y_{\mathbb{Q},1}(N), T_k(\mathbb{Q}))$.

Letting R be as in the previous section, for any $j \in \{0, \dots, k\}$ we can regard $T_{2k-2j}(R)$ as a $\mathrm{SL}_2(\mathbb{Z})$-invariant submodule of the $\mathrm{SL}_2(\mathcal{O}_F)$-module $T_{kk}(R)$, via the Clebsch–Gordan map (normalised as in [KLZ15, §5.1]). Thus we obtain a map

$$(\dagger) \qquad (\iota_{m,\mathfrak{n},a})_* : \mathrm{H}^1\big(Y_{\mathbb{Q}}(m, mN), T_{2k-2j}(\mathbb{Z}_p)\big) \otimes_{\mathbb{Z}_p} R \to \mathrm{H}^2\big(Y_F^*(m, m\mathfrak{n}), T_{kk}(R)\big).$$

Definition 5.3. Let ${}_c\Xi^{k,j}_{m,\mathfrak{n},a} \in \mathrm{H}^2\big(Y^*_{F,1}(\mathfrak{n}), T_{kk}(R)\big)$ be the image of $(\iota_{m,\mathfrak{n},a})_*\left({}_c\mathrm{Eis}^{2k-2j}_{mN}\right)$ under restriction to the identity component followed by $(s_m)_*$. We similarly write $\Xi^{k,j}_{m,\mathfrak{n},a}$ (without c) for the analogous element with L-coefficients, defined using Eis^k_{mN}.

This definition is convenient for p-adic interpolation, but to relate this element to special values it is convenient to have an alternative description involving pushforward along the map $\kappa_{a/m} : Y^*_{F,1}(m^2\mathfrak{n}) \to Y^*_{F,1}(\mathfrak{n})$, as above. (Note that if $p \mid m$, this pushforward map only acts on cohomology with coefficients in $T_{kk}(L)$, not $T_{kk}(R)$, since it corresponds to the action of a matrix whose entries are not p-adically integral.)

Lemma 5.4. *As elements of $\mathrm{H}^2\big(Y^*_{F,1}(\mathfrak{n}), T_{kk}(L)\big)$ we have*

$$\Xi^{k,j}_{m,\mathfrak{n},a} = m^j \cdot (\kappa_{a/m})_*\left(\iota_*(\mathrm{Eis}^{2k-2j}_{m^2N})\right).$$

Proof. This follows in exactly the same way as Proposition 4.3 (which is the case $j = k = 0$), noting that the Clebsch–Gordan maps at levels $Y_F^*(m, m\mathfrak{n})$ and $Y^*_{F,1}(m^2\mathfrak{n})$ differ by the factor m^j; compare the proof of [KLZ17, Theorem 5.4.1]. □

Proposition 5.5. *For any $r \geq 0$ we have*

$$\begin{aligned} {}_c\Phi^{k,r}_{\mathfrak{n},a} &= \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,0}_{p^r,\mathfrak{n},at} \otimes [t] \\ &= \left(c^2 - c^{-2k}[c]^2\langle c\rangle\right) \cdot \sum_{t \in (\mathbb{Z}/p^r)^\times} \Xi^{k,0}_{p^r,\mathfrak{n},at} \otimes [t], \end{aligned}$$

where the first equality takes place in $\mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), T_{kk}(R)) \otimes_R R[(\mathbb{Z}/p^r)^\times]$ *and the second after base-extension to* L.

Proof. This proposition is very close to [KLZ17, Proposition 5.1.2] so we only briefly sketch the proof. There is a $\mathrm{GL}_2/\mathbb{Q}$ moment map mom^k for any $k \geq 0$, and one sees easily that the maps mom^k and mom^{kk} are compatible via the inclusion $Y_{\mathbb{Q},1}(N) \hookrightarrow Y_{F,1}^*(\mathfrak{n})$. However, a theorem due to Kings shows that the higher-weight Eisenstein classes are exactly the moments of the family of Siegel-unit classes ${}_cC_{Np^\infty}$, up to a factor depending on c. □

There is an analogous statement for $j \neq 0$, but this can only be formulated after reduction modulo p^r:

Proposition 5.6. *For* $r \geq 1$, *as classes in* $\mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), T_{kk}(R/p^r))$ *we have*

$$
{}_c\Xi^{k,j}_{p^r,\mathfrak{n},a} = (a - a^\sigma)^j j! \binom{k}{j}^2 {}_c\Xi^{k,0}_{p^r,\mathfrak{n},a}.
$$

Proof. See [KLZ17, §6] and [LLZ16, §8]. □

6. The p-adic Asai L-function

In this short section, we put together the norm-compatibility and p-adic interpolation relations proved above in order to define a measure, or more generally a finite-order distribution, on $\mathbb{Z}_p^\times$ with values in a suitable eigenspace of the Betti H^2. This will be our p-adic L-function.

To ease the notation, we will assume for the rest of the paper that p is odd. Similar arguments – with some additional care – should also hold for $p = 2$, but we leave this case to the interested reader.

6.1. Constructing the measure Let L be a finite extension of $\mathbb{Q}_p$ containing the Hecke field E of the Bianchi modular form Ψ, fix a distinguished embedding of E into L compatibly with the choices in §2 and §5, and write R for the ring of integers in L. In previous sections, we defined the elements

$$
{}_c\Phi^{k,r}_{\mathfrak{n},a} = \sum_{t \in (\mathbb{Z}/p^r)^\times} {}_c\Xi^{k,0}_{p^r,\mathfrak{n},at} \otimes [t] \in \mathrm{H}^2\big(Y_{F,1}^*(\mathfrak{n}), T_{kk}(R)\big) \otimes R[(\mathbb{Z}/p^r)^\times],
$$

for $k \geq 0$ and $r \geq 0$. We also showed that if a is a generator of $\mathcal{O}_F/(p\mathcal{O}_F + \mathbb{Z})$, then under the natural projection maps in the second factor, we have

$$
\pi_{r+1}({}_c\Phi^{r+1}_{\mathfrak{n},a}) = (U_p)_* \left({}_c\Phi^r_{\mathfrak{n},a}\right) \quad \text{for } r \geq 1.
$$

Definition 6.1. Let us write

$$
\begin{aligned}
\mathcal{L}_k(\mathfrak{n}, R) &:= \mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), T_{kk}(R))/(\text{torsion}), \\
\text{and} \quad \mathcal{L}_k^{\mathrm{ord}}(\mathfrak{n}, R) &:= e'_{\mathrm{ord}}\mathcal{L}_k(\mathfrak{n}, R),
\end{aligned}
$$

where $e'_{\mathrm{ord}} := \lim_{n\to\infty}(U_p)_*^{n!}$ is the ordinary projector.

Clearly $e'_{\mathrm{ord}}\mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), T_{kk}(R))$ is an R-direct-summand of $\mathrm{H}^2(Y_{F,1}^*(\mathfrak{n}), T_{kk}(R))$, which is a finitely-generated R-module, since $Y_{F,1}^*(\mathfrak{n})$ is homotopy-equivalent to a finite simplicial complex. On this direct summand, $(U_p)_*$ is invertible, so we may make the following definition:

Definition 6.2. Define

$$
{}_c\Phi^{k,\infty}_{\mathfrak{n},a} := \left[(U_p)_*^{-r} e'_{\mathrm{ord}}({}_c\Phi^{k,r}_{\mathfrak{n},a})\right]_{r\geq 1} \in \mathcal{L}_k^{\mathrm{ord}}(\mathfrak{n}, R) \otimes_R R[[\mathbb{Z}_p^\times]],
$$

where $R[[\mathbb{Z}_p^\times]] = \varprojlim_r R[(\mathbb{Z}/p^r)^\times]$ is the Iwasawa algebra of $\mathbb{Z}_p^\times$ with R-coefficients.

We can interpret $R[[\mathbb{Z}_p^\times]]$ as the dual space of the space of continuous R-valued functions on $\mathbb{Z}_p^\times$. For $\mu \in R[[\mathbb{Z}_p^\times]]$ and f a continuous function, we write this pairing as $(\mu, f) \mapsto \int_{\mathbb{Z}_p^\times} f(x)\mathrm{d}\mu(x)$.

Proposition 6.3. *For j an integer with $0 \le j \le k$, and $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ a finite-order character of conductor p^r with $r \ge 1$, we have*

$$\int_{\mathbb{Z}_p^\times} x^j \chi(x)\,\mathrm{d}\,{}_c\Phi_{\mathfrak{n},a}^{k,\infty}(x) = \frac{1}{(a-a^\sigma)^j j! \binom{k}{j}^2} (U_p)_*^{-r} e'_{\mathrm{ord}} \sum_{t \in (\mathbb{Z}/p^r)^\times} \chi(t) {}_c\Xi^{k,j}_{p^r,\mathfrak{n},at}$$

as elements of $L(\chi) \otimes_R \mathcal{L}_k^{\mathrm{ord}}(\mathfrak{n}, R)$. For χ trivial we have

$$\int_{\mathbb{Z}_p^\times} x^j \,\mathrm{d}\,{}_c\Phi_{\mathfrak{n},a}^{k,\infty}(x) = \frac{1}{(a-a^\sigma)^j j! \binom{k}{j}^2} (1 - p^j (U_p)_*^{-1}) e'_{\mathrm{ord}} {}_c\Xi^{k,j}_{1,\mathfrak{n},a}.$$

Proof. For $j = 0$ this is immediate from the definition of ${}_c\Phi_{\mathfrak{n},a}^{k,\infty}$ (with the Euler factor in the case of trivial χ arising from the fact that the norm of $(U_p)_*^{-1}\,{}_c\Phi_{\mathfrak{n},a}^{k,1}$ is not ${}_c\Phi_{\mathfrak{n},a}^{k,0}$ but $(1 - (U_p)_*^{-1})\,{}_c\Phi_{\mathfrak{n},a}^{k,0}$, by the base case of Theorem 3.13).

The case $j \ge 1$ is more involved. It suffices to show the equality modulo p^h for arbitrarily large h. Modulo p^h with $h \ge r$, we have

$$\begin{aligned}
&(a-a^\sigma)^j j! \binom{k}{j}^2 \int_{\mathbb{Z}_p^\times} x^j \chi(x)\,\mathrm{d}\,{}_c\Phi_{\mathfrak{n},a}^{k,\infty}(x) \\
&= (a-a^\sigma)^j j! \binom{k}{j}^2 (U_p)_*^{-h} e'_{\mathrm{ord}} \sum_{t \in (\mathbb{Z}/p^h)^\times} t^j \chi(t) {}_c\Xi^{k,0}_{p^h,\mathfrak{n},at} \qquad \text{(definition of } {}_c\Phi_{\mathfrak{n},a}^{k,\infty}) \\
&= (U_p)_*^{-h} e'_{\mathrm{ord}} \sum_{t \in (\mathbb{Z}/p^h)^\times} \chi(t) {}_c\Xi^{k,j}_{p^h\mathfrak{n},at} \qquad \text{(Proposition 5.6)} \\
&= (U_p)_*^{-h} e'_{\mathrm{ord}} \sum_{t \in (\mathbb{Z}/p^r)^\times} \chi(t) \Bigg(\sum_{\substack{s \in (\mathbb{Z}/p^h)^\times \\ s = t \bmod p^r}} {}_c\Xi^{k,j}_{p^h,\mathfrak{n},as} \Bigg).
\end{aligned}$$

The bracketed term is $(U_p)_*^{h-r}\,{}_c\Xi^{k,j}_{p^r,\mathfrak{n},at}$ if $r \ge 1$, while for $r = 0$ it is $(U_p)_*^h (1 - p^j (U_p)_*^{-1}) {}_c\Xi^{k,j}_{1,\mathfrak{n},a}$, by the same argument as the proof of Theorem 3.13. □

Now suppose Ψ is a Bianchi modular eigenform of parallel weight (k, k) and level $U_{F,1}(\mathfrak{n})$. Recall that if the Hecke eigenvalues of Ψ lie in a number field E, we attached an element

$$\phi_\Psi^* = \jmath^*(\omega_\Psi)/\Omega_\Psi \in \mathrm{H}^1_{\mathrm{c}}(Y_{F,1}^*(\mathfrak{n}), V_{kk}(\mathcal{O}_E)),$$

well-defined up to p-adically integral elements of $E^\times$. Enlarging L if necessary, we fix an embedding $E \hookrightarrow L$, and regard ϕ_Ψ^* as an element of $\mathrm{H}^1_{\mathrm{c}}(Y_{F,1}^*(\mathfrak{n}), V_{kk}(R))$.

Assumption 1: We shall assume that the Bianchi modular eigenform Ψ is *ordinary* with respect to this embedding, i.e. that the U_p-eigenvalue of Ψ lies in $R^\times$.

Since the adjoint of $(U_p)_*$ is $(U_p)^*$, this assumption implies that the linear functional on $\mathcal{L}_k(\mathfrak{n}, R)$ given by pairing with ϕ_Ψ^* factors through projection to the $(U_p)_*$-ordinary part.

We also need to fix a value of a, which must generate the quotient $\frac{\mathcal{O}_F \otimes \mathbb{Z}_p}{\mathbb{Z}_p}$. It suffices to take $a = \frac{1+\sqrt{-D}}{2}$ if $D = -1 \bmod 4$, and $a = \frac{\sqrt{-D}}{2}$ if $D = 0 \bmod 4$; then we have $\mathcal{O}_F = \mathbb{Z} + \mathbb{Z}a$, and $a - a^\sigma = \sqrt{-D}$.

Definition 6.4. Define the *p-adic Asai L-function* ${}_cL_p^{\mathrm{As}}(\Psi) \in R[[\mathbb{Z}_p^\times]]$ to be

$${}_cL_p^{\mathrm{As}}(\Psi) := \left\langle \phi_\Psi^*, {}_c\Phi_{\mathfrak{n},a}^{k,\infty} \right\rangle$$

where $\langle -, - \rangle$ denotes the (perfect) Poincaré duality pairing

$$\mathrm{H}^1_{\mathrm{c}}(Y^*_{F,1}(\mathfrak{n}), V_{kk}(R)) \times \frac{\mathrm{H}^2(Y^*_{F,1}(\mathfrak{n}), T_{kk}(R))}{\text{(torsion)}} \longrightarrow R. \tag{4}$$

Remark: If we relax the assumption that Ψ be ordinary, and let $h = v_p(\lambda_p(\Psi))$ where $\lambda_p(\Psi)$ is the U_p-eigenvalue (and the valuation is normalised such that $v_p(p) = 1$), then we can still make sense of ${}_cL_p^{\mathrm{As}}(\Psi)$ as long as $h < 1$; however, it is no longer a measure, but a distribution of order h. This can be refined to $h < 1 + k$ using the same techniques as in [LZ16]. However, if $k = 0$ and $h \geq 1$ (as in the case of an eigenform associated to an elliptic curve supersingular at the primes above p) then we are stuck.

Proposition 6.5. *The class ${}_cL_p^{\mathrm{As}}(\Psi)$ is invariant under translation by $[-1] \in \mathbb{Z}_p^\times$.*

Proof. This follows from Proposition 3.10, since $\left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right)_*$ acts trivially on ω_Ψ (and thus on ϕ^*_Ψ). ☐

If we interpret $R[[\mathbb{Z}_p^\times]]$ as the algebra of R-valued rigid-analytic functions on the "weight space" $\mathcal{W} = \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ parametrising characters of $\mathbb{Z}_p^\times$, then this proposition shows that ${}_cL_p^{\mathrm{As}}(\Psi)$ vanishes identically on the subspace $\mathcal{W}^- \subset \mathcal{W}$ parametrising odd characters.

We close this section by giving notation that will be useful when stating the interpolation properties of $L_p^{\mathrm{As}}(\Psi)$.

Notation: Let χ be a Dirichlet character of conductor p^r for some $r \geq 0$, and let j be any integer. We write

$${}_cL_p^{\mathrm{As}}(\Psi, \chi, j) := \int_{\mathbb{Z}_p^\times} \chi(x) x^j \,\mathrm{d}\, {}_cL_p^{\mathrm{As}}(\Psi)(x).$$

6.2. Getting rid of c

Proposition 6.6. *Suppose that the nebentypus character $\varepsilon_\Psi : (\mathcal{O}_F/\mathfrak{n})^\times \to R^\times$ of Ψ has non-trivial restriction to $(\mathbb{Z}/N\mathbb{Z})^\times$, and moreover this restriction does not have p-power conductor. Then there exists a measure $L_p^{\mathrm{As}}(\Psi) \in L \otimes_R R[[\mathbb{Z}_p^\times]]$ such that*

$${}_cL_p^{\mathrm{As}}(\Psi) = (c^2 - c^{-2k}\varepsilon_\Psi(c)[c]^2) L_p^{\mathrm{As}}(\Psi)$$

for all valid integers c.

Proof. Some bookkeeping starting from (1) shows that if c, d are two integers > 1, both coprime to $6Np$, then the element

$$(d^2 - d^{-2k}[d]^2 \varepsilon_\Psi(d)) \cdot {}_cL_p^{\mathrm{As}}(\Psi)$$

is symmetric in c and d. Moreover, we can choose d in such a way that $(d^2 - d^{-2k}[d]^2\varepsilon_\Psi(d))$ is a unit in $L \otimes_R R[[\mathbb{Z}_p^\times]]$. So if we define

$$L_p^{\mathrm{As}}(\Psi) = (d^2 - d^{-2k}\varepsilon_\Psi(d)[d]^2)^{-1} {}_dL_p^{\mathrm{As}}(\Psi),$$

then this is independent of the choice of d and it has the required properties. ☐

If the restriction $\varepsilon_{\Psi,\mathbb{Q}}$ of ε_Ψ has p-power conductor, then the quotient $L_p^{\mathrm{As}}(\Psi)$ is well-defined as an element of the fraction ring of $R[[\mathbb{Z}_p^\times]]$, i.e. as a meromorphic function on $\mathcal{W}$ with coefficients in L. (We shall refer to such elements as *pseudo-measures*.) The only points of $\mathcal{W}$ at which $L_p^{\mathrm{As}}(\Psi)$ may have poles are those corresponding to characters of the form $z \mapsto z^{k+1}\nu(z)$, where $\nu^2 = \varepsilon_{\Psi,\mathbb{Q}}^{-1}$.

Remark: Note that if $p = 1 \bmod 4$ and $\varepsilon_{\Psi,\mathbb{Q}}(\rho) = (-1)^k$, where ρ is either of the square roots of -1 in $\mathbb{Z}_p$, then both of the characters at which L_p^{As} could have a pole actually lie in $\mathcal{W}^-$, so we see immediately that L_p^{As} is a measure.

In the remaining cases, where one or both potential poles are in $\mathcal{W}^+$, we suspect that these potential poles are genuine poles if and only if the corresponding complex-analytic Asai L-functions have poles (which can only occur if Ψ is either of CM type, or a twist of a base-change form). However, we have not proved this.

7. Interpolation of critical L-values

In this section, we want to show that the values of the Asai p-adic L-function at suitable locally-algebraic characters are equal to special values of the complex L-function.

7.1. Twisting maps We let $S_{kk}(U^*_{F,1}(\mathfrak{n}))$ denote the space of automorphic forms of weight (k, k) for the group $\mathrm{GL}_2^*(\mathbb{A}_F)$. These correspond to automorphic representations of the group G^* defined in Remark 2.2, and are defined in the same way as above, with the group $\mathrm{GL}_2^*(\mathbb{A}_F)$ in place of $\mathrm{GL}_2(\mathbb{A}_F)$; that is, they are functions

$$\mathrm{GL}_2^*(F)_+\backslash \mathrm{GL}_2^*(\mathbb{A}_F)/U^*_{F,1}(\mathfrak{n}) \to V_{2k+2}(\mathbb{C})$$

transforming appropriately under $\mathbb{R}_{>0} \cdot \mathrm{SU}_2(\mathbb{C})$, and with suitable harmonicity and growth conditions. Since $Y^*_{F,1}(\mathfrak{n})$ is connected, an element $\mathcal{F} \in S_{kk}(U^*_{F,1}(\mathfrak{n}))$ is uniquely determined by its Fourier–Whittaker coefficients $W_f(\zeta, \mathcal{F})$ for $\zeta \in F^\times$, which are zero unless $\zeta \in \mathcal{D}^{-1}$.

Pullback via $\jmath$ gives a map $\jmath^* : S_{kk}(U_{F,1}(\mathfrak{n})) \to S_{kk}(U^*_{F,1}(\mathfrak{n}))$, whose image is contained in the invariants for the action of the finite group $\begin{pmatrix} \mathcal{O}_F^\times & \\ & 1 \end{pmatrix}$.

Lemma 7.1. *Let $\mathcal{F} \in S_{kk}(U^*_{F,1}(\mathfrak{n}))$, χ a Dirichlet character of conductor m, and $a \in \mathcal{O}_F/m\mathcal{O}_F$. Then the function*

$$R_{a,\chi}\mathcal{F} = \sum_{t \in (\mathbb{Z}/m)^\times} \chi(t) \kappa^*_{at/m}(\mathcal{F})$$

is in $S_{kk}(U^*_{F,1}(m^2\mathfrak{n}))$, and its Fourier–Whittaker coefficients for $\zeta \in \mathcal{D}^{-1}$ are given by

$$W_f(\zeta, R_{a,\chi}\mathcal{F}) = G(\chi)\, \bar{\chi}(\mathrm{tr}_{F/\mathbb{Q}}\, a\zeta)\, W_f(\zeta, \mathcal{F}),$$

where $G(\chi) := \sum_{t\in(\mathbb{Z}/m)^\times} \chi(t) e^{2\pi i t/m}$ is the Gauss sum of χ.

Proof. That $R_{a,\chi}\mathcal{F}$ is an automorphic form of level $U^*_{F,1}(m^2\mathfrak{n})$ is clear. So it suffices to compute its Fourier–Whittaker coefficients. We have

$$\begin{aligned} W_f(\zeta, R_{a,\chi}\mathcal{F}) &= W_f(\zeta, \mathcal{F}) \sum_{t \in (\mathbb{Z}/m)^\times} \chi(t) e_F(\zeta a t/m) \\ &= W_f(\zeta, \mathcal{F}) \sum_{t \in (\mathbb{Z}/m)^\times} \chi(t) e^{2\pi i t\, \mathrm{tr}(a\zeta)/m}. \end{aligned}$$

This is 0 unless the integer $\mathrm{tr}(a\zeta)$ is a unit modulo m, in which case it is $\chi(\mathrm{tr}(a\zeta))^{-1} G(\chi)$, as required. ☐

7.2. An integral formula for the Asai L-function In this section, we describe an integral formula for the Asai L-function of a Bianchi eigenform twisted by a Dirichlet character χ. This is a generalisation of the work of Ghate in [Gha99] (who considers the case where χ is trivial), and we shall prove our theorem by reduction to his setting using the twisting maps $R_{a,\chi}$.

Let $0 \leq j \leq k$, and define

$$I^j_{\Psi,b,m} := \left\langle \phi^*_\Psi,\ (\kappa_{b/m})_*\iota_* F^{(2k-2j+2)}_{1/m^2N}(\tau)\,\mathrm{d}z^{\otimes 2k-2j}\,\mathrm{d}\tau \right\rangle,$$

where $\langle -,- \rangle$ denotes the pairing of equation (4), $\phi^*_\Psi = \jmath^*\phi_\Psi$ as before, and we view the Eisenstein class as an element of the Betti cohomology (with complex coefficients) using the standard comparison isomorphism.

Theorem 7.2. *Let χ be a Dirichlet character of odd conductor m, and let $0 \leq j \leq k$. Let $a \in \mathcal{O}_F$ be the value we chose in the remarks before Definition 6.4 (so that $a - a^\sigma = \sqrt{-D}$). Then*

$$\sum_{t\in(\mathbb{Z}/m\mathbb{Z})^\times} \chi(t) I^j_{\Psi,at,m} = \begin{cases} \frac{C'(k,j)G(\chi)}{(m^2N)^{2k-2j}\Omega_\Psi} L^{\mathrm{As}}(\Psi,\overline{\chi},j+1) & \text{if } (-1)^j\chi(-1) = 1,\\ 0 & \text{if } (-1)^j\chi(-1) = -1,\end{cases}$$

where

$$C'(k,j) = \binom{k}{j}^2 \frac{(k-1)!^2}{(2k-1)!(k-1-j)!}\frac{i^{1+j+2k}(\sqrt{-D})^{j+1}}{\pi^{j+1}2^{2k-j+4}}.$$

We begin by explaining how to reduce the theorem to the case $m = 1$. Note that the definition of the Asai L-series depends only on the pullback $\jmath^*\Psi$, and in fact makes sense for any $\mathcal{F} \in S_{kk}(U^*_{F,1}(\mathfrak{n}))$, whether or not it is in the image of $\jmath^*$, as long as it is an eigenvector for the operators $\langle x \rangle$ for $x \in (\mathbb{Z}/N\mathbb{Z})^\times$. If these operators act on $\mathcal{F}$ via the character ε, then we can define

$$L^{\mathrm{As}}(\mathcal{F},s) := L^{(N)}(\varepsilon_{\mathbb{Q}}, 2s-2k-2)\sum_{n\geq 1} W_f\left(n/\sqrt{-D},\mathcal{F}\right) n^{-s}.$$

One sees easily that if χ is a Dirichlet character of odd conductor, and a is the value we chose above (so that $a - a^\sigma = \sqrt{-D}$), then

$$L^{\mathrm{As}}(R_{a,\chi}\jmath^*\Psi, s) = G(\chi)\cdot L^{\mathrm{As}}(\Psi,\overline{\chi},s).$$

Proposition 7.3. *Let $\mathcal{F} \in S_{kk}(U^*_{F,1}(\mathfrak{n}))$, and let $N = \mathfrak{n}\cap\mathbb{Z}$. Then we have*

$$\left\langle \omega_{\mathcal{F}}, \iota_* F^{(2k-2j+2)}_{1/N}(\tau)\,\mathrm{d}z^{\otimes 2k-2j}\,\mathrm{d}\tau\right\rangle = \begin{cases} \frac{C'(k,j)}{N^{2k-2j}} L^{\mathrm{As}}(\mathcal{F},j+1) & \text{if } \begin{pmatrix} -1 & \\ & 1\end{pmatrix}^* \mathcal{F} = (-1)^j\mathcal{F},\\ 0 & \text{if } \begin{pmatrix} -1 & \\ & 1\end{pmatrix}^* \mathcal{F} = (-1)^{j+1}\mathcal{F}.\end{cases}$$

The proof of the proposition is identical (modulo minor differences of conventions) to the work of Ghate in [Gha99]. Applying this proposition to $R_{a,\chi}\jmath^*\Psi$ and dividing by Ω_Ψ proves Theorem 7.2, since $\begin{pmatrix} -1 & \\ & 1\end{pmatrix}^*$ acts on $R_{a,\chi}\jmath^*\Psi$ as $\chi(-1)$.

7.3. Interpolation of critical values We now use the integral formula of Theorem 7.2 to relate the values of the measure $L^{\mathrm{As}}_p(\Psi)$ to critical values of the classical Asai L-function.

Theorem 7.4. *Let p be an odd prime. Let Ψ be an ordinary Bianchi eigenform of weight (k,k) and level $U_{F,1}(\mathfrak{n})$, where all primes above p divide $\mathfrak{n}$, with U_p-eigenvalue λ_p. Let χ be a Dirichlet character of conductor p^r, and let $0 \leq j \leq k$.*

(a) *If $\chi(-1)(-1)^j = 1$, then*

$$L^{\mathrm{As}}_p(\Psi,\chi,j) = \frac{C(k,j)\mathcal{E}_p(\Psi,\chi,j)G(\chi)}{\Omega_\Psi}\cdot L^{\mathrm{As}}(\Psi,\overline{\chi},j+1),$$

where

$$C(k,j) := \frac{(-1)^{j+k}\sqrt{D}(2\pi i)^{2k-3j-1}(k-1)!^2}{2^{2k-j+2}(2k-1)!(k-1-j)!}\pi, \quad \mathcal{E}_p(\Psi,\chi,j) := \begin{cases}\left(1-\frac{p^j}{\lambda_p}\right) & \text{if } r=0,\\ \left(p^j\lambda_p^{-1}\right)^r & \text{if } r>0.\end{cases}$$

(b) *If* $\chi(-1)(-1)^j = -1$, *then*

$$L_p^{\mathrm{As}}(\Psi,\chi,j) = 0.$$

Proof. For convenience, let $e[r]$ denote the operator $(U_p^{-r})_* e'_{\mathrm{ord}}$ if $r \geq 1$, and $(1-p^j(U_p^{-1})_*)e'_{\mathrm{ord}}$ if $j=0$. By the definition of the measure and Proposition 6.3, we have

$$L_p^{\mathrm{As}}(\Psi,\chi,j) = \frac{1}{\sqrt{-D}^j j! \binom{k}{j}^2} \sum_{t\in(\mathbb{Z}/p^r)^\times} \chi(t) \left\langle \phi_\Psi^*, e[r]\Xi^{k,j}_{p^r,\mathfrak{n},at}\right\rangle.$$

We know that $(U_p)^*$ is the adjoint of $(U_p)_*$, and ϕ_Ψ^* is a $(U_p)^*$ eigenvector with unit eigenvalue λ_p; thus the adjoint of $e[r]$ acts on ϕ_Ψ^* as $p^{-jr}\mathcal{E}_p(\Psi,\chi,j)$, so we have

$$L_p^{\mathrm{As}}(\Psi,\chi,j) = \frac{\mathcal{E}_p(\Psi,\chi,j)}{p^{jr}\sqrt{-D}^j j! \binom{k}{j}^2} \sum_{t\in(\mathbb{Z}/p^r)^\times} \chi(t) \left\langle \phi_\Psi^*, \Xi^{k,j}_{p^r,\mathfrak{n},at}\right\rangle.$$

Now, by Lemma 5.4, we have $\Xi^{k,j}_{p^r,\mathfrak{n},at} = p^{jr}(\kappa_{at/p^r})_*\iota_*\left(\mathrm{Eis}^{2k-2j}_{p^{2r}N}\right)$, and hence

$$\left\langle \phi_\Psi^*, \Xi_{p^r,\mathfrak{n},at}\right\rangle = p^{jr}\left\langle \iota^*\kappa^*_{at/p^r}(\phi_\Psi^*), \mathrm{Eis}^{2k-2j}_{p^{2r}N}\right\rangle,$$

where the first cup product is at the level of $\Gamma^*_{F,1}(\mathfrak{n})\backslash\mathcal{H}_3$, and the second cup product is at the level of $\Gamma_1(p^{2r}N)\backslash\mathcal{H}$. Now work at the level of complex coefficients. We know that

$$\begin{aligned}\mathrm{Eis}^{2k-2j}_{p^{2r}N} &= \mathrm{Eis}^{2k-2j}_{p^{2r}N} \\ &= -(2\pi i)^{(2k-2j+1)}(p^{2r}N)^{2k-2j}F^{(2k-2j+2)}_{1/p^{2r}N}(\tau)\,\mathrm{d}z^{\otimes 2k-2j}\,\mathrm{d}\tau,\end{aligned}$$

Accordingly, we see that

$$L_p^{\mathrm{As}}(\Psi,\chi,j) = -\frac{(2\pi i)^{2k-2j+1}(p^{2r}N)^{2k-2j}\mathcal{E}_p(\Psi,\chi,j)}{\sqrt{-D}^j j! \binom{k}{j}^2} \sum_{t\in(\mathbb{Z}/p^r)^\times} \chi(t) I^j_{\Psi,at,p^r},$$

where $I^j_{\Psi,b,m}$ is as defined in the previous section. Using Theorem 7.2, we see that this expression vanishes unless $\chi(-1)(-1)^j = 1$, in which case we have

$$\begin{aligned}L_p^{\mathrm{As}}(\Psi,\chi,j) &= -\frac{(2\pi i)^{2k-2j+1}\mathcal{E}_p(\Psi,\chi,j)}{\Omega_\Psi\sqrt{-D}^j j! \binom{k}{j}^2} \times C'(k,j)G(\chi)L^{\mathrm{As}}(\Psi,\overline{\chi},j+1) \\ &= \frac{C(k,j)\mathcal{E}_p(\Psi,\chi,j)G(\chi)}{\Omega_\Psi}\cdot L^{\mathrm{As}}(\Psi,\overline{\chi},j+1),\end{aligned}$$

which completes the proof of the theorem. ☐

As an immediate corollary, we get an identical interpolation formula for ${}_cL_p^{\mathrm{As}}(\Psi)$ with the additional factor $(c^2 - c^{2j-2k}\varepsilon_\Psi(c)\chi(c)^2)$.

Remark:

(i) The factor $\mathcal{E}_p(\Psi,\chi,j)$ is non-zero if $r \geq 1$. If $r = 0$ then this factor vanishes if and only if $k = 0$, Ψ is new at the primes above p, and $\varepsilon_\Psi(p) = 1$. In this case the p-adic L-function has an exceptional zero at the trivial character. For exceptional zeroes of the *standard* p-adic L-function of a Bianchi cusp form, a theory of $\mathcal{L}$-invariants was developed in [BSW17]; it would be interesting to investigate analogues of this for the Asai L-function.

(ii) The measure $L_p^{\mathrm{As}}(\Psi)$ depends on the choice of $\sqrt{-D}$ fixed at the start; indeed, this choice was used to pick a value of $a \in \mathcal{O}_F$, which in turn was used to construct the Asai–Eisenstein elements. This choice is further encoded by the appearance of $\sqrt{D} = i\sqrt{-D}$ in the interpolation formula. Choosing the other square root simply scales the measure by -1. The measure also depends on the choice of period Ω_Ψ, and again a different choice changes the measure up to a scalar.

(iii) If the Bianchi eigenform Ψ (or, more precisely, the automorphic representation it generates) is the base-change lift of an elliptic modular eigenform f of weight $k+2$ and character ε_f, then the complex Asai L-function factors as

$$L^{\mathrm{As}}(\Psi, \chi, s) = L(\mathrm{Sym}^2 f, \chi, s) L(\chi\varepsilon_f\varepsilon_F, s-k-1),$$

where ε_K is the quadratic character associated to K. Note that all three L-functions in the above formula have critical values at integer points $s = 1+j$ with $0 \le j \le k$ and $(-1)^j\chi(-1) = 1$. By a comparison of interpolating properties at these points, one can verify that if f is ordinary at p, then there is a corresponding factorisation of $L_p^{\mathrm{As}}(\Psi)$ as a product of a shifted p-adic Dirichlet L-function and Schmidt's p-adic L-function for $\mathrm{Sym}^2 f$.

This factorisation shows, in particular, that the possibility of poles of the p-adic Asai L-function is a genuine aspect of the situation, rather than a shortcoming of our method: if $\varepsilon_f\varepsilon_F = 1$, then one of these factors is the p-adic Riemann zeta function $\zeta_p(s-k-1)$, which has a simple pole at $s = k+2$. If f has CM by an imaginary quadratic field K (with $K \ne F$, so that $\Psi = \mathrm{BC}(f)$ is cuspidal), then there is a second abelian factor $L(\chi\varepsilon_f\varepsilon_K, s-k-1)$; this gives rise to examples where both of the zeros of the factor $c^2 - c^{-2k}\varepsilon_\Psi(c)[c]^2$ correspond to genuine poles of $L_p^{\mathrm{As}}(\Psi)$.

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